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HOKKAIDO UNIVERSITY
Abstract. Let $p$ be a rational prime, $k$ be a perfect field of characteristic $p$, $W = W(k)$ be the ring of Witt vectors, $K$ be a finite totally ramified extension of Frac$(W)$ of degree $e$ and $r$ be a non-negative integer satisfying $r < p - 1$. Let $V$ be a semi-stable $p$-adic $G_K$-representation with Hodge-Tate weights in $\{0, \ldots, r\}$. In this paper, we prove the upper numbering ramification group $G^{(j)}_K$ for $j > u(K, r, n)$ acts trivially on the mod $p^n$ representations associated to $V$, where $u(K, 0, n) = 0$, $u(K, 1, n) = 1 + e(n + 1/(p - 1))$ and $u(K, r, n) = 1 - p^{-e(K(\zeta_p)/K)}^{-1} + e(n + r/(p - 1))$ for $r > 1$.

1. Introduction

Let $p$ be a rational prime, $k$ be a perfect field of characteristic $p$, $W = W(k)$ be the ring of Witt vectors and $K$ be a finite totally ramified extension of $K_0 = \text{Frac}(W)$ of degree $e = e(K)$. Let the maximal ideal of $K$ be denoted by $m_K$, an algebraic closure of $K$ by $\bar{K}$ and the absolute Galois group of $K$ by $G_K = \text{Gal}(\bar{K}/K)$. We normalize the valuation $v_K$ of $K$ as $v_K(p) = e$ and extend this to $\bar{K}$. Let $G^{(j)}_K$ denote the $j$-th upper numbering ramification group in the sense of [7]. Namely, we put $G^{(j)}_K = G^{j-1}_K$, where the latter is the upper numbering ramification group defined in [15].

Let $X_K$ be a proper smooth scheme over $K$ and put $X_{\bar{K}} = X_K \times_K \bar{K}$. Consider the $r$-th étale cohomology group $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ and its $G_K$-stable $\mathbb{Z}_p$-lattices $\mathcal{L} \supseteq \mathcal{L}'$. In [7], Fontaine conjectured the upper numbering ramification group $G^{(j)}_K$ acts trivially on the $G_K$-module $\mathcal{L}/\mathcal{L}'$ for $j > e(n + r/(p-1))$ if $X_K$ has good reduction and this module is killed by $p^n$. For $e = 1$ and $r < p - 1$, this conjecture was proved independently by himself ([8], for $n = 1$) and Abrashkin ([2], for any $n$), using the theory of Fontaine-Laffaille ([10]) and the comparison theorem of Fontaine-Messing ([11]) between the $p$-adic étale cohomology groups of $X_K$ and the crystalline cohomology groups of the reduction of $X_K$. From this result, Fontaine also showed some rareness of a proper smooth scheme over $\mathbb{Q}$ with everywhere good reduction ([8, Théorème 1]). In fact, they proved this ramification bound for the torsion

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representations of the crystalline $p$-adic representations of $G_K$ with Hodge-Tate weights in $\{0, \ldots, r\}$ in the case where $K$ is absolutely unramified.

On the other hand, for a semi-stable $p$-adic representation $V$ with Hodge-Tate weights in the same range, a similar ramification bound for $e = 1$ and $n = 1$ is obtained by Breuil (see [4, Proposition 9.2.2.2]). He showed, assuming the Griffiths transversality which in general does not hold, that if $e = 1$ and $r < p - 1$, then the ramification group $G_K^{(j)}$ acts trivially on the mod $p$ representations of $V$ for $j > 2 + 1/(p - 1)$.

In this paper, we prove a version of the result of Breuil for the case where $K$ is absolutely ramified, under the condition $r < p - 1$. Our main theorem is the following.

**Theorem 1.1.** Let $r$ be a non-negative integer such that $r < p - 1$. Let $V$ be a semi-stable $p$-adic $G_K$-representation with Hodge-Tate weights in $\{0, \ldots, r\}$ and $\mathcal{L} \supseteq \mathcal{L}'$ be $G_K$-stable $\mathbb{Z}_p$-lattices in $V$. Suppose that the quotient $\mathcal{L}/\mathcal{L}'$ is killed by $p^n$. Then the $j$-th upper numbering ramification group $G_K^{(j)}$ acts trivially on the $G_K$-module $\mathcal{L}/\mathcal{L}'$ for $j > u(K, r, n)$, where

$$u(K, r, n) = \begin{cases} 
0 & (r = 0), \\
1 + e(n + \frac{1}{p-1}) & (r = 1), \\
1 - \frac{1}{p^e(K(\zeta_p)/K)} + e(n + \frac{r}{p-1}) & (r > 1)
\end{cases}$$

and $e(K(\zeta_p)/K)$ denotes the relative ramification index of the extension $K(\zeta_p)/K$.

We can check that this bound is sharp for $r \leq 1$ (Remark 5.13). From this theorem and [7, Proposition 1.3], we have the following corollary.

**Corollary 1.2.** Let the notation be as in the theorem and $L$ be the finite extension of $K$ cut out by the $G_K$-module $\mathcal{L}/\mathcal{L}'$. Let $D_{L/K}$ denote the different of the extension $L/K$. Then we have the inequality

$$v_K(D_{L/K}) < u(K, r, n)$$

for $r > 0$ and $v_K(D_{L/K}) = 0$ for $r = 0$.

For the proof of Theorem 1.1, we essentially follow a beautiful argument of Abrashkin ([2]). We may assume $p \geq 3$ and $r \geq 1$. Thanks to Liu's theorem ([14]) on the $G_K$-stable $\mathbb{Z}_p$-lattices in semi-stable $p$-adic representations, it is enough to bound the ramification of the $G_K$-module

$$T_{st, \mathcal{L}}^*(\mathcal{M}_n) = \text{Hom}_{S, \text{Fil}^r, \phi, N}(\mathcal{M}_n, \hat{A}_{st, \infty}),$$

where $\mathcal{M}_n$ is a $p^n$-torsion object of a category $\text{Mod}_{S, \text{Fil}^r, \phi, N}$ of filtered $(\phi, N)$-modules over $S$ defined by Breuil ([3]) and $\hat{A}_{st, \infty}$ is a $p$-adic period ring. We may also assume $\zeta_p \in K$ and consider the finite Galois extension

$$F_n = K(\pi^{1/p^n}, \zeta_p^{p^{n+1}})$$

of $K$ whose upper ramification is bounded by the same value as in the theorem. Let $L_n$ be the finite Galois extension of $F_n$ cut out by $T_{st, \mathcal{L}}^*(\mathcal{M}_n)|_{G_{F_n}}$. 
Then we bound the ramification of $L_n$ over $K$. For this, we show that to study this $G_{F_n}$-module we can use a variant over a smaller coefficient ring $\Sigma$ of filtered $(\phi_\tau, N)$-modules over $S$. In precise, let $E(u)$ be the Eisenstein polynomial of a uniformizer $\pi$ of $K$ over $W$ and we set

$$\Sigma = W[[u, E(u)^p/p]].$$

This ring $\Sigma$ is small enough for the method of Abrashkin, in which he uses filtered modules of Fontaine-Laffaille ([10]) whose coefficient ring is $W$, to work also in the case where $K$ is absolutely ramified.

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2. Filtered $(\phi_\tau, N)$-modules of Breuil

In this section, we recall the theory of filtered $(\phi_\tau, N)$-modules over $S$ of Breuil, which is developed by himself and most recently by Caruso and Liu (see for example [3], [5], [14], [6]). In what follows, we always take the divided power envelope of a $W$-algebra with the compatibility condition with the natural divided power structure on $pW$.

Let $r \geq 3$ be a rational prime and $\sigma$ be the Frobenius endomorphism of $W$. We fix once and for all a uniformizer $\pi$ of $K$ and a system $f_{\pi}^{n}$ of $p$-power roots of $\pi$ such that $\pi_0 = \pi$ and $\pi_n = \pi^{p^n+1}$ for any $n$. Let $E(u)$ be the Eisenstein polynomial of $\pi$ over $W$ and set $S = (W[u]^{PD})^\wedge$, where PD means the divided power envelope and this is taken with respect to the ideal $(E(u))$, and $\wedge$ means the $p$-adic completion. The ring $S$ is endowed with the $\sigma$-semilinear endomorphism $\phi : u \mapsto u^p$ and a natural filtration $\text{Fil}^r S$ induced by the divided power structure such that $\phi(\text{Fil}^r S) \subseteq \text{Fil}^{r+1} S$ for any non-negative integer $r$. We also define a filtration, $\phi, \phi_t, N$ on $S_n = S/p^n S$ similarly.

Let $r \in \{0, \ldots, p-2\}$ be an integer. Set $\text{Mod}_{/S}^{r, \phi, N}$ to be the category consisting of the following data:

- an $S$-module $M$ and its $S$-submodule $\text{Fil}^r S.M$ containing $\text{Fil}^r S.M$,
- a $\phi$-semilinear map $\phi_r : \text{Fil}^r S.M \to M$ satisfying $\phi_r(s_m) = \phi_r(s)\phi(m)$ for any $s_r \in \text{Fil}^r S$ and $m \in M$, where we set $\phi(m) = c^{−r}\phi(E(u)^r m)$,
- a $W$-linear map $N : M \to M$ such that
  - $N(sm) = N(s)m + sN(m)$ for any $s \in S$ and $m \in M$,
  - $E(u)N(\text{Fil}^r S.M) \subseteq \text{Fil}^r S.M$,
the following diagram is commutative:

\[
\begin{align*}
\text{Fil}^r \mathcal{M} & \longrightarrow \mathcal{M} \\
\phi_r & \quad (u)N \\
\text{Fil}^r \mathcal{M} & \longrightarrow \mathcal{M}
\end{align*}
\]

and the morphisms of \(\mathcal{O} \text{Mod}_{/S}^{r,\phi,N}\) are defined to be the \(S\)-linear maps preserving \(\text{Fil}^r\) and commuting with \(\phi_r\) and \(N\). The category defined in the same way but dropping the data \(N\) is denoted by \(\mathcal{O} \text{Mod}_{/S}^{r,\phi}\). These categories have obvious notions of exact sequences. Let \(\mathcal{O} \text{Mod}_{/S_1}^{r,\phi,N}\) denote the full subcategory of \(\mathcal{O} \text{Mod}_{/S}^{r,\phi,N}\) consisting of \(\mathcal{M}\) such that \(\mathcal{M}\) is free of finite rank over \(S_1\) and generated as an \(S_1\)-module by the image of \(\phi_r\). We write \(\mathcal{O} \text{Mod}_{/S_\infty}^{r,\phi,N}\) for the smallest full subcategory which contains \(\mathcal{O} \text{Mod}_{/S_1}^{r,\phi,N}\) and is stable under extensions. We let \(\mathcal{O} \text{Mod}_{/S}^{r,\phi,N}\) denote the full subcategory consisting of \(\mathcal{M}\) such that

- the \(S\)-module \(\mathcal{M}\) is free of finite rank and generated by the image of \(\phi_r\),
- the quotient \(\mathcal{M}/\text{Fil}^r \mathcal{M}\) is \(p\)-torsion free.

We define full subcategories \(\mathcal{O} \text{Mod}_{/S_1}^{r,\phi}, \mathcal{O} \text{Mod}_{/S_\infty}^{r,\phi}\) and \(\mathcal{O} \text{Mod}_{/S}^{r,\phi}\) of \(\mathcal{O} \text{Mod}_{/S}^{r,\phi,N}\) in a similar way. For \(\mathcal{M} \in \mathcal{O} \text{Mod}_{/S_1}^{r,\phi,N}\) (resp. \(\mathcal{O} \text{Mod}_{/S_\infty}^{r,\phi,N}\)), the quotient \(\mathcal{M}/p^n \mathcal{M}\) has a natural structure as an object of \(\mathcal{O} \text{Mod}_{/S_1}^{r,\phi,N}\) (resp. \(\mathcal{O} \text{Mod}_{/S_\infty}^{r,\phi,N}\)).

For \(p\)-torsion objects, we also have the following categories. Consider the \(k\)-algebra \(k[u]/(u^p) \cong S_1/\text{Fil}^p S_1\) and let this algebra be denoted by \(\tilde{S}_1\). The algebra \(\tilde{S}_1\) is equipped with the natural filtration, \(\phi\) and \(N\) induced by those of \(S\). Namely, \(\text{Fil}^r \tilde{S}_1 = u^r \tilde{S}_1\), \(\phi(u) = u^p\) and \(N(u) = -u\). Let \(\mathcal{O} \text{Mod}_{/S_1}^{r,\phi,N}\) denote the category consisting of the following data:

- an \(\tilde{S}_1\)-module \(\tilde{\mathcal{M}}\) and its \(\tilde{S}_1\)-submodule \(\text{Fil}^r \tilde{\mathcal{M}}\) containing \(u^r \tilde{\mathcal{M}}\),
- a \(\phi\)-semilinear map \(\phi_r : \text{Fil}^r \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}\),
- a \(k\)-linear map \(N : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}\) such that
  - \(N(sm) = N(s)m + sN(m)\) for any \(s \in \tilde{S}_1\) and \(m \in \tilde{\mathcal{M}}\),
  - \(u^r N(\text{Fil}^r \tilde{\mathcal{M}}) \subseteq \text{Fil}^r \tilde{\mathcal{M}}\),
- the following diagram is commutative:

\[
\begin{align*}
\text{Fil}^r \tilde{\mathcal{M}} & \longrightarrow \tilde{\mathcal{M}} \\
\phi_r & \quad (u)N \\
\text{Fil}^r \tilde{\mathcal{M}} & \longrightarrow \tilde{\mathcal{M}}
\end{align*}
\]
and whose morphisms are defined as before. Its full subcategory \( \text{Mod}^{r,\phi,N}_{/S_1} \) is defined by the following condition:

- As an \( S_1 \)-module, \( \mathcal{M} \) is free of finite rank and generated by the image of \( \phi_r \).

