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Spacelike surfaces in Anti de Sitter four-space from a contact viewpoint

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Dedicated to Vladimir Igorevich Arnold on the occasion of his 70th birthday

Abstract

We define the notions of $S^1_t \times S^2_s$-nullcone Legendrian Gauss maps and S^2_+-nullcone Lagrangian Gauss maps on spacelike surfaces in Anti de Sitter 4-space. We investigate the relationships between singularities of these maps and geometric properties of surfaces as an application of the theory of Legendrian/Lagrangian singularities. By using S^2_+-nullcone Lagrangian Gauss maps, we define the notion of S^2_+-nullcone Gauss-Kronecker curvatures and show a Gauss-Bonnet type theorem as a global property. We also introduce the notion of horospherical Gauss maps whch has different geometric properties of the above Gauss maps. As a consequence, we can say that Anti de Sitter space has much more rich geometric properties than the other space forms such as Euclidean space, Hyperbolic space, Lorentz-Minkowski space and de Sitter space.

1 Introduction

The study of Anti de Sitter 4-space is of special interested in the theory of relativity, for it represents one of the vacuum solutions of the Einstein equation. We observe that it is a Lorentzian space form with negative curvature. It is well-known that the Lorentzian space form with zero curvature is Lorentz-Minkowski space and with positive curvature is de Sitter space. These Lorentzian space forms have been well studied (cf., [10, 11, 13, 14, 15, 16]). However, there are not much results on submanifolds immersed in Anti de Sitter space, in particular from the viewpoint of singularity theory. We must remark that although Anti de Sitter space is diffeomorphic to de Sitter space, their causalities (i.e., the structure as a Lorentzian manifold) are quite different.

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In the study of submanifolds in Lorentzian-Minkowski space, the codimension two spacelike submanifolds happen to be the most interesting subjects, both from the view point of singularity theory and of the theory of relativity (cf., [10, 11, 13, 14, 15, 16]). For instance, one of the important objects in the theory of relativity are the lightlike hypersurfaces because they provide good models for different types of horizons [6, 20]. A lightlike hypersurface is a ruled hypersurface along a spacelike surface whose rulings are lightlike geodesics. This is one of the motivations for studying the spacelike surfaces in a 4-dimensional Lorentzian space form. In [10] we studied singularities of lightcone Gauss maps of spacelike surface in Minkowski 4-space, and established the relationships between such singularities and geometric invariants of these surfaces under the action of Lorentz group. Our aim in this paper is to develop the analogous study for the spacelike surfaces in Anti de Sitter 4-space. For this purpose, we shall adapt the tools developed in the previous paper for the study of spacelike surfaces in Minkowski space to that of the spacelike surfaces in Anti de Sitter 4-space. To do this we need to develop first the local differential geometry of spacelike surface in Anti de Sitter 4-space in a similar way as the classically done for surfaces in Euclidean 4-space [17] and Lorentz-Minkowski 4-space [10]. As it was to be expected, the situation presents certain peculiarities when compared with the Euclidean case and Lorentz-Minkowski case. For instance, in our case it is always possible to choose two lightlike normal directions along the spacelike surface in a frame of its normal bundle. This is similar to the Lorentz-Minkowski case, but the image is located in three dimensional space $S^1_t \times S^2_s$. For the Lorentz-Mikowski case, the image of the lightcone Gauss map is located in the three dimensional spacelike sphere S^3_+. By using this, we define the *$S^1_t \times S^2_s$-nullcone Legendrian Gauss map* and the *normalized lightlike (null) Gauss-Kronecker curvature* $\widetilde{\mathcal{K}}_\ell(1, \pm 1)$ of the spacelike surface in Anti de Sitter 4-space. We introduce the notion of *nullcone height function* and use it to show that the $S^1_t \times S^2_s$-nullcone Legendrian Gauss map has a singular point if and only if the lightlike Gauss-Kronecker curvature vanishes at such point. Moreover, we show that the $S^1_t \times S^2_s$-nullcone Legendrian Gauss map is a constant map if and only if the spacelike surface is contained in the intersection of a lightlike hyperplane and Anti de Sitter 4-space which we call a *lightlike hyperbolic cylinder* in Anti de Sitter space, so that we can view the singularities of the $S^1_t \times S^2_s$-nullcone Legendrian Gauss map as an estimate of the contacts of the surface with lightlike hyperbolic cylinders.

On the other hand, we introduce a natural S^1-fibration, $\tau : AdS^4 \to H^3_+(-1)$. This fibration induces the *S^2_+-nullcone Lagrangian Gauss map* and the *S^2_+-nullcone Gauss-Kronecker curvature* of the spacelike surface. We show that $S^1_t \times S^2_s$-nullcone Legendrian Gauss map is a Legendrian covering of S^2_+-nullcone Lagrangian Gauss map, so that the singularities of these mappings (the parabolic points with respect to the corresponding curvatures) are the same. Moreover, we show that the Gauss-Bonnet type theorem for S^2_+-nullcone Gauss-Kronecker curvature as a global property.

We also define a *horospherical Gauss map* together with its *horospherical height functions* on the spacelike surfaces in Anti de Sitter 4-space as an application of the horospherical geometry in Hyperbolic space[9, 12]. Such functions measure the contacts of the surface with certain $SO(2)\times SO(3)$ invariant submanifolds that we call here *round horospheres*, where $SO(2)\times SO(3)$ is canonically embedded subgroup of the group of semi-Euclidean motions $SO(2, 3)$. These are obtained as the pull-back by of the horospheres of $H^3_+(-1)$ by $\widehat{\tau}$. This Gauss map has an associated *horospherical Gauss-Kronecker curvature* as well as principal configurations. We state some global properties concerning these, obtained by pull-back of the corresponding results on surfaces in $H^3_+(-1)$ ([12, 14]).

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2 Local differential geometry on spacelike surfaces in Anti de Sitter 4-space

Let $\mathbb{R}^5 = \{(x_1, \dots, x_5) | x_i \in \mathbb{R}\ (i = 1, \dots, 5)\ \}$ be a 5-dimensional vector space. For any vectors $\boldsymbol{x} = (x_1, \dots, x_5)$ and $\boldsymbol{y} = (y_1, \dots, y_5)$ in $\mathbb{R}^5$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 - x_2y_2 + \sum_{i=3}^{5} x_iy_i$. We call $(\mathbb{R}^5, \langle , \rangle)$ a *semi-Euclidean 5-space with index* 2 and write $\mathbb{R}^5_2$ instead of $(\mathbb{R}^5, \langle , \rangle)$.

We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^5_2 \setminus \{\boldsymbol{0}\}$ is *spacelike*, *null* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^5_2$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. For a null vector $\boldsymbol{n} \in \mathbb{R}^5_2$ and a real number c, we define the *null hyperplane* with pseudo normal $\boldsymbol{n}$ by

$$NH(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^5_2 | \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}.$$

We now define *Anti de Sitter* 4-space by

$$AdS^4 = \{\boldsymbol{x} \in \mathbb{R}^5_2 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\}.$$

We also define

$$\Lambda_p = \{\boldsymbol{x} = (x_1, \dots, x_5) \in \mathbb{R}^5_2 | \ -(x_1 - p_1)^2 - (x_2 - p_2)^2 + \sum_{i=3}^{5} (x_i - p_i)^2 = 0\}$$

and

$$S^1_t \times S^2_s = \{\boldsymbol{x} = (x_1, \dots, x_5) \in \Lambda := \Lambda^*_0 \mid x_1^2 + x_2^2 = 1\},$$

where $p = (p_1, \dots, p_5) \in \mathbb{R}^5_2$, S^1_t denotes the *timelike circle* and S^2_s denotes the *spacelike sphere.* We call $\Lambda^*_p = \Lambda_p \setminus \{p\}$ the *nullcone* at the vertex p. Given any null vector $\boldsymbol{x} = (x_1, \dots, x_5)$, we have

$$\overline{\boldsymbol{x}} = (\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \frac{x_3}{\sqrt{x_1^2 + x_2^2}}, \frac{x_4}{\sqrt{x_1^2 + x_2^2}}, \frac{x_5}{\sqrt{x_1^2 + x_2^2}}) \in S^1_t \times S^2_s.$$

It is known that Anti de Sitter 4-space is time orientable, so that we fix a time orientation of Anti de Sitter 4-space. Therefore we can distinguish that any timelike vector has the future direction or the past direction.

On the other hand, we study the local differential geometry of spacelike surface in AdS^4. Let $\boldsymbol{X} : U \to AdS^4$ be an embedding from an open subset $U \subset \mathbb{R}^2$. We denote that $M = \boldsymbol{X}(U)$ and identify M and U through the embedding $\boldsymbol{X}$. We say that M is a *spacelike surface* if the tangent space T_pM of M is a spacelike plane in Anti de Sitter 4-space for any point $p \in M$. In this case, the normal space N_pM is a Lorentzian plane in Anti de Sitter 4-space(cf.,[23]). Let $\{\boldsymbol{e}_3(p), \boldsymbol{e}_4(p); p = \boldsymbol{X}(x, y)\}$ be an orthonormal frame of the tangent space T_pM and $\{\boldsymbol{e}_1(p), \boldsymbol{e}_2(p); p\}$ a pseudo-orthonormal frame of N_pM in Anti de Sitter 4-space, where, $\boldsymbol{e}_1(p)$ are unit timelike vectors and $\boldsymbol{e}_2(p), \boldsymbol{e}_3(p), \boldsymbol{e}_4(p)$ are unit spacelike vectors. Then $\{\boldsymbol{e}_1(p), \boldsymbol{e}_2(p), \boldsymbol{e}_3(p), \boldsymbol{e}_4(p), \boldsymbol{e}_5(p) = \boldsymbol{p} = \boldsymbol{X}(p); p\}$ is a pseudo-orthonormal frame of $\mathbb{R}^5_2$ at p.

We shall now establish the fundamental formula for a spacelike surface in Anti de Sitter 4-space by means of similar notions to those of Little [17].

We can write $d\boldsymbol{X} = \sum_{i=1}^{5} \omega_i \boldsymbol{e}_i$ and $d\boldsymbol{e}_i = \sum_{j=1}^{5} \omega_{ij} \boldsymbol{e}_j;\ i = 1, \ldots, 5.$ where ω_i and ω_{ij} are 1-forms given by $\omega_i = \delta(\boldsymbol{e}_i)\langle d\boldsymbol{X}, \boldsymbol{e}_i\rangle$ and $\omega_{ij} = \delta(\boldsymbol{e}_j)\langle d\boldsymbol{e}_i, \boldsymbol{e}_j\rangle$, whith

$$\delta(\boldsymbol{e}_i) = \operatorname{sign}(\boldsymbol{e}_i) = \begin{cases} 1 & i = 2, 3, 4 \\ -1 & i = 1, 5. \end{cases}$$

We have the Codazzi type equations:

$$\begin{cases} d\omega_i = \sum_{j=1}^{5} \delta(\boldsymbol{e}_i)\delta(\boldsymbol{e}_j)\omega_{ij} \wedge \omega_j \\ d\omega_{ij} = \sum_{k=1}^{5} \omega_{ik} \wedge \omega_{kj}, \end{cases}$$

where d is exterior derivative.

