Time Operators of a Hamiltonian with Purely Discrete Spectrum

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Abstract

We develop a mathematical theory of time operators of a Hamiltonian with purely discrete spectrum. The main results include boundedness, unboundedness and spectral properties of them. In addition, possible connections of a time operator of $H$ with regular perturbation theory are discussed.

Keywords. canonical commutation relation, Hamiltonian, time operator, time-energy uncertainty relation, phase operator, spectrum, regular perturbation theory.

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1 Introduction

This paper is concerned with mathematical theory of time operators in quantum mechanics [2, 3, 4, 6, 10]. There are some types of time operators which are not necessarily equivalent each other. For the reader’s convenience, we first recall the definitions of them with comments.

Let $\mathcal{H}$ be a complex Hilbert space. We denote the inner product and the norm of $\mathcal{H}$ by $\langle \cdot, \cdot \rangle$ (antilinear in the first variable) and $\| \cdot \|$ respectively. For a linear operator $A$ on a Hilbert space, $D(A)$ denotes the domain of $A$.

Let $H$ be a self-adjoint operator on $\mathcal{H}$ and $T$ be a symmetric operator on $\mathcal{H}$.

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The operator \( T \) is called a time operator of \( H \) if there is a dense subspace \( \mathcal{D} \) of \( \mathcal{H} \) such that \( \mathcal{D} \subset D(TH) \cap D(HT) \) and the canonical commutation relation (CCR)

\[
[T, H] := (TH - HT) = i
\]  

(1.1)

holds on \( \mathcal{D} \) (i.e., \([T, H] \psi = i \psi, \forall \psi \in \mathcal{D}\) ), where \( i \) is the imaginary unit. In this case, \( T \) is called a canonical conjugate to \( H \) too.

The name “time operator” for the operator \( T \) comes from the quantum mechanical context where \( H \) is taken to be the Hamiltonian of a quantum system and the heuristic classical-quantum correspondence based on the structure that, in the classical relativistic mechanics, time is a canonical conjugate variable to energy in each Lorentz frame of coordinates. Note also that the dimension of \( T \) is that of time if the dimension of \( H \) is that of energy in the original unit system where the right hand side of (1.1) takes the form \( i \hbar \) with \( \hbar \) being the Planck constant \( h \) divided by \( 2\pi \). We remark, however, that this name is somewhat misleading, because, in the framework of the standard quantum mechanics, time is not an observable, but just a parameter assigning the time when a quantum event is observed. But we follow the convention in this respect. By the same reason as just remarked, \( T \) is not necessarily (essentially) self-adjoint. But this does not mean that it is “unphysical” [2, 10].

From a representation theoretic point of view, the pair \((T, H)\) is a symmetric representation of the CCR with one degree of freedom. But one should remember that, as for this original form of representation of the CCR, the von Neumann uniqueness theorem ([11], [12, Theorem VIII.14]) does not necessarily hold. In other words, \((T, H)\) is not necessarily unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom. Indeed, for example, it is obvious that, if \( T \) or \( H \) is bounded below or bounded above, then \((T, H)\) cannot be unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom.

A classification of pairs \((T, H)\) with \( T \) being a bounded self-adjoint operator has been done by G. Dorfmeister and J. Dorfmeister [7]. We remark, however, that the class discussed in [7] does not cover the pairs \((T, H)\) considered in this paper, because the paper [7] treats only the case where \( T \) is bounded and absolutely continuous.

A weak form of time operator is defined as follows. We say that a symmetric operator \( T \) is a weak time operator of \( H \) if there is a dense subspace \( \mathcal{D}_w \) of \( \mathcal{H} \) such that \( \mathcal{D}_w \subset D(T) \cap D(H) \) and

\[
\langle T \psi, H \phi \rangle - \langle H \psi, T \phi \rangle = \langle \psi, i \phi \rangle, \quad \psi, \phi \in \mathcal{D}_w,
\]

i.e., \((T, H)\) satisfies the CCR in the sense of sesquilinear form on \( \mathcal{D}_w \). Obviously a time operator \( T \) of \( H \) is a weak time operator of \( H \). But the converse is not true (it is easy to see, however, that, if \( T \) is a weak time operator of \( H \) and \( \mathcal{D}_w \subset D(TH) \cap D(HT) \), then \( T \) is a time operator). An important aspect of a weak time operator \( T \) of \( H \) is that a time-energy uncertainty relation is naturally derived [2, Proposition 4.1]: for all unit vectors \( \psi \) in \( \mathcal{D}_w \subset D(T) \cap D(H) \),

\[
(\Delta T)_\psi (\Delta H)_\psi \geq \frac{1}{2},
\]
where, for a linear operator $A$ on $\mathcal{H}$ and $\phi \in D(A)$ with $\|\phi\| = 1$, 
\[
(\Delta A)_\phi := \| (A - \langle \phi, A\phi \rangle) \phi \|,
\]
called the uncertainty of $A$ in the vector $\phi$.

In contrast to the weak form of time operator, there is a strong form. We say that $T$ is a strong time operator of $H$ if, for all $t \in \mathbb{R}$, $e^{-itH}D(T) \subset D(T)$ and
\[
Te^{-itH}\psi = e^{-itH}(T + t)\psi, \quad \psi \in D(T).
\]  
We call (1.2) the weak Weyl relation [2]. From a representation theoretic point of view, we call a pair $(T, H)$ obeying the weak Weyl relation a weak Weyl representation of the CCR. This type of representation of the CCR was extensively studied by Schmüdgen [15, 16]. It is shown that a strong time operator of $H$ is a time operator of $H$ [10]. But the converse is not true. In fact, the time operators considered in the present paper are not strong ones.

There is a generalized version of strong time operator [2]. We say that $T$ is a generalized time operator of $H$ if, for each $t \in \mathbb{R}$, there is a bounded self-adjoint operator $K(t)$ on $\mathcal{H}$ with $D(K(t)) = \mathcal{H}$, $e^{-itH}D(T) \subset D(T)$ and a generalized weak Weyl relation (GWWR)
\[
Te^{-itH}\psi = e^{-itH}(T + K(t))\psi \quad (\forall \psi \in D(T))
\]  
holds. In this case, the bounded operator-valued function $K(t)$ of $t \in \mathbb{R}$ is called the commutation factor of the GWWR under consideration.

We now come to the subject of the present paper. In his interesting paper [8], Galapon showed by an explicit construction that, for every self-adjoint operator $H$ (a Hamiltonian) on an abstract Hilbert space $\mathcal{H}$ which is bounded below and has purely discrete spectrum with some growth condition, there is a time operator $T_1$ on $\mathcal{H}$, which is a bounded self-adjoint operator under an additional condition (for the definition of $T_1$, see (2.12) below). To be definite, we call the operator $T_1$ introduced in [8] the Galapon time operator.

An important point of Galapon’s work [8] is in that it disproved the long-standing belief or folklore among physicists that there is no self-adjoint operator canonically conjugate to a Hamiltonian which is bounded below (for a historical survey, see Introduction of [8]).

Motivated by work of Galapon [8], we investigate, in this paper, properties of time operators of a self-adjoint operator $H$ with purely discrete spectrum. In Section 2, we introduce a densely defined linear operator $T$ whose restriction to a subspace yields the Galapon time operator $T_1$ and prove basic properties of $T$ and $T_1$, in particular the closedness of $T$. It follows that, if $T$ is bounded, then $T$ is self-adjoint with $D(T) = \mathcal{H}$ and a time operator of $H$. We denote by $T^\#$ one of $T_1$, $T$ and $T^*$ (the adjoint of $T$). In Section 3, we discuss some general properties of $T^\#$. Moreover the reflection symmetry of the spectrum of $T^\#$ with respect to the imaginary axis is proved. Sections 4–6 are the main parts of this paper. In Section 4, we present a general criterion for $T$ to be bounded with $D(T) = \mathcal{H}$, while, in Section 5, we give a sufficient condition for $T$ to be unbounded. In Section 6, we present a necessary and sufficient condition for $T$ to be Hilbert-Schmidt. In Section 7, we show that, under some condition, the Galapon time operator is a generalized time operator of $H$, too. We also discuss non-differentiability of the commutation factor $K$ in the GWWR for $(T_1, H)$. In the last section, we consider a perturbation of $H$ by a symmetric operator and try to draw out physical meanings of $T_1$ and $K$ in the context of regular perturbation theory.
2 Time Operators

In this section, we recapitulate some basic aspects of the Galapon time operator in more apparent manner than in [8].

Let $\mathcal{H}$ be a complex Hilbert space and
$H$ be a self-adjoint operator on $\mathcal{H}$ which has the following properties (H.1) and (H.2):

(H.1) The spectrum of $H$, denoted $\sigma(H)$, is purely discrete with $\sigma(H) = \{E_n\}_{n=1}^{\infty}$, where each eigenvalue $E_n$ of $H$ is simple and

$$0 < E_n < E_{n+1}$$

for all $n \in \mathbb{N}$ (the set of positive integers).

(H.2) $\sum_{n=1}^{\infty} \frac{1}{E_n^2} < \infty$.

Throughout the present paper we assume (H.1) and (H.2).