We define categories \( \text{Mod}^{r,\phi} \) and \( \text{Mod}^{r,\phi,N} \) similarly.

Let \( D \) be a weakly admissible filtered \((\phi,N)\)-module over \( K \) satisfying \( \text{Fil}^0 D_K = D_K \) and \( \text{Fil}^{r+1} D_K = 0 \). Set \( S_{K_0} = S \otimes W K_0 \) and \( D = D \otimes_{K_0} S_{K_0} \). Then the \( S_{K_0} \)-module \( D \) is equipped with the natural \( \phi \)-semilinear map \( \phi \otimes \sigma \) and \( K_0 \)-linear derivation \( N \otimes 1 + 1 \otimes N \), which are denoted by \( \phi \) and \( N \), respectively. We define a filtration on \( D \) inductively by \( \text{Fil}^0 D = D \) and

\[
\text{Fil}^{i+1} D = \{ x \in D \mid N(x) \in \text{Fil}^i D \} \text{ and } f_\pi(x) \in \text{Fil}^{i+1} D_K,
\]

where \( f_\pi : D \to D_K \) is induced by the map \( S \to \mathcal{O}_K \) sending \( u \) to \( \pi \). An \( S \)-submodule \( \mathcal{M} \) of \( D \) is said to be a strongly divisible lattice of \( D \) if the following conditions are satisfied:

- the \( S \)-module \( \mathcal{M} \) is free of finite rank,
- \( \mathcal{M} \otimes_W K_0 = D \),
- \( \mathcal{M} \) is stable under \( \phi \) and \( N \),
- \( \phi(\text{Fil}^r \mathcal{M}) \subseteq p^r \mathcal{M} \), where we set \( \text{Fil}^r \mathcal{M} = \mathcal{M} \cap \text{Fil}^r D \).

We put \( \phi_r = p^{-r}\phi|_{\text{Fil}^r \mathcal{M}} \). Then the \( S \)-module \( \mathcal{M} \) is generated by \( \phi_r(\text{Fil}^r \mathcal{M}) \) (see Proposition 2.1.3) and we can consider \( \mathcal{M} \) as an object of \( \text{Mod}^{r,\phi,N}_{/S} \).

Let \( \mathcal{A}_{\text{crys}} \) and \( \mathcal{A}_{\text{st}} \) be \( p \)-adic period rings. These are constructed as follows. Set \( R \) to be the ring

\[
R = \lim_{\rightarrow} (\mathcal{O}_K/p\mathcal{O}_K \leftarrow \mathcal{O}_K/p\mathcal{O}_K \leftarrow \cdots),
\]

where every arrow is the \( p \)-power map. For an element \( x = (x_i)_{i \in \mathbb{Z}_{\geq 0}} \in R \) and an integer \( n \geq 0 \), we set

\[
x^{(n)}_i = \lim_{m \to \infty} x_i^{p^n+m} \in \mathcal{O}_C,
\]

where \( \hat{x}_i \) is a lift of \( x_i \) in \( \mathcal{O}_K \) and \( \hat{\mathcal{O}}_C \) is the \( p \)-adic completion of \( \mathcal{O}_K \). Let \( v_p \) denote the valuation of \( \mathcal{O}_C \) normalized as \( v_p(p) = 1 \). Then the ring \( R \) is a complete valuation ring whose valuation of an element \( x \in R \) is given by \( v_R(x) = v_p(x(0)) \). We define a natural ring homomorphism \( \theta \) by

\[
\theta : W(R) \to \mathcal{O}_C \quad (x_0, x_1, \ldots) \mapsto \sum_{n \geq 0} p^n x^{(n)}_n.
\]

Then \( \mathcal{A}_{\text{crys}} \) is the \( p \)-adic completion of the divided power envelope of \( W(R) \) with respect to the principal ideal \( \ker(\theta) \) and \( \mathcal{A}_{\text{st}} \) is the \( p \)-adic completion of the divided power polynomial ring \( \mathcal{A}_{\text{crys}}(X) \) over \( \mathcal{A}_{\text{crys}} \). We set \( \mathcal{A}_{\text{crys}, \infty} = \mathcal{A}_{\text{crys}} \otimes_W K_0/W \) and \( \mathcal{A}_{\text{st}, \infty} = \mathcal{A}_{\text{st}} \otimes_W K_0/W \). Put \( \pi = (\pi_n)_{n \in \mathbb{Z}_{\geq 0}} \in R \), where we abusively let \( \pi_n \) denote the image of \( \pi_n \in \mathcal{O}_K \) in \( \mathcal{O}_K/p\mathcal{O}_K \). These rings
are considered as $S$-algebras by the ring homomorphisms $S \rightarrow A_{\text{st}}$ and $A_{\text{st}} \rightarrow A_{\text{crys}}$ which are defined by $u \mapsto [\pi]/(1 + X)$ and $X \mapsto 0$, respectively. The ring $A_{\text{crys}}$ is endowed with a natural filtration induced by the divided power structure, a natural Frobenius endomorphism $\phi$ and the $\phi$-semilinear map $\phi_t = p^{-t}\phi|_{\text{Fil}^t A_{\text{crys}}}$. With these structures, $A_{\text{crys}}$ and $A_{\text{crys}, \infty}$ are considered as objects of $'\text{Mod}^r_{/S}$. Moreover, the absolute Galois group $G_K$ acts naturally on these two rings. As for $\hat{A}_{\text{st}}$, its filtration is defined by

$$\text{Fil}^t \hat{A}_{\text{st}} = \left\{ \sum_{i \geq 0} a_i X^i \left| a_i \in \text{Fil}^{t-i} A_{\text{crys}}, \lim_{i \to \infty} a_i = 0 \right. \right\}$$

and the Frobenius structure of $A_{\text{crys}}$ extends to $\hat{A}_{\text{st}}$ by

$$\phi(X) = (1 + X)^p - 1,$$
$$\phi_t = p^{-t}\phi|_{\text{Fil}^t \hat{A}_{\text{st}}}.$$ 

We write $N$ also for the $A_{\text{crys}}$-linear derivation on $\hat{A}_{\text{st}}$ defined by $N(X) = 1 + X$. The rings $\hat{A}_{\text{st}}$ and $\hat{A}_{\text{st}, \infty}$ are objects of $'\text{Mod}^r_{/S}$. The $G_K$-action on $A_{\text{crys}}$ naturally extends to an action on $\hat{A}_{\text{st}}$. Indeed, the action of $g \in G_K$ on $\hat{A}_{\text{st}}$ is defined by the formula

$$g(X) = [\varepsilon(g)](1 + X) - 1,$$

where $g(\pi_n) = \varepsilon_n(g)\pi_n$ and $\varepsilon(g) = (\varepsilon_n(g))_{n \in \mathbb{Z}_{\geq 0}} \in R$ with the abusive notation as above.

These rings have other descriptions, as follows. For an integer $n \geq 1$, put $W_n = W/p^n W$ and let $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ be the ring of Witt vectors of length $n$ associated to $\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$. We define a $W_n$-algebra structure on $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ by twisting the natural $W_n$-algebra structure by $\sigma^{-n}$. Then the natural ring homomorphism

$$\theta_n : W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \rightarrow \mathcal{O}_{\bar{K}}/p^n\mathcal{O}_{\bar{K}}$$

$$(a_0, \ldots, a_{n-1}) \mapsto \sum_{i=0}^{n-1} \hat{a}_i p^{i-1}$$

where $\hat{a}_i$ is a lift of $a_i$ in $\mathcal{O}_{\bar{K}}$, is $W_n$-linear. Let us denote

$$W_n^{PD}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$$

the divided power envelope of $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ with respect to the ideal $\text{Ker}(\theta_n)$. This ring is considered as an $S$-algebra by $u \mapsto [\pi_u]$. This ring also has a natural filtration defined by the divided power structure, and a natural $G_K$-module structure. The Frobenius endomorphism of the ring of Witt vectors
induces on this ring a \( \phi \)-semilinear Frobenius endomorphism, which is denoted also by \( \phi \). Then, by the \( S \)-linear transition maps

\[
W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K) \rightarrow W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K)
\]

\[
(a_0, \ldots, a_n) \mapsto (a_0^p, \ldots, a_{n-1}^p),
\]

these \( S \)-algebras form a projective system compatible with all structures. Using this transition map, a \( \phi \)-semilinear map

\[
\phi_r : \text{Fil}^r W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K) \rightarrow W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K)
\]

is defined by setting \( \phi_r(x) \) to be the image of \( p^{-r}\phi(\hat{x}) \), where \( \hat{x} \) is a lift of \( x \) in \( \text{Fil}^r W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K) \). By definition, the maps \( \phi_r \) are also compatible with the transition maps. The \( S \)-algebra \( W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K) \) is considered as an object of \( \text{Mod}^r_{\phi} \). Then we have a natural isomorphism in \( \text{Mod}^r_{\phi} \)

\[
A_{\text{cryst}}/p^n A_{\text{cryst}} \rightarrow W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K)
\]

\[
(x_0, \ldots, x_{n-1}) \mapsto (x_{0,n}, \ldots, x_{n-1,n}),
\]

where we set \( x_i = (x_{i,k})_{k \in \mathbb{Z}_{\geq 0}} \).

Similarly, the divided power polynomial ring

\[
W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K)(X)
\]

over \( W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K) \) is considered as an \( S \)-algebra by \( u \mapsto [\pi_n]/(1 + X) \). This ring has a natural filtration coming from the divided power structure. We define a \( G_K \)-action on this ring by

\[
g(X) = [\varepsilon_n(g)](1 + X) - 1.
\]

We also define a \( \phi \)-semilinear Frobenius endomorphism, which we also write as \( \phi \), by \( \phi(X) = (1 + X)^r - 1 \) and a \( W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K) \)-linear derivation \( N \) by \( N(X) = 1 + X \). These rings form a projective system of \( S \)-algebras compatible with all structures by the transition maps defined by the maps above and \( X \mapsto X \). We define \( \phi \)-semilinear maps

\[
\phi_r : \text{Fil}^r W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K)(X) \rightarrow W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K)(X)
\]

compatible with the transition maps as before. The \( S \)-algebra \( W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K)(X) \) is considered as an object of \( \text{Mod}^r_{\phi,N} \) and there exists a natural isomorphism in \( \text{Mod}^r_{\phi,N} \)

\[
\hat{A}_{st}/p^n \hat{A}_{st} \rightarrow W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K)(X)
\]

\[
(x_0, \ldots, x_{n-1}) \mapsto (x_{0,n}, \ldots, x_{n-1,n})
\]

\[
X \mapsto X
\]

which is \( G_K \)-linear.

Put \( K_n = K(\pi_n) \) and \( K_\infty = \cup_n K_n \). For \( \mathcal{M} \in \text{Mod}^r_{\phi,N} \), we define a \( G_K \)-module \( T^*_{st,\infty}(\mathcal{M}) \) to be

\[
T^*_{st,\infty}(\mathcal{M}) = \text{Hom}_{S,\text{Fil}^r_{\phi,N}}(\mathcal{M}, \hat{A}_{st,\infty}).
\]
When $\mathcal{M}$ is killed by $p^n$, we have a natural identification of $G_K$-modules

$$T_{\text{st}, \underline{\xi}}^*(\mathcal{M}) = \text{Hom}_{S, \text{Fil}^r, \phi, N}(\mathcal{M}, W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K)(X)).$$

Note that the $G_K$-module on the right-hand side is independent of the choice of $\pi_k$ for $k > n$. Since the natural map

$$W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K)(X) \to W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K)$$

is by definition $G_{K_n}$-linear, we also have a $G_{K_n}$-linear isomorphism ([3, Lemme 2.3.1.1])

$$T_{\text{st}, \underline{\xi}}^*(\mathcal{M})|_{G_{K_n}} \to \text{Hom}_{S, \text{Fil}^r, \phi, N}(\mathcal{M}, W^n_{PD}(\mathcal{O}_K/p\mathcal{O}_K)).$$

A variant of filtered $(\phi, r, N)$-modules over $S$ is also introduced by Breuil and Kisin, and developed also by Caruso and Liu (see for example [12], [13], [14], [6]). Put $\mathcal{S} = W[[u]]$ and let $\phi : \mathcal{S} \to \mathcal{S}$ be the $\sigma$-semilinear Frobenius endomorphism defined by $\phi(u) = u^p$. Let $\text{Mod}^{r, \phi}_S$ denote the category consisting of the following data:

- an $\mathcal{S}$-module $\mathfrak{M}$,
- a $\phi$-semilinear map $\mathfrak{M} \to \mathfrak{M}$, which is denoted also by $\phi$, such that the cokernel of the map $1 \otimes \phi : \phi^\ast \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$,

and whose morphisms are defined as before. The full subcategory of $\text{Mod}^{r, \phi}_S$ consisting of $\mathfrak{M}$ such that $\mathfrak{M}$ is free of finite rank over $\mathcal{S}/p\mathcal{S}$ (resp. over $\mathcal{S}$) is denoted by $\text{Mod}^{r, \phi}_{/\mathcal{S}_1}$ (resp. $\text{Mod}^{r, \phi}_{/\mathcal{S}_{\infty}}$). We let $\text{Mod}^{r, \phi}_{/\mathcal{S}_{\infty}}$ denote the smallest full subcategory which contains $\text{Mod}^{r, \phi}_{/\mathcal{S}_1}$ and is stable under extensions, as before. Then we have an exact functor ([6, Proposition 2.1.2], see also [12, Proposition 1.1.11])

$$\mathcal{M}_{\mathcal{S}_{\infty}} : \text{Mod}^{r, \phi}_{/\mathcal{S}_{\infty}} \to \text{Mod}^{r, \phi}_{/\mathcal{S}_{\infty}}.$$ 

For $\mathfrak{M} \in \text{Mod}^{r, \phi}_{/\mathcal{S}_{\infty}}$, the filtered $\phi_r$-module $\mathcal{M} = \mathcal{M}_{\mathcal{S}_{\infty}}(\mathfrak{M})$ over $S$ is defined as follows:

- $\mathcal{M} = S \otimes_{\phi, \mathcal{S}} \mathfrak{M}$,
- $\text{Fil}^r\mathcal{M} = \text{Ker}(\mathcal{M} \xrightarrow{1 \otimes \phi} S \otimes_{\mathcal{S}} \mathfrak{M} \to (S/\text{Fil}^r S) \otimes_{\mathcal{S}} \mathfrak{M})$,
- $\phi_r : \text{Fil}^r\mathcal{M} \xrightarrow{1 \otimes \phi} \text{Fil}^r S \otimes_{\mathcal{S}} \mathfrak{M} \xrightarrow{1 \otimes 1} S \otimes_{\phi, \mathcal{S}} \mathfrak{M} = \mathcal{M}$.