Since $\langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \delta_{ij}\delta(\boldsymbol{e}_j)$ (where δ_{ij} is Kronecker's delta), we get

$$\omega_{ij} = -\delta(\boldsymbol{e}_i)\delta(\boldsymbol{e}_j)\omega_{ji}.$$

In particular, $\omega_{ii} = 0;\ i = 1, \ldots, 5$. It follows from the fact $\langle d\boldsymbol{X}, \boldsymbol{e}_1\rangle = \langle d\boldsymbol{X}, \boldsymbol{e}_2\rangle = \langle d\boldsymbol{X}, \boldsymbol{X}\rangle = 0$ that

$$\omega_1 = \omega_2 = \omega_5 = 0.$$

Therefore we have

$$\begin{cases} 0 = d\omega_1 = \sum_{j=1}^{4} \delta(\boldsymbol{e}_1)\delta(\boldsymbol{e}_j)\omega_{1j} \wedge \omega_j = \sum_{j=3}^{4} \delta(\boldsymbol{e}_1)\delta(\boldsymbol{e}_j)\omega_{1j} \wedge \omega_j = -\omega_{13} \wedge \omega_3 - \omega_{14} \wedge \omega_4, \\ 0 = d\omega_2 = \sum_{j=1}^{4} \delta(\boldsymbol{e}_2)\delta(\boldsymbol{e}_j)\omega_{2j} \wedge \omega_j = \sum_{j=3}^{4} \delta(\boldsymbol{e}_2)\delta(\boldsymbol{e}_j)\omega_{2j} \wedge \omega_j = \omega_{23} \wedge \omega_3 + \omega_{24} \wedge \omega_4, \\ 0 = d\omega_5 = \sum_{j=1}^{4} \delta(\boldsymbol{e}_5)\delta(\boldsymbol{e}_j)\omega_{5j} \wedge \omega_j = \sum_{j=3}^{4} \delta(\boldsymbol{e}_5)\delta(\boldsymbol{e}_j)\omega_{5j} \wedge \omega_j = -\omega_{35} \wedge \omega_3 - \omega_{45} \wedge \omega_4. \end{cases}$$

By Cartan's lemma, we can write

$$\begin{cases} \omega_{13} = a\omega_3 + b\omega_4,\ \omega_{14} = b\omega_3 + c\omega_4, \\ \omega_{23} = e\omega_3 + f\omega_4,\ \omega_{24} = f\omega_3 + g\omega_4, \\ \omega_{35} = h\omega_3 + m\omega_4,\ \omega_{45} = m\omega_3 + n\omega_4. \end{cases}$$

for appropriate functions a, b, c, e, f, g, h, m and n.

We define $\langle d^2\boldsymbol{X}, \boldsymbol{e}_i\rangle = -\langle d\boldsymbol{X}, d\boldsymbol{e}_i\rangle,\ i = 1, 2, 5$, from which we get a vector-valued quadratic form:

$$-\langle d^2\boldsymbol{X}, \boldsymbol{e}_1\rangle\boldsymbol{e}_1 + \langle d^2\boldsymbol{X}, e_2\rangle\boldsymbol{e}_2 - \langle d^2\boldsymbol{X}, \boldsymbol{e}_5\rangle\boldsymbol{e}_5 = (a\omega_3^2 + 2b\omega_3\omega_4 + c\omega_4^2)\boldsymbol{e}_1 - (e\omega_3^2 + 2f\omega_3\omega_4 + g\omega_4^2)\boldsymbol{e}_2 + (\omega_3^2 + \omega_4^2)\boldsymbol{e}_5,$$

which is called the *second fundamental form* of the spacelike surface in $\mathbb{R}^5_2$. It follows from (2) that

$$d\begin{pmatrix} \boldsymbol{e}_1 + \boldsymbol{e}_2 \\ \boldsymbol{e}_1 - \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \\ \boldsymbol{e}_5 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} + \omega_{23} & \omega_{14} + \omega_{24} & 0 \\ -\omega_{12} & 0 & \omega_{13} - \omega_{23} & \omega_{14} - \omega_{24} & 0 \\ \dfrac{\omega_{13} + \omega_{23}}{2} & \dfrac{\omega_{13} - \omega_{23}}{2} & 0 & \omega_{34} & \omega_{35} \\ \dfrac{\omega_{14} + \omega_{24}}{2} & \dfrac{\omega_{14} - \omega_{24}}{2} & -\omega_{34} & 0 & \omega_{45} \\ 0 & 0 & \omega_{35} & \omega_{45} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_1 - \boldsymbol{e}_2 \\ \boldsymbol{e}_1 + \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \\ \boldsymbol{e}_5 \end{pmatrix}.$$

Let $\boldsymbol{e}_1 = (a_1, \ldots, a_5)$, $\boldsymbol{e}_2 = (b_1, \ldots, b_5)$, $\xi^{\pm} = \sqrt{(a_1-b_1)^2 \pm (a_2-b_2)^2}$, then we have the following fundamental formula:

$$d\begin{pmatrix} \overline{\boldsymbol{e}_1+\boldsymbol{e}_2} \\ \overline{\boldsymbol{e}_1-\boldsymbol{e}_2} \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \\ \boldsymbol{e}_5 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12}-\frac{d\xi^+}{\xi^+} & \frac{\omega_{13}+\omega_{23}}{\xi^+} & \frac{\omega_{14}+\omega_{24}}{\xi^+} & 0 \\ -\omega_{12}-\frac{d\xi^-}{\xi^-} & 0 & \frac{\omega_{13}-\omega_{23}}{\xi^-} & \frac{\omega_{14}-\omega_{24}}{\xi^-} & 0 \\ \dfrac{\omega_{13}+\omega_{23}}{2} & \dfrac{\omega_{13}-\omega_{23}}{2} & 0 & \omega_{34} & \omega_{35} \\ \dfrac{\omega_{14}+\omega_{24}}{2} & \dfrac{\omega_{14}-\omega_{24}}{2} & -\omega_{34} & 0 & \omega_{45} \\ 0 & 0 & \omega_{35} & \omega_{45} & 0 \end{pmatrix} \begin{pmatrix} \overline{\boldsymbol{e}_1-\boldsymbol{e}_2} \\ \overline{\boldsymbol{e}_1+\boldsymbol{e}_2} \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \\ \boldsymbol{e}_5 \end{pmatrix}.$$

Given $\boldsymbol{v} = \xi\boldsymbol{e}_1 + \eta\boldsymbol{e}_2 \in N_pM$ in AdS^4, we have $d\boldsymbol{v} = d\xi\boldsymbol{e}_1 + \xi d\boldsymbol{e}_1 + d\eta\boldsymbol{e}_2 + \eta d\boldsymbol{e}_2$, and

$$\langle d\boldsymbol{v}, \boldsymbol{e}_3\rangle \wedge \langle d\boldsymbol{v}, \boldsymbol{e}_4\rangle = \mathcal{K}_\ell(\xi,\eta)\omega_3\wedge\omega_4,$$

where the function $\mathcal{K}_\ell$ is given by:

$$\mathcal{K}_\ell(\xi,\eta) = (a\xi \pm e\eta)(c\xi \pm g\eta) - (b\xi + f\eta)^2$$

The function $\mathcal{K}_\ell(1,\pm 1)$ is called the *lightlike (or, null) Gauss-Kornecker curvature* of $M = \boldsymbol{X}(U)$. We also have a function

$$\overline{\mathcal{K}_\ell}^{\pm}(p) = \mathcal{K}_\ell\left(\frac{1}{\xi^{\pm}(p)}, \frac{\pm 1}{\xi^{\pm}(p)}\right)(p) = \frac{1}{(\xi^{\pm})^2(p)}\mathcal{K}_\ell(1,\pm 1)(p).$$

We call $\overline{\mathcal{K}_\ell}^{\pm}$ the *normalized lightlike Gauss-Kronecker curvature* of $M = \boldsymbol{X}(U)$.

On the other hand, we choose the frame $\boldsymbol{e}_3, \boldsymbol{e}_4$ of T_pM with the same orientation as the frame $\boldsymbol{X}_x, \boldsymbol{X}_y$. We also choose $\boldsymbol{e}_1$ as the future directed unit normal timelike vector field. Then we can take the spacelike unit vector $\boldsymbol{e}_2$ such that $\det(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5) > 0$. If we have another future directed unit normal timelike vector field $\boldsymbol{e}_1'$, we can choose another $\boldsymbol{e}_2'$. However, since the normal space of M in AdS^4 is a Lorentzian plane, we have $\overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2} = \overline{\boldsymbol{e}_1' \pm \boldsymbol{e}_2'}$. Therefore we have a well defined map, $\mathbb{NG}_M^{\pm} : M \longrightarrow S^1_t \times S^2_s$ given by $\mathbb{NG}_M^{\pm}(p) = \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(p)$. We call it the *$S^1_t \times S^2_s$-nullcone Legendrian Gauss map* of $M = \boldsymbol{X}(U)$. By the above construction, we have

$$\overline{\mathcal{K}_\ell}^{\pm}(p) = \det(\pi^t_p \circ d\mathbb{NG}_M^{\pm})(p),$$

where $\pi^t : T_p\mathbb{R}^5_2 = T_pM \oplus \overline{N}_p(M) \longrightarrow T_pM$ is the canonical projection onto the tangent space of M. We call $\pi^t_p \circ d\mathbb{NG}_M^{\pm}(p)$ the *normalized lightlike Weingarten map* of M at p.

On the other hand, we consider a projection $\tau : \mathbb{R}^2 \setminus \{\boldsymbol{0}\} \to \mathbb{R}$ defined by $\tau(\lambda\cos\theta, \lambda\sin\theta) = \lambda$ for $(\lambda\cos\theta, \lambda\sin\theta) \in \mathbb{R}^2 \setminus \{0\}$. This induces a projection, $\widehat{\tau} : \mathbb{R}^5_{2*} \to \mathbb{R}^4_{1+}$ defined by $\widehat{\tau}(x_1, x_2, x_3, x_4, x_5) = (0, \tau(x_1, x_2), x_3, x_4, x_5)$, where

$$\begin{aligned} \mathbb{R}^5_{2*} &= \{\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5_2 \mid (x_1, x_2) \neq (0,0)\ \} \\ \mathbb{R}^4_{1+} &= \{\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5_2 \mid x_1 = 0, x_2 > 0\ \}. \end{aligned}$$

We can easily show that $\widehat{\tau}$ satisfies $\langle \tau(\boldsymbol{x}), \tau(\boldsymbol{x})\rangle_1 = \langle \boldsymbol{x}, \boldsymbol{x}\rangle_2$, for all $\boldsymbol{x} \in \mathbb{R}^5_{2*}$. That is, it preserves the pseudo-norm of the vectors $\boldsymbol{x} \in \mathbb{R}^5_{2*}$. Consequently it transforms timelike, spacelike and lightlike vectors of $\mathbb{R}^5_{2*}$ respectively, into timelike, spacelike and lightlike vectors in $\mathbb{R}^4_{1+}$. Moreover, it determines a S^1-fibration, $\widehat{\tau} : AdS^4 \to H^3_+(-1)$, since $(\lambda\cos\theta, \lambda\sin\theta, x_3, x_4, x_5) \in AdS^4$, we have that $-\lambda^2 + x_3^2 + x_4^2 + x_5^2 = -1$, and thus $(0, \lambda, x_3, x_4, x_5) \in H^3_+(-1)$. Here we consider $H^3_+(-1) = \{(0, x_2, x_3, x_4, x_5) \in \mathbb{R}^4_{1+} \mid -x_2^2 + x_3^2 + x_4^2 + x_5^2 = -1\ \}$. We also have that $\widehat{\tau}(S^1_t \times S^2_s) = \{\boldsymbol{x} \in \mathbb{R}^4_{1+} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0, x_2 = 1\ \} = S^2_+$ (lightcone unit 2-sphere in $\mathbb{R}^4_1$). The mapping $\widehat{\tau}|S^1_t \times S^2_s : S^1_t \times S^2_s \longrightarrow S^2_+$ defines the S^1-fibration structure over S^2_+. By using these S^1-fibrations, we we define a map $\widehat{\mathbb{NG}}^{\pm}_M : U \longrightarrow S^2_+$ by $\widehat{\mathbb{NG}}^{\pm}_M(p) = \widehat{\tau} \circ \mathbb{NG}^{\pm}_M(p)$, which is called the S^2_+-*nullcome Lagrangian Gauss map* of $M = \boldsymbol{X}(U)$. By taking the derivative of the S^2_+-nullcome Lagrangian Gauss map, we can define the S^2_+-*nullcone Gauss-Kronecker curvature* $\widehat{\mathcal{K}}^{\pm}_\ell$ by

$$\langle d\widehat{\mathbb{NG}}^{\pm}_M, \boldsymbol{e}_3 \rangle \wedge \langle d\widehat{\mathbb{NG}}^{\pm}_M, \boldsymbol{e}_4 \rangle = \widehat{\mathcal{K}}^{\pm}_\ell \omega_3 \wedge \omega_4.$$