Remark 2.1 The positivity condition $E_n > 0$ for the eigenvalues of $H$ does not lose generality, because, if $H$ is bounded below, but not strictly positive, then one needs only to consider, instead of $H$, $\tilde{H} := H + c$ with a constant $c > -\inf \sigma(H)$, which is a strictly positive self-adjoint operator.

Property (H.2) implies that $E_n \to \infty$ ($n \to \infty$). (2.1)

Let $e_n$ be a normalized eigenvector of $H$ belonging to eigenvalue $E_n$:

$$He_n = E_n e_n, \quad n \in \mathbb{N}. \quad (2.2)$$

Then, by property (H.1), the set $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal system (C.O.N.S.) of $\mathcal{H}$.

Lemma 2.1

(i) For all $m \in \mathbb{N}$,

$$\sum_{n \neq m}^{\infty} \frac{1}{(E_n - E_m)^2} < \infty. \quad (2.3)$$

In particular, for each $m \in \mathbb{N}$,

$$\sum_{n \neq m}^{\infty} \frac{1}{E_n - E_m} e_n$$

converges in $\mathcal{H}$. 4
(ii) For all \( n \in \mathbb{N} \) and vectors \( \psi \) in \( \mathcal{H} \), the infinite series
\[
\sum_{m \neq n}^\infty \frac{\langle e_m, \psi \rangle}{E_n - E_m}
\]
absolutely converges.

Proof. (i) By (2.1), we have
\[
C_m := \sup_{n \neq m} \frac{1}{\left(1 - \frac{E_m}{E_n}\right)^2} < \infty.
\]
Hence we have
\[
\sum_{n \neq m}^\infty \frac{1}{|E_n - E_m|^2} \leq C_m \sum_{n \neq m}^\infty \frac{1}{E_n^2} < \infty.
\]

(ii) By the Cauchy-Schwarz inequality, the Parseval equality and part (i), we have
\[
\sum_{m \neq n}^\infty \left| \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right| \leq \left( \sum_{m \neq n}^\infty |\langle e_m, \psi \rangle|^2 \right)^{1/2} \left( \sum_{m \neq n}^\infty \left| \frac{1}{E_n - E_m} \right|^2 \right)^{1/2} \leq \|\psi\| \left( \sum_{m \neq n}^\infty \frac{1}{|E_n - E_m|} \right)^{1/2} < \infty.
\]

By Lemma 2.1-(ii), one can define a linear operator \( T \) on \( \mathcal{H} \) as follows:
\[
D(T) := \left\{ \psi \in \mathcal{H} \mid \sum_{n=1}^\infty \left| \sum_{m \neq n}^\infty \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right|^2 < \infty \right\},
\]
\[
T\psi := i \sum_{n=1}^\infty \left( \sum_{m \neq n}^\infty \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right) e_n, \quad \psi \in D(T).
\]
Note that
\[
\|T\psi\|^2 = \sum_{n=1}^\infty \sum_{m \neq n}^\infty \left| \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right|^2.
\]

For a subset \( \mathcal{D} \subset \mathcal{H} \), we denote by \( \text{l.i.h.}(\mathcal{D}) \) the subspace algebraically spanned by the vectors of \( \mathcal{D} \).

The subspace
\[
\mathcal{D}_0 := \text{l.i.h.} \{ e_n \}_{n=1}^\infty
\]
is dense in \( \mathcal{H} \).
Lemma 2.2  The operator $T$ is densely defined with $D_0 \subset D(T)$ and

$$Te_k = i \sum_{n \neq k}^{\infty} \frac{1}{E_n - E_k} e_n, \quad k \in \mathbb{N}. \quad (2.11)$$

Proof. To prove $D_0 \subset D(T)$, it is sufficient to show that $e_k \in D(T), k \in \mathbb{N}$. Putting

$$c_n(k) := \sum_{m \neq n}^{\infty} \frac{\langle e_m, e_k \rangle}{E_n - E_m},$$

we have $c_k(k) = 0$ and $c_n(k) = 1/(E_n - E_k)$ for $n \neq k$. Hence, by Lemma 2.1-(i), we have

$$\sum_{n=1}^{\infty} |c_n(k)|^2 = \sum_{n \neq k}^{\infty} \frac{1}{(E_n - E_k)^2} < \infty.$$  

Hence $e_k \in D(T)$ and (2.11) holds. \qed

In general, it is not clear whether or not $T$ is a symmetric operator. But a restriction of $T$ to a smaller subspace gives a symmetric operator. Indeed, we have the following fact:

Lemma 2.3 ([8]) The operator

$$T_1 := T|D_0$$

(2.12)

(the restriction of $T$ to $D_0$) is symmetric.

Proof. It is enough to show that, for all $\psi \in D_0$, $\langle \psi, T\psi \rangle$ is real. For a complex number $z \in \mathbb{C}$ (the set of complex numbers), we denote its complex conjugate by $z^*$. We have

$$\langle \psi, T\psi \rangle = i \sum_{n=1}^{\infty} \langle \psi, e_n \rangle \sum_{m \neq n}^{\infty} \frac{\langle e_m, \psi \rangle}{E_n - E_m}. $$

Hence

$$\langle \psi, T\psi \rangle^* = i \sum_{n=1}^{\infty} \langle e_n, \psi \rangle \sum_{m \neq n}^{\infty} \frac{\langle \psi, e_m \rangle}{E_m - E_n}. $$

Since $\psi$ is in $D_0$, the double sum on $m, n$ with $m \neq n$ is a sum consisting of a finite term. Hence we can exchange the sum on $n$ and that on $m$ to obtain

$$\langle \psi, T\psi \rangle^* = i \sum_{m=1}^{\infty} \langle \psi, e_m \rangle \sum_{n \neq m}^{\infty} \frac{\langle e_n, \psi \rangle}{E_m - E_n} = \langle \psi, T\psi \rangle.$$

Hence $\langle \psi, T\psi \rangle$ is real. \qed

The operator $T_1$ defined by (2.12) is the time operator introduced by Galapon in [8]. Obviously we have

$$T_1 \subset T. \quad (2.13)$$

Hence

$$T^* \subset T_1^*. \quad (2.14)$$
Remark 2.2 It is asserted in [8] that $T_1$ is essentially self-adjoint without additional conditions. But, unfortunately, we find that this is not conclusive, because the proof of it given in [8] (pp.2678–2679) has some gap: the interchange of the double sum in Equation (2.30) on p.2678 in [8] may not be justified, at least, by the reasoning given there. The assertion is true in the case where $T_1$ becomes a bounded operator under an additional condition for $f E_n g_1 n = 1$, as we show below in the present paper. But, in the case where $T_1$ is unbounded, it seems to be very difficult to prove or disprove the essential self-adjointness of $T_1$. We leave this problem for future study.

Lemma 2.4 The subspace

$$D_c := \text{i.h.}(\{e_n - e_m \in \mathcal{H}|n, m \geq 1\}).$$

(2.15)

is dense in the Hilbert space $\mathcal{H}$. Moreover

$$D_c \subset D_0 \subset D(T).$$

(2.16)

Proof. Let $\psi \in D_c^\perp$ (the orthogonal complement of $D_c$). Then, for all $m, n \geq 1$, $|\langle e_n, \psi \rangle|^2 = |\langle e_m, \psi \rangle|^2$. By the Parseval equality, $||\psi||^2 = \lim_{N \to \infty} N |\langle e_m, \psi \rangle|^2$. This implies that $|\langle e_m, \psi \rangle|^2 = 0$ for all $m \geq 1$ and $||\psi|| = 0$. Hence $\psi = 0$. Thus $D_c$ is dense in $\mathcal{H}$. Inclusion relation (2.16) is obvious.

Theorem 2.5 ([8]) It holds that

$$D_c \subset D(T_1 H) \cap D(HT_1)$$

(2.17)

and

$$[T_1, H] \psi = i \psi, \quad \psi \in D_c.$$

(2.18)

Theorem 2.5 shows that $T_1$ is a time operator of $H$.

Remark 2.3 It is easy to see that, for all $k \in \mathbb{N}$, $T_1 e_k \notin D(H)$. Hence $D_0 \not\subset D(HT_1)$. Therefore one can not consider the commutation relation $[T_1, H]$ on $D_0$. Moreover, by direct computation, we have

$$\langle T_1 e_k, H e_\ell \rangle - \langle H e_k, T_1 e_\ell \rangle = -i(1 - \delta_{k\ell}), \quad k, \ell \in \mathbb{N}.$$

(2.19)

This means that $(T_1, H)$ does not satisfy the CCR in the sense of sesquilinear form on $D_0$ (a weak form of the CCR), either. These facts suggest that the pair $(T_1, H)$ is very sensitive to the domain on which their commutation relation is applied.

In concluding this section we discuss shortly non-uniqueness of time operators of $H$. We introduce a set of symmetric operators associated with $H$:

$$\{H\}'_{D_c} := \{S|S|S \text{ is a symmetric operator on } \mathcal{H} \text{ such that } D_c \subset D(SH) \cap D(HS) \text{ and } SH \psi = HS \psi, \forall \psi \in D_c\},$$

(2.20)

which may be viewed as a commutant of $\{H\}$ in a restricted sense. It is easy to see that, for all real-valued continuous function $f$ on $\mathbb{R}$, the operator $f(H)$ defined via the functional calculus is in $\{H\}'_{D_c}$.  

