# Time Operators of a Hamiltonian with Purely Discrete Spectrum

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**Abstract**

We develop a mathematical theory of time operators of a Hamiltonian with purely discrete spectrum. The main results include boundedness, unboundedness and spectral properties of them. In addition, possible connections of a time operator of $H$ with regular perturbation theory are discussed.

**Keywords**. canonical commutation relation, Hamiltonian, time operator, time-energy uncertainty relation, phase operator, spectrum, regular perturbation theory.

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## 1 Introduction

This paper is concerned with mathematical theory of time operators in quantum mechanics [2, 3, 4, 6, 10]. There are some types of time operators which are not necessarily equivalent each other. For the reader's convenience, we first recall the definitions of them with comments.

Let $\mathcal{H}$ be a complex Hilbert space. We denote the inner product and the norm of $\mathcal{H}$ by $\langle \cdot, \cdot \rangle$ (antilinear in the first variable) and $\| \cdot \|$ respectively. For a linear operator $A$ on a Hilbert space, $D(A)$ denotes the domain of $A$.

Let $H$ be a self-adjoint operator on $\mathcal{H}$ and $T$ be a symmetric operator on $\mathcal{H}$.

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The operator $T$ is called a *time operator* of $H$ if there is a dense subspace $\mathcal{D}$ of $\mathcal{H}$ such that $\mathcal{D} \subset D(TH) \cap D(HT)$ and the canonical commutation relation (CCR)

$$[T, H] := (TH - HT) = i \tag{1.1}$$

holds on $\mathcal{D}$ (i.e., $[T, H]\psi = i\psi,\ \forall\psi \in \mathcal{D}$), where $i$ is the imaginary unit. In this case, $T$ is called a *canonical conjugate* to $H$ too.

The name "time operator" for the operator $T$ comes from the quantum mechanical context where $H$ is taken to be the Hamiltonian of a quantum system and the heuristic classical-quantum correspondence based on the structure that, in the classical relativistic mechanics, time is a canonical conjugate variable to energy in each Lorentz frame of coordinates. Note also that the dimension of $T$ is that of time if the dimension of $H$ is that of energy in the original unit system where the right hand side of (1.1) takes the form $i\hbar$ with $\hbar$ being the Planck constant $h$ divided by $2\pi$. We remark, however, that this name is somewhat misleading, because, in the framework of the standard quantum mechanics, time is *not an observable*, but just a parameter assigning the time when a quantum event is observed. But we follow the convention in this respect. By the same reason as just remarked, $T$ is not necessarily (essentially) self-adjoint. But this does not mean that it is "unphysical" [2, 10].

From a representation theoretic point of view, the pair $(T, H)$ is a symmetric representation of the CCR with one degree of freedom. But one should remember that, as for this original form of representation of the CCR, the von Neumann uniqueness theorem ([11], [12, Theorem VIII.14]) does not necessarily hold. In other words, $(T, H)$ is not necessarily unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom. Indeed, for example, it is obvious that, if $T$ or $H$ is bounded below or bounded above, then $(T, H)$ cannot be unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom.

A classification of pairs $(T, H)$ with $T$ being a bounded self-adjoint operator has been done by G. Dorfmeister and J. Dorfmeister [7]. We remark, however, that the class discussed in [7] does not cover the pairs $(T, H)$ considered in this paper, because the paper [7] treats only the case where $T$ is bounded and absolutely continuous.

A weak form of time operator is defined as follows. We say that a symmetric operator $T$ is a *weak time operator* of $H$ if there is a dense subspace $\mathcal{D}_{\mathrm{w}}$ of $\mathcal{H}$ such that $\mathcal{D}_{\mathrm{w}} \subset D(T) \cap D(H)$ and

$$\langle T\psi, H\phi\rangle - \langle H\psi, T\phi\rangle = \langle \psi, i\phi\rangle\,, \quad \psi, \phi \in \mathcal{D}_{\mathrm{w}},$$

i.e., $(T, H)$ satisfies the CCR in the sense of sesquilinear form on $\mathcal{D}_{\mathrm{w}}$. Obviously a time operator $T$ of $H$ is a weak time operator of $H$. But the converse is not true (it is easy to see, however, that, if $T$ is a weak time operator of $H$ and $\mathcal{D}_{\mathrm{w}} \subset D(TH) \cap D(HT)$, then $T$ is a time operator). An important aspect of a weak time operator $T$ of $H$ is that a *time-energy uncertainty relation* is naturally derived [2, Proposition 4.1]: for all unit vectors $\psi$ in $\mathcal{D}_{\mathrm{w}} \subset D(T) \cap D(H)$,

$$(\Delta T)_\psi (\Delta H)_\psi \geq \frac{1}{2},$$

where, for a linear operator $A$ on $\mathcal{H}$ and $\phi \in D(A)$ with $\|\phi\| = 1$,

$$(\Delta A)_\phi := \|(A - \langle \phi, A\phi \rangle)\phi\|,$$

called the *uncertainty of $A$ in the vector $\phi$*.

In contrast to the weak form of time operator, there is a strong form. We say that $T$ is a *strong time operator* of $H$ if, for all $t \in \mathbb{R}$, $e^{-itH} D(T) \subset D(T)$ and

$$Te^{-itH}\psi = e^{-itH}(T + t)\psi, \quad \psi \in D(T). \tag{1.2}$$

We call (1.2) the *weak Weyl relation* [2]. From a representation theoretic point of view, we call a pair $(T, H)$ obeying the weak Weyl relation a *weak Weyl representation* of the CCR. This type of representation of the CCR was extensively studied by Schmüdgen [15, 16]. It is shown that a strong time operator of $H$ is a time operator of $H$ [10]. But the converse is not true. In fact, the time operators considered in the present paper are not strong ones.

There is a generalized version of strong time operator [2]. We say that $T$ is a *generalized time operator* of $H$ if, for each $t \in \mathbb{R}$, there is a *bounded self-adjoint* operator $K(t)$ on $\mathcal{H}$ with $D(K(t)) = \mathcal{H}$, $e^{-itH} D(T) \subset D(T)$ and a *generalized weak Weyl relation* (GWWR)

$$Te^{-itH}\psi = e^{-itH}(T + K(t))\psi \quad (\forall \psi \in D(T)) \tag{1.3}$$

holds. In this case, the bounded operator-valued function $K(t)$ of $t \in \mathbb{R}$ is called the *commutation factor* of the GWWR under consideration.

We now come to the subject of the present paper. In his interesting paper [8], Galapon showed by an explicit construction that, for every self-adjoint operator $H$ (a Hamiltonian) on an abstract Hilbert space $\mathcal{H}$ which is *bounded below* and has purely discrete spectrum with some growth condition, there is a time operator $T_1$ on $\mathcal{H}$, which is a bounded self-adjoint operator under an additional condition (for the definition of $T_1$, see (2.12) below). To be definite, we call the operator $T_1$ introduced in [8] the *Galapon time operator*.

An important point of Galapon's work [8] is in that it disproved the long-standing belief or folklore among physicists that there is no self-adjoint operator canonically conjugate to a Hamiltonian which is bounded below (for a historical survey, see Introduction of [8]).

Motivated by work of Galapon [8], we investigate, in this paper, properties of time operators of a self-adjoint operator $H$ with purely discrete spectrum. In Section 2, we introduce a densely defined linear operator $T$ whose restriction to a subspace yields the Galapon time operator $T_1$ and prove basic properties of $T$ and $T_1$, in particular the closedness of $T$. It follows that, if $T$ is bounded, then $T$ is self-adjoint with $D(T) = \mathcal{H}$ and a time operator of $H$. We denote by $T^\#$ one of $T_1$, $T$ and $T^*$ (the adjoint of $T$). In Section 3, we discuss some general properties of $T^\#$. Moreover the reflection symmetry of the spectrum of $T^\#$ with respect to the imaginary axis is proved. Sections 4–6 are the main parts of this paper. In Section 4, we present a general criterion for $T$ to be bounded with $D(T) = \mathcal{H}$, while, in Section 5, we give a sufficient condition for $T$ to be unbounded. In Section 6, we present a necessary and sufficient condition for $T$ to be Hilbert-Schmidt. In Section 7, we show that, under some condition, the Galapon time operator is a generalized time operator of $H$, too. We also discuss non-differentiability of the commutation factor $K$ in the GWWR for $(T_1, H)$. In the last section, we consider a perturbation of $H$ by a symmetric operator and try to draw out physical meanings of $T_1$ and $K$ in the context of regular perturbation theory.

## 2 Time Operators

In this section, we recapitulate some basic aspects of the Galapon time operator in more apparent manner than in [8].

Let $\mathcal{H}$ be a complex Hilbert space and

$H$ be a self-adjoint operator on $\mathcal{H}$ which has the following properties (H.1) and (H.2):

(H.1) The spectrum of $H$, denoted $\sigma(H)$, is purely discrete with $\sigma(H) = \{E_n\}_{n=1}^{\infty}$, where each eigenvalue $E_n$ of $H$ is simple and

$$0 < E_n < E_{n+1}$$

for all $n \in \mathbb{N}$ (the set of positive integers).

(H.2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{{E_n}^2} < \infty.$

*Throughout the present paper we assume (H.1) and (H.2).*

**Remark 2.1** The positivity condition $E_n > 0$ for the eigenvalues of $H$ does not lose generality, because, if $H$ is bounded below, but not strictly positive, then one needs only to consider, instead of $H$, $\tilde{H} := H + c$ with a constant $c > -\inf \sigma(H)$, which is a strictly positive self-adjoint operator.