We write $\mathcal{M}_{/\mathcal{S}}$ for the functor $\text{Mod}^{r, \phi}_{/\mathcal{S}} \to \text{Mod}^{r, \phi}_{/\mathcal{S}}$ defined similarly.

Finally, let $D$ and $\mathcal{D}$ be as above and $\hat{\mathcal{M}}$ be a strongly divisible lattice in $\mathcal{D}$. The $S$-module $\mathcal{M}_n = \hat{\mathcal{M}}/p^n\hat{\mathcal{M}}$ has a natural structure as an object of $\text{Mod}^{r, \phi, N}_{/S_{\infty}}$. We set a $G_K$-module $\hat{T}_{\text{st}, \underline{\xi}}^*(\hat{\mathcal{M}})$ to be

$$\hat{T}_{\text{st}, \underline{\xi}}^*(\hat{\mathcal{M}}) = \text{Hom}_{S, \text{Fil}^r, \phi, N}(\hat{\mathcal{M}}, \hat{\mathcal{A}}_{\text{st}}).$$

Then we have an exact sequence of $G_K$-modules

$$0 \to \hat{T}_{\text{st}, \underline{\xi}}^*(\hat{\mathcal{M}}) \xrightarrow{p^n} \hat{T}_{\text{st}, \underline{\xi}}^*(\hat{\mathcal{M}}) \to \hat{T}_{\text{st}, \underline{\xi}}^*(\mathcal{M}_n) \to 0.$$
The $G_K$-module $\hat{T}_{st,\Xi}^*(\hat{\mathcal{M}})$ is naturally considered as a $G_K$-stable $\mathbb{Z}_p$-lattice in $V_{st}^*(D)$. By Liu’s theorem ([14, Theorem 2.3.5]), the functor $\hat{T}_{st,\Xi}^*$ gives an anti-equivalence of categories between the category of strongly divisible lattices in $\mathcal{D}$ and the category of $G_K$-stable $\mathbb{Z}_p$-lattices in $V_{st}^*(D)$. Moreover, for such a lattice $\mathcal{L}$, its corresponding strongly divisible lattice $\hat{\mathcal{M}}$ in $\mathcal{D}$ is in the essential image of the functor $\mathcal{M}\mathcal{G}_{\xi}$ ([14, Subsection 3.5]).

### 3. Filtered $\phi_r$-modules over $\Sigma$

In this section, we define another variant of filtered $\phi_r$-modules over $S$ and prove its properties.

Let $p \geq 3$ be a rational prime and $r$ be an integer such that $0 \leq r < p - 1$. Consider the $W$-algebra $\Sigma = W[[u, Y]]/(E(u)^p - pY)$ as in [3, Subsection 3.2]. We regard $\Sigma$ as a subring of $S$ by the map sending $Y$ to $E(u)^p/p$. Then the element $c = \phi_1(E(u)) \in S^\times$ is contained in $\Sigma^\times$. We define on $\Sigma$ a $\sigma$-semilinear Frobenius endomorphism $\phi$ by $\phi(u) = u^p$ and $\phi(Y) = p^{p-1}c^p$.

Put $\text{Fil}^r\Sigma = (E(u)^t, Y)$ for $0 \leq t \leq p - 1$ and $\text{Fil}^p\Sigma = (Y)$. Then we have $\phi(\text{Fil}^r\Sigma) \subseteq \text{Fil}^{p^t}\Sigma$ for $0 \leq t \leq p - 1$. We put $\phi_\sigma = p^{-t}\phi|_{\text{Fil}^r\Sigma}$. We also set $\Sigma_{\phi} = \Sigma/p^n\Sigma$ and put on this ring the natural structures induced by those of $\Sigma$.

We define a category $'\text{Mod}^r_{\Sigma, \phi}$ of filtered $\phi_r$-modules over $\Sigma$ to be the category consisting of the following data:

- a $\Sigma$-module $M$ and its $\Sigma$-submodule $\text{Fil}^r\Sigma. M$,
- a $\phi$-semilinear map $\text{Fil}^r\Sigma. M \to M$ satisfying $\phi_r(s_r, m) = \phi_r(s_r)\phi(m)$ for any $s_r \in \text{Fil}^r\Sigma$ and $m \in M$, where we set $\phi(m) = c^{-r}\phi_r(E(u)^r m)$.

and the morphisms are defined in the same manner as $'\text{Mod}^r_{/\Sigma, \phi}$. This category has a natural notion of exact sequences. We define its full subcategory $\text{Mod}^r_{/\Sigma, \phi}$ to be the category consisting of $M$ which is free of finite rank and generated by the image of $\phi_r$ as a $\Sigma_1$-module. We also let $\text{Mod}^r_{/\Sigma, \phi}$ denote the smallest full subcategory of $'\text{Mod}^r_{/\Sigma, \phi}$ which contains $\text{Mod}^r_{/\Sigma_1, \phi}$ and is stable under extensions. Moreover, we define a full subcategory $\text{Mod}^r_{/\Sigma, \phi}$ of $'\text{Mod}^r_{/\Sigma, \phi}$ to be the category consisting of $M$ such that

- the $\Sigma$-module $M$ is free of finite rank and generated by the image of $\phi_r$,
- the quotient $M/\text{Fil}^r\Sigma M$ is $p$-torsion free.

Then we see that for $\hat{M} \in \text{Mod}^r_{/\Sigma, \phi}$, the quotient $\hat{M}/p^n\hat{M}$ is naturally considered as an object of $\text{Mod}^r_{/\Sigma, \phi}$. The natural ring isomorphism $\Sigma_1/\text{Fil}^p\Sigma_1 \cong \tilde{S}_1$ defines a functor $\mathcal{T}_0 : \text{Mod}^r_{/\Sigma_1, \phi} \to \text{Mod}^r_{/\tilde{S}_1, \phi}$ by $M \mapsto M/\text{Fil}^p\Sigma_1 M$. Then [3, Proposition 2.2.1.3] and Nakayama’s lemma shows the following.
Lemma 3.1. Let $M$ be an object of $\text{Mod}^{r,\phi}_{\Sigma_1}$ of rank $d$ over $\Sigma_1$. Then there exists a basis $\{e_1, \ldots, e_d\}$ of $M$ such that $\text{Fil}^r M = \Sigma_1 u^r e_1 + \cdots + \Sigma_1 u^{rd} e_d + \text{Fil}^p \Sigma_1 M$ for some integers $r_1, \ldots, r_d$ with $0 \leq r_i \leq c r$ for any $i$.

Then we can show the following lemma just as in the proof of [3, Lemme 2.3.1.3].

Lemma 3.2. The functor $M \mapsto \text{Hom}_{\Sigma,\text{Fil}^r,\phi} (M, A_{\text{ctys},\infty})$ from $\text{Mod}^{r,\phi}_{\Sigma_1}$ to the category of $G_{K_1}$-modules is exact.

For $M \in \text{Mod}^{r,\phi}_{\Sigma_1}$, we can show as in the case of the category $\text{Mod}^{r,\phi}_{S_1}$ that there is an isomorphism of $G_{K_1}$-modules

$$\text{Hom}_{\Sigma,\text{Fil}^r,\phi} (M, (\mathcal{O}_K/p\mathcal{O}_K)^{\text{PD}}) \rightarrow \text{Hom}_{S_1,\text{Fil}^r,\phi} (T_0(M), \mathcal{O}_K/p\mathcal{O}_K),$$

where $\mathcal{O}_K/p\mathcal{O}_K$ is considered as an object of $\text{Mod}^{r,\phi}_{S_1}$ by the natural isomorphism

$$(\mathcal{O}_K/p\mathcal{O}_K)^{\text{PD}}/\text{Fil}^p (\mathcal{O}_K/p\mathcal{O}_K)^{\text{PD}} \rightarrow \mathcal{O}_K/p\mathcal{O}_K.$$ 

Thus [3, Lemme 2.3.1.2] implies that, for such a $\Sigma_1$-module $M$, we have

$$\#\text{Hom}_{\Sigma,\text{Fil}^r,\phi} (M, (\mathcal{O}_K/p\mathcal{O}_K)^{\text{PD}}) = p^d,$$

where $d = \dim_{\Sigma_1} M$.

For the category $\text{Mod}^{r,\phi}_{\Sigma_1}$, we can show the following lemma just as in the proof of [14, Proposition 4.1.2].

Lemma 3.3. Let $\hat{M}$ be in $\text{Mod}^{r,\phi}_{\Sigma_1}$. Then there exists $\alpha_1, \ldots, \alpha_d \in \hat{M}$ such that $\text{Fil}^r \hat{M} = \Sigma_1 \alpha_1 + \cdots + \Sigma_1 \alpha_d + \text{Fil}^p \Sigma_1 \hat{M}$, $E(u)^r \hat{M} \subseteq \Sigma_1 \alpha_1 + \cdots + \Sigma_1 \alpha_d$ and the elements $e_1 = \phi_r (\alpha_1), \ldots, e_d = \phi_r (\alpha_d)$ form a basis of $\hat{M}$.

Corollary 3.4. Let $\hat{M}$ be in $\text{Mod}^{r,\phi}_{\Sigma_1}$ and $A$ be a $\Sigma$-algebra which has a structure as an object of $\text{Mod}^{r,\phi}_{\Sigma_1}$. Let $C \in M_d(\Sigma)$ be the matrix such that

$$(\alpha_1, \ldots, \alpha_d) = (e_1, \ldots, e_d) C$$

with the notation of the previous lemma. Then a $\Sigma$-linear homomorphism $f : \hat{M} \rightarrow A$ preserving $\text{Fil}^r$ also commutes with $\phi_r$ if and only if

$$\phi_r (f(e_1, \ldots, e_d) C) = (f(e_1), \ldots, f(e_d)).$$

Proof. Suppose that the latter condition holds. By assumption, we have

$E(u)^r (e_1, \ldots, e_d) = (\alpha_1, \ldots, \alpha_d) C'$
for some $C' \in M_d(\Sigma)$. We claim that $f$ commutes with $\phi$. Indeed, we have
\[
\phi(f(\epsilon_1, \ldots, \epsilon_d)) = c^{-7} \phi_r(E(u)^r(f(\epsilon_1, \ldots, \epsilon_d)))
= c^{-r} \phi_r((f(x_1), \ldots, f(x_d))C')
= c^{-r} f(x_1, \ldots, x_d) \phi(C')
= c^{-r} f(\phi_r(\alpha_1, \ldots, \alpha_d)) \phi(C')
= c^{-r} f(\phi_r(E(u)^r(e_1, \ldots, e_d))) = f(\phi(\epsilon_1, \ldots, \epsilon_d)).
\]
This implies $\phi_r \circ f = f \circ \phi_r$ also on $\text{Fil}^p \Sigma \hat{M}$.

\[\square\]

**Corollary 3.5.** Let $\hat{M}$ and $A$ be as above and $J \subseteq \text{Fil}^r A$ be an ideal of $A$ such that $\phi_r(J) \subseteq J$. We can consider the $\Sigma$-algebra $A/J$ naturally as an object of $\text{Mod}^r_{\Sigma}$. Suppose that for any $x \in J$, there exists $t \in \mathbb{Z}_{\geq 0}$ such that $\phi_r^t(x) = 0$. Then we have an isomorphism
\[
\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(\hat{M}, A) \rightarrow \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(\hat{M}, A/J).
\]

**Proof.** Let $f : \hat{M} \rightarrow A/J$ be an element of the abelian group on the right-hand side and $\hat{x}$ be an lift of $f(x)$ in $A$. By the previous corollary, it is enough to show that for any $(\hat{c}_1, \ldots, \hat{c}_d) \in J^d$, there is a unique solution $(\hat{y}_1, \ldots, \hat{y}_d) \in J^d$ of the equation
\[
(\hat{c}_1, \ldots, \hat{c}_d) + (\phi_r(\hat{y}_1), \ldots, \phi_r(\hat{y}_d)) \phi(C) = (\hat{y}_1, \ldots, \hat{y}_d).
\]
By assumption, the $d$-tuple
\[
\sum_{i=0}^{t} (\phi_r^i(\hat{c}_1), \ldots, \phi_r^i(\hat{c}_d)) \phi(C) \phi^{-1}(C) \cdots \phi(C)
\]
is stable for sufficiently large $t$ and we see that this limit gives a unique solution of the equation. \[\square\]

For an $\mathfrak{S}$-module $\mathfrak{M}$ in $\text{Mod}^r_{/\mathfrak{S}_{\infty}}$ (resp. $\text{Mod}^{r, \phi}_{/\mathfrak{S}}$), we associate to it a $\Sigma$-module $M \in \text{Mod}^r_{/\Sigma}$ as follows:

- $M = \Sigma \otimes_{\phi, \mathfrak{S}} \mathfrak{M}$,
- $\text{Fil}^r M = \text{Ker}(M \xrightarrow{1 \otimes \phi} \Sigma \otimes \mathfrak{M} \rightarrow (\Sigma/\text{Fil}^r \Sigma) \otimes \mathfrak{M})$,
- $\phi_r : \text{Fil}^r M \xrightarrow{1 \otimes \phi} \text{Fil}^r \Sigma \otimes \mathfrak{M} \xrightarrow{\phi \otimes 1} \Sigma \otimes_{\phi, \mathfrak{S}} \mathfrak{M} = M$.

