ON A RAMIFICATION BOUND OF SEMI-STABLE
TORSION REPRESENTATIONS OVER A LOCAL FIELD

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Abstract. Let $p$ be a rational prime, $k$ be a perfect field of characteristic $p$, $W = W(k)$ be the ring of Witt vectors, $K$ be a finite totally ramified extension of $\text{Frac}(W)$ of degree $e$ and $r$ be a non-negative integer satisfying $r < p - 1$. Let $V$ be a semi-stable $p$-adic $G_K$-representation with Hodge-Tate weights in $\{0, \ldots, r\}$. In this paper, we prove the upper numbering ramification group $G_{\mathfrak{m}}^{(j)}$ for $j > u(K, r, n)$ acts trivially on the mod $p^n$ representations associated to $V$, where $u(K, 0, n) = 0$, $u(K, 1, n) = 1 + e(n + 1/(p - 1))$ and $u(K, r, n) = 1 - p^{-n}e(K(\zeta_p)/K)^{-1} + e(n + r/(p - 1))$ for $r > 1$.

1. Introduction

Let $p$ be a rational prime, $k$ be a perfect field of characteristic $p$, $W = W(k)$ be the ring of Witt vectors and $K$ be a finite totally ramified extension of $K_0 = \text{Frac}(W)$ of degree $e = e(K)$. Let the maximal ideal of $K$ be denoted by $m_K$, an algebraic closure of $K$ by $\bar{K}$ and the absolute Galois group of $K$ by $G_K = \text{Gal}(\bar{K}/K)$. We normalize the valuation $v_K$ of $K$ as $v_K(p) = e$ and extend this to $\bar{K}$. Let $G_{\mathfrak{m}}^{(j)}$ denote the $j$-th upper numbering ramification group in the sense of [7]. Namely, we put $G_{\mathfrak{m}}^{(j)} = G_{\mathfrak{m}}^{j-1}$, where the latter is the upper numbering ramification group defined in [15].

Let $X_K$ be a proper smooth scheme over $K$ and put $X_{\bar{K}} = X_K \times_K \bar{K}$. Consider the $r$-th étale cohomology group $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ and its $G_K$-stable $\mathbb{Z}_p$-lattices $\mathcal{L} \supseteq \mathcal{L}'$. In [7], Fontaine conjectured the upper numbering ramification group $G_{\mathfrak{m}}^{(j)}$ acts trivially on the $G_K$-module $\mathcal{L}/\mathcal{L}'$ for $j > e(n + r/(p - 1))$ if $X_K$ has good reduction and this module is killed by $p^n$. For $e = 1$ and $r < p - 1$, this conjecture was proved independently by himself ([8], for $n = 1$) and Abrashkin ([2], for any $n$), using the theory of Fontaine-Laffaille ([10]) and the comparison theorem of Fontaine-Messing ([11]) between the $p$-adic étale cohomology groups of $X_K$ and the crystalline cohomology groups of the reduction of $X_K$. From this result, Fontaine also showed some rareness of a proper smooth scheme over $\mathbb{Q}$ with everywhere good reduction ([8, Théorème 1]). In fact, they proved this ramification bound for the torsion
representations of the crystalline $p$-adic representations of $G_K$ with Hodge-Tate weights in $\{0, \ldots, r\}$ in the case where $K$ is absolutely unramified.

On the other hand, for a semi-stable $p$-adic representation $V$ with Hodge-Tate weights in the same range, a similar ramification bound for $e = 1$ and $n = 1$ is obtained by Breuil (see [4, Proposition 9.2.2.2]). He showed, assuming the Griffiths transversality which in general does not hold, that if $e = 1$ and $r < p - 1$, then the ramification group $G_K^{(j)}$ acts trivially on the mod $p$ representations of $V$ for $j > 2 + 1/(p - 1)$.

In this paper, we prove a version of the result of Breuil for the case where $K$ is absolutely ramified, under the condition $r < p - 1$. Our main theorem is the following.

**Theorem 1.1.** Let $r$ be a non-negative integer such that $r < p - 1$. Let $V$ be a semi-stable $p$-adic $G_K$-representation with Hodge-Tate weights in $\{0, \ldots, r\}$ and $L \supseteq \mathcal{L}'$ be $G_K$-stable $\mathbb{Z}_p$-lattices in $V$. Suppose that the quotient $L/\mathcal{L}'$ is killed by $p^n$. Then the $j$-th upper numbering ramification group $G_K^{(j)}$ acts trivially on the $G_K$-module $L/\mathcal{L}'$ for $j > u(K, r, n)$, where

$$u(K, r, n) = \begin{cases} 0 & (r = 0), \\
1 + e(n + \frac{1}{p-1}) & (r = 1), \\
1 - \frac{1}{p^n e(K(\zeta_p)/K)} + e(n + \frac{r}{p-1}) & (r > 1)
\end{cases}$$

and $e(K(\zeta_p)/K)$ denotes the relative ramification index of the extension $K(\zeta_p)/K$.

We can check that this bound is sharp for $r \leq 1$ (Remark 5.13). From this theorem and [7, Proposition 1.3], we have the following corollary.

**Corollary 1.2.** Let the notation be as in the theorem and $L$ be the finite extension of $K$ cut out by the $G_K$-module $L/\mathcal{L}'$. Let $\mathfrak{D}_{L/K}$ denote the different of the extension $L/K$. Then we have the inequality

$$v_K(\mathfrak{D}_{L/K}) < u(K, r, n)$$

for $r > 0$ and $v_K(\mathfrak{D}_{L/K}) = 0$ for $r = 0$.

For the proof of Theorem 1.1, we essentially follow a beautiful argument of Abrashkin ([2]). We may assume $p \geq 3$ and $r \geq 1$. Thanks to Liu’s theorem ([14]) on the $G_K$-stable $\mathbb{Z}_p$-lattices in semi-stable $p$-adic representations, it is enough to bound the ramification of the $G_K$-module

$$T_{st, \mathbb{Z}}^*(\mathcal{M}_n) = \text{Hom}_{S, \text{Fil}_{r, N}}(\mathcal{M}_n, \hat{A}_{st, \infty}),$$

where $\mathcal{M}_n$ is a $p^n$-torsion object of a category $\text{Mod}^{r, N}_{S, \infty}$ of filtered $(\phi_r, N)$-modules over $S$ defined by Breuil ([3]) and $\hat{A}_{st, \infty}$ is a $p$-adic period ring. We may also assume $\zeta_p \in K$ and consider the finite Galois extension

$$F_n = K(\pi^{1/p^n}, \zeta_p^{p^n+1})$$

of $K$ whose upper ramification is bounded by the same value as in the theorem. Let $L_n$ be the finite Galois extension of $F_n$ cut out by $T_{st, \mathbb{Z}}^*(\mathcal{M}_n)|_{G_{F_n}}$.
Then we bound the ramification of $L_n$ over $K$. For this, we show that to study this $G_{F_p}$-module we can use a variant over a smaller coefficient ring $\Sigma$ of filtered $(\phi_r, N)$-modules over $S$. In precise, let $E(u)$ be the Eisenstein polynomial of a uniformizer $\pi$ of $K$ over $W$ and we set

$$\Sigma = W[[u, E(u)^p/p]].$$

This ring $\Sigma$ is small enough for the method of Abrashkin, in which he uses filtered modules of Fontaine-Laffaille ([10]) whose coefficient ring is $W$, to work also in the case where $K$ is absolutely ramified.