Property (H.2) implies that

$$E_n \to \infty \quad (n \to \infty). \tag{2.1}$$

Let $e_n$ be a normalized eigenvector of $H$ belonging to eigenvalue $E_n$:

$$He_n = E_n e_n, \quad n \in \mathbb{N}. \tag{2.2}$$

Then, by property (H.1), the set $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal system (C.O.N.S.) of $\mathcal{H}$.

**Lemma 2.1**

(i) *For all* $m \in \mathbb{N}$,

$$\sum_{n \neq m}^{\infty} \frac{1}{(E_n - E_m)^2} < \infty. \tag{2.3}$$

*In particular, for each* $m \in \mathbb{N}$,

$$\sum_{n \neq m}^{\infty} \frac{1}{E_n - E_m} e_n$$

*converges in* $\mathcal{H}$.

(ii) *For all $n \in \mathbb{N}$ and vectors $\psi$ in $\mathcal{H}$, the infinite series*

$$\sum_{m\neq n}^{\infty} \frac{\langle e_m, \psi\rangle}{E_n - E_m} \tag{2.4}$$

*absolutely converges.*

*Proof.* (i) By (2.1), we have

$$C_m := \sup_{n\neq m} \frac{1}{\left(1 - \frac{E_m}{E_n}\right)^2} < \infty. \tag{2.5}$$

Hence we have

$$\sum_{n\neq m}^{\infty} \frac{1}{|E_n - E_m|^2} \leq C_m \sum_{n\neq m}^{\infty} \frac{1}{E_n^2} < \infty.$$

(ii) By the Cauchy-Schwarz inequality, the Parseval equality and part (i), we have

$$\begin{aligned}\sum_{m\neq n}^{\infty} \left|\frac{\langle e_m, \psi\rangle}{E_n - E_m}\right| &\leq \left(\sum_{m\neq n}^{\infty} |\langle e_m, \psi\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{m\neq n}^{\infty} \left|\frac{1}{E_n - E_m}\right|^2\right)^{\frac{1}{2}} \\ &\leq \|\psi\| \left(\sum_{m\neq n}^{\infty} \left|\frac{1}{E_n - E_m}\right|^2\right)^{\frac{1}{2}} < \infty. \end{aligned} \tag{2.6}$$

$\square$

By Lemma 2.1-(ii), one can define a linear operator $T$ on $\mathcal{H}$ as follows:

$$D(T) := \left\{\psi \in \mathcal{H} \middle| \sum_{n=1}^{\infty} \left|\sum_{m\neq n} \frac{\langle e_m, \psi\rangle}{E_n - E_m}\right|^2 < \infty \right\}, \tag{2.7}$$

$$T\psi := i \sum_{n=1}^{\infty} \left(\sum_{m\neq n}^{\infty} \frac{\langle e_m, \psi\rangle}{E_n - E_m}\right) e_n, \qquad \psi \in D(T). \tag{2.8}$$

Note that

$$\|T\psi\|^2 = \sum_{n=1}^{\infty} \left|\sum_{m\neq n}^{\infty} \frac{\langle e_m, \psi\rangle}{E_n - E_m}\right|^2. \tag{2.9}$$

For a subset $\mathcal{D} \subset \mathcal{H}$, we denote by l.i.h.$(\mathcal{D})$ the subspace algebraically spanned by the vectors of $\mathcal{D}$.

The subspace

$$\mathcal{D}_0 := \text{l.i.h.}(\{e_n\}_{n=1}^{\infty}) \tag{2.10}$$

is dense in $\mathcal{H}$.

**Lemma 2.2** *The operator $T$ is densely defined with $\mathcal{D}_0 \subset D(T)$ and*

$$Te_k = i\sum_{n\neq k}^{\infty} \frac{1}{E_n - E_k} e_n, \quad k \in \mathbb{N}. \tag{2.11}$$

*Proof.* To prove $\mathcal{D}_0 \subset D(T)$, it is sufficient to show that $e_k \in D(T), k \in \mathbb{N}$. Putting

$$c_n(k) := \sum_{m\neq n}^{\infty} \frac{\langle e_m, e_k\rangle}{E_n - E_m},$$

we have $c_k(k) = 0$ and $c_n(k) = 1/(E_n - E_k)$ for $n \neq k$. Hence, by Lemma 2.1-(i), we have

$$\sum_{n=1}^{\infty} |c_n(k)|^2 = \sum_{n\neq k}^{\infty} \frac{1}{(E_n - E_k)^2} < \infty.$$

Hence $e_k \in D(T)$ and (2.11) holds. □

In general, it is not clear whether or not $T$ is a symmetric operator. But a restriction of $T$ to a smaller subspace gives a symmetric operator. Indeed, we have the following fact:

**Lemma 2.3** ([8]) *The operator*

$$T_1 := T|\mathcal{D}_0 \tag{2.12}$$

*(the restriction of $T$ to $\mathcal{D}_0$) is symmetric.*

*Proof.* It is enough to show that, for all $\psi \in \mathcal{D}_0$, $\langle\psi, T\psi\rangle$ is real. For a complex number $z \in \mathbb{C}$ (the set of complex numbers), we denote its complex conjugate by $z^*$. We have

$$\langle\psi, T\psi\rangle = i\sum_{n=1}^{\infty} \langle\psi, e_n\rangle \sum_{m\neq n}^{\infty} \frac{\langle e_m, \psi\rangle}{E_n - E_m}.$$

Hence

$$\langle\psi, T\psi\rangle^* = i\sum_{n=1}^{\infty} \langle e_n, \psi\rangle \sum_{m\neq n}^{\infty} \frac{\langle \psi, e_m\rangle}{E_m - E_n}.$$

Since $\psi$ is in $\mathcal{D}_0$, the double sum on $m, n$ with $m \neq n$ is a sum consisting of a finite term. Hence we can exchange the sum on $n$ and that on $m$ to obtain

$$\langle\psi, T\psi\rangle^* = i\sum_{m=1}^{\infty}\langle\psi, e_m\rangle \sum_{n\neq m}^{\infty} \frac{\langle e_n, \psi\rangle}{E_m - E_n} = \langle\psi, T\psi\rangle.$$

Hence $\langle\psi, T\psi\rangle$ is real. □

The operator $T_1$ defined by (2.12) is the time operator introduced by Galapon in [8]. Obviously we have

$$T_1 \subset T. \tag{2.13}$$

Hence

$$T^* \subset T_1^*. \tag{2.14}$$

**Remark 2.2** It is asserted in [8] that $T_1$ is essentially self-adjoint without additional conditions. But, unfortunately, we find that this is not conclusive, because the proof of it given in [8] (pp.2678–2679) has some gap: the interchange of the double sum in Equation (2.30) on p.2678 in [8] may not be justified, at least, by the reasoning given there. The assertion is true in the case where $T_1$ becomes a bounded operator under an additional condition for $\{E_n\}_{n=1}^{\infty}$, as we show below in the present paper. But, in the case where $T_1$ is unbounded, it seems to be very difficult to prove or disprove the essential self-adjointness of $T_1$. We leave this problem for future study.

**Lemma 2.4** *The subspace*

$$\mathcal{D}_{\mathrm{c}} := \mathrm{l.i.h.}(\{e_n - e_m \in \mathcal{H} | n, m \geq 1\}). \tag{2.15}$$

*is dense in the Hilbert space $\mathcal{H}$. Moreover*

$$\mathcal{D}_{\mathrm{c}} \subset \mathcal{D}_0 \subset D(T). \tag{2.16}$$

*Proof.* Let $\psi \in \mathcal{D}_{\mathrm{c}}^{\perp}$ (the orthogonal complement of $\mathcal{D}_{\mathrm{c}}$). Then, for all $m, n \geq 1$, $|\langle e_n, \psi\rangle|^2 = |\langle e_m, \psi\rangle|^2$. By the Parseval equality, $\|\psi\|^2 = \lim_{N\to\infty} N |\langle e_m, \psi\rangle|^2$. This implies that $|\langle e_m, \psi\rangle|^2 = 0$ for all $m \geq 1$ and $\|\psi\| = 0$. Hence $\psi = 0$. Thus $\mathcal{D}_{\mathrm{c}}$ is dense in $\mathcal{H}$. Inclusion relation (2.16) is obvious. □

**Theorem 2.5** ([8]) *It holds that*

$$\mathcal{D}_{\mathrm{c}} \subset D(T_1 H) \cap D(H T_1) \tag{2.17}$$

*and*

$$[T_1, H]\psi = i\psi, \qquad \psi \in \mathcal{D}_{\mathrm{c}}. \tag{2.18}$$

Theorem 2.5 shows that $T_1$ is a time operator of $H$.

**Remark 2.3** It is easy to see that, for all $k \in \mathbb{N}$, $T_1 e_k \notin D(H)$. Hence $\mathcal{D}_0 \not\subset D(HT_1)$. Therefore one can not consider the commutation relation $[T_1, H]$ on $\mathcal{D}_0$. Moreover, by direct computation, we have

$$\langle T_1 e_k, H e_\ell\rangle - \langle H e_k, T_1 e_\ell\rangle = -i(1 - \delta_{k\ell}), \quad k, \ell \in \mathbb{N}. \tag{2.19}$$

This means that $(T_1, H)$ does not satisfy the CCR in the sense of sesquilinear form on $\mathcal{D}_0$ (a weak form of the CCR), either. These facts suggest that the pair $(T_1, H)$ is very sensitive to the domain on which their commutation relation is applied.

