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ON A RAMIFICATION BOUND OF SEMI-STABLE TORSION REPRESENTATIONS OVER A LOCAL FIELD

SHIN HATTORI

Abstract. Let $p$ be a rational prime, $k$ be a perfect field of characteristic $p$, $W = W(k)$ be the ring of Witt vectors, $K$ be a finite totally ramified extension of $\text{Frac}(W)$ of degree $e$ and $r$ be a non-negative integer satisfying $r < p - 1$. Let $V$ be a semi-stable $p$-adic $G_K$-representation with Hodge-Tate weights in $\{0, \ldots, r\}$. In this paper, we prove the upper numbering ramification group $G^{(j)}_K$ for $j > u(K, r, n)$ acts trivially on the mod $p^n$ representations associated to $V$, where $u(K, 0, n) = 0$, $u(K, 1, n) = 1 + e(n + 1/(p - 1))$ and $u(K, r, n) = 1 - p^{-n}e(K(\zeta_p)/K)^{-1} + e(n + r/(p - 1))$ for $r > 1$.

1. Introduction

Let $p$ be a rational prime, $k$ be a perfect field of characteristic $p$, $W = W(k)$ be the ring of Witt vectors and $K$ be a finite totally ramified extension of $K_0 = \text{Frac}(W)$ of degree $e = e(K)$. Let the maximal ideal of $K$ be denoted by $m_K$, an algebraic closure of $K$ by $\bar{K}$ and the absolute Galois group of $K$ by $G_K = \text{Gal}(\bar{K}/K)$. We normalize the valuation $v_K$ of $K$ as $v_K(p) = e$ and extend this to $\bar{K}$. Let $G^{(j)}_K$ denote the $j$-th upper numbering ramification group in the sense of [7]. Namely, we put $G^{(j)}_K = G^{(j)}_{K, 1}$, where the latter is the upper numbering ramification group defined in [15].

Let $X_K$ be a proper smooth scheme over $K$ and put $X_{\bar{K}} = X_K \times_K \bar{K}$. Consider the $r$-th étale cohomology group $H^r_\text{ét}(X_{\bar{K}}, \mathbb{Q}_p)$ and its $G_K$-stable $\mathbb{Z}_p$-lattices $\mathcal{L} \supsetneq \mathcal{L}'$. In [7], Fontaine conjectured the upper numbering ramification group $G^{(j)}_K$ acts trivially on the $G_K$-module $\mathcal{L}/\mathcal{L}'$ for $j > e(n + r/(p - 1))$ if $X_K$ has good reduction and this module is killed by $p^n$. For $e = 1$ and $r < p - 1$, this conjecture was proved independently by himself ([8], for $n = 1$) and Abrashkin ([2], for any $n$), using the theory of Fontaine-Laffaille ([10]) and the comparison theorem of Fontaine-Messing ([11]) between the $p$-adic étale cohomology groups of $X_K$ and the crystalline cohomology groups of the reduction of $X_K$. From this result, Fontaine also showed some rareness of a proper smooth scheme over $\mathbb{Q}$ with everywhere good reduction ([8, Théorème 1]). In fact, they proved this ramification bound for the torsion

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representations of the crystalline $p$-adic representations of $G_K$ with Hodge-Tate weights in \{0, \ldots, r\} in the case where $K$ is absolutely unramified.

On the other hand, for a semi-stable $p$-adic representation $V$ with Hodge-Tate weights in the same range, a similar ramification bound for $e = 1$ and $n = 1$ is obtained by Breuil (see [4, Proposition 9.2.2.2]). He showed, assuming the Griffiths transversality which in general does not hold, that if $e = 1$ and $r < p - 1$, then the ramification group $G^{(j)}_K$ acts trivially on the mod $p$ representations of $V$ for $j > 2 + 1/(p - 1)$.

In this paper, we prove a version of the result of Breuil for the case where $K$ is absolutely ramified, under the condition $r < p - 1$. Our main theorem is the following.

**Theorem 1.1.** Let $r$ be a non-negative integer such that $r < p - 1$. Let $V$ be a semi-stable $p$-adic $G_K$-representation with Hodge-Tate weights in \{0, \ldots, r\} and $\mathcal{L} \supseteq \mathcal{L}'$ be $G_K$-stable $\mathbb{Z}_p$-lattices in $V$. Suppose that the quotient $\mathcal{L}/\mathcal{L}'$ is killed by $p^n$. Then the $j$-th upper numbering ramification group $G^{(j)}_K$ acts trivially on the $G_K$-module $\mathcal{L}/\mathcal{L}'$ for $j > u(K, r, n)$, where

$$u(K, r, n) = \begin{cases} 
0 & (r = 0), \\
1 + e(n + \frac{1}{p-1}) & (r = 1), \\
1 - \frac{1}{p^n e(K(\zeta_p)/K)} + e(n + \frac{r}{p-1}) & (r > 1)
\end{cases}$$

and $e(K(\zeta_p)/K)$ denotes the relative ramification index of the extension $K(\zeta_p)/K$.

We can check that this bound is sharp for $r \leq 1$ (Remark 5.13). From this theorem and [7, Proposition 1.3], we have the following corollary.

**Corollary 1.2.** Let the notation be as in the theorem and $L$ be the finite extension of $K$ cut out by the $G_K$-module $\mathcal{L}/\mathcal{L}'$. Let $D_{L/K}$ denote the different of the extension $L/K$. Then we have the inequality

$$v_K(D_{L/K}) < u(K, r, n)$$

for $r > 0$ and $v_K(D_{L/K}) = 0$ for $r = 0$.

For the proof of Theorem 1.1, we essentially follow a beautiful argument of Abrashkin ([2]). We may assume $p \geq 3$ and $r \geq 1$. Thanks to Liu’s theorem ([14]) on the $G_K$-stable $\mathbb{Z}_p$-lattices in semi-stable $p$-adic representations, it is enough to bound the ramification of the $G_K$-module

$$T^*_\text{st,}\mathbb{Z}(\mathcal{M}_n) = \text{Hom}_{S,\text{Fil}^r,\phi,N}(\mathcal{M}_n, \hat{A}_{\text{st,}\infty}),$$

where $\mathcal{M}_n$ is a $p^n$-torsion object of a category $\text{Mod}^\dagger_{S,\text{Fil}^r,\phi,N}$ of filtered $(\phi_r, N)$-modules over $S$ defined by Breuil ([3]) and $\hat{A}_{\text{st,}\infty}$ is a $p$-adic period ring. We may also assume $\zeta_p \in K$ and consider the finite Galois extension

$$F_n = K(\pi^{1/p^n}, \zeta_{p^{n+1}})$$

of $K$ whose upper ramification is bounded by the same value as in the theorem. Let $L_n$ be the finite Galois extension of $F_n$ cut out by $T^*_\text{st,}\mathbb{Z}(\mathcal{M}_n)|_{G_{F_n}}$. 

Then we bound the ramification of $L_n$ over $K$. For this, we show that to study this $G_{F_n}$-module we can use a variant over a smaller coefficient ring $\Sigma$ of filtered $(\phi_r, N)$-modules over $S$. In precise, let $E(u)$ be the Eisenstein polynomial of a uniformizer $\pi$ of $K$ over $W$ and we set

$$\Sigma = W[[u, E(u)^p/p]].$$

This ring $\Sigma$ is small enough for the method of Abrashkin, in which he uses filtered modules of Fontaine-Laffaille ([10]) whose coefficient ring is $W$, to work also in the case where $K$ is absolutely ramified.

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2. Filtered $(\phi_r, N)$-modules of Breuil

In this section, we recall the theory of filtered $(\phi_r, N)$-modules over $S$ of Breuil, which is developed by himself and most recently by Caruso and Liu (see for example [3], [5], [14], [6]). In what follows, we always take the divided power envelope of a $W$-algebra with the compatibility condition with the natural divided power structure on $pW$.

Let $r \in \{0, \ldots, p - 2\}$ be an integer. Set $\text{Mod}^{\phi_r, N}_{/S}$ to be the category consisting of the following data:

- an $S$-module $\mathcal{M}$ and its $S$-submodule $\text{Fil}^r \mathcal{M}$ containing $\text{Fil}^r S \mathcal{M}$,
- a $\phi$-semilinear map $\phi_r : \text{Fil}^r \mathcal{M} \to \mathcal{M}$ satisfying $\phi_r(s_r m) = \phi_r(s_r)\phi(m)$ for any $s_r \in \text{Fil}^r S$ and $m \in \mathcal{M}$, where we set $\phi(m) = c^{-r} \phi_r(E(u)^r m)$,
- a $W$-linear map $N : \mathcal{M} \to \mathcal{M}$ such that
  - $N(sm) = N(s)m + sN(m)$ for any $s \in S$ and $m \in \mathcal{M}$,
  - $E(u)N(\text{Fil}^r \mathcal{M}) \subseteq \text{Fil}^r \mathcal{M}$,
– the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Fil}^r M & \xrightarrow{\phi_r} & M \\
E(u)N & \downarrow & eN \\
\text{Fil}^r M & \xrightarrow{\phi_r} & M,
\end{array}
\]

and the morphisms of \(\text{Mod}^{r,\phi,N}_{/S}\) are defined to be the \(S\)-linear maps preserving \(\text{Fil}^r\) and commuting with \(\phi_r\) and \(N\). The category defined in the same way but dropping the data \(N\) is denoted by \(\text{Mod}^{r,\phi}_{/S}\). These categories have obvious notions of exact sequences. Let \(\text{Mod}^{r,\phi,N}_{/S_1}\) denote the full subcategory of \(\text{Mod}^{r,\phi,N}_{/S}\) consisting of \(M\) such that \(M\) is free of finite rank over \(S_1\) and generated as an \(S_1\)-module by the image of \(\phi_r\). We write \(\text{Mod}^{r,\phi,N}_{/S_\infty}\) for the smallest full subcategory which contains \(\text{Mod}^{r,\phi,N}_{/S_1}\) and is stable under extensions. We let \(\text{Mod}^{r,\phi,N}_{/S}\) denote the full subcategory consisting of \(M\) such that

- the \(S\)-module \(M\) is free of finite rank and generated by the image of \(\phi_r\),
- the quotient \(M/\text{Fil}^r M\) is \(p\)-torsion free.

We define full subcategories \(\text{Mod}^{r,\phi}_{/S_1}\), \(\text{Mod}^{r,\phi}_{/S_\infty}\) and \(\text{Mod}^{r,\phi}_{/S}\) of \(\text{Mod}^{r,\phi,N}_{/S}\) in a similar way. For \(\tilde{\mathcal{M}} \in \text{Mod}^{r,\phi,N}_{/S_1}\) (resp. \(\text{Mod}^{r,\phi}_{/S_\infty}\)), the quotient \(\tilde{\mathcal{M}}/p^n\tilde{\mathcal{M}}\) has a natural structure as an object of \(\text{Mod}^{r,\phi,N}_{/S_1}\) (resp. \(\text{Mod}^{r,\phi}_{/S_\infty}\)).

For \(p\)-torsion objects, we also have the following categories. Consider the \(k\)-algebra \(k[u]/(u^p) \cong S_1/\text{Fil}^p S_1\) and let this algebra be denoted by \(\tilde{S}_1\). The algebra \(\tilde{S}_1\) is equipped with the natural filtration, \(\phi\) and \(N\) induced by those of \(S\). Namely, \(\text{Fil}^r \tilde{S}_1 = u^r \tilde{S}_1\), \(\phi(u) = u^p\) and \(N(u) = -u\). Let \(\text{Mod}^{r,\phi,N}_{/\tilde{S}_1}\) denote the category consisting of the following data:

- an \(\tilde{S}_1\)-module \(\tilde{\mathcal{M}}\) and its \(\tilde{S}_1\)-submodule \(\text{Fil}^r \tilde{\mathcal{M}}\) containing \(u^r \tilde{\mathcal{M}}\),
- a \(\phi\)-semilinear map \(\phi_r : \text{Fil}^r \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}\),
- a \(k\)-linear map \(N : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}\) such that
  - \(N(sm) = N(s)m + sN(m)\) for any \(s \in \tilde{S}_1\) and \(m \in \tilde{\mathcal{M}}\),
  - \(u^r N(\text{Fil}^r \tilde{\mathcal{M}}) \subseteq \text{Fil}^r \tilde{\mathcal{M}}\),
- the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Fil}^r \tilde{\mathcal{M}} & \xrightarrow{\phi_r} & \tilde{\mathcal{M}} \\
u^*N & \downarrow & eN \\
\text{Fil}^r \tilde{\mathcal{M}} & \xrightarrow{\phi_r} & \tilde{\mathcal{M}}.
\end{array}
\]
and whose morphisms are defined as before. Its full subcategory $\text{Mod}_{S_1}^{r, N}$ is defined by the following condition:

- As an $S_1$-module, $\hat{\mathcal{M}}$ is free of finite rank and generated by the image of $\phi_r$.

We define categories $\text{Mod}_{S_1}^{r, \phi}$ and $\text{Mod}_{S_1}^{r, \phi, N}$ similarly.

