Singularities of lightcone Gauss images of spacelike hypersurfaces in de Sitter space

Masaki Kasedou

Abstract

We define the notions of lightcone Gauss images of spacelike hypersurfaces in de Sitter space. We investigate the relationships between singularities of these maps and geometric properties of spacelike hypersurfaces as an application of the theory of Legendrian singularities. We classify the singularities and give some examples in the generic case in de Sitter 3-space.
Mathematics Subject Classification (2000): 53A35, 53B30, 58C25.
Key words: de Sitter space, Gauss image, spacelike hypersurface, lightcone.

1 Introduction

In this paper, we discuss the extrinsic differential geometry of the spacelike hypersurfaces in de Sitter space. Bleeker and Wilson [3] studied the singularities of the Gauss map of a surface in Euclidean 3-space. In their paper, the main theorem asserts that the generic singularities of Gauss maps are folds or cusps. Banchoff *et al.* [2], Landis [6] and Platnova [8] studied geometric meanings of cusps of the Gauss map of a surface. Bruce [4] and Romero-Fuster [9] have also independently studied the singularities of the Gauss map and the dual of hypersurface in Euclidean space. The main tool of Bruce and Romero-Fuster for the study is the family of height functions on a hypersurface. Izumiya *et al.* [5] studied the extrinsic differential geometry on hypersurfaces in hyperbolic space as an application of the theory of Legendrian singularities.

The investigation in this paper is the analogue of that in [5] for spacelike hypersurfaces in de Sitter space. It is known that de Sitter space is a Lorentzian space form with positive curvature which is one of the vacuum solutions of the Einstein equations. In §2 we introduce the notion of the lightcone Gauss image and the lightcone Gauss-Kronecker curvature. The lightcone Gauss-Kronecker curvature is a invariant under the Lorentzian transformation in de Sitter space. In §3,4 we introduce a family of functions that is called the lightcone height function on the spacelike hypersurface. The singular set of the lightcone Gauss image is the lightcone parabolic set of the spacelike hypersurface and this can be interpreted as the discriminant set of the family of height functions. In §5,6 we discuss the contact between hypersurfaces and de Sitter hyperhorospheres. We apply the theory of Legendrian singularities for the study of lightcone Gauss images of generic hypersurfaces. In §8 we classify the singularities of lightcone Gauss images of generic spacelike surfaces in de Sitter 3-space. Here, we have two singularity types of lightcone Gauss images, which are cuspidal edges and swallowtails. In §7,9 we construct the spacelike Monge form in de Sitter space. It makes us to give examples which are corresponding to generic singularities of lightcone Gauss images.

2 Hypersurfaces in de Sitter space

In this section we introduce the local differential geometry in the explicit way. Let $\mathbb{R}^{n+1} = \{\mathbf{x} = (x_0, \cdots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, \cdots, n)\}$ be an $(n+1)$-dimensional vector space. For any vectors $\mathbf{x} = (x_0, \cdots, x_n)$, $\mathbf{y} = (y_0, \cdots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i.$$

We call $(\mathbb{R}^{n+1}, \langle , \rangle)$ a *Minkowski* $(n+1)$*-space* and write $\mathbb{R}^{n+1}_1$ instead of $(\mathbb{R}^{n+1}, \langle , \rangle)$.

We say that a vector $\mathbf{x} \in \mathbb{R}^{n+1}_1 \setminus \{\mathbf{0}\}$ is *spacelike*, *timelike* or *lightlike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ respectively. The norm of the vector $\mathbf{x} \in \mathbb{R}^{n+1}_1$ is defined by $||x|| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$.

For a vector $\mathbf{v} \in \mathbb{R}^{n+1}_1 \setminus \{\mathbf{0}\}$ and a real number c, we define a *hyperplane with pseudo-normal* $\mathbf{v}$ by

$$\mathrm{HP}(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{v} \rangle = c\}.$$

We call $\mathrm{HP}(\mathbf{v}, c)$ a *spacelike hyperplane*, *timelike hyperplane* or *lightlike hyperplane* if $\mathbf{v}$ is timelike, spacelike or lightlike respectively.

We now define *hyperbolic* n*-space* by

$$H^n_+(-1) = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1, x_0 \geq 1\}$$

and *de Sitter* n*-space* by

$$S^n_1 = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}.$$

For any $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \in \mathbb{R}^{n+1}_1$, we define a vector $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_n$ by

$$\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_n = \begin{vmatrix} -\mathbf{e}_0 & \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ x^1_0 & x^1_1 & \cdots & x^1_n \\ \vdots & \vdots & \ddots & \vdots \\ x^n_0 & x^n_1 & \cdots & x^n_n \end{vmatrix},$$

where $\mathbf{e}_0, \mathbf{e}_1, \cdots \mathbf{e}_n$ are the canonical basis of $\mathbb{R}^{n+1}_1$ and $\mathbf{x}_i = (x^i_0, x^i_1, \cdots, x^i_n)$ which satisfies that $\langle \mathbf{x}, \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_n \rangle = \det(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n)$, so that $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_n$ is pseudo-orthogonal to any $\mathbf{x}_i$ (for $i = 1, \cdots, n$).

We also define a set $LC_a = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle\}$, which is called a *closed lightcone* with vertex $\mathbf{a}$. We denote

$$LC^*_{\pm} = \{\mathbf{x} = (x_0, \cdots, x_n) \in LC_{\mathbf{0}} \mid x_0 > 0\ (x_0 < 0)\}$$

and call it the *future* (resp. *past*) *lightcone* at the origin.

We now study the extrinsic differential geometry of spacelike hypersurfaces in S^n_1. Let $\mathbf{X}: U \longrightarrow S^n_1$ be an embedding, where $U \subset \mathbb{R}^{n-1}$ is an open subset. We say $\mathbf{X}$ is a *spacelike hypersurface* in S^n_1 if every non zero vector generated by $\{\mathbf{X}_{u_i}(u)\}_{i=1}^{n-1}$ is always spacelike, where $u = (u_1, \cdots, u_{n-1})$ is an element of U and $\mathbf{X}_{u_i}$ is a partial derivative of $\mathbf{X}$ with respect to u_i. We denote $M = \mathbf{X}(U)$ and identify M with U through the embedding $\mathbf{X}$. Since $\langle \mathbf{X}, \mathbf{X}\rangle \equiv 1$, we have $\langle \mathbf{X}_{u_i}, \mathbf{X}\rangle \equiv 0$ (for $i = 1, \cdots, n-1$). It follows that a hyperplane spanned by $\{\mathbf{X}, \mathbf{X}_{u_1}, \cdots, \mathbf{X}_{u_{n-1}}\}$ is spacelike. We define a vector

$$\mathbf{e}(u) = \frac{\mathbf{X}(u) \wedge \mathbf{X}_{u_1}(u) \wedge \cdots \wedge \mathbf{X}_{u_{n-1}}(u)}{||\mathbf{X}(u) \wedge \mathbf{X}_{u_1}(u) \wedge \cdots \wedge \mathbf{X}_{u_{n-1}}(u)||}.$$

Then we have

$$\langle \mathbf{e}, \mathbf{X}_{u_i}\rangle \equiv \langle \mathbf{e}, \mathbf{X}\rangle \equiv 0, \qquad \langle \mathbf{e}, \mathbf{e}\rangle \equiv -1. \quad (\text{for } i = 1, \cdots, n-1),$$

Therefore the vector $\mathbf{X} \pm \mathbf{e}$ is lightlike. Since $\mathbf{X}(u) \in S^n_1$ and $\mathbf{e}(u) \in H^n_+(-1)$, we can show that $\mathbf{X}(u) \pm \mathbf{e}(u) \in LC^*_\pm$. We define a map

$$\mathbb{L}^\pm : U \longrightarrow LC^*_\pm$$

by $\mathbb{L}^\pm(u) = \mathbf{X}(u) \pm \mathbf{e}(u)$, which is called the *lightcone Gauss image* of $\mathbf{X}$.

We now define the lightcone Gauss-Kronecker curvature and the lightcone mean curvature of the hypersurface $M = \mathbf{X}(U)$. Since $\{\mathbf{X}(u_0), \mathbf{e}(u_0), \mathbf{X}_{u_1}(u_0), \cdots, \mathbf{X}_{u_{n-1}}(u_0)\}$ is a basis of the vector space $T_p\mathbb{R}^{n+1}_1$, we have the following lemma analogous to ([5], Lemma 2.1).