In concluding this section we discuss shortly non-uniqueness of time operators of $H$. We introduce a set of symmetric operators associated with $H$:

$$\begin{aligned} \{H\}'_{\mathcal{D}_c} := \{S | S &\text{ is a symmetric operator on } \mathcal{H} \text{ such that} \\ &\mathcal{D}_c \subset D(SH) \cap D(HS) \text{ and } SH\psi = HS\psi, \forall \psi \in \mathcal{D}_c\}, \end{aligned} \tag{2.20}$$

which may be viewed as a commutant of $\{H\}$ in a restricted sense. It is easy to see that, for all real-valued continuous function $f$ on $\mathbb{R}$, the operator $f(H)$ defined via the functional calculus is in $\{H\}'_{\mathcal{D}_c}$.

**Proposition 2.6** *For all $S \in \{H\}'_{\mathcal{D}_c}$, $\mathcal{D}_c \subset D((T_1+S)H) \cap D(H(T_1+S))$ and*

$$[T_1 + S, H]\psi = i\psi, \quad \psi \in \mathcal{D}_c. \tag{2.21}$$

*Proof.* A direct computation using Theorem 2.5 and (2.20). □

**Proposition 2.7** *Let $T_2$ be a time operator of $H$ such that $\mathcal{D}_c \subset D(T_2H) \cap D(HT_2)$ and*

$$[T_2, H]\psi = i\psi, \quad \forall\psi \in \mathcal{D}_c.$$

*Then $T_2 = T_1 + S$ with some $S \in \{H\}'_{\mathcal{D}_c}$.*

*Proof.* We need only to show that $S := T_2 - T_1$ is in $\{H\}'_{\mathcal{D}_c}$. But this is obvious. □

# 3 General Properties

## 3.1 Closedness of $T$ and symmetricity of $T^*$

**Lemma 3.1** *$\mathcal{D}_0 \subset D(T^*)$ and $T^*|\mathcal{D}_0 = T_1$, i.e., $T_1 \subset T^*$.*

*Proof.* It is enough to show that, for all $k \in \mathbb{N}$, $e_k \in D(T^*)$ and $T^*e_k = Te_k (= T_1 e_k)$. It is easy to see that, for all $\psi \in D(T)$,

$$\langle e_k, T\psi\rangle = i \sum_{m\neq k}^{\infty} \frac{\langle e_m, \psi\rangle}{E_k - E_m}. \tag{3.1}$$

By Lemma 2.2, the right hand side is equal to $\langle Te_k, \psi\rangle$. Hence $e_k \in D(T^*)$ and $T^*e_k = Te_k$. □

**Proposition 3.2** *The operator $T$ is closed and*

$$T^* \subset T. \tag{3.2}$$

*In particular, if $T$ is bounded, then $T$ is self-adjoint with $D(T) = \mathcal{H}$.*

*Proof.* Let $\psi_k \in D(T)$, $k \in \mathbb{N}$ and $\psi_k \to \psi \in \mathcal{H}, T\psi_k \to \phi \in \mathcal{H}$ as $k \to \infty$. Then $\sup_{k\geq 1} \|T\psi_k\| < \infty$. Hence, by (2.9), there exists a constant $C > 0$ independent of $k \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \left| \sum_{n\neq m} \frac{\langle e_m, \psi_k\rangle}{E_n - E_m} \right|^2 \leq C.$$

By (2.6), we have

$$\lim_{k\to\infty} \sum_{n\neq m}^{\infty} \frac{\langle e_m, \psi_k\rangle}{E_n - E_m} = \sum_{n\neq m}^{\infty} \frac{\langle e_m, \psi\rangle}{E_n - E_m}. \tag{3.3}$$

Hence it follows that

$$\sum_{n=1}^{\infty}\left|\sum_{n\neq m}^{\infty}\frac{\langle e_m,\psi\rangle}{E_n-E_m}\right|^2\leq C.$$

Therefore $\psi\in D(T)$. By (3.1) and (3.3), we have for all $\ell\in\mathbb{N}$

$$\lim_{k\to\infty}\langle e_\ell, T\psi_k\rangle=\langle e_\ell, T\psi\rangle\,.$$

Hence $\langle e_\ell,\phi\rangle=\langle e_\ell,T\psi\rangle$ , $\ell\in\mathbb{N}$, implying $\phi=T\psi$. Thus $T$ is closed.

To prove (3.2), let $\psi\in D(T^*)$. Putting $\eta=T^*\psi$, we have $\langle\eta,\chi\rangle=\langle\psi,T\chi\rangle$ for all $\chi\in D(T)$. Taking $\chi=e_k$ ($k\in\mathbb{N}$), we have

$$\langle\eta,e_k\rangle=i\sum_{n\neq k}^{\infty}\frac{\langle\psi,e_n\rangle}{E_n-E_k},\tag{3.4}$$

which implies that

$$\sum_{k=1}^{\infty}\left|\sum_{n\neq k}^{\infty}\frac{\langle\psi,e_n\rangle}{E_n-E_k}\right|^2=\|\eta\|^2<\infty.$$

Hence $\psi\in D(T)$. Then, by (3.1), the right hand side of (3.4) is equal to $\langle T\psi,e_k\rangle$. Hence $\eta=T\psi$. Thus (3.2) holds.

Let $T$ be bounded. Then, by the deneseness of $D(T)$ and the closedness of $T$, $D(T)=\mathcal{H}$. Hence $D(T^*)=\mathcal{H}$. Thus, by (3.2), $T^*=T$, i.e., $T$ is self-adjoint. □

**Corollary 3.3** *The operator $T^*$ is symmetric.*

*Proof.* By Lemma 3.1, $T^*$ is densely defined. Hence, by Proposition 3.2, $T^*\subset T=(T^*)^*$. Thus $T^*$ is symmetric. □

Thus we have

$$T_1\subset T^*\subset T.\tag{3.5}$$

Corollary 3.3 shows that $T^*$ also is a time operator of $H$.

For a closable operator $A$ on a Hilbert space, we denote its closure by $\bar{A}$.

**Proposition 3.4** $\bar{T_1}=T^*$.

*Proof.* Note that $\bar{T_1}=T^*$ if and only if ${T_1}^*=T$. By (3.5), we have $\bar{T_1}\subset T^*$. Hence $T\subset {T_1}^*$. Thus it is enough to show that $D({T_1}^*)\subset D(T)$. For all $\psi\in D({T_1}^*)$, we have

$$\langle {T_1}^*\psi,e_l\rangle=\langle\psi,T_1e_l\rangle=i\sum_{n\neq l}\frac{\langle\psi,e_n\rangle}{E_n-E_l}.$$

Hence we obtain

$$\infty>\|{T_1}^*\psi\|^2=\sum_{l=1}^{\infty}|\langle {T_1}^*\psi,e_l\rangle|^2=\sum_{l=1}^{\infty}\left|\sum_{n\neq l}\frac{\langle\psi,e_n\rangle}{E_n-E_l}\right|^2,$$

implying that $\psi\in D(T)$. Thus $D({T_1}^*)\subset D(T)$. □

## 3.2 Absence of invariant dense domains for $T$

We first note the following general fact:

**Proposition 3.5** *Let $Q$ be a bounded self-adjoint operator on $\mathcal{H}$ and $P$ be a self-adjoint operator on $\mathcal{H}$. Suppose that there is a dense subspace $\mathcal{D}$ in $\mathcal{H}$ such that the following* (i)–(iii) *hold:*

(i) $Q\mathcal{D} \subset \mathcal{D} \subset D(P)$.

(ii) $\mathcal{D}$ *is a core of* $P$.

(iii) *The pair $(Q, P)$ obeys the CCR on $\mathcal{D}$ : $[Q, P]\psi = i\psi, \forall \psi \in \mathcal{D}$.*

*Then $\sigma(P) = \mathbb{R}$.*

*Proof.* Since $Q$ is a bounded self-adjoint operator, we have for all $t \in \mathbb{R}$

$$e^{itQ} = \sum_{k=0}^{\infty} \frac{(itQ)^k}{k!}$$

in operator norm. Conditions (i) and (iii) imply that, for all $k \in \mathbb{N}$ and $\psi \in \mathcal{D}$

$$Q^k P\psi - PQ^k\psi = ikQ^{k-1}\psi.$$

Hence, for all $t \in \mathbb{R}$ and vectors $\psi$ in $\mathcal{D}$, we have

$$\begin{aligned} e^{itQ}P\psi &= P\psi + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} Q^k P\psi \\ &= P\psi + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} (PQ^k + ikQ^{k-1})\psi \\ &= P\psi + \sum_{k=1}^{\infty} \left\{ P\frac{(itQ)^k}{k!}\psi - t\frac{(itQ)^{k-1}}{(k-1)!}\psi \right\}. \end{aligned}$$

It follows from the closedness of $P$ that $e^{itQ}\psi$ is in $D(P)$ and

$$Pe^{itQ}\psi = e^{itQ}P\psi + te^{itQ}\psi. \tag{3.6}$$

By condition (ii), this equality extends to all $\psi \in D(P)$ with $e^{itQ}\psi \in D(P), \forall t \in \mathbb{R}, \forall \psi \in D(P)$. Hence the operator equality $e^{-itQ}Pe^{itQ} = P + t$ follows. Thus $\sigma(P) = \sigma(P + t)$ for all $t \in \mathbb{R}$. This implies that $\sigma(P) = \mathbb{R}$. $\square$

**Theorem 3.6** *If $T$ is bounded (hence self-adjoint by Proposition 3.2), then there is no dense subspace $\mathcal{D}$ in $\mathcal{H}$ such that the following* (i)–(iii) *hold:*

(i) $T\mathcal{D} \subset \mathcal{D} \subset D(H)$.

