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ON A RAMIFICATION BOUND OF SEMI-STABLE TORSION REPRESENTATIONS OVER A LOCAL FIELD

SHIN HATTORI

Abstract. Let \( p \) be a rational prime, \( k \) be a perfect field of characteristic \( p \), \( W = W(k) \) be the ring of Witt vectors, \( K \) be a finite totally ramified extension of \( \text{Frac}(W) \) of degree \( e \) and \( r \) be a non-negative integer satisfying \( r < p - 1 \). Let \( V \) be a semi-stable \( p \)-adic \( G_K \)-representation with Hodge-Tate weights in \( \{0, \ldots, r\} \). In this paper, we prove the upper numbering ramification group \( G^j_K \) for \( j > u(K, r, n) \) acts trivially on the mod \( p^n \) representations associated to \( V \), where \( u(K, 0, n) = 0 \), \( u(K, 1, n) = 1 + e(n + 1/(p - 1)) \) and \( u(K, r, n) = 1 - p^{-n} e(K(\zeta_p)/K)^{-1} + e(n + r/(p - 1)) \) for \( r > 1 \).

1. Introduction

Let \( p \) be a rational prime, \( k \) be a perfect field of characteristic \( p \), \( W = W(k) \) be the ring of Witt vectors and \( K \) be a finite totally ramified extension of \( K_0 = \text{Frac}(W) \) of degree \( e = e(K) \). Let the maximal ideal of \( K \) be denoted by \( m_K \), an algebraic closure of \( K \) by \( \bar{K} \) and the absolute Galois group of \( K \) by \( G_K = \text{Gal}(\bar{K}/K) \). We normalize the valuation \( v_K \) of \( K \) as \( v_K(p) = e \) and extend this to \( \bar{K} \). Let \( G^{(j)}_K \) denote the \( j \)-th upper numbering ramification group in the sense of [7]. Namely, we put \( G^{(j)}_K = G_{K, j}^{L_0} \), where the latter is the upper numbering ramification group defined in [15].

Let \( X_K \) be a proper smooth scheme over \( K \) and put \( X_\bar{K} = X_K \times_K \bar{K} \). Consider the \( r \)-th étale cohomology group \( H^r_\text{ét}(X_\bar{K}, \mathbb{Q}_p) \) and its \( G_K \)-stable \( \mathbb{Z}_p \)-lattices \( \mathcal{L} \supseteq \mathcal{L}' \). In [7], Fontaine conjectured the upper numbering ramification group \( G^{(j)}_K \) acts trivially on the \( G_K \)-module \( \mathcal{L}/\mathcal{L}' \) for \( j > e(n + r/(p - 1)) \) if \( X_K \) has good reduction and this module is killed by \( p^n \). For \( e = 1 \) and \( r < p - 1 \), this conjecture was proved independently by himself ([8], for \( n = 1 \)) and Abrashkin ([2], for any \( n \)), using the theory of Fontaine-Laffaille ([10]) and the comparison theorem of Fontaine-Messing ([11]) between the \( p \)-adic étale cohomology groups of \( X_\bar{K} \) and the crystalline cohomology groups of the reduction of \( X_K \). From this result, Fontaine also showed some rareness of a proper smooth scheme over \( \mathbb{Q} \) with everywhere good reduction ([8, Théorème 1]). In fact, they proved this ramification bound for the torsion...
representations of the crystalline \( p \)-adic representations of \( G_K \) with Hodge-Tate weights in \( \{0, \ldots, r\} \) in the case where \( K \) is absolutely unramified.

On the other hand, for a semi-stable \( p \)-adic representation \( V \) with Hodge-Tate weights in the same range, a similar ramification bound for \( e = 1 \) and \( n = 1 \) is obtained by Breuil (see [4, Proposition 9.2.2.2]). He showed, assuming the Griffiths transversality which in general does not hold, that if \( e = 1 \) and \( r < p - 1 \), then the ramification group \( G_K^{(j)} \) acts trivially on the mod \( p \) representations of \( V \) for \( j > 2 + 1/(p - 1) \).

In this paper, we prove a version of the result of Breuil for the case where \( K \) is absolutely ramified, under the condition \( r < p - 1 \). Our main theorem is the following.

**Theorem 1.1.** Let \( r \) be a non-negative integer such that \( r < p - 1 \). Let \( V \) be a semi-stable \( p \)-adic \( G_K \)-representation with Hodge-Tate weights in \( \{0, \ldots, r\} \) and \( L \supseteq L' \) be \( G_K \)-stable \( \mathbb{Z}_p \)-lattices in \( V \). Suppose that the quotient \( L/L' \) is killed by \( p^n \). Then the \( j \)-th upper numbering ramification group \( G_K^{(j)} \) acts trivially on the \( G_K \)-module \( L/L' \) for \( j > u(K, r, n) \), where

\[
u(K, r, n) = \begin{cases} 0 & (r = 0), \\ 1 + e(n + \frac{1}{p-1}) & (r = 1), \\ 1 - \frac{p^n e(K(\zeta_p)/K)}{1 - \frac{1}{p-1}} + e(n + \frac{r}{p-1}) & (r > 1) \end{cases}
\]

and \( e(K(\zeta_p)/K) \) denotes the relative ramification index of the extension \( K(\zeta_p)/K \).

We can check that this bound is sharp for \( r \leq 1 \) (Remark 5.13). From this theorem and [7, Proposition 1.3], we have the following corollary.

**Corollary 1.2.** Let the notation be as in the theorem and \( L \) be the finite extension of \( K \) cut out by the \( G_K \)-module \( L/L' \). Let \( D_{L/K} \) denote the different of the extension \( L/K \). Then we have the inequality

\[
v_K(D_{L/K}) < u(K, r, n)
\]

for \( r > 0 \) and \( v_K(D_{L/K}) = 0 \) for \( r = 0 \).

For the proof of Theorem 1.1, we essentially follow a beautiful argument of Abrashkin ([2]). We may assume \( p \geq 3 \) and \( r \geq 1 \). Thanks to Liu’s theorem ([14]) on the \( G_K \)-stable \( \mathbb{Z}_p \)-lattices in semi-stable \( p \)-adic representations, it is enough to bound the ramification of the \( G_K \)-module

\[
T_{st, \mathbb{Z}}^*(\mathcal{M}_n) = \text{Hom}_{S, \text{Fil}^r, \phi, N}(\mathcal{M}_n, \hat{A}_{st, \infty}),
\]

where \( \mathcal{M}_n \) is a \( p^n \)-torsion object of a category \( \text{Mod}^{r, \phi, N}_{S, \infty} \) of filtered \( (\phi, N) \)-modules over \( S \) defined by Breuil ([3]) and \( \hat{A}_{st, \infty} \) is a \( p \)-adic period ring. We may also assume \( \zeta_p \in K \) and consider the finite Galois extension

\[
F_n = K(\pi^{1/p^n}, \zeta_{p^{n+1}})
\]

of \( K \) whose upper ramification is bounded by the same value as in the theorem. Let \( L_n \) be the finite Galois extension of \( F_n \) cut out by \( T_{st, \mathbb{Z}}^*(\mathcal{M}_n) \) over \( F_n \).
Then we bound the ramification of $L_n$ over $K$. For this, we show that to study this $G_{F_n}$-module we can use a variant over a smaller coefficient ring $\Sigma$ of filtered $(\phi_r, N)$-modules over $S$. In precise, let $E(u)$ be the Eisenstein polynomial of a uniformizer $\pi$ of $K$ over $W$ and we set

$$\Sigma = W[[u, E(u)^p/p]].$$

This ring $\Sigma$ is small enough for the method of Abrashkin, in which he uses filtered modules of Fontaine-Laffaille ([10]) whose coefficient ring is $W$, to work also in the case where $K$ is absolutely ramified.

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2. Filtered $(\phi_r, N)$-modules of Breuil

In this section, we recall the theory of filtered $(\phi_r, N)$-modules over $S$ of Breuil, which is developed by himself and most recently by Caruso and Liu (see for example [3], [5], [14], [6]). In what follows, we always take the divided power envelope of a $W$-algebra with the compatibility condition with the natural divided power structure on $pW$.

Let $p \geq 3$ be a rational prime and $\sigma$ be the Frobenius endomorphism of $W$. We fix once and for all a uniformizer $\pi$ of $K$ and a system $\left\{ \pi^n \right\}_{n \geq 0}$ of $p$-power roots of $\pi$ such that $\pi_0 = \pi$ and $\pi_n = \pi_n^p$ for any $n$. Let $E(u)$ be the Eisenstein polynomial of $\pi$ over $W$ and set $S = (W[u]^{PD})^\wedge$, where $PD$ means the divided power envelope and this is taken with respect to the ideal $(E(u))$, and $\wedge$ means the $p$-adic completion. The ring $S$ is endowed with the $\sigma$-semilinear endomorphism $\phi : u \mapsto u^p$ and a natural filtration Fil$^rS$ induced by the divided power structure such that $\phi(Fil^rS) \subseteq p^rS$ for any non-negative integer $t$. We set $\phi_t = p^{-t}\phi|_{Fil^rS}$ and $c = \phi_1(E(u)) \in S^\times$. Let $N$ denote the $W$-linear derivation on $S$ defined by the formula $N(u) = -u$. We also define a filtration, $\phi$, $\phi_t$, $N$ on $S_n = S/p^nS$ similarly.

Let $r \in \{0, \ldots, p-2\}$ be an integer. Set $\text{Mod}^{r, \phi, N}_{/S}$ to be the category consisting of the following data:

- an $S$-module $\mathcal{M}$ and its $S$-submodule $\text{Fil}^r\mathcal{M}$ containing $\text{Fil}^rS.\mathcal{M}$,
- a $\phi$-semilinear map $\phi_r : \text{Fil}^r\mathcal{M} \to \mathcal{M}$ satisfying
  $$\phi_r(s_m) = \phi_r(s)\phi(m)$$
  for any $s_r \in \text{Fil}^rS$ and $m \in \mathcal{M}$, where we set $\phi(m) = c^{-r}\phi_r(E(u)^r m)$,
- a $W$-linear map $N : \mathcal{M} \to \mathcal{M}$ such that
  - $N(sm) = N(s)m + sN(m)$ for any $s \in S$ and $m \in \mathcal{M}$,
  - $E(u)N(\text{Fil}^r\mathcal{M}) \subseteq \text{Fil}^r\mathcal{M}$,
the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Fil}^r M & \xrightarrow{\phi_r} & M \\
E(u)N \downarrow & & \downarrow cN \\
\text{Fil}^r M & \xrightarrow{\phi_r} & M,
\end{array}
\]

and the morphisms of 'Mod\(^r,\phi,N\) are defined to be the \(S\)-linear maps preserving Fil' and commuting with \(\phi_r\) and \(N\). The category defined in the same way but dropping the data \(N\) is denoted by 'Mod\(^r,\phi\). These categories have obvious notions of exact sequences. Let Mod\(^r,\phi,N\) denote the full subcategory of 'Mod\(^r,\phi,N\) consisting of \(M\) such that \(M\) is free of finite rank over \(S\) and generated as an \(S_1\)-module by the image of \(\phi_r\). We write Mod\(^r,\phi,N\) for the smallest full subcategory which contains Mod\(^r,\phi,N\) and is stable under extensions. We let Mod\(^r,\phi\) denote the full subcategory consisting of \(M\) such that

- the \(S\)-module \(M\) is free of finite rank and generated by the image of \(\phi_r\),
- the quotient \(M/\text{Fil}^r M\) is \(p\)-torsion free.

We define full subcategories Mod\(^r,\phi,N\)\(_{S_1}\), Mod\(^r,\phi,N\)\(_{S_\infty}\) and Mod\(^r,\phi\)\(_{S_1}\) of 'Mod\(^r,\phi,N\)\(_{S}\) in a similar way. For \(\tilde{\mathcal{M}} \in \text{Mod}\(^r,\phi,N\)\(_{S_1}\) (resp. \(\text{Mod}\(^r,\phi,N\)\(_{S_\infty}\)), the quotient \(\tilde{\mathcal{M}}/p^n\tilde{\mathcal{M}}\) has a natural structure as an object of \(\text{Mod}\(^r,\phi,N\)\(_{S_1}\) (resp. \(\text{Mod}\(^r,\phi\)\(_{S_1}\)).