Lemma 2.1. *For any $p = \mathbf{X}(u_0) \in M$ and $\mathbf{v} \in T_pM$, we have $D_\mathbf{v}\mathbf{e} \in T_pM$, so that $D_\mathbf{v}\mathbb{L}^\pm \in T_pM$. Here, $D_\mathbf{v}$ denotes the covariant derivative with respect to the tangent vector $\mathbf{v}$.*

We now consider a hypersurface defined by $HP(\mathbf{v}, c) \cap S^n_1$. We say that $HP(\mathbf{v}, c) \cap S^n_1$ is an *elliptic* hyperquadric or a *hyperbolic* hyperquadric if $HP(\mathbf{v}, c)$ is spacelike or timelike respectively. We say that $HP(\mathbf{v}, 1) \cap S^n_1$ is a *de Sitter hyperhorosphere* if $HP(\mathbf{v}, 1)$ is lightlike. Then we have the following proposition.

Proposition 2.2. *Let $\mathbf{X}: U \longrightarrow S^n_1$ be a spacelike hypersurface in S^n_1. The lightcone Gauss image $\mathbb{L}^\pm$ is constant if and only if the spacelike hypersurface $M = \mathbf{X}(U)$ is a part of a de Sitter hyperhorosphere.*

Proof. Since $\mathbb{L}^\pm(u)$ is constant $\mathbb{L}^\pm$, so we have $\langle \mathbf{X}(u), \mathbb{L}^\pm\rangle = \langle \mathbf{X}(u), \mathbf{X}(u) \pm \mathbf{e}(u)\rangle = 1$ for any $u \in U$. Therefore, we have $\mathbf{X}(U) \subset HP(\mathbb{L}^\pm, +1) \cap S^n_1$.

If $\mathbf{X}(U) \subset HP(\mathbf{v}, c) \cap S^n_1$ for some $\mathbf{v} \in LC^*$ and $c \neq 0$, then we have $\langle \mathbf{X}(u), \mathbf{v}\rangle = r$ and $\langle \mathbf{X}_{u_i}(u), \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{v}\rangle = 0$. This means that $\mathbf{v} = r\mathbb{L}^+(u)$ or $\mathbf{v} = r\mathbb{L}^+(u)$. Therefore, $\mathbb{L}^\pm$ is a constant vector $(1/c)\mathbf{v}$. $\square$

Under the identification of U and M, the derivative $d\mathbf{X}(u_o)$ can be identified with the identity mapping id_{T_pM} on the tangent space T_pM, where $p = \mathbf{X}(u_o)$. This means that

$$d\mathbb{L}^\pm(u_0) = \mathrm{id}_{T_pM} \pm d\mathbf{e}(u_0).$$

By Lemma 2.1, $d\mathbf{e}(u_0)$ is a linear transformation on the tangent space T_pM, so that $d\mathbb{L}^\pm(u_0)$ is also a linear transformation on T_pM. We respectively call $S_p^\pm = -d\mathbb{L}^\pm(u_0) : T_pM \longrightarrow T_pM$ the *lightcone shape operator* of $M = \mathbf{X}(U)$ of at $p = \mathbf{X}(u_0)$ and $A_p = -d\mathbf{e} : T_pM \longrightarrow T_pM$ the *shape operator* of $M = \mathbf{X}(U)$ of at $p = \mathbf{X}(u_0)$. We denote the eigenvalue of $S_p^\pm$ by $\bar{\kappa}_p^\pm$ and the eigenvalue of A_p by κ_p. By the relation $S_p^\pm = -\mathrm{id}_{T_pM} \pm A_p$, $S_p^\pm$ and A_p have the common eigenvectors and we have a relation $\bar{\kappa}_p^\pm = -1 \pm \kappa_p$.

The *lightcone Gauss-Kronecker curvature* of $M = \mathbf{X}(U)$ at $p = \mathbf{X}(u_0)$ is defined to be

$$K_\ell^\pm(u_0) = \det S_p^\pm.$$

Since A_p is the shape operator with respect to the Riemannian metric on M induced from the Lorentzian metric on $\mathbb{R}^{n+1}_1$, we define the Gauss-Kronecker curvature on M by $K(u_0) = \det A_p$.

We say that a point $u \in U$ or $p = \mathbf{X}(u)$ is an *umbilic point* if $S_p^\pm = \bar{\kappa}_p^\pm \mathrm{id}_{T_pM}$. Since the eigenvectors of $S_p^\pm$ and A_p are the same, the above condition is equivalent to the condition $A_p = \kappa_p \mathrm{id}_{T_pM}$. We say that $M = \mathbf{X}(U)$ is *totally umbilic* if all points on M are umbilic.

Proposition 2.3. Suppose that $M = \mathbf{X}(U)$ is totally umbilic. Then $\bar{\kappa}_p^\pm$, κ_p are constant $\bar{\kappa}^\pm$, κ. Under this condition, we have the following classification.

(1) Suppose that $\bar{\kappa}^\pm \neq 0$.

  (a) If $0 < |\kappa| = |\bar{\kappa}^\pm + 1| < 1$, then M is a part of a hyperbolic hyperquadric $HP(\mathbf{v}, 1) \cap S^n_1$.

  (b) If $1 < |\kappa| = |\bar{\kappa}^\pm + 1|$, then M is a part of an elliptic hyperquadric $HP(\mathbf{v}, 1) \cap S^n_1$.

  (c) If $\kappa = \bar{\kappa}^\pm + 1 = 0$, then M is a part of a hyperbolic hyperquadric $HP(\mathbf{v}, 0) \cap S^n_1$.

(2) If $\bar{\kappa}^\pm = 0$, then M is a part of a de Sitter hyperhorosphere.

Proof. By definition, we have $-\mathbb{L}^\pm_{u_i}(u) = \bar{\kappa}_p^\pm \mathbf{X}_{u_i}(u)$ (for $i = 1, \cdots, n-1$) for any $p = \mathbf{X}(u) \in M$. Therefore, we have

$$\mathbb{L}^\pm_{u_i,u_j}(u) = \bar{\kappa}^\pm_{u_j\,p}\mathbf{X}_{u_i}(u) + \bar{\kappa}_p^\pm \mathbf{X}_{u_i,u_j}(u).$$

Since $\mathbb{L}^\pm_{u_i,u_j} = \mathbb{L}^\pm_{u_j,u_i}$ and $\mathbf{X}_{u_i,u_j} = \mathbf{X}_{u_j,u_i}$, we have $\bar{\kappa}^\pm_{u_j\,p}\mathbf{X}_{u_i}(u) - \bar{\kappa}^\pm_{u_i\,p}\mathbf{X}_{u_j}(u) = 0$. On the other hand, $\mathbf{X}_{u_i}$ $(i = 1, \cdots, n-1)$ are linearly independent, so that $\bar{\kappa}_p^\pm$ is constant $\bar{\kappa}^\pm$. Since $\bar{\kappa}^\pm = \pm\kappa_p - 1$, this means that $\kappa_p^\pm$ is constant $\kappa^\pm$.

We now assume that $\bar{\kappa}^\pm \neq 0$. By the assumption, we have $-\mathbf{e}_{u_i} = \kappa \mathbf{X}_{u_i}$ (for $i = 1, \cdots, n-1$), so that there exists a constant vector $\mathbf{a}$ such that, $\mathbf{a} = \kappa\mathbf{X}(u) + \mathbf{e}(u)$ for any $u \in U$. If $|\kappa| = |\bar{\kappa}^\pm + 1| \neq 0$, then the vector $\mathbf{v} = (1/\kappa)\mathbf{a}$ satisfies $\langle \mathbf{v}, \mathbf{v} \rangle = 1 - 1/\kappa^2$ and $\langle \mathbf{X}, \mathbf{v} \rangle = +1$, so that the assertion (a), (b) follows. If $\kappa = 0$, then $\mathbf{v} = \mathbf{a}$ satisfies $\langle \mathbf{v}, \mathbf{v} \rangle = -1$, $\langle \mathbf{X}, \mathbf{v} \rangle = 0$. so that the assertion (c) follows.