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2. Filtered $(\phi_r, N)$-modules of Breuil

In this section, we recall the theory of filtered $(\phi_r, N)$-modules over $S$ of Breuil, which is developed by himself and most recently by Caruso and Liu (see for example [3], [5], [14], [6]). In what follows, we always take the divided power envelope of a $W$-algebra with the compatibility condition with the natural divided power structure on $pW$.

Let $p \geq 3$ be a rational prime and $\sigma$ be the Frobenius endomorphism of $W$. We fix once and for all a uniformizer $\pi$ of $K$ and a system $\{\pi_n\}_{n \geq 0}$ of $p$-power roots of $\pi$ such that $\pi_0 = \pi$ and $\pi_n = \pi_{n+1}^p$ for any $n$. Let $E(u)$ be the Eisenstein polynomial of $\pi$ over $W$ and set $S = (W[u]^{\text{PD}})\wedge$, where PD means the divided power envelope and this is taken with respect to the ideal $(E(u))$, and $\wedge$ means the $p$-adic completion. The ring $S$ is endowed with the $\sigma$-semilinear endomorphism $\phi : u \mapsto u^p$ and a natural filtration $\text{Fil}^t S$ induced by the divided power structure such that $\phi(\text{Fil}^t S) \subseteq p^t S$ for any non-negative integer $t$. We set $\phi_t = p^{-t}\phi|_{\text{Fil}^t S}$ and $c = \phi_1(E(u)) \in S^\times$. Let $N$ denote the $W$-linear derivation on $S$ defined by the formula $N(u) = -u$. We also define a filtration, $\phi, \phi_t, N$ on $S_n = S/p^n S$ similarly.

Let $r \in \{0, \ldots, p-2\}$ be an integer. Set $\text{Mod}_{/S}^{\phi, N}$ to be the category consisting of the following data:

- an $S$-module $\mathcal{M}$ and its $S$-submodule $\text{Fil}^r S.\mathcal{M}$ containing $\text{Fil}^r S.\mathcal{M}$,
- a $\phi$-semilinear map $\phi_r : \text{Fil}^r S.\mathcal{M} \rightarrow \mathcal{M}$ satisfying
  $$\phi_r(s_m) = \phi_r(s) \phi(m)$$
  for any $s_r \in \text{Fil}^r S$ and $m \in \mathcal{M}$, where we set $\phi(m) = c^{-r} \phi_r(E(u)^r m)$,
- a $W$-linear map $N : \mathcal{M} \rightarrow \mathcal{M}$ such that
  $$N(sm) = N(s)m + sN(m)$$
  for any $s \in S$ and $m \in \mathcal{M}$,
  $$E(u)N(\text{Fil}^r S.\mathcal{M}) \subseteq \text{Fil}^r S.\mathcal{M},$$
the following diagram is commutative:

\[
\begin{array}{ccc}
\Fil^r \mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M} \\
E(u)N & \downarrow & eN \\
\Fil^r \mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M},
\end{array}
\]

and the morphisms of \(\text{Mod}^r_{/\mathcal{S}}\) are defined to be the \(S\)-linear maps preserving \(\Fil^r\) and commuting with \(\phi_r\) and \(N\). The category defined in the same way but dropping the data \(N\) is denoted by \(\text{Mod}^r_{/\mathcal{S}}\). These categories have obvious notions of exact sequences. Let \(\text{Mod}^{r,N}_{/\mathcal{S}_1}\) denote the full subcategory of \(\text{Mod}^{r,N}_{/\mathcal{S}}\) consisting of \(\mathcal{M}\) such that \(\mathcal{M}\) is free of finite rank over \(\mathcal{S}_1\) and generated as an \(\mathcal{S}_1\)-module by the image of \(\phi_r\). We write \(\text{Mod}^{r,N}_{/\mathcal{S}_\infty}\) for the smallest full subcategory which contains \(\text{Mod}^{r,N}_{/\mathcal{S}_1}\) and is stable under extensions. We let \(\text{Mod}^{r,N}_{/\mathcal{S}}\) denote the full subcategory consisting of \(\mathcal{M}\) such that

- the \(\mathcal{S}\)-module \(\mathcal{M}\) is free of finite rank and generated by the image of \(\phi_r\),
- the quotient \(\mathcal{M}/\Fil^r \mathcal{M}\) is \(p\)-torsion free.

We define full subcategories \(\text{Mod}^r_{/\mathcal{S}_1}\), \(\text{Mod}^r_{/\mathcal{S}_\infty}\) and \(\text{Mod}^r_{/\mathcal{S}}\) of \(\text{Mod}^{r,N}_{/\mathcal{S}}\) in a similar way. For \(\tilde{\mathcal{M}} \in \text{Mod}^{r,N}_{/\mathcal{S}_1}\) (resp. \(\text{Mod}^{r,N}_{/\mathcal{S}_\infty}\)), the quotient \(\tilde{\mathcal{M}}/p^n \mathcal{M}\) has a natural structure as an object of \(\text{Mod}^{r,N}_{/\mathcal{S}_1}\) (resp. \(\text{Mod}^{r,N}_{/\mathcal{S}_\infty}\)).

For \(p\)-torsion objects, we also have the following categories. Consider the \(k\)-algebra \(k[u]/(u^p) \cong \mathcal{S}_1/\Fil^p \mathcal{S}_1\) and let this algebra be denoted by \(\tilde{\mathcal{S}}_1\). The algebra \(\tilde{\mathcal{S}}_1\) is equipped with the natural filtration, \(\phi\) and \(N\) induced by those of \(\mathcal{S}\). Namely, \(\Fil^r \tilde{\mathcal{S}}_1 = u^{e_r} \tilde{\mathcal{S}}_1\), \(\phi(u) = u^p\) and \(N(u) = -u\). Let \(\text{Mod}^{r,N}_{/\tilde{\mathcal{S}}_1}\) denote the category consisting of the following data:

- an \(\tilde{\mathcal{S}}_1\)-module \(\tilde{\mathcal{M}}\) and its \(\tilde{\mathcal{S}}_1\)-submodule \(\Fil^r \tilde{\mathcal{M}}\) containing \(u^{e_r} \tilde{\mathcal{M}}\),
- a \(\phi\)-semilinear map \(\phi_r : \Fil^r \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}\),
- a \(k\)-linear map \(N : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}\) such that
  - \(N(sm) = N(s)m + sN(m)\) for any \(s \in \tilde{\mathcal{S}}_1\) and \(m \in \tilde{\mathcal{M}}\),
  - \(u^{e_r} N(\Fil^r \tilde{\mathcal{M}}) \subseteq \Fil^r \tilde{\mathcal{M}}\),
- the following diagram is commutative:

\[
\begin{array}{ccc}
\Fil^r \tilde{\mathcal{M}} & \xrightarrow{\phi_r} & \tilde{\mathcal{M}} \\
u^{e_r}N & \downarrow & eN \\
\Fil^r \tilde{\mathcal{M}} & \xrightarrow{\phi_r} & \tilde{\mathcal{M}},
\end{array}
\]
and whose morphisms are defined as before. Its full subcategory \( \text{Mod}^\phi_{/S_1} \) is defined by the following condition:

- As an \( \hat{S}_1 \)-module, \( \hat{\mathcal{M}} \) is free of finite rank and generated by the image of \( \phi_r \).