It follows that

$$\widehat{\mathcal{K}}^{\pm}_\ell(p) = \det(\pi^t_p \circ d\widehat{\mathbb{NG}}^{\pm}_M)(p).$$

We respectively denote the Riemannian metrics g_M and $g_{S^2_+}$ on M and S^2_+ which are induced from the semi-Euclidean scalar product $\langle , \rangle$. Moreover, we denote the area forms on M and S^2_+ by $d\mathfrak{a}_M$ and $d\mathfrak{a}_{S^2_+}$ respectively. By definition, we have

$$(\widehat{\mathbb{NG}}^{\pm}_M)^* d\mathfrak{a}_{S^2_+} = \widehat{\mathcal{K}}^{\pm}_\ell d\mathfrak{a}_M.$$

3 Nullcone height functions

In this section we introduce the notion of nullcone height functions on spacelike surfaces in AdS^4 which is useful for the study of singularities of $S^1_t \times S^2_s$-nullcone Legendrian Gauss maps. Given a spacelike surface $M(= \boldsymbol{X}(U)) \subset AdS^4 \subset \mathbb{R}^5_2$, we define a function $H : U \times S^1_t \times S^2_s \longrightarrow \mathbb{R}$ by $H((x, y), \boldsymbol{w}) = \langle \boldsymbol{X}(x, y), \boldsymbol{w} \rangle$, where $\boldsymbol{w} = (\sin\theta, \cos\theta, w_3, w_4, w_5) \in S^1_t \times S^2_s,\ 0 \le \theta < \pi$. We call H the *nullcone height function* on the spacelike surface M. We denote $h_{w_0}(x, y) = H(x, y, \boldsymbol{w}_0)$, for any fixed $\boldsymbol{w}_0 \in S^1_t \times S^2_s$. Then we have the following proposition.

Proposition 3.1 *Let M be a spacelike surface in $AdS^4 \subset \mathbb{R}^5_2$ and $H : U \times S^1_t \times S^2_s \longrightarrow \mathbb{R}$ a nullcone height function. Then we have the following assertions:*
(1) $h_w(p_0) = (\partial h_w/\partial x)(p_0) = (\partial h_w/\partial y)(p_0) = 0$ *if and only if* $\boldsymbol{w} = \mu(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)(p_0) = \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(p_0)$, *where*
$\boldsymbol{e}_1(p_0) = (a_1, \ldots, a_5),\ \boldsymbol{e}_2(p_0) = (b_1, \ldots, b_5),\ \mu = \frac{1}{\sqrt{(a_1 \pm b_1)^2 + (a_2 \pm b_2)^2}}$ *and* $p_0 = (x_0, y_0) \in M$.
(2) $h_w(p_0) = (\partial h_w/\partial x)(p_0) = (\partial h_w/\partial y)(p_0) = \det\mathcal{H}(h_w)(p_0) = 0$ *if and only if* $\boldsymbol{w} = \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(p_0)$ *and* $\mathcal{K}_\ell(1, \pm 1)(p_0) = 0$. *Here,* $\det\mathcal{H}(h_w)(p_0)$ *is the determinant of the Hessian matrix of* h_w *at* p_0.

Proof. By a straight forward calculation, $h_w(p_0) = (\partial h_w/\partial x)(p_0) = (\partial h_w/\partial y)(p_0) = 0$ if and only if

$$\langle \boldsymbol{X}, \boldsymbol{w} \rangle(p_0) = \langle \boldsymbol{X}_x, \boldsymbol{w} \rangle(p_0) = \langle \boldsymbol{X}_y, \boldsymbol{w} \rangle(p_0) = 0.$$

The above condition is equivalent to the condition that $\boldsymbol{w} \in N_{p_0}M$ and $\boldsymbol{w} \in S^1_t \times S^2_s$. This means that $\boldsymbol{w} = \mu(\boldsymbol{e}_1 \pm \boldsymbol{e}_2) = \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}$.

By a Lorentzian motion on AdS^4, we may assume that $p_0 = (0, 1, 0, 0, 0) \in AdS^4$. We can choose local coordinates such that $\boldsymbol{X}$ is given by the Monge form

$$\boldsymbol{X}(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y), x, y),$$

with $f_1(0,0) = f_3(0,0) = 0, f_2(0,0) = 1$ and $f_{1_x}(0,0) = f_{1_y}(0,0) = f_{2_x}(0,0) = f_{2_y}(0,0) = f_{3_x}(0,0) = f_{3_y}(0,0) = 0$, so that we have $\boldsymbol{e}_1(p_0) = (1,0,0,0,0)$, $\boldsymbol{e}_2(p_0) = (0,0,1,0,0)$. In this case we have

$$\begin{array}{ccc} f_{1_{xx}}(0,0) = -a(p_0), & f_{1_{xy}}(0,0) = -b(p_0), & f_{1_{yy}}(0,0) = -c(p_0), \\ f_{2_{xx}}(0,0) = e(p_0), & f_{2_{xy}}(0,0) = f(p_0), & f_{2_{yy}}(0,0) = g(p_0), \\ f_{3_{xx}}(0,0) = -1, & f_{3_{xy}}(0,0) = 0, & f_{3_{yy}}(0,0) = -1. \end{array}$$

Under the asumption (1) and taking into account that

$$\det \mathcal{H}(h_w)(x,y) = \begin{vmatrix} \langle \boldsymbol{X}_{xx}, \boldsymbol{w} \rangle & \langle \boldsymbol{X}_{xy}, \boldsymbol{w} \rangle \\ \langle \boldsymbol{X}_{xy}, \boldsymbol{w} \rangle & \langle \boldsymbol{X}_{yy}, \boldsymbol{w} \rangle \end{vmatrix} = 0$$

and $\boldsymbol{w}(p_0) = (1,0,\pm 1,0,0)$, we have

$$\begin{aligned} &\begin{vmatrix} \langle (f_{1_{xx}}, 0, f_{2_{xx}}, f_{3_{xx}}, 0), \boldsymbol{w}(p_0) \rangle & \langle (f_{1_{xy}}, 0, f_{2_{xy}}, f_{3_{xy}}, 0), \boldsymbol{w}(p_0) \rangle \\ \langle (f_{1_{xy}}, 0, f_{2_{xy}}, f_{3_{xy}}, 0), \boldsymbol{w}(p_0) \rangle & \langle (f_{1_{yy}}, 0, f_{2_{yy}}, f_{3_{yy}}, 0), \boldsymbol{w}(p_0) \rangle \end{vmatrix} \\ = &\begin{vmatrix} \langle (-a,0,e,-1,0), (1,0,\pm 1,0,0) \rangle & \langle (-b,0,f,0,0), (1,0,\pm 1,0,0) \rangle \\ \langle (-b,0,f,0,0), (1,0,\pm 1,0,0) \rangle & \langle (-c,0,g,-1,0), (1,0,\pm 1,0,0) \rangle \end{vmatrix} \\ = &\begin{vmatrix} a \pm e & b \pm f \\ b \pm f & c \pm g \end{vmatrix} = (a \pm e)(c \pm g) - (b \pm f)^2 = 0. \end{aligned}$$

This is equivalent to the condition that $\mathcal{K}_\ell(1, \pm 1)(p_0) = 0$ and $\boldsymbol{w}(p_0) = (1,0,\pm 1,0,0)$. □

We now consider a point $p \in M$. As a corollary of the above proposition, we have the following theorem.

Theorem 3.2 *Under the assumptions of Proposition 3.1, the following conditions are equivalent:*
(1) *There exists a $\boldsymbol{w} \in S^1_t \times S^2_s$ such that $p \in M$ is a degenerate singular point of nullcone height function h_w.*
(2) *The point $p \in M$ is a singular point of the $S^1_t \times S^2_s$-nullcone Legendrian Gauss map $\mathbb{NG}^{\pm}_M$.*
(3) $\mathcal{K}_\ell(1, \pm 1)(p) = 0$.

Proof. By the general theory of unfoldings (cf., [18, 3]), (1) and (2) are equivalent. By the assertion (2) of Proposition 3.1, (1) and (3) are equivalent. □

We say that a point $p_0 = (x_0, y_0)$ is a *lightlike parabolic point* of M if $\mathcal{K}_\ell(1,1)(p_0) = 0$ or $\mathcal{K}_\ell(1,-1)(p_0) = 0$. We study next the case when the $S^1_t \times S^2_s$-nullcone Legendrian Gauss map has the most degenerate singularities (i.e., it is constant).

Theorem 3.3 *Let M be a spacelike surface in $AdS^4 \subset \mathbb{R}^5_2$.*
(1) *The $S^1_t \times S^2_s$-nullcone Legendrian Gauss map $\mathbb{NG}^+_M$ (respectively, $\mathbb{NG}^-_M$) is constant if and only if there exists a unique null hyperplane $NH_M(\boldsymbol{v}^+, 0)$ (respectively, $NH_M(\boldsymbol{v}^-, 0)$) in $\mathbb{R}^5_2$, such that $M \subset NH(\boldsymbol{v}^+, 0) \cap AdS^4$ (respectively, $M \subset NH(\boldsymbol{v}^-, 0) \cap AdS^4$), where $\boldsymbol{v}^\pm = \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(x,y)$ for any $(x,y) \in M$.*
(2) *Both of the $S^1_t \times S^2_s$-nullcone Legendrian Gauss maps $\mathbb{NG}^+_M$ and $\mathbb{NG}^-_M$ are constant if and only if M is an open subset of*

$$NH(\overline{\boldsymbol{e}_1 + \boldsymbol{e}_2}, 0) \cap NH_M(\overline{\boldsymbol{e}_1 - \boldsymbol{e}_2}, 0) \cap AdS^4.$$

Proof. (1) For convenience, we only consider the case when $\mathbb{NG}_M^+(x,y) = \overline{\boldsymbol{e}_1 + \boldsymbol{e}_2}(x,y)$ is constant, so we have $d\langle \boldsymbol{X}, \overline{\boldsymbol{e}_1+\boldsymbol{e}_2}\rangle = \langle d\boldsymbol{X}, \overline{\boldsymbol{e}_1+\boldsymbol{e}_2}\rangle + \langle \boldsymbol{X}, d(\overline{\boldsymbol{e}_1+\boldsymbol{e}_2})\rangle = 0$. Therefore, $\langle \boldsymbol{X}, \overline{\boldsymbol{e}_1+\boldsymbol{e}_2}\rangle \equiv 0$ in AdS^4. This means that $M = \boldsymbol{X}(U) \subset NH_M(\boldsymbol{v}^+, 0)$, where $\boldsymbol{v}^+ = \overline{\boldsymbol{e}_1+\boldsymbol{e}_2}(x,y)$. For the converse assertion, suppose that there exists a null vector $\boldsymbol{v}$ and a real number c such that $\boldsymbol{X}(U) = M \subset NH(\boldsymbol{v}, 0)$. Since $\langle \boldsymbol{X}(x,y), \boldsymbol{v}\rangle = 0$, we have $d\langle \boldsymbol{X}(x,y), \boldsymbol{v}\rangle = 0$. This means that $\boldsymbol{v}$ is a null normal vector of M. Thus we have $\overline{\boldsymbol{v}} = \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(x,y)$. This completes the proof of the assertion (1).