Finally, we assume that $\bar{\kappa}^\pm = 0$. In this case, we have $\mathbb{L}^\pm_{u_i} = 0$ (for $i = 1, \cdots, n-1$), so that $\mathbb{L}^\pm$ is constant. Therefore we apply Proposition 2.2. This completes the proof. $\square$

Let $p = \mathbf{X}(u_0) \in M$ be an umbilic point, we say that p is a *positive* (or *negative*) *lightcone flat point* (or, briefly an $L^\pm$-*flat point*) if $\bar{\kappa}^\pm = 0$.

Since $\mathbf{X}_{u_i}$ (for $i = 1, \cdots, n-1$) are spacelike vectors, we have the Riemannian metric (the *first fundamental form*) $ds^2 = \sum_{i,j=1}^{n-1} g_{ij} du_i du_j$ on $M = \mathbf{X}(U)$, where $g_{ij}(u) = \langle \mathbf{X}_{u_i}(u), \mathbf{X}_{u_j}(u) \rangle$ for any $u \in U$. We also define a *positive* (or *negative*) *lightcone second fundamental form* by $\bar{h}^{\pm}_{ij}(u) = \langle -\mathbb{L}^{\pm}_{u_i}(u), \mathbf{X}_{u_j}(u) \rangle$, and a *second fundamental invariant* by $h_{ij}(u) = \langle -\mathbf{e}_{u_i}(u), \mathbf{X}_{u_j}(u) \rangle$ for any $u \in U$. By definition, we have the following relation:

$$\bar{h}^{\pm}_{ij}(u) = -g_{ij}(u) \pm h_{ij}(u).$$

The following proposition is analogous to ([5], Proposition 2.4):

Proposition 2.4. Under the above notation, we have the following Weingarten formula:

$$\mathbb{L}^{\pm}_{u_i} = -\sum_{j=1}^{n-1} (\bar{h}^{\pm})^j_i \mathbf{X}_{u_j},$$

where $((\bar{h}^{\pm})^j_i) = (\bar{h}^{\pm}_{ik})(g^{kj})$ and $(g^{kj}) = (g_{kj})^{-1}$.

As a corollary of the above proposition, we have an explicit expression for the lightcone Gauss-Kronecker curvature by Riemannian metric and the lightcone second fundamental invariant.

Corollary 2.5. With the same notation as in the above proposition, the lightcone Gauss-Kronecker curvature is given by

$$K^{\pm}_{\ell} = \det(\bar{h}^{\pm}_{ij}) / \det(g_{\alpha\beta}).$$

We say that $p = \mathbf{X}(u_0)$ is a *positive* (or *negative*) *lightcone parabolic point* (or, briefly an $L^{\pm}$-parabolic point) of $\mathbf{X}$ if $K^{\pm}_{\ell}(u_0) = 0$.

3 Lightcone height functions

In this section we introduce families of functions on a spacelike hypersurface in de Sitter space, which are useful for the study of singularities of lightcone Gauss images and lightcone Gauss maps. Let $\mathbf{X} : U \longrightarrow S^n_1$ be a hypersurface. We define a family of functions

$$H : U \times LC^* \longrightarrow \mathbb{R}$$

by $H(u, \mathbf{v}) = \langle \mathbf{X}(u), \mathbf{v} \rangle - 1$. We call H a *lightcone height function* on $\mathbf{X} : U \longrightarrow S^n_1$. we have the following proposition:

Proposition 3.1. Let $H : U \times LC^* \longrightarrow \mathbb{R}$ be a lightcone height function on $\mathbf{X}$. Then

(1) H(u,$\mathbf{v}$)=0 if and only if there exist real numbers $\mu, \xi_i \in \mathbb{R}$ $(i = 1, \cdots, n-1)$ such that $\mathbf{v} = \mathbf{X}(u) + \mu \mathbf{e}(u) + \xi_1 \mathbf{X}_{u_1}(u) + \cdots + \xi_{n-1} \mathbf{X}_{u_{n-1}}(u)$.

(2) $H(u, \mathbf{v}) = \partial H(u, \mathbf{v}) / \partial u_i = 0$ (for $i = 1, \cdots, n-1$) if and only if $\mathbf{v} = \mathbb{L}^{\pm}(u)$.

We denote the Hessian matrix of the lightcone height function $h^{\pm}_{\mathbf{v_0}}(u) = H(u, \mathbf{v_0})$ at u_0 by Hess $(h^{\pm}_{\mathbf{v_0}})(u_0)$.

Proposition 3.2. Let $\mathbf{X} : U \longrightarrow S^n_1$ be a hypersurface in S^n_1 and $\mathbf{v_0}^{\pm} = \mathbb{L}^{\pm}(u_0)$. Then

(1) $p = \mathbf{X}(u_0)$ is an $L^{\pm}$-parabolic point if and only if $\det \mathrm{Hess}\,(h^{\pm}_{\mathbf{v_0}})(u_0) = 0$.

(2) $p = \mathbf{X}(u_0)$ is an $L^{\pm}$-flat point if and only if $\mathrm{rank}\,\mathrm{Hess}\,(h^{\pm}_{\mathbf{v_0}})(u_0) = 0$.

The proofs for the above propositions are parallel to those of Propositions 3.1, 3.2 in [5], so that we omit these.

4 Lightcone Gauss images as wave fronts

In this section we naturally interpret the lightcone Gauss image of a spacelike hypersurface in S^n_1 as a wave front set in the theory of Legendrian singularities.

Let $\pi^{\pm} : PT(LC^*_{\pm}) \longrightarrow LC^*_{\pm}$ be the projective cotangent bundles with canonical contact structures. Consider the tangent bundle $\tau^{\pm} : TPT^*(LC^*_{\pm}) \longrightarrow PT^*(LC^*_{\pm})$ and the differential map $d\pi^{\pm} : TPT(LC^*_{\pm}) \longrightarrow T(LC^*_{\pm})$ of $\pi^{\pm}$. For any $X \in TPT^*(LC^*_{\pm})$, there exists an element $\alpha \in T^*(LC^*_{\pm})$ such that $\tau^{\pm}(X) = [\alpha]$. For an element $V \in T_x(LC^*_{\pm})$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus, we can define the canonical contact structure on $PT^*(LC^*_{\pm})$ by

$$K = \{X \in TPT^*(LC^*_{\pm}) \mid \tau^{\pm}(X)(d\pi^{\pm}(X)) = 0\}.$$

On the other hand, we consider a point $\mathbf{v} = (v_0, v_1, \cdots, v_n) \in LC^*_{\pm}$, then we have the relation $v_0 = \pm\sqrt{v_1^2 + \cdots + v_n^2}$. So we adopt the coordinate system $(v_1, \cdots, v_n)$ of the manifold $LC^*_{\pm}$. Then we have the trivialization $PT^*(LC^*_{\pm}) \equiv LC^*_{\pm} \times P\mathbb{R}^{n-1}$, and call $((v_0, \cdots, v_n), [\xi_1 : \cdots : \xi_n])$ homogeneous coordinates of $PT^*(LC^*_{\pm})$, where $[\xi_1 : \cdots : \xi_n]$ are the homogeneous coordinates of the dual projective space $P\mathbb{R}^{n-1}$.

It is easy to show that $X_\bullet \in K^{\pm}_\bullet$ if and only if $\sum_{i=1}^n \mu_i \xi_i = 0$, where $\bullet = (x, [\xi])$ and $d\pi^{\pm}_\bullet(X_\bullet) = \sum_{i=1}^n \mu_i \partial/\partial v_i \in T_\bullet LC^*_{\pm}$. An immersion $i : L \longrightarrow PT^*(LC^*_{\pm})$ is said to be a *Legendrian immersion* if $\dim L = n-1$ and $di_q(T_q L) \subset K_{i(q)}$ for any $q \in L$. The map $\pi \circ i$ is also called the *Legendrian map* and the image $W(i) = \mathrm{image}(\pi \circ i)$, the *wave front* of i. Moreover, i (or the image of i) is called the *Legendrian lift* of $W(i)$.

Let $F : (\mathbb{R}^{n-1} \times \mathbb{R}^k, (u_0, \mathbf{v}_0)) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family* of hypersurfaces if the map germ $\Delta^* F : (\mathbb{R}^{n-1} \times \mathbb{R}^k, (u_0, \mathbf{v}_0)) \longrightarrow (\mathbb{R}^n, \mathbf{0})$ defined by

$$\Delta^* F = \left(F, \frac{\partial F}{\partial u_1}, \cdots, \frac{\partial F}{\partial u_{n-1}} \right)$$

is non singular.

