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# Another form of the reciprocity law of Dedekind sum

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## Abstract

We propose a certain formula of Dedekind sum by using two mutually distinct methods. The one is an elementary number theoretic method and the other is a geometric method. Moreover, we re-prove the reciprocity law by the formula.

## 1 Introduction

Let  $(\lambda, \sigma)$  be any pair of mutually prime natural numbers. Let

$$s(\sigma, \lambda) := \sum_{k=1}^{\lambda-1} \left( \left( \frac{k}{\lambda} \right) \right) \left( \left( \frac{k\sigma}{\lambda} \right) \right)$$

be the Dedekind sum, where  $((x)) = x - [x] - 1/2$  for a non-integer  $x$  and  $((x)) = 0$  for an integer  $x$ . ( $[x]$  is the greatest integer not exceeding  $x$ .) This notion was introduced by Dedekind in 19-th century in order to describe the behavior under the action of the modular transformation of the Dedekind eta function  $\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau})$ , where  $\tau$  is a complex variable with  $\text{Im } \tau > 0$ . Dedekind also proved the beautiful reciprocity law

$$s(\sigma, \lambda) + s(\lambda, \sigma) = -\frac{1}{4} + \frac{1}{12} \left( \frac{\sigma}{\lambda} + \frac{\lambda}{\sigma} + \frac{1}{\lambda\sigma} \right). \quad (1)$$

Nowadays many proofs of (1) are known via number theoretic methods (e.g.[9] Chap. 2).

On the other hand, Dedekind sum historically appeared again in 20-th century in the field of topology related to the signature theorem, namely as the term of the equivariant L-class at an isolated fixed point of a group action to a four-dimensional manifold (e.g. [5]). Especially, among many other results, Hirzebruch-Zagier [5] proved (1) via this context and also by using quotient singularities, which we call the first geometric proof of the reciprocity law.

Now the aim of this paper is to present another geometric proof of (1) along this context. At the same time, we re-formulate (1) into a new form.

Assume  $1 \leq \sigma \leq \lambda - 1$ , and let  $\delta$  be the integer with  $\sigma\delta \equiv 1 \pmod{\lambda}$ ,  $1 \leq \delta \leq \lambda - 1$ .

Let

$$\frac{\lambda}{\sigma} = K_1 - \frac{1}{K_2 - \frac{1}{K_3 - \cdots - \frac{1}{K_r}}} \quad (2)$$

be the continued linear fraction of negative type. Then;

**Theorem 1.1**

$$s(\sigma, \lambda) = -\frac{r}{4} + \frac{1}{12} \left( \frac{\sigma + \delta}{\lambda} + \sum_{j=1}^r K_j \right). \quad (3)$$

**Proposition 1.2**     *The formula (3) implies the reciprocity law (1).*

We prove Theorem 1.1 by two mutually distinct ways. In the first proof in §2, we use the reciprocity law successively and also use the Matsumoto-Montesinos formula [7], which is proved by purely elementary number theoretic method (see also Remark 2.2). The proof of Proposition 1.2 is elementary, which is given in §3.

Therefore the formula (3) is, in some sense, “another form” of the reciprocity law.

The second proof of Theorem 1.1 given in §4 is independent of the reciprocity law, and is geometrical as follows; We construct a fibered complex surface as the resolution space of some cyclic quotient singularities, and compute its signature by two ways. The one is the use of the equivariant signature theorem, in which the Dedekind sum appears. The other is the use of the global signature theorem of Hirzebruch, and the calculation of the signature is reduced to that of the self-intersection numbers of the canonical bundle, which can be done by the formula in Ishida [6]. By comparing them, the proof is completed.

From Proposition 1.2 and the second proof of Theorem 1.1, we complete our geometric proof of the reciprocity law.

Note that the formula (3) is applied to the localization problem of the signature of fibered complex surfaces ([1]).

## 2 The first proof of Theorem 1.1

We consider the continued linear fraction (2), and set  $n_0 = \lambda, n_1 = \sigma$  and  $n_i = K_{i-1}n_{i-1} - n_{i-2}$  ( $2 \leq i \leq r$ ). For the proof of the following theorem, see [7] §4.

**Theorem 2.1** (*Matsumoto-Montesinos*)

$$\sum_{i=1}^r \frac{1}{n_{i-1}n_i} = \frac{\delta}{\lambda}.$$

**Remark 2.2** Although the proof of Matsumoto-Montesins ([7] §4) is done via elementary number theoretic way, their motivation of this formula comes from geometry, which relates the analysis of the local monodromy of degenerations of complex curves. Namely, the formula means the decomposition of the  $2\pi\delta/\lambda$  rotation to the composition of fractional Dehn twists, see [7] §8.