We can check that this defines an exact functor $\text{Mod}^r_{/\mathfrak{S}_{\infty}} \rightarrow \text{Mod}^r_{/\Sigma}$ (resp. $\text{Mod}^{r, \phi}_{/\mathfrak{S}} \rightarrow \text{Mod}^{r, \phi}_{/\Sigma}$) as in the proof of [12, Proposition 1.1.11]. We let this functor be denoted by $M_{\mathfrak{S}_{\infty}}$ (resp. $M_{\mathfrak{S}}$).

**Proposition 3.6.** Let $\mathfrak{M}$ be an object of $\text{Mod}^r_{/\mathfrak{S}_{\infty}}$ which is killed by $p^n$. Set $M = M_{\mathfrak{S}_{\infty}}(\mathfrak{M})$ and $\mathcal{M} = M_{\mathfrak{S}_{\infty}}(\mathfrak{M})$. Then there exists a natural isomorphism of $G_{K_n}$-modules
\[
\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, W^P_n(\mathcal{O}_K/p\mathcal{O}_K)) \rightarrow \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(\mathcal{M}, W^P_n(\mathcal{O}_K/p\mathcal{O}_K)).
\]
Proof. By definition, \( \mathcal{M} = S \otimes \Sigma M \) and we have a natural isomorphism

\[
\text{Hom}_\Sigma(M, W_n^{\text{PD}}(O_K/pO_K)) \to \text{Hom}_S(\mathcal{M}, W_n^{\text{PD}}(O_K/pO_K)).
\]

Let \( f \) be an element of \( \text{Hom}_\Sigma,\text{Fil}^r,\phi_r(M, W_n^{\text{PD}}(O_K/pO_K)) \) and \( f' \) be the image of \( f \) in the right-hand side of the above isomorphism. Let us check that \( f' \) preserves \( \text{Fil}^r \) and commutes with \( \phi_r \). Since \( f' \) is \( S \)-linear, it maps \( \text{Fil}^r S, \mathcal{M} \) into \( \text{Fil}^r W_n^{\text{PD}}(O_K/pO_K) \). For \( x \in \text{Fil}^r \mathcal{M} \cap \text{Im}(M \to \mathcal{M}) \), the commutative diagram whose right vertical arrow is an isomorphism

\[
\begin{array}{ccc}
M = \Sigma \otimes \phi_r, \mathcal{M} & \xrightarrow{1 \otimes \phi} & \Sigma / \text{Fil}^r \Sigma \otimes \phi_r, \mathcal{M} \\
\downarrow & & \downarrow \\
\mathcal{M} = S \otimes \phi_r, \mathcal{M} & \xrightarrow{1 \otimes \phi} & S / \text{Fil}^r S \otimes \phi_r, \mathcal{M}
\end{array}
\]

implies \( x \in \text{Im}(\text{Fil}^r M \to \text{Fil}^r \mathcal{M}) \) and thus \( f'(x) \in \text{Fil}^r W_n^{\text{PD}}(O_K/pO_K) \).

As for the compatibility with \( \phi_r \), again by the \( S \)-linearity of \( f' \) it suffices to show \( f'(\phi_r(x)) = \phi_r(f'(x)) \) for \( x \in \text{Fil}^r \mathcal{M} \cap \text{Im}(M \to \mathcal{M}) = \text{Im}(\text{Fil}^r M \to \text{Fil}^r \mathcal{M}) \). This follows from the commutative diagram

\[
\begin{array}{ccc}
\text{Fil}^r M & \xrightarrow{\phi_r} & M \\
\downarrow & & \downarrow \\
\text{Fil}^r \mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M}
\end{array}
\]

\[
\begin{array}{ccc}
& \xrightarrow{f} & \xrightarrow{f'} \\
\downarrow & & \downarrow \\
\text{Fil}^r W_n^{\text{PD}}(O_K/pO_K) & & \text{Fil}^r W_n^{\text{PD}}(O_K/pO_K)
\end{array}
\]

Hence the map in the proposition is well-defined and injective. To prove the bijectivity, by devissage we may assume that \( p\mathcal{M} = 0 \). Then both sides of this injection have the same cardinality by the above remark. Thus the proposition follows. \( \square \)

4. A METHOD OF ABRASHKIN

In this section, we study the \( G_{K^p_n} \)-module \( \text{Hom}_\Sigma,\text{Fil}^r,\phi_r(M, W_n^{\text{PD}}(O_K/pO_K)) \) following Abrashkin ([2]).

Let \( p \geq 3 \) and \( 0 \leq r < p - 1 \) be as before. Consider the Lubin-Tate logarithm

\[
l(X) = X + \frac{X^p}{p} + \frac{X^{p^2}}{p^2} + \cdots
\]

and put \( \psi(X) = l^{-1}(\log(1 + X)) \). Then \( \psi \) gives a strict isomorphism of formal groups between the formal group associated to the logarithm \( l(X) \) and the multiplicative group \( \hat{G}_m \) over \( \mathbb{Z}_p \). We fix a system of \( p \)-power roots of unity \( \{ \zeta_n \}_{n \in \mathbb{Z}_p} \) such that \( \zeta_p \neq 1 \) and \( \zeta_p^n = \zeta_{p^{n+1}} \) for any \( n \), and set an element \( \varepsilon \) of \( R \) to be \( (\zeta_p^n)_{n \in \mathbb{Z}_p} \). Then the elements \( [\varepsilon] - 1 \) and \( [\varepsilon^{1/p}] - 1 \) are topologically nilpotent in \( W(R) \) and the element of \( W(R) \)

\[
t = \psi([\varepsilon] - 1)/\psi([\varepsilon^{1/p}] - 1)
\]
is a generator of the principal ideal $\text{Ker}(\theta)$. The element $Z = \psi([z] - 1)^{p-1}/p$ of $A_{\text{crys}}$ is topologically nilpotent and $\phi(t)$ is contained in the subset $p(1 + ZW(R)[[Z]])$

of $A_{\text{crys}}$ ([2, Subsection 1.8]). We set

$$\hat{A} = W(R)[[Z]] \subseteq A_{\text{crys}}.$$  

**Lemma 4.1.** The element $t^p/p$ of $A_{\text{crys}}$ is contained in the subring $\hat{A}$ and topologically nilpotent in this subring.

**Proof.** Put $t' = (\lceil z \rceil - 1)/([z]^{1/p} - 1)$. This is another generator of $\text{Ker}(\theta)$. We have

$$\frac{([z] - 1)^{p-1}}{p} = \frac{(t')^{p-1}}{p} \cdot ([z]^{1/p} - 1)^{p-1}$$

and $\theta([z]^{1/p} - 1) = \zeta_p - 1$. Take an element $a \in W(R)^\times$ such that $\theta(a) = (\zeta_p - 1)^{p-1}/p$. Then we have

$$\frac{([z] - 1)^{p-1}}{p} = a(t')^{p-1} + b(t')^p/p$$

for some $b \in W(R)^\times$. Indeed, to show $b \in W(R)^\times$, it suffices to check that the element $([z]^{1/p} - 1)^{p-1} - pa$ of $\text{Ker}(\theta)$ also generates this ideal. This follows from the fact that the 0-th entry $(\lceil z \rceil^{1/p} - 1)^{p-1}$ of this element satisfies $v_R((\lceil z \rceil^{1/p} - 1)^{p-1}) = 1$. Then we see that $(t')^p/p$ is topologically nilpotent because so is $t'$ in $W(R)$.

In the following, we set the element $a$ in the proof of the lemma to be

$$a = \sum_{k=1}^{p-2} \left((-1)^{p-1-k}p_{-1}C_k - 1\right)\lceil z \rceil^{1/p\cdot k},$$

where $p_{-1}C_k = (p-1)!/(k!(p-1-k)!)$. Note that the coefficient of $\lceil z \rceil^{1/p\cdot k}$ in each term is an integer.

From this lemma, we can consider the ring $\hat{A}$ as a $\Sigma$-algebra by $u \mapsto \lceil z \rceil$. Put $\text{Fil}^i\hat{A} = (t', Z)$ for $0 \leq i \leq p - 1$. The Frobenius endomorphism $\phi$ of $A_{\text{crys}}$ preserves $\hat{A}$ and satisfies $\phi(\text{Fil}^i\hat{A}) \subseteq p^i\hat{A}$ for $0 \leq i \leq p - 1$. Set $\phi_r = p^{-r}\phi|_{\text{Fil}^r\hat{A}}$. Then we can consider the ring $\hat{A}$ also as an object of the category $\textbf{Mod}^{\Sigma}_{\phi}$, and similarly for $\hat{A}_n = \hat{A}/p^n\hat{A}$ and $\hat{A}_\infty = \hat{A} \otimes W K_0/W$.

The absolute Galois group $G_{K_\infty}$ acts naturally on these $\Sigma$-algebras. The following lemma is used implicitly in [2].

**Lemma 4.2.** We have a natural decomposition

$$\hat{A}_1 = R/(t^p) \oplus (Z).$$

**Proof.** Consider the natural inclusion $W(R) \to \hat{A}$. First we claim that this induces an injection $R/(t^p) \to \hat{A}_1$. Let $x$ be in the ring $R$. If the element
$[x] \in W(R)$ is contained in $p\hat{A}$, then its image in $A_{\text{crys}}/pA_{\text{crys}}$ is zero. We have an isomorphism of $R$-algebras

$$R[Y_1, Y_2, \ldots]/(t^p, Y_1^p, Y_2^p, \ldots) \to A_{\text{crys}}/pA_{\text{crys}}$$

which sends $Y_i$ to the image of $t^p/(p^i)!$. Thus the inequality $v_R(x) \geq p$ holds. Conversely, if $v_R(x) \geq p$, then we have

$$[x] = w(\psi([\varepsilon] - 1)^{p-1}) + pw'$$

for some $w, w' \in W(R)$ and this implies $[x] \in p\hat{A}$.

Let us consider the commutative diagram of $R$-algebras

$$
\begin{array}{ccc}
R/(t^p) & \longrightarrow & \hat{A}_1 \\
\downarrow & & \downarrow \\
\hat{A}_1/(Z).
\end{array}
$$

By definition, the left downward arrow is surjective. We claim that this arrow is an isomorphism. Indeed, let $x$ be in the kernel of this surjection. From the proof of Lemma 4.1, we see that the image of $Z$ in the ring on the left-hand side of the above isomorphism can be written as $a't^{p-1} + b'Y_1$ for some $a', b' \in R^\times$. By assumption, in this ring, we have

$$x = c_1(a't^{p-1} + b'Y_1) + c_2(a't^{p-1} + b'Y_1)^2 + \cdots + c_{p-1}(a't^{p-1} + b'Y_1)^{p-1}$$

for some elements $c_1, \ldots, c_{p-1}$ of $R$. Then we see that $c_i = 0$ for any $i$ and $v_R(x) \geq p$. This concludes the proof.

Since $r < p - 1$, from this lemma we can show the following lemma as in the proof of [3, Lemme 2.3.1.3].

**Lemma 4.3.** The functor

$$M \mapsto \text{Hom}_{\Sigma, \text{Fil}, \phi_r}(M, \hat{A}_\infty)$$

from $\text{Mod}_{/\Sigma_\infty}^{r, \phi}$ to the category of $G_{K_\infty}$-modules is exact.

**Corollary 4.4.** For any $M \in \text{Mod}_{/\Sigma_\infty}^{r, \phi}$, the natural map

$$\text{Hom}_{\Sigma, \text{Fil}, \phi_r}(M, \hat{A}_\infty) \to \text{Hom}_{\Sigma, \text{Fil}, \phi_r}(M, A_{\text{crys}, \infty})$$

is an isomorphism of $G_{K_\infty}$-modules.

**Proof.** By Lemma 3.2 and Lemma 4.3, we may assume $pM = 0$. Consider the commutative diagram of rings

$$
\begin{array}{ccc}
\hat{A}_1 & \longrightarrow & A_{\text{crys}}/pA_{\text{crys}} \\
\downarrow & & \downarrow \\
R/(t^{p-1}).
\end{array}
$$
whose downward arrows are defined by modulo $\text{Fil}^{p-1}$ of the rings $\hat{A}_1$ and $A_{\text{crys}}/pA_{\text{crys}}$, respectively. Since $r < p - 1$, we have $\phi_r(\text{Fil}^{p-1}\hat{A}_1) = 0$ and similarly for the ring $A_{\text{crys}}/pA_{\text{crys}}$. Thus these two surjections induce on the ring $R/(p-1)$ the same structure of a filtered $\phi_r$-module over $\Sigma$. Hence, as in the proof of Corollary 3.5, we see from Lemma 3.1 that we have a commutative diagram

$$
\xymatrix{
\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, \hat{A}_1) \ar[r] \ar[d] & \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, A_{\text{crys}}/pA_{\text{crys}}) \ar[d] \\
\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, R/(p-1)) & }
$$

whose downward arrows are isomorphisms. This concludes the proof. \hfill \Box

We sketch the proof of the following lemma stated in [2, Subsection 3.2].

**Lemma 4.5.** The natural inclusion $W(R) \to \hat{A}$ induces an isomorphism of $W(R)$-algebras $W_n(R)/(\psi(\varepsilon) - 1)^{p-1} \to \hat{A}_n/(Z)$.