Let $D$ be a weakly admissible filtered ($\phi, N$)-module over $K$ satisfying $\text{Fil}^0D_K = D_K$ and $\text{Fil}^{r+1}D_K = 0$. Set $S_{K_0} = S \otimes W K_0$ and $D = D \otimes_{K_0} S_{K_0}$. Then the $S_{K_0}$-module $D$ is equipped with the natural $\phi$-semilinear map $\phi \otimes \sigma$ and $K_0$-linear derivation $N \otimes 1 + 1 \otimes N$, which are denoted by $\phi$ and $N$, respectively. We define a filtration on $D$ inductively by $\text{Fil}^0D = D$ and

$$\text{Fil}^{i+1}D = \{ x \in D \mid N(x) \in \text{Fil}^iD \text{ and } f_\pi(x) \in \text{Fil}^{i+1}D_K \},$$

where $f_\pi : D \to D_K$ is induced by the map $S \to \mathcal{O}_K$ sending $u$ to $\pi$. An $S$-submodule $\hat{\mathcal{M}}$ of $D$ is said to be a strongly divisible lattice of $D$ if the following conditions are satisfied:

- the $S$-module $\hat{\mathcal{M}}$ is free of finite rank,
- $\hat{\mathcal{M}} \otimes W K_0 = D$,
- $\hat{\mathcal{M}}$ is stable under $\phi$ and $N$,
- $\phi(\text{Fil}^r\hat{\mathcal{M}}) \subseteq p^r\hat{\mathcal{M}}$, where we set $\text{Fil}^r\hat{\mathcal{M}} = \hat{\mathcal{M}} \cap \text{Fil}^rD$.

We put $\phi_r = p^{-r} \phi |_{\text{Fil}^r\hat{\mathcal{M}}}$. Then the $S$-module $\hat{\mathcal{M}}$ is generated by $\phi_r(\text{Fil}^r\hat{\mathcal{M}})$ ([3, Proposition 2.1.3]) and we can consider $\hat{\mathcal{M}}$ as an object of $\text{Mod}_{S_1}^{r, \phi, N}$.

Let $A_{\text{crys}}$ and $\hat{A}_{\text{st}}$ be $p$-adic period rings. These are constructed as follows. Set $R$ to be the ring

$$R = \varprojlim (\mathcal{O}_K / p\mathcal{O}_K \leftarrow \mathcal{O}_K / p\mathcal{O}_K \leftarrow \cdots),$$

where every arrow is the $p$-power map. For an element $x = (x_i)_{i \in \mathbb{Z}_{\geq 0}} \in R$ and an integer $n \geq 0$, we set

$$x^{(n)} = \lim_{m \to \infty} \hat{x}^{p^m + m} \in \mathcal{O}_C,$$

where $\hat{x}_i$ is a lift of $x_i$ in $\mathcal{O}_K$ and $\mathcal{O}_C$ is the $p$-adic completion of $\mathcal{O}_K$. Let $v_p$ denote the valuation of $\mathcal{O}_C$ normalized as $v_p(p) = 1$. Then the ring $R$ is a complete valuation ring whose valuation of an element $x \in R$ is given by $v_R(x) = v_p(x^{(0)})$. We define a natural ring homomorphism $\theta$ by

$$\theta : W(R) \to \mathcal{O}_C$$

$$(x_0, x_1, \ldots) \mapsto \sum_{n \geq 0} p^n x_n^{(n)}.$$

Then $A_{\text{crys}}$ is the $p$-adic completion of the divided power envelope of $W(R)$ with respect to the principal ideal $\text{Ker}(\theta)$ and $\hat{A}_{\text{st}}$ is the $p$-adic completion of the divided power polynomial ring $A_{\text{crys}}(X)$ over $A_{\text{crys}}$. We set $A_{\text{crys}} = A_{\text{crys}} \otimes W K_0 / W$ and $\hat{A}_{\text{st}} = \hat{A}_{\text{st}} \otimes W K_0 / W$. Put $\pi = (\pi_n)_{n \in \mathbb{Z}_{\geq 0}} \in R$, where we abusively let $\pi_n$ denote the image of $\pi_n \in \mathcal{O}_K$ in $\mathcal{O}_K / p\mathcal{O}_K$. These rings
are considered as $S$-algebras by the ring homomorphisms $S \to \hat{A}_{st}$ and $\hat{A}_{st} \to A_{crys}$ which are defined by $u \mapsto \lfloor \pi \rfloor/(1 + X)$ and $X \mapsto 0$, respectively. The ring $A_{crys}$ is endowed with a natural filtration induced by the divided power structure, a natural Frobenius endomorphism $\phi$ and the $\phi$-semilinear map $\phi_t = p^{-t}\phi|_{\text{Fil}^t A_{crys}}$. With these structures, $A_{crys}$ and $A_{crys, \infty}$ are considered as objects of $\text{Mod}^S_{/S}$. Moreover, the absolute Galois group $G_K$ acts naturally on these two rings. As for $\hat{A}_{st}$, its filtration is defined by

$$\text{Fil}^t \hat{A}_{st} = \left\{ \sum_{i \geq 0} a_i X^i \middle| a_i \in \text{Fil}^{t-i} A_{crys}, \lim_{i \to \infty} a_i = 0 \right\}$$

and the Frobenius structure of $A_{crys}$ extends to $\hat{A}_{st}$ by

$$\phi(X) = (1 + X)^p - 1,$$

$$\phi_t = p^{-t}\phi|_{\text{Fil}^t \hat{A}_{st}}.$$ 

We write $N$ also for the $A_{crys}$-linear derivation on $\hat{A}_{st}$ defined by $N(X) = 1 + X$. The rings $\hat{A}_{st}$ and $\hat{A}_{st, \infty}$ are objects of $\text{Mod}^S_{/S}$. The $G_K$-action on $A_{crys}$ naturally extends to an action on $\hat{A}_{st}$. Indeed, the action of $g \in G_K$ on $\hat{A}_{st}$ is defined by the formula

$$g(X) = [\varepsilon(g)](1 + X) - 1,$$

where $g(\pi_n) = \varepsilon_n(g)\pi_n$ and $\varepsilon(g) = (\varepsilon_n(g))_{n \in \mathbb{Z}_{>0}} \in R$ with the abusive notation as above.

These rings have other descriptions, as follows. For an integer $n \geq 1$, put $W_n = W/p^nW$ and let $W_n(\bar{O}_K/p\bar{O}_K)$ be the ring of Witt vectors of length $n$ associated to $\bar{O}_K/p\bar{O}_K$. We define a $W_n$-algebra structure on $W_n(\bar{O}_K/p\bar{O}_K)$ by twisting the natural $W_n$-algebra structure by $\sigma^{-n}$. Then the natural ring homomorphism

$$\theta_n : W_n(\bar{O}_K/p\bar{O}_K) \to \bar{O}_K/p^n\bar{O}_K$$

$$(a_0, \ldots, a_{n-1}) \mapsto \sum_{i=0}^{n-1} p^i \hat{a}_i p^{-n-i},$$

where $\hat{a}_i$ is a lift of $a_i$ in $\bar{O}_K$, is $W_n$-linear. Let us denote

$$W_n^{PD}(\bar{O}_K/p\bar{O}_K)$$

the divided power envelope of $W_n(\bar{O}_K/p\bar{O}_K)$ with respect to the ideal $\text{Ker}(\theta_n)$. This ring is considered as an $S$-algebra by $u \mapsto [\pi_n]$. This ring also has a natural filtration defined by the divided power structure, and a natural $G_K$-module structure. The Frobenius endomorphism of the ring of Witt vectors
induces on this ring a $\phi$-semilinear Frobenius endomorphism, which is denoted also by $\phi$. Then, by the $S$-linear transition maps
\[
W_{n+1}^{\text{PD}}(O_K/pO_K) \to W_n^{\text{PD}}(O_K/pO_K)
\]
\[(a_0, \ldots, a_n) \mapsto (a_0^p, \ldots, a_n^p),\]
these $S$-algebras form a projective system compatible with all structures. Using this transition map, a $\phi$-semilinear map
\[
\phi_r : \text{Fil}^r W_n^{\text{PD}}(O_K/pO_K) \to W_n^{\text{PD}}(O_K/pO_K)
\]
is defined by setting $\phi_r(x)$ to be the image of $p^{-r}\phi(\hat{x})$, where $\hat{x}$ is a lift of $x$ in $\text{Fil}^r W_{n+r}(O_K/pO_K)$. By definition, the maps $\phi_r$ are also compatible with the transition maps. The $S$-algebra $W_n^{\text{PD}}(O_K/pO_K)$ is considered as an object of $'\text{Mod}^r_*/S$. Then we have a natural isomorphism in $'\text{Mod}^r_*/S$
\[
A_{\text{cris}}/p^n A_{\text{cris}} \to W_n^{\text{PD}}(O_K/pO_K)
\]
\[(x_0, \ldots, x_{n-1}) \mapsto (x_{0,n}, \ldots, x_{n-1,n}),\]
where we set $x_i = (x_i)_k \in \mathbb{Z}_p$.

Similarly, the divided power polynomial ring
\[
W_n^{\text{PD}}(O_K/pO_K)(X)
\]
over $W_n^{\text{PD}}(O_K/pO_K)$ is considered as an $S$-algebra by $u \mapsto [\pi_n]/(1 + X)$. This ring has a natural filtration coming from the divided power structure. We define a $G_K$-action on this ring by
\[
g(X) = [\varepsilon_n(g)](1 + X) - 1.
\]
We also define a $\phi$-semilinear Frobenius endomorphism, which we also write as $\phi$, by $\phi(X) = (1 + X)^r - 1$ and a $W_n^{\text{PD}}(O_K/pO_K)$-linear derivation $N$ by $N(X) = 1 + X$. These rings form a projective system of $S$-algebras compatible with all structures by the transition maps defined by the maps above and $X \mapsto X$. We define $\phi$-semilinear maps
\[
\phi_r : \text{Fil}^r W_n^{\text{PD}}(O_K/pO_K)(X) \to W_n^{\text{PD}}(O_K/pO_K)(X)
\]
compatible with the transition maps as before. The $S$-algebra $W_n^{\text{PD}}(O_K/pO_K)(X)$ is considered as an object of $'\text{Mod}^r_*/S$ and there exists a natural isomorphism in $'\text{Mod}^r_*/S$
\[
\hat{A}_{\text{st}}/p^n \hat{A}_{\text{st}} \to W_n^{\text{PD}}(O_K/pO_K)(X)
\]
\[(x_0, \ldots, x_{n-1}) \mapsto (x_{0,n}, \ldots, x_{n-1,n})
\]
\[X \mapsto X\]
which is $G_K$-linear.

Put $K_n = K(\pi_n)$ and $K_\infty = \bigcup_n K_n$. For $M \in \text{Mod}^r_*/S_\infty$, we define a $G_K$-module $T^*_\text{st,}M(M)$ to be
\[
T^*_\text{st,}M(M) = \text{Hom}_{S,\text{Fil}^r_*,N}(M, \hat{A}_{\text{st,}\infty}).
\]
When $\mathcal{M}$ is killed by $p^n$, we have a natural identification of $G_K$-modules

$$T^\ast_{\text{st}, \underline{\mathbb{Z}}}(\mathcal{M}) = \text{Hom}_{\text{S,Fil}, r, \phi, N}(\mathcal{M}, W_n^\text{PD}(\mathcal{O}_K/\mathcal{O}_K)(X)).$$

Note that the $G_K$-module on the right-hand side is independent of the choice of $\pi_k$ for $k > n$. Since the natural map

$$W_n^\text{PD}(\mathcal{O}_K/\mathcal{O}_K)(X) \to W_n^\text{PD}(\mathcal{O}_K/\mathcal{O}_K)$$

$X \mapsto 0$

is by definition $G_{K_n}$-linear, we also have a $G_{K_n}$-linear isomorphism ([3, Lemme 2.3.1.1])

$$T^\ast_{\text{st}, \underline{\mathbb{Z}}}(\mathcal{M})|_{G_{K_n}} \to \text{Hom}_{\text{S,Fil}, r, \phi, N}(\mathcal{M}, W_n^\text{PD}(\mathcal{O}_K/\mathcal{O}_K)).$$

A variant of filtered $(\phi, r, N)$-modules over $S$ is also introduced by Breuil and Kisin, and developed also by Caruso and Liu (see for example [12], [13], [14], [6]). Put $\mathcal{S} = W[[u]]$ and let $\phi : \mathcal{S} \to \mathcal{S}$ be the $\sigma$-semilinear Frobenius endomorphism defined by $\phi(u) = u^p$. Let $'\text{Mod}^r_{/\mathcal{S}}$ denote the category consisting of the following data:

- an $\mathcal{S}$-module $\mathfrak{M}$,
- a $\phi$-semilinear map $\mathfrak{M} \to \mathfrak{M}$, which is denoted also by $\phi$, such that the cokernel of the map $1 \otimes \phi : \phi^\ast \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$,

and whose morphisms are defined as before. The full subcategory of $'\text{Mod}^r_{/\mathcal{S}}$ consisting of $\mathfrak{M}$ such that $\mathfrak{M}$ is free of finite rank over $\mathcal{S}/p\mathcal{S}$ (resp. over $\mathcal{S}$) is denoted by $\text{Mod}^r_{/\mathcal{S}_1}$ (resp. $\text{Mod}^r_{/\mathcal{S}}$). We let $\text{Mod}^r_{/\mathcal{S}}$ denote the smallest full subcategory which contains $\text{Mod}^r_{/\mathcal{S}_1}$ and is stable under extensions, as before. Then we have an exact functor ([6, Proposition 2.1.2], see also [12, Proposition 1.1.11])

$$\mathcal{M}_{\mathcal{S}_\infty} : \text{Mod}^r_{/\mathcal{S}_\infty} \to \text{Mod}^r_{/\mathcal{S}}.$$

For $\mathfrak{M} \in \text{Mod}^r_{/\mathcal{S}_\infty}$, the filtered $\phi$-module $\mathcal{M} = \mathcal{M}_{\mathcal{S}_\infty}(\mathfrak{M})$ over $S$ is defined as follows:

- $\mathcal{M} = S \otimes_{\phi, \mathcal{S}} \mathfrak{M}$,
- $\text{Fil}^r_\mathcal{M} = \text{Ker}(\mathcal{M} \xrightarrow{1\otimes \phi} S \otimes_{\mathcal{S}} \mathfrak{M} \to (S/\text{Fil}^r S) \otimes_{\mathcal{S}} \mathfrak{M})$,
- $\phi_r : \text{Fil}^r_\mathcal{M} \xrightarrow{1\otimes \phi} \text{Fil}^r S \otimes_{\mathcal{S}} \mathfrak{M}$.