Now we prove Theorem 1.1. Since  $s(\lambda, \sigma) = s(-n_2, \sigma) = -s(n_2, \sigma)$  by  $\lambda = K_1\sigma - n_2$ , it follows from the reciprocity law that

$$s(\sigma, \lambda) = s(n_2, \sigma) - \frac{1}{4} + \frac{1}{12} \left( \frac{\sigma}{\lambda} - \frac{n_2}{\sigma} + K_1 + \frac{1}{n_0 n_1} \right).$$

By  $\sigma = K_2 n_2 - n_3$  and also the reciprocity law, we have

$$s(\sigma, \lambda) = s(n_3, n_2) - \frac{1}{2} + \frac{1}{12} \left( \frac{\sigma}{\lambda} - \frac{n_3}{n_2} + K_1 + K_2 + \frac{1}{n_0 n_1} + \frac{1}{n_1 n_2} \right).$$

By using the same argument inductively, we have

$$s(\sigma, \lambda) = s(1, K_r) - \frac{r-1}{4} + \frac{1}{12} \left( \frac{\sigma}{\lambda} - \frac{1}{K_r} + \sum_{i=1}^{r-1} K_i + \sum_{i=1}^{r-1} \frac{1}{n_{i-1} n_i} \right).$$

Therefore, by Theorem 2.1 and  $s(1, K_r) = (K_r - 1)(K_r - 2)/(12K_r)$ , we have the assertion.

### 3 Proof of Proposition 1.2

Let  $m_2$  be the integer with  $n_2 m_2 \equiv 1 \pmod{\sigma}$ ,  $1 \leq m_2 \leq \sigma - 1$ . Since  $s(\lambda, \sigma) = s(-n_2, \sigma) = -s(n_2, \sigma)$  and

$$\frac{\sigma}{n_2} = K_2 - \frac{1}{K_3 - \frac{1}{K_4 - \cdots - \frac{1}{K_r}}}$$

it follows from Theorem 1.1 that

$$\begin{aligned} s(\sigma, \lambda) + s(\lambda, \sigma) &= -\frac{r}{4} + \frac{1}{12} \left( \frac{\sigma + \delta}{\lambda} + \sum_{j=1}^r K_j \right) - \left\{ -\frac{r-1}{4} + \frac{1}{12} \left( \frac{n_2 + m_2}{\sigma} + \sum_{j=2}^r K_j \right) \right\} \\ &= -\frac{1}{4} + \frac{1}{12} \cdot \frac{\sigma^2 + \lambda^2 + (\sigma\delta - \lambda m_2)}{\lambda\sigma}. \end{aligned}$$

From [6] Lemma 1.7, we have

$$\sigma\delta - \lambda m_2 = 1.$$

Therefore we complete the proof of Proposition 1.2.

## 4 The second proof of Theorem 1.1

First, we review some facts about the cyclic quotient singularity. Put  $\zeta = e^{2\pi i/\lambda}$ . We call a germ  $(X', o)$  a *cyclic quotient singularity of type  $(\sigma, \lambda)$*  if  $(X', o)$  is isomorphic to the germ at the origin of the quotient analytic space  $\mathbb{C}^2/G$  where  $G$  is the group of automorphisms of  $\mathbb{C}^2 = \{(z_1, z_2)\}$  generated by the map  $(z_1, z_2) \mapsto (\zeta z_1, \zeta^\sigma z_2)$ . The exceptional set of the minimal resolution  $\tilde{X} \rightarrow X'$  is a  $\mathbf{P}^1$  chain, so that the self intersection numbers of the components are  $-K_i$  ( $1 \leq i \leq r$ ) in the continued linear fraction (2). The following description of the difference of the self-intersection numbers of the canonical divisors  $K_{\tilde{X}}^2 - K_{X'}^2$  is due to Ishida [6].

**Proposition 4.1** ([6], Proposition 1.9, p. 43)

$$K_{\tilde{X}}^2 - K_{X'}^2 = 2(r+1) - \frac{\sigma + \delta + 2}{\lambda} - \sum_{i=1}^r K_i.$$