In this case, we have a smooth $(k-1)$-dimensional smooth submanifold,

$$\Sigma_*(F) = \left\{ (u, \mathbf{v}) \in (\mathbb{R}^{n-1} \times \mathbb{R}^k, (u_0, \mathbf{v}_0) \,\middle|\, F(u, \mathbf{v}) = \frac{\partial F}{\partial u_1}(u, \mathbf{v}) = \cdots = \frac{\partial F}{\partial u_{n-1}}(u, \mathbf{v}) = 0 \right\},$$

and the map germ $\mathcal{L}_F : (\Sigma_*(F), (u_0, \mathbf{v}_0)) \longrightarrow PT^*\mathbb{R}^k$ defined by

$$\mathcal{L}_F(u, \mathbf{v}) = \left(v, \left[\frac{\partial F}{\partial u_1}(u, \mathbf{v}) : \cdots : \frac{\partial F}{\partial u_{n-1}}(u, \mathbf{v}) \right] \right)$$

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd and Zakalyukin [1, 11].

Proposition 4.1. All Legendrian submanifold germs in $PT^*\mathbb{R}^k$ are constructed by the above method.

We call F a generating family of $\mathcal{L}_F(\Sigma_*(F))$. Therefore the wave front is

$$W(\mathcal{L}_F) = \left\{ \mathbf{v} \in \mathbb{R}^k \;\middle|\; \exists u \in \mathbb{R}^{n-1} \text{ such that } F(u, \mathbf{v}) = \frac{\partial F}{\partial u_1}(u, \mathbf{v}) = \cdots = \frac{\partial F}{\partial u_{n-1}}(u, \mathbf{v}) = 0 \right\}.$$

We call it the *discriminant set* of F.

Proposition 4.2. The lightcone height function $H : U \times LC^* \longrightarrow \mathbb{R}$ is a Morse family.

Proof. For any $\mathbf{v} = (v_0, \cdots, v_n) \in LC^*_\pm$, we have $v_0 \neq 0$. Without loss of generality, we assume that $v_0 = \sqrt{v_1^2 + \cdots + v_n^2} \geq 0$, so that we have

$$H(u, \mathbf{v}) = -x_0(u)\sqrt{v_1^2 + \cdots + v_n^2} + x_1(u)v_1 + \cdots + x_n(u)v_n - 1,$$

where $\mathbf{X}(u) = (x_0(u), \cdots, x_n(u))$. We have to prove that the mapping $\Delta^* H : U \times LC^*_\pm \longrightarrow \mathbb{R}^n$ is non-singular on $(\Delta^* H)^{-1}(0)$. But this computation is similar to [5] (see Proposition 4.2), so that we omit it. $\square$

Since H is a Morse family of hypersurfaces, we apply the previous arguments. So that we have the Legendrian immersion germ $\mathcal{L}^\pm : (\Sigma^\pm_*(H), (u_0, \mathbf{v}^\pm_0)) \longrightarrow PT^*(LC^*_\pm)$ by

$$\mathcal{L}^\pm(u) = \left(\mathbf{v}^\pm, \left[\frac{\partial H}{\partial v_1}(u, \mathbf{v}^\pm) : \cdots : \frac{\partial H}{\partial v_n}(u, \mathbf{v}^\pm) \right] \right),$$

where $\mathbf{v}^\pm = \mathbb{L}^\pm(u)$ and $\Sigma^\pm_*(H)$ is a singular set of H

$$\Sigma^\pm_*(H) = (\Delta^* H)^{-1}(0) = \{(u, \mathbf{v}) \in U \times LC^*_\pm \mid \mathbf{v} = \mathbb{L}^\pm(u)\}.$$

Therefore, we have the Legendrian immersion $\mathcal{L}^\pm$ whose wave front set is the lightcone Gauss image $\mathbb{L}^\pm$.

5 Contact with de Sitter hyperhorospheres

In this section we start to review the theory of contact due to Montaldi [7]. Let X_i and Y_i (for $i = 1, 2$) be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the contact of X_1 and Y_1 at y_1 is the same type as the contact of X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$.

Two function germs $g_1, g_2 : (\mathbb{R}^n, a_i) \longrightarrow (\mathbb{R}, 0)$ $(i = 1, 2)$ are $\mathcal{K}$*-equivalent* if there are a diffeomorphism germ $\Phi : (\mathbb{R}^n, \mathbf{a_1}) \longrightarrow (\mathbb{R}^n, \mathbf{a_2})$, and a function germ $\lambda : (\mathbb{R}^n, \mathbf{a_1}) \longrightarrow \mathbb{R}$ with $\lambda(\mathbf{a_1}) \neq 0$ such that $f_1 = \lambda \cdot (g_2 \circ \Phi)$. In [7] Montaldi has shown the following theorem.

Theorem 5.1. (Montaldi [7]) Let X_i and Y_i (for $i = 1, 2$) be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_1, x_1) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, \mathbf{0})$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(\mathbf{0}), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.

We can apply the above theorem to our case. Let $\mathbf{v_0} \in LC^*$. We define $\mathfrak{h}_{\mathbf{v_0}} : S^n_1 \longrightarrow \mathbb{R}$ by $\mathfrak{h}_{\mathbf{v_0}}(w) = \langle w, \mathbf{v_0} \rangle - 1$. Then we have a de Sitter hyperhorosphere $\mathfrak{h}^{-1}_{\mathbf{v_0}}(0) = HP(\mathbf{v_0}, +1) \cap S^n_1$. We write $HS(\mathbf{v_0}, +1) = HP(\mathbf{v_0}, +1) \cap S^n_1$. For any $u_0 \in U$, we consider the lightlike vector $\mathbf{v_0}^{\pm} = \mathbb{L}^{\pm}(u_0)$. Then we have

$$\mathfrak{h}_{\mathbf{v}^{\pm}_{\mathbf{0}}} \circ \mathbf{X}(u_0) = H(u_0, \mathbb{L}^{\pm}(u_0)) = 0.$$

We also have relations

$$\frac{\partial \mathfrak{h}_{\mathbf{v}^{\pm}_{\mathbf{0}}} \circ \mathbf{X}}{\partial u_i}(u_0) = \langle \mathbf{X}_{u_i}(u_0), \mathbb{L}^{\pm}(u_0) \rangle = 0,$$

for $i = 1, \cdots, n-1$. This means that the de Sitter hyperhorosphere $\mathfrak{h}^{-1}_{\mathbf{v}^{\pm}_{\mathbf{0}}}(0) = HS(\mathbf{v}^{\pm}_{\mathbf{0}}, +1)$ is tangent to $M = \mathbf{X}(U)$ at $p = \mathbf{X}(u_0)$. In this case, we call $HS(\mathbf{v}^{\pm}_{\mathbf{0}}, +1)$ the *tangent de Sitter hyperhorosphere* of $M = \mathbf{X}$ at $p = \mathbf{X}(u)$ (or u_0), which we write $HS^{\pm}(\mathbf{X}, u_0)$. Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be lightlike vectors. We say that $HS(\mathbf{v_1}, +1)$ and $HS(\mathbf{v_2}, +1)$ are *parallel* if $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent. Then we have the following proposition.

Proposition 5.2. Let $\mathbf{X} : U \longrightarrow S^n_1$ be a hypersurface. Consider two points $u_1, u_2 \in U$. Then $\mathbb{L}^{\pm}(u_1) = \mathbb{L}^{\pm}(u_2)$ if and only if $HS^{\pm}(\mathbf{X}, u_1) = HS^{\pm}(\mathbf{X}, u_2)$.

We now review some notions of Legendrian singularity theory to study the contact between hypersurfaces and de Sitter hyperhorospheres. We say that Legendrian immersion germs $i_j : (U_j, u_j) \longrightarrow (PT^*\mathbb{R}^n, p_j)$ $(j = 1, 2)$ are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p_1) \longrightarrow (PT^*\mathbb{R}^n, p_2)$ such that H preserves fibers of π and $H(U_1) = U_2$. A Legendrian immersion germ i_1 is said *Legendrian stable* if there are a neighborhood U in the map space of Legendrian immersions (in the Whitney C^∞ topology) such that for any elements of U are Legendrian equivalent to i_1.