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Proposition 2.6 For all $S \in \{H\}_D$, $D_c \subset D((T_1 + S)H) \cap D(H(T_1 + S))$ and

$$[T_1 + S, H] \psi = i \psi, \quad \psi \in D_c. \quad (2.21)$$

Proof. A direct computation using Theorem 2.5 and (2.20).

Proposition 2.7 Let $T_2$ be a time operator of $H$ such that $D_c \subset D(T_2H) \cap D(HT_2)$ and

$$[T_2, H] \psi = i \psi, \quad \forall \psi \in D_c.$$  

Then $T_2 = T_1 + S$ with some $S \in \{H\}_D$.

Proof. We need only to show that $S := T_2 - T_1$ is in $\{H\}_D$. But this is obvious. \hfill \qed

3 General Properties

3.1 Closedness of $T$ and symmetricity of $T^*$

Lemma 3.1 $D_0 \subset D(T^*)$ and $T^*|D_0 = T_1$, i.e., $T_1 \subset T^*$.

Proof. It is enough to show that, for all $k \in \mathbb{N}$, $e_k \in D(T^*)$ and $T^*e_k = Te_k (= T_1 e_k)$, It is easy to see that, for all $\psi \in D(T)$,

$$\langle e_k, T \psi \rangle = i \sum_{m \neq k} \frac{\langle e_m, \psi \rangle}{E_k - E_m}. \quad (3.1)$$

By Lemma 2.2, the right hand side is equal to $\langle Te_k, \psi \rangle$. Hence $e_k \in D(T^*)$ and $T^*e_k = Te_k$. \hfill \qed

Proposition 3.2 The operator $T$ is closed and

$$T^* \subset T. \quad (3.2)$$

In particular, if $T$ is bounded, then $T$ is self-adjoint with $D(T) = \mathcal{H}$.

Proof. Let $\psi_k \in D(T)$, $k \in \mathbb{N}$ and $\psi_k \rightarrow \psi \in \mathcal{H}, T \psi_k \rightarrow \phi \in \mathcal{H}$ as $k \rightarrow \infty$. Then $\sup_{k \geq 1} \|T \psi_k\| < \infty$. Hence, by (2.9), there exists a constant $C > 0$ independent of $k \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \left| \sum_{n \neq m} \frac{\langle e_m, \psi_k \rangle}{E_n - E_m} \right|^2 \leq C.$$  

By (2.6), we have

$$\lim_{k \rightarrow \infty} \sum_{n \neq m} \frac{\langle e_m, \psi_k \rangle}{E_n - E_m} = \sum_{n \neq m} \frac{\langle e_m, \psi \rangle}{E_n - E_m}. \quad (3.3)$$
Hence it follows that
\[ \sum_{n=1}^{\infty} \left| \sum_{n \neq m} \frac{\langle e_n, \psi \rangle}{E_n - E_m} \right|^2 \leq C. \]

Therefore \( \psi \in D(T) \). By (3.1) and (3.3), we have for all \( \ell \in \mathbb{N} \)
\[ \lim_{k \to \infty} \langle e_\ell, T\psi_k \rangle = \langle e_\ell, T\psi \rangle. \]

Hence \( \langle e_\ell, \phi \rangle = \langle e_\ell, T\psi \rangle \), \( \ell \in \mathbb{N} \), implying \( \phi = T\psi \). Thus \( T \) is closed.

To prove (3.2), let \( \psi \in D(T^*) \). Putting \( \eta = T^*\psi \), we have \( \langle \eta, \chi \rangle = \langle \psi, T\chi \rangle \) for all \( \chi \in D(T) \). Taking \( \chi = e_k \) \((k \in \mathbb{N})\), we have
\[ \langle \eta, e_k \rangle = i \sum_{n \neq k} \frac{\langle \psi, e_n \rangle}{E_n - E_k}, \]which implies that
\[ \sum_{k=1}^{\infty} \left| \sum_{n \neq k} \frac{\langle \psi, e_n \rangle}{E_n - E_k} \right|^2 = \| \eta \|^2 < \infty. \]

Hence \( \psi \in D(T) \). Then, by (3.1), the right hand side of (3.4) is equal to \( \langle T\psi, e_k \rangle \). Hence \( \eta = T\psi \). Thus (3.2) holds.

Let \( T \) be bounded. Then, by the denseness of \( D(T) \) and the closedness of \( T, D(T) = \mathcal{H} \). Hence \( D(T^*) = \mathcal{H} \). Thus, by (3.2), \( T^* = T \), i.e., \( T \) is self-adjoint.

**Corollary 3.3** The operator \( T^* \) is symmetric.

**Proof.** By Lemma 3.1, \( T^* \) is densely defined. Hence, by Proposition 3.2, \( T^* \subset T = (T^*)^* \). Thus \( T^* \) is symmetric.

Thus we have
\[ T_1 \subset T^* \subset T. \]

Corollary 3.3 shows that \( T^* \) also is a time operator of \( H \).

For a closable operator \( A \) on a Hilbert space, we denote its closure by \( \bar{A} \).

**Proposition 3.4** \( \bar{T}_1 = T^* \).

**Proof.** Note that \( \bar{T}_1 = T^* \) if and only if \( T_1^* = T \). By (3.5), we have \( \bar{T}_1 \subset T^* \). Hence \( T \subset T_1^* \). Thus it is enough to show that \( D(T_1^*) \subset D(T) \). For all \( \psi \in D(T_1^*) \), we have
\[ \langle T_1^*\psi, e_\ell \rangle = \langle \psi, T_1 e_\ell \rangle = i \sum_{n \neq \ell} \frac{\langle \psi, e_n \rangle}{E_n - E_\ell}. \]

Hence we obtain
\[ \infty > \| T_1^*\psi \|^2 = \sum_{l=1}^{\infty} \left| \langle T_1^*\psi, e_l \rangle \right|^2 = \sum_{l=1}^{\infty} \sum_{n \neq l} \left| \frac{\langle \psi, e_n \rangle}{E_n - E_l} \right|^2, \]
implicating that \( \psi \in D(T) \). Thus \( D(T_1^*) \subset D(T) \).
3.2 Absence of invariant dense domains for $T$

We first note the following general fact:

**Proposition 3.5** Let $Q$ be a bounded self-adjoint operator on $\mathcal{H}$ and $P$ be a self-adjoint operator on $\mathcal{H}$. Suppose that there is a dense subspace $D$ in $\mathcal{H}$ such that the following (i)–(iii) hold:

(i) $QD \subset D \subset D(P)$.

(ii) $D$ is a core of $P$.

(iii) The pair $(Q, P)$ obeys the CCR on $D$: $[Q, P]\psi = i\psi, \forall \psi \in D$.

Then $\sigma(P) = \mathbb{R}$.

**Proof.** Since $Q$ is a bounded self-adjoint operator, we have for all $t \in \mathbb{R}$

$$e^{itQ} = \sum_{k=0}^{\infty} \frac{(itQ)^k}{k!}$$

in operator norm. Conditions (i) and (iii) imply that, for all $k \in \mathbb{N}$ and $\psi \in D$

$$Q^k P\psi - PQ^k \psi = i k Q^{k-1} \psi.$$

Hence, for all $t \in \mathbb{R}$ and vectors $\psi$ in $D$, we have

$$e^{itQ} P\psi = P\psi + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} Q^k P\psi$$

$$= P\psi + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} (PQ^k + ik Q^{k-1}) \psi$$

$$= P\psi + \sum_{k=1}^{\infty} \left\{ \frac{P (itQ)^k}{k!} \psi - t \frac{Q^{k-1}}{(k-1)!} \psi \right\}.$$

It follows from the closedness of $P$ that $e^{itQ}\psi$ is in $D(P)$ and

$$Pe^{itQ} \psi = e^{itQ} P\psi + te^{itQ} \psi.$$  \hspace{1cm} (3.6)

By condition (ii), this equality extends to all $\psi \in D(P)$ with $e^{itQ}\psi \in D(P), \forall t \in \mathbb{R}, \forall \psi \in D(P)$. Hence the operator equality $e^{-itQ} Pe^{itQ} = P + t$ follows. Thus $\sigma(P) = \sigma(P + t)$ for all $t \in \mathbb{R}$. This implies that $\sigma(P) = \mathbb{R}$. \hfill $\square$

**Theorem 3.6** If $T$ is bounded (hence self-adjoint by Proposition 3.2), then there is no dense subspace $D$ in $\mathcal{H}$ such that the following (i)–(iii) hold:

(i) $T D \subset D \subset D(H)$.

(ii) $D$ is a core of $H$. 

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The pair \((T, H)\) obeys the CCR on \(\mathcal{D}\).