For \(p\)-torsion objects, we also have the following categories. Consider the \(k\)-algebra \(k[u]/(u^{ep}) \cong S_1/\text{Fil}^p S_1\) and let this algebra be denoted by \(\tilde{S}_1\). The algebra \(\tilde{S}_1\) is equipped with the natural filtration, \(\phi\) and \(N\) induced by those of \(S\). Namely, Fil'\(\tilde{S}_1 = u^{ep}\tilde{S}_1\), \(\phi(u) = u^p\) and \(N(u) = -u\). Let 'Mod\(^r,\phi,N\)\(_{\tilde{S}_1}\) denote the category consisting of the following data:

- an \(\tilde{S}_1\)-module \(\tilde{\mathcal{M}}\) and its \(\tilde{S}_1\)-submodule Fil'\(\tilde{\mathcal{M}}\) containing \(u^{ep}\tilde{\mathcal{M}}\),
- a \(\phi\)-semilinear map \(\phi_r : \text{Fil}^r\tilde{\mathcal{M}} \to \tilde{\mathcal{M}}\),
- a \(k\)-linear map \(N : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}\) such that
  - \(N(sm) = N(s)m + sN(m)\) for any \(s \in \tilde{S}_1\) and \(m \in \tilde{\mathcal{M}}\),
  - \(u^{ep}N(\text{Fil}^r\tilde{\mathcal{M}}) \subseteq \text{Fil}^r\tilde{\mathcal{M}}\),
- the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Fil}^r\tilde{\mathcal{M}} & \xrightarrow{\phi_r} & \tilde{\mathcal{M}} \\
\bigcup u^s N \downarrow & & \downarrow cN \\
\text{Fil}^r\tilde{\mathcal{M}} & \xrightarrow{\phi_r} & \tilde{\mathcal{M}}.
\end{array}
\]
and whose morphisms are defined as before. Its full subcategory $\text{Mod}_S^{f,\phi,N}$ is defined by the following condition:

- As an $S_1$-module, $\mathcal{M}$ is free of finite rank and generated by the image of $\phi_r$.

We define categories $\text{Mod}_S^{f,\phi}$ and $\text{Mod}_S^{f,\phi,N}$ similarly.

Let $D$ be a weakly admissible filtered $(\phi,N)$-module over $K$ satisfying $\text{Fil}^0 D_K = D_K$ and $\text{Fil}^{r+1} D_K = 0$. Set $S_{K_0} = S \otimes W K_0$ and $D = D \otimes_{K_0} S_{K_0}$. Then the $S_{K_0}$-module $D$ is equipped with the natural $\phi$-semilinear map $\phi \otimes \sigma$ and $K_0$-linear derivation $N \otimes 1 + 1 \otimes N$, which are denoted by $\phi$ and $N$, respectively. We define a filtration on $D$ inductively by $\text{Fil}^0 D = D$ and

$$\text{Fil}^{i+1} D = \{ x \in D \mid N(x) \in \text{Fil}^i D \text{ and } f_\pi(x) \in \text{Fil}^{i+1} D_K \},$$

where $f_\pi : D \to D_K$ is induced by the map $S \to O_K$ sending $u$ to $\pi$. An $S$-submodule $\mathcal{M}$ of $D$ is said to be a strongly divisible lattice of $D$ if the following conditions are satisfied:

- the $S$-module $\mathcal{M}$ is free of finite rank,
- $\mathcal{M} \otimes W K_0 = D$,
- $\mathcal{M}$ is stable under $\phi$ and $N$,
- $\phi(\text{Fil}^r \mathcal{M}) \subseteq p^r \mathcal{M}$, where we set $\text{Fil}^r \mathcal{M} = \mathcal{M} \cap \text{Fil}^r D$.

We put $\phi_r = p^{-r} \phi |_{\text{Fil}^r \mathcal{M}}$. Then the $S$-module $\mathcal{M}$ is generated by $\phi_r(\text{Fil}^r \mathcal{M})$ ([3, Proposition 2.1.3]) and we can consider $\mathcal{M}$ as an object of $\text{Mod}_S^{f,\phi,N}$.

Let $A_{\text{crys}}$ and $\hat{A}_{\text{st}}$ be $p$-adic period rings. These are constructed as follows. Set $R$ to be the ring

$$R = \varprojlim (O_K/p O_K \leftarrow O_K/p O_K \leftarrow \cdots),$$

where every arrow is the $p$-power map. For an element $x = (x_i)_{i \in \mathbb{Z}_{\geq 0}} \in R$ and an integer $n \geq 0$, we set

$$x^{(n)} = \varprojlim_{m \to \infty} \hat{x}^{p^n} x^{n+m} \in O_C,$$

where $\hat{x}_i$ is a lift of $x_i$ in $O_K$ and $O_C$ is the $p$-adic completion of $O_K$. Let $v_p$ denote the valuation of $O_C$ normalized as $v_p(p) = 1$. Then the ring $R$ is a complete valuation ring whose valuation of an element $x \in R$ is given by $v_R(x) = v_p(x^{(0)})$. We define a natural ring homomorphism $\theta$ by

$$\theta : W(R) \to O_C$$

$$(x_0, x_1, \ldots) \mapsto \sum_{n \geq 0} p^n x^{(n)}.$$

Then $A_{\text{crys}}$ is the $p$-adic completion of the divided power envelope of $W(R)$ with respect to the principal ideal $\ker(\theta)$ and $\hat{A}_{\text{st}}$ is the $p$-adic completion of the divided power polynomial ring $A_{\text{crys}}(X)$ over $A_{\text{crys}}$. We set $A_{\text{crys}, \infty} = A_{\text{crys}} \otimes_W K_0/W$ and $\hat{A}_{\text{st}, \infty} = \hat{A}_{\text{st}} \otimes_W K_0/W$. Put $\pi = (\pi_n)_{n \in \mathbb{Z}_{\geq 0}} \in R$, where we abusively let $\pi_n$ denote the image of $\pi_n \in O_K$ in $O_K/p O_K$. These rings
are considered as $S$-algebras by the ring homomorphisms $S \to \hat{A}_{\text{st}}$ and $A_{\text{crys}} \to A_{\text{crys}}$ which are defined by $u \mapsto [\pi]/(1 + X)$ and $X \mapsto 0$, respectively. The ring $A_{\text{crys}}$ is endowed with a natural filtration induced by the divided power structure, a natural Frobenius endomorphism $\phi$ and the $\phi$-semilinear map $\phi_t = p^{-t}\phi|_{\text{Fil}_t A_{\text{crys}}}$. With these structures, $A_{\text{crys}}$ and $A_{\text{crys}, \infty}$ are considered as objects of $\text{Mod}^r_{/S}$. Moreover, the absolute Galois group $G_K$ acts naturally on these two rings. As for $\hat{A}_{\text{st}}$, its filtration is defined by $\text{Fil}_t \hat{A}_{\text{st}} = \left\{ \sum_{i \geq 0} a_i \frac{X^i}{i!} \left| a_i \in \text{Fil}^{t-i} A_{\text{crys}}, \lim_{i \to \infty} a_i = 0 \right. \right\}$ and the Frobenius structure of $A_{\text{crys}}$ extends to $\hat{A}_{\text{st}}$ by $\phi(X) = (1 + X)^p - 1,$ $\phi_t = p^{-t}\phi|_{\text{Fil}_t \hat{A}_{\text{st}}}.$

We write $N$ also for the $A_{\text{crys}}$-linear derivation on $\hat{A}_{\text{st}}$ defined by $N(X) = 1 + X$. The rings $\hat{A}_{\text{st}}$ and $\hat{A}_{\text{st}, \infty}$ are objects of $\text{Mod}^r_{/S}$. The $G_K$-action on $A_{\text{crys}}$ naturally extends to an action on $\hat{A}_{\text{st}}$. Indeed, the action of $g \in G_K$ on $\hat{A}_{\text{st}}$ is defined by the formula $g(X) = \varepsilon(g)(1 + X) - 1,$ where $g(\pi_n) = \varepsilon_n(g)\pi_n$ and $\varepsilon(g) = (\varepsilon_n(g))_{n \geq 0} \in R$ with the abusive notation as above.

These rings have other descriptions, as follows. For an integer $n \geq 1$, put $W_n = W/p^nW$ and let $W_n(O_{\bar{K}}/pO_{\bar{K}})$ be the ring of Witt vectors of length $n$ associated to $O_{\bar{K}}/pO_{\bar{K}}$. We define a $W_n$-algebra structure on $W_n(O_{\bar{K}}/pO_{\bar{K}})$ by twisting the natural $W_n$-algebra structure by $\sigma^{-n}$. Then the natural ring homomorphism $\theta_n : W_n(O_{\bar{K}}/pO_{\bar{K}}) \to O_{\bar{K}}/p^nO_{\bar{K}}$ $(a_0, \ldots, a_{n-1}) \mapsto \sum_{i=0}^{n-1} p^i a_i p^{-n-i},$ where $\hat{a}_i$ is a lift of $a_i$ in $O_{\bar{K}}$, is $W_n$-linear. Let us denote $W_n^{\text{PD}}(O_{\bar{K}}/pO_{\bar{K}})$ the divided power envelope of $W_n(O_{\bar{K}}/pO_{\bar{K}})$ with respect to the ideal $\text{Ker}(\theta_n)$. This ring is considered as an $S$-algebra by $u \mapsto [\pi_n]$. This ring also has a natural filtration defined by the divided power structure, and a natural $G_K$-module structure. The Frobenius endomorphism of the ring of Witt vectors
induces on this ring a $\phi$-semilinear Frobenius endomorphism, which is denoted also by $\phi$. Then, by the $S$-linear transition maps
\[ W_{n+1}^{\text{PD}}(O_K/pO_K) \to W_n^{\text{PD}}(O_K/pO_K) \]
\[ (a_0, \ldots, a_n) \mapsto (a_0^n, \ldots, a_n^n), \]
these $S$-algebras form a projective system compatible with all structures. Using this transition map, a $\phi$-semilinear map
\[ \phi_r : \text{Fil}^r W_n^{\text{PD}}(O_K/pO_K) \to W_n^{\text{PD}}(O_K/pO_K), \]
which is \(\phi\)-linear. Similarly, the divided power polynomial ring
\[ W_n^{\text{PD}}(O_K/pO_K)(X) \]
over \(W_n^{\text{PD}}(O_K/pO_K)\) is considered as an \(S\)-algebra by \(u \mapsto [\pi_n]/(1 + X)\). This ring has a natural filtration coming from the divided power structure. We define a $G_K$-action on this ring by
\[ g(X) = [\varepsilon_n(g)](1 + X) - 1. \]
We also define a $\phi$-semilinear Frobenius endomorphism, which we also write as $\phi$, by $\phi(X) = (1 + X)^r - 1$ and a $W_n^{\text{PD}}(O_K/pO_K)$-linear derivation $N$ by $N(X) = 1 + X$. These rings form a projective system of $S$-algebras compatible with all structures by the transition maps defined by the maps above and $X \mapsto X$. We define $\phi$-semilinear maps
\[ \phi_r : \text{Fil}^r W_n^{\text{PD}}(O_K/pO_K)(X) \to W_n^{\text{PD}}(O_K/pO_K)(X) \]
compatible with the transition maps as before. The $S$-algebra $W_n^{\text{PD}}(O_K/pO_K)(X)$ is considered as an object of $'\text{Mod}^{r,\phi}_S$ and there exists a natural isomorphism in $'\text{Mod}^{r,\phi, N}_S$
\[ \hat{A}_{\text{st}}/p^n \hat{A}_{\text{st}} \to W_n^{\text{PD}}(O_K/pO_K)(X) \]
\[ (x_0, \ldots, x_{n-1}) \mapsto (x_0, \ldots, x_{n-1}, 0) \]
\[ X \mapsto X \]
which is $G_K$-linear.

Put $K_n = K(\pi_n)$ and $K_{\infty} = \bigcup_n K_n$. For $\mathcal{M} \in \text{Mod}^{r,\phi, N}_S$, we define a $G_K$-module $T^{*}_{\text{st, } \mathcal{M}}(\mathcal{M})$ to be
\[ T^{*}_{\text{st, } \mathcal{M}}(\mathcal{M}) = \text{Hom}_{S, \text{Fil}^r, \phi, N}(\mathcal{M}, \hat{A}_{\text{st, } \mathcal{M}}). \]
When $\mathcal{M}$ is killed by $p^n$, we have a natural identification of $G_K$-modules

$$T^*_{st,\mathbb{Z}}(\mathcal{M}) = \text{Hom}_{S,\text{Fil}^r,\phi,\mathcal{M}}(\mathcal{M}, W_n^{PD}(\mathcal{O}_K/p\mathcal{O}_K)(X)).$$

Note that the $G_K$-module on the right-hand side is independent of the choice of $\pi_k$ for $k > n$. Since the natural map

$$W_n^{PD}(\mathcal{O}_K/p\mathcal{O}_K)(X) \to W_n^{PD}(\mathcal{O}_K/p\mathcal{O}_K)$$

is by definition $G_{K_n}$-linear, we also have a $G_{K_n}$-linear isomorphism ([3, Lemme 2.3.1.1])

$$T^*_{st,\mathbb{Z}}(\mathcal{M})|_{G_{K_n}} \to \text{Hom}_{S,\text{Fil}^r,\phi,\mathcal{M}}(\mathcal{M}, W_n^{PD}(\mathcal{O}_K/p\mathcal{O}_K)).$$

A variant of filtered $(\phi, N)$-modules over $S$ is also introduced by Breuil and Kisin, and developed also by Caruso and Liu (see for example [12], [13], [14], [6]). Put $\mathfrak{S} = W[[u]]$ and let $\phi : \mathfrak{S} \to \mathfrak{S}$ be the $\sigma$-semilinear Frobenius endomorphism defined by $\phi(u) = u^p$. Let $\text{Mod}_{/\mathfrak{S}}^{r,\phi}$ denote the category consisting of the following data:

- an $\mathfrak{S}$-module $\mathfrak{M}$,
- a $\phi$-semilinear map $\mathfrak{M} \to \mathfrak{M}$, which is denoted also by $\phi$, such that the cokernel of the map $1 \otimes \phi : \phi^*\mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$,

and whose morphisms are defined as before. The full subcategory of $\text{Mod}_{/\mathfrak{S}}^{r,\phi}$ consisting of $\mathfrak{M}$ such that $\mathfrak{M}$ is free of finite rank over $\mathfrak{S}/p\mathfrak{S}$ (resp. over $\mathfrak{S}$) is denoted by $\text{Mod}_{/\mathfrak{S}_1}^{r,\phi}$ (resp. $\text{Mod}_{/\mathfrak{S}}^{r,\phi}$). We let $\text{Mod}_{/\mathfrak{S}_\infty}^{r,\phi}$ denote the smallest full subcategory which contains $\text{Mod}_{/\mathfrak{S}_1}^{r,\phi}$ and is stable under extensions, as before. Then we have an exact functor ([6, Proposition 2.1.2], see also [12, Proposition 1.1.11])

$$\mathcal{M}_{\mathfrak{S}_\infty} : \text{Mod}_{/\mathfrak{S}_\infty}^{r,\phi} \to \text{Mod}_{/\mathfrak{S}_\infty}^{r,\phi}.$$

For $\mathfrak{M} \in \text{Mod}_{/\mathfrak{S}_\infty}^{r,\phi}$, the filtered $\phi$-module $\mathcal{M} = \mathcal{M}_{\mathfrak{S}_\infty}(\mathfrak{M})$ over $S$ is defined as follows:

- $\mathcal{M} = S \otimes_{\phi, \mathfrak{S}} \mathfrak{M}$,
- $\text{Fil}^r\mathcal{M} = \text{Ker}(\mathcal{M} \xrightarrow{1 \otimes \phi} S \otimes_{\mathfrak{S}} \mathfrak{M} \to (S/\text{Fil}^r S) \otimes_{\mathfrak{S}} \mathfrak{M})$,
- $\phi_r : \text{Fil}^r\mathcal{M} \xrightarrow{1 \otimes \phi} \phi_r^*\text{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\phi_r \otimes 1} S \otimes_{\phi, \mathfrak{S}} \mathfrak{M} = \mathcal{M}$.