Since $\boldsymbol{v}^+ \notin NH(\boldsymbol{v}^-, 0)$ and $\boldsymbol{v}^- \notin NH(\boldsymbol{v}^+, 0)$, $NH(\boldsymbol{v}^-, 0)$ and $NH(\boldsymbol{v}^+, 0)$ intersect transversally. By the assertion (1), both of the $S^1_t \times S^2_s$-nullcone Legendrian Gauss maps $\mathbb{NG}_M^+$ and $\mathbb{NG}_M^-$ are constant if and only if $M \subset NH(\boldsymbol{v}^+, 0) \cap NH(\boldsymbol{v}^-, 0)$. Here, the intersection is a spacelike affine 3-space. Thus we have the assertion (2). □

We analyze the intersection $NH(\boldsymbol{v}^\pm, 0) \cap AdS^4$ when $\boldsymbol{v}^\pm$ is a null vector. Here we assume that both vectors $\boldsymbol{v}^\pm$ are null and such that $\langle \boldsymbol{v}^+, \boldsymbol{v}^-\rangle = -2$. By applying a semi-Euclidean motion on $\mathbb{R}^5_2$ if necessary, we may assume that $\boldsymbol{v}^\pm = (1, 0, \pm 1, 0, 0)$. For $\boldsymbol{v}^+ = (1,0,1,0,0)$ the intersection is given by the equations $-x_1^2 - x_2^2 + x_3^2 + x_4^2 + x_5^2 = -1$, $-x_1 + x_3 = 0$, and we get

$$NH(\boldsymbol{v}^+, 0) \cap AdS^4 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5_2 \mid -x_2^2 + x_4^2 + x_5^2 = -1 \}.$$

This is $H^2(-1) \times \mathbb{R}$, where $H^2(-1)$ is a spacelike hyperboloid and the direction of the ruling $\mathbb{R}$ is $\boldsymbol{v}^+$. Therefore we call $NH(\boldsymbol{v}^+, 0) \cap AdS^4$ a *lightlike hyperbolic cylinder*. For $\boldsymbol{v}^- = (1, 0, -1, 0, 0)$, we have a similar lightlike hyperbolic cylinder but with the lightlike direction $\boldsymbol{v}^-$. The intersection $NH(\boldsymbol{v}^+, 0) \cap NH(\boldsymbol{v}^-, 0) \cap AdS^4$ is a spacelike hyperboloid. For $\boldsymbol{v}^\pm = (1, 0, \pm 1, 0, 0)$, this is given by

$$\{(0, x_2, 0, x_3, x_4, x_5) \in \mathbb{R}^5_2 \mid -x_2^2 + x_4^2 + x_5^2 = -1 \}.$$

4 $S^1_t \times S^2_s$-nullcone Legendrian Gauss maps

In this section we consider the $S^1_t \times S^2_s$-nullcone Legendrian Gauss map of M from the view point of Legendrian singularity theory. We give a brief review on Legendrian singularity theory mainly due to Arnol'd-Zakalyukin [1, 26]. Although the general theory has been described for any dimension, we shall just be concerned here with the 3-dimensional case. Since we only study local properties, we can consider the projective cotangent bundle $\pi : PT^*(\mathbb{R}^3) \longrightarrow \mathbb{R}^3$ with the canonical contact structure K. An immersion $i : L \to PT^*(\mathbb{R}^3)$ is said to be *a Legendrian immersion* if $\dim L = 2$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. We also call the map $\pi \circ i$ *the Legendrian map* and the set $W(i) = \operatorname{image} \pi \circ i$, *the wave front* of i. Moreover, i (or, the image of i) is called *the Legendrian lift* of $W(i)$. The main tool of the theory of Legendrian singularities is the notion of *generating families*. Let $F : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is *a Morse family of hypersurfaces* if the mapping

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k}\right) : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where $(q, x) = (q_1, \dots, q_k, x_1, x_2, x_3) \in (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0})$. In this case we have a smooth 2-dimensional submanifold

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \mid F(q,x) = \frac{\partial F}{\partial q_1}(q,x) = \dots = \frac{\partial F}{\partial q_k}(q,x) = 0 \right\}$$

and the map germ $\mathcal{L}_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^3$ defined by

$$\mathcal{L}_F(q,x) = \left(x, [\frac{\partial F}{\partial x_1}(q,x) : \frac{\partial F}{\partial x_2}(q,x) : \frac{\partial F}{\partial x_3}(q,x)] \right)$$

is a Legendrian immersion. Then we have the following fundamental theorem of Arnol'd-Zakalyukin [1, 26].

Proposition 4.1 *All Legendrian submanifold germs in $PT^*\mathbb{R}^4$ are constructed by the above method.*

We call F *a generating family* of $\mathcal{L}_F$. Therefore the corresponding wave front is

$$W(\mathcal{L}_F) = \left\{ x \in \mathbb{R}^4 \mid \exists q \in \mathbb{R}^k \text{ such that } F(q,x) = \frac{\partial F}{\partial q_1}(q,x) = \cdots = \frac{\partial F}{\partial q_k}(q,x) = 0 \right\}.$$

We write $\mathcal{D}_F = W(\mathcal{L}_F)$ and we call it the *discriminant set* of F. By Proposition 3.1, the image of $S^1_t \times S^2_s$-nullcone Legendrian Gauss map $\mathbb{NG}^{\pm}_M(U)$ is the discriminant set of the nullcone height function H. We have the following proposition.

Proposition 4.2 *The nullcone height function H is a Morse family of hypersurfaces.*

Proof. We consider the nullcone height function $H : M \times S^1_t \times S^2_s \longrightarrow \mathbb{R}$. For any $\boldsymbol{w} = (\cos\theta, \sin\theta, w_3, w_4, w_5) \in S^1_t \times S^2_s$, we assume that $w_3 > 0$, so we can write

$$H(p, \boldsymbol{w}) = -x_1(p)\cos\theta - x_2(p)\sin\theta + x_3(p)\sqrt{1 - w_4^2 - w_5^2} + x_4(p)w_4 + x_5(p)w_5,$$

where $\boldsymbol{X}(p) = (x_1(p), x_2(p), x_3(p), x_4(p), x_5(p))$. We now prove that the mapping

$$\Delta^* H = \left(H, \frac{\partial H}{\partial x}, \frac{\partial H}{\partial y} \right) : U \times S^1_t \times S^2_s \longrightarrow \mathbb{R}^3$$

is non-singular at any point in $(\Delta^*)^{-1}(0) = \Sigma_*(H)$. The Jacobian matrix of $\Delta^* H$ is given by

$$\begin{pmatrix} \langle \boldsymbol{X}_x, \boldsymbol{w} \rangle & \langle \boldsymbol{X}_y, \boldsymbol{w} \rangle & x_1 \sin\theta - x_2 \cos\theta & -x_3 \frac{w_4}{w_3} + x_4 & -x_3 \frac{w_5}{w_3} + x_5 \\ \langle \boldsymbol{X}_{xx}, \boldsymbol{w} \rangle & \langle \boldsymbol{X}_{xy}, \boldsymbol{w} \rangle & x_{1,x} \sin\theta - x_{2,x} \cos\theta & -x_{3,x} \frac{w_4}{w_3} + x_{4,x} & -x_{3,x} \frac{w_5}{w_3} + x_{5,x} \\ \langle \boldsymbol{X}_{xy}, \boldsymbol{w} \rangle & \langle \boldsymbol{X}_{yy}, \boldsymbol{w} \rangle & x_{1,y} \sin\theta - x_{2,y} \cos\theta & -x_{3,y} \frac{w_4}{w_3} + x_{4,y} & -x_{3,y} \frac{w_5}{w_3} + x_{5,y} \end{pmatrix}.$$

By a straight forward calculation, the determinant of the matrix

$$A = \begin{pmatrix} -x_{3,x}\dfrac{w_4}{w_3} + x_{4,x} & -x_{3,x}\dfrac{w_5}{w_3} + x_{5,x} \\ -x_{3,y}\dfrac{w_4}{w_3} + x_{4,y} & -x_{3,y}\dfrac{w_5}{w_3} + x_{5,y} \end{pmatrix}.$$

can be seen to be equal to

$$\frac{w_5}{w_3}(x_{3,x}x_{4,y} - x_{4,x}x_{3,y}) + \frac{w_4}{w_3}(x_{5,x}x_{3,y} - x_{3,x}x_{5,y}) + (x_{4,x}x_{5,y} - x_{5,x}x_{4,x}).$$

Since $\boldsymbol{X}(U) = M$ is a spacelike surface, the surface in Euclidean space $\{(0,0)\} \times \mathbb{R}^3$ parameterized by $(0, 0, x_3(x,y), x_4(x,y), x_5(x,y))$ is a regular surface. It follows that the above determinant vanishes if and only if the vector $(0, w_3, w_4, w_5)$ is tangent to the surface $M_0 = \{(0, 0, x_3(x, y), x_4(x, y), x_5(x, y)) \mid (x, y) \in U\}$ in $\{(0, 0)\} \times \mathbb{R}^3$. We consider the canonical projection $\pi_3 : \mathbb{R}^5_2 \longrightarrow \{(0, 0)\} \times \mathbb{R}^3$. Since $M = \boldsymbol{X}(U)$ is a spacelike surface, $\pi_3|M$ is a diffeomorphism onto M_0. The vector $\boldsymbol{w} = (\cos\theta, \sin\theta, w_1, w_2, w_3)$ is a null normal vector of M and does not belong to $\mathrm{Ker} d\pi_3|M$, therefore $d\pi_3(\boldsymbol{w}) = (0, 0, w_1, w_2, w_3)$ is not tangent to M_0. □

It follows from Proposition 4.2 that the images of the $S^1_t \times S^2_s$-nullcone Legendrian Gauss maps $\mathbb{NG}^{\pm}_M(U)$are wave fronts and the nullcone height function H is a generating families of the Legendrian lifts of $\mathbb{NG}^{\pm}_M(U)$, at least locally. By the assertion (1) of Proposition 3.1, we have

$$\Sigma_*(H) = \{(p, \boldsymbol{w}) \in M \times S^1_t \times S^2_s \mid \boldsymbol{w} = \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(p)\ \}.$$

We now consider the coordinate neighborhood

$$I \times U^+_3 = \{(\cos\overline{\theta}, \sin\overline{\theta}, w_3, w_4, w_5) \in S^1_t \times S^2_s\ |\overline{\theta} \in I = (0, 2\pi),\ w_3 > 0\}$$

of $S^1_t \times S^2_s$. By the construction of the Legendrian immersion in Proposition 4.1, we have the Legendrian immersion $\mathcal{L}_H : V \longrightarrow PT^*(S^1_t \times S^2_s)|I \times U^+_3$ defined by

$$\mathcal{L}_H(p) = (\overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(p), [\lambda(p)\sin(\theta(p) - \overline{\theta}(p)) : -\frac{w_4(p)}{w_3(p)}x_3(p) + x_4(p) : -\frac{w_5(p)}{w_3(p)}x_3(p) + x_5(p)]),$$

where $V = (\mathbb{NG}^{\pm}_M)^{-1}(I \times U^+_3)$ and we write

$$\overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(p) = (\cos\overline{\theta}(p), \sin\overline{\theta}(p), w_3(p), w_4(p), w_5(p))$$

and

$$\boldsymbol{X}(p) = (\lambda(p)\cos\theta(p), \lambda(p)\sin\theta(p), x_3(p), x_4(p), x_5(p)).$$

In the other coordinate neighborhoods, we have a similar expression for the Legendrian lift. This expression will be used in the next section.