**Proof.** Suppose that there were such a dense subspace \(\mathcal{D}\) as stated above. Then we can apply Proposition 3.5 with \((Q, P) = (T, H)\) to conclude that \(\sigma(H) = \mathbb{R}\). But this is a contradiction. \(\Box\)

**Remark 3.1** A special case of this theorem was established in [7, Theorem 9.5].

### 3.3 Reflection symmetry of the spectrum of \(T_1, T^*\) and \(T\)

We first recall the definition of the spectrum of a general linear operator (not necessarily closed). For a linear operator \(A\) on a Hilbert space \(\mathcal{K}\), the resolvent set of \(A\), denoted \(\rho(A)\), is defined by

\[
\rho(A) := \{ z \in \mathbb{C} | \text{Ran}(A - z) \text{ is dense in } \mathcal{K} \text{ and } A - z \text{ is injective with } (A - z)^{-1} \text{ bounded} \}.
\]

Then the set \(\sigma(A) := \mathbb{C} \setminus \rho(A)\) is called the spectrum of \(A\).

We denote by \(T^\#\) any of \(T_1, T^*\) and \(T\).

**Proposition 3.7** The spectrum \(\sigma(T^\#)\) of \(T^\#\) is reflection symmetric with respect to the imaginary axis, i.e., if \(z \in \sigma(T^\#)\), then \(-z^* \in \sigma(T^\#)\). In particular, if \(T\) is self-adjoint, then \(\sigma(T)\) is reflection symmetric with respect to the origin of the real axis.

**Proof.** We define a conjugation \(J\) on \(\mathcal{K}\) by

\[
J\psi := \sum_{n=1}^{\infty} \langle \psi, e_n \rangle e_n, \quad \psi \in \mathcal{K}.
\] (3.7)

It is easy to see that operator equality \(JT^\#J = -T^\#\) holds \((JD(T^\#) = D(T^\#))\). Hence, for all \(z \in \mathbb{C}\), we have \(J(T^\# - z)J = -(T^\# + z^*)\). This implies that, if \(z \in \rho(T^\#)\), then \(-z^* \in \rho(T^\#)\). Thus the same holds for the spectrum \(\sigma(T^\#) = \mathbb{C} \setminus \rho(T^\#)\). \(\Box\)

### 4 Boundedness of \(T\)

In this section we present a general criterion for the operator \(T\) to be bounded. For mathematical generality and for later use, we consider a more general class of operators than that of \(T\). Let \(b := \{b_{nm}\}_{n,m=1}^{\infty}\) be a double sequence of complex numbers such that

\[
\|b\|_\infty := \sup_{n,m \geq 1} |b_{nm}| < \infty.
\] (4.1)
Then, in the same way as in Lemma 2.1-(ii), for all \( \psi \in \mathcal{H} \), the infinite series
\[
\sum_{m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle e_m, \psi \rangle
\]
absolutely converges. Hence one can define a linear operator \( T_b \) on \( \mathcal{H} \) as follows:
\[
\begin{align*}
D(T_b) & := \left\{ \psi \in \mathcal{H} \left| \sum_{n=1}^{\infty} \left| \sum_{m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle e_m, \psi \rangle \right|^2 < \infty \right\} \right., \\
T_b \psi & := i \sum_{n=1}^{\infty} \left( \sum_{m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle e_m, \psi \rangle \right) e_n, \quad \psi \in D(T_b).
\end{align*}
\] (4.2)

Obviously \( T = T_b \) with \( b \) satisfying \( b_{nm} = 1 \) for all \( n, m \in \mathbb{N} \). In the same way as in the case of \( T \), one can prove the following fact:

**Lemma 4.1** The operator \( T_b \) is closed.

The following lemma is probably well known (but, for the completeness, we give a proof):

**Lemma 4.2** Let \( A \) be a densely defined linear operator on a Hilbert space \( \mathcal{K} \). Suppose that there exist a dense subspace \( \mathcal{D} \) in \( \mathcal{K} \) and a constant \( C > 0 \) such that \( \mathcal{D} \subseteq D(A) \) and
\[
| \langle \psi, A \phi \rangle | \leq C \| \psi \|^{2}, \quad \psi \in \mathcal{D}.
\]
Then \( A \) is bounded with \( \| \tilde{A} \| \leq 2C \), where \( \tilde{A} \) is the closure of \( A \).

If \( A \) is symmetric in addition, then \( \| \tilde{A} \| \leq C \).

**Proof.** Let \( \psi, \phi \in \mathcal{D} \). Then, by the polarization identity
\[
\langle \psi, A \phi \rangle = \frac{1}{4} \left( \langle \psi + \phi, A(\psi + \phi) \rangle - \langle \psi - \phi, A(\psi - \phi) \rangle + i \langle \psi - i\phi, A(\psi - i\phi) \rangle - i \langle \psi + i\phi, A(\psi + i\phi) \rangle \right),
\]
we have
\[
| \langle \psi, A \phi \rangle | \leq \frac{C}{4} (\| \psi + \phi \|^2 + \| \psi - \phi \|^2 + \| \psi - i\phi \|^2 + \| \psi + i\phi \|^2) = C (\| \psi \|^2 + \| \phi \|^2).
\]
Replacing \( \psi \neq 0 \) by \( \| \phi \|/\| \psi \| \) we have
\[
| \langle \psi, A \phi \rangle | \leq 2C \| \psi \| \| \phi \|.
\]
For \( \psi = 0 \), this inequality trivially holds. Since \( \mathcal{D} \) is dense, it follows from the Riesz representation theorem that \( \| A \phi \| \leq 2C \| \phi \|, \phi \in \mathcal{D} \). Thus the first half of the lemma follows.
Let $A$ be symmetric. Then, $\langle \psi, A\psi \rangle \in \mathbb{R}$ for all $\psi \in D(A)$. Hence

$$|\Re \langle \psi, A\phi \rangle| = \frac{1}{4} |\langle \psi + \phi, A(\psi + \phi) \rangle - \langle \psi - \phi, A(\psi - \phi) \rangle| \leq \frac{C}{2}(\|\psi\|^2 + \|\phi\|^2), \quad \psi \in D.$$

We write $\langle \psi, A\phi \rangle = |\langle \psi, A\phi \rangle|e^{i\theta}$ with $\theta \in \mathbb{R}$. Then $|\langle \psi, A\phi \rangle| = \langle e^{i\theta} \psi, A\phi \rangle$. Hence

$$|\langle \psi, A\phi \rangle| = \Re \langle e^{i\theta} \psi, A\phi \rangle \leq \frac{C}{2}(\|e^{i\theta} \psi\|^2 + \|\phi\|^2) = \frac{C}{2}(\|\psi\|^2 + \|\phi\|^2).$$

Thus, in the same manner as above, we can obtain $|\langle \psi, A\phi \rangle| \leq C\|\psi\|\|\phi\|$, $\psi, \phi \in D$. □

**Lemma 4.3** For all $s > 1$ and $n \geq 2$,

$$\sum_{m=1}^{n-1} \frac{1}{n^s - m^s} \leq \frac{\log n}{n^{s-1}} + \frac{1}{s(n-1)^{s-1}}. \quad (4.4)$$

**Proof.** It easy to see that

$$\sum_{m=1}^{n-1} \frac{1}{n^s - m^s} \leq \int_1^{n-1} \frac{1}{n^s - x^s} dx + \frac{1}{n^s - (n-1)^s}. \quad (4.5)$$

By the change of variable $x = ny$, we have

$$\int_1^{n-1} \frac{1}{n^s - x^s} dx = \frac{1}{n^{s-1}} \int_{1/n}^{(n-1)/n} \frac{1}{1 - y^s} dy \quad (4.6)$$

$$\leq \frac{1}{n^{s-1}} \int_0^{(n-1)/n} \frac{1}{1 - y} dy \quad (4.7)$$

$$\leq \frac{\log n}{n^{s-1}}. \quad (4.8)$$

We have

$$\frac{1}{n^s - (n-1)^s} = \frac{1}{(n-1)^s} \cdot \frac{1}{\left(\frac{n}{n-1}\right)^s - 1}.$$

By the well known inequality

$$x^s - 1 \geq s(x - 1), \quad x > 0, s \geq 1, \quad (4.9)$$

we obtain

$$\frac{1}{n^s - (n-1)^s} \leq \frac{1}{s(n-1)^{s-1}}. \quad (4.10)$$

Thus (4.4) holds. □
Lemma 4.4 Let \( s > 1 \). Then

\[
\sup_{n \geq 2} \left( \sum_{m=1}^{n-1} \frac{1}{n^s - m^s} \right) < \infty \tag{4.11}
\]

and

\[
\sup_{m \geq 1} \left( \sum_{n=m+1}^{\infty} \frac{1}{n^s - m^s} \right) < \infty. \tag{4.12}
\]

Proof. Property (4.11) follows from Lemma 4.3.