We write $\mathcal{M}_{\mathfrak{S}}$ for the functor $\text{Mod}_{/\mathfrak{S}}^{r,\phi} \to \text{Mod}_{/\mathfrak{S}}^{r,\phi}$ defined similarly.

Finally, let $D$ and $\mathfrak{D}$ be as above and $\hat{\mathcal{M}}$ be a strongly divisible lattice in $\mathfrak{D}$. The $S$-module $\mathcal{M}_n = \hat{\mathcal{M}}/p^n\hat{\mathcal{M}}$ has a natural structure as an object of $\text{Mod}_{/\mathfrak{S}_\infty}^{r,\phi,N}$. We set a $G_K$-module $\hat{T}^*_{st,\mathbb{Z}}(\hat{\mathcal{M}})$ to be

$$\hat{T}^*_{st,\mathbb{Z}}(\hat{\mathcal{M}}) = \text{Hom}_{S,\text{Fil}^r,\phi,\mathcal{M}}(\mathcal{M}, \hat{\mathcal{A}}_{\text{st}}).$$

Then we have an exact sequence of $G_K$-modules

$$0 \to \hat{T}^*_{st,\mathbb{Z}}(\hat{\mathcal{M}}) \xrightarrow{p^n} \hat{T}^*_{st,\mathbb{Z}}(\hat{\mathcal{M}}) \to T^*_{st,\mathbb{Z}}(\mathcal{M}_n) \to 0.$$
The $G_K$-module $\hat{T}_{st,\Sigma}^\ast(M)$ is naturally considered as a $G_K$-stable $\mathbb{Z}_p$-lattice in $V_{st}^\ast(D)$. By Liu’s theorem ([14, Theorem 2.3.5]), the functor $\hat{T}_{st,\Sigma}^\ast$ gives an anti-equivalence of categories between the category of strongly divisible lattices in $D$ and the category of $G_K$-stable $\mathbb{Z}_p$-lattices in $V_{st}^\ast(D)$. Moreover, for such a lattice $\mathcal{L}$, its corresponding strongly divisible lattice $\hat{\mathcal{L}}$ in $D$ is in the essential image of the functor $\mathcal{M}_{\Sigma}$ ([14, Subsection 3.5]).

3. Filtered $\phi_r$-modules over $\Sigma$

In this section, we define another variant of filtered $\phi_r$-modules over $S$ and prove its properties.

Let $p \geq 3$ be a rational prime and $r$ be an integer such that $0 \leq r < p - 1$. Consider the $W$-algebra $\Sigma = W[[u,Y]]/(E(u)^p - pY)$ as in [3, Subsection 3.2]. We regard $\Sigma$ as a subring of $S$ by the map sending $Y$ to $E(u)^p/p$. Then the element $e = \phi_1(E(u)) \in S^\times$ is contained in $\Sigma^\times$. We define on $\Sigma$ a $\sigma$-semilinear Frobenius endomorphism $\phi$ by $\phi(u) = u^p$ and $\phi(Y) = p^{p-1}e^p$. Put $\text{Fil}^t\Sigma = (E(u)^t,Y)$ for $0 \leq t \leq p - 1$ and $\text{Fil}^p\Sigma = (Y)$. Then we have $\phi(\text{Fil}^t\Sigma) \subseteq p^t\Sigma$ for $0 \leq t \leq p - 1$. We put $\phi_t = p^{-t}\phi|_{\text{Fil}^t\Sigma}$. We also set $\Sigma_n = \Sigma/p^n\Sigma$ and put on this ring the natural structures induced by those of $\Sigma$.

We define a category $\text{Mod}^r_{/\Sigma}^\phi$ of filtered $\phi_r$-modules over $\Sigma$ to be the category consisting of the following data:

- a $\Sigma$-module $M$ and its $\Sigma$-submodule $\text{Fil}^r\Sigma.M$,
- a $\phi$-semilinear map $\text{Fil}^r\Sigma.M \to M$ satisfying $\phi_r(s,m) = \phi_r(s_r)\phi(m)$ for any $s_r \in \text{Fil}^r\Sigma$ and $m \in M$, where we set $\phi(m) = c^{-r}\phi_r(E(u)^r m)$.

and the morphisms are defined in the same manner as $\text{Mod}^r_{/\Sigma}^\phi$. This category has a natural notion of exact sequences. We define its full subcategory $\text{Mod}^r_{/\Sigma}^\phi$ to be the category consisting of $M$ which is free of finite rank and generated by the image of $\phi_r$ as a $\Sigma_1$-module. We also let $\text{Mod}^r_{/\Sigma}^\phi_{\Sigma_\infty}$ denote the smallest full subcategory of $\text{Mod}^r_{/\Sigma}^\phi$ which contains $\text{Mod}^r_{/\Sigma_1}$ and is stable under extensions. Moreover, we define a full subcategory $\text{Mod}^r_{/\Sigma}^\phi$ of $\text{Mod}^r_{/\Sigma}^\phi$ to be the category consisting of $M$ such that

- the $\Sigma$-module $M$ is free of finite rank and generated by the image of $\phi_r$,
- the quotient $M/\text{Fil}^r\Sigma.M$ is $p$-torsion free.

Then we see that for $\hat{M} \in \text{Mod}^r_{/\Sigma}^\phi$, the quotient $\hat{M}/p^n\hat{M}$ is naturally considered as an object of $\text{Mod}^r_{/\Sigma}^\phi_{\Sigma_\infty}$.

The natural ring isomorphism $\Sigma_1/\text{Fil}^p\Sigma_1 \cong \tilde{S}_1$ defines a functor $T_0 : \text{Mod}^r_{/\Sigma_1}^\phi \to \text{Mod}^r_{/\Sigma_1}^\phi$ by $M \mapsto M/\text{Fil}^p\Sigma_1.M$. Then [3, Proposition 2.2.1.3] and Nakayama’s lemma shows the following.
Lemma 3.1. Let $M$ be an object of $\text{Mod}_{/\Sigma_1}^{r,\phi}$ of rank $d$ over $\Sigma_1$. Then there exists a basis $\{e_1, \ldots, e_d\}$ of $M$ such that $\text{Fil}^r M = \Sigma_1 u^r e_1 + \cdots + \Sigma_1 u^r e_d + \text{Fil}^p \Sigma_1 M$ for some integers $r_1, \ldots, r_d$ with $0 \leq r_i \leq er$ for any $i$.

Then we can show the following lemma just as in the proof of [3, Lemme 2.3.1.3].

Lemma 3.2. The functor $\phi_r \colon M \mapsto \text{Hom}_{\Sigma_1}^{\Sigma_1} (M, \text{A}_{\text{cr}, \infty})$ from $\text{Mod}_{/\Sigma_1}^{r,\phi}$ to the category of $G_{K_\infty}$-modules is exact.

For $M \in \text{Mod}_{/\Sigma_1}^{r,\phi}$, we can show as in the case of the category $\text{Mod}_{/\Sigma_1}^{r,\phi}$ that there is an isomorphism of $G_{K_1}$-modules

$\text{Hom}_{\Sigma_1}^{\Sigma_1} (M, (\text{O}_K/p\text{O}_K)^{\text{PD}}) \rightarrow \text{Hom}_{\Sigma_1}^{\Sigma_1} (T_0 (M), \text{O}_K/p\text{O}_K),$

where $\text{O}_K/p\text{O}_K$ is considered as an object of $\text{Mod}_{/\Sigma_1}^{r,\phi}$ by the natural isomorphism

$(\text{O}_K/p\text{O}_K)^{\text{PD}}/\text{Fil}^p (\text{O}_K/p\text{O}_K) \rightarrow \text{O}_K/p\text{O}_K.$

Thus [3, Lemme 2.3.1.2] implies that, for such a $\Sigma_1$-module $M$, we have

$\# \text{Hom}_{\Sigma_1}^{\Sigma_1} (M, (\text{O}_K/p\text{O}_K)^{\text{PD}}) = p^d,$

where $d = \dim_{\Sigma_1} M$.

For the category $\text{Mod}_{/\Sigma_1}^{r,\phi}$, we can show the following lemma just as in the proof of [14, Proposition 4.1.2].

Lemma 3.3. Let $\hat{M}$ be in $\text{Mod}_{/\Sigma_1}^{r,\phi}$. Then there exists $\alpha_1, \ldots, \alpha_d \in \hat{M}$ such that $\text{Fil}^r \hat{M} = \Sigma_1 \alpha_1 + \cdots + \Sigma_1 \alpha_d + \text{Fil}^p \Sigma_1 \hat{M}$, $E(u)^r \hat{M} \subseteq \Sigma_1 \alpha_1 + \cdots + \Sigma_1 \alpha_d$ and the elements $e_1 = \phi_r (\alpha_1), \ldots, e_d = \phi_r (\alpha_d)$ form a basis of $\hat{M}$.

Corollary 3.4. Let $\hat{M}$ be in $\text{Mod}_{/\Sigma_1}^{r,\phi}$ and $A$ be a $\Sigma$-algebra which has a structure as an object of $\text{Mod}_{/\Sigma_1}^{r,\phi}$. Let $C \in M_d (\Sigma)$ be the matrix such that

$(\alpha_1, \ldots, \alpha_d) = (e_1, \ldots, e_d) C$

with the notation of the previous lemma. Then a $\Sigma$-linear homomorphism $f : \hat{M} \rightarrow A$ preserving $\text{Fil}^r$ also commutes with $\phi_r$ if and only if

$\phi_r (f (e_1, \ldots, e_d) C) = (f (e_1), \ldots, f (e_d)).$

Proof. Suppose that the latter condition holds. By assumption, we have

$E(u)^r (e_1, \ldots, e_d) = (\alpha_1, \ldots, \alpha_d) C'$
for some \( C' \in M_d(\Sigma) \). We claim that \( f \) commutes with \( \phi \). Indeed, we have
\[
\phi(f(e_1, \ldots, e_d)) = e^{-r} \phi_r(E(u)^r(f(e_1), \ldots, f(e_d))) \\
= e^{-r} \phi_r((f(\alpha_1), \ldots, f(\alpha_d))C') \\
= e^{-r} f(e_1, \ldots, e_d) \phi(C') \\
= e^{-r} f(\phi_r(\alpha_1, \ldots, \alpha_d)) \phi(C') \\
= e^{-r} f(\phi_r(E(u)^r(e_1, \ldots, e_d))) = f(\phi(e_1, \ldots, e_d)).
\]
This implies \( \phi_r \circ f = f \circ \phi_r \) also on \( \Fil^p \Sigma \hat{M} \).