(ii) $\mathcal{D}$ *is a core of* $H$.

(iii) *The pair* $(T, H)$ *obeys the CCR on* $\mathcal{D}$,

*Proof.* Suppose that there were such a dense subspace $\mathcal{D}$ as stated above. Then we can apply Proposition 3.5 with $(Q, P) = (T, H)$ to conclude that $\sigma(H) = \mathbb{R}$. But this is a contradiction. □

**Remark 3.1** A special case of this theorem was established in [7, Theorem 9.5].

### 3.3 Reflection symmetry of the spectrum of $T_1, T^*$ and $T$

We first recall the definition of the spectrum of a general linear operator (not necessarily closed). For a linear operator $A$ on a Hilbert space $\mathcal{K}$, the resolvent set of $A$, denoted $\rho(A)$, is defined by

$$\begin{aligned} \rho(A) \;:=\; & \{z \in \mathbb{C} | \mathrm{Ran}(A - z) \text{ (the range of } A - z) \text{ is dense in } \mathcal{K} \\ & \text{and } A - z \text{ is injective with } (A - z)^{-1} \text{ bounded}\}. \end{aligned}$$

Then the set

$$\sigma(A) := \mathbb{C} \setminus \rho(A)$$

is called the spectrum of $A$.

We denote by $T^\#$ any of $T_1, T^*$ and $T$.

**Proposition 3.7** *The spectrum* $\sigma(T^\#)$ *of* $T^\#$ *is reflection symmetric with respect to the imaginary axis, i.e., if* $z \in \sigma(T^\#)$, *then* $-z^* \in \sigma(T^\#)$. *In particular, if* $T$ *is self-adjoint, then* $\sigma(T)$ *is reflection symmetric with respect to the origin of the real axis.*

*Proof.* We define a conjugation $J$ on $\mathcal{H}$ by

$$J\psi := \sum_{n=1}^{\infty} \langle \psi, e_n \rangle e_n, \quad \psi \in \mathcal{H}. \tag{3.7}$$

It is easy to see that operator equality $JT^\# J = -T^\#$ holds ($JD(T^\#) = D(T^\#)$). Hence, for all $z \in \mathbb{C}$, we have $J(T^\# - z)J = -(T^\# + z^*)$. This implies that, if $z \in \rho(T^\#)$, then $-z^* \in \rho(T^\#)$. Thus the same holds for the spectrum $\sigma(T^\#) = \mathbb{C} \setminus \rho(T^\#)$. □

## 4 Boundedness of $T$

In this section we present a general criterion for the operator $T$ to be bounded. For mathematical generality and for later use, we consider a more general class of operators than that of $T$. Let $b := \{b_{nm}\}_{n,m=1}^{\infty}$ be a double sequence of complex numbers such that

$$\|b\|_\infty := \sup_{n,m \geq 1} |b_{nm}| < \infty. \tag{4.1}$$

Then, in the same way as in Lemma 2.1-(ii), for all $\psi \in \mathcal{H}$, the infinite series

$$\sum_{m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle e_m, \psi \rangle$$

absolutely converges. Hence one can define a linear operator $T_b$ on $\mathcal{H}$ as follows:

$$D(T_b) := \left\{ \psi \in \mathcal{H} | \sum_{n=1}^{\infty} \left| \sum_{m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle e_m, \psi \rangle \right|^2 < \infty \right\}, \tag{4.2}$$

$$T_b \psi := i \sum_{n=1}^{\infty} \left( \sum_{m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle e_m, \psi \rangle \right) e_n, \quad \psi \in D(T_b). \tag{4.3}$$

Obviously $T = T_b$ with $b$ satisfying $b_{nm} = 1$ for all $n, m \in \mathbb{N}$. In the same way as in the case of $T$, one can prove the following fact:

**Lemma 4.1** *The operator $T_b$ is closed.*

The following lemma is probably well known (but, for the completeness, we give a proof):

**Lemma 4.2** *Let $A$ be a densely defined linear operator on a Hilbert space $\mathcal{K}$. Suppose that there exist a dense subspace $\mathcal{D}$ in $\mathcal{K}$ and a constant $C > 0$ such that $\mathcal{D} \subset D(A)$ and*

$$|\langle \psi, A\psi \rangle| \leq C \|\psi\|^2, \quad \psi \in \mathcal{D}.$$

*Then $A$ is bounded with $\|\bar{A}\| \leq 2C$, where $\bar{A}$ is the closure of $A$.*
*If $A$ is symmetric in addition, then $\|\bar{A}\| \leq C$.*

*Proof.* Let $\psi, \phi \in \mathcal{D}$. Then, by the polarization identity

$$\begin{aligned} \langle \psi, A\phi \rangle &= \frac{1}{4}(\langle \psi + \phi, A(\psi + \phi) \rangle - \langle \psi - \phi, A(\psi - \phi) \rangle \\ &\quad + i \langle \psi - i\phi, A(\psi - i\phi) \rangle - i \langle \psi + i\phi, A(\psi + i\phi) \rangle), \end{aligned}$$

we have

$$|\langle \psi, A\phi \rangle| \leq \frac{C}{4}(\|\psi + \phi\|^2 + \|\psi - \phi\|^2 + \|\psi - i\phi\|^2 + \|\psi + i\phi\|^2) = C(\|\psi\|^2 + \|\phi\|^2).$$

Replacing $\psi \neq 0$ by $\|\phi\|\psi/\|\psi\|$ we have

$$|\langle \psi, A\phi \rangle| \leq 2C\|\psi\|\|\phi\|.$$

For $\psi = 0$, this inequality trivially holds. Since $\mathcal{D}$ is dense, it follows from the Riesz representation theorem that $\|A\phi\| \leq 2C\|\phi\|$, $\phi \in \mathcal{D}$. Thus the first half of the lemma follows.

Let $A$ be symmetric. Then, $\langle \psi, A\psi \rangle \in \mathbb{R}$ for all $\psi \in D(A)$. Hecne

$$|\Re \langle \psi, A\phi \rangle| = \frac{1}{4} |\langle \psi + \phi, A(\psi + \phi) \rangle - \langle \psi - \phi, A(\psi - \phi) \rangle| \leq \frac{C}{2}(\|\psi\|^2 + \|\phi\|^2), \quad \psi \in \mathcal{D}.$$

We write $\langle \psi, A\phi \rangle = |\langle \psi, A\phi \rangle| e^{i\theta}$ with $\theta \in \mathbb{R}$. Then $|\langle \psi, A\phi \rangle| = \langle e^{i\theta}\psi, A\phi \rangle$. Hence

$$\begin{aligned} |\langle \psi, A\phi \rangle| &= \Re \langle e^{i\theta}\psi, A\phi \rangle \leq \frac{C}{2}(\|e^{i\theta}\psi\|^2 + \|\phi\|^2) \\ &= \frac{C}{2}(\|\psi\|^2 + \|\phi\|^2). \end{aligned}$$

Thus, in the same manner as above, we can obtain $|\langle \psi, A\phi \rangle| \leq C\|\psi\|\|\phi\|$, $\psi, \phi \in \mathcal{D}$. □

**Lemma 4.3** *For all $s > 1$ and $n \geq 2$,*

$$\sum_{m=1}^{n-1} \frac{1}{n^s - m^s} \leq \frac{\log n}{n^{s-1}} + \frac{1}{s(n-1)^{s-1}}. \tag{4.4}$$

*Proof.* It easy to see that

$$\sum_{m=1}^{n-1} \frac{1}{n^s - m^s} \leq \int_1^{n-1} \frac{1}{n^s - x^s} dx + \frac{1}{n^s - (n-1)^s}. \tag{4.5}$$

By the change of variable $x = ny$, we have

$$\begin{aligned} \int_1^{n-1} \frac{1}{n^s - x^s} dx &= \frac{1}{n^{s-1}} \int_{1/n}^{(n-1)/n} \frac{1}{1 - y^s} dy & (4.6) \\ &\leq \frac{1}{n^{s-1}} \int_0^{(n-1)/n} \frac{1}{1-y} dy & (4.7) \\ &\leq \frac{\log n}{n^{s-1}}. & (4.8) \end{aligned}$$

We have

$$\frac{1}{n^s - (n-1)^s} = \frac{1}{(n-1)^s} \cdot \frac{1}{\left(\frac{n}{n-1}\right)^s - 1}.$$

By the well known inequality

$$x^s - 1 \geq s(x-1), \quad x > 0, s \geq 1, \tag{4.9}$$

we obtain

$$\frac{1}{n^s - (n-1)^s} \leq \frac{1}{s(n-1)^{s-1}}. \tag{4.10}$$

Thus (4.4) holds. □

**Lemma 4.4** *Let $s > 1$. Then*

$$\sup_{n\geq 2}\left(\sum_{m=1}^{n-1}\frac{1}{n^s-m^s}\right)<\infty \tag{4.11}$$

*and*

$$\sup_{m\geq 1}\left(\sum_{n=m+1}^{\infty}\frac{1}{n^s-m^s}\right)<\infty. \tag{4.12}$$

*Proof.* Property (4.11) follows from Lemma 4.3.