5 S^2_+-nullcone Lagrangian Gauss maps

In this section we describe some of the geometric properties of the S^2_+-nullcone Lagrangian Gauss map $\widehat{\mathbb{NG}}^{\pm}_M$ of a spacelike surface in $M = \boldsymbol{X}(U)$. We fix the coordinate neighborhood, $I \times U^+_3$, as in the last paragraph of the previous section. By the local triviality of the projective cotangent bundle, we have $PT^*(S^1_t \times S^2_s)|I \times U^+_3 \equiv (I \times U^+_3) \times \mathbb{R}P^*(\mathbb{R} \times \mathbb{R}^2)$. We assume that $\xi_3 \neq 0$ for $(\boldsymbol{w}, [\xi]) = ((\overline{\theta}, \sqrt{1 - w_4^2 - w_5^2}, w_4, w_5), [\xi_3 : \xi_4 : \xi_5]) \in (I \times U^+_3) \times \mathbb{R}P^*(\mathbb{R} \times \mathbb{R}^2)$. Under this assumption, we have $(\boldsymbol{w}, [\xi_3 : \xi_4 : \xi_5]) = (\boldsymbol{w}, [1 : \frac{\xi_4}{\xi_3} : \frac{\xi_5}{\xi_3}])$. Therefore the canonical contact form on $PT^*(S^1_t \times S^2_s)|I \times U^+_3$ is given by $\alpha = d\overline{\theta} - \sum_{i=4}^5 \eta_i dw_i$, where $\eta_i = \frac{\xi_i}{\xi_3}$. It follows that there exists a contact morphism $\Phi : PT^*(S^1_t \times S^2_s)|(I \times U^+_3) \cap \{\xi_3 \neq 0\} \longrightarrow I \times T^*S^2_+|\{w_3 > 0\}$ defined by

$$\Phi((\theta, w_3, w_4, w_5), [\xi_3 : \xi_4 : \xi_5]) = \left(\theta, (0, 1, w_3, w_4, w_5), \left(\frac{\xi_4}{\xi_3}, \frac{\xi_5}{\xi_3}\right)\right),$$

where the contact structure on $I \times T^*S^2_+|\{w_3 > 0\}$ is also given by α. We now consider the symplectic manifold $T^*S^2_+|\{w_3 > 0\}$ with the canonical symplectic structure $\omega = \sum_{i=4}^5 d\eta_i \wedge dw_i$.

On the other hand, we have $\Sigma_*(H) = \{(p, \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(p)) \mid p \in M\}$ by Propsition 3.1. By the definition of $\Sigma_*(H)$, we have

$$0 = H(p, \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(p)) = \langle \boldsymbol{X}(p), \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(p)\rangle = -\lambda(p)\cos(\theta(p) - \overline{\theta}(p)) + (\boldsymbol{X}_E(p), \boldsymbol{w}_E(p)),$$

where $\boldsymbol{X}_E(p) = (x_3(p), x_4(p), x_5(p))$, $\boldsymbol{w}_E(p) = (w_3(p), w_4(p), w_5(p))$ and $(\cdot,\cdot)$ is the canonical Euclidean scalar product of $\mathbb{R}^3$. Therefore, we have

$$\cos(\theta(p) - \overline{\theta}(p)) \leq \frac{\sqrt{\lambda^2(p) - 1}}{\lambda(p)} < 1,$$

and thus $\lambda(p)\sin(\theta(p) - \overline{\theta}(p)) \neq 0$, so that $\mathcal{L}_H(V) \subset PT^*(S^1_t \times S^2_s)|(I \times U^+_3) \cap \{\xi_3 \neq 0\}$. Let $\Pi : I \times T^*S^2_+ \longrightarrow T^*S^2_+$ be the canonical projection. We define a mapping $L_H : V \longrightarrow T^*S^2_+$ by $L_H(p) = \Pi \circ \Phi \circ \mathcal{L}_H(p)$. Since Φ is a contact morphism, $\Phi \circ \mathcal{L}_H$ is a Legendrian immersion. It follows that L_H is a Lagrangian immersion. Moreover, we have

$$L_H(p) = \left(\widehat{\tau} \circ \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(p), \left(\frac{\xi_4}{\xi_3}, \frac{\xi_5}{\xi_3}\right)\right) = \left(\widehat{\mathbb{NG}}^{\pm}_M(p), \left(\frac{\xi_4}{\xi_3}, \frac{\xi_5}{\xi_3}\right)\right).$$

This means that $\widehat{\mathbb{NG}}^{\pm}_M$ is a Lagrangian map. Since $L_H = \Pi \circ \Phi \circ \mathcal{L}_H$, it is easy to show that $p \in M$ is a singular point of $\widehat{\mathbb{NG}}^{\pm}_M$ if and only if it is a singular point of $\mathbb{NG}^{\pm}_M$. This fact is a well-known and simple fact on the relation between Lagrangian singularities and Legendrian singularities in general. Therefore we have the following proposition.

Proposition 5.1 *For a point $p \in M$, the following conditions are equivalent:*

(1) $\mathcal{K}_\ell(1, \pm 1)(p) = 0$.
(2) $\overline{\mathcal{K}}_\ell(1, \pm 1)(p) = 0$.
(3) *The point $p \in M$ is a singular point of* $\mathbb{NG}^{\pm}_M$.
(4) *The point $p \in M$ is a singular point of* $\widehat{\mathbb{NG}}^{\pm}_M$.
(5) $\widehat{\mathcal{K}}^{\pm}_\ell(p) = 0$.

6 Contact with lightlike hyperbolic cylinders

We analyze in this section the geometrical meaning of the singularities of the $S^1_t \times S^2_s$-nullcone Legendrian Gauss map on a spacelike surface $\boldsymbol{X}(U) = M$ in Anti de Sitter 4-space. For this purpose we shall study the contacts between spacelike surfaces and lightlike hyperbolic cylinders as in the classical differential geometry. In the first place, we briefly review the theory of contact due to Montaldi [21]. Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that *the contact of* X_1 and Y_1 at y_1 is of the same type than *the contact of* X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in this definition $\mathbb{R}^n$ could be replaced by any manifold. In [21], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory.

Theorem 6.1 *Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent. For the definition of the $\mathcal{K}$-equivalence among smooth map-germs, see* [18, 19].

Consider the function $\mathcal{H} : AdS^4 \times (S^1_t \times S^2_s) \longrightarrow \mathbb{R}$ defined by $\mathcal{H}(\boldsymbol{x}, \boldsymbol{v}) = \langle \boldsymbol{x}, \boldsymbol{v} \rangle$, where $(v_1, v_2, v_3, v_4, v_5)$. For any $\boldsymbol{v}_0 = (v_{0,1}, v_{0,2}, v_{0,3}, v_{0,4}, v_{0,5}) \in S^1_t \times S^2_s$, we denote that $\mathfrak{h}_{v_0}(\boldsymbol{x}) = \mathcal{H}(\boldsymbol{x}, \boldsymbol{v}_0)$. Then the subset $\mathfrak{h}^{-1}_{v_0}(0) = NH(\boldsymbol{v}_0, 0) \cap AdS^4$ is lightlike hyperbolic cylinder. Given $p_0 = \boldsymbol{X}(x_0, y_0) \in U$, we can take the lightlike vector $\boldsymbol{v}^{\pm}_0 = \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(p_0)$, then we have

$$\mathfrak{h}_{v^{\pm}_0} \circ \boldsymbol{X}(p_0) = \mathcal{H} \circ (\boldsymbol{X} \times id_{LC^*_0})((p_0), \boldsymbol{v}^{\pm}_0) = H((x_0, y_0), \boldsymbol{v}^{\pm}_0) = 0.$$

We also have the relations

$$\frac{\partial \mathfrak{h}_{v^{\pm}_0} \circ \boldsymbol{X}}{\partial x}(p_0) = \frac{\partial H}{\partial x}((p_0), \boldsymbol{v}^{\pm}_0) = 0$$

and

$$\frac{\partial \mathfrak{h}_{v^{\pm}_0} \circ \boldsymbol{X}}{\partial y}(p_0) = \frac{\partial H}{\partial y}((p_0), \boldsymbol{v}^{\pm}_0) = 0.$$

This means that the lightlike hyperbolic cylinder $\mathfrak{h}^{-1}_{v^{\pm}_0}(0) = NH(\boldsymbol{v}^{\pm}_0, 0) \cap AdS^4$ is tangent to $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(x_0, y_0)$ in the Anti de Sitter 4-space AdS^4. In this case, we call each $NH(\boldsymbol{v}^{\pm}_0, 0) \cap AdS^4$ a *tangent lightlike hyperbolic cylinder* of $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(x_0, y_0)$ in AdS^4. We denote it as $THC(\boldsymbol{v}^{\pm}, M, p_0)$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be two null vectors. The intersection $NH(\boldsymbol{v}^{+}_0, 0) \cap NH(\boldsymbol{v}^{-}_0,) \cap AdS^4$ is a spacelike hyperboloid tangent to $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(x_0, y_0)$. We call it the *tangent spacelike hyperboloid* of $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(x_0, y_0)$. We write it as $TSH(\boldsymbol{v}^{\pm}, M, p_0)$. If $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly dependent, then the corresponding null hyperplanes $NH(\boldsymbol{v}_1, 0)$ and $NH(\boldsymbol{v}_2, 0)$ are coincident. Then we have the following lemma, whose proof is traightforward.

Lemma 6.2 *Let $\boldsymbol{X} : U \longrightarrow AdS^4$ be an immersion such that $M = \boldsymbol{X}(U)$ is a spacelike surface and $\sigma = \pm$. Consider two points $p_1 = \boldsymbol{X}(x_1, y_1), p_2 = \boldsymbol{X}(x_2, y_2)$. Then $\mathbb{NG}^{\sigma}_M(p_1) = \mathbb{NG}^{\sigma}_M(p_2)$ if and only if $THC(\boldsymbol{v}^{\sigma}_1, M, p_1) = THC(\boldsymbol{v}^{\sigma}_2, M, p_2)$. Here, $v^{\pm}_i = \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(p_i)$ for $i = 1, 2$.*

We denote $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, \boldsymbol{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0}) \longrightarrow (\mathbb{R}, \boldsymbol{0})$ be function germs. We say that F and G are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0})$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0})$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