We write $\mathcal{M}_{\mathcal{S}}$ for the functor $\text{Mod}^r_{/\mathcal{S}} \to \text{Mod}^r_{/\mathcal{S}}$ defined similarly.

Finally, let $D$ and $\mathcal{D}$ be as above and $\hat{\mathcal{M}}$ be a strongly divisible lattice in $\mathcal{D}$. The $S$-module $\mathcal{M}_n = \hat{\mathcal{M}}/p^n \hat{\mathcal{M}}$ has a natural structure as an object of $\text{Mod}^r_{/S_\infty}$. We set a $G_K$-module $\tilde{T}^\ast_{\text{st}, \underline{\mathbb{Z}}}(\hat{\mathcal{M}})$ to be

$$\tilde{T}^\ast_{\text{st}, \underline{\mathbb{Z}}}(\hat{\mathcal{M}}) = \text{Hom}_{\text{S,Fil}, r, \phi, N}(\hat{\mathcal{M}}, \hat{A}_{\text{st}}).$$

Then we have an exact sequence of $G_K$-modules

$$0 \to \tilde{T}^\ast_{\text{st}, \underline{\mathbb{Z}}}(\hat{\mathcal{M}}) \xrightarrow{p^n} \tilde{T}^\ast_{\text{st}, \underline{\mathbb{Z}}}(\hat{\mathcal{M}}) \to \tilde{T}^\ast_{\text{st}, \underline{\mathbb{Z}}}(\mathcal{M}_n) \to 0.$$
The $G_K$-module $\hat{T}^*_{st, \xi}(\mathcal{M})$ is naturally considered as a $G_K$-stable $\mathbb{Z}_p$-lattice in $V^*_\xi(D)$. By Liu’s theorem ([14, Theorem 2.3.5]), the functor $\hat{T}^*_{st, \xi}$ gives an anti-equivalence of categories between the category of strongly divisible lattices in $\mathcal{D}$ and the category of $G_K$-stable $\mathbb{Z}_p$-lattices in $V^*_\xi(D)$. Moreover, for such a lattice $\mathcal{L}$, its corresponding strongly divisible lattice $\hat{\mathcal{M}}$ in $\mathcal{D}$ is in the essential image of the functor $\mathcal{M}_\xiS$ ([14, Subsection 3.5]).

3. Filtered $\phi\tau$-modules over $\Sigma$

In this section, we define another variant of filtered $\phi\tau$-modules over $S$ and prove its properties.

Let $p \geq 3$ be a rational prime and $\tau$ be an integer such that $0 \leq \tau < p - 1$. Consider the $W$-algebra $\Sigma = W[[u, Y]]/(E(u)p - pY)$ as in [3, Subsection 3.2]. We regard $\Sigma$ as a subring of $S$ by the map sending $Y$ to $E(u)p/p$. Then the element $c = \phi_1(E(u)) \in S^\times$ is contained in $\Sigma^\times$. We define on $\Sigma$ a $\sigma$-semilinear Frobenius endomorphism $\phi$ by $\phi(u) = u^p$ and $\phi(Y) = p^{\tau-1}c^\phi$. Put $\text{Fil}^t\Sigma = (E(u)^t, Y)$ for $0 \leq t \leq p - 1$ and $\text{Fil}^p\Sigma = (Y)$. Then we have $\phi(\text{Fil}^t\Sigma) \subseteq p^t\Sigma$ for $0 \leq t \leq p - 1$. We put $\phi_t = p^{-\tau}\phi|_{\text{Fil}^t\Sigma}$. We also set $\Sigma_t = \Sigma/p^n\Sigma$ and put on this ring the natural structures induced by those of $\Sigma$.

We define a category $\text{Mod}^r_{\Sigma_S}$ of filtered $\phi\tau$-modules over $\Sigma$ to be the category consisting of the following data:

- a $\Sigma$-module $M$ and its $\Sigma$-submodule $\text{Fil}^r\Sigma M$ containing $\text{Fil}^r\Sigma M$,
- a $\phi$-semilinear map $\text{Fil}^rM \to M$ satisfying $\phi_r(s_r m) = \phi_r(s_r)\phi(m)$ for any $s_r \in \text{Fil}^r\Sigma$ and $m \in M$, where we set $\phi(m) = c^{-\tau}\phi_r(E(u)^r m)$.

and the morphisms are defined in the same manner as $\text{Mod}^r_{/S}$. This category has a natural notion of exact sequences. We define its full subcategory $\text{Mod}^r_{/\Sigma}$ to be the category consisting of $M$ which is free of finite rank and generated by the image of $\phi_r$ as a $\Sigma_1$-module. We also let $\text{Mod}^r_{/\Sigma_{\infty}}$ denote the smallest full subcategory of $\text{Mod}^r_{/\Sigma}$ which contains $\text{Mod}^r_{/\Sigma_1}$ and is stable under extensions. Moreover, we define a full subcategory $\text{Mod}^r_{/\Sigma}$ of $\text{Mod}^r_{/\Sigma}$ to be the category consisting of $M$ such that

- the $\Sigma$-module $M$ is free of finite rank and generated by the image of $\phi_r$,
- the quotient $M/\text{Fil}^rM$ is $p$-torsion free.

Then we see that for $\hat{M} \in \text{Mod}^r_{/\Sigma}$, the quotient $\hat{M}/p^n\hat{M}$ is naturally considered as an object of $\text{Mod}^r_{/\Sigma_{\infty}}$.

The natural ring isomorphism $\Sigma_1/\text{Fil}^p\Sigma_1 \cong \tilde{S}_1$ defines a functor $T_0 : \text{Mod}^r_{/\Sigma_1} \to \text{Mod}^r_{/\tilde{S}_1}$ by $M \mapsto M/\text{Fil}^p\Sigma_1 M$. Then [3, Proposition 2.2.1.3] and Nakayama’s lemma shows the following.
Lemma 3.1. Let $M$ be an object of $\text{Mod}^{r,\phi}_{/\Sigma_1}$ of rank $d$ over $\Sigma_1$. Then there exists a basis $\{e_1, \ldots, e_d\}$ of $M$ such that $\text{Fil}^r M = \Sigma_1 u_1^{r_1} e_1 + \cdots + \Sigma_1 u_d^{r_d} e_d + \text{Fil}^p \Sigma_1 M$ for some integers $r_1, \ldots, r_d$ with $0 \leq r_i \leq c r$ for any $i$.

Then we can show the following lemma just as in the proof of [3, Lemme 2.3.1.3].

Lemma 3.2. The functor $\xrightarrow{\text{Hom}}_{\Sigma} M \mapsto \text{Hom}_{\Sigma} \text{Fil}^r_{/\Sigma} (M, A_{\text{crys},\infty})$ from $\text{Mod}^{r,\phi}_{/\Sigma_1}$ to the category of $G_{K_1}$-modules is exact.

For $M \in \text{Mod}^{r,\phi}_{/\Sigma_1}$, we can show as in the case of the category $\text{Mod}^{r,\phi}_{/S_1}$ that there is an isomorphism of $G_{K_1}$-modules

$$\text{Hom}_{\Sigma} \text{Fil}^r_{/\Sigma} (M, (\mathcal{O}_K/p\mathcal{O}_K)^{\text{PD}}) \to \text{Hom}_{S_1} \text{Fil}^r_{/\Sigma} (T_0(M), \mathcal{O}_K/p\mathcal{O}_K),$$

where $\mathcal{O}_K/p\mathcal{O}_K$ is considered as an object of $\text{Mod}^{r,\phi}_{S_1}$ by the natural isomorphism

$$(\mathcal{O}_K/p\mathcal{O}_K)^{\text{PD}} / \text{Fil}^p (\mathcal{O}_K/p\mathcal{O}_K)^{\text{PD}} \to \mathcal{O}_K/p\mathcal{O}_K.$$ 

Thus [3, Lemme 2.3.1.2] implies that, for such a $\Sigma_1$-module $M$, we have

$$\# \text{Hom}_{\Sigma} \text{Fil}^r_{/\Sigma} (M, (\mathcal{O}_K/p\mathcal{O}_K)^{\text{PD}}) = p^d,$$

where $d = \dim_{\Sigma_1} M$.

For the category $\text{Mod}^{r,\phi}_{/\Sigma}$, we can show the following lemma just as in the proof of [14, Proposition 4.1.2].

Lemma 3.3. Let $\hat{M}$ be in $\text{Mod}^{r,\phi}_{/\Sigma}$. Then there exists $\alpha_1, \ldots, \alpha_d \in \hat{M}$ such that $\text{Fil}^r \hat{M} = \Sigma_1 \alpha_1 + \cdots + \Sigma_1 \alpha_d + \text{Fil}^p \Sigma \hat{M}$, $E(u)^r \hat{M} \subseteq \Sigma_1 \alpha_1 + \cdots + \Sigma_1 \alpha_d$ and the elements $e_1 = \phi_r (\alpha_1), \ldots, e_d = \phi_r (\alpha_d)$ form a basis of $\hat{M}$.

Corollary 3.4. Let $\hat{M}$ be in $\text{Mod}^{r,\phi}_{/\Sigma}$ and $A$ be a $\Sigma$-algebra which has a structure as an object of $\text{Mod}^{r,\phi}_{/\Sigma}$. Let $C \in M_d(\Sigma)$ be the matrix such that

$$(\alpha_1, \ldots, \alpha_d) = (e_1, \ldots, e_d) C$$

with the notation of the previous lemma. Then a $\Sigma$-linear homomorphism $f : \hat{M} \to A$ preserving $\text{Fil}^r$ also commutes with $\phi_r$ if and only if

$$\phi_r (f (e_1, \ldots, e_d) C) = (f (e_1), \ldots, f (e_d)).$$

Proof. Suppose that the latter condition holds. By assumption, we have

$$E(u)^r (e_1, \ldots, e_d) = (\alpha_1, \ldots, \alpha_d) C'.$$
Let \( \phi \) be denoted by \( \hat{\phi} \).

**Proposition 3.6.** We can check that this defines an exact functor \( \text{Mod}^r \Sigma \rightarrow \text{Mod}^r \Sigma \) such that \( \phi_r(J) \subseteq J \). We can consider the \( \Sigma \)-algebra \( A/J \) naturally as an object of \( \text{Mod}^r \Sigma \). Suppose that for any \( x \in J \), there exists \( t \in \mathbb{Z}_{\geq 0} \) such that \( \phi^t_r(x) = 0 \). Then we have an isomorphism

\[
\text{Hom}^r_{\Sigma, \text{Fil}^r\Sigma, \hat{\phi}}(\hat{M}, A) \rightarrow \text{Hom}^r_{\Sigma, \text{Fil}^r\Sigma, \hat{\phi}}(\hat{M}, A/J).
\]

**Proof.** Let \( f : \hat{M} \rightarrow A/J \) be an element of the abelian group on the right-hand side and \( \hat{x} \) be an lift of \( f(e_i) \) in \( A \). By the previous corollary, it is enough to show that for any \( (\hat{c}_1, \ldots, \hat{c}_d) \in J^d \), there is a unique solution \( (\hat{y}_1, \ldots, \hat{y}_d) \in J^d \) of the equation

\[
(\hat{c}_1, \ldots, \hat{c}_d) + (\phi_r(\hat{y}_1), \ldots, \phi_r(\hat{y}_d))\phi(C) = (\hat{y}_1, \ldots, \hat{y}_d).
\]

By assumption, the \( d \)-tuple

\[
\sum_{i=0}^{t}(\phi^t_i(\hat{c}_1), \ldots, \phi^t_i(\hat{c}_d))\phi(C)\phi^{i-1}(C) \cdots \phi(C)
\]

is stable for sufficiently large \( t \) and we see that this limit gives a unique solution of the equation. \( \square \)

For an \( \mathcal{S} \)-module \( \mathfrak{M} \) in \( \text{Mod}^r_{/\mathcal{S}, \Sigma} \) (resp. \( \text{Mod}^r_{/\mathcal{S}} \)), we associate to it a \( \Sigma \)-module \( M \in \text{Mod}^r_{/\Sigma} \) as follows:

- \( M = \Sigma \otimes_{\phi_r, \mathcal{S}} \mathfrak{M} \),
- \( \text{Fil}^r M = \text{Ker}(M \xrightarrow{1 \otimes \phi_r} \Sigma \otimes_{\mathcal{S}} \mathfrak{M} \rightarrow (\Sigma/\text{Fil}^r \Sigma) \otimes_{\mathcal{S}} \mathfrak{M}) \),
- \( \phi_r : \text{Fil}^r M \xrightarrow{1 \otimes \phi_r} \Sigma \otimes_{\phi_r, \mathcal{S}} \mathfrak{M} = M \).

We can check that this defines an exact functor \( \text{Mod}^r_{/\mathcal{S}, \Sigma} \rightarrow \text{Mod}^r_{/\Sigma} \) (resp. \( \text{Mod}^r_{/\mathcal{S}} \rightarrow \text{Mod}^r_{/\Sigma} \)) as in the proof of [12, Proposition 1.1.11]. We let this functor be denoted by \( M_{\mathcal{S}, \Sigma} \). 