To prove (4.12), we write

$$\sum_{n=m+1}^{\infty}\frac{1}{n^s-m^s}\leq\int_{m+1}^{\infty}\frac{1}{x^s-m^s}dx+\frac{1}{(m+1)^s-m^s}.$$

We fix a constant $R > 2(\geq (m+1)/m)$. By the change of variable $x = my$, we have

$$\int_{m+1}^{\infty}\frac{1}{x^s-m^s}dx \;=\; \frac{1}{m^{s-1}}\left(\int_{(m+1)/m}^{R}\frac{1}{y^s-1}ds+C_R\right),$$

where

$$C_R:=\int_R^{\infty}\frac{1}{y^s-1}ds<\infty.$$

Using (4.9) we have

$$\begin{aligned}\int_{(m+1)/m}^{R}\frac{1}{y^s-1}dy &\leq \int_{(m+1)/m}^{R}\frac{1}{s(y-1)}dy\\ &= \frac{1}{s}\left(\log(R-1)+\log m\right).\end{aligned}$$

Hence

$$\int_{m+1}^{\infty}\frac{1}{x^s-m^s}dx\leq\frac{\log m}{sm^{s-1}}+\frac{\log(R-1)}{sm^{s-1}}+\frac{1}{m^{s-1}}C_R.$$

Thus (4.12) follows. □

Let

$$c_H(n):=\sum_{m=1}^{n-1}\frac{E_n}{(E_n-E_m)E_m},\quad n\geq 2, \tag{4.13}$$

$$d_H(m):=\sum_{n=m+1}^{\infty}\frac{E_m}{(E_n-E_m)E_n},\quad m\geq 1. \tag{4.14}$$

Since $c_H(n)$ and $d_H(m)$ are positive (recall that $E_n > 0, \forall n \in \mathbb{N}$), one can define constants

$$c_H := \sup_{n\geq 2} c_H(n), \tag{4.15}$$

$$d_H := \sup_{m\geq 1} d_H(m), \tag{4.16}$$

which are finite or infinite.

**Theorem 4.5** *Suppose that there exist constants $\alpha > 1$, $C > 0$ and $a > 0$ such that*

$$E_n - E_m \geq C(n^\alpha - m^\alpha), \quad n > m > a. \tag{4.17}$$

*Then $T_b$ is a bounded operator with $D(T_b) = \mathcal{H}$ and*

$$\|T_b\| \leq 4\|b\|_\infty \sqrt{c_H d_H}. \tag{4.18}$$

*Moreover, if $b_{nm}^* = b_{mn}$ for all $m, n \in \mathbb{N}$, then $T_b$ is a bounded self-adjoint operator with $D(T_b) = \mathcal{H}$ and*

$$\|T_b\| \leq 2\|b\|_\infty \sqrt{c_H d_H}. \tag{4.19}$$

*In particular, $T$ is a bounded self-adjoint operator with $D(T) = \mathcal{H}$.*

*Proof.* By Lemma 4.2, it is enough to show that $c_H$ and $d_H$ are finite and

$$|\langle \psi, T_b \psi \rangle| \leq 2\|b\|_\infty \sqrt{c_H d_H} \|\psi\|^2, \quad \psi \in \mathcal{D}_0. \tag{4.20}$$

Then $T_b$ is bounded with (4.18). Since $T_b$ is densely defined and closed, it follows that $D(T_b) = \mathcal{H}$. As in the case of $T$, one can show that, if $b_{nm}^* = b_{mn}$ for all $m, n \in \mathbb{N}$, then $T_b|\mathcal{D}_0$ is symmetric and hence $T_b$ is a bouned self-adjoint operator with $D(T_b) = \mathcal{H}$ and (4.19) holds. Therefore the desired result follows.

To prove (4.20), we first note that, for $\psi \in \mathcal{D}_0$,

$$\langle \psi, T_b \psi \rangle = i \sum_{m,n=1, m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle \psi, e_n \rangle \langle e_m, \psi \rangle .$$

Hence

$$|\langle \psi, T_b \psi \rangle| \leq 2\|b\|_\infty A(\psi),$$

where

$$A(\psi) := \sum_{n > m \geq 1} \frac{|\langle e_m, \psi \rangle| |\langle \psi, e_n \rangle|}{E_n - E_m}.$$

Inserting $1 = \sqrt{E_m/E_n} \cdot \sqrt{E_n/E_m}$ into the summand on the right hand side and using the Cauchy-Schwarz inequality, we have

$$A(\psi)^2 \leq B(\psi) C(\psi)$$

with

$$B(\psi) = \sum_{n > m \geq 1} \frac{|\langle e_n, \psi \rangle|^2}{E_n - E_m} \cdot \frac{E_n}{E_m},$$

$$C(\psi) = \sum_{n > m \geq 1} \frac{|\langle e_m, \psi \rangle|^2}{E_n - E_m} \cdot \frac{E_m}{E_n}.$$

One can rewrite and estimate $B(\psi)$ as follows:

$$\begin{aligned} B(\psi) &= \sum_{n=2}^{\infty} |\langle e_n, \psi\rangle|^2 c_H(n) \\ &\leq \|\psi\|^2 c_H. \end{aligned}$$

Similarly we have

$$C(\psi) \leq \|\psi\|^2 d_H. \tag{4.21}$$

Hence

$$|\langle \psi, T_b\psi\rangle| \leq 2\|b\|_\infty \sqrt{c_H d_H}\|\psi\|^2. \tag{4.22}$$

Therefore we need only to prove that $c_H$ and $d_H$ are finite.

We can write

$$c_H(n) = \sum_{m=1}^{n-1} \frac{1}{E_n - E_m} + \sum_{m=1}^{n-1} \frac{1}{E_m}.$$

By assumption (4.17), we have

$$\frac{1}{E_n - E_m} \leq \frac{1}{C(n^\alpha - m^\alpha)}, \quad n > m > a. \tag{4.23}$$

Since we have

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty,$$

it follows that

$$\varepsilon_1 := \sum_{m=1}^{\infty} \frac{1}{E_m} < \infty.$$

Thus

$$c_H(n) \leq \sum_{m=1}^{n-1} \frac{1}{E_n - E_m} + \varepsilon_1.$$

Let $n_0 \geq 2$ be a natural number such that $n_0 > a$. Then, for all $n > n_0$

$$\sum_{m=1}^{n-1} \frac{1}{E_n - E_m} \leq \sum_{m=1}^{n_0-1} \frac{1}{E_n - E_m} + \frac{1}{C} \sum_{m=n_0}^{n-1} \frac{1}{n^\alpha - m^\alpha}.$$

By (4.4), the right hand side is uniformly bounded in $n$. Thus we have $c_H < \infty$.

To prove $d_H < \infty$, we write for $m > a$

$$\begin{aligned} d_H(m) &= \sum_{n=m+1}^{\infty} \frac{1}{(E_n - E_m)} - \sum_{n=m+1}^{\infty} \frac{1}{E_n} \\ &\leq \sum_{n=m+1}^{\infty} \frac{1}{(E_n - E_m)} \\ &\leq \frac{1}{C} \sum_{n=m+1}^{\infty} \frac{1}{n^\alpha - m^\alpha}. \end{aligned}$$

Hence, by (4.12) in Lemma 4.4, we have

$$\sup_{m>a} d_H(m) < \infty.$$

Thus it follows that $d_H < \infty$. $\square$

**Example 4.1** Let $\lambda > 0, \alpha > 1$ and $P(x)$ be a real polynomial of $x \in \mathbb{R}$ with degree $p < \alpha$. Then it is easy to see that the sequense $\{E_n\}_{n=1}^{\infty}$ defined by

$$E_n := \lambda n^{\alpha} + P(n)$$

satisfies the assumptions (H.1), (H.2) and (4.17). Thus, by Theorem 4.5, in the present example, $T_b$ is bounded.

We remark that Theorem 4.5 does not cover the case $E_n = \lambda n + \mu$ with constants $\lambda > 0$ and $\mu \in \mathbb{R}$. For this case, we have the following theorem:

**Theorem 4.6** *Suppose that there exist constants $\lambda > 0, \mu \in \mathbb{R}$ and $a > 0$ such that*

$$E_n = \lambda n + \mu, \quad n > a. \tag{4.24}$$

*Then $T$ is a bounded self-adjoint operator with $D(T) = \mathcal{H}$.*

*Proof.* Let $k_0$ be the greatest integer such that $k_0 \leq a$. Let $a_n := \langle e_n, \psi\rangle$ ($\psi \in \mathcal{H}$). Then, by the Parseval equality, we have $\sum_{n=1}^{\infty} |a_n|^2 = \|\psi\|^2$. Let $\psi \in \mathcal{D}_0$. Then we can write:

$$\langle \psi, T\psi\rangle = S_I + S_2 + S_3 + S_4,$$

where

$$\begin{aligned}
S_1 &:= i \sum_{n=1}^{k_0} \sum_{m\neq n}^{k_0} \frac{a_n^* a_m}{E_n - E_m}, \\
S_2 &:= i \sum_{n=1}^{k_0} \sum_{m\geq k_0+1}^{\infty} \frac{a_n^* a_m}{E_n - E_m}, \\
S_3 &:= i \sum_{n\geq k_0+1}^{\infty} \sum_{m=1}^{k_0} \frac{a_n^* a_m}{E_n - E_m}, \\
S_4 &:= i\frac{1}{\lambda} \sum_{n=k_0+1}^{\infty} \sum_{m\neq n, m\geq k_0+1}^{\infty} \frac{a_n^* a_m}{n-m}.
\end{aligned}$$