**Proof.** For a subring $B$ of $A_{\text{crys}}$, put

$$I^{[s]}B = \{ x \in B \mid \phi^i(x) \in \text{Fil}^s A_{\text{crys}} \text{ for any } i \}$$

as in [9, Subsection 5.3]. Then we have $I^{[s]}W(R) = ([\varepsilon] - 1)^sW(R)$ and the natural ring homomorphism

$$W(R)/I^{[s]}W(R) \to A_{\text{crys}}/I^{[s]}A_{\text{crys}}$$

is an injection ([9, Proposition 5.1.3, Proposition 5.3.5]). Since the element $Z$ is contained in the ideal $I^{[p-1]}A_{\text{crys}}$, this injection factors as

$$W(R)/I^{[p-1]}W(R) \to \hat{A}/(Z) \to A_{\text{crys}}/I^{[p-1]}A_{\text{crys}}.$$ 

Hence the former arrow is an isomorphism and the lemma follows. \hfill \Box

Since the ideal $(Z)$ of $\hat{A}_n$ satisfies the condition of Corollary 3.5, the $\Sigma$-algebra $\hat{A}_n/(Z)$ is naturally considered as an object of $\text{Mod}_{\Sigma}^{\phi_r}$. We also give the ring $W_n(R)/(\psi([\varepsilon] - 1)^{p-1})$ the structures of a $\Sigma$-algebra and a filtered $\phi_r$-module over $\Sigma$ induced by those of $\hat{A}_n/(Z)$. The map

$$\Sigma \to W_n(R)/(\psi([\varepsilon] - 1)^{p-1})$$

sends the element $u \in \Sigma$ to the image of $[\pi]$ in the ring on the right-hand side. Put $v = t'/E([\pi]) \in W(R)\times$ with the notation of Lemma 4.1. As for the element $Y \in \Sigma$, the equality

$$Y = -ab^{-1}v^{-1}E([\pi])^{p-1} +wb^{-1}v^{-p}Z$$

holds in $\hat{A}$, where $a$ and $b$ are the elements in $W(R)\times$ as in the proof of Lemma 4.1 and the remark after this lemma, and $w \in W(R)\times$ is a power series of $[\varepsilon] - 1$. Hence the above homomorphism sends the element $Y$ to the image of $-ab^{-1}v^{-1}E([\pi])^{p-1}$. 


Consider the surjective ring homomorphism
\[ R \to \mathcal{O}_K/p\mathcal{O}_K \]
\[ x = (x_0, x_1, \ldots) \mapsto x_n \]
and the induced surjection \( W_n(R) \to W_n(\mathcal{O}_K/p\mathcal{O}_K) \). Let
\[ J = \{ (x_0, \ldots, x_{n-1}) \in W_n(R) \mid v_R(x_i) \geq p^n \text{ for any } i \} \]
be the kernel of the latter surjection.

**Lemma 4.6.** The ideal \( J \) is contained in the ideal \( (\psi([-z] - 1)p^{-1}) \) of the ring \( W_n(R) \).

*Proof.* Write \( ([z] - 1)p^{-1} \) also as \( x = (x_0, \ldots, x_{n-1}) \in W_n(R) \) with \( v_R(x_0) = p \). Take an element \( z = (z_0, \ldots, z_{n-1}) \) in the ideal \( J \). We construct \( y \in W_n(R) \) such that \( xy = z \). By induction, it is enough to show that if \( z_0 = \cdots = z_{i-1} = 0 \) for some \( 0 \leq i \leq n - 1 \) and \( (x_0, \ldots, x_i)(0, \ldots, 0, y_i) = (0, \ldots, 0, z_i) \) in \( W_{i+1}(R) \), then \( (x_0, \ldots, 0, y_i, 0, \ldots, 0) \in J \). Let us write this element as \( (0, \ldots, 0, w_i, \ldots, w_{n-1}) \) with \( w_i = z_i \). We have \( v_R(y_i) \geq p^n - p^{i+1} \). In the ring of Witt vectors \( W_n(\mathbb{F}_p[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}]) \), the \( k \)-th entry of the vector
\[ (X_0, \ldots, X_{n-1})(0, \ldots, 0, Y_i, 0, \ldots, 0) \]
is \( X_{k-i}^p Y_i^{p^{k-i}} \) for any \( k \geq i \). Thus we have \( v_R(w_k) \geq p^n \). \( \Box \)

Note that the elements \( [\zeta^p - 1] \) and \( [\zeta^{p+1} - 1] \) is nilpotent in \( W_n(\mathcal{O}_K/p\mathcal{O}_K) \).

By the above lemma, we have an isomorphism of rings
\[ W_n(R)/(\psi([-z] - 1)p^{-1}) \to W_n(\mathcal{O}_K/p\mathcal{O}_K)/(\psi([\zeta^p - 1])p^{-1}). \]

We give the ring on the right-hand side the structure of a filtered \( \phi_r \)-module over \( \Sigma \) induced by this isomorphism.

Put \( F_n = K_n(\zeta^{p+1}) \). For an algebraic extension \( F \) of \( F_n \), let us consider the ideals
\[ m_{n,F} = \{ (x_0, \ldots, x_{n-1}) \in W_n(\mathcal{O}_F/p\mathcal{O}_F) \mid x_i \in m_F/p\mathcal{O}_F \text{ for any } i \} \]
\[ m_{n,F} = \{ (x_0, \ldots, x_{n-1}) \in W_n(\mathcal{O}_F) \mid x_i \in m_F \text{ for any } i \} \]
of \( W_n(\mathcal{O}_F/p\mathcal{O}_F) \) and \( W_n(\mathcal{O}_F) \), respectively. The elements \( [\zeta^p] - 1 \) and \( [\zeta^{p+1}] - 1 \) are topologically nilpotent in \( W_n(\mathcal{O}_F) \) and we define an element \( \hat{t} \in W_n(\mathcal{O}_F) \) to be
\[ \hat{t} = \psi([\zeta^p] - 1)/\psi([\zeta^{p+1}] - 1). \]

Note that these elements are non-zero divisors of \( W_n(\mathcal{O}_F) \). Let the ring
\[ W_n(\mathcal{O}_F/p\mathcal{O}_F)/(\psi([\zeta^p] - 1))\hat{m}_{n,F} \]
be denoted by \( \hat{\mathcal{A}}_{n,F,r+} \). We also put \( \tilde{m}_n = \hat{m}_{n}, m_n = m_{n,K} \) and \( \hat{\mathcal{A}}_{n,r+} = \hat{\mathcal{A}}_{n,K,r+} \).

For an algebraic extension \( F \) of \( K \), we put
\[ B_F = \{ x \in \mathcal{O}_F \mid v_K(x) > cr/(p-1) \}. \]
Note that the ring $O_F/b_F$ is killed by $p$. When $F$ contains $F_n$, we also put
\[ A'_{n,F,r+} = W_n(O_F/b_F)/\psi([\zeta_p^n] - 1)^{\infty} \bar{m}_{n,F} \).

Then, for $0 \leq r < p - 1$, we have natural isomorphisms of rings
\[ W_n(O_F)/\psi([\zeta_p^n] - 1)^r \bar{m}_{n,F} \to \bar{A}_{n,F,r+} \to \bar{A}'_{n,F,r+} \]
Indeed, as in the proof of Lemma 4.6, we can show that both of the kernels of the maps $W_n(O_F) \to W_n(O_F/pO_F)$ and $W_n(O_F) \to W_n(O_F/b_F)$ are contained in the ideal $\psi([\zeta_p^n] - 1)^r \bar{m}_{n,F}$ of the ring $W_n(O_F)$. We often identify these rings. We also put $\bar{A}_{0,n,r+} = \bar{A}_{n,K,r+}$.

Write $Z_n$ for the image of the element $Z$ of $A_{\text{crys}}$ in $W_n^{PD}(O_K/pO_K)$. Then we have a commutative diagram of $\Sigma$-algebras

\[
\begin{array}{ccc}
\hat{A}_n & \longrightarrow & A_{\text{crys}}/p^n A_{\text{crys}} \\
\downarrow \quad & & \quad \downarrow \\
W_n^{PD}(O_K/pO_K) & \quad & \hat{A}_n/(Z) \\
\downarrow \quad & & \quad \downarrow \\
W_n(O_K/pO_K)/\psi([\zeta_p^n] - 1)^{p-1} & \sim & \hat{A}_{n,r+} \\
\downarrow \quad & & \quad \downarrow \\
\hat{A}_{n,r+} \\
\end{array}
\]

where all vertical arrows are surjections satisfying the condition of Corollary 3.5. Thus we see that this is also a commutative diagram in $\text{Mod}_{\Sigma}^{r,\phi}$. Note that these rings and homomorphisms are independent of the choice of a system $\{\zeta_p^n\}_{n \in \mathbb{Z}_{\geq 0}}$. Moreover, let $M$ be in $\text{Mod}_{\Sigma}^{r,\phi}$ and put $M_n = M/p^n M$. Then, by Corollary 3.5 and Corollary 4.4, we have a natural isomorphism of abelian groups
\[
\text{Hom}_{\Sigma,F^\circ} (M_n, W_n^{PD}(O_K/pO_K)) \simeq \text{Hom}_{\Sigma,F^\circ} (M_n, \hat{A}_{n,r+}).
\]

To study $G_{F_n}$-actions on both sides of this isomorphism, we need the following proposition.

**Proposition 4.7.** The image of the element $Y \in \Sigma$ in the ring $\hat{A}_{n,r+}$ is contained in its subring $\hat{A}_{n,F_{n,r+}}$. 

Proof. We have the equality
\[(\varepsilon^{1/p} - 1)^{p-1} = pa - E(\varepsilon)bv\]
in $W(R)$. By definition, we see that the images of the elements \(a\), \(\varepsilon^{1/p}\) and \(E(\varepsilon)\) in $W_n(\mathcal{O}_K/p\mathcal{O}_K)$ are contained in the subring $W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$. Write \(\beta\) for the product $bv \in W(R)$. Let $a_n$, $\beta_n$ and $\alpha_n$ denote the images of the elements $a$, $\beta$ and $pa - (\varepsilon^{1/p} - 1)^{p-1}$ in the ring $W_n(\mathcal{O}_K/p\mathcal{O}_K)$, respectively. Then the element $\alpha_n$ is also contained in the subring $W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$. Now we have the equality
\[E(\varepsilon^{1/p})\beta_n = \alpha_n.\]

Note that any element $\beta'_n \in W_n(\mathcal{O}_K/p\mathcal{O}_K)$ satisfying the same equality is invertible and thus the elements $(\beta'_n)^{-1} E(\varepsilon^{1/p})$ are equal to each other. Since $Y = -a_n\beta_n^{-1} E(\varepsilon^{1/p})$ in $A_{n,r+}$, it suffices to construct an element $\beta'_n$ in the ring $W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$ such that the equality $E(\varepsilon^{1/p})\beta'_n = \alpha_n$ holds. Take a lift $\bar{\alpha}_n$ of $\alpha_n$ in $W_n(\mathcal{O}_{F_n})$. Since $E(\varepsilon^{1/p}) \in W_n(\mathcal{O}_K)$ divides every element in the kernel of the surjection $W_n(\mathcal{O}_K) \to W_n(\mathcal{O}_K/p\mathcal{O}_K)$, we have $E(\varepsilon^{1/p})\bar{\alpha}_n = \alpha_n$ for some $\beta'_n \in W_n(\mathcal{O}_K)$. Then the element $\beta'_n$ is contained in the subring $W_n(\mathcal{O}_{F_n})$ and we set $\beta'_n$ to be the image of $\beta'_n$.

By a similar argument, we can also check that the ring $\bar{A}_{n,F,r+}$ is a subring of $\bar{A}_{n,r+}$ and coincides with the image of $W_n(\mathcal{O}_F)$ in $\bar{A}_{n,r+}$. This concludes the proof. \(\square\)

Let $t_n$ and $\bar{t}_n$ be the images of $t \in W(R)$ in $W_n(\mathcal{O}_K/p\mathcal{O}_K)$ and $\bar{A}_{n,r+}$ (or $\bar{A}'_{n,r+}$), respectively. Then $t$ is a lift of $t_n$ and $\bar{t}_n$ to $W_n(\mathcal{O}_K)$ by the natural surjections
\[W_n(\mathcal{O}_K) \to W_n(\mathcal{O}_K/p\mathcal{O}_K) \to \bar{A}_{n,r+}.\]

Note that we defined the filtration of $A_{n,r+}$ as $\text{Fil}^r \bar{A}_{n,r+} = \bar{t}_n^r \bar{A}_{n,r+}$.

Lemma 4.8. Let $\bar{x}$ be in $\text{Fil}^r \bar{A}_{n,r+}$. Then we have
\[\phi_r(\bar{x}) = \phi(y) \mod \psi(\zeta^p) - 1)^{r} m_n,\]
where $y$ is any element of $W_n(\mathcal{O}_K/p\mathcal{O}_K)$ such that the element $t^n y$ is a lift of $\bar{x}$. In particular, the right-hand side of the above equality is independent of the choice of a system $\{\zeta^p\}_{n \in \mathbb{Z}_{\geq 0}}$. Similar assertions also hold for the ring
\[W_n(\mathcal{O}_K/p\mathcal{O}_K)/\psi(\zeta^p) - 1)^{p-1}.\]

Proof. Since the filtered \(\phi_r\)-module structure of $\bar{A}_{n,r+}$ is induced from that of $\bar{A}$ and $\phi_r(t^n) \equiv 1 \mod Z$ in $\bar{A}$, we see that there exists an element $y$ as in the lemma.

To prove the independence of the choice of a lift, let $z = (z_0, \ldots, z_{n-1})$ be an element of $W_n(\mathcal{O}_K/p\mathcal{O}_K)$ killed by $t_n$. The element $z$ is also killed by
Let \( p - [p_n], \) where \( p_k = p^{1/p^k} \). This implies

\[
\begin{align*}
    z_0^p + p_{n-1}z_1 & \in p\mathcal{O}_K \\
    z_n^p + p_1z_n & \in p\mathcal{O}_K
\end{align*}
\]

and \( v_p(z_k) \geq 1 - 1/p^{n-k} \) for \( 0 \leq k \leq n - 1 \). Repeating this, we see that if \( z \) is killed by \( t_r^p \), then \( v_p(z_k) \geq 1 - r/p^{n-k} \). For such an element \( z \), we have \( \phi(z) = 0 \) in the ring \( W_n(\mathcal{O}_K/p\mathcal{O}_K) \).