Next we review the terminology introduced by Nielsen [8]. (Although the original situation in [8] is topological, we present it in analytic situation.) For an analytic automorphism  $\phi: \Sigma_g \rightarrow \Sigma_g$  of a Riemann surface  $\Sigma_g$  of genus  $g$ , we call the smallest positive integer  $\Lambda$  satisfying  $\phi^\Lambda = \text{id}_{\Sigma_g}$  the *period*. For each point  $P$  on  $\Sigma_g$ , we denote by  $r_P$  the cardinality of the orbit of  $P$  under  $\phi$ , and set  $\lambda_P := \Lambda/r_P$ . Let  $\sigma_P$  be the smallest nonnegative integer such that  $\phi^{r_P}$  is the rotation of angle  $2\pi\sigma_P/\lambda_P$  near  $P$ . Namely, by a suitable coordinate  $t$  of a neighborhood  $U$  of  $P$ , the restriction map  $\phi^{r_P}|_U$  is written by  $t \mapsto e^{2\pi\sigma_P/\lambda_P}t$ . Denote by  $\delta_P$  the smallest positive integer satisfying  $\sigma_P\delta_P \equiv 1 \pmod{\lambda_P}$  if  $\sigma_P \neq 0$ , and set  $\delta_P := 0$  if  $\sigma_P = 0$ . The rational number  $\delta_P/\lambda_P$  is called the *valency* of the orbit of  $P$ .

The valencies of all but a finite number of orbits are zero and the sum of all valencies is an integer. The set of the positive valencies is called the *total valency* of  $\phi$ . By Hurwitz formula, the genus  $g'$  of the quotient Riemann surface  $\Sigma_g/\langle\phi\rangle$  coincides with  $1 + (1/2\Lambda)\{2g - 2 - \sum_{i=1}^l(\lambda_i - 1)\}$ .

Conversely, there exists an analytic automorphism  $\phi: \Sigma_g \rightarrow \Sigma_g$  whose period and total valency are  $\Lambda$ ,  $\{\delta_i/\lambda_i\}_{(i=1, \dots, l)}$  ( $l \neq 1$ ), respectively, if  $\sum_{i=1}^l \delta_i/\lambda_i$  is an integer and the following Harvey conditions are satisfied ([3], see also [4]).

- (1)  $L = \text{lcm}(\lambda_1, \dots, \lambda_l) = \text{lcm}(\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_l)$  for all  $i$ , where  $\hat{\lambda}_i$  denotes the omission of  $\lambda_i$ .
- (2)  $L$  divides  $\Lambda$ , and if  $g' = 0$ , then  $L = \Lambda$  and  $l \geq 3$ .
- (3) If  $2|L$ , the number of  $\lambda_1, \dots, \lambda_l$  which are divisible by the maximal power of 2 dividing  $L$  is even.

We express the total valency as the formal symbol  $\sigma_1/\lambda_1 + \cdots + \sigma_l/\lambda_l$  as in [2] §1.

**Lemma 4.2** *Let  $\lambda$  be a positive integer which is greater than or equal to two. If we choose suitable integers  $l$  and  $g$ , there exist a Riemann surface  $C_g$  of genus  $g$  and an automorphism  $\phi_{l,\lambda}: C_g \rightarrow C_g$  with the period  $\lambda$  and the total valency  $\underbrace{1/\lambda + \cdots + 1/\lambda}_l$ .*

**PROOF** Let  $l$  ( $> 2$ ) be an integer so that  $l(\lambda - 1)$  is even. We choose an integer  $g'$  satisfying the Hurwitz formula, i.e.,  $2g - 2 = \lambda(2g' - 2) + l(\lambda - 1)$ . Then the Harvey conditions (1) and (2) are satisfied. Moreover, if 2 divides  $\lambda$ , then the above integer  $l$  is even. Then the Harvey condition (3) is also clear. q.e.d.

Let  $\{P_i\}$  be the set of the fixed points of  $\phi_{l,\lambda}$ . Denote a neighborhood of  $P_i$  and its parameter centered at  $P_i$  by  $U_i$  and  $t_i$ , respectively. We set  $M = \Sigma_g \times \Sigma_g$ ,  $\Phi = \phi_{l,\lambda}^\sigma \times \phi_{l,\lambda}: M \rightarrow M$  and  $M' = M/\langle \Phi \rangle$ . Let  $\pi': M' \rightarrow \Sigma_{g'}$  be the natural morphism induced from the second projection  $M \rightarrow \Sigma_g$ . Let  $f: \widetilde{M} \rightarrow M'$  be the minimal resolution of the singularities on  $M'$ , and  $\widetilde{\pi}: \widetilde{M} \rightarrow \Sigma_{g'}$  be the natural morphism induced from  $\pi'$ .

Since  $\Phi|_{U_i \times U_j}(t_i, t_j) = (e^{2\pi\sigma i/\lambda} t_i, e^{2\pi i/\lambda} t_j)$ , there exist  $l$  cyclic quotient singularities of type  $(\sigma, \lambda)$  on each singular fiber of  $\pi'$ . From the construction, the irreducible decomposition of each singular fiber of  $\widetilde{\pi}$  is written by  $\lambda\Sigma_{g'} + \sum_{j=1}^l D_j$ , where  $\Sigma_{g'}$  is a Riemann surface of genus  $g'$  and  $D_j$  is the exceptional set of the minimal resolution of each singular point so that  $\Sigma_{g'} D_j = 1$  and  $D_j D_k = 0$  ( $j \neq k$ ).