**Proposition 3.6.** Let \( \mathfrak{M} \) be an object of \( \text{Mod}^r_{/\mathcal{S}, \Sigma} \) which is killed by \( p^n \). Set \( M = M_{\mathcal{S}, \Sigma, \Sigma}(\mathfrak{M}) \) and \( M = M_{\mathcal{S}, \Sigma}(\mathfrak{M}) \). Then there exists a natural isomorphism of \( G_K \)-modules

\[
\text{Hom}^r_{\Sigma, \text{Fil}^r, \hat{\phi}}(M, W^p_{/\mathcal{O}_K}) \rightarrow \text{Hom}^r_{\Sigma, \text{Fil}^r, \hat{\phi}}(M, W^p_{/\mathcal{O}_K}).
\]
Proof. By definition, \( \mathcal{M} = S \otimes_{\Sigma} M \) and we have a natural isomorphism
\[
\Hom_{\Sigma}(M, W_n^{\PD}(\mathcal{O}_K/p\mathcal{O}_K)) \to \Hom_{\Sigma}(\mathcal{M}, W_n^{\PD}(\mathcal{O}_K/p\mathcal{O}_K)).
\]
Let \( f \) be an element of \( \Hom_{\Sigma}^{\Fil'}(M, W_n^{\PD}(\mathcal{O}_K/p\mathcal{O}_K)) \) and \( f' \) be the image of \( f \) in the right-hand side of the above isomorphism. Let us check that \( f' \) preserves \( \Fil' \) and commutes with \( \phi_{r} \). Since \( f' \) is \( S \)-linear, it maps \( \Fil' S \mathcal{M} \) into \( \Fil' W_n^{\PD}(\mathcal{O}_K/p\mathcal{O}_K) \). For \( x \in \Fil' \mathcal{M} \cap \text{Im}(M \to \mathcal{M}) \), the commutative diagram whose right vertical arrow is an isomorphism
\[
\begin{array}{ccc}
M = \Sigma \otimes_{\phi_{r},\Sigma} \mathfrak{M} & \xrightarrow{1 \otimes \phi} & \Sigma \otimes_{\phi_{r},\Sigma} \mathfrak{M} \\
\downarrow & & \downarrow \\
\mathcal{M} = S \otimes_{\phi_{r},\Sigma} \mathfrak{M} & \xrightarrow{1 \otimes \phi} & S \otimes_{\phi_{r},\Sigma} \mathfrak{M}
\end{array}
\]
implies \( x \in \text{Im}(\Fil' M \to \Fil' \mathcal{M}) \) and thus \( f'(x) \in \Fil' W_n^{\PD}(\mathcal{O}_K/p\mathcal{O}_K) \).

As for the compatibility with \( \phi_{r} \), again by the \( S \)-linearity of \( f' \) it suffices to show \( f'(\phi_{r}(x)) = \phi_{r}(f'(x)) \) for \( x \in \Fil' \mathcal{M} \cap \text{Im}(M \to \mathcal{M}) = \text{Im}(\Fil' M \to \Fil' \mathcal{M}) \). This follows from the commutative diagram
\[
\begin{array}{ccc}
\Fil' M & \xrightarrow{\phi_{r}} & M \\
\downarrow & & \downarrow \\
\Fil' \mathcal{M} & \xrightarrow{\phi_{r}} & \mathcal{M}
\end{array}
\]
and \( f' \). Hence the map in the proposition is well-defined and injective. To prove the bijectivity, by devissage we may assume that \( p\mathfrak{M} = 0 \). Then both sides of this injection have the same cardinality by the above remark. Thus the proposition follows. \( \square \)

4. A METHOD OF ABRASHKIN

In this section, we study the \( G_{K_{s}} \)-module \( \Hom_{\Sigma}^{\Fil'_{r},\phi_{r}}(M, W_n^{\PD}(\mathcal{O}_K/p\mathcal{O}_K)) \) following Abrashkin ([2]).

Let \( p \geq 3 \) and \( 0 \leq r < p - 1 \) be as before. Consider the Lubin-Tate logarithm
\[
\log(X) = X + \frac{X^p}{p} + \frac{X^{p^2}}{p^2} + \cdots
\]
and put \( \psi(X) = \log^{-1}(1 + X) \). Then \( \psi \) gives a strict isomorphism of formal groups between the formal group associated to the logarithm \( \log(X) \) and the multiplicative group \( \mathbb{G}_{m} \) over \( \mathbb{Z}_{p} \). We fix a system of \( p \)-power roots of unity \( \{ \zeta_{p}^{n} \}_{n \in \mathbb{Z}_{\geq 0}} \) such that \( \zeta_{p} \neq 1 \) and \( \zeta_{p}^{n} = \zeta_{p}^{n+1} \) for any \( n \), and set an element \( \varepsilon \) of \( R \) to be \( (\zeta_{p}^{n})_{n \in \mathbb{Z}_{\geq 0}} \). Then the elements \( [\varepsilon] - 1 \) and \( [\varepsilon^{1/p}] - 1 \) are topologically nilpotent in \( W(R) \) and the element of \( W(R) \)
\[
t = \psi([\varepsilon] - 1)/\psi([\varepsilon^{1/p}] - 1)
\]
is a generator of the principal ideal \( \ker(\phi) \). The element \( Z = \psi(\frac{[\xi] - 1}{p}) \) of \( A_{\text{crys}} \) is topologically nilpotent and \( \phi(t) \) is contained in the subset
\[
p(1 + Z W(R)[[Z]])
\]
of \( A_{\text{crys}} ([2, \text{Subsection 1.8}]) \). We set
\[
\hat{A} = W(R)[[Z]] \subseteq A_{\text{crys}}.
\]

**Lemma 4.1.** The element \( tp/p \) of \( A_{\text{crys}} \) is contained in the subring \( \hat{A} \) and topologically nilpotent in this subring.

**Proof.** Put \( t' = (\frac{[\xi] - 1}{(\frac{[\xi]}{p}) - 1}) \). This is another generator of \( \ker(\phi) \). We have
\[
\frac{(\frac{[\xi] - 1}{p})^{p-1}}{p} = \frac{(t')^{p-1}}{p} \cdot (\frac{[\xi]}{p})^{p-1}
\]
and \( \theta((\frac{[\xi]}{p}) - 1) = \zeta_p - 1 \). Take an element \( a \in W(R)^\times \) such that \( \theta(a) = (\zeta_p - 1)^{p-1}/p \). Then we have
\[
\frac{(\frac{[\xi] - 1}{p})^{p-1}}{p} = a(t')^{p-1} + b(t')^p/p
\]
for some \( b \in W(R)^\times \). Indeed, to show \( b \in W(R)^\times \), it suffices to check that the element \( (\frac{[\xi]}{p})^{p-1} - pa \) of \( \ker(\phi) \) also generates this ideal. This follows from the fact that the 0-th entry \( (\frac{[\xi]}{p} - 1)^{p-1} \) of this element satisfies \( v_R((\frac{[\xi]}{p} - 1)^{p-1}) = 1 \). Then we see that \( (t')^p/p \) is topologically nilpotent because so is \( t' \) in \( W(R) \).

In the following, we set the element \( a \) in the proof of the lemma to be
\[
a = \sum_{k=0}^{p-2} p^{-1}(\frac{1}{p-1} - k)C_k - 1)\frac{[\xi]}{p}^k,
\]
where \( p^{-1}C_k = (p-1)!/(k!(p-1-k)!) \) is the binomial coefficient. Note that the coefficient of \( \frac{[\xi]}{p}^k \) in each term is an integer.

From this lemma, we can consider the ring \( \hat{A} \) as a \( \Sigma \)-algebra by \( u \mapsto [\xi] \). Put \( \text{Fil}'\hat{A} = (t', Z) \) for \( 0 \leq i \leq p-1 \). The Frobenius endomorphism \( \phi \) of \( A_{\text{crys}} \) preserves \( \hat{A} \) and satisfies \( \phi(\text{Fil}'\hat{A}) \subseteq p'\hat{A} \) for \( 0 \leq i \leq p-1 \). Set \( \phi_r = p^{-r}\phi|_{\text{Fil}'\hat{A}} \). Then we can consider the ring \( \hat{A} \) also as an object of the category \( \text{Mod}_{\Sigma}^{r}\phi \), and similarly for \( \hat{A}_n = \hat{A}/p^n\hat{A} \) and \( \hat{A}_\infty = \hat{A} \otimes W K_0/W \).

The absolute Galois group \( G_{K_\infty} \) acts naturally on these \( \Sigma \)-algebras. The following lemma is used implicitly in \([2]\).

**Lemma 4.2.** We have a natural decomposition
\[
\hat{A}_1 = R/(tp) \oplus (Z).
\]

**Proof.** Consider the natural inclusion \( W(R) \to \hat{A} \). First we claim that this induces an injection \( R/(tp) \to \hat{A}_1 \). Let \( x \) be in the ring \( R \). If the element
Let us consider the commutative diagram of $R$-algebras

\[
\begin{array}{ccc}
R/(p) & \longrightarrow & \hat{A}_1 \\
& \downarrow & \\
& \hat{A}_1/(Z).
\end{array}
\]

By definition, the left downward arrow is surjective. We claim that this arrow is an isomorphism. Indeed, let $x$ be in the kernel of this surjection. From the proof of Lemma 4.1, we see that the image of $Z$ in the ring on the left-hand side of the above isomorphism can be written as $a't^{p-1} + b'Y_1$ for some $a', b' \in R^\times$. By assumption, in this ring, we have

\[
x = c_1(a't^{p-1} + b'Y_1) + c_2(a't^{p-1} + b'Y_1)^2 + \cdots + c_{p-1}(a't^{p-1} + b'Y_1)^{p-1}
\]

for some elements $c_1, \ldots, c_{p-1}$ of $R$. Then we see that $c_i = 0$ for any $i$ and $v_R(x) \geq p$. This concludes the proof.

Since $r < p - 1$, from this lemma we can show the following lemma as in the proof of [3, Lemme 2.3.1.3].

**Lemma 4.3.** The functor

\[
M \mapsto \text{Hom}_{\Sigma, \text{Fil}, \phi_r}(M, \hat{A}_\infty)
\]

from $\text{Mod}_{\Sigma, \phi_r}^r$ to the category of $G_{K_\infty}$-modules is exact.

**Corollary 4.4.** For any $M \in \text{Mod}_{\Sigma, \phi_r}^r$, the natural map

\[
\text{Hom}_{\Sigma, \text{Fil}, \phi_r}(M, \hat{A}_\infty) \to \text{Hom}_{\Sigma, \text{Fil}, \phi_r}(M, A_{\text{crys}, \infty})
\]

is an isomorphism of $G_{K_\infty}$-modules.

**Proof.** By Lemma 3.2 and Lemma 4.3, we may assume $pM = 0$. Consider the commutative diagram of rings

\[
\begin{array}{ccc}
\hat{A}_1 & \longrightarrow & A_{\text{crys}}/pA_{\text{crys}} \\
& \downarrow & \\
& R/(t^{p-1}).
\end{array}
\]
whose downward arrows are defined by modulo $\text{Fil}^{p-1}$ of the rings $\hat{A}_1$ and $A_{\text{crys}}/pA_{\text{crys}}$, respectively. Since $r < p - 1$, we have $\phi_r(\text{Fil}^{p-1} \hat{A}_1) = 0$ and similarly for the ring $A_{\text{crys}}/pA_{\text{crys}}$. Thus these two surjections induce on the ring $R/(p^{r-1})$ the same structure of a filtered $\phi_r$-module over $\Sigma$. Hence, as in the proof of Corollary 3.5, we see from Lemma 3.1 that we have a commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, \hat{A}_1) & \longrightarrow & \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, A_{\text{crys}}/pA_{\text{crys}}) \\
\downarrow & & \downarrow \\
\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, R/(p^{r-1}))
\end{array}
$$

whose downward arrows are isomorphisms. This concludes the proof. \qed

We sketch the proof of the following lemma stated in [2, Subsection 3.2].

**Lemma 4.5.** The natural inclusion $W(R) \to \hat{A}$ induces an isomorphism of $W(R)$-algebras $W_n(R)/(\psi([\varepsilon]) - 1)^{p-1} \to \hat{A}_n/(Z)$.

**Proof.** For a subring $B$ of $A_{\text{crys}}$, put

$$I^{[s]}B = \{ x \in B \mid \phi^i(x) \in \text{Fil}^s A_{\text{crys}} \text{ for any } i \}$$

as in [9, Subsection 5.3]. Then we have $I^{[s]}W(R) = ([\varepsilon] - 1)^s W(R)$ and the natural ring homomorphism

$$W(R)/I^{[s]}W(R) \to A_{\text{crys}}/I^{[s]} A_{\text{crys}}$$

is an injection ([9, Proposition 5.1.3, Proposition 5.3.5]). Since the element $Z$ is contained in the ideal $I^{[p-1]} A_{\text{crys}}$, this injection factors as

$$W(R)/I^{[p-1]}W(R) \to \hat{A}/(Z) \to A_{\text{crys}}/I^{[p-1]} A_{\text{crys}}.$$ 

Hence the former arrow is an isomorphism and the lemma follows. \qed

Since the ideal $(Z)$ of $\hat{A}_n$ satisfies the condition of Corollary 3.5, the $\Sigma$-algebra $\hat{A}_n/(Z)$ is naturally considered as an object of $\text{"{M}od}^\phi_{\Sigma}$. We also give the ring $W_n(R)/(\psi([\varepsilon]) - 1)^{p-1}$ the structures of a $\Sigma$-algebra and a filtered $\phi_r$-module over $\Sigma$ induced by those of $\hat{A}_n/(Z)$. The map

$$\Sigma \to W_n(R)/(\psi([\varepsilon]) - 1)^{p-1}$$

sends the element $u \in \Sigma$ to the image of $[\pi]$ in the ring on the right-hand side. Put $v = v'/E([\pi]) \in W(R)\times$ with the notation of Lemma 4.1. As for the element $Y \in \Sigma$, the equality

$$Y = -ab^{-1}v^{-1}E([\pi])^{p-1} + wb^{-1}v^{-p}Z$$

holds in $\hat{A}$, where $a$ and $b$ are the elements in $W(R)\times$ as in the proof of Lemma 4.1 and the remark after this lemma, and $w \in W(R)\times$ is a power series of $[\varepsilon] - 1$. Hence the above homomorphism sends the element $Y$ to the image of $-ab^{-1}v^{-1}E([\pi])^{p-1}$. 