Let \( y_1 \) and \( y_2 \) be elements as in the lemma. Then we have

\[ y_1 - y_2 = \psi([\zeta_{p^{n+1}}] - 1)^rw + z, \]

where \( w \in \bar{m}_n \) and \( z \) is an element as above. The Frobenius endomorphism \( \phi \) sends the element on the right-hand side to an element which is contained in the ideal \( \psi([\zeta_{p^n}] - 1)^r\bar{m}_n \). Thus the assertions for the ring \( \hat{A}_{n,r+} \) follows. We can show the assertion for the ring \( W_n(\mathcal{O}_K/p\mathcal{O}_K)/(\psi([\zeta_{p^n}] - 1)^{r-1}) \) similarly.

From this lemma and Proposition 4.7, we see that the natural \( G_{F_n} \)-actions on the rings \( W_n(\mathcal{O}_K/p\mathcal{O}_K)/(\psi([\zeta_{p^n}] - 1)^{r-1}), \hat{A}_{n,r+} \) and \( A'_{n,r+} \) are compatible with the filtered \( \phi_r \)-module structures over \( \Sigma \). In the commutative diagram above, the lowest horizontal arrow and lower right vertical arrow are \( G_{F_n} \)-linear by definition. Hence we have shown the following proposition.

**Proposition 4.9.** Let \( \hat{M} \) be in \( \text{Mod}^{r,\phi}_\Sigma \) and put \( M_n = \hat{M}/p^n\hat{M} \). Then we have an isomorphism of \( G_{F_n} \)-modules

\[
\text{Hom}_{\Sigma,\text{Fil}r,\phi_r}(M_n, \hat{A}_{n,r+}) \simeq \text{Hom}_{\Sigma,\text{Fil}r,\phi_r}(M_n, W_{n,\text{PD}}(\mathcal{O}_K/p\mathcal{O}_K)).
\]

Let \( e_1, \ldots, e_d \) be a basis of \( \hat{M} \) as in Lemma 3.3 and \( C = (c_{i,j}) \in M_d(\Sigma) \) be the associated matrix representing \( \phi_r \) as in Corollary 3.4. Then the underlying \( G_{F_n} \)-set of the \( G_{F_n} \)-module

\[
\text{Hom}_{\Sigma,\text{Fil}r,\phi_r}(M_n, \hat{A}_{n,r+})
\]

is identified with the set of \( d \)-tuples \( (\bar{x}_1, \ldots, \bar{x}_d) \) in \( \hat{A}_{n,r+} \) such that \( c_{1,i}\bar{x}_1 + \cdots + c_{d,i}\bar{x}_d \in \text{Fil}r\hat{A}_{n,r+} \) for any \( i \) and the following equality holds:

\[
\begin{align*}
    \phi_r(c_{1,1}\bar{x}_1 + \cdots + c_{d,1}\bar{x}_d) &= \bar{x}_1 \\
    \vdots \\
    \phi_r(c_{1,d}\bar{x}_1 + \cdots + c_{d,d}\bar{x}_d) &= \bar{x}_d.
\end{align*}
\]

We choose a lift \( (\bar{c}_{i,j}) \in M_d(W_n(\mathcal{O}_{F_n})) \) of the image of \( C \) in \( M_d(\hat{A}_{n,r+}) \) by the natural ring homomorphism

\[
W_n(\mathcal{O}_K) \to W_n(\mathcal{O}_K/p\mathcal{O}_K) \to \hat{A}_{n,r+}.
\]

Fix a polynomial \( \Phi_1 \in \mathbb{Z}[X_0, \ldots, X_{n-1}] \) such that \( \Phi_1 \equiv X_0^p \mod p \). This induces for any commutative ring \( B \) a map \( \Phi = (\Phi_0, \ldots, \Phi_{n-1}) : W_n(B) \to W_n(B) \) which is a lift of the Frobenius endomorphism on \( W_n(B/pB) \). In
particular, set $B$ to be the polynomial ring $\mathbb{Z}[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}]$. 
Put $X = (X_0, \ldots, X_{n-1})$ and $Y = (Y_0, \ldots, Y_{n-1})$ in the ring $W_n(B)$. 
Then we see that there exists elements $U_0, \ldots, U_{n-1}$ and $U'_0, \ldots, U'_{n-1}$ of the polynomial ring $B$ such that 
\[
\Phi(X + Y) = \Phi(X) + \Phi(Y) + (pU_0, \ldots, pU_{n-1}), \\
\Phi(XY) = \Phi(X)\Phi(Y) + (pU'_0, \ldots, pU'_{n-1})
\]
in the ring $W_n(B)$.

**Proposition 4.10.** Every solution $(\hat{x}_1, \ldots, \hat{x}_d)$ in $\hat{A}_{n,r+}$ of the equation (1) such that $c_{1,i}\hat{x}_1 + \cdots + c_{d,i}\hat{x}_d \in \text{Fil}^r\hat{A}_{n,r+}$ for any $i$ uniquely lifts to a $d$-tuple $(\hat{x}_1, \ldots, \hat{x}_d)$ in $W_n(\mathcal{O}_R)$ such that $\hat{c}_{1,i}\hat{x}_1 + \cdots + \hat{c}_{d,i}\hat{x}_d \in \text{Fil}^r W_n(\mathcal{O}_R)$ for any $i$ and the following equality holds:

\[
\begin{cases}
\Phi((\hat{c}_{1,1}\hat{x}_1 + \cdots + \hat{c}_{d,1}\hat{x}_d)/\hat{\nu}^r) = \hat{x}_1 \\
\vdots \\
\Phi((\hat{c}_{1,d}\hat{x}_1 + \cdots + \hat{c}_{d,d}\hat{x}_d)/\hat{\nu}^r) = \hat{x}_d.
\end{cases}
\]

Proof. Fix a lift $\hat{x}_i$ of $\bar{x}_i$ to $W_n(\mathcal{O}_R)$. Then we have
\[
\begin{cases}
\Phi((\hat{c}_{1,1}\hat{x}_1 + \cdots + \hat{c}_{d,1}\hat{x}_d)/\hat{\nu}^r) = \hat{x}_1 + (\zeta_{\nu}^\rho - 1)^r \hat{\delta}_1 \\
\vdots \\
\Phi((\hat{c}_{1,d}\hat{x}_1 + \cdots + \hat{c}_{d,d}\hat{x}_d)/\hat{\nu}^r) = \hat{x}_d + (\zeta_{\nu}^\rho - 1)^r \hat{\delta}_d
\end{cases}
\]
for some $\hat{\delta}_1, \ldots, \hat{\delta}_d \in m_n$. It suffices to show that there exists a unique $d$-tuple $(\hat{y}_1, \ldots, \hat{y}_d)$ in $m_n$ such that
\[
\Phi((\hat{c}_{1,i}(\hat{x}_1 + (\zeta_{\nu}^\rho - 1)^r \hat{y}_1) + \cdots + \hat{c}_{d,i}(\hat{x}_d + (\zeta_{\nu}^\rho - 1)^r \hat{y}_d))/\hat{\nu}^r) = \hat{x}_i + (\zeta_{\nu}^\rho - 1)^r \hat{y}_i
\]
for any $i$. For this, we need the following lemma.

**Lemma 4.11.** Let $N$ be a complete discrete valuation field and $m_N$ be the maximal ideal of $N$. Let $\epsilon_1, \ldots, \epsilon_d$ be in $m_N$. Let $P_1, \ldots, P_d$ and $P'_1, \ldots, P'_d$ be elements of $\mathcal{O}_N[[Y_1, \ldots, Y_d]]$ such that $P_i \in (Y_1, \ldots, Y_d)^2$. Then the equation
\[
\begin{cases}
Y_1 - P_1(Y_1, \ldots, Y_d) - \epsilon_1 P'_1(Y_1, \ldots, Y_d) = 0 \\
\vdots \\
Y_d - P_d(Y_1, \ldots, Y_d) - \epsilon_d P'_d(Y_1, \ldots, Y_d) = 0
\end{cases}
\]
has a unique solution in $m_N$.

Proof. By assumption, we see that for any integer $l \geq 1$, a $d$-tuple $y = (y_1, \ldots, y_d)$ in $m_N^l/m_N^{l+1}$ satisfying the above equation lifts uniquely to a $d$-tuple in $m_N/m_N^{l+1}$ satisfying the same equation. Thus the lemma follows.
Let us write as \( \hat{y}_i = (\hat{y}_{i,0}, \ldots, \hat{y}_{i,n-1}) \). Since the image of \( \Phi((\zeta_{p^n} - 1)^r) \) in \( \bar{A}_{n,r+} \) is divisible by \( ([\zeta_{p^n}] - 1)^r \), we can find \( \hat{b} \in W_n(O_{\bar{K}}) \) such that
\[
\Phi(([\zeta_{p^n}] - 1)^r) = ([\zeta_{p^n}] - 1)^r \hat{b}.
\]
Then there exists polynomials \( U_{i,m} \) over \( O_{\bar{K}} \) of the indeterminates \( \underline{Y} = (Y_{i,m})_{1 \leq i \leq d, 0 \leq m \leq n-1} \) such that the equation we have to solve is
\[
\hat{x}_i + ([\zeta_{p^n}] - 1)^r \hat{y}_i = \hat{x}_i + ([\zeta_{p^n}] - 1)^r \hat{\delta}_i
\]
\[
+ ([\zeta_{p^n}] - 1)^r \hat{b}(\Phi(\hat{c}_{1,1})\Phi(\hat{y}_1) + \cdots + \Phi(\hat{c}_{i,d})\Phi(\hat{y}_d))
\]
\[
+ (pU_{i,0}(\hat{y}), \ldots, pU_{i,n-1}(\hat{y}))
\]
for any \( i \), where we put \( \hat{y} = (\hat{y}_{i,m})_{1 \leq i \leq d, 0 \leq m \leq n-1} \). Note that, for any elements \( P_0, \ldots, P_{n-1} \) of the polynomial ring \( O_{\bar{K}}[\underline{Y}] \), we can uniquely find elements \( Q_0, \ldots, Q_{n-1} \) of this ring such that the coefficients of these polynomials are in the maximal ideal \( m_{\bar{K}} \) and the equality
\[
(pP_0, \ldots, pP_{n-1}) = ([\zeta_{p^n}] - 1)^r(Q_0, \ldots, Q_{n-1})
\]
holds in the ring of Witt vectors \( W_n(O_{\bar{K}}[\underline{Y}]) \). Therefore, this equation is equivalent to the equation
\[
\hat{y}_i = \hat{\delta}_i + \hat{b}(\Phi(\hat{c}_{1,1})\Phi(\hat{y}_1) + \cdots + \Phi(\hat{c}_{i,d})\Phi(\hat{y}_d))
\]
\[
+ (V_{i,0}(\hat{y}), \ldots, V_{i,n-1}(\hat{y}))
\]
where \( V_{i,m} \) is a polynomial of \( \underline{Y} \) over \( O_{\bar{K}} \) whose coefficients are in the maximal ideal \( m_{\bar{K}} \). From the definition of \( \Phi \), we see that the elements \( \hat{y}_{i,m} \) is a solution of a system of equations
\[
Y_{i,m} - P_{i,m}(\underline{Y}) - \epsilon_{i,m}P'_{i,m}(\underline{Y}) = 0
\]
satisfying the condition of Lemma 4.11 for a sufficiently large finite extension \( N \) of \( K \). Then, by this lemma, we can solve the equation uniquely in \( m_{\bar{K}} \). \( \square \)

Let \( F \) be an algebraic extension of \( F_n = K_n(\zeta_{p^n+1}) \) and consider the ring \( A_{n,F,r+} \). By Lemma 4.7, we can consider this ring as a \( \Sigma \)-algebra and also as an object of \( \text{Mod}_{\Sigma}^{\text{cris}} \) by putting \( \text{Fil}^r A_{n,F,r+} = \bar{t}_n A_{n,F,r+} \) and for \( \bar{x} \in \text{Fil}^r A_{n,F,r+} \)
\[
\phi_r(\bar{x}) = \phi(y) \mod \psi([\zeta_{p^n}] - 1)^r \bar{m}_{n,F},
\]
where \( y \) is any element of \( W_n(O_F/pO_F) \) such that the element \( t^r y \) is a lift of \( \bar{x} \). For \( M_n = \bar{M}/p^n\bar{M} \in \text{Mod}_{\Sigma}^{\text{cris}} \) as before, let us set
\[
T^r_{\text{cris},\pi_n,F}(M_n) = \text{Hom}_{\Sigma,\text{Fil}^r \phi_r}(M_n, A_{n,F,r+}).
\]
We see that
\[
A_{n,r+} = A_{n,K,r+} = \bigcup_{F/F_n} A_{n,F,r+}
\]
in $\text{Mod}^{r,\phi}_{/\Sigma}$ and thus we have a natural identification of abelian groups
\[ T^*_{\text{crys}, \pi_n, K}(M_n) = \bigcup_{F/F_n} T^*_{\text{crys}, \pi_n, F}(M_n). \]

The absolute Galois group $G_{F_n}$ acts on the abelian group on the left-hand side.

**Lemma 4.12.** For an algebraic extension $F$ of $F_n$, the fixed part $T^*_{\text{crys}, \pi_n, K}(M_n)^{G_F}$ is equal to $T^*_{\text{crys}, \pi_n, F}(M_n)$.

**Proof.** From Proposition 4.10, we see that the elements of $T^*_{\text{crys}, \pi_n, K}(M_n)$ correspond bijectively to the solutions of the equation (2) in $W_n(O_K)$ satisfying the condition on $\text{Fil}^+$.