**Corollary 3.5.** Let \( \hat{M} \) and \( A \) be as above and \( J \subseteq \Fil^r A \) be an ideal of \( A \) such that \( \phi_r(J) \subseteq J \). We can consider the \( \Sigma \)-algebra \( A/J \) naturally as an object of \( \Mod^{r, \phi}_{/\Sigma} \). Suppose that for any \( x \in J \), there exists \( t \in \mathbb{Z}_{\geq 0} \) such that \( \phi^t_r(x) = 0 \). Then we have an isomorphism
\[
\Hom_{\Sigma, \Fil^r, \phi}(\hat{M}, A) \to \Hom_{\Sigma, \Fil^r, \phi} (\hat{M}, A/J).
\]

**Proof.** Let \( f : \hat{M} \to A/J \) be an element of the abelian group on the right-hand side and \( \hat{x} \) be an lift of \( f(e_i) \) in \( A \). By the previous corollary, it is enough to show that for any \( (\hat{c}_1, \ldots, \hat{c}_d) \in J^d \), there is a unique solution \((\hat{y}_1, \ldots, \hat{y}_d) \in J^d \) of the equation
\[(\hat{c}_1, \ldots, \hat{c}_d) + (\phi_r(\hat{y}_1), \ldots, \phi_r(\hat{y}_d))\phi(C) = (\hat{y}_1, \ldots, \hat{y}_d).
\]
By assumption, the \( d \)-tuple
\[
\sum_{i=0}^{t} (\phi^i_r(\hat{c}_1), \ldots, \phi^i_r(\hat{c}_d)) \phi^i(C) \phi^{i-1}(C) \cdots \phi(C)
\]
is stable for sufficiently large \( t \) and we see that this limit gives a unique solution of the equation. \( \square \)

For an \( \mathcal{E} \)-module \( \mathfrak{M} \) in \( \Mod_{/\mathcal{E}}^{r, \phi} \) (resp. \( \Mod_{/\mathcal{E}}^{r, \phi} \)), we associate to it a \( \Sigma \)-module \( M \in \Mod^{r, \phi}_{/\Sigma} \) as follows:
- \( M = \Sigma \otimes_{\phi, \mathcal{E}} \mathfrak{M} \),
- \( \Fil^r M = \text{Ker}(M \xrightarrow{1 \otimes \phi} \Sigma \otimes_{\mathcal{E}} \mathfrak{M} \to (\Sigma/\Fil^r \Sigma) \otimes_{\mathcal{E}} \mathfrak{M}) \),
- \( \phi_r : \Fil^r M \xrightarrow{1 \otimes \phi} \Fil^r \Sigma \otimes_{\mathcal{E}} \mathfrak{M} \xrightarrow{\phi_{r, 1}} \Sigma \otimes_{\phi, \mathcal{E}} \mathfrak{M} = M \).

We can check that this defines an exact functor \( \Mod_{/\mathcal{E}}^{r, \phi} \to \Mod_{/\Sigma}^{r, \phi} \) (resp. \( \Mod_{/\mathcal{E}}^{r, \phi} \to \Mod_{/\Sigma}^{r, \phi} \)) as in the proof of [12, Proposition 1.1.11]. We let this functor be denoted by \( M_{\mathcal{E}} \) (resp. \( M_{\mathcal{E}} \)).

**Proposition 3.6.** Let \( \mathfrak{M} \) be an object of \( \Mod_{/\mathcal{E}}^{r, \phi} \) which is killed by \( p^n \). Set \( M = M_{\mathcal{E}}(\mathfrak{M}) \) and \( \mathcal{M} = M_{\mathcal{E}}(\mathfrak{M}) \). Then there exists a natural isomorphism of \( G_{K_n} \)-modules
\[
\Hom_{\Sigma, \Fil^r, \phi_r}(M, W_n^{PD}(\mathcal{O}_K/p\mathcal{O}_K)) \to \Hom_{\Sigma, \Fil^r, \phi_r}(\mathcal{M}, W_n^{PD}(\mathcal{O}_K/p\mathcal{O}_K)).
\]
Proof. By definition, \( M = S \otimes_{\Sigma} M \) and we have a natural isomorphism
\[
\text{Hom}_{\Sigma}(M, W_{n}^{\text{PD}}(O_{K}/pO_{K})) \to \text{Hom}_{\Sigma}(M, W_{n}^{\text{PD}}(O_{K}/pO_{K})).
\]
Let \( f \) be an element of \( \text{Hom}_{\Sigma}^{\text{Fil}^{r}, \phi_{r}}(M, W_{n}^{\text{PD}}(O_{K}/pO_{K})) \) and \( f' \) be the image of \( f \) in the right-hand side of the above isomorphism. Let us check that \( f' \) preserves \( \text{Fil}^{r} \) and commutes with \( \phi_{r} \). Since \( f' \) is \( S \)-linear, it maps \( \text{Fil}^{r}S.M \) into \( \text{Fil}^{r}W_{n}^{\text{PD}}(O_{K}/pO_{K}) \). For \( x \in \text{Fil}^{r}M \cap \text{Im}(M \to M) \), the commutative diagram whose right vertical arrow is an isomorphism
\[
M = \Sigma \otimes_{\phi_{r}, \Sigma} M \xrightarrow{1 \otimes \phi} \Sigma \otimes_{\phi_{r}, \Sigma} M \xrightarrow{f} \Sigma/\text{Fil}^{r}\Sigma \otimes_{\phi_{r}, \Sigma} M
\]
implies \( x \in \text{Im}(\text{Fil}^{r}M \to \text{Fil}^{r}M) \) and thus \( f'(x) \in \text{Fil}^{r}W_{n}^{\text{PD}}(O_{K}/pO_{K}) \).

As for the compatibility with \( \phi_{r} \), again by the \( S \)-linearity of \( f' \) it suffices to show \( f'(\phi_{r}(x)) = \phi_{r}(f'(x)) \) for \( x \in \text{Fil}^{r}M \cap \text{Im}(M \to M) = \text{Im}(\text{Fil}^{r}M \to \text{Fil}^{r}M) \). This follows from the commutative diagram
\[
\begin{array}{ccc}
\text{Fil}^{r}M & \xrightarrow{\phi_{r}} & M \\
\downarrow & & \downarrow \phi_{r} \\
\text{Fil}^{r}M & \xrightarrow{f} & W_{n}^{\text{PD}}(O_{K}/pO_{K})
\end{array}
\]

Hence the map in the proposition is well-defined and injective. To prove the bijectivity, by devissage we may assume that \( pM = 0 \). Then both sides of this injection have the same cardinality by the above remark. Thus the proposition follows.

\[\square\]

4. A METHOD OF ABRASHKIN

In this section, we study the \( G_{K_{r}} \)-module \( \text{Hom}_{\Sigma}^{\text{Fil}^{r}, \phi_{r}}(M, W_{n}^{\text{PD}}(O_{K}/pO_{K})) \)
following Abrashkin ([2]).

Let \( p \geq 3 \) and \( 0 \leq r < p - 1 \) be as before. Consider the Lubin-Tate logarithm
\[
l(X) = X + \frac{X^{p}}{p} + \cdots + \frac{X^{p^{n}}}{p^{n}} + \cdots
\]
and put \( \psi(X) = l^{-1}(\log(1 + X)) \). Then \( \psi \) gives a strict isomorphism of formal groups between the formal group associated to the logarithm \( l(X) \) and the multiplicative group \( \mathbb{G}_{m} \) over \( \mathbb{Z}_{p} \). We fix a system of \( p \)-power roots of unity \( \{\zeta_{p^{n}}\}_{n \in \mathbb{Z}_{\geq 0}} \) such that \( \zeta_{p} \neq 1 \) and \( \zeta_{p^{n}} = \zeta_{p^{n+1}} \) for any \( n \), and set an element \( \xi \) of \( \mathbb{R} \) to be \( (\zeta_{p^{n}})_{n \in \mathbb{Z}_{\geq 0}} \). Then the elements \( [\xi] - 1 \) and \( [\xi^{1/p}] - 1 \) are topologically nilpotent in \( W(R) \) and the element of \( W(R) \)
\[
t = \psi([\xi] - 1)/\psi([\xi^{1/p}] - 1)
\]
is a generator of the principal ideal Ker(θ). The element \( Z = \psi([x] - 1)^{p-1}/p \) of \( A_{\text{crys}} \) is topologically nilpotent and \( \phi(t) \) is contained in the subset

\[
p(1 + ZW(R)[[Z]])
\]

of \( A_{\text{crys}} \) ([2, Subsection 1.8]). We set

\[
\hat{A} = W(R)[[Z]] \subseteq A_{\text{crys}}.
\]

**Lemma 4.1.** The element \( tp/p \) of \( A_{\text{crys}} \) is contained in the subring \( \hat{A} \) and topologically nilpotent in this subring.

**Proof.** Put \( t' = ([x] - 1)/([x]^{1/p} - 1) \). This is another generator of Ker(θ). We have

\[
\frac{([x] - 1)^{p-1}}{p} = (t')^{p-1} \cdot ([x]^{1/p} - 1)^{p-1}
\]

and \( \theta([x]^{1/p} - 1) = \zeta_p - 1 \). Take an element \( a \in W(R)^\times \) such that \( \theta(a) = (\zeta_p - 1)^{p-1}/p \). Then we have

\[
\frac{([x] - 1)^{p-1}}{p} = a(t')^{p-1} + b(t')^p/p
\]

for some \( b \in W(R)^\times \). Indeed, to show \( b \in W(R)^\times \), it suffices to check that the element \( ([x]^{1/p} - 1)^{p-1} - pa \) of Ker(θ) also generates this ideal. This follows from the fact that the 0-th entry \( ([x]^{1/p} - 1)^{p-1} \) of this element satisfies \( \psi_R([x]^{1/p} - 1) = 1 \). Then we see that \( (t')^p/p \) is topologically nilpotent because so is \( t' \) in \( W(R) \). \( \square \)

In the following, we set the element \( a \) in the proof of the lemma to be

\[
a = \sum_{k=1}^{p-2} p^{-1}((-1)^{p-1-k}p_{-1}C_k - 1)[x]^{1/p^k},
\]

where \( p_{-1}C_k = (p-1)!/(k!(p-1-k)!) \) is the binomial coefficient. Note that the coefficient of \( [x]^{1/p^k} \) in each term is an integer.

From this lemma, we can consider the ring \( \hat{A} \) as a \( \Sigma \)-algebra by \( u \mapsto [x] \). Put Fil\(^i\hat{A} = (t^i, Z) \) for \( 0 \leq i \leq p - 1 \). The Frobenius endomorphism \( \phi \) of \( A_{\text{crys}} \) preserves \( \hat{A} \) and satisfies \( \phi(\text{Fil}^i\hat{A}) \subseteq p^i\hat{A} \) for \( 0 \leq i \leq p - 1 \). Set \( \phi_r = p^{-r}\phi|_{\text{Fil}^r\hat{A}} \). Then we can consider the ring \( \hat{A} \) also as an object of the category \( \text{Mod}_{[\Sigma]}^{\phi,r} \), and similarly for \( A_{n} = \hat{A}/p^n\hat{A} \) and \( A_{\infty} = \hat{A} \otimes W K_0/W \).

The absolute Galois group \( G_{K_0} \) acts naturally on these \( \Sigma \)-algebras. The following lemma is used implicitly in [2].

**Lemma 4.2.** We have a natural decomposition

\[
\hat{A}_1 = R/(t^p) \oplus (Z).
\]

**Proof.** Consider the natural inclusion \( W(R) \to \hat{A} \). First we claim that this induces an injection \( R/(t^p) \to \hat{A}_1 \). Let \( x \) be in the ring \( R \). If the element
$[x] \in W(R)$ is contained in $p\hat{A}$, then its image in $A_{\text{crys}}/pA_{\text{crys}}$ is zero. We have an isomorphism of $R$-algebras

$$R[Y_1, Y_2, \ldots]/(t^p, Y_1^p, Y_2^p, \ldots) \to A_{\text{crys}}/pA_{\text{crys}}$$

which sends $Y_i$ to the image of $t^p/(p^i)$. Thus the inequality $v_R(x) \geq p$ holds. Conversely, if $v_R(x) \geq p$, then we have

$$[x] = w(\psi([\varepsilon] - 1)^{p-1}) + pw'$$

for some $w, w' \in W(R)$ and this implies $[x] \in p\hat{A}$.

Let us consider the commutative diagram of $R$-algebras

$$\begin{array}{ccc}
R/(t^p) & \longrightarrow & \hat{A}_1 \\
\downarrow & & \downarrow \\
\hat{A}_1/(Z).
\end{array}$$

By definition, the left downward arrow is surjective. We claim that this arrow is an isomorphism. Indeed, let $x$ be in the kernel of this surjection. From the proof of Lemma 4.1, we see that the image of $Z$ in the ring on the left-hand side of the above isomorphism can be written as $a't^{p-1} + bY_1$ for some $a', b' \in R^\times$. By assumption, in this ring, we have

$$x = c_1(a't^{p-1} + bY_1) + c_2(a't^{p-1} + bY_1)^2 + \cdots + c_{p-1}(a't^{p-1} + bY_1)^{p-1}$$

for some elements $c_1, \ldots, c_{p-1}$ of $R$. Then we see that $c_i = 0$ for any $i$ and $v_R(x) \geq p$. This concludes the proof.

Since $r < p - 1$, from this lemma we can show the following lemma as in the proof of [3, Lemme 2.3.1.3].

**Lemma 4.3.** The functor

$$M \mapsto \text{Hom}_{\Sigma, \text{Fil}^r, \phi}(M, \hat{A}_\infty)$$

from $\text{Mod}_{\Sigma, \text{Fil}^r, \phi}$ to the category of $G_{K_\infty}$-modules is exact.

**Corollary 4.4.** For any $M \in \text{Mod}_{\Sigma, \text{Fil}^r, \phi}$, the natural map

$$\text{Hom}_{\Sigma, \text{Fil}^r, \phi}(M, \hat{A}_\infty) \to \text{Hom}_{\Sigma, \text{Fil}^r, \phi}(M, A_{\text{crys}, \infty})$$

is an isomorphism of $G_{K_\infty}$-modules.