Consider the surjective ring homomorphism
\[ R \to \mathcal{O}_K/p\mathcal{O}_K \]
\[ x = (x_0, x_1, \ldots) \mapsto x_n \]
and the induced surjection \( W_n(R) \to W_n(\mathcal{O}_K/p\mathcal{O}_K) \). Let
\[ J = \{ (x_0, \ldots, x_{n-1}) \in W_n(R) \mid v_R(x_i) \geq p^n \text{ for any } i \} \]
be the kernel of the latter surjection.

**Lemma 4.6.** The ideal \( J \) is contained in the ideal \((\psi([\varepsilon] - 1)p^{-1})\) of the ring \( W_n(R) \).

**Proof.** Write \((\varepsilon - 1)p^{-1}\) also as \( x = (x_0, \ldots, x_{n-1}) \in W_n(R) \) with \( v_R(x_0) = p \). Take an element \( z = (z_0, \ldots, z_{n-1}) \) in the ideal \( J \). We construct \( y \in W_n(R) \) such that \( xz = z \). By induction, it is enough to show that if \( z_0 = \cdots = z_{i-1} = 0 \) for some \( 0 \leq i \leq n - 1 \) and \( (x_0, \ldots, x_i)(0, \ldots, 0, y_i) = (0, \ldots, 0, z_i) \) in \( W_i+1(R) \), then \( x(0, \ldots, 0, y_i, 0, \ldots, 0) \in J \). Let us write this element as \((0, \ldots, 0, w_i, \ldots, w_{n-1})\) with \( w_i = z_i \). We have \( v_R(y_i) \geq p^n - p^{i+1} \).

In the ring of Witt vectors \( \mathbb{W}_n(F_p[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}]) \), the \( k \)-th entry of the vector
\[ (X_0, \ldots, X_{n-1})(0, \ldots, 0, Y_1, 0, \ldots, 0) \]
is \( X_{k-i}^{p^i}Y_i^{p^k-1} \) for any \( k \geq i \). Thus we have \( v_R(w_k) \geq p^n \).

Note that the elements \([\zeta_{p^n}] - 1\) and \([\zeta_{p^{n+1}}] - 1\) is nilpotent in \( W_n(\mathcal{O}_K/p\mathcal{O}_K) \).

By the above lemma, we have an isomorphism of rings
\[ W_n(R)/(\psi([\varepsilon] - 1)p^{-1}) \cong W_n(\mathcal{O}_K/p\mathcal{O}_K)/(\psi([\zeta_{p^n}] - 1)p^{-1}). \]

We give the ring on the right-hand side the structure of a filtered \( \phi_r \)-module over \( \Sigma \) induced by this isomorphism.

Put \( F_n = K_n(\zeta_{p^{n+1}}) \). For an algebraic extension \( F \) of \( F_n \), let us consider the ideals
\[ m_{n,F} = \{ (x_0, \ldots, x_{n-1}) \in W_n(\mathcal{O}_F/p\mathcal{O}_F) \mid x_i \in m_{F}/p\mathcal{O}_F \text{ for any } i \} \]
\[ m_{n,F} = \{ (x_0, \ldots, x_{n-1}) \in W_n(\mathcal{O}_F) \mid x_i \in m_{F} \text{ for any } i \} \]
of \( W_n(\mathcal{O}_F/p\mathcal{O}_F) \) and \( W_n(\mathcal{O}_F) \), respectively. The elements \([\zeta_{p^n}] - 1\) and \([\zeta_{p^{n+1}}] - 1\) are topologically nilpotent in \( W_n(\mathcal{O}_F) \) and we define an element \( \hat{t} \in W_n(\mathcal{O}_F) \) to be
\[ \hat{t} = \psi([\zeta_{p^n}] - 1)/\psi([\zeta_{p^{n+1}}] - 1). \]

Note that these elements are non-zero divisors of \( W_n(\mathcal{O}_F) \). Let the ring
\[ W_n(\mathcal{O}_F/p\mathcal{O}_F)/(\psi([\zeta_{p^n}] - 1)p^n) \]
be denoted by \( \tilde{A}_{n,F,r} \). We also put \( m_n = m_{n,F} \) and \( m_n = m_{n,K} \) and \( \tilde{A}_{n,r} = \tilde{A}_{n,K,r} \).

For an algebraic extension \( F \) of \( K \), we put
\[ b_F = \{ x \in \mathcal{O}_F \mid v_K(x) > cr/(p - 1) \}. \]
Note that the ring $\mathcal{O}_F/b_F$ is killed by $p$. When $F$ contains $F_n$, we also put $\bar{A}_{n, F, r+} = W_n(\mathcal{O}_F/b_F)/\psi([\zeta_{p^n}] - 1)^r \bar{m}_{n, F}$. Then, for $0 \leq r < p - 1$, we have natural isomorphisms of rings $W_n(\mathcal{O}_F)/\psi([\zeta_{p^n}] - 1)^r m_{n, F} \to \bar{A}_{n, F, r+} \to \bar{A}_{n, F, r+}$. Indeed, as in the proof of Lemma 4.6, we can show that both of the kernels of the maps $W_n(\mathcal{O}_F) \to W_n(\mathcal{O}_F/p\mathcal{O}_F)$ and $W_n(\mathcal{O}_F) \to W_n(\mathcal{O}_F/b_F)$ are contained in the ideal $\psi([\zeta_{p^n}] - 1)^r m_{n, F}$ of the ring $W_n(\mathcal{O}_F)$. We often identify these rings. We also put $\bar{A}_{0, n, F, r} = \bar{A}_{n, F, r+}$. Write $Z_n$ for the image of the element $Z$ of $A_{\text{crys}}$ in $W_{n}^{\text{PD}}(\mathcal{O}_K/p\mathcal{O}_K)$. Then we have a commutative diagram of $\Sigma$-algebras

$$
\begin{array}{ccc}
\hat{A}_n & \longrightarrow & A_{\text{crys}}/p^n A_{\text{crys}} \\
\downarrow & & \downarrow \\
W_{n}^{\text{PD}}(\mathcal{O}_K/p\mathcal{O}_K) & & \\
\downarrow & & \downarrow \\
W_n(\mathcal{O}_K/p\mathcal{O}_K)/([\zeta_{p^n}] - 1)^{p-1} & \longrightarrow & \hat{A}_n/(Z) \\
\downarrow & & \downarrow \\
W_n(\mathcal{O}_K/p\mathcal{O}_K)/([\zeta_{p^n}] - 1)^{p-1} & \longrightarrow & W_n^{\text{PD}}(\mathcal{O}_K/p\mathcal{O}_K)/(Z_n), \\
\downarrow & & \downarrow \\
\hat{A}_{n, r+} & \longrightarrow & \hat{A}_{n, r+} \\
\downarrow & & \downarrow \\
\hat{A}_{n, r+} & \longrightarrow & \hat{A}_{n, r+}
\end{array}
$$

where all vertical arrows are surjections satisfying the condition of Corollary 3.5. Thus we see that this is also a commutative diagram in $\text{Mod}_{\Sigma}^{r, \phi}$. Note that these rings and homomorphisms are independent of the choice of a system $\{\zeta_{p^n}\}_{n \in \mathbb{Z}_{\geq 0}}$. Moreover, let $M$ be in $\text{Mod}_{\Sigma}^{r, \phi}$ and put $M_n = M/p^n M$. Then, by Corollary 3.5 and Corollary 4.4, we have a natural isomorphism of abelian groups $\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M_n, W_{n}^{\text{PD}}(\mathcal{O}_K/p\mathcal{O}_K)) \simeq \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M_n, A_{n, r+})$. To study $G_{F_n}$-actions on both sides of this isomorphism, we need the following proposition.

**Proposition 4.7.** The image of the element $Y \in \Sigma$ in the ring $\hat{A}_{n, r+}$ is contained in its subring $\bar{A}_{n, F_n, r+}$. 

We have the equality
\[(\varepsilon^{1/p} - 1)^{p-1} = pa - E(\varepsilon)bv\]
in \(W(R)\). By definition, we see that the images of the elements \(a, \varepsilon^{1/p}\) and \(E(\varepsilon)\) in \(W_n(O_K/pO_K)\) are contained in the subring \(W_n(O_{F_n}/pO_{F_n})\). Write \(\beta\) for the product \(bv \in W(R)\). Let \(a_n, \beta_n\) and \(\alpha_n\) denote the images of the elements \(a, \beta\) and \(pa - (\varepsilon^{1/p} - 1)^{p-1}\) in the ring \(W_n(O_K/pO_K)\), respectively. Then the element \(\alpha_n\) is also contained in the subring \(W_n(O_{F_n}/pO_{F_n})\). Now we have the equality
\[E(\varepsilon^n)\beta_n = \alpha_n.\]
Note that any element \(\beta_n' \in W_n(O_K/pO_K)\) satisfying the same equality is invertible and thus the elements \(\beta_n^{-1}E(\varepsilon^n)\) are equal to each other. Since \(Y = -a_n\beta_n^{-1}E(\varepsilon^n)\) in \(A_{n,r+}\), it suffices to construct an element \(\beta_n'\) in the ring \(W_n(O_{F_n}/pO_{F_n})\) such that the equality \(E(\varepsilon^n)\beta_n' = \alpha_n\) holds. Take a lift \(\hat{a}_n\) of \(a_n\) in \(W_n(O_{F_n})\). Since \(E(\varepsilon^n)\) divides every element in the kernel of the surjection \(W_n(O_K) \to W_n(O_K/pO_K)\), we have \(E(\varepsilon^n)\beta_n' = \hat{a}_n\) for some \(\beta_n' \in W_n(O_K)\). Then the element \(\beta_n'\) is contained in the subring \(W_n(O_{F_n})\) and we set \(\beta_n'\) to be the image of \(\beta_n'\).

By a similar argument, we can also check that the ring \(\hat{A}_{n,r+}\) is a subring of \(\hat{A}_{n,r+}\) and coincides with the image of \(W_n(O_F)\) in \(\hat{A}_{n,r+}\). This concludes the proof. 

Let \(t_n\) and \(\bar{t}_n\) be the images of \(t \in W(R)\) in \(W_n(O_K/pO_K)\) and \(\hat{A}_{n,r+}\) (or \(\hat{A}_{n,r+}'\)), respectively. Then \(t\) is a lift of \(t_n\) and \(\bar{t}_n\) to \(W_n(O_K)\) by the natural surjections
\[W_n(O_K) \to W_n(O_K/pO_K) \to \hat{A}_{n,r+}.\]
Note that we defined the filtration of \(A_{n,r+}\) as \(\Phi^r\hat{A}_{n,r+} = \bar{t}_n\hat{A}_{n,r+}\).

**Lemma 4.8.** Let \(\bar{x}\) be in \(\text{Fil}^r\hat{A}_{n,r+}\). Then we have
\[\phi_r(\bar{x}) = \phi(y) \mod \psi(\varepsilon^n - 1)^{r \bar{m}_n},\]
where \(y\) is any element of \(W_n(O_K/pO_K)\) such that the element \(t_n^r y\) is a lift of \(\bar{x}\). In particular, the right-hand side of the above equality is independent of the choice of a system \(\{\zeta^n\}_n \in \mathbb{Z}_{\geq 0}\). Similar assertions also hold for the ring
\[W_n(O_K/pO_K) \to W_n(O_K/pO_K) \to (\varepsilon^n - 1)^{p-1}.\]

**Proof.** Since the filtered \(\phi_r\)-module structure of \(\hat{A}_{n,r+}\) is induced from that of \(\hat{A}\) and \(\phi_r(t_n^r) \equiv 1 \mod \mathbb{Z}\) in \(\hat{A}\), we see that there exists an element \(y\) as in the lemma.

To prove the independence of the choice of a lift, let \(z = (z_0, \ldots, z_{n-1})\) be an element of \(W_n(O_K/pO_K)\) killed by \(t_n\). The element \(z\) is also killed by
Let $\phi$ be the associated matrix representing $\hat{\phi}$ in the ideal $\ker \nabla$. From this lemma and Proposition 4.7, we see that the natural ring homomorphism $\phi$ sends the element on the right-hand side to an element which is contained in the ideal $\psi([\zeta_p^n]-1)^r \bar{m}_n$. Thus the assertions for the ring $\bar{A}_{n,r+}$ follows. We can show the assertion for the ring $\bar{W}_n(\mathcal{O}_K/p\mathcal{O}_K)/(\psi([\zeta_p^n]-1)^{r-1})$ similarly. 