Proposition 5.3. (Zakalyukin [12]) Let i_1, i_2 be Legendrian immersion germs such that regular sets of $\pi \circ i_1$ and $\pi \circ i_2$ are respectively dense. Then i_1, i_2 are Legendrian equivalent if and only if corresponding wave front sets $W(i_1)$ and $W(i_2)$ are diffeomorphic as set germs.

Let $F_i : (\mathbb{R}^n \times \mathbb{R}^k, (a_i, b_i)) \longrightarrow (\mathbb{R}, c)$ $(k = 1, 2)$ be k-parameter unfoldings of function germs f_i, we say F_1 and F_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}^k, (a_1, b_1)) \longrightarrow (\mathbb{R}^n \times \mathbb{R}^k, (a_2, b_2))$ of the form $\Phi(u, x) = (\phi_1(u, x), \phi_2(x))$ for $(u, x) \in \mathbb{R}^n \times \mathbb{R}^k$ and a function germ $\lambda : (\mathbb{R}^n \times \mathbb{R}^k, (a_1, b_1)) \longrightarrow \mathbb{R}$ such that $\lambda(a_1, b_1) \neq 0$ and $F_1(u, x) = \lambda(u, x) \cdot (F_2 \circ \Phi)(u, x)$.

Theorem 5.4. (Arnol'd, Zakalyukin [1, 11]) Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be Morse families and denote their Legendrian immersion by $\mathcal{L}_F, \mathcal{L}_G$. Then

(1) $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent if and only if F and G are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(2) $\mathcal{L}_F$ is Legendrian stable if and only if F is $\mathcal{K}$-versal deformation.

Let $\mathbb{L}_i^{\pm} : (U, u_i) \longrightarrow (LC_{\pm}^*, \mathbf{v_i}^{\pm})$ (for $i = 1, 2$) be de Sitter Gauss image germs of hypersurface germs $\mathbf{X}_i : (U, u_i) \longrightarrow (S_1^n, p_i)$. We say $\mathbb{L}_1^{\pm}$ and $\mathbb{L}_2^{\pm}$ are $\mathcal{A}$-equivalent if and only if there exist diffeomorphism germs $\phi : (U, u_1) \longrightarrow (U, u_2)$ and $\Phi : (S_1^n, \mathbf{v}_1^{\pm}) \longrightarrow (S_1^n, \mathbf{v}_2^{\pm})$ such that $\Phi \circ \mathbb{L}_1^{\pm} = \mathbb{L}_2^{\pm} \circ \phi$.

We denote $h_{i, \mathbf{v_i}^{\pm}} : (U, u_i) \longrightarrow (\mathbb{R}, \mathbf{0})$ by $h_{i, \mathbf{v_i}^{\pm}}(u) = H_i(u, \mathbf{v_i}^{\pm})$; Then we have $h_{i, \mathbf{v_i}^{\pm}}(u) = (\mathfrak{h}_{\mathbf{v_i}^{\pm}}) \circ \mathbf{X}_i(u)$. By Theorem 5.1,

$$K(\mathbf{X_1}(U), HS^{\pm}(\mathbf{X_1}, u_1), \mathbf{v}_1^{\pm}) = K(\mathbf{X_2}(U), HS^{\pm}(\mathbf{X_2}, u_2), \mathbf{v}_2^{\pm})$$

if and only if $h_{1, \mathbf{v}_1^{\pm}}$ and $h_{2, \mathbf{v}_2^{\pm}}$ are $\mathcal{K}$-equivalent.

We denote $Q^{\pm}(\mathbf{X}, u_0)$ the local ring of the function germ $h_{\mathbf{v}_0^{\pm}} : (U, u_0) \longrightarrow \mathbb{R}$ by

$$Q^{\pm}(\mathbf{X}, u_0) = C_{u_0}^{\infty}(U) / \langle h_{\mathbf{v}_0} \rangle_{C_{u_0}^{\infty}},$$

where $\mathbf{v}_0^{\pm} = \mathbb{L}^{\pm}(u_0)$ and $C_{u_0}^{\infty}(U)$ is the local ring of function germs at u_0 with the unique maximal ideal $\mathfrak{M}$.

Proposition 5.5. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be Morse families. Suppose that Legendrian immersion germs $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian stable, then the following conditions are equivalent:

(1) $(W(\mathcal{L}_F), \mathbf{0})$ and $(W(\mathcal{L}_G), \mathbf{0})$ are diffeomorphic as set germs.

(2) $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent.

(3) $Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f = F\mid_{\mathbb{R}^k \times \{\mathbf{0}\}}$ and $g = G\mid_{\mathbb{R}^k \times \{\mathbf{0}\}}$.

By the above propositions, we have following theorem.

Theorem 5.6. Let $\mathbf{X}_i : (U, u_i) \longrightarrow (S^n_1, p_i)$ (for $i = 1, 2$) be hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. Then the following conditions are equivalent:

(1) Lightcone Gauss image germs $\mathbb{L}^\pm_1$ and $\mathbb{L}^\pm_2$ are $\mathcal{A}$-equivalent.

(2) Legendrian immersion germs $\mathcal{L}_1$ and $\mathcal{L}_2$ are Legendrian equivalent.

(3) Lightcone height function germs H_1 and H_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent.

(4) $h_{1,\mathbf{v}^\pm_1}$ and $h_{2,\mathbf{v}^\pm_2}$ are $\mathcal{K}$-equivalent.

(5) $K(\mathbf{X}_1(U), HS^\pm(\mathbf{X}_1, u_1), \mathbf{v}^\pm_1) = K(\mathbf{X}_2(U), HS^\pm(\mathbf{X}_2, u_2), \mathbf{v}^\pm_2)$

(6) Local rings $Q^\pm(\mathbf{X}_1, u_1)$ and $Q^\pm(\mathbf{X}_2, u_2)$ are isomorphic as $\mathbb{R}$-algebras.

Proof. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are Legendrian stable, regular sets of $\mathbb{L}_1$ and $\mathbb{L}_2$ are respectively dense, by Proposition 5.3, the conditions (1) and (2) are equivalent. And we apply Theorem 5.4, the conditions (2) and (3) are equivalent. By the previous arguments from Theorem 5.1, the conditions (4) and (5) are equivalent.

If we assume the condition (3), then $\mathcal{P}$-$\mathcal{K}$-equivalence preserves the $\mathcal{K}$-equivalence, so that the condition (4) holds. Since the local ring $Q^\pm(\mathbf{X}_i, u_i)$ is $\mathcal{K}$-invariant, this means that the condition (6) holds. By Proposition 5.5, the condition (6) implies the condition (2). □

In the next section, we will prove that the assumption of the Theorem 5.6 is generic property in the case when $n \leq 6$. In general we have the following proposition.

Proposition 5.7. Let $\mathbf{X}_i : (U, u_i) \longrightarrow (S^n_1, p_i)$ (for $i = 1, 2$) be hypersurface germs such that their $L^\pm$-parabolic sets have no interior points as subspaces of U. If lightcone Gauss image germs $\mathbb{L}^\pm_1$ and $\mathbb{L}^\pm_2$ are $\mathcal{A}$-equivalent, then

$$K(\mathbf{X}_1(U), HS^\pm(\mathbf{X}_1, u_1), \mathbf{v}^\pm_1) = K(\mathbf{X}_2(U), HS^\pm(\mathbf{X}_2, u_2), \mathbf{v}^\pm_2).$$

In this case, $(\mathbf{X}_1^{-1}(HS(\mathbb{L}^\pm_1(u_1), +1)), u_1)$ and $(\mathbf{X}_2^{-1}(HS(\mathbb{L}^\pm_2(u_2), +1)), u_2)$ are diffeomorphic as set germs.

Proof. Since the $L^\pm$-parabolic point set is a singular set of the lightcone Gauss image, the corresponding Legendrian immersion germs $\mathcal{L}_i$ satisfy the hypothesis of Proposition 5.3. If $\mathbb{L}^\pm_1$ and $\mathbb{L}^\pm_2$ are $\mathcal{A}$-equivalent, then $\mathcal{L}_1$ and $\mathcal{L}_2$ are Legendrian equivalent. By Theorem 5.4, H_1 and H_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent, so that $h_{1,\mathbf{v}^\pm_1}$ and $H_{2,\mathbf{v}^\pm_2}$ are $\mathcal{K}$-equivalent. Applying Theorem 5.1, the first assertion holds.