The uniqueness assertion of this proposition shows that $g \in G_F$ fixes a solution in $W_n(O_K)$ if and only if $g$ fixes its image in $A_{n, r^+}$. Hence a solution is fixed by $G_F$ if and only if this solution is contained in the image of $W_n(O_F)$. Thus the lemma follows. \[ \square \]

**Corollary 4.13.** Let $L_n$ be the finite Galois extension of $F_n$ corresponding to the kernel of the map $G_{F_n} \to \text{Aut}(T^*_{\text{crys}, \pi_n, K}(M_n))$. Then an algebraic extension $F$ of $F_n$ contains $L_n$ if and only if $\#T^*_{\text{crys}, \pi_n, F}(M_n) = \#T^*_{\text{crys}, \pi_n, K}(M_n)$.

**Proof.** An algebraic extension $F$ of $F_n$ contains $L_n$ if and only if the action of $G_F$ on $T^*_{\text{crys}, \pi_n, K}(M_n)$ is trivial. By Lemma 4.12, this is equivalent to $T^*_{\text{crys}, \pi_n, F}(M_n) = T^*_{\text{crys}, \pi_n, K}(M_n)$. \[ \square \]

### 5. Ramification bound

In this section, we prove Theorem 1.1. Take $G_K$-stable $\mathbb{Z}_p$-lattices $\mathcal{L} \supseteq \mathcal{L}'$ in $V$ such that $\mathcal{L}' \supseteq p^n \mathcal{L}$. Since the $G_K$-module $\mathcal{L}/\mathcal{L}'$ is a quotient of $\mathcal{L}/p^n \mathcal{L}$, we may assume $\mathcal{L}' = p^n \mathcal{L}$. If $r = 0$, then the $G_K$-module $V$ is unramified and the theorem is trivial. Thus we may assume $r \geq 1$ and $p \geq 3$. Let $L$ be the finite Galois extensions of $K$ corresponding to the kernel of the map
\[ G_K \to \text{Aut}(\mathcal{L}/p^n \mathcal{L}). \]

It is enough to show that, for the greatest upper ramification break $u_{L(\zeta_p)/K}$ of the Galois extension $L(\zeta_p)/K$, the inequality
\[ u_{L(\zeta_p)/K} \leq u(K, r, n) \]
holds. Since the Herbrand function is transitive and the finite Galois extension $K(\zeta_p)$ is tamely ramified over $K$, we may assume $\zeta_p \in K$. We fix a uniformizer $\pi$ of $K$ and a system $\{\pi_n\}_{n \in \mathbb{Z}_>}$ as before. Then, by Liu’s theorem ([14, Theorem 2.3.5]), it suffices to show the following.

**Theorem 5.1.** Let $r$ be an integer such that $1 \leq r < p - 1$ and $\mathcal{M}$ be the strongly divisible lattice corresponding to $\mathcal{L}$. Put $\mathcal{M}_n = \mathcal{M}/p^n \mathcal{M} \in \text{Mod}^{r,\phi,N}_{/S_{\infty}}$. Then $G_K^{(j)}$ acts trivially on the $G_K$-module $T^*_{\text{st}, \Sigma}(\mathcal{M}_n)$ for $j > u(K, r, n)$. 

Let $L_n$ be the finite Galois extension of $F_n = K_n(\zeta_{p^n+1})$ corresponding to the kernel of the map
\[ G_{F_n} \to \text{Aut}(T_{\text{st},\mathbb{Z}}(\mathcal{M}_n)). \]
Since $F_n$ is Galois over $K$, the extension $L_n$ is also a Galois extension of $K$. Let $\hat{M}$ be the object of the category $\text{Mod}_{\phi}^{r,\mathbb{S}}$ such that $M_{\phi}(\hat{M})' = \hat{M}$. From Proposition 3.6 and Proposition 4.9, we see that $L_n$ is also the finite extension of $F_n$ cut out by the $G_{F_n}$-module $T_{\text{crys},\pi_n,K}(M_n)$ for $M_n = M_{\phi}(\hat{M})/p^nM_{\phi}(\hat{M})$. It is enough to prove the inequality
\[ u_{L_n/K} \leq u(K, r, n) = \begin{cases} 1 + e(n + \frac{1}{p-1}) & (r = 1), \\ \frac{p^n-1}{p^n} + e(n + \frac{r}{p-1}) & (r > 1). \end{cases} \]
Before proving this, we state some general lemmas to calculate the ramification bound. Let $N$ be a complete discrete valuation field of positive residue characteristic, $v_N$ be its valuation normalized as $v_N(N^\times) = \mathbb{Z}$ and $N^{\text{sep}}$ be its separable closure.

**Lemma 5.2.** Let $f(T) \in \mathcal{O}_N[T]$ be a separable monic polynomial and $z_1, \ldots, z_d$ be the zeros of $f$ in $\mathcal{O}_N^{\text{sep}}$. Suppose that the set $\{v_N(z_k - z_i) \mid k = 1, \ldots, d, k \neq i\}$ is independent of $i$. Put
\[ s = \sum_{k=1}^{d} v_N(z_k - z_i) \quad \text{and} \quad \alpha = \sup_{k=1, \ldots, d, k \neq i} v_N(z_k - z_i), \]
which are independent of $i$ by assumption. If $j > s + \alpha$, then we have the decomposition
\[ \{x \in \mathcal{O}_N^{\text{sep}} \mid v_N(f(x)) \geq j\} = \prod_{i=1, \ldots, d} \{x \in \mathcal{O}_N^{\text{sep}} \mid v_N(x - z_i) \geq j - s\}. \]
Otherwise, the set on the left-hand side contains
\[ \{x \in \mathcal{O}_N^{\text{sep}} \mid v_N(x - z_i) \geq \alpha\}, \]
which contains at least two zeros of $f$.

**Proof.** A verbatim argument in the proof of [1, Lemma 6.6] shows the claim. \hfill \Box

**Corollary 5.3.** Let $f(T)$ be as above and put $B = \mathcal{O}_N[T]/(f(T))$. Suppose that the algebra $B$ is finite flat and of relative complete intersection over $\mathcal{O}_N$. Let us write the $N$-algebra $N' = B \otimes_{\mathcal{O}_N} N$ as the product $N_1 \times \cdots \times N_t$ of finite separable extensions $N_1, \ldots, N_t$ of $N$. If $j > s + \alpha$, then the $j$-th upper numbering ramification group $G_N^{(j)}$, which we let be denoted by $G_N^{(j)}$, is contained in $G_{N_i}$ for any $i$. Moreover, if $N'$ is a field and $B$ coincides with $\mathcal{O}_N$, then $j > s + \alpha$ if and only if $G_N^{(j)} \subseteq G_N$. \hfill \Box
Proof. From the previous lemma, the conductor $c(B)$ of the $\mathcal{O}_N$-algebra $B$ ([1, Proposition 6.4]) is equal to $s + \alpha$. Thus we have the inequality
$$c(\mathcal{O}_{N_1} \times \cdots \times \mathcal{O}_{N_i}) \leq c(B) = s + \alpha$$
by the definition of the conductor and a functoriality of the functor $\mathcal{F}^j$ defined in [1]. This implies the corollary.

Corollary 5.4. Consider the finite Galois extension $F_n = K_n(\zeta_{p^n+1})$ of $K$ and let $u_{F_n/K}$ denote the greatest upper ramification break of $F_n/K$. Then we have the equality
$$u_{F_n/K} = 1 + e\left(n + \frac{1}{p-1}\right).$$

Proof. Note that we are assuming that $\zeta_p$ is contained in $K$. Applying the previous corollary to the Eisenstein polynomial $f(T) = T^{p^n} - \pi$ shows that $j > 1 + e(n + 1/(p - 1))$ if and only if $G_K^{(j)} \subseteq G_{K_n}$. Similarly, putting $f(T) = T^{p^n} - \zeta_p$, we see that if $j > e(n + 1/(p - 1))$, then $G_K^{(j)} \subseteq G_K(\zeta_{p^n+1})$. Since $G_{F_n} = G_{K_n} \cap G_K(\zeta_{p^n+1})$, we conclude that $j > 1 + e(n + 1/(p - 1))$ if and only if $G_K^{(j)} \subseteq G_{F_n}$.

Remark 5.5. Note that this argument also shows the equality
$$u_{K_n(\zeta_{p^n})/K} = 1 + e\left(n + \frac{1}{p-1}\right)$$
without assuming $\zeta_p \in K$.

Next we assume that the residue field of $N$ is perfect. For an algebraic extension $F$ of $N$, we put
$$a^{\mathfrak{f}}_{F/N} = \{ x \in \mathcal{O}_F \mid v_N(x) \geq j \}.$$
For a finite Galois extension $Q$ of $N$, we write $u_{Q/N}$ for the greatest upper ramification break ([7]) of $Q/N$. Let us consider the property
$$(P_j) \begin{cases} 
\text{for any algebraic extension } F \text{ of } N, \text{ if there exists an } \mathcal{O}_N\text{-algebra homomorphism } \mathcal{O}_Q \to \mathcal{O}_F/a^{\mathfrak{f}}_{F/N}, \\
\text{then there exists an } N\text{-algebra injection } Q \to F \end{cases}$$
for $j \in \mathbb{R}_{\geq 0}$, as in [7, Proposition 1.5]. Then we have the following proposition, which is due to Yoshida. Here we reproduce his proof for the convenience of the reader.

Proposition 5.6 ([16]).
$$u_{Q/N} = \inf\{ j \in \mathbb{R}_{\geq 0} \mid \text{the property } (P_j) \text{ holds } \}.$$

Proof. By [7, Proposition 1.5 (i)], it is enough to show that the property $(P_j)$ does not hold for $j = u_{Q/N} - (e')^{-1}$ with an arbitrarily large $e' > 0$. As in the proof of [7, Proposition 1.5 (ii)], we may assume that $Q$ is totally and wildly ramified over $N$. Let $N'$ be a finite tamely ramified Galois
extension of $N$ such that $Q \cap N' = N$ and put $Q' = QN'$. Note that we have $u_{Q'/N} = u_{Q/N}$ by assumption. From this proposition in [7], we see that for some algebraic extension $F$ of $N$, there exists an $O_N$-algebra homomorphism $O_{Q'} \to O_F/a_{F/K}^j$ for $j = u_{Q/N} - e(Q'/N)^{-1}$ but no $N$-algebra injection $Q' \to F$. Since $Q/N$ is wildly ramified, we see that $e(Q/N)u_{Q/N} - 1 > e(Q/N)$.

Hence we have $u_{Q/N} - e(Q'/N)^{-1} > 1 = u_{N'/N}$ and there exists an $N$-algebra injection $N' \to F$ also by this proposition. Thus there exists no $N$-algebra injection $Q \to F$ and the property $(P_j)$ for $Q/N$ does not hold. Since we can choose an arbitrarily large $N'$ as above, the proposition follows.

We see from Proposition 5.6 that to bound the greatest upper ramification break $u_{L_n/K}$ it is enough to show the following proposition.

**Proposition 5.7.** Let $F$ be an algebraic extension of $K$. If $j > u(K, r, n)$ and there exists an $O_K$-algebra homomorphism

$$
\eta : O_{L_n} \to O_F/a^j_{F/K},
$$

then there exists a $K$-algebra injection $L_n \to F$.

**Proof.** By assumption, we have $j > er/(p - 1)$ and $b_F \supset a^j_{F/K}$. Thus $\eta$ induces an $O_K$-algebra homomorphism

$$
O_{L_n} \to O_F/b_F.
$$

Since $\eta$ also induces an $O_K$-algebra homomorphism $O_{F_n} \to O_F/a^j_{F/K}$ and $r \geq 1$, from Corollary 5.4 and [7, Proposition 1.5] we get a $K$-linear injection $F_n \to F$. Thus we see that $F$ contains $\pi_n$ and $\zeta_{p^{n+1}}$. More precisely, we have the following lemma.

**Lemma 5.8.** For some integers $i$ and $i'$ such that $i' \equiv 1 \mod p$, we have $\eta(\pi_n) \equiv \pi_n\zeta_{p^n}^i \mod b_F$ and $\eta(\zeta_{p^{n+1}}) \equiv \zeta_{p^{n+1}}^i \mod b_F$. Moreover, there exists $g \in G_K$ such that $g(\pi_n) = \pi_n\zeta_{p^n}^i$ and $g(\zeta_{p^{n+1}}) = \zeta_{p^{n+1}}^i$.

**Proof.** Since the map $\eta$ is $O_K$-linear, the equality $\eta(\pi_n)^{p^n} = \pi$ holds in $O_F/a^j_{F/K}$. Set $\hat{x}$ to be a lift of $\eta(\pi_n)$ in $O_F$. Then we have

$$
v_K(\hat{x}^{p^n} - \pi) = \sum_{i=0}^{p^n-1} v_K(\hat{x} - \pi_n\zeta_{p^n}^i) \geq j.
$$

Let us apply Lemma 5.2 to $f(T) = T^{p^n} - \pi$. Then, with the notation of the lemma, we have

$$
s = ne + \frac{p^n - 1}{p^n} \quad \text{and} \quad \alpha = \frac{1}{p^n} + \frac{e}{p - 1}.
$$

Since $j - s > er/(p - 1)$ by assumption, we have

$$
\hat{x} \equiv \pi_n\zeta_{p^n}^i \mod b_F
$$

for some $i$. 

Let \( h(T) \) be the minimal polynomial of \( \zeta_{p^{n+1}} \) over \( \mathcal{O}_K \). Since \( h \) divides \( T^{p^n} - \zeta_p \), the \( \mathcal{O}_K \)-algebra \( B' = \mathcal{O}_K[T]/(h(T)) \) is also finite flat of relative complete intersection and the \( \mathcal{O}_K \)-algebra \( B' \otimes_{\mathcal{O}_K} K \) is étale. The Galois group \( \text{Gal}(K(\zeta_{p^{n+1}})/K) \) acts transitively on the set of zeros of \( h \). Hence \( h \) also satisfies the conditions of Lemma 5.2. Let \( s' \) and \( \alpha' \) be as in this lemma for \( h \). Then we have \( s' \leq n e \) and \( \alpha' \leq e/(p-1) \). This implies \( j-s' > er/(p-1) \).