By the Schwarz inequality, we have

$$|S_j| \leq C_j \|\psi\|^2, \quad j = 1, 2, 3,$$

where $C_j > 0$ is a constant. To estimate $|S_4|$, we use the following well known inequality [9, Theorem 294]:

$$\left|\sum_{n,m=1,n\neq m}^{\infty}\frac{x_n y_m}{n-m}\right| \leq \pi\sqrt{\sum_{n=1}^{\infty}x_n^2}\sqrt{\sum_{m=1}^{\infty}y_m^2}$$

for all real sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$. Hence

$$|S_4| \leq \pi\|\psi\|^2.$$

Therefore it follows that $|\langle\psi, T\psi\rangle| \leq \text{const.}\|\psi\|^2$. Thus $T$ is bounded. □

**Example 4.2** A physically interesting case is the case where $E_n = \omega(n+\frac{1}{2}), \quad n \in \{0\}\cup\mathbb{N}$ with a constant $\omega > 0$. In this case, by Theorem 4.6, $T$ is a bounded self-adjoint operator with $D(T) = \mathcal{H}$ and takes the form

$$T\psi = \frac{i}{\omega}\sum_{n=1}^{\infty}\left(\sum_{m\neq n}^{\infty}\frac{\langle e_m, \psi\rangle}{n-m}\right)e_n, \quad \psi \in \mathcal{H}.$$

Moreover, one can prove that

$$\sigma(T) = [-\pi/\omega, \pi/\omega]$$

([4, Theorem 2.1]).

Let $\hat{N} := \omega^{-1}H - 1/2$ and $\hat{\theta} := \omega T$. Then it follows that

$$\sigma(\hat{N}) = \{0\}\cup\mathbb{N}, \quad \sigma(\hat{\theta}) = [-\pi, \pi], \tag{4.25}$$

$$[\hat{\theta}, \hat{N}]\psi = i\psi, \quad \psi \in \mathcal{D}_{\mathrm{c}}. \tag{4.26}$$

As is well known, in the context of quantum mechanics, the sequence $\{\omega(n+\frac{1}{2})\}_{n=1}^{\infty}$ appears as the spectrum of the one-dimensional quantum harmonic oscillator Hamiltonian with mass $m > 0$

$$H_{\mathrm{os}} := \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

in the Schrödinger representation $(q, p)$ of the CCR, where $p := -iD$ with $D$ being the generalized partial differential operator on $L^2(\mathbb{R})$ and $q$ is the multiplication operator by the variable $x \in \mathbb{R}$. In this context, the operator $\hat{N}$ is called the number operator and, in view of (4.25) and (4.26), the operator $\hat{\theta}$ is interpreted as a *phase operator* [7].

# 5 Unboundedness of $T$

As for the unboundedness of $T$, we have the following fact:

**Theorem 5.1** *If $\{E_n\}_{n=1}^{\infty}$ satisfies*

$$\inf_{n\in\mathbb{N}}(E_{n+1} - E_n) = 0, \tag{5.1}$$

*then $T$ is unbounded.*

*Proof.* By (5.1), there exists a subsequence $\{E_{p_k}\}_{k=1}^{\infty}$ of $\{E_p\}_{p=1}^{\infty}$ such that

$$\lim_{k\to\infty}(E_{p_k+1}-E_{p_k})=0. \tag{5.2}$$

Hence we have

$$\begin{aligned}\|Te_{p_k}\|^2 &= \sum_{n=1}^{\infty}\left|\sum_{m\neq n}^{\infty}\frac{\langle e_m, e_{p_k}\rangle}{E_n-E_m}\right|^2 = \sum_{n\neq p_k}^{\infty}\left|\frac{1}{E_n-E_{p_k}}\right|^2 \\ &\geq \left|\frac{1}{E_{p_k+1}-E_{p_k}}\right|^2 \to \infty \ (k\to\infty).\end{aligned}$$

Thus $T$ is unbounded. □

**Example 5.1** Let

$$E_n = \lambda n^{\alpha} + \mu$$

with constants $\lambda > 0$, $\alpha > 1/2$ and $\mu \in \mathbb{R}$. Then $\{E_n\}_{n=1}^{\infty}$ satisfies the assumptions (H.1) and (H.2). As we have already seen, $T$ is bounded if $\alpha \geq 1$

Let $1/2 < \alpha < 1$. Then one easily sees that

$$\lim_{n\to\infty}(E_{n+1}-E_n)=0.$$

Hence $\inf_{n\in\mathbb{N}}(E_{n+1}-E_n)=0$. Therefore, in this case, $T$ is unbounded. Thus $T$ is bounded if and only if $\alpha \geq 1$.

# 6 Hilbert-Schmidtness of $T$

In this section we investigate Hilbert-Schmidtness of the operator $T$.

**Proposition 6.1** *The operator $T$ is Hilbert-Schmidt if and only if*

$$\sum_{n=1}^{\infty}\sum_{m\neq n}^{\infty}\frac{1}{(E_n-E_m)^2} < \infty. \tag{6.1}$$

*In that case, $T$ is self-adjoint with*

$$\|T\|_2^2 = \sum_{n=1}^{\infty}\sum_{m\neq n}^{\infty}\frac{1}{(E_n-E_m)^2}, \tag{6.2}$$

*where $\|\cdot\|_2$ denotes Hilbert-Schmidt norm. In particular, there exist a C.O.N.S. $\{f_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ and real numbers $t_n$, $n \in \mathbb{N}$ such that $Tf_n = t_n f_n$ and $t_n \to 0$ $(n \to \infty)$.*

*Proof.* Suppose that $T$ is Hilbert-Schmidt. Then $\sum_{n=1}^{\infty}\|Te_n\|^2 < \infty$. On the other hand, we have

$$\sum_{n=1}^{\infty}\|Te_n\|^2 = \sum_{n=1}^{\infty}\sum_{m\neq n}^{\infty}\frac{1}{(E_n-E_m)^2} \tag{6.3}$$

Hence (6.1) follows with (6.2).

Conversely, (6.1) holds. Hence, by (6.3), $\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty$. Therefore $T$ is Hilbert-Schmidt. The last statement follows from the Hilbert-Schmidt theorem (e.g., [12, Theorem VI.16]). $\square$

**Remark 6.1** In Proposition 6.1, the number $t_n \neq 0$ is an eigenvalue of $T$ with a finite multiplicity. Since $T$ is self-adjoint in the present case, it may be an observable in the context of quantum mechanics. If this is the case, then Proposition 6.1 shows that the observable described by $T$ ("time" in any sense ?) is quantized (discretized) in the quantum system whose Hamiltonian is $H$ with eigenvalues $\{E_n\}_{n=1}^{\infty}$ satisfying (6.1).

The next theorem gives a class of $H$ such that $T$ is Hilbert-Schmidt:

**Theorem 6.2** *Suppose that (4.17) in Theorem 4.5 holds with $\alpha > 3/2$. Then $T$ is Hilbert-Schmidt and self-adjoint.*

*Proof.* Since $1/(E_n - E_m)^2$ is symmetric in $n$ and $m$, it is sufficient to show that $\sum_{n>m\geq 1} 1/(E_n - E_m)^2 < \infty$. By the present assumption, we need only to show that

$$\Sigma := \sum_{n>m\geq 1} \frac{1}{(n^\alpha - m^\alpha)^2} < \infty$$

for all $\alpha > 3/2$. We have

$$\begin{aligned} \Sigma &= \sum_{n=2}^{\infty}\sum_{m=1}^{n-1} \frac{1}{(n^\alpha - m^\alpha)^2} \\ &\leq \sum_{n=2}^{\infty} \frac{1}{n^\alpha - (n-1)^\alpha} \cdot \sum_{m=1}^{n-1} \frac{1}{(n^\alpha - m^\alpha)}. \end{aligned}$$

Using (4.4) and (4.10), we obtain

$$\Sigma \leq \sum_{n=2}^{\infty} \frac{1}{\alpha(n-1)^{\alpha-1}} \frac{\log n}{n^{\alpha-1}} + \sum_{n=2}^{\infty} \frac{1}{\alpha^2 (n-1)^{2(\alpha-1)}}.$$

Each infinite series on the right hand side converges for all $\alpha > 3/2$. Thus the desired result follows. $\square$

# 7 The Galapon Time Operator as a Generalized Time Operator

It is shown that every self-adjoint operator which has a strong time operator is absolutely continuous [10]. Hence the Galapon time operator $T_1$ is not a strong time operator of $H$. But it may be a generalized time operator of $H$. In this section we investigate this aspect.