On the other hand, we have $(\mathbf{X}_i^{-1}(HS(\mathbb{L}^\pm_i(u_i), +1)), u_i) = (h^{-1}_{i,\mathbf{v}^\pm_i}(0), u_i)$. Since $\mathcal{K}$-equivalence preserves the zero level sets, $(\mathbf{X}_1^{-1}(HS(\mathbb{L}^\pm_1(u_1), +1)), u_1)$ and $(\mathbf{X}_2^{-1}(HS(\mathbb{L}^\pm_2(u_2), +1)), u_2)$ are diffeomorphic as set germs. □

For a hypersurface germ $\mathbf{X}$, we call $(\mathbf{X}^{-1}(HS(\mathbb{L}^{\pm}(u_0), +1)), u_0)$ the *tangent de Sitter hyper-horospherical indicatrix germ* of $\mathbf{X}$. By Proposition 5.7, the diffeomorphic type of the tangent de Sitter horospherical indicatrix germ is an invariant under the $\mathcal{A}$-equivalence among lightcone Gauss image germs. We define the $\mathcal{K}$-codimension (or Tyurina number) of the function germ $h_{\mathbf{v}_0^{\pm}}$ by

$$\text{H-ord}^{\pm}(\mathbf{X}, u_0) = \dim C^{\infty}_{u_0}/\langle h_{\mathbf{v}_0^{\pm}}(u_0), \partial h_{\mathbf{v}_0^{\pm}}(u_0)/\partial u_i\rangle_{C^{\infty}_{u_0}},$$

where $\mathbf{v}_0^{\pm} = \mathbb{L}^{\pm}(u_0)$. We also have the notion of corank of the function germ:

$$\text{H-corank}^{\pm}(\mathbf{X}, u_0) = (n-1) - \operatorname{rank}\operatorname{Hess}(h_{\mathbf{v}_0^{\pm}}(u_0)).$$

By Proposition 3.2, $p = \mathbf{X}(u_0)$ is a $L^{\pm}$-parabolic point if and only if $\text{H-corank}^{\pm}(\mathbf{X}, u_0) \geq 1$. Moreover p is a $L^{\pm}$-flat point if and only if $\text{H-ord}^{\pm}(\mathbf{X}, u_0) = n-1$.

We say a function germ $f : (\mathbb{R}^{n-1}, \mathbf{a}) \longrightarrow \mathbb{R}$ has the $\mathcal{A}_k$-type singularity at $\mathbf{a}$ if f is $\mathcal{K}$-equivalent to the germ

$$g(u_1, \cdots, u_{n-1}) = \pm u_1 \pm \cdots \pm u_{n-2} + u_{n-1}^{k+1}.$$

6 Generic properties

In this section we consider generic properties of hypersurfaces in S^n_1. We consider the map space of spacelike embeddings $\text{Sp-Emb}(U, S^n_1)$ with Whitney C^{∞}-topology. We define the function $\mathcal{H} : S^n_1 \times LC^* \longrightarrow \mathbb{R}$ by

$$\mathcal{H}(\mathbf{x}, \mathbf{v}) = \langle \mathbf{x}, \mathbf{v}\rangle - 1,$$

and denote $\mathcal{H}_{\mathbf{x}}(\mathbf{v}) = \mathcal{H}(\mathbf{x}, \mathbf{v})$. Then $\mathcal{H}_{\mathbf{x}}$ is a submersion for any $\mathbf{x} \in S^n_1$. For any spacelike hypersurface $\mathbf{X} \in \text{Sp-Emb}(U, S^n_1)$, we have $H = \mathcal{H} \circ (\mathbf{X} \times \text{id}_{LC^*})$. We also have the ℓ-jet extension

$$j^{\ell}_1\mathcal{H} : U \times LC^* \longrightarrow J^{\ell}(U, \mathbb{R})$$

defined by $j^{\ell}_1\mathcal{H}(\mathbf{x}, \mathbf{v}) = j^{\ell}h_{\mathbf{v}}(u)$. We consider the trivialization

$$J^{\ell}(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^{\ell}(n-1, 1).$$

For any submanifold $Q \subset J^{\ell}(n-1, 1)$, we denote $\widetilde{Q} = U \times \{0\} \times Q$. Then we have the following proposition as a corollary of Lemma 6 of Wassermann.

Proposition 6.1. (Wassermann [10]) Let Q be a submanifold of $J^{\ell}(n-1, 1)$. Then the set

$$T_Q = \{\mathbf{X} \in \text{Sp-Emb}(U, S^n_1) \mid j^{\ell}_1 H \text{ is transversal to } \widetilde{Q}\}$$

is a residual subset of $\text{Sp-Emb}(U, S^n_1)$. If Q is a closed subset, then T_Q is open.

We remark that if corresponding height function $h_{\mathbf{v}_0}$ is ℓ-determined relative to $\mathcal{K}$, then H is a $\mathcal{K}$-versal deformation if and only if H is transversal to $\widetilde{\mathcal{K}}^{\ell}_{h_{v_0}}$, where $\mathcal{K}^{\ell}_{h_{v_0}}$ is the $\mathcal{K}$-orbit through $j^{\ell}h_{\mathbf{v}_0}(\mathbf{0}) \in J^{\ell}(n-1,1)$. Applying Theorem 5.4, this condition is equivalent to the condition that corresponding Legendrian immersion germ is Legendrian stable. From the previous arguments and the appendix of [5], we have following proposition. (See also [1].)

Theorem 6.2. if $n \leq 6$, there exists an open subset $\mathcal{O} \subset \text{Sp-Emb}(U, S^n_1)$ such that for any $\mathbf{X} \in \mathcal{O}$, corresponding Legendrian immersion germ $\mathcal{L}$ is Legendrian stable.

7 Spacelike de Sitter Monge form

In this section we consider the analogous notion for a spacelike hypersurface in de Sitter n-space. We now consider the function $f(u_1, \cdots, u_{n-1})$ with $f(0) = 0$ and $f_{u_i}(0) = 0$. Then we have a spacelike hypersurface in S^n_1 by

$$\mathbf{X}_f(u) = \left(-f(u), -\sqrt{1 + f^2(u) - u_1^2 - \cdots - u_{n-1}^2}, u_1, \cdots, u_{n-1}\right).$$

where U is an open neighborhood at $\mathbf{0}$. We can easily calculate $\mathbf{e}(\mathbf{0}) = (1, 0, \cdots, 0)$; therefore $\mathbb{L}^{\pm}(\mathbf{0}) = (\pm 1, -1, 0, \cdots, 0)$. We call $\mathbf{X}_f$ a *spacelike de Sitter Monge form* (briefly, spacelike Monge form). Then we have the following proposition.

Proposition 7.1. Any spacelike surface in S^n_1 is locally given by the spacelike Monge form.

Proof. Let $\mathbf{X} : U \longrightarrow S^n_1$ be a spacelike hypersurface. Since we can apply Lorentzian motions of Minkowski $(n+1)$-space such that S^n_1 is the invariant set, without loss of generality, we assume that $p = \mathbf{X}(\mathbf{0}) = (0, -1, 0, \cdots, 0)$. We have a basis $\{\mathbf{X}(\mathbf{0}), \mathbf{e}(\mathbf{0}), \mathbf{X}_{u_1}(\mathbf{0}), \cdots, \mathbf{X}_{u_{n-1}}(\mathbf{0})\}$ of $T_p\mathbb{R}^{n+1}$ such that $T_pM = \langle \mathbf{X}_{u_1}(\mathbf{0}), \cdots, \mathbf{X}_{u_{n-1}}(\mathbf{0})\rangle_{\mathbb{R}}$. Applying the Gram-Schmidt procedure, we have a pseudo orthonormal basis $\{\mathbf{X}(\mathbf{0}), \mathbf{e}(\mathbf{0}), \mathbf{e}_1, \cdots, \mathbf{e}_{n-1}\}$ of $\mathbb{R}^{n+1}_1$ such that $T_pM = \langle \mathbf{e}_1, \cdots, \mathbf{e}_{n-1}\rangle_{\mathbb{R}}$.

In particular, $\{\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}\}$ is an orthonormal basis of spacelike subspace T_pM, T_pM is considered to be a subspace of $\mathbb{R}^n_0 = \{(0, x_1, \cdots, x_n) \mid x_i \in \mathbb{R}\}$. By a rotation of the space $\mathbb{R}^n_0$, we assume that $T_pM = \{(0, 0, x_2, \cdots, x_n) \mid x_i \in \mathbb{R}\}$. We remark that this rotation can be considered to be a Lorentzian motion of $\mathbb{R}^{n+1}_1$.