By this lemma, there exists an element \( g' \in \text{Gal}(K(\zeta_{p^{n+1}})/K) \) such that the element \( g' (\zeta_{p^{n+1}}) = \zeta_{p^{n+1}}' \) satisfies

\[
\eta(\zeta_{p^{n+1}}) \equiv g' (\zeta_{p^{n+1}}) \mod b_F.
\]

Since \( K_n \cap K(\zeta_{p^{n+1}}) = K \) (see for example [14, Lemma 5.1.2]), we can find an element \( g \in G_K \) such that \( g(\pi_n) = \pi_n \zeta_p^j \) and \( g(\zeta_{p^{n+1}}) = g' (\zeta_{p^{n+1}}) \). This concludes the proof.

**Lemma 5.9.** The \( \mathcal{O}_K \)-algebra homomorphism \( \eta \) induces an \( \mathcal{O}_K \)-algebra injection

\[
\eta_b : \mathcal{O}_{L_n}/b_{L_n} \to \mathcal{O}_F/b_F.
\]

**Proof.** We write the Eisenstein polynomial of a uniformizer \( \pi_{L_n} \) of \( L_n \) over \( \mathcal{O}_K \) as

\[
P(T) = T^{e'} + c_1 T^{e'-1} + \cdots + c_{e'-1} T + c_{e'},
\]

where \( e' = e(L_n/K) \). Then \( z = \eta(\pi_{L_n}) \) satisfies \( P(z) = 0 \) in \( \mathcal{O}_F / a_F^2/K \).

Let \( \tilde{z} \) be a lift of \( z \) in \( \mathcal{O}_F \). Since \( j > 1 \), we have \( v_F(\tilde{z}^i) = e(F/K)/e' \). The condition \( i > e(L_n)r/(p-1) \) is equivalent to the condition

\[
v_F(\tilde{z}^i) > \frac{e(L_n)r}{p-1} \cdot \frac{e(F/K)}{e'} = \frac{e(F)r}{p-1}.
\]

Thus the claim follows. \( \square \)

Since \( L_n \) contains \( F_n \), we can consider the ring

\[
\tilde{\mathcal{A}}'_{n,L_n,r+} = W_n(\mathcal{O}_{L_n}/b_{L_n})/\psi([\zeta_p^n] - 1)^{r} \tilde{m}_{n,L_n}
\]

and similarly \( \tilde{\mathcal{A}}'_{n,F,r+} \) for \( F \). We give these rings structures of \( \Sigma \)-algebras as follows. The ring \( \tilde{\mathcal{A}}'_{n,L_n,r+} \) is considered as a \( \Sigma \)-algebra by using the system \( \{\pi_n\}_{n \geq 0} \) which we chose of \( p \)-power roots of \( \pi \), as in the previous section. On the other hand, using \( i \) and \( i' \) in Lemma 5.8, put \( \tilde{\pi}_n = \pi_n \zeta_p^j \) and \( \tilde{\zeta}_{p^{n+1}} = \zeta_{p^{n+1}}' \). Then we consider the ring \( \tilde{\mathcal{A}}'_{n,F,r+} \) as a \( \Sigma \)-algebra by using a system of \( p \)-power roots of \( \pi \) containing \( \tilde{\pi}_n \). We define \( \text{Fil}^r \) and \( \phi_r \) of these rings in the same way as before.

**Lemma 5.10.** The induced ring homomorphism

\[
\tilde{\eta} : \tilde{\mathcal{A}}'_{n,L_n,r+} \to \tilde{\mathcal{A}}'_{n,F,r+}
\]

is a morphism of the category \( \text{Mod}_{\Sigma}^{r,\phi} \).
Proof. Firstly, we check that \( \bar{\eta} \) is \( \Sigma \)-linear. By definition, this homomorphism commutes with the action of the element \( u \in \Sigma \). To show the compatibility with the element \( Y \in \Sigma \), let us consider the commutative diagram

\[
\begin{array}{ccc}
W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n}) & \rightarrow & W_n(\mathcal{O}_F/p\mathcal{O}_F) \\
\downarrow & & \downarrow \\
W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n}) & \xrightarrow{\bar{\eta}} & W_n(\mathcal{O}_F/\mathfrak{b}_F) \\
\downarrow & & \downarrow \\
\bar{A}_{n,L_n,r} & \xrightarrow{\bar{\eta}} & \bar{A}_{n,F,r} 
\end{array}
\]

where the horizontal arrows are induced by \( \eta \). Note that we have \( \eta_p(\pi_n) = \bar{\pi}_n \) and \( \eta_p(\zeta_{p^{n+1}}) = \bar{\zeta}_{p^{n+1}} \). Put \( \beta \in W(R)^\times \) as in the proof of Proposition 4.7. Namely, the element \( \beta \) is the solution in \( W(R) \) of the equation

\[
E([\xi])\beta = pa - ([\xi^{1/p}] - 1)^{p-1},
\]

where the element \( a \in W(R) \) is as in the remark after Lemma 4.1. Let \( a_n = \bar{a}_n \) and \( \beta_n = \bar{\beta}_n \) denote the images of \( a \) and \( \beta \) in \( W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n}) \), respectively. Then the element \( \beta_n \) is a solution of the equation

\[
E([\bar{\pi}_n])\beta_n = p\bar{a}_n - ([\zeta_{p^{n+1}}] - 1)^{p-1}.
\]

Similarly, we define elements \( \tilde{a}_n \) and \( \tilde{\beta}_n \) of \( W_n(\mathcal{O}_F/p\mathcal{O}_F) \) using \( \tilde{\pi}_n \) and \( \tilde{\zeta}_{p^{n+1}} \). By definition, the element \( \beta_n \) is a solution of the equation

\[
E([\tilde{\pi}_n])\tilde{\beta}_n = p\tilde{a}_n - ([\tilde{\zeta}_{p^{n+1}}] - 1)^{p-1}.
\]

Now what we have to show is the equality

\[
\bar{\eta}(a_n\beta_n^{-1}E([\pi_n])^{p-1}) = \tilde{a}_n\tilde{\beta}_n^{-1}E([\bar{\pi}_n])^{p-1}
\]

in the ring \( \bar{A}_{n,F,r} \). Since the element \( a_n \) of \( W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n}) \) is a linear combination of the elements \( 1, [\zeta_{p^{n+1}}], \ldots, [\zeta_{p^{n+1}}]^{p-1} \) over \( \mathbb{Z} \), we have \( \bar{\eta}(a_n) = \tilde{a}_n \) in \( \bar{A}_{n,F,r} \). Thus the elements \( \tilde{\beta}_n \) and \( \bar{\eta}(\beta_n) \) satisfy the same equation in \( \bar{A}_{n,F,r} \). Since these two elements are invertible, we see that \( \bar{\eta}(\bar{\beta}_n)^{-1}E([\bar{\pi}_n]) = \tilde{\beta}_n^{-1}E([\bar{\pi}_n]) \) and the equality holds. Since the diagram above is compatible with the Frobenius endomorphisms, we see from the definition that \( \bar{\eta} \) also preserves Fil\( r \) and commutes with \( \phi_r \) of both sides. \( \square \)

Thus the homomorphism \( \bar{\eta} \) induces a homomorphism of abelian groups

\[
T_{\text{crys},L_n,\pi_n}(M_n) \rightarrow T_{\text{crys},F,\tilde{\pi}_n}(M_n).
\]

Then the following lemma, whose proof is omitted in [2, Subsection 3.13], implies that this homomorphism is an injection. We insert here a proof of this lemma for the convenience of the reader.

**Lemma 5.11.** The ring homomorphism \( \bar{\eta} : \bar{A}_{n,L_n,r} \rightarrow \bar{A}_{n,F,r} \) is an injection.
Proof. For an algebraic extension $N$ of $F_n$, let us write $\mathcal{A}'_{n}$ for the ring $\mathcal{A}'_{n,N,r}$. Note that the ring $\mathcal{A}'_{n}/p\mathcal{A}'_{n}$ is isomorphic to the ring
\[
\mathcal{O}_N/\{x \in \mathcal{O}_N \mid v_p(x) > \frac{r}{p^{n-1}(p-1)}\}.
\]
As in the proof of Lemma 5.9, we see that the homomorphism $\bar{\eta}$ induces an injection
\[
\mathcal{A}'_{L_n}/p\mathcal{A}'_{L_n} \to \mathcal{A}'_{F}/p\mathcal{A}'_{F}.
\]
Thus it is enough to show the exactness of the sequence
\[
0 \to \mathcal{A}'_{n}/p^{m}\mathcal{A}'_{n} \xrightarrow{\times p} \mathcal{A}'_{n}/p^{m+1}\mathcal{A}'_{n} \to \mathcal{A}'_{n}/p\mathcal{A}'_{n} \to 0.
\]
Let $\bar{x}$ and $\bar{y}$ be in $\mathcal{A}'_{n}$ such that $px = p^{m+1}\bar{y}$. Let $\bar{x} = (\bar{x}_0, \ldots, \bar{x}_{n-1})$ and $\bar{y} = (\bar{y}_0, \ldots, \bar{y}_{n-1})$ be lifts of $\bar{x}$ and $\bar{y}$ in the ring $W_n(\mathcal{O}_N)$, respectively. In this ring, we have
\[
(0, \bar{x}_0, \bar{x}_1', \ldots) = (0, \ldots, 0, \bar{y}_0^{p^{m+1}}, \bar{y}_1^{p^{m+1}}, \ldots) + ([\zeta_{p^n}] - 1)^r \bar{z},
\]
where $\bar{z}$ is in the ideal $m_{n,N}$. From this equality we see that $v_p(\bar{x}_0) > r/(p^{n-1}(p-1))$ and
\[
\hat{x} = ([\zeta_{p^n}] - 1)^r \hat{w} + (0, \bar{x}_1', \bar{x}_2', \ldots)
\]
for some $\hat{w} \in m_{n,N}$ and $\bar{x}_1' \in \mathcal{O}_N$. The image of the first term on the right-hand side in $\mathcal{A}'_{n}$ is zero. Hence we may assume $\bar{x}_0 = 0$. Repeating this, we can see that $\bar{x} \in p^{m}\mathcal{A}'_{n}$ and the above sequence is exact. \hfill $\square$

Now Corollary 4.13 shows that the abelian group $T^*_{\text{crys},L_n,\pi_n}(M_n)$ is of order $p^nd$, where $d = \dim_{\mathbb{Q}_p} V$. This implies that the the abelian group $T^*_{\text{crys},F,\pi_n}(M_n)$ is also of order $p^nd$. Let $g \in G_K$ be as in Lemma 5.8. Then we have the following lemma.

**Lemma 5.12.** The $G_{F_n}$-action on $T^*_{\text{crys},K,\pi_n}(M_n)$ is the conjugate of the action on $T^*_{\text{crys},K,\pi_n}(M_n)$ by the element $g$.

**Proof.** Let $a_n, \tilde{a}_n$ and $\beta_n, \tilde{\beta}_n$ be the elements of $W_n(\mathcal{O}_K/p\mathcal{O}_K)$ as in the proof of Lemma 5.10. Let us consider the composite
\[
\Sigma \to \mathcal{A}'_{n,r} \xrightarrow{g} \mathcal{A}'_{n,r}
\]
of the ring homomorphism defined by $u \mapsto [\pi_n]$ and $y \mapsto -a_n\beta_n^{-1}E([\pi_n])^{p-1}$, and the map induced by $g$. We claim that this is the natural ring homomorphism defined by $\tilde{\eta}$. For this, we only have to check that this composite sends the element $Y \in \Sigma$ to $-\tilde{a}_n\beta_n^{-1}E([\tilde{\pi}_n])$. Since the equality
\[
E([\pi_n])\beta_n = pa_n - ([\zeta_{p^{n+1}}] - 1)^{p-1}
\]
holds in the ring $\mathcal{A}'_{n,r}$ on the source of the above map $g$, we have
\[
E([\tilde{\pi}_n])g(\beta_n) = p\tilde{a}_n - ([\tilde{\zeta}_{p^{n+1}}] - 1)^{p-1}
\]
in the ring $A'_{n,r+}$ on the target. Since the elements $g(\beta_n)$ and $\tilde{\beta}_n$ are invertible, we have $g(\beta_n)^{-1}E(\tilde{n}_n) = \tilde{\beta}_n^{-1}E(\tilde{n}_n)$ and the claim follows. Thus we have an isomorphism of abelian groups

$$\text{Hom}_\Sigma(M_n, \tilde{A}'_{n,r+}) \rightarrow \text{Hom}_\Sigma(M_n, \tilde{A}'_{n,r+})$$

$$f \mapsto g \circ f,$$

where we consider on the ring $\tilde{A}'_{n,r+}$ on the right-hand side the filtered $\phi_r$-module structure over $\Sigma$ defined by $\tilde{n}_n$. Since $g(t_n) = \tilde{t}_n$, we can check that this isomorphism induces an injection

$$\text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M_n, \tilde{A}'_{n,r+}) \rightarrow \text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M_n, \tilde{A}'_{n,r+}).$$

Since these abelian groups have the same cardinality, this is also an isomorphism.

Since $L_n$ is Galois over $K$, the above lemma shows that the finite Galois extension of $F_n$ cut out by the action on $T^*_{\text{crys},K,\tilde{n}_n}(M_n)$ is also $L_n$. Hence we see from Corollary 4.13 that $F$ also contains $L_n$ and Proposition 5.7 follows. This concludes the proof of Theorem 1.1.

**Remark 5.13.** The ramification bound in Theorem 1.1 is sharp for $r \leq 1$. Indeed, the greatest upper ramification break $1 + c(n + 1/(p - 1))$ for $r = 1$ is obtained by the $p^n$-torsion of the Tate curve $\tilde{K}^\times / \pi^Z$ (see Remark 5.5). The author does not know whether this bound is sharp also for $r > 1$.

**References**


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