**Proof.** By Lemma 3.2 and Lemma 4.3, we may assume $pM = 0$. Consider the commutative diagram of rings

$$\begin{array}{ccc}
\hat{A}_1 & \longrightarrow & A_{\text{crys}}/pA_{\text{crys}} \\
\downarrow & & \downarrow \\
R/(t^{p-1}).
\end{array}$$
whose downward arrows are defined by modulo $\text{Fil}^{p-1}$ of the rings $\hat{A}_1$ and $A_{\text{crys}}/pA_{\text{crys}}$, respectively. Since $r < p - 1$, we have $\phi_r(\text{Fil}^{p-1}\hat{A}_1) = 0$ and similarly for the ring $A_{\text{crys}}/pA_{\text{crys}}$. Thus these two surjections induce on the ring $R/((p-1))$ the same structure of a filtered $\phi_r$-module over $\Sigma$. Hence, as in the proof of Corollary 3.5, we see from Lemma 3.1 that we have a commutative diagram

$$\begin{array}{ccc}
\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, \hat{A}_1) & \longrightarrow & \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, A_{\text{crys}}/pA_{\text{crys}}) \\
\downarrow & & \downarrow \\
\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, R/((p-1))) & \longrightarrow & \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, R/((p-1)))
\end{array}$$

whose downward arrows are isomorphisms. This concludes the proof. \hfill \Box

We sketch the proof of the following lemma stated in [2, Subsection 3.2].

**Lemma 4.5.** The natural inclusion $W(R) \rightarrow \hat{A}$ induces an isomorphism of $W(R)$-algebras $W_n(R)/((\psi([z]) - 1)^{p-1}) \rightarrow \hat{A}_n/([z])$.

**Proof.** For a subring $B$ of $A_{\text{crys}}$, put

$I^{[s]}B = \{x \in B \mid \phi^i(x) \in \text{Fil}^s A_{\text{crys}} \text{ for any } i \}$

as in [9, Subsection 5.3]. Then we have $I^{[s]}W(R) = ([z] - 1)^s W(R)$ and the natural ring homomorphism

$W(R)/I^{[s]}W(R) \rightarrow A_{\text{crys}}/I^{[s]}A_{\text{crys}}$

is an injection ([9, Proposition 5.1.3, Proposition 5.3.5]). Since the element $Z$ is contained in the ideal $I^{[p-1]}A_{\text{crys}}$, this injection factors as

$W(R)/I^{[p-1]}W(R) \rightarrow \hat{A}/([z]) \rightarrow A_{\text{crys}}/I^{[p-1]}A_{\text{crys}}$.

Hence the former arrow is an isomorphism and the lemma follows. \hfill \Box

Since the ideal $([z])$ of $\hat{A}_n$ satisfies the condition of Corollary 3.5, the $\Sigma$-algebra $\hat{A}_n/([z])$ is naturally considered as an object of $\text{Mod}_{\Sigma}^{\phi_r}$. We also give the ring $W_n(R)/((\psi([z]) - 1)^{p-1})$ the structures of a $\Sigma$-algebra and a filtered $\phi_r$-module over $\Sigma$ induced by those of $\hat{A}_n/([z])$. The map

$\Sigma \rightarrow W_n(R)/((\psi([z]) - 1)^{p-1})$

sends the element $u \in \Sigma$ to the image of $[\pi]$ in the ring on the right-hand side. Put $v = t'/E([\pi]) \in W(R)^\times$ with the notation of Lemma 4.1. As for the element $Y \in \Sigma$, the equality

$Y = -ab^{-1}v^{-1}E((\pi))^{p-1} + wb^{-1}v^{-p}Z$

holds in $\hat{A}$, where $a$ and $b$ are the elements in $W(R)^\times$ as in the proof of Lemma 4.1 and the remark after this lemma, and $w \in W(R)^\times$ is a power series of $[\pi] - 1$. Hence the above homomorphism sends the element $Y$ to the image of $-ab^{-1}v^{-1}E((\pi))^{p-1}$. 


Consider the surjective ring homomorphism

\[ R \to \mathcal{O}_K/p\mathcal{O}_K \]

\[ x = (x_0, x_1, \ldots) \mapsto x_n \]

and the induced surjection \( W_n(R) \to W_n(\mathcal{O}_K/p\mathcal{O}_K) \). Let

\[ J = \{(x_0, \ldots, x_{n-1}) \in W_n(R) \mid v_R(x_i) \geq p^n \text{ for any } i \} \]

be the kernel of the latter surjection.

**Lemma 4.6.** The ideal \( J \) is contained in the ideal \((\psi([\varepsilon] - 1)p^{-1})\) of the ring \( W_n(R) \).

**Proof.** Write \((\varepsilon - 1)p^{-1}\) also as \( x = (x_0, \ldots, x_{n-1}) \in W_n(R) \) with \( v_R(x_0) = p \). Take an element \( z = (z_0, \ldots, z_{n-1}) \) in the ideal \( J \). We construct \( y \in W_n(R) \) such that \( xy = z \). By induction, it is enough to show that if \( z_0 = \cdots = z_{i-1} = 0 \) for some \( 0 \leq i \leq n - 1 \) and \((x_0, \ldots, x_i)(0, \ldots, 0, y_i) = (0, \ldots, 0, z_i) \) in \( W_{i+1}(R) \), then \( x(0, \ldots, 0, y_i, 0, \ldots, 0) \in J \). Let us write this element as \((0, \ldots, 0, w_i, \ldots, w_{n-1}) \) with \( w_i = z_i \). We have \( v_R(y_i) \geq p^n - p^{i+1} \).

In the ring of Witt vectors \( W_n(F_\ell[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}]) \), the \( k \)-th entry of the vector

\[(X_0, \ldots, X_{n-1})(0, \ldots, 0, Y_i, 0, \ldots, 0)\]

is \( X_{k-i}^{p^i}Y_i^{p^{k-i}} \) for any \( k \geq i \). Thus we have \( v_R(w_k) \geq p^n \).

Note that the elements \([\zeta_p^{p^n}] - 1\) and \([\zeta_{p^n+1}] - 1\) is nilpotent in \( W_n(\mathcal{O}_K/p\mathcal{O}_K) \).

By the above lemma, we have an isomorphism of rings

\[ W_n(R)/([\varepsilon] - 1)p^{-1}) \to W_n(\mathcal{O}_K/p\mathcal{O}_K)/([\zeta_p^n] - 1)p^{-1}). \]

We give the ring on the right-hand side the structure of a filtered \( \phi_r \)-module over \( \Sigma \) induced by this isomorphism.

Put \( F_n = K_n([\zeta_p^{p^n+1}] \right. \). For an algebraic extension \( F \) of \( F_n \), let us consider the ideals

\[ m_{n,F} = \{(x_0, \ldots, x_{n-1}) \in W_n(\mathcal{O}_F/p\mathcal{O}_F) \mid x_i \in m_F/p\mathcal{O}_F \text{ for any } i \} \]

\[ m_{n,F} = \{(x_0, \ldots, x_{n-1}) \in W_n(\mathcal{O}_F) \mid x_i \in m_F \text{ for any } i \} \]

of \( W_n(\mathcal{O}_F/p\mathcal{O}_F) \) and \( W_n(\mathcal{O}_F) \), respectively. The elements \([\zeta_p^n] - 1\) and \([\zeta_{p^n+1}] - 1\) are topologically nilpotent in \( W_n(\mathcal{O}_F) \) and we define an element \( \tilde{\ell} \in W_n(\mathcal{O}_F) \) to be

\[ \tilde{\ell} = \psi([\zeta_p^n] - 1)/\psi([\zeta_{p^n+1}] - 1). \]

Note that these elements are non-zero divisors of \( W_n(\mathcal{O}_F) \). Let the ring

\[ W_n(\mathcal{O}_F/p\mathcal{O}_F)/\psi([\zeta_p^n] - 1))m_{n,F} \]

be denoted by \( \tilde{A}_{n,F,r+} \). We also put \( \tilde{m}_n = \tilde{m}_{n,K} \), \( m_n = m_{n,K} \) and \( \tilde{A}_{n,r+} = \tilde{A}_{n,K,r+} \).

For an algebraic extension \( F \) of \( K \), we put

\[ b_F = \{x \in \mathcal{O}_F \mid v_K(x) > cr/(p-1)\} \].
Note that the ring $\mathcal{O}_F / b_F$ is killed by $p$. When $F$ contains $F_n$, we also put
\[ A'_{n,F_r} = W_n(\mathcal{O}_F / b_F) / \psi([\zeta_{p^n}] - 1)^r m_{n,F}. \]

Then, for $0 \leq r < p - 1$, we have natural isomorphisms of rings
\[ W_n(\mathcal{O}_F) / \psi([\zeta_{p^n}] - 1)^r m_{n,F} \to \hat{A}_{n,F,r} \to \bar{A}_{n,F,r}. \]

Indeed, as in the proof of Lemma 4.6, we can show that both of the kernels of the maps $W_n(\mathcal{O}_F) \to W_n(\mathcal{O}_F / p\mathcal{O}_F)$ and $W_n(\mathcal{O}_F) \to W_n(\mathcal{O}_F / b_F)$ are contained in the ideal $\psi([\zeta_{p^n}] - 1)^r m_{n,F}$ of the ring $W_n(\mathcal{O}_F)$. We often identify these rings. We also put $\bar{A}_{n,F,r} = \bar{A}_{n,F,r}$. Write $Z_n$ for the image of the element $Z$ of $A_{\text{crys}}$ in $W_n^{\text{PD}}(\mathcal{O}_{\bar{K}} / p\mathcal{O}_{\bar{K}})$. Then we have a commutative diagram of $\Sigma$-algebras

```
\[
\begin{array}{c}
\hat{A}_n \longrightarrow A_{\text{crys}} / p^n A_{\text{crys}} \\
\downarrow \downarrow
\end{array}
\]

```

where all vertical arrows are surjections satisfying the condition of Corollary 3.5. Thus we see that this is also a commutative diagram in $\text{Mod}_{r,\phi}^\Sigma$. Note that these rings and homomorphisms are independent of the choice of a system $\{\zeta_{p^n}\}_{n \in \mathbb{Z}_{\geq 0}}$. Moreover, let $M$ be in $\text{Mod}_{r,\phi}^\Sigma$ and put $M_n = M / p^n M$. Then, by Corollary 3.5 and Corollary 4.4, we have a natural isomorphism of abelian groups
\[
\text{Hom}_{\Sigma, \text{Fil}_{r,\phi}}(M_n, W_n^{\text{PD}}(\mathcal{O}_{\bar{K}} / p\mathcal{O}_{\bar{K}})) \simeq \text{Hom}_{\Sigma, \text{Fil}_{r,\phi}}(M_n, \bar{A}_{n,F,r}).
\]

To study $G_{F_n}$-actions on both sides of this isomorphism, we need the following proposition.

**Proposition 4.7.** The image of the element $Y \in \Sigma$ in the ring $\hat{A}_{n,F,r}$ is contained in its subring $\bar{A}_{n,F,n,r}$.  


Proof. We have the equality
\[(\varepsilon^{1/p} - 1)^{p-1} = pa - E([\varepsilon])bv\]
in $W(R)$. By definition, we see that the images of the elements $a_i$, $[\varepsilon^{1/p}]$ and $E([\varepsilon])$ in $W_n(O_{\mathcal{K}}/pO_{\mathcal{K}})$ are contained in the subring $W_n(O_{F_n}/pO_{F_n})$. Write $\beta$ for the product $bv \in W(R)$. Let $a_n$, $\beta_n$ and $\alpha_n$ denote the images of the elements $a_i$, $\beta$ and $pa - ([\varepsilon^{1/p}] - 1)^{p-1}$ in the ring $W_n(O_{\mathcal{K}}/pO_{\mathcal{K}})$, respectively. Then the element $\alpha_n$ is also contained in the subring $W_n(O_{F_n}/pO_{F_n})$. Now we have the equality
\[E([\pi_n])\beta_n = \alpha_n.\]
Note that any element $\beta'_n \in W_n(O_{\mathcal{K}}/pO_{\mathcal{K}})$ satisfying the same equality is invertible and thus the elements $(\beta'_n)^{-1}E([\pi_n])$ are equal to each other. Since $Y = -a_n\beta_n^{-1}E([\pi_n])^{p-1}$ in $A_{n,r+}$, it suffices to construct an element $\beta'_n$ in the ring $W_n(O_{F_n}/pO_{F_n})$ such that the equality $E([\pi_n])\beta'_n = \alpha_n$ holds.
Take a lift $\hat{\alpha}_n$ of $\alpha_n$ in $W_n(O_{F_n})$. Since $E([\pi_n]) \in W_n(O_{\mathcal{K}})$ divides every element in the kernel of the surjection $W_n(O_{\mathcal{K}}) \to W_n(O_{\mathcal{K}}/pO_{\mathcal{K}})$, we have $E([\pi_n])\hat{\beta}'_n = \hat{\alpha}_n$ for some $\hat{\beta}'_n \in W_n(O_{\mathcal{K}})$. Then the element $\hat{\beta}'_n$ is contained in the subring $W_n(O_{F_n})$ and we set $\beta'_n$ to be the image of $\hat{\beta}'_n$.