We calculate some invariants of  $M$ ,  $M'$  and  $\widetilde{M}$ . Since  $M$  is a direct product, the second Chern number  $c_2(M)$  of  $M$  coincides with  $(2 - 2g)^2$ . The Euler number of each singular fiber of  $\widetilde{\pi}$  coincides with  $2 - 2g' + rl$ . Therefore

$$c_2(\widetilde{M}) = (2 - 2g)(2 - 2g') + l(lr + 2(g - g')). \quad (4)$$

Set  $D = K_{\widetilde{M}} - f^*K_{M'}$ . Let  $\text{Sign}M$  (resp.  $\text{Sign}M'$ , resp.  $\text{Sign}\widetilde{M}$ ) be the signature of the intersection form on the 2-homology of  $M$  (resp.  $M'$ , resp.  $\widetilde{M}$ ). (Note that, since  $M'$  is a rationally homology manifold,  $\text{Sign}M'$  is also well-defined.) Then the Hirzebruch signature theorem says that  $\text{Sign}\widetilde{M} = (1/3)(K_{\widetilde{M}}^2 - 2c_2(\widetilde{M}))$ . Since the number of the components of the exceptional set of  $f$  coincides with  $rl^2$ , we have  $\text{Sign}M' = \text{Sign}\widetilde{M} + rl^2$ . Therefore

$$\text{Sign}M' = \frac{1}{3}(K_{M'}^2 + D^2 - 2c_2(\widetilde{M})) + rl^2.$$

From  $\lambda K_{M'}^2 - K_M^2 = 0$ ,  $c_2(M) = (2 - 2g)^2$  and (4), we obtain

$$\begin{aligned} \text{Sign}M' - \frac{1}{\lambda}\text{Sign}M &= \frac{1}{3}(K_{M'}^2 + D^2 - 2c_2(\widetilde{M})) + rl^2 - \frac{1}{3\lambda}(K_M^2 - 2c_2(M)) \\ &= \frac{1}{3}D^2 + \frac{1}{3}rl^2 - \frac{2}{3} \left\{ (2 - 2g) \left( 2 - 2g' - \frac{1}{\lambda}(2 - 2g) \right) + 2l(g - g') \right\}. \end{aligned}$$

By the Hurwitz formula, we have

$$\text{Sign}M' - \frac{1}{\lambda}\text{Sign}M = \frac{1}{3}D^2 + \frac{1}{3}rl^2 - \frac{2l^2}{3}\left(\frac{\lambda-1}{\lambda}\right). \quad (5)$$

On the other hand, the defect formula in Hirzebruch-Zagier [5], p.181, Theorem 2 implies that

$$\text{Sign}M' - \frac{1}{\lambda}\text{Sign}M = -4l^2s(\sigma, \lambda). \quad (6)$$

Since  $M'$  has  $l^2$  cyclic quotient singularity, we have  $D^2 = l^2(K_{\tilde{X}}^2 - K_{X'}^2)$ . Thus, combining the above (5), (6) and Proposition 4.1, we complete our geometric proof of Theorem 1.1.

## References

- [1] T. Ashikaga, Local signature defect of fibered complex surfaces via monodormy and stable reduction, Preprint (2007).
- [2] T. Ashikaga and M. Ishizaka, Classification of degenerations of curves of genus 3 via Matsumoto-Montesinos' theorem, *Tohoku Math. J.* **53** (2001), 369–394.
- [3] J. Harvey, Cyclic groups of automorphisms of compact Riemann surfaces, *Quart. J. Math. Oxford Ser. (2)* **17** (1966), 86–97.
- [4] S. Hirose, Presentations of periodic maps on oriented closed surface of genera up to 4, preprint.
- [5] F. Hirzebruch and D. B. Zagier, *The Atiyah-Singer index theorem and elementary number theory*, Publish or Perish, Wilmington, 1974.
- [6] M. Ishida, The algebraic surface defined by a sum of four monomials, *Comment. Math. Univ. Sancti Pauli* **40** (1991), 39–51.
- [7] Y. Matsumoto and J. M. Montesinos-Amilibia, Pseudo-periodic maps and degeneration of Riemann surfaces, I, II, preprints, Univ. of Tokyo and Univ. Complutense de Madrid, (Old version:1991/1992, New version:2007).
- [8] J. Nielsen, Die Structur periodischer Transformationen von Flächen, *Mat.-Fys. Medd. Danske Vid. Selsk.* **15** (1937). English translation: in *Collected Papers 2*, Birkhäuser, 1986.
- [9] H. Rademacher and E. Grosswald, *Dedekind Sums*, The Carus Math. Mono. **16** (1972).

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