We define categories \( \text{Mod}^\phi_{/\hat{S}_1} \) and \( \text{Mod}^\phi_{/S_1} \) similarly.

Let \( D \) be a weakly admissible filtered \((\phi, N)\)-module over \( K \) satisfying \( \text{Fil}^0 D_K = D_K \) and \( \text{Fil}^{r+1} D_K = 0 \). Set \( S_{K_0} = S \otimes_W K_0 \) and \( D = D \otimes_{K_0} S_{K_0} \). Then the \( S_{K_0} \)-module \( D \) is equipped with the natural \( \phi \)-semilinear map \( \phi \otimes \sigma \) and \( K_0 \)-linear derivation \( N \otimes 1 + 1 \otimes N \), which are denoted by \( \phi \) and \( N \), respectively. We define a filtration on \( D \) inductively by \( \text{Fil}^0 D = D \) and

\[
\text{Fil}^{i+1} D = \{ x \in D \mid N(x) \in \text{Fil}^i D \text{ and } f_\pi(x) \in \text{Fil}^{i+1} D_K \},
\]

where \( f_\pi : D \to D_K \) is induced by the map \( S \to \mathcal{O}_K \) sending \( u \) to \( \pi \). An \( S \)-submodule \( \hat{\mathcal{M}} \) of \( D \) is said to be a strongly divisible lattice of \( D \) if the following conditions are satisfied:

1. the \( S \)-module \( \hat{\mathcal{M}} \) is free of finite rank,
2. \( \hat{\mathcal{M}} \otimes_W K_0 = D \),
3. \( \hat{\mathcal{M}} \) is stable under \( \phi \) and \( N \),
4. \( \phi(\text{Fil}^r \hat{\mathcal{M}}) \subseteq p^r \hat{\mathcal{M}} \), where we set \( \text{Fil}^r \hat{\mathcal{M}} = \hat{\mathcal{M}} \cap \text{Fil}^r D \).

We put \( \phi_r = p^{-r} \phi |_{\text{Fil}^r \hat{\mathcal{M}}} \). Then the \( S \)-module \( \hat{\mathcal{M}} \) is generated by \( \phi_r(\text{Fil}^r \hat{\mathcal{M}}) \) ([3, Proposition 2.1.3]) and we can consider \( \hat{\mathcal{M}} \) as an object of \( \text{Mod}^\phi_{/S_1} \).

Let \( \mathcal{A}_\text{crys} \) and \( \mathcal{A}_\text{st} \) be \( p \)-adic period rings. These are constructed as follows. Set \( R \) to be the ring

\[
R = \varprojlim (\mathcal{O}_K / p \mathcal{O}_K \leftarrow \mathcal{O}_K / p \mathcal{O}_K \leftarrow \cdots ),
\]

where every arrow is the \( p \)-power map. For an element \( x = (x_i)_{i \in \mathbb{Z}_{\geq 0}} \in R \) and an integer \( n \geq 0 \), we set

\[
x^{(n)} = \lim_{m \to \infty} \tilde{x}^{p^n m} \in \mathcal{O}_\mathbb{C},
\]

where \( \tilde{x} \) is a lift of \( x \) in \( \mathcal{O}_K \) and \( \mathcal{O}_\mathbb{C} \) is the \( p \)-adic completion of \( \mathcal{O}_K \). Let \( v_p \) denote the valuation of \( \mathcal{O}_\mathbb{C} \) normalized as \( v_p(p) = 1 \). Then the ring \( R \) is a complete valuation ring whose valuation of an element \( x \in R \) is given by \( v_R(x) = v_p(x^{(0)}) \). We define a natural ring homomorphism \( \theta \) by

\[
\theta : W(R) \to \mathcal{O}_\mathbb{C} \\
(x_0, x_1, \ldots) \mapsto \sum_{n \geq 0} p^n x^{(n)}.
\]

Then \( \mathcal{A}_\text{crys} \) is the \( p \)-adic completion of the divided power envelope of \( W(R) \) with respect to the principal ideal \( \text{Ker} (\theta) \) and \( \mathcal{A}_\text{st} \) is the \( p \)-adic completion of the divided power polynomial ring \( \mathcal{A}_\text{crys} (X) \) over \( \mathcal{A}_\text{crys} \). We set \( \mathcal{A}_\text{crys,}\infty = \mathcal{A}_\text{crys} \otimes_W K_0 / W \) and \( \mathcal{A}_\text{st,}\infty = \mathcal{A}_\text{st} \otimes_W K_0 / W \). Put \( \pi = (\pi_n)_{n \in \mathbb{Z}_{\geq 0}} \in R \), where we abusively let \( \pi_n \) denote the image of \( \pi_n \in \mathcal{O}_K \) in \( \mathcal{O}_K / p \mathcal{O}_K \). These rings
are considered as $S$-algebras by the ring homomorphisms $S \to \hat{A}_{\text{st}}$ and $\hat{A}_{\text{st}} \to A_{\text{crys}}$ which are defined by $u \mapsto [\pi]/(1 + X)$ and $X \mapsto 0$, respectively. The ring $A_{\text{crys}}$ is endowed with a natural filtration induced by the divided power structure, a natural Frobenius endomorphism $\phi$ and the $\phi$-semilinear map $\phi_t = p^{-t} \phi|_{\text{Fil}^t A_{\text{crys}}}$. With these structures, $A_{\text{crys}}$ and $A_{\text{crys}, 1}$ are considered as objects of $\text{Mod}_{r, \phi, N}/S$. Moreover, the absolute Galois group $G_K$ acts naturally on these two rings. As for $\hat{A}_{\text{st}}$, its filtration is defined by

$$\text{Fil}^t \hat{A}_{\text{st}} = \left\{ \sum_{i \geq 0} a_i \frac{X^i}{i!} \bigg| a_i \in \text{Fil}^{t-i} A_{\text{crys}}, \lim_{i \to \infty} a_i = 0 \right\}$$

and the Frobenius structure of $A_{\text{crys}}$ extends to $\hat{A}_{\text{st}}$ by

$$\phi(X) = (1 + X)^p - 1,$$

$$\phi_t = p^{-t} \phi|_{\text{Fil}^t \hat{A}_{\text{st}}}.$$  

We write $N$ also for the $A_{\text{crys}}$-linear derivation on $\hat{A}_{\text{st}}$ defined by

$$N(X) = 1 + X.$$  

The rings $\hat{A}_{\text{st}}$ and $\hat{A}_{\text{st}, 1}$ are objects of $\text{Mod}_{r, \phi, N}/S$. The $G_K$-action on $A_{\text{crys}}$ naturally extends to an action on $\hat{A}_{\text{st}}$. Indeed, the action of $g \in G_K$ on $\hat{A}_{\text{st}}$ is defined by the formula

$$g(X) = [\varepsilon(g)](1 + X) - 1,$$

where $g(\pi_n) = \varepsilon_n(g) \pi_n$ and $\varepsilon(g) = (\varepsilon_n(g))_{n \in \mathbb{Z} \geq 0} \in R$ with the abusive notation as above.