## 7.1 An operator-valued function on $\mathbb{R}$

In the same way as in Lemma 2.1-(ii), one can show that, for all $\psi \in \mathcal{H}$, $n \in \mathbb{N}$ and all $t \in \mathbb{R}$, the infinite series

$$\sum_{m \neq n}^{\infty} \frac{e^{it(E_n - E_m)} - 1}{E_n - E_m} \langle e_m, \psi \rangle$$

absolutely converges. Hence, for each $t \in \mathbb{R}$, one can define a linear operator $K(t)$ as follows:

$$D(K(t)) := \left\{ \psi \in \mathcal{H} \middle| \sum_{n=1}^{\infty} \left| \sum_{m \neq n}^{\infty} \frac{e^{it(E_n - E_m)} - 1}{E_n - E_m} \langle e_m, \psi \rangle \right|^2 < \infty \right\}, \tag{7.1}$$

$$K(t)\psi := i \sum_{n=1}^{\infty} \left( \sum_{m \neq n}^{\infty} \frac{e^{it(E_n - E_m)} - 1}{E_n - E_m} \langle e_m, \psi \rangle \right) e_n, \quad \psi \in D(K(t)). \tag{7.2}$$

It is easy to see that, for all $t \in \mathbb{R}$,

$$\mathcal{D}_0 \subset D(K(t)) \tag{7.3}$$

and

$$K(t)e_k = i \sum_{n \neq k} \frac{e^{it(E_n - E_k)} - 1}{E_n - E_k} e_n, \quad k \in \mathbb{N}. \tag{7.4}$$

The correspondence $K : \mathbb{R} \ni t \mapsto K(t)$ gives an operator-valued function on $\mathbb{R}$. In the notation in Section 4, $K(t)$ is the operator $T_b$ with $b_{nm} = e^{it(E_n - E_m)} - 1,\ n, m \in \mathbb{N}$.

**Remark 7.1** Equation (7.4) shows that $K(t) \neq tI|\mathcal{D}_0$. Hence $T$ cannot be a strong time operator of $H$, as already remarked based on the general theory of strong time operators.

**Proposition 7.1** *For all $t \in \mathbb{R}$, $K(t)$ is a densely defined closed operator.*

*Proof.* Similar to the proof of Proposition 3.2. □

**Proposition 7.2** *For all $t \in \mathbb{R}$, $K(t)|\mathcal{D}_0$ is symmetric.*

*Proof.* Similar to the proof of Lemma 2.3. □

**Theorem 7.3** *For all $\psi \in D(T_1)(= \mathcal{D}_0)$ and $t \in \mathbb{R}$, $e^{-itH}\psi \in D(T_1)$ and*

$$T_1 e^{-itH}\psi = e^{-itH}(T_1 + K(t))\psi. \tag{7.5}$$

*Proof.* We need only to prove the statement in the case $\psi = e_k$ ($\forall k \in \mathbb{N}$). Since $e^{-itH}e_k = e^{-itE_k}e_k$, it follows that $e^{-itH}e_k \in D(T_1)$ with

$$T_1 e^{-itH} e_k = e^{-itE_k} \sum_{n \neq k}^{\infty} \frac{i}{E_n - E_k} e_n.$$

We have

$$e^{-itH}T_1e_k = i\sum_{n\neq k}^{\infty}\frac{e^{-itE_n}}{E_n-E_k}e_n.$$

It follows from these equations that

$$T_1e^{-itH}e_k - e^{-itH}T_1e_k = e^{-itH}K(t)e_k.$$

Thus the desired result follows. □

**Corollary 7.4** *Suppose that, for all $t\in\mathbb{R}$, $K(t)$ is bounded. Then $T_1$ is a generalized time operator of $H$ with commutation factor $K$.*

*Proof.* This follows from Theorem 7.3, Proposition 7.1 and Proposition 7.2. □

In view of Corollary 7.4, we need to investigate conditions for $K(t)$ to be bounded.

**Proposition 7.5** *Suppose that (4.17) holds with $\alpha > 1$. Then, for all $t\in\mathbb{R}$, $K(t)$ is a bounded self-adjoint operator with $D(K(t)) = \mathcal{H}$.*

*Proof.* This follows from an application of Theorem 4.5 to the case where $b_{nm} = e^{it(E_n-E_m)} - 1,\ n,m\in\mathbb{N}$. □

**Proposition 7.6** *Suppose that (6.1) holds. Then, for all $t\in\mathbb{R}$, $K(t)$ is Hilbert-Schmidt and self-adjoint with*

$$\|K(t)\|_2^2 = \sum_{k=1}^{\infty}\sum_{n\neq k}^{\infty}\left|\frac{e^{it(E_n-E_k)}-1}{E_n-E_k}\right|^2. \tag{7.6}$$

*Proof.* Similar to the proof of Proposition 6.1. □

## 7.2 Non-differentiability of $K$

From the view-point of the theory of generalized time operators [2], it is interesting to examine differentiability of the operator-valued function $K$.

**Proposition 7.7** *For all $k\in\mathbb{N}$, the $\mathcal{H}$-valued function : $\mathbb{R}\ni t\mapsto K(t)e_k$ is not strongly differentiable on $\mathbb{R}$.*

*Proof.* We first show that $K(t)e_k$ is not strongly differentiable at $t=0$. Since $K(0)e_k = 0$, we have for all $t\in\mathbb{R}\setminus\{0\}$ and $N>k$

$$\begin{aligned}\left\|\frac{K(t)e_k-K(0)e_k}{t}\right\|^2 &= \sum_{n\neq k}^{\infty}\frac{|e^{it(E_n-E_k)}-1|^2}{t^2|E_n-E_k|^2}\\ &\geq \sum_{n\neq k}^{N+1}\frac{|e^{it(E_n-E_k)}-1|^2}{t^2|E_n-E_k|^2}.\end{aligned}$$

Hence

$$\liminf_{t\to 0}\left\|\frac{K(t)e_k - K(0)e_k}{t}\right\|^2 \geq \sum_{n\neq k}^{N+1} 1 = N.$$

Since $N > k$ is arbitrary, it follows that

$$\lim_{t\to 0}\left\|\frac{K(t)e_k - K(0)e_k}{t}\right\|^2 = +\infty.$$

This implies that $K(t)e_k$ is not strongly differentiable at $t = 0$.

We next show that $K(t)e_k$ is not strongly differentiable at each $t \neq 0$. By (7.5), we have for all $s \in \mathbb{R}\setminus\{0\}$

$$\frac{K(t+s)e_k - K(t)e_k}{s} = e^{it(H-E_k)}\frac{K(s)e_k}{s}.$$

Hence

$$\left\|\frac{K(t+s)e_k - K(t)e_k}{s}\right\| = \left\|\frac{K(s)e_k}{s}\right\|.$$

By the preceding result, the right hand side diverges to $+\infty$ as $s \to 0$. Therefore $K(t)e_k$ is not strongly differentiable at $t$. $\square$

**Remark 7.2** We have

$$\langle e_\ell, K(t)e_k\rangle = \begin{cases} i\frac{e^{it(E_\ell - E_k)}-1}{E_\ell - E_k} & ;\ \ell \neq k \\ 0 & ;\ \ell = k \end{cases} \tag{7.7}$$

Hence, for all $k, \ell \in \mathbb{N}$, $\langle e_\ell, K(t)e_k\rangle$ is differentiable in $t \in \mathbb{R}$ and

$$\frac{d}{dt}\langle e_\ell, K(t)e_k\rangle = (\delta_{\ell k} - 1)e^{it(E_\ell - E_k)}. \tag{7.8}$$

Proposition 7.7 tells us some singularity of $K(t)$ acting on $\mathcal{D}_0$. But, as shown in the next proposition, $K(t)$ restricted to $\mathcal{D}_c$ is strongly differentiable at $t = 0$.

**Proposition 7.8** *For all $\psi \in \mathcal{D}_c$, the $\mathcal{H}$-valued function $K(t)\psi$ is strongly differentiable at $t = 0$ with*

$$\left.\frac{d}{dt}K(t)\psi\right|_{t=0} = \psi. \tag{7.9}$$

*Proof.* We need only to prove the statement for $\psi$ of the form $\psi = e_k - e_\ell$ ($k, \ell \in \mathbb{N}, k \neq \ell$). For all $t \in \mathbb{R}\setminus\{0\}$, we have

$$\frac{K(t)(e_k - e_\ell)}{t} = A(t) + B(t),$$

where

$$A(t) := i\frac{e^{it(E_\ell - E_k)} - 1}{t(E_\ell - E_k)} e_\ell - i\frac{e^{it(E_k - E_\ell)} - 1}{t(E_k - E_\ell)} e_k,$$
$$B(t) := i\sum_{n \neq k,\ell}^{\infty} \left( \frac{e^{it(E_n - E_k)} - 1}{t(E_n - E_k)} - \frac{e^{it(E_n - E_\ell)} - 1}{t(E_n - E_\ell)} \right) e_n.$$

It is easy to see that

$$\lim_{t \to 0} A(t) = e_k - e_\ell.$$

As for $B(t)$, we have

$$\|B(t)\|^2 = \sum_{n \neq k,\ell}^{\infty} |F_n(t)|^2,$$

where

$$F_n(t) := \frac{e^{it(E_n - E_k)} - 1}{t(E_n - E_k)} - \frac{e^{it(E_n - E_\ell)} - 1}{t(E_n - E_\ell)}.$$

It is easy to see that

$$\lim_{t \to 0} F_n(t) = 0.$$

Moreover, one can show that

$$|F_n(t)| \leq \frac{C}{|E_n - E_k|}, \quad n \neq k, \ell,$$

where $C > 0$ is a constant independent of $n$ and $t$. Since $\sum_{n \neq k}^{\infty} 1/|E_n - E_k|^2 < \infty$, one can apply the dominated convergence theorem to conclude that $\lim_{t \to 0} \|B(t)\|^2 = 0$. Thus $K(t)(e_k - e_\ell)$ is strongly differentiable at $t = 0$ and (7.9) with $\psi = e_k - e_\ell$ holds. □

**Proposition 7.9** *For all $k, \ell \in \mathbb{N}$ with $k \neq \ell$, the $\mathcal{H}$-valued function $K(t)(e_k - e_\ell)$ is not strongly differentiable at $t \notin \{2\pi n/(E_k - E_\ell) | n \in \mathbb{Z}\}$.*