To prove (4.12), we write

\[
\int_{m+1}^{\infty} \frac{1}{x^s - m^s} \, dx = \frac{1}{m^{s-1}} \left( \int_{(m+1)/m}^{R} \frac{1}{y^s - 1} \, dy + C_R \right),
\]

where

\[
C_R := \int_{R}^{\infty} \frac{1}{y^s - 1} \, dy < \infty.
\]

We fix a constant \( R > 2(\geq (m + 1)/m) \). By the change of variable \( x = my \), we have

\[
\int_{m+1}^{\infty} \frac{1}{x^s - m^s} \, dx \leq \int_{(m+1)/m}^{R} \frac{1}{y^s - 1} \, dy 
\]

Hence

\[
\int_{m+1}^{\infty} \frac{1}{x^s - m^s} \, dx \leq \frac{\log m}{sm^{s-1}} + \frac{\log(R - 1)}{sm^{s-1}} + \frac{1}{m^{s-1}}C_R.
\]

Thus (4.12) follows. \( \square \)

Let

\[
c_H(n) := \sum_{m=1}^{n-1} \frac{E_n}{(E_n - E_m)E_m}, \quad n \geq 2, \tag{4.13}
\]

\[
d_H(m) := \sum_{n=m+1}^{\infty} \frac{E_m}{(E_n - E_m)E_n}, \quad m \geq 1. \tag{4.14}
\]

Since \( c_H(n) \) and \( d_H(m) \) are positive (recall that \( E_n > 0, \forall n \in \mathbb{N} \)), one can define constants

\[
c_H := \sup_{n \geq 2} c_H(n), \tag{4.15}
\]

\[
d_H := \sup_{m \geq 1} d_H(m), \tag{4.16}
\]

which are finite or infinite.
Theorem 4.5 Suppose that there exist constants \( \alpha > 1, C > 0 \) and \( a > 0 \) such that
\[
E_n - E_m \geq C(n^\alpha - m^\alpha), \quad n > m > a.
\] (4.17)

Then \( T_b \) is a bounded operator with \( D(T_b) = \mathcal{H} \) and
\[
\|T_b\| \leq 4\|b\|_\infty \sqrt{c_H d_H}.
\] (4.18)

Moreover, if \( b^*_{nm} = b_{nm} \) for all \( m, n \in \mathbb{N} \), then \( T_b \) is a bounded self-adjoint operator with \( D(T_b) = \mathcal{H} \) and
\[
\|T_b\| \leq 2\|b\|_\infty \sqrt{c_H d_H}.
\] (4.19)

In particular, \( T \) is a bounded self-adjoint operator with \( D(T) = \mathcal{H} \).

Proof. By Lemma 4.2, it is enough to show that \( c_H \) and \( d_H \) are finite and
\[
| \langle \psi, T_b \psi \rangle | \leq 2\|b\|_\infty \sqrt{c_H d_H} \|\psi\|^2, \quad \psi \in \mathcal{D}_0.
\] (4.20)

Then \( T_b \) is bounded with (4.18). Since \( T_b \) is densely defined and closed, it follows that \( D(T_b) = \mathcal{H} \). As in the case of \( T \), one can show that, if \( b^*_{nm} = b_{nm} \) for all \( m, n \in \mathbb{N} \), then \( T_b|\mathcal{D}_0 \) is symmetric and hence \( T_b \) is a bounded self-adjoint operator with \( D(T_b) = \mathcal{H} \) and (4.19) holds. Therefore the desired result follows.

To prove (4.20), we first note that, for \( \psi \in \mathcal{D}_0 \),
\[
\langle \psi, T_b \psi \rangle = i \sum_{m,n=1, m\neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle \psi, e_n \rangle \langle e_m, \psi \rangle.
\]

Hence
\[
| \langle \psi, T_b \psi \rangle | \leq 2\|b\|_\infty A(\psi),
\]

where
\[
A(\psi) := \sum_{n>m \geq 1} \left| \frac{\langle e_m, \psi \rangle \langle \psi, e_n \rangle}{E_n - E_m} \right|.
\]

Inserting \( 1 = \sqrt{E_m/E_n} \cdot \sqrt{E_n/E_m} \) into the summand on the right hand side and using the Cauchy-Schwarz inequality, we have
\[
A(\psi)^2 \leq B(\psi) C(\psi)
\]

with
\[
B(\psi) = \sum_{n>m \geq 1} \frac{|\langle e_n, \psi \rangle|^2}{E_n - E_m} \cdot \frac{E_n}{E_m},
\]
\[
C(\psi) = \sum_{n>m \geq 1} \frac{|\langle e_m, \psi \rangle|^2}{E_n - E_m} \cdot \frac{E_m}{E_n}.
\]
One can rewrite and estimate $B(\psi)$ as follows:

$$B(\psi) = \sum_{n=2}^{\infty} |\langle e_n, \psi \rangle|^2 c_H(n) \leq \|\psi\|^2 c_H.$$ 

Similarly we have

$$C(\psi) \leq \|\psi\|^2 d_H. \quad (4.21)$$

Hence

$$|\langle \psi, T_b \psi \rangle| \leq 2\|b\|_{\infty} \sqrt{c_H d_H} |\psi|^2. \quad (4.22)$$

Therefore we need only to prove that $c_H$ and $d_H$ are finite.

We can write

$$c_H(n) = \sum_{m=1}^{n-1} \frac{1}{E_n - E_m} + \sum_{m=1}^{n-1} \frac{1}{E_m}.$$ 

By assumption (4.17), we have

$$\frac{1}{E_n - E_m} \leq \frac{1}{C (n^\alpha - m^\alpha)}, \quad n > m > a. \quad (4.23)$$

Since we have

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty,$$

it follows that

$$\varepsilon_1 := \sum_{m=1}^{\infty} \frac{1}{E_m} < \infty.$$ 

Thus

$$c_H(n) \leq \sum_{m=1}^{n-1} \frac{1}{E_n - E_m} + \varepsilon_1.$$ 

Let $n_0 \geq 2$ be a natural number such that $n_0 > a$. Then, for all $n > n_0$

$$\sum_{m=1}^{n-1} \frac{1}{E_n - E_m} \leq \sum_{m=1}^{n_0-1} \frac{1}{E_n - E_m} + \frac{1}{C} \sum_{m=n_0}^{n-1} \frac{1}{n^\alpha - m^\alpha}. $$

By (4.4), the right hand side is uniformly bounded in $n$. Thus we have $c_H < \infty$.

To prove $d_H < \infty$, we write for $m > a$

$$d_H(m) = \sum_{n=m+1}^{\infty} \frac{1}{(E_n - E_m)} - \sum_{n=m+1}^{\infty} \frac{1}{E_n} \leq \sum_{n=m+1}^{\infty} \frac{1}{E_n - E_m} \leq \frac{1}{C} \sum_{n=m+1}^{\infty} \frac{1}{n^\alpha - m^\alpha}.$$ 

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Hence, by (4.12) in Lemma 4.4, we have
\[ \sup_{m > a} d_H(m) < \infty. \]
Thus it follows that \(d_H < \infty.\)

**Example 4.1** Let \(\lambda > 0, \alpha > 1\) and \(P(x)\) be a real polynomial of \(x \in \mathbb{R}\) with degree \(p < \alpha\). Then it is easy to see that the sequence \(\{E_n\}_{n=1}^\infty\) defined by
\[ E_n := \lambda n^\alpha + P(n) \]
satisfies the assumptions (H.1), (H.2) and (4.17). Thus, by Theorem 4.5, in the present example, \(T_b\) is bounded.

We remark that Theorem 4.5 does not cover the case \(E_n = \lambda n + \mu\) with constants \(\lambda > 0\) and \(\mu \in \mathbb{R}\). For this case, we have the following theorem:

**Theorem 4.6** Suppose that there exist constants \(\lambda > 0, \mu \in \mathbb{R}\) and \(a > 0\) such that
\[ E_n = \lambda n + \mu, \quad n > a. \tag{4.24} \]
Then \(T\) is a bounded self-adjoint operator with \(D(T) = \mathcal{H}\).

**Proof.** Let \(k_0\) be the greatest integer such that \(k_0 \leq a\). Let \(a_n := \langle e_n, \psi \rangle (\psi \in \mathcal{H})\). Then, by the Parseval equality, we have \(\sum_{n=1}^\infty |a_n|^2 = ||\psi||^2\). Let \(\psi \in \mathcal{D}_0\). Then we can write:
\[ \langle \psi, T\psi \rangle = S_1 + S_2 + S_3 + S_4, \]
where
\[
S_1 := i \sum_{n=1}^{k_0} \sum_{m \neq n}^{k_0} \frac{a_n^* a_m}{E_n - E_m}, \\
S_2 := i \sum_{n=1}^{k_0} \sum_{m=k_0+1}^{\infty} \frac{a_n^* a_m}{E_n - E_m}, \\
S_3 := i \sum_{n=k_0+1}^{\infty} \sum_{m=1}^{k_0} \frac{a_n^* a_m}{E_n - E_m}, \\
S_4 := \frac{1}{\lambda} \sum_{n=k_0+1}^{\infty} \sum_{m \neq n, m \geq k_0+1}^{\infty} \frac{a_n^* a_m}{n - m}. \\
\]
By the Schwarz inequality, we have
\[ |S_j| \leq C_j ||\psi||^2, \quad j = 1, 2, 3, \]
where $C_j > 0$ is a constant. To estimate $|S_4|$, we use the following well known inequality [9, Theorem 294]:

$$\left| \sum_{n,m=1,n\neq m}^{\infty} \frac{x_n y_m}{n-m} \right| \leq \pi \sqrt{\sum_{n=1}^{\infty} x_n^2} \sqrt{\sum_{m=1}^{\infty} y_m^2}$$

for all real sequences \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \). Hence

$$|S_4| \leq \pi \|\psi\|^2.$$ 

Therefore it follows that $|\langle \psi, T\psi \rangle| \leq \text{const.} \|\psi\|^2$. Thus $T$ is bounded. \(\square\)

**Example 4.2** A physically interesting case is the case where $E_n = \omega(n + \frac{1}{2})$, \( n \in \{0\} \cup \mathbb{N} \) with a constant $\omega > 0$. In this case, by Theorem 4.6, $T$ is a bounded self-adjoint operator with $D(T) = \mathcal{H}$ and takes the form

$$T\psi = \frac{i}{\omega} \sum_{n=1}^{\infty} \left( \sum_{m \neq n}^{\infty} \frac{\langle e_m, \psi \rangle}{n-m} \right) e_n, \quad \psi \in \mathcal{H}.$$ 

Moreover, one can prove that

$$\sigma(T) = [-\pi/\omega, \pi/\omega]$$

([4, Theorem 2.1]).