These rings have other descriptions, as follows. For an integer $n \geq 1$, put $W_n = W/p^n W$ and let $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ be the ring of Witt vectors of length $n$ associated to $\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$. We define a $W_n$-algebra structure on $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ by twisting the natural $W_n$-algebra structure by $\sigma^{-n}$. Then the natural ring homomorphism

$$\theta_n : W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \to \mathcal{O}_{\bar{K}}/p^n \mathcal{O}_{\bar{K}}$$

$$(a_0, \ldots, a_{n-1}) \mapsto \sum_{i=0}^{n-1} \hat{a}_i p^{n-i},$$

where $\hat{a}_i$ is a lift of $a_i$ in $\mathcal{O}_{\bar{K}}$, is $W_n$-linear. Let us denote

$$W_{n, \text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$$

the divided power envelope of $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ with respect to the ideal $\text{Ker}(\theta_n)$. This ring is considered as an $S$-algebra by $u \mapsto [\pi_n]$. This ring also has a natural filtration defined by the divided power structure, and a natural $G_K$-module structure. The Frobenius endomorphism of the ring of Witt vectors
induces on this ring a $\phi$-semilinear Frobenius endomorphism, which is denoted also by $\phi$. Then, by the $S$-linear transition maps

$$W^\text{PD}_{n+1}(\mathcal{O}_K/p\mathcal{O}_K) \to W^\text{PD}_n(\mathcal{O}_K/p\mathcal{O}_K)$$

$$(a_0, \ldots, a_n) \mapsto (a_0^p, \ldots, a_{n-1}^p),$$

these $S$-algebras form a projective system compatible with all structures. Using this transition map, a $\phi$-semilinear map

$$\phi_r : \text{Fil}^r W^\text{PD}_n(\mathcal{O}_K/p\mathcal{O}_K) \to W^\text{PD}_n(\mathcal{O}_K/p\mathcal{O}_K)$$

is defined by setting $\phi_r(x)$ to be the image of $p^{-r}\phi(\hat{x})$, where $\hat{x}$ is a lift of $x$ in $\text{Fil}^r W^\text{PD}_{n+r}(\mathcal{O}_K/p\mathcal{O}_K)$. By definition, the maps $\phi_r$ are also compatible with the transition maps. The $S$-algebra $W^\text{PD}_n(\mathcal{O}_K/p\mathcal{O}_K)$ is considered as an object of $\text{Mod}^r_{/S}$. Then we have a natural isomorphism in $\text{Mod}^r_{/S}$

$$\text{A}_{\text{cris}}/p^n \text{A}_{\text{cris}} \to W^\text{PD}_n(\mathcal{O}_K/p\mathcal{O}_K)$$

$$(x_0, \ldots, x_{n-1}) \mapsto (x_{0,n}, \ldots, x_{n-1,n}),$$

where we set $x_i = (x_{i,k})_{k \in \mathbb{Z}_{\geq 0}}$.

Similarly, the divided power polynomial ring

$$W^\text{PD}_n(\mathcal{O}_K/p\mathcal{O}_K)(X)$$

over $W^\text{PD}_n(\mathcal{O}_K/p\mathcal{O}_K)$ is considered as an $S$-algebra by $u \mapsto [\pi_n]/(1 + X)$. This ring has a natural filtration coming from the divided power structure. We define a $G_K$-action on this ring by

$$g(X) = [\varepsilon_n(g)](1 + X) - 1.$$  

We also define a $\phi$-semilinear Frobenius endomorphism, which we also write as $\phi$, by $\phi(X) = (1 + X)^7 - 1$ and a $W^\text{PD}_n(\mathcal{O}_K/p\mathcal{O}_K)$-linear derivation $N$ by $N(X) = 1 + X$. These rings form a projective system of $S$-algebras compatible with all structures by the transition maps defined by the maps above and $X \mapsto X$. We define $\phi$-semilinear maps

$$\phi_r : \text{Fil}^r W^\text{PD}_n(\mathcal{O}_K/p\mathcal{O}_K)(X) \to W^\text{PD}_n(\mathcal{O}_K/p\mathcal{O}_K)(X)$$

compatible with the transition maps as before. The $S$-algebra $W^\text{PD}_n(\mathcal{O}_K/p\mathcal{O}_K)(X)$ is considered as an object of $\text{Mod}^r_{/S}$ and there exists a natural isomorphism in $\text{Mod}^r_{/S}$

$$\hat{\text{A}}_{\text{st}}/p^n \hat{\text{A}}_{\text{st}} \to W^\text{PD}_n(\mathcal{O}_K/p\mathcal{O}_K)(X)$$

$$(x_0, \ldots, x_{n-1}) \mapsto (x_{0,n}, \ldots, x_{n-1,n})$$

$$X \mapsto X$$

which is $G_K$-linear.

Put $K_n = K(\pi_n)$ and $K_\infty = \bigcup_n K_n$. For $\mathcal{M} \in \text{Mod}^r_{/S_\infty}$, we define a $G_K$-module $T^*_{\text{st,}\mathbb{Z}}(\mathcal{M})$ to be

$$T^*_{\text{st,}\mathbb{Z}}(\mathcal{M}) = \text{Hom}_{S, \text{Fil}^r_{/\mathbb{Z}}}(\mathcal{M}, \hat{\text{A}}_{\text{st,}\infty}).$$
When $\mathcal{M}$ is killed by $p^n$, we have a natural identification of $G_K$-modules
\[ T_{st,\mathbb{Z}}^*(\mathcal{M}) = \text{Hom}_{S,\text{Fil}^r,\phi_r,N}(\mathcal{M}, W_n^{PD}(\mathcal{O}_K/p\mathcal{O}_K) \langle X \rangle). \]
Note that the $G_K$-module on the right-hand side is independent of the choice of $\pi_k$ for $k > n$. Since the natural map
\[ W_n^{PD}(\mathcal{O}_K/p\mathcal{O}_K) \langle X \rangle \rightarrow W_n^{PD}(\mathcal{O}_K/p\mathcal{O}_K) \]
\[ X \mapsto 0 \]
is by definition $G_{K_n}$-linear, we also have a $G_{K_n}$-linear isomorphism ([3, Lemme 2.3.1.1])
\[ T_{st,\mathbb{Z}}^*(\mathcal{M})|_{G_{K_n}} \rightarrow \text{Hom}_{S,\text{Fil}^r,\phi_r}(\mathcal{M}, W_n^{PD}(\mathcal{O}_K/p\mathcal{O}_K)). \]