We now apply the usual tools for the study of the contacts between spacelike surfaces and lightlike hyperbolic cylinders. Let $\mathbb{NG}^{\sigma}_{M,i} : (U, (x_i, y_i)) \longrightarrow (S^1_t \times S^2_s, \boldsymbol{v}^{\sigma}_i)$ $(i = 1, 2)$ be two $(S^1_t \times S^2_s)$-nullcone Legendrian Gauss map germs of spacelike surface germs $\boldsymbol{X}_i : (U, (x_i, y_i)) \longrightarrow (AdS^4, p_i)$, where $\sigma = \pm$. We say that $\mathbb{NG}^{\sigma}_{M,1}$ and $\mathbb{NG}^{\sigma}_{M,2}$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\phi : (U, (x_1, y_1)) \longrightarrow (U, x_2, y_2))$ and $\Phi : (S^1_t \times S^2_s, \boldsymbol{v}^{\sigma}_1) \longrightarrow (S^1_t \times S^2_s, \boldsymbol{v}^{\sigma}_2)$ such that $\Phi \circ \mathbb{NG}^{\sigma}_{M,1} = \mathbb{NG}^{\sigma}_{M,2} \circ \phi$. Let $H_i : (U \times S^1_t \times S^2_s, ((x_i, y_i), \boldsymbol{v}^{\sigma}_i)) \longrightarrow \mathbb{R}$ be the nullcone height function germ of $\boldsymbol{X}_i$. We denote $h_{i,v^{\sigma}_i}(u) = H_i(u, \boldsymbol{v}^{\sigma}_i)$, then we have $h_{i,v^{\pm}_i}(u) = \mathfrak{h}_{v^{\pm}_i} \circ \boldsymbol{X}_i(u)$. By Theorem 5.1, $K(\boldsymbol{X}_1(U), NH(\boldsymbol{v}^{\sigma}, -1), \boldsymbol{v}^{\sigma}_1) = K(\boldsymbol{X}_2(U), NH(\boldsymbol{v}^{\sigma}, -1), \boldsymbol{v}^{\sigma}_2)$ if and only if h_{1,v_1} and h_{1,v_2} are $\mathcal{K}$-equivalent. We define the local ring by

$$Q^{\pm}(\boldsymbol{X}, (x_0, y_0)) = \frac{C^{\infty}_{(x_0,y_0)}(U)}{\langle \langle \boldsymbol{X}(x, y), \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(x_0, y_0) \rangle \rangle_{C^{\infty}_{(x_0,y_0)}(U)}},$$

where $C^{\infty}_{(x_0,y_0)}(U)$ is the local ring of function germs at (x_0, y_0) with the unique maximal ideal $\mathfrak{M}_{(x_0,y_0)}(U)$. By similar arguments to those of the proof for Theorem 6.3 in [9], we can show the following theorem:

Theorem 6.3 *Let* $\boldsymbol{X}_i : (U, (x_i, y_i)) \longrightarrow (AdS^4, \boldsymbol{X}_i((x_i, y_i)))$ $(i = 1, 2)$ *be spacelike surface germs such that the corresponding Legendrian lift germs of the* $S^1_t \times S^2_s$*-nullcone Legendrian Gauss map germs are Legendrian stable and* $\sigma = \pm$*. Then the following conditions are equivalent:*

(1) $S^1_t \times S^2_s$*-nullcone Legendrian Gauss map germs* $\mathbb{NG}^\sigma_1$ *and* $\mathbb{NG}^\sigma_2$ *are* $\mathcal{A}$*-equivalent.*

(2) H_1 *and* H_2 *are* P-$\mathcal{K}$*-equivalent.*

(3) h_{1,v_1} *and* h_{1,v_2} *are* $\mathcal{K}$*-equivalent.*

(4) $K(\boldsymbol{X}_1(U), THC(\boldsymbol{v}^\sigma, \boldsymbol{X}_1(U_1)), \boldsymbol{X}(x_1, y_1)) = K(\boldsymbol{X}_2(U), THC(\boldsymbol{v}^\sigma, \boldsymbol{X}_2(U), \boldsymbol{X}_2(x_2, y_2)))$.

(5) $Q^\sigma(\boldsymbol{X}_1, (x_1, y_1))$ *and* $Q^\sigma(\boldsymbol{X}_2, (x_2, y_2))$ *are isomorphic as* $\mathbb{R}$*-algebras.*

For a spacelike surface germ $\boldsymbol{X} : (U, (x_0, y_0)) \longrightarrow (AdS^4, \boldsymbol{X}(x_0, y_0))$, we call each set

$$(\boldsymbol{X}^{-1}(THC(\boldsymbol{v}^\sigma, \boldsymbol{X}(U), \boldsymbol{X}(x_0, y_0))), (x_0, y_0))$$

a *tangent lightlike hyperpbolic cylinder indicatrix germ* of $\boldsymbol{X}$, where $\boldsymbol{v}^\pm = \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(x_0, y_0)$. Moreover, by the above results, we adopt some basic invariants from the singularity theory on function germs. We need $\mathcal{K}$-invariants for function germ. The local ring of a function germ is a complete $\mathcal{K}$-invariant for generic function germs. It is, however, not a numerical invariant. The $\mathcal{K}$-codimension (or, Tyurina number) of a function germ is a numerical $\mathcal{K}$-invariant of function germs [18]. We denote it by

$$\text{L-ord}^\pm(\boldsymbol{X}, (x_0, y_0)) = \dim_{\mathbb{R}} \frac{C^\infty_{(x_0,y_0)}(U)}{\langle h_{v_0^\pm}(x, y), h_{v_0^\pm, x}(x, y), h_{v_0^\pm, y}(x, y)\rangle}.$$

Usually L-ord$^\sigma(\boldsymbol{X}, u_0)$ is called *the* $\mathcal{K}$*-codimension of* $h_{v_0^\sigma}$, where $\sigma = \pm$. However, we call it the *order of contact with the tangent lightlike hyperbolic cylinder* at $\boldsymbol{X}(x_0, y_0)$. We also have the notion of corank of function germs.

$$\text{L-corank}^\sigma(\boldsymbol{X}, (x_0, y_0)) = 2 - \operatorname{rank} \operatorname{Hess}(h_{v_0^\sigma}(x_0, y_0)),$$

where $\boldsymbol{v}_0^\pm = \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(x_0, y_0)$. By Proposition 2.1, $\boldsymbol{X}(x_0, y_0)$ is a L^σ-parabolic point if and only if L-corank$^\sigma(\boldsymbol{X}, (x_0, y_0)) \geq 1$. Moreover $\boldsymbol{X}(x_0, y_0)$ is a lightlike umbilic point if and only if L-corank$^\sigma(\boldsymbol{X}, (x_0, y_0)) = 2$.

On the other hand, a function germ $f : (\mathbb{R}^{n-1}, \boldsymbol{a}) \longrightarrow \mathbb{R}$ has the A_k-type singularity if f is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 \pm \cdots \pm u_{n-2}^2 + u_{n-1}^{k+1}$. If L-corank$^\sigma(\boldsymbol{X}, (x_0, y_0)) = 1$, the nullcone height function $h_{v_0^\sigma}$ has the A_k-type singularity at (x_0, y_0) in generic. In this case we have H-ord$^\sigma(\boldsymbol{X}, u_0) = k$. This number k is equal to the order of contact in the classical sense (cf., [4]). This is the reason why we call L-ord$^\sigma(\boldsymbol{X}, (x_0, y_0))$ the order of contact with the tangent lightlike hyperbolic cylinder at $\boldsymbol{X}(x_0, y_0)$.

7 Singularities of $S^1_t \times S^2_s$-nullcone Legendrian Gauss maps

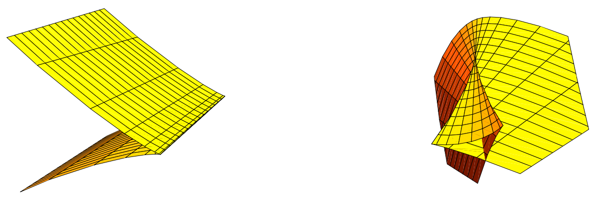
In this section we study the generic singularities of $S^1_t \times S^2_s$-nullcone Legendrian Gauss maps. We consider the space of spacelike embeddings $\mathrm{Emb}_s\,(U, AdS^4)$ with the Whitney C^∞-topology, where $U \subset \mathbb{R}^2$ is an open subset. By standard arguments analogous to those used in [9, 13], we obtain the following theorem as a corollary of Lemma 6 in Wassermann [24]. (See also Montaldi [22]).

Theorem 7.1 *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_s(U, AdS^4)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the following conditions hold:*

(1) *Each lightlike parabolic set $\mathcal{K}_\ell(1, \sigma 1)^{-1}(0)$ is a regular curve. We call such a curve the lightlike parabolic curve.*

(2) *The image of the $S^1_t \times S^2_s$-nullcone Legendrian Gauss map $\mathbb{NG}^\sigma_M(U)$ along the lightlike parabolic curve is a cuspidaledge except at isolated points. At these points $\mathbb{NG}^\sigma_M(U)$ is a swallowtail.*

Here, $\sigma = \pm$ and a map germ $f : (\mathbb{R}^2, \boldsymbol{a}) \longrightarrow (\mathbb{R}^3, \boldsymbol{b})$ is called a cuspidaledge if it is $\mathcal{A}$-equivalent to the germ (u_1, u_2^2, u_2^3) (cf., Fig. 1) *and a swallowtail if it is $\mathcal{A}$-equivalent to the germ $(3u_1^4 + u_1^2 u_2, 4u_1^3 + 2u_1 u_2, u_2)$ (cf.,* Fig.1).



cuspidaledge swallowtail

Fig. 1.

Following the terminology of Whitney [25], we say that a surface $\boldsymbol{X} : U \longrightarrow AdS^4$ has *an excellent $S^1_t \times S^2_s$-nullcone Legendrian Gauss map* $\mathbb{NG}^\sigma_M$ if the Legendrian lift of $\mathbb{NG}^\sigma_M$ is a stable Legendrian immersion at each point. In this case, the $S^1_t \times S^2_s$-nullcone Legendrian Gauss map $\mathbb{NG}^\sigma_M$ has only cuspidaledges and swallowtails as singularities. We have the following results analogous to the results of Banchoff et al [2].

Theorem 7.2 *Let $\mathbb{NG}^\sigma_M : (U, (x_0, y_0)) \longrightarrow (AdS^4, p_0)$ be the excellent $S^1_t \times S^2_s$-nullcone Legendrian Gauss map of a spacelike surface $\boldsymbol{X}$ and $h_{v_0^\sigma} : (U, (x_0, y_0)) \longrightarrow \mathbb{R}$ be the nullcone height function germ at $\boldsymbol{v}_0^\pm = \overline{\boldsymbol{e}_1 \pm \boldsymbol{e}_2}(p_0)$, where $\sigma = \pm$. Then we have the following:*

(1) *(x_0, y_0) is a lightlike parabolic point of $\boldsymbol{X}$ if and only if* $\text{L-corank}^\sigma(\boldsymbol{X}, (x_0, y_0)) = 1$.

(2) *If (x_0, y_0) is a lightlike parabolic point of $\boldsymbol{X}$, then $h_{v_0^\sigma}$ has the A_k-type singularity for $k = 2, 3$.*

(3) *Suppose that (x_0, y_0) is a lightlike parabolic point of $\boldsymbol{X}$. Then the following conditions are equivalent:*

(a) $\mathbb{NG}^\sigma_M$ *has a cuspidaledge at (x_0, y_0)*

(b) *$h_{v_0^\sigma}$ has the A_2-type singularity.*

(c) $\text{L-ord}^\sigma(\boldsymbol{X}, (x_0, y_0)) = 2$.

(d) *The tangent lightlike hyperbolic cylinder indicatrix is an ordinary cusp, where a curve $C \subset \mathbb{R}^2$ is called a ordinary cusp if it is diffeomorphic to the curve given by $\{(u_1, u_2) \mid u_1^2 - u_2^3 = 0\}$.*

(e) $\widehat{\mathbb{NG}}^\sigma_M$ *has a fold at (x_0, y_0).*

(4) *Suppose that (x_0, y_0) is a lightlike parabolic point of $\boldsymbol{X}$. Then the following conditions are equivalent:*

(a) $\mathbb{NG}^\sigma_M$ *has a swallowtail at (x_0, y_0).*

(b) *$h_{v_0^\sigma}$ has the A_3-type singularity.*

(c) L-ord$^\sigma(\boldsymbol{X},(x_0,y_0))=3$.