Let $\hat{N} := \omega^{-1}H - 1/2$ and $\hat{\theta} := \omega T$. Then it follows that

$$\sigma(\hat{N}) = \{0\} \cup \mathbb{N}, \quad \sigma(\hat{\theta}) = [-\pi, \pi],$$

$$[\hat{\theta}, \hat{N}]\psi = i\psi, \quad \psi \in \mathcal{D}_c.$$ (4.25) (4.26)

As is well known, in the context of quantum mechanics, the sequence \( \{\omega(n + \frac{1}{2})\}_{n=1}^{\infty} \) appears as the spectrum of the one-dimensional quantum harmonic oscillator Hamiltonian with mass $m > 0$

$$H_{os} := \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

in the Schrödinger representation $(q, p)$ of the CCR, where $p := -iD$ with $D$ being the generalized partial differential operator on $L^2(\mathbb{R})$ and $q$ is the multiplication operator by the variable $x \in \mathbb{R}$. In this context, the operator $\hat{N}$ is called the number operator and, in view of (4.25) and (4.26), the operator $\hat{\theta}$ is interpreted as a *phase operator* [7].

## 5 Unboundedness of $T$

As for the unboundedness of $T$, we have the following fact:

**Theorem 5.1** If \( \{E_n\}_{n=1}^{\infty} \) satisfies

$$\inf_{n \in \mathbb{N}} (E_{n+1} - E_n) = 0,$$

(5.1) then $T$ is unbounded.
Proof. By (5.1), there exists a subsequence \( \{E_{p_k}\}_{k=1}^\infty \) of \( \{E_p\}_{p=1}^\infty \) such that
\[
\lim_{k \to \infty} (E_{p_{k+1}} - E_{p_k}) = 0. \tag{5.2}
\]
Hence we have
\[
\|Te_{p_k}\|^2 = \sum_{n=1}^{\infty} \sum_{m \neq n} \frac{1}{(E_n - E_m)^2} \leq \sum_{n \neq p_k} \frac{1}{E_{p_{k+1}} - E_{p_k}} \to \infty \quad (k \to \infty).
\]
Thus \( T \) is unbounded. \( \square \)

Example 5.1 Let
\[
E_n = \lambda n^\alpha + \mu
\]
with constants \( \lambda > 0, \alpha > 1/2 \) and \( \mu \in \mathbb{R} \). Then \( \{E_n\}_{n=1}^{\infty} \) satisfies the assumptions (H.1) and (H.2). As we have already seen, \( T \) is bounded if \( \alpha \geq 1 \).

Let \( 1/2 < \alpha < 1 \). Then one easily sees that
\[
\lim_{n \to \infty} (E_{n+1} - E_n) = 0.
\]
Hence \( \inf_{n \in \mathbb{N}} (E_{n+1} - E_n) = 0. \) Therefore, in this case, \( T \) is unbounded. Thus \( T \) is bounded if and only if \( \alpha \geq 1 \).

6 Hilbert-Schmidtness of \( T \)

In this section we investigate Hilbert-Schmidtness of the operator \( T \).

Proposition 6.1 The operator \( T \) is Hilbert-Schmidt if and only if
\[
\sum_{n=1}^{\infty} \sum_{m \neq n} \frac{1}{(E_n - E_m)^2} < \infty. \tag{6.1}
\]
In that case, \( T \) is self-adjoint with
\[
\|T\|_2^2 = \sum_{n=1}^{\infty} \sum_{m \neq n} \frac{1}{(E_n - E_m)^2}, \tag{6.2}
\]
where \( \| \cdot \|_2 \) denotes Hilbert-Schmidt norm. In particular, there exist a C.O.N.S. \( \{f_n\}_{n=1}^{\infty} \) of \( \mathcal{H} \) and real numbers \( t_n, n \in \mathbb{N} \) such that \( Tf_n = t_n f_n \) and \( t_n \to 0 \) \( (n \to \infty) \).

Proof. Suppose that \( T \) is Hilbert-Schmidt. Then \( \sum_{n=1}^{\infty} \|Te_n\|^2 < \infty. \) On the other hand, we have
\[
\sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{n=1}^{\infty} \sum_{m \neq n} \frac{1}{(E_n - E_m)^2}. \tag{6.3}
\]
Hence (6.1) follows with (6.2).

Conversely, (6.1) holds. Hence, by (6.3), $\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty$. Therefore $T$ is Hilbert-Schmidt. The last statement follows from the Hilbert-Schmidt theorem (e.g., [12, Theorem VI.16]). \hfill \Box

**Remark 6.1** In Proposition 6.1, the number $t_n \neq 0$ is an eigenvalue of $T$ with a finite multiplicity. Since $T$ is self-adjoint in the present case, it may be an observable in the context of quantum mechanics. If this is the case, then Proposition 6.1 shows that the observable described by $T$ ("time" in any sense?) is quantized (discretized) in the quantum system whose Hamiltonian is $H$ with eigenvalues $\{E_n\}_{n=1}^{\infty}$ satisfying (6.1).

The next theorem gives a class of $H$ such that $T$ is Hilbert-Schmidt:

**Theorem 6.2** Suppose that (4.17) in Theorem 4.5 holds with $\alpha > 3/2$. Then $T$ is Hilbert-Schmidt and self-adjoint.

**Proof.** Since $1/(E_n - E_m)^2$ is symmetric in $n$ and $m$, it is sufficient to show that $\sum_{n>m \geq 1} 1/(E_n - E_m)^2 < \infty$. By the present assumption, we need only to show that

$$\Sigma := \sum_{n>m \geq 1} \frac{1}{(n^\alpha - m^\alpha)^2} < \infty$$

for all $\alpha > 3/2$. We have

$$\Sigma = \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \frac{1}{(n^\alpha - m^\alpha)^2} \leq \sum_{n=2}^{\infty} \frac{1}{n^\alpha - (n-1)^\alpha} \cdot \sum_{m=1}^{n-1} \frac{1}{(n^\alpha - m^\alpha)}.$$

Using (4.4) and (4.10), we obtain

$$\Sigma \leq \sum_{n=2}^{\infty} \frac{1}{\alpha(n-1)^{\alpha-1}n^{\alpha-1}} \log n + \sum_{n=2}^{\infty} \frac{1}{\alpha^2(n-1)^{2(\alpha-1)}}.$$

Each infinite series on the right hand side converges for all $\alpha > 3/2$. Thus the desired result follows. \hfill \Box

**7 The Galapon Time Operator as a Generalized Time Operator**

It is shown that every self-adjoint operator which has a strong time operator is absolutely continuous [10]. Hence the Galapon time operator $T_1$ is not a strong time operator of $H$. But it may be a generalized time operator of $H$. In this section we investigate this aspect.
7.1 An operator-valued function on \( \mathbb{R} \)

In the same way as in Lemma 2.1-(ii), one can show that, for all \( \psi \in \mathcal{H} \), \( n \in \mathbb{N} \) and all \( t \in \mathbb{R} \), the infinite series

\[
\sum_{m \neq n} ^\infty \frac{e^{it(E_n - E_m)} - 1}{E_n - E_m} \langle e_m, \psi \rangle
\]

absolutely converges. Hence, for each \( t \in \mathbb{R} \), one can define a linear operator \( K(t) \) as follows:

\[
D(K(t)) := \left\{ \psi \in \mathcal{H} \left| \sum_{n=1} ^\infty \left| \sum_{m \neq n} \frac{e^{it(E_n - E_m)} - 1}{E_n - E_m} \langle e_m, \psi \rangle \right|^2 < \infty \right. \right\}, \tag{7.1}
\]

\[
K(t)\psi := i \sum_{n=1} ^\infty \left( \sum_{m \neq n} \frac{e^{it(E_n - E_m)} - 1}{E_n - E_m} \langle e_m, \psi \rangle \right) e_n, \quad \psi \in D(K(t)). \tag{7.2}
\]

It is easy to see that, for all \( t \in \mathbb{R} \),

\[
\mathcal{D}_0 \subset D(K(t)) \tag{7.3}
\]

and

\[
K(t) e_k = i \sum_{n \neq k} \frac{e^{it(E_n - E_k)} - 1}{E_n - E_k} e_n, \quad k \in \mathbb{N}. \tag{7.4}
\]

The correspondence \( K : \mathbb{R} \ni t \mapsto K(t) \) gives an operator-valued function on \( \mathbb{R} \). In the notation in Section 4, \( K(t) \) is the operator \( T_b \) with \( b_{nm} = e^{it(E_n - E_m)} - 1, \ n, m \in \mathbb{N} \).