By a similar argument, we can also check that the ring $\bar{A}_{n,F,r,+} = \bar{A}_{n,r+}$ is a subring of $\bar{A}_{n,r+}$ and coincides with the image of $W_n(O_F)$ in $\bar{A}_{n,r+}$. This concludes the proof. □

Let $t_n$ and $\bar{t}_n$ be the images of $t \in W(R)$ in $W_n(O_{\mathcal{K}}/pO_{\mathcal{K}})$ and $\bar{A}_{n,r+}$ (or $\bar{A}'_{n,r+}$), respectively. Then $t$ is a lift of $t_n$ and $\bar{t}_n$ to $W_n(O_{\mathcal{K}})$ by the natural surjections
\[W_n(O_{\mathcal{K}}) \to W_n(O_{\mathcal{K}}/pO_{\mathcal{K}}) \to \bar{A}_{n,r+}.
\]
Note that we defined the filtration of $A_{n,r+}$ as $\text{Fil}'\bar{A}_{n,r+} = t_n\bar{A}_{n,r+}$.

Lemma 4.8. Let $\bar{x}$ be in $\text{Fil}'\bar{A}_{n,r+}$. Then we have
\[\phi_r(\bar{x}) = \phi(y) \text{ mod } \psi([\zeta^{p^n}] - 1)^r\bar{m}_n,\]
where $y$ is any element of $W_n(O_{\mathcal{K}}/pO_{\mathcal{K}})$ such that the element $t_n^r y$ is a lift of $\bar{x}$. In particular, the right-hand side of the above equality is independent of the choice of a system $\{\zeta^{p^n}\}_{n \geq 0}$. Similar assertions also hold for the ring
\[W_n(O_{\mathcal{K}}/pO_{\mathcal{K}})/(\psi([\zeta^{p^n}] - 1)^{p-1}).\]

Proof. Since the filtered $\phi_r$-module structure of $\bar{A}_{n,r+}$ is induced from that of $A$ and $\phi_r(t^n) \equiv 1 \text{ mod } Z$ in $\bar{A}$, we see that there exists an element $y$ as in the lemma.

To prove the independence of the choice of a lift, let $z = (z_0, \ldots, z_{n-1})$ be an element of $W_n(O_{\mathcal{K}}/pO_{\mathcal{K}})$ killed by $t_n$. The element $z$ is also killed by
Let $\phi$ be the associated matrix representing $\phi$ on the rings $W_n(\mathcal{O}_K/p\mathcal{O}_K)$. We can show the assertion for the ring $\phi$ sends the element on the right-hand side to an element which is contained in the ideal $\psi([\zeta^n] - 1)^r\bar{m}_n$. Thus the assertions for the ring $\bar{A}_{n,r+}$ follows. We can show the assertion for the ring $W_n(\mathcal{O}_K/p\mathcal{O}_K)/(\psi([\zeta^n] - 1)^{p-1})$ similarly.

From this lemma and Proposition 4.7, we see that the natural $G_{F_n}$-actions on the rings $W_n(\mathcal{O}_K/p\mathcal{O}_K)/(\psi([\zeta^n] - 1)^{p-1})$, $\bar{A}_{n,r+}$ and $\bar{A}'_{n,r+}$ are compatible with the filtered $\phi_r$-module structures over $\Sigma$. In the commutative diagram above, the lowest horizontal arrow and lower right vertical arrow are $G_{F_n}$-linear by definition. Hence we have shown the following proposition.

**Proposition 4.9.** Let $\hat{M}$ be in $\text{Mod}_{\Sigma}^{F_n}$ and put $M_n = \hat{M}/p^n\hat{M}$. Then we have an isomorphism of $G_{F_n}$-modules

$$\text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M_n, \bar{A}_{n,r+}) \simeq \text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M_n, W_n^\text{PD}(\mathcal{O}_K/p\mathcal{O}_K)).$$

Let $e_1, \ldots, e_d$ be a basis of $\hat{M}$ as in Lemma 3.3 and $C = (c_{i,j}) \in M_d(\Sigma)$ be the associated matrix representing $\phi_r$ as in Corollary 3.4. Then the underlying $G_{F_n}$-set of the $G_{F_n}$-module

$$\text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M_n, \bar{A}_{n,r+})$$

is identified with the set of $d$-tuples $(\bar{x}_1, \ldots, \bar{x}_d)$ in $\bar{A}_{n,r+}$ such that $c_{1,i}\bar{x}_1 + \cdots + c_{d,i}\bar{x}_d \in \text{Fil}^r\bar{A}_{n,r+}$ for any $i$ and the following equality holds:

$$\phi_r(c_{1,1}\bar{x}_1 + \cdots + c_{d,1}\bar{x}_d) = \bar{x}_1$$

$$\vdots$$

$$\phi_r(c_{1,d}\bar{x}_1 + \cdots + c_{d,d}\bar{x}_d) = \bar{x}_d.$$

We choose a lift $(\bar{e}_{i,j}) \in M_d(W_n(\mathcal{O}_F))$ of the image of $C$ in $M_d(\bar{A}_{n,r+})$ by the natural ring homomorphism

$$W_n(\mathcal{O}_K) \rightarrow W_n(\mathcal{O}_K/p\mathcal{O}_K) \rightarrow \bar{A}_{n,r+}.$$

Fix a polynomial $\Phi_i \in \mathbb{Z}[X_0, \ldots, X_{n-1}]$ such that $\Phi_i \equiv X_i^p \mod p$. This induces for any commutative ring $B$ a map $\Phi = (\Phi_0, \ldots, \Phi_{n-1}) : W_n(B) \rightarrow W_n(B)$ which is a lift of the Frobenius endomorphism on $W_n(B/pB)$. In
particular, set $B$ to be the polynomial ring $\mathbb{Z}[X_0,\ldots,X_{n-1},Y_0,\ldots,Y_{n-1}]$. Put $X = (X_0,\ldots,X_{n-1})$ and $Y = (Y_0,\ldots,Y_{n-1})$ in the ring $W_n(B)$. Then we see that there exists elements $U_0,\ldots,U_{n-1}$ and $U'_0,\ldots,U'_{n-1}$ of the polynomial ring $B$ such that

$$
\Phi(X + Y) = \Phi(X) + \Phi(Y) + (pU_0,\ldots,pU_{n-1}),
\Phi(XY) = \Phi(X)\Phi(Y) + (pU'_0,\ldots,pU'_{n-1})$

in the ring $W_n(B)$.

**Proposition 4.10.** Every solution $(\bar{x}_1,\ldots,\bar{x}_d)$ in $\hat{A}_{n,r}$ of the equation (1) such that $c_{1,i}\bar{x}_1 + \cdots + c_{d,i}\bar{x}_d \in \text{Fil}^r \hat{A}_{n,r}$ for any $i$ uniquely lifts to a $d$-tuple $(\bar{x}_1,\ldots,\bar{x}_d)$ in $W_n(O_R)$ such that $\hat{c}_{1,i}\bar{x}_1 + \cdots + \hat{c}_{d,i}\bar{x}_d \in \bar{t}^r_1 W_n(O_R)$ for any $i$ and the following equality holds:

$$
\Phi((\hat{c}_{1,1}\bar{x}_1 + \cdots + \hat{c}_{d,1}\bar{x}_d)/\bar{t}^r_1) = \bar{x}_1
$$

(2)

$$
\Phi((\hat{c}_{1,d}\bar{x}_1 + \cdots + \hat{c}_{d,d}\bar{x}_d)/\bar{t}^r_d) = \bar{x}_d.
$$

**Proof.** Fix a lift $\hat{x}_i$ of $\bar{x}_i$ to $W_n(O_R)$. Then we have

$$
\Phi((\hat{c}_{1,1}\hat{x}_1 + \cdots + \hat{c}_{d,1}\hat{x}_d)/\bar{t}^r_1) = \hat{x}_1 + ((\zeta^{\rho n})^r - 1)^r \hat{\delta}_1
$$

$$
\Phi((\hat{c}_{1,d}\hat{x}_1 + \cdots + \hat{c}_{d,d}\hat{x}_d)/\bar{t}^r_d) = \hat{x}_d + ((\zeta^{\rho n})^r - 1)^r \hat{\delta}_d
$$

for some $\hat{\delta}_1,\ldots,\hat{\delta}_d \in m_n$. It suffices to show that there exists a unique $d$-tuple $(\bar{y}_1,\ldots,\bar{y}_d)$ in $m_n$ such that

$$
\Phi((\hat{c}_{1,i}(\hat{x}_1 + ((\zeta^{\rho n})^r - 1)^r \bar{y}_1) + \cdots + \hat{c}_{d,i}(\hat{x}_d + ((\zeta^{\rho n})^r - 1)^r \bar{y}_d))/\bar{t}^r_i)
$$

$$
= \hat{x}_i + ((\zeta^{\rho n})^r - 1)^r \bar{y}_i
$$

for any $i$. For this, we need the following lemma.

**Lemma 4.11.** Let $N$ be a complete discrete valuation field and $m_N$ be the maximal ideal of $N$. Let $\epsilon_1,\ldots,\epsilon_d$ be in $m_N$. Let $P_1,\ldots,P_d$ and $P'_1,\ldots,P'_d$ be elements of $O_N[[Y_1,\ldots,Y_d]]$ such that $P_i \in (Y_1,\ldots,Y_d)^2$. Then the equation

$$
\begin{align*}
Y_1 - P_1(Y_1,\ldots,Y_d) - \epsilon_1 P'_1(Y_1,\ldots,Y_d) &= 0 \\
\vdots \\
Y_d - P_d(Y_1,\ldots,Y_d) - \epsilon_d P'_d(Y_1,\ldots,Y_d) &= 0
\end{align*}
$$

has a unique solution in $m_N$.

**Proof.** By assumption, we see that for any integer $l \geq 1$, a $d$-tuple $y = (y_1,\ldots,y_d)$ in $m_N/m_N^{l+1}$ satisfies the above equation lifts uniquely to a $d$-tuple in $m_N/m_N^{l+1}$ satisfying the same equation. Thus the lemma follows. \(\square\)
Let us write as \( \hat{y}_i = (\hat{y}_{i,0}, \ldots, \hat{y}_{i,n-1}) \). Since the image of \( \Phi((\zeta_{p^r+1} - 1)^{r'}) \) in \( \tilde{A}_{n,r^+} \) is divisible by \( ([\zeta_{p^r}] - 1)^{r'} \), we can find \( \hat{b} \in W_n(\mathcal{O}_K) \) such that
\[
\Phi(([\zeta_{p^r}] - 1)^{r'/p^r}) = ([\zeta_{p^r}] - 1)^{r'} \hat{b}.
\]
Then there exists polynomials \( U_{i,m} \) over \( \mathcal{O}_K \) of the indeterminates \( Y = (Y_{i,m})_{1 \leq i \leq d, 0 \leq m \leq n-1} \) such that the equation we have to solve is
\[
\hat{x}_i + ([\zeta_{p^r}] - 1)^{r'} \hat{y}_i = \hat{x}_i + ([\zeta_{p^r}] - 1)^{r'} \delta_i + ([\zeta_{p^r}] - 1)^{r'} \hat{b}(\Phi(\hat{c}_{i,1})\Phi(\hat{y}_1) + \cdots + \Phi(\hat{c}_{i,d})\Phi(\hat{y}_d)) + (pU_{i,0}(\hat{y}), \ldots, pU_{i,n-1}(\hat{y}))
\]
for any \( i \), where we put \( \hat{y} = (\hat{y}_{i,m})_{1 \leq i \leq d, 0 \leq m \leq n-1} \). Note that, for any elements \( P_0, \ldots, P_{n-1} \) of the polynomial ring \( \mathcal{O}_K[\hat{Y}] \), we can uniquely find elements \( Q_0, \ldots, Q_{n-1} \) of this ring such that the coefficients of these polynomials are in the maximal ideal \( m_K \) and the equality
\[
(pP_0, \ldots, pP_{n-1}) = ([\zeta_{p^r}] - 1)^{r'}(Q_0, \ldots, Q_{n-1})
\]
holds in the ring of Witt vectors \( W_n(\mathcal{O}_K[\hat{Y}]) \). Therefore, this equation is equivalent to the equation
\[
\hat{y}_i = \hat{\delta}_i + \hat{b}(\Phi(\hat{c}_{i,1})\Phi(\hat{y}_1) + \cdots + \Phi(\hat{c}_{i,d})\Phi(\hat{y}_d)) + (V_{i,0}(\hat{y}), \ldots, V_{i,n-1}(\hat{y}))
\]
where \( V_{i,m} \) is a polynomial of \( Y \) over \( \mathcal{O}_K \) whose coefficients are in the maximal ideal \( m_K \). From the definition of \( \Phi \), we see that the elements \( \hat{y}_{i,m} \) is a solution of a system of equations
\[
Y_{i,m} - P_{i,m}(Y) - \epsilon_{i,m}P'_{i,m}(Y) = 0
\]
satisfying the condition of Lemma 4.11 for a sufficiently large finite extension \( N \) of \( K \). Then, by this lemma, we can solve the equation uniquely in \( m_K \). \( \square \)

Let \( F \) be an algebraic extension of \( F_n = K_n(\zeta_{p^r+1}) \) and consider the ring \( \tilde{A}_{n,F,r^+} \). By Lemma 4.7, we can consider this ring as a \( \Sigma \)-algebra and also as an object of \( \text{Mod}^{\phi,0}_{/\Sigma} \) by putting \( \text{Fil}' \tilde{A}_{n,F,r^+} = \tilde{r}_{n\cdot}^* \tilde{A}_{n,F,r^+} \) and for \( \bar{x} \in \text{Fil}' \tilde{A}_{n,F,r^+} \),
\[
\phi_r(\bar{x}) = \phi(y) \mod \psi(([\zeta_{p^r}] - 1)^{r'}\bar{m}_{n,F}, F)
\]
where \( y \) is any element of \( W_n(\mathcal{O}_F/p\mathcal{O}_F) \) such that the element \( t^r_p y \) is a lift of \( \bar{x} \). For \( M_n = \tilde{M}/p^n\tilde{M} \in \text{Mod}^{\phi,0}_{/\Sigma_{\infty}} \) as before, let us set
\[
T_{\text{cryst}, \pi_n, F}(M_n) = \text{Hom}_{\Sigma, \text{Fil}', \phi_r}(M_n, \tilde{A}_{n,F,r^+}).
\]
We see that \( \tilde{A}_{n,r^+} = \tilde{A}_{n,K,r^+} = \bigcup_{F/F_n} \tilde{A}_{n,F,r^+} \).
From Proposition 4.10, we see that the elements of $A_n$ algebraic extension $K$ and the theorem is trivial. Thus we may assume we may assume in $V$

Proof.