A variant of filtered $(\phi_r,N)$-modules over $S$ is also introduced by Breuil and Kisin, and developed also by Caruso and Liu (see for example [12, 13, 14, 6]). Put $\mathcal{S} = W[[u]]$ and let $\phi : \mathcal{S} \rightarrow \mathcal{S}$ be the $\sigma$-semilinear Frobenius endomorphism defined by $\phi(u) = u^p$. Let $'\text{Mod}_{\mathcal{S}}^{r,\phi}$ denote the category consisting of the following data:
- an $\mathcal{S}$-module $\mathfrak{M}$,
- a $\phi$-semilinear map $\mathfrak{M} \rightarrow \mathfrak{M}$, which is denoted also by $\phi$, such that the cokernel of the map $1 \otimes \phi : \phi^*\mathfrak{M} \rightarrow \mathfrak{M}$ is killed by $E(u)^r$,
and whose morphisms are defined as before. The full subcategory of $'\text{Mod}_{\mathcal{S}}^{r,\phi}$ consisting of $\mathfrak{M}$ such that $\mathfrak{M}$ is free of finite rank over $\mathcal{S}/p\mathcal{S}$ (resp. over $\mathcal{S}$) is denoted by $\text{Mod}_{\mathcal{S}}^{r,\phi}$ (resp. $\text{Mod}_{\mathcal{S}_1}^{r,\phi}$). We let $\text{Mod}_{\mathcal{S}_\infty}^{r,\phi}$ denote the smallest full subcategory which contains $\text{Mod}_{\mathcal{S}_1}^{r,\phi}$ and is stable under extensions, as before. Then we have an exact functor ([6, Proposition 2.1.2], see also [12, Proposition 1.1.11])
\[ \mathcal{M}_{\mathcal{S}_\infty} : \text{Mod}_{\mathcal{S}_\infty}^{r,\phi} \rightarrow \text{Mod}_{\mathcal{S}_\infty}^{r,\phi}. \]

For $\mathfrak{M} \in \text{Mod}_{\mathcal{S}_\infty}^{r,\phi}$, the filtered $\phi_r$-module $\mathcal{M} = \mathcal{M}_{\mathcal{S}_\infty}(\mathfrak{M})$ over $S$ is defined as follows:
- $\mathcal{M} = S \otimes_{\phi,\mathcal{S}} \mathfrak{M}$,
- $\text{Fil}^r\mathcal{M} = \text{Ker}(\mathcal{M} \rightarrow S \otimes_{\mathcal{S}} \mathfrak{M} \rightarrow (S/\text{Fil}^rS) \otimes_{\mathcal{S}} \mathfrak{M})$,
- $\phi_r : \text{Fil}^r\mathcal{M} \rightarrow \text{Fil}^rS \otimes_{\mathcal{S}} \mathfrak{M} \rightarrow 1 : S \otimes_{\phi,\mathcal{S}} \mathfrak{M} = \mathcal{M}$.

We write $\mathcal{M}_{\mathcal{S}}$ for the functor $\text{Mod}_{\mathcal{S}}^{r,\phi} \rightarrow \text{Mod}_{\mathcal{S}}^{r,\phi}$ defined similarly.

Finally, let $D$ and $\mathcal{D}$ be as above and $\mathbf{M}$ be a strongly divisible lattice in $\mathcal{D}$. The $S$-module $\mathcal{M}_n = \mathbf{M}/p^n\mathbf{M}$ has a natural structure as an object of $\text{Mod}_{/S_\infty}^{r,\phi,N}$. We set a $G_K$-module $\hat{T}_{st,\mathbb{Z}}^*(\mathbf{M})$ to be
\[ \hat{T}_{st,\mathbb{Z}}^*(\mathbf{M}) = \text{Hom}_{S,\text{Fil}^r,\phi_r,N}(\mathbf{M}, \mathbf{A}_{st}). \]
Then we have an exact sequence of $G_K$-modules
\[ 0 \rightarrow \hat{T}_{st,\mathbb{Z}}^*(\mathbf{M}) \xrightarrow{p^n} \hat{T}_{st,\mathbb{Z}}^*(\mathbf{M}) \rightarrow T_{st,\mathbb{Z}}^*(\mathcal{M}_n) \rightarrow 0. \]
The $G_K$-module $\hat{T}_{st, \mathcal{M}}^*(\mathcal{M})$ is naturally considered as a $G_K$-stable $\mathbb{Z}_p$-lattice in $V_{st}^*(D)$. By Liu’s theorem ([14, Theorem 2.3.5]), the functor $\hat{T}_{st, \mathcal{M}}^*$ gives an anti-equivalence of categories between the category of strongly divisible lattices in $\mathcal{D}$ and the category of $G_K$-stable $\mathbb{Z}_p$-lattices in $V_{st}^*(D)$. Moreover, for such a lattice $\mathcal{L}$, its corresponding strongly divisible lattice $\hat{\mathcal{M}}$ in $\mathcal{D}$ is in the essential image of the functor $\mathcal{M}_{\mathcal{L}}$ ([14, Subsection 3.5]).

3. Filtered $\phi_r$-modules over $\Sigma$

In this section, we define another variant of filtered $\phi_r$-modules over $S$ and prove its properties.

Let $p \geq 3$ be a rational prime and $r$ be an integer such that $0 \leq r < p - 1$. Consider the $W$-algebra $\Sigma = W[[u,Y]]/(E(u)p - pY)$ as in [3, Subsection 3.2]. We regard $\Sigma$ as a subring of $S$ by the map sending $Y$ to $E(u)p/p$. Then the element $c = \phi_1(E(u)) \in \Sigma^\times$ is contained in $\Sigma^\times$. We define on $\Sigma$ a $\sigma$-semilinear Frobenius endomorphism $\phi$ by $\phi(u) = u^p$ and $\phi(Y) = p^{p-1}c^\phi$. Put $\text{Fil}^r\Sigma = (E(u)^r, Y)$ for $0 \leq t \leq p - 1$ and $\text{Fil}^p\Sigma = (Y)$. Then we have $\phi(\text{Fil}^r\Sigma) \subseteq p^t\Sigma$ for $0 \leq t \leq p - 1$. We put $\phi_t = p^{-t}\phi|_{\text{Fil}^t\Sigma}$. We also set $\Sigma_m = \Sigma/p^n\Sigma$ and put on this ring the natural structures induced by those of $\Sigma$.

We define a category $'\text{Mod}^r_{\Sigma}$ of filtered $\phi_r$-modules over $\Sigma$ to be the category consisting of the following data:

- a $\Sigma$-module $M$ and its $\Sigma$-submodule $\text{Fil}^r\Sigma.M$,
- a $\phi$-semilinear map $\text{Fil}^rM \to M$ satisfying $\phi_r(s_r m) = \phi_r(s_r)\phi(m)$ for any $s_r \in \text{Fil}^r\Sigma$ and $m \in M$, where we set $\phi(m) = c^{-r}\phi_r(E(u)^r m)$.

and the morphisms are defined in the same manner as $'\text{Mod}^p_{/\Sigma}$. This category has a natural notion of exact sequences. We define its full subcategory $\text{Mod}^r_{/\Sigma_{\Sigma}}$ to be the category consisting of $M$ which is free of finite rank and generated by the image of $\phi_r$ as a $\Sigma_1$-module. We also let $\text{Mod}^{r}_{/\Sigma_{\infty}}$ denote the smallest full subcategory of $'\text{Mod}^r_{/\Sigma}$ which contains $\text{Mod}^r_{/\Sigma_1}$ and is stable under extensions. Moreover, we define a full subcategory $\text{Mod}^r_{/\Sigma}\phi$ of $'\text{Mod}^r_{/\Sigma}$ to be the category consisting of $M$ such that

- the $\Sigma$-module $M$ is free of finite rank and generated by the image of $\phi_r$,
- the quotient $M/\text{Fil}^rM$ is $p$-torsion free.