(d) *The tangent lightlike hyperbolic cylinder indicatrix is either a point, or a tachnodal, where a curve $C\subset\mathbb{R}^2$ is called a tachnodal if it is diffeomorphic to the curve given by* $\{(u_1,u_2)\mid u_1^2-u_2^4=0\ \}$.

(e) *For each $\varepsilon>0$, there exist two distinct points $(x_i,y_i)\in U$ $(i=1,2)$ such that*

$$\|(x_0,y_0)-(x_i,y_i)\|<\varepsilon$$

for $i=1,2$, and the tangent lightlike hyperbolic cylinder to $M=\boldsymbol{X}(U)$ at (x_i,y_i) coincide.

(f) $\widehat{\mathbb{NG}}^\sigma_M$ *has a cusp at* (x_0,y_0).

Proof. We have shown that (x_0,y_0) is a lightlike parabolic point if and only if

$$\text{L-corank}^\sigma(\boldsymbol{X},(x_0,y_0))\geq 1.$$

We have L-corank$^\sigma(\boldsymbol{X},(x_0,y_0))\leq 2$. Since the nullcone height function germ $H:(U\times S^1_t\times S^2_s,((x_0,y_0),\boldsymbol{v}_0))\longrightarrow\mathbb{R}$ can be considered as a generating family of the Legendrian lift of $\mathbb{NG}^\sigma_M$, $h_{v_0^\sigma}$ has only the A_k-type singularities $(k=1,2,3)$. This means that the corank of the Hessian matrix of $h_{v_0^\sigma}$ at a lightlike parabolic point is 1. The assertion (2) also follows. By the same reason, the conditions (3);(a),(b),(c) (respectively, (4); (a),(b),(c)) are equivalent. If the height function germ $h_{v_0^\sigma}$ has the A_2-type singularity, then it is $\mathcal{K}$-equivalent to the germ $\pm u_1^2+u_2^3$. Since the $\mathcal{K}$-equivalence preserve the diffeomorphism type of zero level sets, the tangent lightlike hyperbolic cylinder indicatrix is diffeomorphic to the curve given by $\pm u_1^2+u_2^3=0$. This is the ordinary cusp. The normal form for the A_3-type singularity is given by $\pm u_1^2+u_2^4$, so that the tangent lightlike hyperbolic cylinder indicatrix is diffeomorphic to the curve $\pm u_1^2+u_2^4=0$. This means that the condition (3),(d) (respectively, (4),(d)) is also equivalent to the other conditions.

Suppose that (x_0,y_0) is a lightlike parabolic point, the singularities of the $S^1_t\times S^2_s$-nullcone Legendrian Gauss map are cuspidaledges or swallowtails. For the swallowtail point (x_0,y_0), there is a self intersection curve (cf., Fig. 1) approaching to (x_0,y_0). On this curve, there are two distinct points (x_i,y_i) $(i=1,2)$ such that $\mathbb{NG}^\sigma_M(x_1,y_1)=\mathbb{NG}^\sigma_M(x_2,y_2)$. By Lemma 5.2, this means that the tangent lightlike hyperbolic cylinders to $M=\boldsymbol{X}(U)$ at (x_i,y_i) coincide. Since there are no other singularities in this case, the condition (4),(e) characterize a swallowtail point of $\mathbb{NG}^\sigma_M$.

Finally, we know from the theory of Legendrian/Lagrangian singularities that a Legendrian map has a swallowtail (resp., cuspidaledge) if and only if the corresponding Lagrangian map has the cusp (resp., fold). This completes the proof. $\square$

8 The Gauss-Bonnet type theorem

In this section we give the definition of the global S^2_+-nullcone Gauss-Kronecker curvature and show a nullcone Gauss-Bonnet type theorem. Let M be a closed orientable 2-dimensional manifold and $f:M\to AdS^4$ a spacelike embedding.

Since AdS^4 is time-oriented, we can globally choose future directions in the normal bundle $N(M)$ of $f(M)$. Let $\boldsymbol{e}_1$ be a timelike unit normal vector field along $f(M)$ directing towards the future direction at every point. We can now construct a spacelike unit normal vector field

$\boldsymbol{e}_2$ as in §2. In this way we have a S^2_+-nullcone Lagrangian Gauss map globally defined on M, $\widehat{\mathbb{NG}}^{\pm}_M : M \longrightarrow S^2_+$.

The *global S^2_+-nullcone Gauss-Kronecker curvature function* $\widehat{\mathcal{K}}^{\pm}_{\ell} : M \to \mathbb{R}$ is then defined in the usual way in terms of the global S^2_+-nullcone Lagrangian Gauss map $\widehat{\mathbb{NG}}^{\pm}_M$. By the equality (12), we have

$$\widehat{\mathcal{K}}^{\pm}_{\ell} d\mathfrak{a}_M = (\widehat{\mathbb{NG}}^{\pm}_M)^* d\mathfrak{a}_{S^2_+},$$

We can now prove the following Gauss-Bonnet type theorem.

Theorem 8.1 *If M is a closed orientable spacelike surface in AdS^4, then*

$$\int_M \widehat{\mathcal{K}}^{\pm}_{\ell} d\mathfrak{a}_M = 2\pi \boldsymbol{\chi}(M)$$

where $d\mathfrak{a}_M$ is the area form and $\boldsymbol{\chi}(M)$ is the Euler number of M.

The principal idea of the proof is almost the same as that of the Gauss-Bonnet type theorem in [12]. Let $\pi^5_3 : \mathbb{R}^5_2 \longrightarrow \mathbb{R}^3$ be the canonical projection defined by $\pi^5_3(x_1, x_2, x_3, x_4, x_5) = (0, 0, x_3, x_4, x_5)$. Since $f(M)$ is a spacelike surface in $AdS^4 \subset \mathbb{R}^5_2$, $\pi^5_3(f(M)$ is a smooth surface in the Euclidean space $\mathbb{R}^3$. Let $\mathbb{N} : M \longrightarrow S^2$ be the Euclidean Gauss map on $\pi^5_3(f(M))$. We show the following lemma.

Lemma 8.2 *For a suitable choice of the direction $\mathbb{N}$, $\pi^5_3 \circ \widehat{\mathbb{NG}}^{\pm}_M$ and $\mathbb{N}$ are homotopic.*

Proof. Since $\mathbb{NG}^{\pm}_M(p)$ is a lightlike vector, $\mathbb{NG}^{\pm}_M(p) \notin \mathrm{Ker}\pi^5_3$. Moreover, $\mathrm{Ker} d\widehat{\tau}_p \subset \mathrm{Ker}\pi^5_3$, so that $\mathbb{NG}^{\pm}_M(p) \notin \mathrm{Ker} d\widehat{\tau}_p$. By definition, we can identify $d\widehat{\tau}_p(\mathbb{NG}^{\pm}_M(p))$ with $\widehat{\mathbb{NG}}^{\pm}_M(p)$. Therefore, the fact $\mathbb{NG}^{\pm}_M(p) \notin T_p f(M)$ induces that $\pi^5_3 \circ \widehat{\mathbb{NG}}^{\pm}_M(p) \notin T_{\pi^5_3(p)} \pi^5_3(f(M))$. This means that $\langle \pi^5_3 \circ \widehat{\mathbb{NG}}^{\pm}_M(p), \mathbb{N}(p) \rangle \neq 0$. We now choose the direction $\mathbb{N}$ such that $\langle \pi^5_3 \circ \pi^5_3 \circ \widehat{\mathbb{NG}}^{\pm}_M(p), \mathbb{N}(p) \rangle \not> 0$.

On the other hand, we define a mapping $F : M \times [0,1] \longrightarrow S^2$ by

$$F(p,t) = \frac{t\mathbb{N}(p) + (1-t)\pi^5_3 \circ \widehat{\mathbb{NG}}^{\pm}_M(p)}{\|t\mathbb{N}(p) + (1-t)\pi^5_3 \circ \widehat{\mathbb{NG}}^{\pm}_M(p)\|},$$

where $\|\cdot\|$ is the Euclidean norm. If there exist $t' \in [0,1]$ and $p' \in M$ such that $t'\mathbb{N}(p') + (1-t')\pi^5_3 \circ \widehat{\mathbb{NG}}^{\pm}_M(p') = 0$, then $\mathbb{N}(p') = -\pi^5_3 \circ \widehat{\mathbb{NG}}^{\pm}_M(p')$, contrary to the assumption that $\langle \pi^5_3 \circ \widehat{\mathbb{NG}}^{\pm}_M(p), \mathbb{N}(p) \rangle \not> 0$. Therefore F is continuous and $F(p,0) = \pi^5_3 \circ \widehat{\mathbb{NG}}^{\pm}_M(p)$ and $F(p,1) = \mathbb{N}(p)$. $\square$

It follows easily from the definitions of π^5_3 and $\widehat{\mathbb{NG}}^{\pm}_M$ that we may identify $\pi^5_3 \circ \widehat{\mathbb{NG}}^{\pm}_M$ with $\widehat{\mathbb{NG}}^{\pm}_M$. Since the mapping degree is a homotopy invariant, we obtain (cf., [8], Chapter 4, §9):

$$\deg \widehat{\mathbb{NG}}^{\pm}_M = \frac{1}{2}\boldsymbol{\chi}(M).$$

Therefore we obtain

$$\int_M \widehat{\mathcal{K}}^{\pm}_{\ell} d\mathfrak{a}_M = \int_M (\widehat{\mathbb{NG}}^{\pm}_M)^* d\mathfrak{a}_{S^2_+} = \deg \widehat{\mathbb{NG}}^{\pm}_M \int_{S^2_+} d\mathfrak{a}_{S^2_+} = \deg \widehat{\mathbb{NG}}^{\pm}_M \times 4\pi = 2\pi \boldsymbol{\chi}(M).$$

This completes the proof of Theorem 8.1.