From this lemma and Proposition 4.7, we see that the natural $G_{F_n}$-actions on the rings $W_n(\mathcal{O}_K/p\mathcal{O}_K)/(\psi([\zeta_p^n]-1)^r)$, $\bar{A}_{n,r+}$ and $\bar{A}_{n,r+}'$ are compatible with the filtered $\phi_r$-module structures over $\Sigma$. In the commutative diagram above, the lowest horizontal arrow and lower right vertical arrow are $G_{F_n}$-linear by definition. Hence we have shown the following proposition.

**Proposition 4.9.** Let $\hat{M}$ be in $\text{Mod}^{\Gamma}_{\Sigma}$ and put $M_n = \hat{M}/p^r\hat{M}$. Then we have an isomorphism of $G_{F_n}$-modules

$$\text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M_n, \bar{A}_{n,r+}) \simeq \text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M_n, W_n^{\text{Fil}}(\mathcal{O}_K/p\mathcal{O}_K)).$$

Let $e_1, \ldots, e_d$ be a basis of $\hat{M}$ as in Lemma 3.3 and $C = (c_{i,j}) \in M_d(\Sigma)$ be the associated matrix representing $\phi_r$ as in Corollary 3.4. Then the underlying $G_{F_n}$-set of the $G_{F_n}$-module

$$\text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M_n, \bar{A}_{n,r+})$$

is identified with the set of $d$-tuples $(\bar{x}_1, \ldots, \bar{x}_d)$ in $\bar{A}_{n,r+}$ such that $c_{1,i}\bar{x}_1 + \cdots + c_{d,i}\bar{x}_d \in \text{Fil}^r\bar{A}_{n,r+}$ for any $i$ and the following equality holds:

\[
\begin{align*}
\phi_r(c_{1,1}\bar{x}_1 + \cdots + c_{d,1}\bar{x}_d) &= \bar{x}_1 \\
& \vdots \\
\phi_r(c_{1,d}\bar{x}_1 + \cdots + c_{d,d}\bar{x}_d) &= \bar{x}_d.
\end{align*}
\]

(1)

We choose a lift $(c_{i,j}) \in M_d(W_n(\mathcal{O}_{F_n}))$ of the image of $C$ in $M_d(\bar{A}_{n,r+})$ by the natural ring homomorphism

$$W_n(\mathcal{O}_K) \to W_n(\mathcal{O}_K/p\mathcal{O}_K) \to \bar{A}_{n,r+}.$$ 

Fix a polynomial $\Phi_t \in \mathbb{Z}[X_0, \ldots, X_{n-1}]$ such that $\Phi_t \equiv X_p^t \mod p$. This induces for any commutative ring $B$ a map $\Phi = (\Phi_0, \ldots, \Phi_{n-1}) : W_n(B) \to W_n(B)$ which is a lift of the Frobenius endomorphism on $W_n(B/pB)$. In
particular, set \( B \) to be the polynomial ring \( \mathbb{Z}[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}] \). Put \( X = (X_0, \ldots, X_{n-1}) \) and \( Y = (Y_0, \ldots, Y_{n-1}) \) in the ring \( W_n(B) \). Then we see that there exists elements \( U_0, \ldots, U_{n-1} \) and \( U'_0, \ldots, U'_n \) of the polynomial ring \( B \) such that

\[
\Phi(X + Y) = \Phi(X) + \Phi(Y) + (pU_0, \ldots, pU_{n-1}),
\]
\[
\Phi(XY) = \Phi(X)\Phi(Y) + (pU'_0, \ldots, pU'_{n-1})
\]
in the ring \( W_n(B) \).

**Proposition 4.10.** Every solution \((\hat{x}_1, \ldots, \hat{x}_d)\) in \( \hat{A}_{n,r+} \) of the equation (1) such that \( c_{1,i}\hat{x}_1 + \cdots + c_{d,i}\hat{x}_d \in \text{Fil}^r \hat{A}_{n,r+} \) for any \( i \) uniquely lifts to a \( d \)-tuple \((\hat{x}_1, \ldots, \hat{x}_d)\) in \( W_n(\mathcal{O}_K) \) such that \( \hat{c}_{1,i}\hat{x}_1 + \cdots + \hat{c}_{d,i}\hat{x}_d \in \hat{\mathfrak{p}}^r W_n(\mathcal{O}_K) \) for any \( i \) and the following equality holds:

\[
\begin{align*}
\Phi((\hat{c}_{1,1}\hat{x}_1 + \cdots + \hat{c}_{d,1}\hat{x}_d)/\hat{\mathfrak{p}}^r) &= \hat{x}_1 \\
&\quad \vdots \\
\Phi((\hat{c}_{1,d}\hat{x}_1 + \cdots + \hat{c}_{d,d}\hat{x}_d)/\hat{\mathfrak{p}}^r) &= \hat{x}_d.
\end{align*}
\]

**Proof.** Fix a lift \( \hat{x}_i \) of \( \hat{x}_i \) to \( W_n(\mathcal{O}_K) \). Then we have

\[
\begin{align*}
\Phi((\hat{c}_{1,1}\hat{x}_1 + \cdots + \hat{c}_{d,1}\hat{x}_d)/\hat{\mathfrak{p}}^r) &= \hat{x}_1 + ([\zeta_{\rho^n}^r] - 1)^r \hat{\delta}_1 \\
&\quad \vdots \\
\Phi((\hat{c}_{1,d}\hat{x}_1 + \cdots + \hat{c}_{d,d}\hat{x}_d)/\hat{\mathfrak{p}}^r) &= \hat{x}_d + ([\zeta_{\rho^n}^r] - 1)^r \hat{\delta}_d
\end{align*}
\]

for some \( \hat{\delta}_1, \ldots, \hat{\delta}_d \in m_\mathbb{D} \). It suffices to show that there exists a unique \( d \)-tuple \((\hat{y}_1, \ldots, \hat{y}_d)\) in \( m_\mathbb{D} \) such that

\[
\Phi((\hat{c}_{1,1}\hat{x}_1 + ([\zeta_{\rho^n}^r] - 1)^r \hat{y}_1) + \cdots + \hat{c}_{d,1}\hat{x}_d + ([\zeta_{\rho^n}^r] - 1)^r \hat{y}_d)/\hat{\mathfrak{p}}^r) = \hat{x}_1 + ([\zeta_{\rho^n}^r] - 1)^r \hat{y}_i
\]

for any \( i \). For this, we need the following lemma.

**Lemma 4.11.** Let \( N \) be a complete discrete valuation field and \( m_N \) be the maximal ideal of \( N \). Let \( \epsilon_1, \ldots, \epsilon_d \) be in \( m_N \). Let \( P_1, \ldots, P_d \) and \( P'_1, \ldots, P'_d \) be elements of \( \mathcal{O}_N[[Y_1, \ldots, Y_d]] \) such that \( P_i \in (Y_1, \ldots, Y_d)^2 \). Then the equation

\[
\begin{align*}
Y_1 - P_1(Y_1, \ldots, Y_d) - \epsilon_1 P'_1(Y_1, \ldots, Y_d) &= 0 \\
&\quad \vdots \\
Y_d - P_d(Y_1, \ldots, Y_d) - \epsilon_d P'_d(Y_1, \ldots, Y_d) &= 0
\end{align*}
\]

has a unique solution in \( m_N \).

**Proof.** By assumption, we see that for any integer \( l \geq 1 \), a \( d \)-tuple \( y = (y_1, \ldots, y_d) \) in \( m_N/m_N^{l+1} \) satisfying the above equation lifts uniquely to a \( d \)-tuple in \( m_N/m_N^{l+1} \) satisfying the same equation. Thus the lemma follows.
Let us write as \( \hat{y}_i = (\hat{y}_{i,0}, \ldots, \hat{y}_{i,n-1}) \). Since the image of \( \Phi((\zeta_p)_{p+1} - 1)^r \) in \( \bar{A}_{n,r^+} \) is divisible by \( (\zeta_p^n - 1)^r \), we can find \( \hat{b} \in W_n(\mathcal{O}_{\bar{K}}) \) such that
\[
\Phi((\zeta_p^n - 1)^r) = (\zeta_p^n - 1)^r \hat{b}.
\]
Then there exists polynomials \( U_{i,m} \) over \( \mathcal{O}_{\bar{K}} \) of the indeterminates \( Y = (Y_{i,m})_{1 \leq i \leq d, 0 \leq m \leq n-1} \) such that the equation we have to solve is
\[
\hat{x}_i + ((\zeta_p^n - 1)^r \hat{y}_i = \hat{x}_i + ((\zeta_p^n - 1)^r \hat{\delta}_i + ((\zeta_p^n - 1)^r \hat{b}(\Phi(\hat{c}_{i,1})\Phi(\hat{y}_i) + \cdots + \Phi(\hat{c}_{i,d})\Phi(\hat{y}_d))
+ (pU_{i,0}(\hat{y}), \ldots, pU_{i,n-1}(\hat{y}))
\]
for any \( i \), where we put \( \hat{y} = (\hat{y}_{i,m})_{1 \leq i \leq d, 0 \leq m \leq n-1} \). Note that, for any elements \( P_0, \ldots, P_{n-1} \) of the polynomial ring \( \mathcal{O}_{\bar{K}}[Y] \), we can uniquely find elements \( Q_0, \ldots, Q_{n-1} \) of this ring such that the coefficients of these polynomials are in the maximal ideal \( m_{\bar{K}} \) and the equality
\[
(pP_0, \ldots, pP_{n-1}) = ((\zeta_p^n - 1)^r(Q_0, \ldots, Q_{n-1})
\]
holds in the ring of Witt vectors \( W_n(\mathcal{O}_{\bar{K}}[Y]) \). Therefore, this equation is equivalent to the equation
\[
\hat{y}_i = \hat{\delta}_i + \hat{b}(\Phi(\hat{c}_{i,1})\Phi(\hat{y}_i) + \cdots + \Phi(\hat{c}_{i,d})\Phi(\hat{y}_d))
+ (V_{i,0}(\hat{y}), \ldots, V_{i,n-1}(\hat{y}))
\]
where \( V_{i,m} \) is a polynomial of \( Y \) over \( \mathcal{O}_{\bar{K}} \) whose coefficients are in the maximal ideal \( m_{\bar{K}} \). From the definition of \( \Phi \), we see that the elements \( \hat{y}_{i,m} \) is a solution of a system of equations
\[
Y_{i,m} - P_{i,m}(Y_{i,m}) - \epsilon_{i,m}P'_{i,m}(Y_{i,m}) = 0
\]
satisfying the condition of Lemma 4.11 for a sufficiently large finite extension \( N \) of \( K \). Then, by this lemma, we can solve the equation uniquely in \( m_{\bar{K}} \). \( \square \)

Let \( F \) be an algebraic extension of \( F_n = K_n(\zeta_p)_{p+1} \) and consider the ring \( A_{n,F,r^+} \). By Lemma 4.7, we can consider this ring as a \( \Sigma \)-algebra and also as an object of \( '\text{Mod}^{\phi}_{\Sigma} \) by putting \( \text{Fil}^r A_{n,F,r^+} = \hat{P}_n A_{n,F,r^+} \) and for \( \bar{x} \in \text{Fil}^r A_{n,F,r^+} \)
\[
\phi_r(\bar{x}) = \phi(y) \mod \psi((\zeta_p^n - 1)^r m_{n,F},
\]
where \( y \) is any element of \( W_n(\mathcal{O}_F/p\mathcal{O}_F) \) such that the element \( t^r_y \) is a lift of \( \bar{x} \). For \( M_n = M/p^n M \in \text{Mod}^{\phi}_{\Sigma_{\infty}} \) as before, let us set
\[
T_{\text{cris}, \pi_n,F}(M_n) = \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M_n, A_{n,F,r^+}).
\]
We see that
\[
\bar{A}_{n,r^+} = \bar{A}_{n,K,r^+} = \bigcup_{F/F_n} \bar{A}_{n,F,r^+}
\]
in \( '\text{Mod}^r_{/\Sigma} \) and thus we have a natural identification of abelian groups

\[
T^*_{\text{crys},\pi_0,\tilde{K}}(M_n) = \bigcup_{F/F_n} T^*_\text{crys},\pi_n,F(M_n).
\]

The absolute Galois group \( G_{F_n} \) acts on the abelian group on the left-hand side.

**Lemma 4.12.** For an algebraic extension \( F \) of \( F_n \), the fixed part \( T^*_{\text{crys},\pi_0,\tilde{K}}(M_n)^{G_F} \) is equal to \( T^*_\text{crys},\pi_n,F(M_n) \).

**Proof.** From Proposition 4.10, we see that the elements of \( T^*_{\text{crys},\pi_0,\tilde{K}}(M_n) \) correspond bijectively to the solutions of the equation (2) in \( W_n(\mathcal{O}_K) \) satisfying the condition on \( \text{Fil}^\ell \). The uniqueness assertion of this proposition shows that \( g \in G_F \) fixes a solution in \( W_n(\mathcal{O}_K) \) if and only if \( g \) fixes its image in \( A_{n,r^+} \). Hence a solution is fixed by \( G_F \) if and only if this solution is contained in the image of \( W_n(\mathcal{O}_F) \). Thus the lemma follows. \( \square \)

**Corollary 4.13.** Let \( L_n \) be the finite Galois extension of \( F_n \) corresponding to the kernel of the map \( G_{F_n} \to \text{Aut}(T^*_{\text{crys},\pi_0,\tilde{K}}(M_n)) \). Then an algebraic extension \( F \) of \( F_n \) contains \( L_n \) if and only if \( \# T^*_{\text{crys},\pi_n,F}(M_n) = \# T^*_{\text{crys},\pi_0,\tilde{K}}(M_n) \).