*Proof.* Let $t \neq 2\pi n/(E_k - E_\ell)$ $(n \in \mathbb{Z})$ and $s \in \mathbb{R} \setminus \{0\}$. Then, by (7.5), we have

$$\frac{(K(t+s) - K(t))(e_k - e_\ell)}{s} = e^{itH} \frac{K(s)}{s} e^{-itH}(e_k - e_\ell).$$

Hence

$$\left\| \frac{(K(t+s) - K(t))(e_k - e_\ell)}{s} \right\| = \|u(s)\|$$

with

$$u(s) := \frac{K(s)}{s}(e^{-itE_k} e_k - e^{-itE_\ell} e_\ell).$$

We write

$$u(s) = u_1(s) + u_2(s)$$

with

$$u_1(s) := e^{-itE_k}\frac{K(s)}{s}(e_k - e_\ell), \quad u_2(s) := (e^{-itE_k} - e^{-itE_\ell})\frac{K(s)}{s}e_\ell.$$

By Proposition 7.8, we have $\lim_{s\to 0} u_1(s) = e^{-itE_k}(e_k - e_\ell)$. On the other hand, we have from the proof of Proposition 7.7 and the assumed condition for $t$

$$\lim_{s\to 0} \|u_2(s)\| = +\infty.$$

Hence $\lim_{s\to 0} \|u(s)\| = +\infty$. Thus the desired result follows. □

# 8 Possible Connections with Regular Perturbation Theory

We consider a perturbation of $H$ by a symmetric operator $H_I$ on $\mathcal{H}$:

$$H(\lambda) := H + \lambda H_I, \tag{8.1}$$

where $\lambda \in \mathbb{R}$ is a perturbation parameter. For simplicity, we assume that $H_I$ is $H$-bounded: $D(H) \subset D(H_I)$ and there exist constants $a, b \geq 0$ such that

$$\|H_I\psi\| \leq a\|H\psi\| + b\|\psi\|, \quad \psi \in D(H).$$

Then, by the Kato-Rellich theorem (e.g., [13, Theorem X.12]), for all $\lambda \in \mathbb{R}$ satisfying

$$a|\lambda| < 1, \tag{8.2}$$

$H(\lambda)$ is self-adjoint and bounded below. In what follows, we assume (8.2).

## 8.1 Eigenvalues of $H(\lambda)$

We fix $n \in \mathbb{N}$ arbitrarily. By a general theorem in regular perturbation theory (e.g., [14, Theorem XII.9]), there exists a constant $c_n > 0$ such that, for all $|\lambda| < c_n$, $H$ has a unique, isolated non-degenerate eigenvalue $E_n(\lambda)$ near $E_n$. Moreover, $E_n(\lambda)$ is analytic in $\lambda$ with Taylor expansion

$$E_n(\lambda) = E_n + E_n^{(1)}\lambda + E_n^{(2)}\lambda^2 + \cdots, \tag{8.3}$$

where

$$E_n^{(1)} := \langle e_n, H_I e_n\rangle, \quad E_n^{(2)} := \sum_{m\neq n}^{\infty} \frac{|\langle e_n, H_I e_m\rangle|^2}{E_n - E_m}. \tag{8.4}$$

As an eigenvector of $H(\lambda)$ with eigenvalue $E_n(\lambda)$, one can take a vector $\psi_n(\lambda)$ analytic in $\lambda$ with Taylor expansion

$$\psi_n(\lambda) = e_n + e_n^{(1)}\lambda + \cdots, \tag{8.5}$$

where

$$e_n^{(1)} := \sum_{m\neq n}^{\infty} \frac{\langle e_m, H_I e_n\rangle}{E_n - E_m} e_m. \tag{8.6}$$

By Lemma 2.2, we have

$$\langle e_n, Te_m \rangle = \begin{cases} \dfrac{i}{E_n - E_m} & ; n \neq m \\ 0 & ; n = m \end{cases}$$

Hence $E_n^{(2)}$ can be written

$$E_n^{(2)} = (-i) \sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^2 \langle e_n, Te_m \rangle . \tag{8.7}$$

To rewrite the right hand side only in terms of $e_n$ and linear operators on $\mathcal{H}$, we note that

$$\sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^2 = \|H_I e_n\|^2 < \infty$$

by the Parseval equality. Hence

$$\sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^4 < \infty.$$

Therefore the infinite series

$$f_n := \sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^2 e_m \tag{8.8}$$

strongly converges and defines a vector in $\mathcal{H}$. Thus we can define a linear operator $V$ on $\mathcal{H}$ as follows:

$$D(V) := \mathcal{D}_0, \tag{8.9}$$

$$V\psi := -i \sum_{n=1}^{\infty} \langle e_n, \psi \rangle f_n, \quad \psi \in \mathcal{D}_0 \tag{8.10}$$

where the right hand side of (8.10) is a sum over a finite term. It is easy to see that $V$ is a symmetric operator.

**Proposition 8.1** *For all $n \in \mathbb{N}$,*

$$E_n^{(2)} = \langle Te_n, Ve_n \rangle . \tag{8.11}$$

*Proof.* We have $Ve_n = -if_n$. Hence $\langle Te_n, Ve_n \rangle = -i \langle Te_n, f_n \rangle$, which is equal to the right hand side of (8.7). □

This proposition suggests some role of the time operator $T_1 = T|\mathcal{D}_0$ in the perturbation expansions of the eigenvalues of $H$.

As for the first order term $e_n^{(1)}\lambda$ of the eigenvector $\psi_n(\lambda)$, we have

$$e_n^{(1)} = (-i) \sum_{m=1}^{\infty} \langle e_m, H_I e_n \rangle \langle e_n, Te_m \rangle e_m. \tag{8.12}$$

## 8.2 Transition probability amplitudes

In the context of quantum mechanics where $H(\lambda)$ is the Hamiltonian of a quantum system, the complex number $\langle \phi, e^{-itH(\lambda)}\psi \rangle$ with unit vectors $\phi, \psi \in \mathcal{H}$ is called the *transition probability amplitude* for the probability such that the state of the quantum system at time $t$ is found in the state $\phi$ under the condition that the state of the quantum system at time zero is $\psi$.

**Lemma 8.2** *Let $\phi, \psi \in D(H)$. Then, for all $t \in \mathbb{R}$,*

$$\langle \phi, e^{-itH(\lambda)}\psi \rangle = \langle \phi, e^{-itH}\psi \rangle - i\lambda \int_0^t \langle e^{i(t-s)H}\phi, H_I e^{-isH}\psi \rangle \, ds + O(\lambda^2). \tag{8.13}$$

*Proof.* By a simple application of a general formula for the unitary group generated by a self-adjoint operator ([5, Lemma 5.9]), we have

$$e^{-itH(\lambda)}\psi = e^{-itH}\psi - i\lambda \int_0^t e^{-i(t-s)H(\lambda)} H_I e^{-isH}\psi ds, \tag{8.14}$$

where the integral is taken in the strong sense. Hence

$$\begin{aligned} \langle \phi, e^{-itH(\lambda)}\psi \rangle &= \langle \phi, e^{-itH}\psi \rangle - i\lambda \int_0^t \langle e^{i(t-s)H(\lambda)}\phi, H_I e^{-isH}\psi \rangle \, ds \\ &= \langle \phi, e^{-itH}\psi \rangle - i\lambda \int_0^t \langle e^{i(t-s)H}\phi, H_I e^{-isH}\psi \rangle \, ds + R(\lambda), \end{aligned}$$

where

$$R(\lambda) := -i\lambda \int_0^t \left\langle \left( e^{i(t-s)H(\lambda)} - e^{i(t-s)H} \right) \phi, H_I e^{-isH}\psi \right\rangle ds.$$

Using (8.14) again, we have

$$R(\lambda) = -\lambda^2 \int_0^t ds \int_0^{-(t-s)} ds' \left\langle e^{i(t-s+s')H(\lambda)} H_I e^{-is'H}\phi, H_I e^{-isH}\psi \right\rangle .$$

Hence

$$|R(\lambda)| \le \lambda^2 \int_0^{|t|} ds \int_0^{|t-s|} ds' \|H_I e^{-is'H}\phi\| \|H_I e^{-isH}\psi\|$$

Therefore $R(\lambda) = O(\lambda^2)$. Thus (8.13) holds. □

Applying (8.13) with $\phi = e_m$ and $\psi = e_n$ $(n \neq m)$, we have

$$\langle e_m, e^{-itH(\lambda)} e_n \rangle = -\lambda \frac{e^{-itE_n} - e^{-itE_m}}{E_m - E_n} \langle e_m, H_I e_n \rangle + O(\lambda^2), \tag{8.15}$$

which, combined with (7.7), gives

$$\langle e_m, e^{-itH(\lambda)} e_n \rangle = i\lambda \langle e_n, e^{-itH} K(t) e_m \rangle \langle e_m, H_I e_n \rangle + O(\lambda^2). \quad m \neq n. \tag{8.16}$$

This suggests a physical meaning of the commutation factor $K$.

By Theorem 7.3, one can rewrite the first term on the right hand side in terms of $T_1$ and $e^{-itH}$, obtaining

$$\langle e_m, e^{-itH(\lambda)} e_n \rangle = i\lambda \langle e_n, [T_1, e^{-itH}] e_m \rangle \langle e_m, H_I e_n \rangle + O(\lambda^2), \quad m \neq n. \tag{8.17}$$

This also is suggestive on physical meaning of the time operator $T_1$.

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