9 Horospherical Gauss maps

In this section we consider the geometric meaning of the singularities of the *hyperbolic Gauss map* of $\widehat{M} = \widehat{\boldsymbol{X}}(U)$ in $H^3_+(-1)$, which was introduced in [12]. Here we denote that $\widehat{\boldsymbol{X}} = \widehat{\tau} \circ \boldsymbol{X} : U \longrightarrow H^3_+(-1)$. The hyperbolic Gauss map of $\widehat{M}$ is defined as follows: Take a pseudo-orthogonal frame $\{\widehat{\boldsymbol{X}}, \widehat{\boldsymbol{E}}, \widehat{e}_3, \widehat{e}_4\}$ along $\widehat{M}$ in such a way that $\{\widehat{e}_3, \widehat{e}_4\}$ is a tangent frame. Then we have that $\widehat{\boldsymbol{X}} \pm \widehat{\boldsymbol{E}}$ is a normal lightlike vector and hence we can define a map $\mathbb{L}^\pm : \widehat{M} \longrightarrow LC^*_+$ by $\mathbb{L}^\pm(\hat{p}) = \widehat{\boldsymbol{X}}(u) \pm \widehat{\boldsymbol{E}}(u)$ which is called *the hyperbolic Gauss indicatrix* (or *the lightcone dual*) of $\boldsymbol{X}$. The linear transformation $S^\pm_{\hat{p}} = -d\mathbb{L}^\pm(u) : T_{\hat{p}}\widehat{M} \longrightarrow T_{\hat{p}}\widehat{M}$ is called *the hyperbolic shape operator* of $\widehat{M} = \widehat{\boldsymbol{X}}(U)$ at $\hat{p} = \widehat{\boldsymbol{X}}(u)$. *The hyperbolic Gauss-Kronecker curvature* of $\widehat{M} = \widehat{\boldsymbol{X}}(U)$ at $\hat{p} = \widehat{\boldsymbol{X}}(u)$ is defined to be $K^\pm_h(u) = \det S^\pm_{\hat{p}}$. If $\boldsymbol{x} = (x_1, x_1, x_3, x_4) \in \mathbb{R}^4_{1+}$ is a lightlike vector, then $x_1 \neq 0$. Therefore we have $\tilde{\boldsymbol{x}} = \left(1, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_0}\right) \in S^2_+$. The *the hyperbolic Gauss map* $\widetilde{\mathbb{L}}^\pm : \widehat{M} \longrightarrow S^2_+$ on $\widehat{M}$ is defined by $\widetilde{\mathbb{L}}^\pm(\hat{p}) = \widetilde{\mathbb{L}^\pm(\hat{p})}$. We remark that this definition of hyperbolic Gauss map is equivalent to the one introduced in [5, 7]. Let $T_p\widehat{M}$ be the tangent space of $\widehat{M}$ at $\hat{p}$ and $N_{\hat{p}}\widehat{M}$ be the pseudo-normal space of $T_{\hat{p}}\widehat{M}$ in $T_{\hat{p}}\mathbb{R}^4_1$. We have a linear transformation $\pi^t_{\hat{p}} \circ d\widetilde{\mathbb{L}}^\pm(u) : T_{\hat{p}}\widehat{M} \longrightarrow T_{\hat{p}}\widehat{M}$ by the identification of U and $\widehat{\boldsymbol{X}}(U) = \widehat{M}$ via the embedding $\widehat{\boldsymbol{X}}$. We call the linear transformation $\widetilde{S}^\pm_{\hat{p}} = -\pi^t_{\hat{p}} \circ d\widetilde{\mathbb{L}}^\pm$ the *horospherical shape operator* of $\widehat{M}$. The *principal horospherical curvatures*, $\widetilde{\kappa}^\pm_i(\hat{p}), i = 1, 2$, are the eigenvalue of $\widetilde{S}^\pm_{\hat{p}}$, the corresponding eigenvectors are called *principal horospherical directions*. These determine a couple of orthogonal line foliations on $\widehat{M}$, except at those points, called *horoumbilics*, for which both principal horospherical curvatures coincide. The *horospherical principal configuration* is composed by the principal horospherical curvature lines and the horoumbilic points. The *horospherical Gauss-Kronecker curvature* of $\widehat{M}$ is defined to be $\widetilde{K}^\pm_h(\hat{p}) = \det \widetilde{S}^\pm_{\hat{p}}$.

We can use this setting in order to associate a *hyperbolic Gauss map* $\mathbb{HG}^\pm_M$ to any spacelike surface $M \subset AdS^4$ by putting $\mathbb{HG}^\pm_M = \widetilde{\mathbb{L}}^\pm \circ \widehat{\tau} : M \to S^2_+$. We observe that $\mathbb{HG}^\pm(p) \neq \widehat{\mathbb{NG}}^\pm_M(p)$.

On the other hand, the intersection of a lightlike hyperplane $H(\boldsymbol{n}, c)$, $\boldsymbol{n} = (0, 1, n_3, n_4, n_5) \in S^2_+ \subset \mathbb{R}^4_{1+}$ and $c \in \mathbb{R}_+$, of $\mathbb{R}^4_{1+}$ with $H^3_+(-1)$ is a horosphere $h(\boldsymbol{n}, c) = \{\boldsymbol{x} \in H^3_+(-1) | -x_2 + n_3x_3 + n_4x_4 + n_5x_5 = c\}$. The pull-back by $\widehat{\tau}$ of a horosphere $h(\boldsymbol{n}, c)$ in $H^3_+(-1)$ will be called *round horosphere* in AdS^4. We observe that $\widehat{\tau}^1(h(\boldsymbol{n}, c)) = \{(\lambda\cos\theta, \lambda\sin\theta, x_3, x_4, x_5) \in AdS^4 : x_3n_3 + x_4n_4 + x_5n_5 = c + \lambda\}$, is an invariant subset through the action of the group $SO(2) \times SO(3)$ over $\mathbb{R}^5_2$. Topologically, we can see it as a product $S^1 \times h(\boldsymbol{n}, c)$ (with the radii of the fibers varying along the points of $h(\boldsymbol{n}, c)$). In [12] we observed that $\widetilde{\mathbb{L}}^\pm$ is a constant vector if and only if $\widehat{M}$ is a part of a horosphere in $H^3_+(-1)$. Therefore, $\mathbb{HG}^\pm$ is constant if and only if M is a subset of a round horosphere in AdS^4. Moreover,the pull-back by $\widehat{\tau}$ of the horospherical principal configuration on $\widehat{M}$ is, by definition, the *horospherical principal configuration* on M. We can interpret this as follows: For any $p \in M$, consider the lightlike vector $\mathbb{HG}^\pm_M(p) = (1, w_3, w_4, w_5)^\pm \in S^2_+ \subset \mathbb{R}^4_1$. Then $\widehat{\tau}^{-1}((1, w_3, w_4, w_5)^\pm)$ is a circle $S^1_t \times \{(w_3, w_4, w_5)^\pm\} \subset S^1_t \times S^2_s$. Consider now this circle in $T_p\mathbb{R}^5_{2*} \equiv \mathbb{R}^5_2$. Then we have that the intersection $S^1_t \times \{(w_3, w_4, w_5)\} \times N_pM$ determines a normal lightlike direction $\ell^\pm(p) \in S^1_t \times S^2_s$ for M at p. In this way we obtain a lightlike normal field $\ell^\pm$ on M that satisfies $\mathbb{HG}^\pm(p) \neq \widehat{\tau} \circ \ell^\pm$. Moreover, if $S_{\ell^\pm}(p)$ is the shape operator associated to the normal direction $\ell^\pm(p)$, we have that $\widetilde{S}^\pm_p = d_p\widehat{\tau} \circ S_{\ell^\pm}(p)$. Therefore, the principal configuration associated to the normal field $\ell^\pm$ coincides with the horospherical principal configuration on M. The *horospherical Gauss-*

Kronecker curvature of M is defined as $\widetilde{K}^{\pm}_{hM}(p) = \widetilde{K}^{\pm}_{h}(\tau(p))$. We say that a point p is *a (positive or negative) horoparabolic point* of M if $\widetilde{K}^{+}_{hM}(p) = 0$ or $\widetilde{K}^{-}_{hM}(p) = 0$. Moreover, a point p is said to be a *horospherical point* if it is both horoumbilic and horoparabolic. We observe that the *horoflatness* is invariant through motions in $SO(2) \times SO(3)$.

The *Horospherical height functions family* $\widetilde{H} : U \times S^2_+ \longrightarrow \mathbb{R}$ on $M = \boldsymbol{X}(U) \subset AdS^4$ is defined by $\widetilde{H}(u, \boldsymbol{v}) = \langle \widehat{\boldsymbol{X}}(u), \boldsymbol{v} \rangle$. We denote the Hessian matrix of the horospherical height function $\widetilde{h}_{v_0}(u) = \widetilde{H}(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}(\widetilde{h}_{v_0})(u_0)$. The following are immediate consequences of ([12] Proposition 3.4 and Corollary 3.5).

Proposition 9.1 *Let $\widetilde{H} : U \times S^2_+ \longrightarrow \mathbb{R}$ be a lightcone height function on $\boldsymbol{X} : U \longrightarrow AdS^4$. Then $(u, \boldsymbol{v}) \in U \times S^2_+$ is a critical point of $\widetilde{H}$ if and only if $\boldsymbol{v} = \mathbb{HG}^{\pm}(u)$.*
Moreover, provided $\boldsymbol{v}_0 = \mathbb{HG}^{\pm}(u_0)$. Then we have

(1) $p = \boldsymbol{X}(u_0)$ *is a horoparabolic point if and only if* $\det \mathrm{Hess}(\widetilde{h}_{v_0})(u_0) = 0$.

(2) $p = \boldsymbol{X}(u_0)$ *is a horospherical point if and only if* $\mathrm{rank}\, \mathrm{Hess}(\widetilde{h}_{v_0})(u_0) = 0$.

Corollary 9.2 *For a point $p = \boldsymbol{X}(u_0) \in M$, the following conditions are equivalent:*
(1) *The point $p \in M$ is a horoparabolic point (i.e., $\widetilde{K}^{\pm}_{hM}(p) = 0$).*
(2) *The point $p \in M$ is a singular point of the hyperbolic Gauss map $\mathbb{HG}^{\pm}$.*
(3) $\det \mathrm{Hess}(\widetilde{h}_{v_0})(u_0) = 0$ *for* $\boldsymbol{v}_0 = \mathbb{HG}^{\pm}(u_0)$.

We observe that the horospherical height functions family measures the contacts of the surface M with round horospheres in AdS^4. The following result is also an immediate consequence of the above considerations.

Corollary 9.3 *For $M \subset AdS^4$, the following conditions are equivalent:*
(1) *The horospherical Gauss-Kronecker curvature $\widetilde{K}^{\pm}_{hM}$ vanishes identically on M.*
(2) *The hyperbolic Gauss map $\mathbb{HG}^{\pm}$ is constant over M.*
(3) *M lies in a round horosphere.*

We say that the embedding $\boldsymbol{X}$ is generic if its associated horospherical height functions family is structurally stable (see [14]), in other words, if $\widehat{X}$ is a generic embedding in $H^3_+(-1)$. The results obtained in [14] concerning the horospherical points of generically immersed surfaces in $H^3_+(-1)$ allow us to state:

Theorem 9.4 *(1) The horospherical configurations in a neighbourhood of a horoumbilical point in a generic surface M in AdS^4 are of Darbouxian type $D_i, i = 1, 2, 3$. Therefore, the index of the horospherical principal direction fields at a horospherical point of a surface generically immersed in AdS^4 is $\pm\frac{1}{2}$.*

(2) The number of horospherical points of any closed surface M generically immersed in AdS^4 is greater or equal than $2|\chi(M)|$, where $\chi(M)$ denotes the Euler number of M.

(3) Any 2-sphere generically immersed in AdS^4 has at least 4 horoumbilical points.

On the other hand, as seen in [12], when $\widehat{M}$ is a closed orientable surface in $H^3_+(-1)$, we can consider a globally defined hyperbolic Gauss map on $\widehat{M}$ (and thus on M), and consequently, a globally defined Gauss-Kronecker curvature function on $\widehat{M}$ (and thus on M). For the purposes of the following result, we can either fix the superindex + or − in the above arguments and denote by $\widetilde{\mathcal{K}}_{hM}$ the globally defined Gauss-Kronecker curvature function on M and by $\widetilde{\mathcal{K}}_h$ the globally defined Gauss-Kronecker curvature function on $\widehat{M}$. We first observe that if $d\mathfrak{v}_M$ and $d\mathfrak{v}_{\widehat{M}}$ represent respectively the volume forms of M and $\widehat{M}$. If M is a closed orientable spacelike surface in AdS^4, then

$$\int_M \widetilde{\mathcal{K}}_{hM} d\mathfrak{a}_M = \int_{\widehat{M}} \widetilde{\mathcal{K}}_h d\mathfrak{a}_{\widehat{M}}.$$

Then, since $\widehat{\tau}$ determines a diffeomorphism between M and $\widehat{M}$, we have that their Euler characteristics coincide. Therefore, as a consequence of the Gauss-Bonnet Theorem obtained in [12] for the closed orientable surfaces immersed in Hyperbolic 3-space, we can state:

Theorem 9.5 *If M is a closed orientable spacelike surface in AdS^4, then*

$$\int_M \widetilde{\mathcal{K}}_{hM} d\mathfrak{a}_M = 2\pi\boldsymbol{\chi}(M)$$

where $d\mathfrak{a}_M$ is the area form and $\boldsymbol{\chi}(M)$ is the Euler number of M.

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