Corollary 4.13. For an algebraic extension $F$ of $F_n$, the fixed part $T^*_{crys, \pi_n, K}(M_n)^{G_F}$ is equal to $T^*_{crys, \pi_n, F}(M_n)$.

Proof. From Proposition 4.10, we see that the elements of $T^*_{crys, \pi_n, K}(M_n)$ correspond bijectively to the solutions of the equation (2) in $W_n(\mathcal{O}_K)$ satisfying the condition on Fil $T^*$. The uniqueness assertion of this proposition shows that $g \in G_F$ fixes a solution in $W_n(\mathcal{O}_K)$ if and only if $g$ fixes its image in $\mathcal{A}_n, r_u$. Hence a solution is fixed by $G_F$ if and only if this solution is contained in the image of $W_n(\mathcal{O}_F)$. Thus the lemma follows.

Corollary 4.13. Let $L_n$ be the finite Galois extension of $F_n$ corresponding to the kernel of the map $G_{F_n} \to \text{Aut}(T^*_{crys, \pi_n, K}(M_n))$. Then an algebraic extension $F$ of $F_n$ contains $L_n$ if and only if $T^*_{crys, \pi_n, F}(M_n) = T^*_{crys, \pi_n, K}(M_n)$.

Proof. An algebraic extension $F$ of $F_n$ contains $L_n$ if and only if the action of $G_F$ on $T^*_{crys, \pi_n, K}(M_n)$ is trivial. By Lemma 4.12, this is equivalent to $T^*_{crys, \pi_n, F}(M_n) = T^*_{crys, \pi_n, K}(M_n)$.

5. Ramification bound

In this section, we prove Theorem 1.1. Take $G_K$-stable $\mathbb{Z}_p$-lattices $L \supseteq \mathcal{L}'$ in $V$ such that $\mathcal{L}' \supseteq \mathcal{L}'^p \mathcal{L}$. Since the $G_K$-module $\mathcal{L}/\mathcal{L}'$ is a quotient of $\mathcal{L}/\mathcal{L}'^p$, we may assume $\mathcal{L}' = \mathcal{L}'^p \mathcal{L}$. If $r = 0$, then the $G_K$-module $V$ is unramified and the theorem is trivial. Thus we may assume $r \geq 1$ and $p \geq 3$. Let $L$ be the finite Galois extensions of $K$ corresponding to the kernel of the map $G_K \to \text{Aut}(\mathcal{L}/\mathcal{L}')$.

It is enough to show that, for the greatest upper ramification break $u_{L(\zeta_p)/K}$ of the Galois extension $L(\zeta_p)/K$, the inequality

$$u_{L(\zeta_p)/K} \leq u(K, r, n)$$

holds. Since the Herbrand function is transitive and the finite Galois extension $K(\zeta_p)$ is tamely ramified over $K$, we may assume $\zeta_p \in K$. We fix a uniformizer $\pi$ of $K$ and a system $\{\pi_n\}_{n \in \mathbb{Z}_{\geq 0}}$ as before. Then, by Liu’s theorem ([14, Theorem 2.3.5]), it suffices to show the following.

Theorem 5.1. Let $r$ be an integer such that $1 \leq r < p - 1$ and $\mathcal{M}$ be the strongly divisible lattice corresponding to $L$. Put $\mathcal{M}_n = \mathcal{M}/p^n \mathcal{M} \in \text{Mod}^{r, \phi, N}_{/S\infty}$. Then $G_K(n)^j$ acts trivially on the $G_K$-module $T^*_{st, A}(\mathcal{M}_n)$ for $j > u(K, r, n)$. 
Let $L_n$ be the finite Galois extension of $F_n = K_n(\zeta_{p^n+1})$ corresponding to the kernel of the map

$$G_{F_n} \to \text{Aut}(T^*_{\text{st}, \mathbb{Z}}(\mathcal{M}_n)).$$

Since $F_n$ is Galois over $K$, the extension $L_n$ is also a Galois extension of $K$. Let $\mathcal{M}$ be the object of the category $\text{Mod}_{/\mathfrak{S}}$ such that $\mathcal{M}_{/\mathfrak{S}} \simeq \mathcal{M}$. From Proposition 3.6 and Proposition 4.9, we see that $L_n$ is also the finite extension of $F_n$ cut out by the $G_{F_n}$-module $T_{\text{crys}, \pi_n, K}(M_n)$ for $M_n = M_{/\mathfrak{S}}(\mathcal{M})/p^n M_{/\mathfrak{S}}(\mathcal{M})$. It is enough to prove the inequality

$$u_{L_n/K} \leq u(K, r, n) = \begin{cases} 1 + e(n + \frac{1}{p-1}) & (r = 1), \\ \frac{p^n-1}{p^r} + e(n + \frac{r}{p-1}) & (r > 1). \end{cases}$$

Before proving this, we state some general lemmas to calculate the ramification bound. Let $\mathcal{N}$ be a complete discrete valuation field of positive residue characteristic, $v_{\mathcal{N}}$ be its valuation normalized as $v_{\mathcal{N}}(N) = \mathbb{Z}$ and $\mathcal{N}_{/\mathfrak{S}}$ be its separable closure.

**Lemma 5.2.** Let $f(T) \in \mathcal{O}_N[T]$ be a separable monic polynomial and $z_1, \ldots, z_d$ be the zeros of $f$ in $\mathcal{O}_{/\mathfrak{S}}$. Suppose that the set $\{v_{\mathcal{N}}(z_k - z_i) | k = 1, \ldots, d, k \neq i\}$ is independent of $i$. Put

$$s = \sum_{k=1}^{d} \sum_{k \neq i} v_{\mathcal{N}}(z_k - z_i) \quad \text{and} \quad \alpha = \sup_{k=1, \ldots, d} v_{\mathcal{N}}(z_k - z_i),$$

which are independent of $i$ by assumption. If $j > s + \alpha$, then we have the decomposition

$$\{x \in \mathcal{O}_{/\mathfrak{S}} \mid v_{\mathcal{N}}(f(x)) \geq j\} = \coprod_{i=1, \ldots, d} \{x \in \mathcal{O}_{/\mathfrak{S}} \mid v_{\mathcal{N}}(x - z_i) \geq j - s\}.$$

Otherwise, the set on the left-hand side contains

$$\{x \in \mathcal{O}_{/\mathfrak{S}} \mid v_{\mathcal{N}}(x - z_i) \geq \alpha\},$$

which contains at least two zeros of $f$.

**Proof.** A verbatim argument in the proof of [1, Lemma 6.6] shows the claim. \qed

**Corollary 5.3.** Let $f(T)$ be as above and put $B = \mathcal{O}_N[T]/(f(T))$. Suppose that the algebra $B$ is finite flat and of relative complete intersection over $\mathcal{O}_N$. Let us write the $\mathcal{N}$-algebra $N' = B \otimes_{\mathcal{O}_N} N$ as the product $N_1 \times \cdots \times N_t$ of finite separable extensions $N_1, \ldots, N_t$ of $\mathcal{N}$. If $j > s + \alpha$, then the $j$-th upper numbering ramification group ([1]), which we let be denoted by $G_N^{(j)}$, is contained in $G_N$, for any $i$. Moreover, if $N'$ is a field and $B$ coincides with $\mathcal{O}_N$, then $j > s + \alpha$ if and only if $G_N^{(j)} \leq G_N$. \qed
Proof. From the previous lemma, the conductor $c(B)$ of the $\mathcal{O}_N$-algebra $B$ ([1, Proposition 6.4]) is equal to $s + \alpha$. Thus we have the inequality
$$c(\mathcal{O}_{N_1} \times \cdots \times \mathcal{O}_{N_t}) \leq c(B) = s + \alpha$$
by the definition of the conductor and a functoriality of the functor $\mathcal{F}^j$ defined in [1]. This implies the corollary. \qed

**Corollary 5.4.** Consider the finite Galois extension $F_n = K_n(\zeta_p^{n+1})$ of $K$ and let $u_{F_n/K}$ denote the greatest upper ramification break of $F_n/K$. Then we have the equality
$$u_{F_n/K} = 1 + e(n + \frac{1}{p-1}).$$

Proof. Note that we are assuming that $\zeta_p$ is contained in $K$. Applying the previous corollary to the Eisenstein polynomial $f(T) = T^{p^n} - \pi$ shows that $j > 1 + e(n + 1/(p-1))$ if and only if $G_K^{(j)} \subseteq G_{K_n}$. Similarly, putting $f(T) = T^{p^n} - \zeta_p$, we see that if $j > e(n + 1/(p-1))$, then $G_K^{(j)} \subseteq G_K(\zeta_p^{n+1})$. Since $G_{F_n} = G_{K_n} \cap G_{K(\zeta_p^{n+1})}$, we conclude that $j > 1 + e(n + 1/(p-1))$ if and only if $G_K^{(j)} \subseteq G_{F_n}$. \qed

**Remark 5.5.** Note that this argument also shows the equality
$$u_{K_n(\zeta_p^{n+1})/K} = 1 + e(n + \frac{1}{p-1})$$
without assuming $\zeta_p \in K$.

Next we assume that the residue field of $N$ is perfect. For an algebraic extension $F$ of $N$, we put
$$\alpha^j_{F/N} = \{ x \in \mathcal{O}_F \mid v_N(x) \geq j \}.$$For a finite Galois extension $Q$ of $N$, we write $u_{Q/N}$ for the greatest upper ramification break ([7]) of $Q/N$. Let us consider the property
$$(P_j) \left\{ \begin{array}{ll}
\text{for any algebraic extension } F \text{ of } N, \text{ if there exists an } \mathcal{O}_N \text{-algebra homomorphism } \mathcal{O}_Q \to \mathcal{O}_F/\alpha^j_{F/N}, \\
\text{then there exists an } N \text{-algebra injection } Q \to F 
\end{array} \right.$$for $j \in \mathbb{R}_{\geq 0}$, as in [7, Proposition 1.5]. Then we have the following proposition, which is due to Yoshida. Here we reproduce his proof for the convenience of the reader.

**Proposition 5.6 ([16]).**
$$u_{Q/N} = \inf \{ j \in \mathbb{R}_{\geq 0} \mid \text{the property (P_j) holds} \}.$$Proof. By [7, Proposition 1.5 (i)], it is enough to show that the property (P_j) does not hold for $j = u_{Q/N} - (e')^{-1}$ with an arbitrarily large $e' > 0$. As in the proof of [7, Proposition 1.5 (ii)], we may assume that $Q$ is totally and wildly ramified over $N$. Let $N'$ be a finite tamely ramified Galois
Let us apply Lemma 5.2 to \( f(T) = T^{p^n} - \pi \). Then, with the notation of the lemma, we have

\[
s = ne + \frac{p^n - 1}{p^n} \quad \text{and} \quad \alpha = \frac{1}{p^n} + \frac{e}{p - 1}.
\]

Since \( j - s > er/(p - 1) \) by assumption, we have

\[
x \equiv \pi_n \zeta_{p^n}^i \mod b_F
\]

for some \( i \).
Let $h(T)$ be the minimal polynomial of $\zeta_{p^{n+1}}$ over $O_K$. Since $h$ divides $T^{p^n} - \zeta_p$, the $O_K$-algebra $B' = O_K[T]/(h(T))$ is also finite flat of relative complete intersection and the $K$-algebra $B' \otimes_K K$ is étale. The Galois group $\text{Gal}(K(\zeta_{p^{n+1}})/K)$ acts transitively on the set of zeros of $h$. Hence $h$ also satisfies the conditions of Lemma 5.2. Let $s'$ and $\alpha'$ be as in this lemma for $h$. Then we have $s' \leq ne$ and $\alpha' \leq e/(p-1)$. This implies $j - s' > e/(p-1)$. By this lemma, there exists an element $g' \in \text{Gal}(K(\zeta_{p^{n+1}})/K)$ such that the element $g'(\zeta_{p^{n+1}}) = \zeta'_{p^{n+1}}$ satisfies
\[
\eta(\zeta_{p^{n+1}}) \equiv g'(\zeta_{p^{n+1}}) \mod b_F.
\]
Since $K_n \cap K(\zeta_{p^{n+1}}) = K$ (see for example [14, Lemma 5.1.2]), we can find an element $g \in G_K$ such that $g(\pi_n) = \pi_n \zeta_{p^n}$ and $g(\zeta_{p^{n+1}}) = g'(\zeta_{p^{n+1}})$. This concludes the proof.