**Remark 7.1** Equation (7.4) shows that \( K(t) \neq tI |\mathcal{D}_0 \). Hence \( T \) cannot be a strong time operator of \( H \), as already remarked based on the general theory of strong time operators.

**Proposition 7.1** For all \( t \in \mathbb{R} \), \( K(t) \) is a densely defined closed operator.

*Proof.* Similar to the proof of Proposition 3.2. \( \square \)

**Proposition 7.2** For all \( t \in \mathbb{R} \), \( K(t) |\mathcal{D}_0 \) is symmetric.

*Proof.* Similar to the proof of Lemma 2.3. \( \square \)

**Theorem 7.3** For all \( \psi \in D(T_1) (= \mathcal{D}_0) \) and \( t \in \mathbb{R} \), \( e^{-itH} \psi \in D(T_1) \) and

\[
T_1 e^{-itH} \psi = e^{-itH}(T_1 + K(t))\psi. \tag{7.5}
\]

*Proof.* We need only to prove the statement in the case \( \psi = e_k (\forall k \in \mathbb{N}) \). Since \( e^{-itH} e_k = e^{-itE_k} e_k \), it follows that \( e^{-itH} e_k \in D(T_1) \) with

\[
T_1 e^{-itH} e_k = e^{-itE_k} \sum_{n \neq k} ^\infty \frac{i}{E_n - E_k} e_n.
\]
We have
\[ e^{-itH}T_1 e_k = i \sum_{n \neq k} \frac{e^{-itE_n} - 1}{E_n - E_k} e_n. \]

It follows from these equations that
\[ T_1 e^{-itH} e_k - e^{-itH} T_1 e_k = e^{-itH} K(t) e_k. \]

Thus the desired result follows. \( \square \)

**Corollary 7.4** Suppose that, for all \( t \in \mathbb{R} \), \( K(t) \) is bounded. Then \( T_1 \) is a generalized time operator of \( H \) with commutation factor \( K \).

**Proof.** This follows from Theorem 7.3, Proposition 7.1 and Proposition 7.2. \( \square \)

In view of Corollary 7.4, we need to investigate conditions for \( K(t) \) to be bounded.

**Proposition 7.5** Suppose that (4.17) holds with \( \alpha > 1 \). Then, for all \( t \in \mathbb{R} \), \( K(t) \) is a bounded self-adjoint operator with \( D(K(t)) = \mathcal{H} \).

**Proof.** This follows from an application of Theorem 4.5 to the case where \( b_{nm} = e^{it(E_n - E_m)} - 1 \), \( n, m \in \mathbb{N} \). \( \square \)

**Proposition 7.6** Suppose that (6.1) holds. Then, for all \( t \in \mathbb{R} \), \( K(t) \) is Hilbert-Schmidt and self-adjoint with
\[ \| K(t) \|_2^2 = \sum_{k=1}^{\infty} \sum_{n \neq k} \left| \frac{e^{it(E_n - E_k)} - 1}{E_n - E_k} \right|^2. \]

**Proof.** Similar to the proof of Proposition 6.1. \( \square \)

### 7.2 Non-differentiability of \( K \)

From the viewpoint of the theory of generalized time operators [2], it is interesting to examine differentiability of the operator-valued function \( K \).

**Proposition 7.7** For all \( k \in \mathbb{N} \), the \( \mathcal{H} \)-valued function \( : \mathbb{R} \ni t \mapsto K(t)e_k \) is not strongly differentiable on \( \mathbb{R} \).

**Proof.** We first show that \( K(t)e_k \) is not strongly differentiable at \( t = 0 \). Since \( K(0)e_k = 0 \), we have for all \( t \in \mathbb{R} \setminus \{0\} \) and \( N > k \)
\[
\left\| \frac{K(t)e_k - K(0)e_k}{t} \right\|^2 \geq \sum_{n \neq k}^{N+1} \frac{|e^{it(E_n - E_k)} - 1|^2}{t^2 |E_n - E_k|^2}. 
\]
Hence
\[
\liminf_{t \to 0} \left\| \frac{K(t)e_k - K(0)e_k}{t} \right\|^2 \geq \sum_{n \neq k}^{N+1} 1 = N.
\]
Since \(N > k\) is arbitrary, it follows that
\[
\lim_{t \to 0} \left\| \frac{K(t)e_k - K(0)e_k}{t} \right\|^2 = +\infty.
\]
This implies that \(K(t)e_k\) is not strongly differentiable at \(t = 0\).

We next show that \(K(t)e_k\) is not strongly differentiable at each \(t \neq 0\). By (7.5), we have for all \(s \in \mathbb{R} \setminus \{0\}\)
\[
\frac{K(t+s)e_k - K(t)e_k}{s} = e^{it(H-E_k)} \frac{K(s)e_k}{s}.
\]
Hence
\[
\left\| \frac{K(t+s)e_k - K(t)e_k}{s} \right\| = \left\| \frac{K(s)e_k}{s} \right\|.
\]
By the preceding result, the right hand side diverges to \(+\infty\) as \(s \to 0\). Therefore \(K(t)e_k\) is not strongly differentiable at \(t\).

\[\square\]

**Remark 7.2** We have
\[
\langle e_\ell, K(t)e_k \rangle = \begin{cases} 
\frac{e^{it(E_\ell-E_k)}-1}{E_\ell-E_k} & ; \ell \neq k \\
0 & ; \ell = k
\end{cases} \quad (7.7)
\]
Hence, for all \(k, \ell \in \mathbb{N}\), \(\langle e_\ell, K(t)e_k \rangle\) is differentiable in \(t \in \mathbb{R}\) and
\[
\frac{d}{dt} \langle e_\ell, K(t)e_k \rangle = (\delta_{\ell k} - 1)e^{it(E_\ell-E_k)}. \quad (7.8)
\]

Proposition 7.7 tells us some singularity of \(K(t)\) acting on \(D_0\). But, as shown in the next proposition, \(K(t)\) restricted to \(D_c\) is strongly differentiable at \(t = 0\).

**Proposition 7.8** For all \(\psi \in D_c\), the \(\mathcal{H}\)-valued function \(K(t)\psi\) is strongly differentiable at \(t = 0\) with
\[
\frac{d}{dt}K(t)\psi \bigg|_{t=0} = \psi. \quad (7.9)
\]

**Proof.** We need only to prove the statement for \(\psi\) of the form \(\psi = e_k - e_\ell\) \((k, \ell \in \mathbb{N}, k \neq \ell)\). For all \(t \in \mathbb{R} \setminus \{0\}\), we have
\[
\frac{K(t)(e_k - e_\ell)}{t} = A(t) + B(t),
\]
where
\[
A(t) := i e^{it(E_k - E_{E_k})} - 1 \frac{e^{it(E_k - E_{E_k})}}{t(E_k - E_{E_k})} e_{E_k},
\]
\[
B(t) := i \sum_{n \neq k, \ell} \left( \frac{e^{it(E_n - E_k)} - 1}{t(E_n - E_k)} - \frac{e^{it(E_n - E_{E_k})} - 1}{t(E_n - E_{E_k})} \right) e_n.
\]

It is easy to see that
\[
\lim_{t \to 0} A(t) = e_k - e_{E_k}.
\]
As for \(B(t)\), we have
\[
\|B(t)\|^2 = \sum_{n \neq k, \ell} |F_n(t)|^2,
\]
where
\[
F_n(t) := \frac{e^{it(E_n - E_k)} - 1}{t(E_n - E_k)} - \frac{e^{it(E_n - E_{E_k})} - 1}{t(E_n - E_{E_k})}.
\]
It is easy to see that
\[
\lim_{t \to 0} F_n(t) = 0.
\]
Moreover, one can show that
\[
|F_n(t)| \leq \frac{C}{|E_n - E_k|}, \quad n \neq k, \ell,
\]
where \(C > 0\) is a constant independent of \(n\) and \(t\). Since \(\sum_{n \neq k} 1/|E_n - E_k|^2 < \infty\), one can apply the dominated convergence theorem to conclude that \(\lim_{t \to 0} \|B(t)\|^2 = 0\). Thus \(K(t)(e_k - e_{E_k})\) is strongly differentiable at \(t = 0\) and (7.9) with \(\psi = e_k - e_{E_k}\) holds.

\[\Box\]

**Proposition 7.9** For all \(k, \ell \in \mathbb{N}\) with \(k \neq \ell\), the \(\mathcal{H}\)-valued function \(K(t)(e_k - e_{E_k})\) is not strongly differentiable at \(t \notin \{2\pi n/(E_k - E_{E_k}) | n \in \mathbb{Z}\}\).