Therefore, the hypersurface germ (M, p) is written in the form

$$\mathbf{X}(u) = (-f(u), -g(u), u_1, \cdots, u_{n-1}).$$

with function germs $f(u), g(u)$. Since $M \subset S^n_1$, we have a relation $g^2(u) = 1 + f^2(u) - u_1^2 - \cdots - u_{n-1}^2$. By a rotation of the space $\mathbb{R}^n_0$, we can assume $g(u) \geq 0$, so that we have

$$g(u) = \sqrt{1 + f^2(u) - u_1^2 - \cdots - u_{n-1}^2}.$$

Since $T_pM = \{(0, 0, x_2, \cdots, x_n) \mid x_i \in \mathbb{R}\}$, the conditions $f(\mathbf{0}) = f_{u_i}(\mathbf{0}) = 0$ $(i = 1, \cdots, n-1)$ are automatically satisfied. This completes the proof. □

For the lightlike vector $\mathbf{v_0} = (\pm 1, -1, 0, \cdots, 0)$, we consider the de Sitter hyperhorosphere $HS(\mathbf{v_0}, +1)$. Then we have the spacelike Monge form of $HS(\mathbf{v_0}, +1)$.

$$\mathbf{X}_{HS^{\pm}} = \left(\mp\frac{1}{2}(u_1^2 + \cdots + u_{n-1}^2), -1 + \frac{1}{2}(u_1^2 + \cdots + u_{n-1}^2), u_1, \cdots, u_{n-1} \right).$$

Here we can check the relation $\langle \mathbf{v}_0^{\pm}, \mathbf{X}_{HS^{\pm}}(u) \rangle = 1$. On the other hand, we have $\mathbf{X}_{HS^{\pm}}(\mathbf{0}) = (0, -1, 0, \cdots, 0)$ and $\mathbf{X}_{HS^{\pm}, u_i}(\mathbf{0})$ is the x_{i+1}-axis for $i = 1, \cdots, n-1$. This means that $T_pM = T_p(\mathbf{X}_{HS^{\pm}}(U))$. Therefore $\mathbf{X}_{HS^{\pm}}(U) \subset HS(\mathbf{v}_0^{\pm}, +1)$ is the tangent de Sitter hyperhorosphere of $M = \mathbf{X}(U)$ at $p = \mathbf{X}(\mathbf{0})$. It follows from this fact that the tangent de Sitter hyperhorospherical indicatrix germ of the spacelike Monge form $\mathbf{X}_f$ is given as follows:

$$\mathbf{X}_f^{-1}(HS(\mathbf{v}_0^{\pm}, +1)) = \{(u_1, \cdots, u_{n-1}) \mid \pm 2f(u) = u_1^2 + \cdots + u_{n-1}^2\}.$$

Since the lightcone height function of $\mathbf{X}_f$ at $\mathbf{v}_0^{\pm}$ is

$$h_{\mathbf{v}_0^{\pm}}(u) = \pm f(u) + \sqrt{1 + f^2(u) - u_1^2 - \cdots - u_{n-1}^2} - 1,$$

we can calculate the Hessian matrix; then we have $\mathrm{Hess}\, h_{\mathbf{v}_0^{\pm}}(\mathbf{0}) = \pm\mathrm{Hess}\,(f(\mathbf{0})) - \mathrm{I}_{n-1}$, where I_{n-1} is an identity matrix.

On the other hand, since $f(\mathbf{0}) = f_{u_i}(\mathbf{0}) = 0$, We may write

$$f(u) = \frac{1}{2}\kappa_1 u_1^2 + \cdots \frac{1}{2}\kappa_{n-1} u_{n-1}^2 + g(u),$$

where $g \in \mathfrak{M}_{n-1}^3$ and $\kappa_1, \cdots, \kappa_{n-1}$ are eigenvalues of $\mathrm{Hess}\,(f(\mathbf{0}))$. Under this representation, we can easily calculate $(\mathbf{X}_f)_{u_i, u_j}(\mathbf{0}) = (-f_{u_i, u_j}(\mathbf{0}), \delta_{ij}, 0, \cdots, 0)$. It follows from that this fact

$$\bar{h}_{ij}^{\pm}(\mathbf{0}) = \pm f_{u_i, u_j}(\mathbf{0}) - \delta_{ij} = \delta_{ij}(\pm\kappa_i - 1),$$

and $g_{ij}(\mathbf{0}) = \delta_{ij}$. Therefore, we have $\bar{\kappa}_i^{\pm}(\mathbf{0}) = -1 \pm \kappa_i$ and

$$K_\ell^{\pm}(\mathbf{0}) = \prod_{i=1}^{n-1} \bar{\kappa}_i^{\pm}(\mathbf{0}) = \prod_{i=1}^{n-1}(-1 \pm \kappa_i).$$

The tangent de Sitter hyperhorospherical indicatrix germ is given by

$$\mathbf{X}_f^{-1}(HS(\mathbf{v}_0^{\pm}, +1)) = \{(u_1, \cdots, u_{n-1}) \mid \bar{\kappa}_1^{\pm}(\mathbf{0}) u_1^2 + \cdots + \bar{\kappa}_{n-1}^{\pm}(\mathbf{0}) u_{n-1}^2 \pm 2g(u) = 0\}.$$

8 Spacelike surfaces in de Sitter 3-space

In this section we consider $n = 3$. In this case we call $\mathbf{X} : U \longrightarrow S_1^3$ a *spacelike surface* and $HS(\mathbf{v}_0, +1)$ a de Sitter *horosphere* etc. By Theorem 6.2 and the classification of function germs [1], we have he following theorem.

Theorem 8.1. There exists an open dense subset $\mathcal{O} \subset \text{Sp-Emb}(U, S^n_1)$ such that for any $\mathbf{X} \in \mathcal{O}$, the following conditions holds.

(1) The $L^{\pm}$-parabolic set $K_\ell^{-1}(0)$ is a regular curve. We call such a curve the *$L^{\pm}$-parabolic curve.*

(2) The lightcone Gauss image $\mathbb{L}^{\pm}$ along the $L^{\pm}$-parabolic curve is a cuspidal edge except at isolated points. At this points $\mathbb{L}^{\pm}$ is swallowtail.

Here, a map germ $L : (\mathbb{R}^2, \mathbf{a}) \longrightarrow (\mathbb{R}^3, \mathbf{b})$ is called the *cuspidal edge* if it is $\mathcal{A}$-equivalent to the germ (u_1, u_2^2, u_2^3) and the swallowtail if it is $\mathcal{A}$-equivalent to the germ $(3u_1^4+u_1^2u_2, 4u_1^3+2u_1u_2, u_2)$.

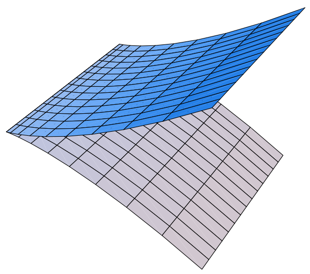

Figure 1: Cuspidal edge

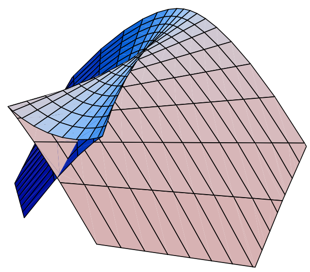

Figure 2: Swallowtail

The assertion of Theorem 8.1 can be interpreted as saying that the Legendrian lift $\mathcal{L}^{\pm}$ of the lightcone Gauss image $\mathbb{L}^{\pm}$ is Legendrian stable at each point. In this case, the lightcone Gauss image $\mathbb{L}^{\pm}$ has only cuspidal edges and swallowtails as singularities.

Corollary 8.2. Let $\mathcal{O} \subset \text{Sp-Emb}(U, S^n_1)$ be the same open dense subset as in Theorem 8.1. Let $\mathbf{X} \in \mathcal{O}$, $\mathbf{v}_0^{\pm} = \mathbb{L}^{\pm}(u_0)$ and $h_{\mathbf{v}_0^{\pm}} : (U, u_0) \longrightarrow \mathbb{R}$ be the lightcone height function germ at u_0. Then we have the following.