**Proof.** An algebraic extension \( F \) of \( F_n \) contains \( L_n \) if and only if the action of \( G_F \) on \( T^*_{\text{crys},\pi_0,\tilde{K}}(M_n) \) is trivial. By Lemma 4.12, this is equivalent to \( T^*_{\text{crys},\pi_n,F}(M_n) = T^*_{\text{crys},\pi_0,\tilde{K}}(M_n) \). \( \square \)

### 5. Ramification bound

In this section, we prove Theorem 1.1. Take \( G_K \)-stable \( \mathbb{Z}_p \)-lattices \( \mathcal{L} \supseteq \mathcal{L}' \) in \( V \) such that \( \mathcal{L}' \supseteq p^n \mathcal{L} \). Since the \( G_K \)-module \( \mathcal{L}/\mathcal{L}' \) is a quotient of \( \mathcal{L}/p^n \mathcal{L} \), we may assume \( \mathcal{L}' = p^n \mathcal{L} \). If \( r = 0 \), then the \( G_K \)-module \( V \) is unramified and the theorem is trivial. Thus we may assume \( r \geq 1 \) and \( p \geq 3 \). Let \( L \) be the finite Galois extensions of \( K \) corresponding to the kernel of the map

\[
G_K \to \text{Aut}(\mathcal{L}/p^n \mathcal{L}).
\]

It is enough to show that, for the greatest upper ramification break \( u_{L(\zeta_p)/K} \) of the Galois extension \( L(\zeta_p)/K \), the inequality

\[
u_{L(\zeta_p)/K} \leq u(K,r,n)
\]

holds. Since the Herbrand function is transitive and the finite Galois extension \( K(\zeta_p) \) is tamely ramified over \( K \), we may assume \( \zeta_p \in K \). We fix a uniformizer \( \pi \) of \( K \) and a system \( \{ \pi_n \}_{n \in \mathbb{Z}_{\geq 0}} \) as before. Then, by Liu’s theorem ([14, Theorem 2.3.5]), it suffices to show the following.

**Theorem 5.1.** Let \( r \) be an integer such that \( 1 \leq r < p - 1 \) and \( \mathcal{M} \) be the strongly divisible lattice corresponding to \( \mathcal{L} \). Put \( \mathcal{M}_n = \mathcal{M}/p^n \mathcal{M} \in \text{Mod}_{r,\phi} \). Then \( G_K^{(j)} \) acts trivially on the \( G_K \)-module \( T^*_{\text{crys},\Sigma}(\mathcal{M}_n) \) for \( j > u(K,r,n) \).
Let $L_n$ be the finite Galois extension of $F_n = K_n(\zeta_{p^n+1})$ corresponding to the kernel of the map

$$G_{F_n} \to \text{Aut}(T^*_{\text{st},\mathbb{Z}}(\mathcal{M}_n)).$$

Since $F_n$ is Galois over $K$, the extension $L_n$ is also a Galois extension of $K$. Let $\bar{M}$ be the object of the category $\text{Mod}_{/\phi}^{/S}$ such that $\mathcal{M}(\mathfrak{M}) \simeq \bar{M}$. From Proposition 3.6 and Proposition 4.9, we see that $L_n$ is also the finite extension of $F_n$ cut out by the $G_{F_n}$-module $T^*_{\text{crys},\pi_n,K}(M_n)$ for $M_n = M_{/S}(\hat{M})/p^nM_{/S}(\hat{M})$. It is enough to prove the inequality

$$u_{L_n/K} \leq u(K, r, n) = \begin{cases} 1 + e(n + \frac{1}{p-1}) & (r = 1), \\ \frac{p^n-1}{p^n} + e(n + \frac{r}{p-1}) & (r > 1). \end{cases}$$

Before proving this, we state some general lemmas to calculate the ramification bound. Let $N$ be a complete discrete valuation field of positive residue characteristic, $v_N$ be its valuation normalized as $v_N(N) = \mathbb{Z}$ and $N^\text{sep}$ be its separable closure.

**Lemma 5.2.** Let $f(T) \in \mathcal{O}_N[T]$ be a separable monic polynomial and $z_1, \ldots, z_d$ be the zeros of $f$ in $\mathcal{O}_N^{\text{sep}}$. Suppose that the set $\{v_N(z_k - z_i) | k = 1, \ldots, d, k \neq i\}$ is independent of $i$. Put

$$s = \sum_{k=1, \ldots, d, k \neq i} v_N(z_k - z_i) \quad \text{and} \quad \alpha = \sup_{k=1, \ldots, d, k \neq i} v_N(z_k - z_i),$$

which are independent of $i$ by assumption. If $j > s + \alpha$, then we have the decomposition

$$\{x \in \mathcal{O}_N^{\text{sep}} | v_N(f(x)) \geq j\} = \prod_{i=1, \ldots, d} \{x \in \mathcal{O}_N^{\text{sep}} | v_N(x - z_i) \geq j - s\}.$$

Otherwise, the set on the left-hand side contains

$$\{x \in \mathcal{O}_N^{\text{sep}} | v_N(x - z_i) \geq \alpha\},$$

which contains at least two zeros of $f$.

**Proof.** A verbatim argument in the proof of [1, Lemma 6.6] shows the claim. \hfill \square

**Corollary 5.3.** Let $f(T)$ be as above and put $B = \mathcal{O}_N[T]/(f(T))$. Suppose that the algebra $B$ is finite flat and of relative complete intersection over $\mathcal{O}_N$. Let us write the $N$-algebra $N' = B \otimes_{\mathcal{O}_N} N$ as the product $N_1 \times \cdots \times N_t$ of finite separable extensions $N_1, \ldots, N_t$ of $N$. If $j > s + \alpha$, then the $j$-th upper numbering ramification group ($[1]$), which we let be denoted by $G_N^{(j)}$, is contained in $G_{N_i}$ for any $i$. Moreover, if $N'$ is a field and $B$ coincides with $\mathcal{O}_N$, then $j > s + \alpha$ if and only if $G_N^{(j)} \subseteq G_{N'}$. \hfill \square
Proof. From the previous lemma, the conductor $c(B)$ of the $\mathcal{O}_N$-algebra $B$ ([1, Proposition 6.4]) is equal to $s + \alpha$. Thus we have the inequality 

$$c(\mathcal{O}_{N_1} \times \cdots \times \mathcal{O}_{N_t}) \leq c(B) = s + \alpha$$

by the definition of the conductor and a functoriality of the functor $\mathcal{F}^j$ defined in [1]. This implies the corollary. 

\[ \square \]

Corollary 5.4. Consider the finite Galois extension $F_n = K_n(\zeta_{p^n+1})$ of $K$ and let $u_{F_n/K}$ denote the greatest upper ramification break of $F_n/K$. Then we have the equality 

$$u_{F_n/K} = 1 + e(n + \frac{1}{p-1}).$$

Proof. Note that we are assuming that $\zeta_p$ is contained in $K$. Applying the previous corollary to the Eisenstein polynomial $f(T) = T^{p^n} - \pi$ shows that $j > 1 + e(n + 1/(p - 1))$ if and only if $G_K^{(j)} \subseteq G_{K_n}$. Similarly, putting $f(T) = T^{p^n} - \zeta_p$, we see that if $j > e(n + 1/(p - 1))$, then $G_K^{(j)} \subseteq G_K(\zeta_{p^n+1})$. Since $G_{F_n} = G_{K_n} \cap G_K(\zeta_{p^n+1})$, we conclude that $j > 1 + e(n + 1/(p - 1))$ if and only if $G_K^{(j)} \subseteq G_{F_n}$. 

\[ \square \]

Remark 5.5. Note that this argument also shows the equality 

$$u_{K_n(\zeta_{p^n})/K} = 1 + e(n + \frac{1}{p-1})$$

without assuming $\zeta_p \in K$.

Next we assume that the residue field of $N$ is perfect. For an algebraic extension $F$ of $N$, we put 

$$\mathfrak{a}_F^N = \{ x \in \mathcal{O}_F \mid v_N(x) \geq j \}.$$ 

For a finite Galois extension $Q$ of $N$, we write $u_{Q/N}$ for the greatest upper ramification break ([7]) of $Q/N$. Let us consider the property 

$(P_j) \left\{ \begin{array}{l}
\text{for any algebraic extension } F \text{ of } N, \text{ if there exists}
\text{an } \mathcal{O}_N\text{-algebra homomorphism } \mathcal{O}_Q \to \mathcal{O}_F/\mathfrak{a}_F^N,
\text{then there exists an } N\text{-algebra injection } Q \to F
\end{array} \right.$

for $j \in \mathbb{R}_{\geq 0}$, as in [7, Proposition 1.5]. Then we have the following proposition, which is due to Yoshida. Here we reproduce his proof for the convenience of the reader.

Proposition 5.6 ([16]).

$$u_{Q/N} = \inf \{ j \in \mathbb{R}_{\geq 0} \mid \text{the property } (P_j) \text{ holds} \}.$$ 

Proof. By [7, Proposition 1.5 (i)], it is enough to show that the property $(P_j)$ does not hold for $j = u_{Q/N} - (e')^{-1}$ with an arbitrarily large $e' > 0$. As in the proof of [7, Proposition 1.5 (ii)], we may assume that $Q$ is totally and wildly ramified over $N$. Let $N'$ be a finite tamely ramified Galois
extension of $N$ such that $Q \cap N' = N$ and put $Q' = QN'$. Note that we have $u_{Q'/N} = u_{Q/N}$ by assumption. From this proposition in [7], we see that for some algebraic extension $F$ of $N$, there exists an $O_N$-algebra homomorphism $O_Q \rightarrow O_F/a_{F/K}^{\eta}$ for $j = u_{Q/N} - e(Q'/N)^{-1}$ but no $N$-algebra injection $Q' \rightarrow F$. Since $Q/N$ is wildly ramified, we see that $e(Q/N)u_{Q/N} - 1 > e(Q/N)$.

Hence we have $u_{Q/N} - e(Q'/N)^{-1} > 1 = u_{N'/N}$ and there exists an $N$-algebra injection $N' \rightarrow F$ also by this proposition. Thus there exists no $N$-algebra injection $Q \rightarrow F$ and the property ($P_j$) for $Q/N$ does not hold. Since we can choose an arbitrarily large $N'$ as above, the proposition follows. \qed

We see from Proposition 5.6 that to bound the greatest upper ramification break $u_{L_n/K'}$, it is enough to show the following proposition.

**Proposition 5.7.** Let $F$ be an algebraic extension of $K$. If $j > u(K,r,n)$ and there exists an $O_K$-algebra homomorphism

$$\eta : O_{L_n} \rightarrow O_F/a_{F/K}^{\eta},$$

then there exists a $K$-algebra injection $L_n \rightarrow F$.

**Proof.** By assumption, we have $j > er/(p-1)$ and $b_F \supseteq a_{F/K}^{\eta}$. Thus $\eta$ induces an $O_K$-algebra homomorphism

$$O_{L_n} \rightarrow O_F/b_F.$$

Since $\eta$ also induces an $O_K$-algebra homomorphism $O_{L_n} \rightarrow O_F/a_{F/K}^{\eta}$ and $r \geq 1$, from Corollary 5.4 and [7, Proposition 1.5] we get a $K$-linear injection $F_n \rightarrow F$. Thus we see that $F$ contains $\pi_n$ and $\zeta_{p^{n+1}}$. More precisely, we have the following lemma.

**Lemma 5.8.** For some integers $i$ and $i'$ such that $i' \equiv 1 \mod p$, we have $\eta(\pi_n) \equiv \pi_n \zeta_{p^n}^i \mod b_F$ and $\eta(\zeta_{p^{n+1}}) \equiv \zeta_{p^{n+1}}^{i'} \mod b_F$. Moreover, there exists $g \in G_K$ such that $g(\pi_n) = \pi_n \zeta_{p^n}^i$ and $g(\zeta_{p^{n+1}}) = \zeta_{p^{n+1}}^{i'}$.

**Proof.** Since the map $\eta$ is $O_K$-linear, the equality $\eta(\pi_n)^{p^n} = \pi$ holds in $O_F/a_{F/K}^{\eta}$. Set $\hat{x}$ to be a lift of $\eta(\pi_n)$ in $O_F$. Then we have

$$v_K(\hat{x}^{p^n} - \pi) = \sum_{i=0}^{p^n-1} v_K(\hat{x} - \pi_n \zeta_{p^n}^i) \geq j.$$

Let us apply Lemma 5.2 to $f(T) = T^{p^n} - \pi$. Then, with the notation of the lemma, we have

$$s = ne + \frac{p^n - 1}{p^n} \text{ and } \alpha = \frac{1}{p^n} + \frac{e}{p-1}.$$

Since $j - s > er/(p-1)$ by assumption, we have

$$\hat{x} \equiv \pi_n \zeta_{p^n}^i \mod b_F$$

for some $i$. 
Let $h(T)$ be the minimal polynomial of $\zeta_{p^{n+1}}$ over $\mathcal{O}_K$. Since $h$ divides $T^{p^n} - \zeta_p$, the $\mathcal{O}_K$-algebra $B' = \mathcal{O}_K[T]/(h(T))$ is also finite flat of relative complete intersection and the $K$-algebra $B' \otimes_K K$ is étale. The Galois group $\text{Gal}(K(\zeta_{p^{n+1}})/K)$ acts transitively on the set of zeros of $h$. Hence $h$ also satisfies the conditions of Lemma 5.2. Let $s'$ and $\alpha'$ be as in this lemma for $h$. Then we have $s' \leq ne$ and $\alpha' \leq e/(p-1)$. This implies $j - s' > \alpha'/(p-1)$. By this lemma, there exists an element $g' \in \text{Gal}(K(\zeta_{p^{n+1}})/K)$ such that the element $g'(\zeta_{p^{n+1}}) = \zeta_{p^{n+1}}'$ satisfies

$$\eta(\zeta_{p^{n+1}}) \equiv g'(\zeta_{p^{n+1}}) \mod b_F.$$ 

Since $K_n \cap K(\zeta_{p^{n+1}}) = K$ (see for example [14, Lemma 5.1.2]), we can find an element $g \in G_K$ such that $g(\pi_n) = \pi_n \zeta_{p^n}$ and $g(\zeta_{p^{n+1}}) = g'(\zeta_{p^{n+1}})$. This concludes the proof. 