**Lemma 5.9.** The $O_K$-algebra homomorphism $\eta$ induces an $O_K$-algebra injection
\[
\eta_b : O_{L_n}/\mathfrak{b}_{L_n} \to O_F/\mathfrak{b}_F.
\]

**Proof.** We write the Eisenstein polynomial of a uniformizer $\pi_{L_n}$ of $L_n$ over $O_K$ as
\[
P(T) = T^{e'} + c_1 T^{e'-1} + \cdots + c_{e'-1} T + c_{e'},
\]
where $e' = e(L_n/K)$. Then $z = \eta(\pi_{L_n})$ satisfies $P(z) = 0$ in $O_F/\mathfrak{a}_F^2 /K$.

Let $\hat{z}$ be a lift of $z$ in $O_F$. Since $j > 1$, we have $v_F(\hat{z}) = e(F/K)/e'$. The condition $i > e(L_n)r/(p-1)$ is equivalent to the condition
\[
v_F(\hat{z}^i) > \frac{e(L_n)r}{p-1} \cdot \frac{e(F/K)}{e'} = \frac{e(F)r}{p-1}.
\]
Thus the claim follows. \qed

Since $L_n$ contains $F_n$, we can consider the ring
\[
\tilde{A}'_{n,L_n,r^+} = W_n(O_{L_n}/\mathfrak{b}_{L_n})/\psi([\zeta_{p^n}] - 1)^{r} \bar{m}_{n,L_n}
\]
and similarly $\tilde{A}'_{n,F,r^+}$ for $F$. We give these rings structures of $\Sigma$-algebras as follows. The ring $\tilde{A}'_{n,L_n,r^+}$ is considered as a $\Sigma$-algebra by using the system $\{\pi_n\}_{n \in \mathbb{Z}_{\geq 0}}$ we chose of $p$-power roots of $\pi$, as in the previous section. On the other hand, using $i$ and $i'$ in Lemma 5.8, put $\bar{\pi}_n = \pi_n \zeta_{p^n}$ and $\zeta'_{p^{n+1}} = \zeta'_{p^{n+1}}$. Then we consider the ring $\tilde{A}'_{n,F,r^+}$ as a $\Sigma$-algebra by using a system of $p$-power roots of $\pi$ containing $\bar{\pi}_n$. We define $\text{Fil}'$ and $\phi_r$ of these rings in the same way as before.

**Lemma 5.10.** The induced ring homomorphism
\[
\bar{\eta} : \tilde{A}'_{n,L_n,r^+} \to \tilde{A}'_{n,F,r^+}
\]
is a morphism of the category $\text{Mod}^{\Sigma}_{\psi}$. 
Proof. Firstly, we check that $\bar{\eta}$ is $\Sigma$-linear. By definition, this homomorphism commutes with the action of the element $u \in \Sigma$. To show the compatibility with the element $Y \in \Sigma$, let us consider the commutative diagram

$$
\begin{array}{ccc}
W_n(O_{L_n}/pO_{L_n}) & \longrightarrow & W_n(O_F/pO_F) \\
\downarrow & & \downarrow \\
W_n(O_{L_n}/b_{L_n}) & \xrightarrow{\eta_b} & W_n(O_F/b_F) \\
\downarrow & & \downarrow \\
\bar{A}'_{n,L_n,r+} & \xrightarrow{\bar{\eta}} & \bar{A}'_{n,F,r+},
\end{array}
$$

where the horizontal arrows are induced by $\eta$. Note that we have $\eta_b(\pi_n) = \tilde{\pi}_n$ and $\eta_b(\zeta_{p^{n+1}}) = \tilde{\zeta}_{p^{n+1}}$. Put $\beta \in W(R)^\times$ as in the proof of Proposition 4.7. Namely, the element $\beta$ is the solution in $W(R)$ of the equation

$$E([\pi_n])\beta = p a_n - ([\zeta_{p^{n+1}}] - 1)^{p-1},$$

where the element $a \in W(R)$ is as in the remark after Lemma 4.1. Let $a_n$ and $\beta_n$ denote the images of $a$ and $\beta$ in $W_n(O_{L_n}/pO_{L_n})$, respectively. Then the element $\beta_n$ is a solution of the equation

$$E([\pi_n])\beta_n = p a_n - ([\zeta_{p^{n+1}}] - 1)^{p-1}.$$

Similarly, we define elements $\tilde{a}_n$ and $\tilde{\beta}_n$ of $W_n(O_F/pO_F)$ using $\tilde{\pi}_n$ and $\tilde{\zeta}_{p^{n+1}}$. By definition, the element $\tilde{\beta}_n$ is a solution of the equation

$$E([\tilde{\pi}_n])\tilde{\beta}_n = p \tilde{a}_n - ([\tilde{\zeta}_{p^{n+1}}] - 1)^{p-1}.$$

Now what we have to show is the equality

$$\bar{\eta}(a_n \beta_n^{-1} E([\pi_n])^{p-1}) = \tilde{a}_n \tilde{\beta}_n^{-1} E([\tilde{\pi}_n])^{p-1}$$

in the ring $\bar{A}'_{n,F,r+}$. Since the element $a_n$ of $W_n(O_{L_n}/pO_{L_n})$ is a linear combination of the elements $1, [\zeta_{p^{n+1}}], \ldots, [\zeta_{p^{n+1}}]^{p-1}$ over $\mathbb{Z}$, we have $\bar{\eta}(a_n) = \tilde{a}_n$ in $\bar{A}'_{n,F,r+}$. Thus the elements $\tilde{\beta}_n$ and $\bar{\eta}(\beta_n)$ satisfy the same equation in $\bar{A}'_{n,F,r+}$. Since these two elements are invertible, we see that $\bar{\eta}(\beta_n)^{-1} E([\tilde{\pi}_n]) = \tilde{\beta}_n^{-1} E([\tilde{\pi}_n])$ and the equality holds. Since the diagram above is compatible with the Frobenius endomorphisms, we see from the definition that $\bar{\eta}$ also preserves $\text{Fil}^r$ and commutes with $\phi_r$ of both sides. 

Thus the homomorphism $\bar{\eta}$ induces a homomorphism of abelian groups

$$T^*_{\text{crys},L_n,\pi_n}(M_n) \rightarrow T^*_{\text{crys},F,\tilde{\pi}_n}(M_n).$$

Then the following lemma, whose proof is omitted in [2, Subsection 3.13], implies that this homomorphism is an injection. We insert here a proof of this lemma for the convenience of the reader.

**Lemma 5.11.** The ring homomorphism $\bar{\eta} : \bar{A}'_{n,L_n,r+} \rightarrow \bar{A}'_{n,F,r+}$ is an injection.
Proof. For an algebraic extension $N$ of $F_n$, let us write $\tilde{A}'_N$ for the ring $\tilde{A}'_{n,N,r}$. Note that the ring $\tilde{A}'_{n,N}/p\tilde{A}'_{N}$ is isomorphic to the ring

$$O_N/\{x \in O_N | v_p(x) > \frac{r}{p^{n-1}(p-1)}\}.$$

As in the proof of Lemma 5.9, we see that the homomorphism $\overline{\eta}$ induces an injection

$$\tilde{A}'_{L_n}/p\tilde{A}'_{L_n} \rightarrow \tilde{A}'_{F}/p\tilde{A}'_{F}.$$

Thus it is enough to show the exactness of the sequence

$$0 \rightarrow \tilde{A}'_{n}/p^m\tilde{A}'_{n} \rightarrow \tilde{A}'_{F}/p\tilde{A}'_{F} \rightarrow 0.$$

Let $\tilde{x}$ and $\tilde{y}$ be in $\tilde{A}'_{n}$ such that $p\tilde{x} = p^{m+1}\tilde{y}$. Let $\tilde{x} = (\tilde{x}_0, \ldots, \tilde{x}_{n-1})$ and $\tilde{y} = (\tilde{y}_0, \ldots, \tilde{y}_{n-1})$ be lifts of $\tilde{x}$ and $\tilde{y}$ in the ring $W_n(O_N)$, respectively. In this ring, we have

$$(0, \tilde{x}_0^p, \tilde{x}_1^p, \ldots) = (0, \ldots, 0, \tilde{y}_0^{m+1}, \tilde{y}_1^{m+1}, \ldots) + ([\zeta_{pn}] - 1)^r \tilde{z},$$

where $\tilde{z}$ is in the ideal $m_{n,N}$. From this equality we see that $v_p(\tilde{x}_0) > r/(p^{n-1}(p-1))$ and

$$\tilde{x} = ([\zeta_{pn}] - 1)^r \tilde{w} + (0, \tilde{x}_1', \tilde{x}_2', \ldots)$$

for some $\tilde{w} \in m_{n,N}$ and $\tilde{x}_i' \in O_N$. The image of the first term on the right-hand side in $\tilde{A}'_{n}$ is zero. Hence we may assume $\tilde{x}_0 = 0$. Repeating this, we can see that $\tilde{x} \in p^m\tilde{A}'_{n}$ and the above sequence is exact. \hfill $\Box$

Now Corollary 4.13 shows that the abelian group $T^{\text{crys},L_n,\pi_n}(M_n)$ is of order $p^d$, where $d = \dim_{q_p} V$. This implies that the the abelian group $T^{\text{crys},F,\pi_n}(M_n)$ is also of order $p^d$. Let $g \in G_K$ be as in Lemma 5.8. Then we have the following lemma.

**Lemma 5.12.** The $G_{F_n}$-action on $T^{\text{crys},K,\pi_n}(M_n)$ is the conjugate of the action on $T^{\text{crys},K,\pi_n}(M_n)$ by the element $g$.

**Proof.** Let $a_n, \tilde{a}_n$ and $\beta_n, \tilde{\beta}_n$ be the elements of $W_n(O_K/pO_K)$ as in the proof of Lemma 5.10. Let us consider the composite

$$\Sigma \rightarrow \tilde{A}'_{n,r} \rightarrow \tilde{A}'_{n,r+}$$

of the ring homomorphism defined by $u \mapsto [\pi_n]$ and $Y \mapsto -a_n\beta_n^{-1}E([\pi_n])p^{-1}$, and the map induced by $g$. We claim that this is the natural ring homomorphism defined by $\pi_n$. For this, we only have to check that this composite sends the element $Y \in \Sigma$ to $-\tilde{a}_n\tilde{\beta}_n^{-1}E([\pi_n])$. Since the equality

$$E([\pi_n])\beta_n = pa_n - ([\zeta_{pn+1}] - 1)^{p-1}$$

holds in the ring $\tilde{A}'_{n,r+}$ on the source of the above map $g$, we have

$$E([\pi_n])g(\beta_n) = p\tilde{a}_n - ([\tilde{\zeta}_{pn+1}] - 1)^{p-1}$$
in the ring $A'_{n,r^+}$ on the target. Since the elements $g(\beta_n)$ and $\tilde{\beta}_n$ are invertible, we have $g(\beta_n)^{-1}E([\tilde{\pi}_n]) = \tilde{\beta}_n^{-1}E([\tilde{\pi}_n])$ and the claim follows. Thus we have an isomorphism of abelian groups

$$\text{Hom}_\Sigma(M_n, \tilde{A}_{n,r}^I) \to \text{Hom}_\Sigma(M_n, \tilde{A}_{n,r^+}^I)$$

where we consider on the ring $\tilde{A}_{n,r^+}^I$ on the right-hand side the filtered $\phi_r$-module structure over $\Sigma$ defined by $\tilde{\pi}_n$. Since $g(t_n) = \tilde{t}_n$, we can check that this isomorphism induces an injection

$$\text{Hom}_{\Sigma, Fil^r, \phi_r}(M_n, \tilde{A}_{n,r}^I) \to \text{Hom}_{\Sigma, Fil^r, \phi_r}(M_n, \tilde{A}_{n,r^+}^I).$$

Since these abelian groups have the same cardinality, this is also an isomorphism.

Since $L_n$ is Galois over $K$, the above lemma shows that the finite Galois extension of $F_n$ cut out by the action on $T_{\text{crys}, K}^* (M_n)$ is also $L_n$. Hence we see from Corollary 4.13 that $F$ also contains $L_n$ and Proposition 5.7 follows. This concludes the proof of Theorem 1.1.

**Remark 5.13.** The ramification bound in Theorem 1.1 is sharp for $r \leq 1$. Indeed, the greatest upper ramification break $1 + e(n + 1/(p - 1))$ for $r = 1$ is obtained by the $p^n$-torsion of the Tate curve $K^* / \pi Z$ (see Remark 5.5). The author does not know whether this bound is sharp also for $r > 1$.

**References**


E-mail address: shin-h@math.sci.hokudai.ac.jp

Department of Mathematics, Hokkaido University