(1) The point u_0 is an $L^{\pm}$-parabolic point of $\mathbf{X}$ if and only if $\text{H-corank}^{\pm}(\mathbf{X}, u_0) = 1$ (that is, u_0 is not an $L^{\pm}$-flat point). In this case, $h_{\mathbf{v}_0^{\pm}}$ has the $\mathcal{A}_k$-type singularity for $k = 2, 3$.

(2) Suppose that u_0 is an $L^{\pm}$-parabolic point of $\mathbf{X}$. Then the following conditions are equivalent:

  (a) $\mathbb{L}^{\pm}$ has the cuspidal edge at u_0;

  (b) $h_{\mathbf{v}_0^{\pm}}$ has the $\mathcal{A}_2$-type singularity;

  (c) $\text{H-ord}^{\pm}(\mathbf{X}, u_0) = 2$;

  (d) The tangent de Sitter horospherical indicatrix germ is an ordinary cusp, where a curve $C \subset \mathbb{R}^2$ is called an ordinary cusp if it is diffeomorphic to the curve given by $\{(u_1, u_2) \mid u_1^2 - u_2^3 = 0\}$;

(3) Suppose that u_0 is an $L^{\pm}$-parabolic point of $\mathbf{X}$. Then the following conditions are equivalent:

(a) $\mathbb{L}^\pm$ has the swallowtail at u_0;

(b) $h_{\mathbf{v}_0^\pm}$ has the $\mathcal{A}_3$-type singularity;

(c) $\text{H-ord}^\pm(\mathbf{X}, u_0) = 3$;

(d) The tangent de Sitter horospherical indicatrix germ is a point or a tachnodal, where a curve $C \subset \mathbb{R}^2$ is called an tachnodal if it is diffeomorphic to the curve given by $\{(u_1, u_2) \mid u_1^2 - u_2^4 = 0\}$;

(e) For each $\varepsilon > 0$, there exist $L^\pm$-non-parabolic points $u_1, u_2 \in U$ such that $||u_0 - u_i|| < \varepsilon$ for $i = 1, 2$, and the tangent de Sitter horospheres to $M = \mathbf{X}(U)$ at u_1 and u_2 are equal.

Proof. Since $n = 3$, u_0 is $L^\pm$-parabolic point if and only if $\text{H-corank}^\pm(\mathbf{X}, u_0) \geq 1$. By classification of singularities, $h_{\mathbf{v}_0^\pm}$ has only the $\mathcal{A}_2$ or $\mathcal{A}_3$-type singularities. We can avoid the case that u_0 is $L^\pm$-flat point, so that $\text{H-corank}^\pm(\mathbf{X}, u_0) = 1$.

By Theorem 5.6, the conditions of (2) are equivalent. Similarly, the conditions (a),(b),(c),(d) of (3) are also equivalent. Suppose that corresponding Gauss image has swallowtail at u_0. We can observe that there is a self-intersection curve approaching u_0. (cf. Figure 2.) On this curve, there are two distinct points u_1 and u_2 such that $\mathbb{L}^\pm(u_1) = \mathbb{L}^\pm(u_2)$. By Lemma 5.2, this means that tangent de Sitter horospheres to $M = \mathbf{X}(U)$ at u_1 and u_2 are equal. On the other hand, if the Gauss image has cuspidal edge at u_0, there are no self-intersection on $\mathbb{L}^\pm$. (cf. Figure 1.) This means that (3)(a) is equivalent to (3)(e). This completes the proof. $\square$

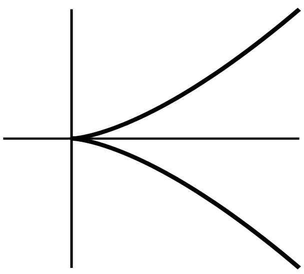

Figure 3: ordinary cusp

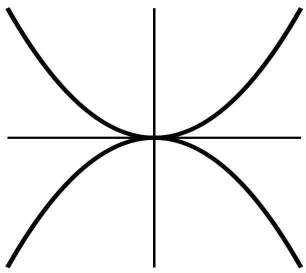

Figure 4: tachnodal

9 Examples in de Sitter 3-space

In this section we give some examples using the spacelike Monge form in de Sitter space.

Example 9.1. If $f(u_1, u_2) = \frac{1}{3}u_1^3 + \frac{1}{2}u_1^2$, then

$$\mathbf{X}_f(u_1, u_2) = \left(-\frac{1}{3}u_1^3 - \frac{1}{2}u_1^2, -\sqrt{1 + \left(\frac{1}{3}u_1^3 + \frac{1}{2}u_1^2\right)^2 - u_1^2 - u_2^2}, u_1, u_2 \right),$$

and $\kappa_1 = 1, \kappa_2 = 0$. Then we have $\bar{\kappa}_1^+(\mathbf{0}) = 0$, $\bar{\kappa}_2^+(\mathbf{0}) = -1$, $\bar{\kappa}_1^-(\mathbf{0}) = -2$ and $\bar{\kappa}_2^-(\mathbf{0}) = -1$. So the origin is not L^--parabolic point but a L^+-parabolic point. The positive tangent de Sitter horospherical indicatrix germ is the ordinary cusp $\{(u_1, u_2) \mid 2u_1^3 = 3u_2^2\}$. Therefore, the lightcone Gauss image $\mathbb{L}^-$ is non-singular at the origin and $\mathbb{L}^+$ is a cuspidal edge at the origin.

Example 9.2. If $f(u_1, u_2) = \frac{1}{2}u_1^4 + \frac{1}{2}u_1^2$, then

$$\mathbf{X}_f(u_1, u_2) = \left(-\frac{1}{2}u_1^4 - \frac{1}{2}u_1^2, -\sqrt{1 + \left(\frac{1}{2}u_1^4 + \frac{1}{2}u_1^2\right)^2 - u_1^2 - u_2^2}, u_1, u_2 \right),$$

and $\kappa_1 = 1, \kappa_2 = 0$. For the same reason as in the previous example, the origin is not L^--parabolic point but a L^+-parabolic point. The positive tangent de Sitter horospherical indicatrix germ is the tachnodal $\{(u_1, u_2) \mid u_1^4 = u_2^2\}$. Therefore, the lightcone Gauss image $\mathbb{L}^-$ is non-singular at the origin and $\mathbb{L}^+$ is a swallowtail at the origin.

Acknowledgment

The author wishes to express his thanks to Professor Shyuichi Izumiya, Kentaro Saji and Takashi Sano for good advice and valuable discussions.

References

[1] V.I. Arnold, S.M. Gusein-Zade and A.N. Varchenko, Singularities of Differential Maps, Volume I, Birkhäuser, Basel, 1986.

[2] T. Banchoff, T. Gaffney and C. McCrory, Cusps of Gauss mappings, Research Notes in Mathematics 55, Pitman, London, 1982.

[3] D. Bleeker and L. Wilson, Stability of Gauss maps, Illinois J. Math. 22 (1978) 279–289.

[4] J. W. Bruce, The dual of generic hypersurfaces, Math. Scand. 49 (1981) 36–60.

[5] S. Izumiya, D. Pei and T. Sano, Singularities of hyperbolic Gauss maps, Proc. London Math Soc. 86 (2003) 485–512.

[6] E.E. Landis, Tangential singularities, Funct. Anal. Appl. 15 (1981) 103–114.

[7] J.A. Montaldi, On contact between submanifolds, Michigan Math. J. 33 (1986) 195–199.

[8] O.A. Platonova, Singularities in the problem of the quickest way round an obstruct, Funct. Anal. Appl. 15 (1981) 147–148.

[9] M.C. Romero-Fuster, Sphere stratifications and the Gauss map, Proc. Roy. Soc. Edinburgh Sect. A 95 (1983) 115–136.

[10] G. Wassermann, Stability of Caustics, Math. Ann. 216 (1975) 43–50.

[11] V.M. Zakalyukin, Lagrangian and Legendrian singularities, Funct. Anal. Appl. 10 (1976) 26–36.

[12] V.M. Zakalyukin, Reconstructions of fronts and caustics depending one parameter and versality of mappings, J. Soviet. Math. 27 (1984) 2713–2735.

Masaki Kasedou,
Department of Mathematics, Hokkaido University,
Sapporo 060-0810, Japan.
e-mail: kasedou@math.sci.hokudai.ac.jp