Then we see that for $\hat{M} \in \text{Mod}^r_{/\Sigma}\phi$, the quotient $\hat{M}/p^n\hat{M}$ is naturally considered as an object of $\text{Mod}^r_{/\Sigma_{\infty}}$.

The natural ring isomorphism $\Sigma_1/\text{Fil}^p\Sigma_1 \cong \tilde{S}_1$ defines a functor $T_0 : \text{Mod}^r_{/\Sigma_1} \to \text{Mod}^r_{/\tilde{S}_1}$ by $M \mapsto M/\text{Fil}^p\Sigma_1.M$. Then [3, Proposition 2.2.1.3] and Nakayama’s lemma shows the following.
Lemma 3.1. Let $M$ be an object of $\text{Mod}_{r,\phi}^\Sigma_1$ of rank $d$ over $\Sigma_1$. Then there exists a basis $\{e_1, \ldots, e_d\}$ of $M$ such that $\text{Fil}^r M = \Sigma_1 u^r e_1 + \cdots + \Sigma_1 u^r e_d + \text{Fil}^p \Sigma_1 M$ for some integers $r_1, \ldots, r_d$ with $0 \leq r_i \leq e_r$ for any $i$.

Then we can show the following lemma just as in the proof of [3, Lemme 2.3.1.3].

**Lemma 3.2.** The functor $M \mapsto \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, A_{\text{crys}, \infty})$ from $\text{Mod}_{r,\phi}^\Sigma_1$ to the category of $G_{K_1}$-modules is exact.

For $M \in \text{Mod}_{r,\phi}^\Sigma_1$, we can show as in the case of the category $\text{Mod}_{r,\phi}^S_1$ that there is an isomorphism of $G_{K_1}$-modules

$$\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, (\mathcal{O}_K/p\mathcal{O}_K)^{\text{PD}}) \rightarrow \text{Hom}_{\Sigma_1, \text{Fil}^r, \phi_r}(T_0(M), \mathcal{O}_K/p\mathcal{O}_K),$$

where $\mathcal{O}_K/p\mathcal{O}_K$ is considered as an object of $\mathcal{M}_{r,\phi}^\Sigma_1$ by the natural isomorphism

$$(\mathcal{O}_K/p\mathcal{O}_K)^{\text{PD}}/\text{Fil}^p(\mathcal{O}_K/p\mathcal{O}_K)^{\text{PD}} \rightarrow \mathcal{O}_K/p\mathcal{O}_K.$$

Thus [3, Lemme 2.3.1.2] implies that, for such a $\Sigma_1$-module $M$, we have

$$\#\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, (\mathcal{O}_K/p\mathcal{O}_K)^{\text{PD}}) = p^d,$$

where $d = \dim_{\Sigma_1} M$.

For the category $\text{Mod}_{r,\phi}^\Sigma_1$, we can show the following lemma just as in the proof of [14, Proposition 4.1.2].

**Lemma 3.3.** Let $\hat{M}$ be in $\text{Mod}_{r,\phi}^\Sigma_1$. Then there exists $\alpha_1, \ldots, \alpha_d \in \hat{M}$ such that $\text{Fil}^r \hat{M} = \Sigma_1 \alpha_1 + \cdots + \Sigma_1 \alpha_d + \text{Fil}^p \Sigma \hat{M}$, $E(u)^r \hat{M} \subseteq \Sigma_1 \alpha_1 + \cdots + \Sigma_1 \alpha_d$ and the elements $e_1 = \phi_r(\alpha_1), \ldots, e_d = \phi_r(\alpha_d)$ form a basis of $\hat{M}$.

**Corollary 3.4.** Let $\hat{M}$ be in $\text{Mod}_{r,\phi}^\Sigma_1$ and $A$ be a $\Sigma$-algebra which has a structure as an object of $\mathcal{M}_{r,\phi}^\Sigma$. Let $C \in M_d(\Sigma)$ be the matrix such that

$$(\alpha_1, \ldots, \alpha_d) = (e_1, \ldots, e_d)C$$

with the notation of the previous lemma. Then a $\Sigma$-linear homomorphism $f : \hat{M} \rightarrow A$ preserving $\text{Fil}^r$ also commutes with $\phi_r$ if and only if

$$\phi_r(f(e_1, \ldots, e_d)C) = (f(e_1), \ldots, f(e_d)).$$

**Proof.** Suppose that the latter condition holds. By assumption, we have

$$E(u)^r(e_1, \ldots, e_d) = (\alpha_1, \ldots, \alpha_d)C'.$$
for some \( C' \in \text{Mod}_d(\Sigma) \). We claim that \( f \) commutes with \( \phi \). Indeed, we have
\[
\phi(f(e_1, \ldots, e_d)) = e^{-r}\phi_r(E(u)^r(f(e_1), \ldots, f(e_d))) = e^{-r}\phi_r((f(\alpha_1), \ldots, f(\alpha_d))C') = e^{-r}f(e_1, \ldots, e_d)\phi(C') = e^{-r}f(\phi_r(\alpha_1, \ldots, \alpha_d))\phi(C') = e^{-r}f(\phi_r(E(u)^r(e_1, \ldots, e_d))) = f(\phi(e_1, \ldots, e_d)).
\]
This implies \( \phi_r \circ f = f \circ \phi_r \) also on \( \text{Fil}^p \Sigma \hat{M} \).

**Corollary 3.5.** Let \( \hat{M} \) and \( A \) be as above and \( J \subseteq \text{Fil}^r A \) be an ideal of \( A \) such that \( \phi_r(J) \subseteq J \). We can consider the \( \Sigma \)-algebra \( A/J \) naturally as an object of \( \text{Mod}^{r, \phi}_\Sigma \). Suppose that for any \( x \in J \), there exists \( t \in \mathbb{Z}_{\geq 0} \) such that \( \phi_r^t(x) = 0 \). Then we have an isomorphism
\[
\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(\hat{M}, A) \rightarrow \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(\hat{M}, A/J).
\]

**Proof.** Let \( f : \hat{M} \rightarrow A/J \) be an element of the abelian group on the right-hand side and \( \hat{x}_i \) be an lift of \( f(e_i) \) in \( A \). By the previous corollary, it is enough to show that for any \( (\hat{c}_1, \ldots, \hat{c}_d) \in J^d \), there is a unique solution \( (\hat{y}_1, \ldots, \hat{y}_d) \in J^d \) of the equation
\[
(\hat{c}_1, \ldots, \hat{c}_d) + (\phi_r(\hat{y}_1), \ldots, \phi_r(\hat{y}_d))\phi(C) = (\hat{y}_1, \ldots, \hat{y}_d).
\]
By assumption, the \( d \)-tuple
\[
\sum_{i=0}^{t}(\phi_r(\hat{c}_1), \ldots, \phi_r(\hat{c}_d))\phi(C)^i(-1)\cdots\phi(C)
\]
is stable for sufficiently large \( t \) and we see that this limit gives a unique solution of the equation. \( \square \)

For an \( \mathcal{E} \)-module \( \mathfrak{M} \) in \( \text{Mod}^{r, \phi}_{/\mathcal{E}^\infty} \) (resp. \( \text{Mod}^{r, \phi}_{/\mathcal{E}} \)), we associate to it a \( \Sigma \)-module \( M \in \text{Mod}^{r, \phi}_\Sigma \) as follows:

- \( M = \Sigma \otimes_{\phi, \mathcal{E}} \mathfrak{M} \),
- \( \text{Fil}^r M = \text{Ker}(M \xrightarrow{1^r\phi} \Sigma \otimes_{\mathcal{E}} \mathfrak{M} \rightarrow (\Sigma/\text{Fil}^r \Sigma) \otimes_{\mathcal{E}} \mathfrak{M}) \),
- \( \phi_r : \text{Fil}^r M \xrightarrow{1^r\phi} \Sigma \otimes_{\phi, \mathcal{E}} \mathfrak{M} \xrightarrow{\phi_r+1} \Sigma \otimes_{\phi, \mathcal{E}} \mathfrak{M} = M \).