**Lemma 5.9.** The $\mathcal{O}_K$-algebra homomorphism $\eta$ induces an $\mathcal{O}_K$-algebra injection

$$\eta_b : \mathcal{O}_{L_n}/b_{L_n} \rightarrow \mathcal{O}_F/b_F.$$

**Proof.** We write the Eisenstein polynomial of a uniformizer $\pi_{L_n}$ of $L_n$ over $\mathcal{O}_K$ as

$$P(T) = T^{e'} + c_1 T^{e'-1} + \cdots + c_{e'-1} T + c_{e'},$$

where $e' = e(L_n/K)$. Then $z = \eta(\pi_{L_n})$ satisfies $P(z) = 0$ in $\mathcal{O}_F/a_F^\delta/K$.

Let $\tilde{z}$ be a lift of $z$ in $\mathcal{O}_F$. Since $j > 1$, we have $v_F(\tilde{z}) = e(F/K)/e'$. The condition $i > e(L_n)r/(p-1)$ is equivalent to the condition

$$v_F(\tilde{z}^i) > \frac{e(L_n)r}{p-1} \cdot \frac{e(F/K)}{e'} = \frac{e(F)r}{p-1}.$$ 

Thus the claim follows. 

Since $L_n$ contains $F_n$, we can consider the ring

$$\tilde{A}_{n,L_n,r^+} = W_n(\mathcal{O}_{L_n}/b_{L_n})/\psi([\zeta_p^n] - 1)^r \tilde{m}_{n,L_n}$$

and similarly $\tilde{A}_{n,F,r^+}$ for $F$. We give these rings structures of $\Sigma$-algebras as follows. The ring $\tilde{A}_{n,L_n,r^+}$ is considered as a $\Sigma$-algebra by using the system $\{\pi_n\}_{n \in \mathbb{Z}_{\geq 0}}$ we chose of $p$-power roots of $\pi$, as in the previous section. On the other hand, using $i$ and $i'$ in Lemma 5.8, put $\tilde{\pi}_n = \pi_n \zeta_{p^n}$ and $\tilde{\zeta}_{p^{n+1}} = \zeta_{p^{n+1}}'$. Then we consider the ring $\tilde{A}_{n,F,r^+}$ as a $\Sigma$-algebra by using a system of $p$-power roots of $\pi$ containing $\tilde{\pi}_n$. We define $\text{Fil}^r$ and $\phi_r$ of these rings in the same way as before.

**Lemma 5.10.** The induced ring homomorphism

$$\tilde{\eta} : \tilde{A}_{n,L_n,r^+} \rightarrow \tilde{A}_{n,F,r^+}$$

is a morphism of the category $\text{Mod}_{\Sigma}^r$. 

Proof. Firstly, we check that $\tilde{\eta}$ is $\Sigma$-linear. By definition, this homomorphism commutes with the action of the element $u \in \Sigma$. To show the compatibility with the element $Y \in \Sigma$, let us consider the commutative diagram

$$
\begin{array}{c}
W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n}) \longrightarrow W_n(\mathcal{O}_F/p\mathcal{O}_F) \\
\bigl\downarrow\bigl\downarrow \\
W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n}) \longrightarrow W_n(\mathcal{O}_F/\mathfrak{b}_F) \\
\bigl\downarrow\bigl\downarrow \\
\bar{A}'_{n,L_n,r^+} \longrightarrow \bar{A}'_{n,F,r^+},
\end{array}
$$

where the horizontal arrows are induced by $\eta$. Note that we have $\eta_b(\pi_n) = \tilde{\pi}_n$ and $\eta_b(\zeta_{p^n+1}) = \tilde{\zeta}_{p^n+1}$. Put $\beta \in W(R)^\times$ as in the proof of Proposition 4.7. Namely, the element $\beta$ is the solution in $W(R)$ of the equation

$$
E([\xi])\beta = pa - ([\xi^{-1}\eta] - 1)^{p-1},
$$

where the element $a \in W(R)$ is as in the remark after Lemma 4.1. Let $a_n$ and $\beta_n$ denote the images of $a$ and $\beta$ in $W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n})$, respectively. Then the element $\beta_n$ is a solution of the equation

$$
E(\pi_n)\beta_n = pa_n - ([\zeta_{p^n+1}] - 1)^{p-1}.
$$

Similarly, we define elements $\tilde{a}_n$ and $\tilde{\beta}_n$ of $W_n(\mathcal{O}_F/p\mathcal{O}_F)$ using $\tilde{\pi}_n$ and $\tilde{\zeta}_{p^n+1}$. By definition, the element $\tilde{\beta}_n$ is a solution of the equation

$$
E([\tilde{\pi}_n])\tilde{\beta}_n = p\tilde{a}_n - ([\tilde{\zeta}_{p^n+1}] - 1)^{p-1}.
$$

Now what we have to show is the equality

$$
\tilde{\eta}(a_n\beta_n^{-1}E([\pi_n])^{p-1}) = \tilde{a}_n\tilde{\beta}_n^{-1}E([\tilde{\pi}_n])^{p-1}
$$

in the ring $\bar{A}'_{n,F,r^+}$. Since the element $a_n$ of $W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n})$ is a linear combination of the elements $1, [\zeta_{p^n+1}], \ldots, [\zeta_{p^n+1}]^{p-1}$ over $\mathbb{Z}$, we have $\tilde{\eta}(a_n) = \tilde{a}_n$ in $\bar{A}'_{n,F,r^+}$. Thus the elements $\tilde{\beta}_n$ and $\tilde{\eta}(\beta_n)$ satisfy the same equation in $\bar{A}'_{n,F,r^+}$. Since these two elements are invertible, we see that $\tilde{\eta}(\beta_n)^{-1}E([\tilde{\pi}_n]) = \tilde{\beta}_n^{-1}E([\tilde{\pi}_n])$ and the equality holds. Since the diagram above is compatible with the Frobenius endomorphisms, we see from the definition that $\tilde{\eta}$ also preserves $\text{Fil}^r$ and commutes with $\phi_r$ of both sides. \hfill $\square$

Thus the homomorphism $\tilde{\eta}$ induces a homomorphism of abelian groups

$$
T_{\text{crys},L_n,\pi_n}(M_n) \rightarrow T_{\text{crys},F,\tilde{\pi}_n}(M_n).
$$

Then the following lemma, whose proof is omitted in [2, Subsection 3.13], implies that this homomorphism is an injection. We insert here a proof of this lemma for the convenience of the reader.

Lemma 5.11. The ring homomorphism $\tilde{\eta} : \bar{A}'_{n,L_n,r^+} \rightarrow \bar{A}'_{n,F,r^+}$ is an injection.
For an algebraic extension \( N \) of \( F_n \), let us write \( \bar{A}'_N \) for the ring \( A'_{n,N,r} \). Note that the ring \( \bar{A}'_N / p\bar{A}'_N \) is isomorphic to the ring
\[
\mathcal{O}_N / \{ x \in \mathcal{O}_N \mid v_p(x) > \frac{r}{p^{n-1}(p-1)} \}.
\]
As in the proof of Lemma 5.9, we see that the homomorphism \( \bar{\eta} \) induces an injection
\[
\bar{A}'_{L_n} / p\bar{A}'_{L_n} \rightarrow \bar{A}'_F / p\bar{A}'_F.
\]
Thus it is enough to show the exactness of the sequence
\[
0 \rightarrow \bar{A}'_N / p^m \bar{A}'_N \overset{\times p}{\rightarrow} \bar{A}'_N / p^{m+1} \bar{A}'_N \rightarrow \bar{A}'_N / p\bar{A}'_N \rightarrow 0.
\]
Let \( \bar{x} \) and \( \bar{y} \) be in \( \bar{A}'_N \) such that \( p\bar{x} = p^{m+1}\bar{y} \). Let \( \bar{x} = (\bar{x}_0, \ldots, \bar{x}_{n-1}) \) and \( \bar{y} = (\bar{y}_0, \ldots, \bar{y}_{n-1}) \) be lifts of \( \bar{x} \) and \( \bar{y} \) in the ring \( W_n(\mathcal{O}_N) \), respectively. In this ring, we have
\[
(0, \bar{x}_0, \bar{x}_1, \ldots, ) = (0, \ldots, 0, \bar{y}_0^{m+1}, \bar{y}_1^{m+1}, \ldots) + ([\zeta_{pn}] - 1)r \hat{\bar{z}},
\]
where \( \hat{\bar{z}} \) is in the ideal \( m_{n,N} \). From this equality we see that \( v_p(\bar{x}_0) \geq r/(p^{n-1}(p-1)) \) and
\[
\hat{\bar{x}} = ([\zeta_{pn}] - 1)^r \hat{\bar{w}} + (0, \bar{x}_1', \bar{x}_2', \ldots)
\]
for some \( \hat{\bar{w}} \in m_{n,N} \) and \( \bar{x}_i' \in \mathcal{O}_N \). The image of the first term on the right-hand side in \( \bar{A}'_N \) is zero. Hence we may assume \( \bar{x}_0 = 0 \). Repeating this, we can see that \( \bar{x} \in p^m\bar{A}'_N \) and the above sequence is exact. \( \square \)

Now Corollary 4.13 shows that the abelian group \( T^*_{\text{crys}.L,\bar{\pi}_n}(M_n) \) is of order \( p^{nd} \), where \( d = \dim_{\mathbb{Q}_p} V \). This implies that the the abelian group \( T^*_{\text{crys}.F,\bar{\pi}_n}(M_n) \) is also of order \( p^{nd} \). Let \( g \in G_K \) be as in Lemma 5.8. Then we have the following lemma.

**Lemma 5.12.** The \( G_{F_n} \)-action on \( T^*_{}{}_{\text{crys}.K,\bar{\pi}_n}(M_n) \) is the conjugate of the action on \( T^*_{\text{crys}.K,\bar{\pi}_n}(M_n) \) by the element \( g \).

**Proof.** Let \( a_n, \tilde{a}_n \) and \( \beta_n, \tilde{\beta}_n \) be the elements of \( W_n(\mathcal{O}_K / p\mathcal{O}_K) \) as in the proof of Lemma 5.10. Let us consider the composite
\[
\Sigma \rightarrow A'_{n,r} \rightarrow A'_{n,r+}
\]
of the ring homomorphism defined by \( u \mapsto [\pi_n] \) and \( Y \mapsto -a_n\beta^{-1}_n E([\pi_n])^{p-1} \), and the map induced by \( g \). We claim that this is the natural ring homomorphism defined by \( \bar{\pi}_n \). For this, we only have to check that this composite sends the element \( Y \in \Sigma \) to \( -a_n\beta^{-1}_n E([\bar{\pi}_n]) \). Since the equality
\[
E([\pi_n])\beta_n = pa_n - (\zeta_{pn+1})-1)^{p-1}
\]
holds in the ring \( A'_{n,r+} \) on the source of the above map \( g \), we have
\[
E([\bar{\pi}_n])g(\beta_n) = p\tilde{a}_n - (\zeta_{pn+1})-1)^{p-1}
\]
in the ring $A'_{n,r+}$ on the target. Since the elements $g(\beta_n)$ and $\tilde{\beta}_n$ are invertible, we have $g(\beta_n)^{-1}E([\pi_n]) = \tilde{\beta}_n^{-1}E([\pi_n])$ and the claim follows. Thus we have an isomorphism of abelian groups

$$\text{Hom}_\Sigma(M_n, \bar{A}_{n,r+}) \rightarrow \text{Hom}(M_n, \bar{A}'_{n,r+})$$

where we consider on the ring $A'_{n,r+}$ on the right-hand side the filtered $\phi_r$ module structure over $\Sigma$ defined by $\tilde{\pi}_n$. Since $g(t_n) = \tilde{t}_n$, we can check that this isomorphism induces an injection

$$\text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M_n, \bar{A}'_{n,r+}) \rightarrow \text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M_n, \bar{A}_{n,r+}).$$

Since these abelian groups have the same cardinality, this is also an isomorphism. 

Since $L_n$ is Galois over $K$, the above lemma shows that the finite Galois extension of $F_n$ cut out by the action on $T^*_{\text{cris}, K, \tilde{\pi}_n}(M_n)$ is also $L_n$. Hence we see from Corollary 4.13 that $F$ also contains $L_n$ and Proposition 5.7 follows. This concludes the proof of Theorem 1.1. 

**Remark 5.13.** The ramification bound in Theorem 1.1 is sharp for $r \leq 1$. Indeed, the greatest upper ramification break $1 + e(n + 1/(p - 1))$ for $r = 1$ is obtained by the $p^n$-torsion of the Tate curve $K^\times/\pi^Z$ (see Remark 5.5). The author does not know whether this bound is sharp also for $r > 1$.

**References**


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