**Proof.** Let \(t \neq 2\pi n/(E_k - E_{E_k}) (n \in \mathbb{Z})\) and \(s \in \mathbb{R} \setminus \{0\}\). Then, by (7.5), we have
\[
\frac{(K(t + s) - K(t))(e_k - e_{E_k})}{s} = e^{itH} K(s) e^{-itH}(e_k - e_{E_k}).
\]
Hence
\[
\left\| \frac{(K(t + s) - K(t))(e_k - e_{E_k})}{s} \right\| = \|u(s)\|
\]
with
\[
u(s) = \frac{K(s)}{s} (e^{-itE_k} e_k - e^{-itE_{E_k}} e_{E_k}).
\]
We write
\[
u(s) = u_1(s) + u_2(s)
\]
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with
\[ u_1(s) := e^{-itE_k} \frac{K(s)}{s} (e_k - e_\ell), \quad u_2(s) := (e^{-itE_k} - e^{-itE_\ell}) \frac{K(s)}{s} e_\ell. \]

By Proposition 7.8, we have \( \lim_{s \to 0} u_1(s) = e^{-itE_k}(e_k - e_\ell) \). On the other hand, we have from the proof of Proposition 7.7 and the assumed condition for \( t \)
\[ \lim_{s \to 0} \|u_2(s)\| = +\infty. \]
Hence \( \lim_{s \to 0} \|u(s)\| = +\infty \). Thus the desired result follows. \( \square \)

8 Possible Connections with Regular Perturbation Theory

We consider a perturbation of \( H \) by a symmetric operator \( H_I \) on \( \mathcal{H} \):
\[ H(\lambda) := H + \lambda H_I, \quad (8.1) \]
where \( \lambda \in \mathbb{R} \) is a perturbation parameter. For simplicity, we assume that \( H_I \) is \( H \)-bounded: \( D(H) \subseteq D(H_I) \) and there exist constants \( a, b \geq 0 \) such that
\[ \|H_I \psi\| \leq a \|H \psi\| + b \|\psi\|, \quad \psi \in D(H). \]
Then, by the Kato-Rellich theorem (e.g., [13, Theorem X.12]), for all \( \lambda \in \mathbb{R} \) satisfying
\[ a|\lambda| < 1, \quad (8.2) \]
\( H(\lambda) \) is self-adjoint and bounded below. In what follows, we assume (8.2).

8.1 Eigenvalues of \( H(\lambda) \)

We fix \( n \in \mathbb{N} \) arbitrarily. By a general theorem in regular perturbation theory (e.g., [14, Theorem XII.9]), there exists a constant \( c_n > 0 \) such that, for all \( |\lambda| < c_n \), \( H \) has a unique, isolated non-degenerate eigenvalue \( E_n(\lambda) \) near \( E_n \). Moreover, \( E_n(\lambda) \) is analytic in \( \lambda \) with Taylor expansion
\[ E_n(\lambda) = E_n + E_n^{(1)} \lambda + E_n^{(2)} \lambda^2 + \cdots, \quad (8.3) \]
where
\[ E_n^{(1)} := \langle e_n, H_I e_n \rangle, \quad E_n^{(2)} := \sum_{m \neq n} \frac{\left| \langle e_n, H_I e_m \rangle \right|^2}{E_n - E_m}. \quad (8.4) \]

As an eigenvector of \( H(\lambda) \) with eigenvalue \( E_n(\lambda) \), one can take a vector \( \psi_n(\lambda) \) analytic in \( \lambda \) with Taylor expansion
\[ \psi_n(\lambda) = e_n + e_n^{(1)} \lambda + \cdots, \quad (8.5) \]
where
\[ e_n^{(1)} := \sum_{m \neq n} \frac{\langle e_m, H_I e_n \rangle}{E_n - E_m} e_m. \quad (8.6) \]
By Lemma 2.2, we have

$$\langle e_n, T e_m \rangle = \begin{cases} \frac{i}{E_n - E_m} & ; n \neq m \\ 0 & ; n = m \end{cases}$$

Hence $E_n^{(2)}$ can be written

$$E_n^{(2)} = (-i) \sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^2 \langle e_n, T e_m \rangle. \quad (8.7)$$

To rewrite the right hand side only in terms of $e_n$ and linear operators on $\mathcal{H}$, we note that

$$\sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^2 = \|H_I e_n\|^2 < \infty$$

by the Parseval equality. Hence

$$\sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^4 < \infty.$$

Therefore the infinite series

$$f_n := \sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^2 e_m \quad (8.8)$$

strongly converges and defines a vector in $\mathcal{H}$. Thus we can define a linear operator $V$ on $\mathcal{H}$ as follows:

$$D(V) := D_0, \quad (8.9)$$

$$V \psi := -i \sum_{n=1}^{\infty} \langle e_n, \psi \rangle f_n, \quad \psi \in D_0 \quad (8.10)$$

where the right hand side of (8.10) is a sum over a finite term. It is easy to see that $V$ is a symmetric operator.

**Proposition 8.1** For all $n \in \mathbb{N}$,

$$E_n^{(2)} = \langle T e_n, V e_n \rangle. \quad (8.11)$$

**Proof.** We have $V e_n = -i f_n$. Hence $\langle T e_n, V e_n \rangle = -i \langle T e_n, f_n \rangle$, which is equal to the right hand side of (8.7). \qed

This proposition suggests some role of the time operator $T_1 = T |D_0$ in the perturbation expansions of the eigenvalues of $H$.

As for the first order term $e_n^{(1)} \lambda$ of the eigenvector $\psi_n(\lambda)$, we have

$$e_n^{(1)} = (-i) \sum_{m=1}^{\infty} \langle e_m, H_I e_n \rangle \langle e_n, T e_m \rangle e_m. \quad (8.12)$$
8.2 Transition probability amplitudes

In the context of quantum mechanics where \( H(\lambda) \) is the Hamiltonian of a quantum system, the complex number \( \langle \phi, e^{-itH(\lambda)} \psi \rangle \) with unit vectors \( \phi, \psi \in \mathcal{H} \) is called the transition probability amplitude for the probability such that the state of the quantum system at time \( t \) is found in the state \( \psi \) under the condition that the state of the quantum system at time zero is \( \psi \).

**Lemma 8.2** Let \( \phi, \psi \in D(H) \). Then, for all \( t \in \mathbb{R} \),

\[
\langle \phi, e^{-itH(\lambda)} \psi \rangle = \langle \phi, e^{-itH} \psi \rangle - i\lambda \int_0^t \langle e^{i(t-s)H} \phi, H_I e^{-isH} \psi \rangle \, ds + O(\lambda^2). \tag{8.13}
\]

**Proof.** By a simple application of a general formula for the unitary group generated by a self-adjoint operator ([5, Lemma 5.9]), we have

\[
e^{-itH(\lambda)} \psi = e^{-itH} \psi - i\lambda \int_0^t e^{-i(t-s)H(\lambda)} H_I e^{-isH} \psi \, ds,
\]

where the integral is taken in the strong sense. Hence

\[
\langle \phi, e^{-itH(\lambda)} \psi \rangle = \langle \phi, e^{-itH} \psi \rangle - i\lambda \int_0^t \langle e^{i(t-s)H(\lambda)} \phi, H_I e^{-isH} \psi \rangle \, ds \nonumber
\]

\[
= \langle \phi, e^{-itH} \psi \rangle - i\lambda \int_0^t \langle e^{i(t-s)H} \phi, H_I e^{-isH} \psi \rangle \, ds + R(\lambda),
\]

where

\[
R(\lambda) := -i\lambda \int_0^t \left( \langle e^{i(t-s)H(\lambda)} - e^{i(t-s)H} \rangle \phi, H_I e^{-isH} \psi \right) \, ds.
\]

Using (8.14) again, we have

\[
R(\lambda) = -\lambda^2 \int_0^t \lambda \int_0^t ds \int_0^{t-s} ds' \langle e^{i(t-s+s')H(\lambda)} H_I e^{-is'H} \phi, H_I e^{-isH} \psi \rangle.
\]

Hence

\[
|R(\lambda)| \leq \lambda^2 \int_0^{|t|} ds \int_0^{|t-s|} ds' \|H_I e^{-is'H} \phi\| \|H_I e^{-isH} \psi\|
\]

Therefore \( R(\lambda) = O(\lambda^2) \). Thus (8.13) holds. \( \Box \)

Applying (8.13) with \( \phi = e_m \) and \( \psi = e_n \) \((n \neq m)\), we have

\[
\langle e_m, e^{-itH(\lambda)} e_n \rangle = -\lambda \frac{e^{-itE_n} - e^{-itE_m}}{E_m - E_n} \langle e_m, H_I e_n \rangle + O(\lambda^2), \tag{8.15}
\]

which, combined with (7.7), gives

\[
\langle e_m, e^{-itH(\lambda)} e_n \rangle = i\lambda \langle e_n, e^{-itH} K(t) e_m \rangle \langle e_m, H_I e_n \rangle + O(\lambda^2). \tag{8.16} \quad m \neq n.
\]

This suggests a physical meaning of the commutation factor \( K \).

By Theorem 7.3, one can rewrite the first term on the right hand side in terms of \( T_1 \) and \( e^{-itH} \), obtaining

\[
\langle e_m, e^{-itH(\lambda)} e_n \rangle = i\lambda \langle e_n, [T_1, e^{-itH}] e_m \rangle \langle e_m, H_I e_n \rangle + O(\lambda^2), \quad m \neq n. \tag{8.17}
\]

This also is suggestive on physical meaning of the time operator \( T_1 \).
References