We can check that this defines an exact functor \( \text{Mod}^{r, \phi}_{/\mathcal{E}^\infty} \rightarrow \text{Mod}^{r, \phi}_{/\mathcal{E}} \) (resp. \( \text{Mod}^{r, \phi}_{/\mathcal{E}} \rightarrow \text{Mod}^{r, \phi}_{/\mathcal{E}^\infty} \)) as in the proof of [12, Proposition 1.1.11]. We let this functor be denoted by \( M_{\mathcal{E}^\infty} \) (resp. \( M_{\mathcal{E}} \)).

**Proposition 3.6.** Let \( \mathfrak{M} \) be an object of \( \text{Mod}^{r, \phi}_{/\mathcal{E}^\infty} \) which is killed by \( p^n \). Set \( M = M_{\mathcal{E}^\infty} (\mathfrak{M}) \) and \( \mathcal{M} = M_{\mathcal{E}^\infty} (\mathfrak{M}) \). Then there exists a natural isomorphism of \( G_{K_n} \)-modules
\[
\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, W_n^{\text{PD}}(O_K/pO_K)) \rightarrow \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(\mathcal{M}, W_n^{\text{PD}}(O_K/pO_K)).
\]
Proof. By definition, $\mathcal{M} = S \otimes_{\Sigma} M$ and we have a natural isomorphism

$$\text{Hom}_{\Sigma}(M, W_n^{\text{PD}}(\mathcal{O}_R/p\mathcal{O}_R)) \to \text{Hom}_S(\mathcal{M}, W_n^{\text{PD}}(\mathcal{O}_R/p\mathcal{O}_R)).$$

Let $f$ be an element of $\text{Hom}_{\Sigma}\Fil^r,\phi_r(M, W_n^{\text{PD}}(\mathcal{O}_R/p\mathcal{O}_R))$ and $f'$ be the image of $f$ in the right-hand side of the above isomorphism. Let us check that $f'$ preserves $\Fil^r$ and commutes with $\phi_r$. Since $f'$ is $S$-linear, it maps $\Fil^r S M$ into $\Fil^r W_n^{\text{PD}}(\mathcal{O}_R/p\mathcal{O}_R)$. For $x \in \Fil^r \mathcal{M} \cap \text{Im}(M \to \mathcal{M})$, the commutative diagram whose right vertical arrow is an isomorphism

$$M = \Sigma \otimes_{\phi_r} M \xrightarrow{1 \otimes \phi} \Sigma \otimes_{\phi_r} M \xrightarrow{\phi} \Sigma/\Fil^r \Sigma \otimes_{\phi_r} M \xrightarrow{1 \otimes \phi} \Sigma/\Fil^r S \otimes_{\phi_r} M \xrightarrow{1 \otimes \phi} S/\Fil^r S \otimes_{\phi_r} M$$

implies $x \in \text{Im}(\Fil^r M \to \Fil^r \mathcal{M})$ and thus $f'(x) \in \Fil^r W_n^{\text{PD}}(\mathcal{O}_R/p\mathcal{O}_R)$.

As for the compatibility with $\phi_r$, again by the $S$-linearity of $f'$ it suffices to show $f'(x) = \phi_r(f'(x))$ for $x \in \Fil^r \mathcal{M} \cap \text{Im}(M \to \mathcal{M}) = \text{Im}(\Fil^r M \to \Fil^r \mathcal{M})$. This follows from the commutative diagram

$$\Fil^r M \xrightarrow{\phi_r} M \xrightarrow{f} W_n^{\text{PD}}(\mathcal{O}_R/p\mathcal{O}_R) \xrightarrow{\phi} \Fil^r \mathcal{M} \xrightarrow{f'} W_n^{\text{PD}}(\mathcal{O}_R/p\mathcal{O}_R).$$

Hence the map in the proposition is well-defined and injective. To prove the bijectivity, by devissage we may assume that $p\mathfrak{m} = 0$. Then both sides of this injection have the same cardinality by the above remark. Thus the proposition follows. \qed

4. A METHOD OF ABRASHKIN

In this section, we study the $G_{K_n}$-module $\text{Hom}_{\Sigma}\Fil^r,\phi_r(M, W_n^{\text{PD}}(\mathcal{O}_R/p\mathcal{O}_R))$ following Abrashkin ([2]).

Let $p \geq 3$ and $0 \leq r < p - 1$ be as before. Consider the Lubin-Tate logarithm

$$l(X) = X + \frac{X^p}{p} + \cdots + \frac{X^{p^n}}{p^n} + \cdots$$

and put $\psi(X) = l^{-1}(\log(1 + X))$. Then $\psi$ gives a strict isomorphism of formal groups between the formal group associated to the logarithm $l(X)$ and the multiplicative group $\mathbb{G}_m$ over $\mathbb{Z}_p$. We fix a system of $p$-power roots of unity $\{\zeta_p^n\}_{n \in \mathbb{Z}_{>0}}$ such that $\zeta_p \neq 1$ and $\zeta_p^n = \zeta_{p^{n+1}}$ for any $n$, and set an element $\xi$ of $R$ to be $\langle \zeta_p^n \rangle_{n \in \mathbb{Z}_{>0}}$. Then the elements $[\xi] - 1$ and $[\xi^{1/p}] - 1$ are topologically nilpotent in $W(R)$ and the element of $W(R)$

$$t = \psi([\xi] - 1)/\psi([\xi^{1/p}] - 1)$$

is a generator of the principal ideal \( \text{Ker}(\theta) \). The element \( Z = \psi([\bar{e}] - 1)^{p-1}/p \) of \( A_{\text{crys}} \) is topologically nilpotent and \( \phi(t) \) is contained in the subset

\[
p(1 + ZW(R)[[Z]])
\]

of \( A_{\text{crys}} \) ([2, Subsection 1.8]). We set

\[
\hat{A} = W(R)[[Z]] \subseteq A_{\text{crys}}.
\]

**Lemma 4.1.** The element \( tp/p \) of \( A_{\text{crys}} \) is contained in the subring \( \hat{A} \) and topologically nilpotent in this subring.

**Proof.** Put \( t' = ([\bar{e}] - 1)/([\bar{e}]^{1/p} - 1) \). This is another generator of \( \text{Ker}(\theta) \). We have

\[
\frac{([\bar{e}] - 1)^{p-1}}{p} = \frac{(t')^{p-1}}{p} \cdot ([\bar{e}]^{1/p} - 1)^{p-1}
\]

and \( \theta([\bar{e}]^{1/p} - 1) = \zeta_p - 1 \). Take an element \( a \in W(R) \) such that \( \theta(a) = (\zeta_p - 1)^{p-1}/p \). Then we have

\[
\frac{([\bar{e}] - 1)^{p-1}}{p} = a(t')^{p-1} + b(t')^p/p
\]

for some \( b \in W(R) \). Indeed, to show \( b \in W(R) \), it suffices to check that the element \( ([\bar{e}]^{1/p} - 1)^{p-1} - pa \) of \( \text{Ker}(\theta) \) also generates this ideal. This follows from the fact that the 0-th entry of \( ([\bar{e}]^{1/p} - 1)^{p-1} \) of this element satisfies \( v_R((\zeta_p - 1)^{p-1}) = 1 \). Then we see that \( (t')^p/p \) is topologically nilpotent because so is \( t' \) in \( W(R) \).

In the following, we set the element \( a \) in the proof of the lemma to be

\[
a = \sum_{k=1}^{p-2} p^{-1}((-1)^{p-1-k}p^{-k-1}C_k - 1)[\bar{e}]^{1/p}^k,
\]

where \( p^{-k}C_k = (p-1)!/(k!(p-1-k))! \) is the binomial coefficient. Note that the coefficient of \( [\bar{e}]^{1/p}^k \) in each term is an integer.

From this lemma, we can consider the ring \( \hat{A} \) as a \( \Sigma \)-algebra by \( u \mapsto [\bar{u}] \). Put \( \text{Fil}^i\hat{A} = (t^i, Z) \) for \( 0 \leq i \leq p - 1 \). The Frobenius endomorphism \( \phi \) of \( A_{\text{crys}} \) preserves \( \hat{A} \) and satisfies \( \phi(\text{Fil}^i\hat{A}) \subseteq p^i\hat{A} \) for \( 0 \leq i \leq p - 1 \). Set \( \phi_r = p^{-r}\phi|_{\text{Fil}^r\hat{A}} \). Then we can consider the ring \( \hat{A} \) also as an object of the category \( \text{Mod}_{\text{crys}}^r \), and similarly for \( \hat{A}_n = \hat{A}/p^n\hat{A} \) and \( \hat{A}_\infty = \hat{A} \otimes W K_0/W \).

The absolute Galois group \( G_{K_0} \) acts naturally on these \( \Sigma \)-algebras. The following lemma is used implicitly in [2].

**Lemma 4.2.** We have a natural decomposition

\[
\hat{A}_1 = R/(t^p) \oplus (Z).
\]

**Proof.** Consider the natural inclusion \( W(R) \hookrightarrow \hat{A} \). First we claim that this induces an injection \( R/(t^p) \hookrightarrow \hat{A}_1 \). Let \( x \) be in the ring \( R \). If the element
$[x] \in W(R)$ is contained in $p\hat{A}$, then its image in $A_{\text{crys}}/pA_{\text{crys}}$ is zero. We have an isomorphism of $R$-algebras

$$R[Y_1, Y_2, \ldots]/(t^p, Y_1^p, Y_2^p, \ldots) \to A_{\text{crys}}/pA_{\text{crys}}$$

which sends $Y_i$ to the image of $t^p/(p^i)$! Thus the inequality $v_R(x) \geq p$ holds. Conversely, if $v_R(x) \geq p$, then we have

$$[x] = w(\psi([x]) - 1)^{p-1} + pw'$$

for some $w, w' \in W(R)$ and this implies $[x] \in p\hat{A}$.

Let us consider the commutative diagram of $R$-algebras

$$
\begin{array}{c}
R/(t^p) \twoheadrightarrow \hat{A}_1 \\
\downarrow \\
\hat{A}_1/(Z).
\end{array}
$$

By definition, the left downward arrow is surjective. We claim that this arrow is an isomorphism. Indeed, let $x$ be in the kernel of this surjection. From the proof of Lemma 4.1, we see that the image of $Z$ in the ring on the left-hand side of the above isomorphism can be written as $a't^{b-1} + b'Y_1$ for some $a', b' \in R^\times$. By assumption, in this ring, we have

$$x = c_1(a't^{b-1} + b'Y_1) + c_2(a't^{b-1} + b'Y_1)^2 + \cdots + c_{p-1}(a't^{b-1} + b'Y_1)^{p-1}$$

for some elements $c_1, \ldots, c_{p-1}$ of $R$. Then we see that $c_i = 0$ for any $i$ and $v_R(x) \geq p$. This concludes the proof. □

Since $r < p - 1$, from this lemma we can show the following lemma as in the proof of [3, Lemme 2.3.1.3].

Lemma 4.3. The functor

$$M \mapsto \text{Hom}_{\Sigma, \text{Fil}^r, \phi}(M, \hat{A}_\infty)$$

from $\text{Mod}_{/\Sigma_\infty}^{r, \phi}$ to the category of $G_{K_\infty}$-modules is exact.

Corollary 4.4. For any $M \in \text{Mod}_{/\Sigma_\infty}^{r, \phi}$, the natural map

$$\text{Hom}_{\Sigma, \text{Fil}^r, \phi}(M, \hat{A}_\infty) \to \text{Hom}_{\Sigma, \text{Fil}^r, \phi}(M, A_{\text{crys}, \infty})$$

is an isomorphism of $G_{K_\infty}$-modules.

Proof. By Lemma 3.2 and Lemma 4.3, we may assume $pM = 0$. Consider the commutative diagram of rings

$$
\begin{array}{c}
\hat{A}_1 \longrightarrow A_{\text{crys}}/pA_{\text{crys}} \\
\downarrow \\
R/(t^{p-1})
\end{array}
$$
whose downward arrows are defined by modulo Fil_{p-1} of the rings \( \hat{A}_1 \) and \( A_{\text{crys}}/pA_{\text{crys}} \), respectively. Since \( r < p - 1 \), we have \( \phi_r(\text{Fil}_{p-1} \hat{A}_1) = 0 \) and similarly for the ring \( A_{\text{crys}}/pA_{\text{crys}} \). Thus these two surjections induce on the ring \( R/(p^{p-1}) \) the same structure of a filtered \( \phi_r \)-module over \( \Sigma \). Hence, as in the proof of Corollary 3.5, we see from Lemma 3.1 that we have a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, \hat{A}_1) & \rightarrow & \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, A_{\text{crys}}/pA_{\text{crys}}) \\
\downarrow & & \downarrow \\
\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, R/(p^{p-1})) & \rightarrow & \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, (R/(p^{p-1})))
\end{array}
\]

whose downward arrows are isomorphisms. This concludes the proof. \( \square \)

We sketch the proof of the following lemma stated in [2, Subsection 3.2].

**Lemma 4.5.** The natural inclusion \( W(R) \rightarrow \hat{A} \) induces an isomorphism of \( W(R) \)-algebras \( W_n(R)/(\psi(\varepsilon) - 1)^{p-1} \rightarrow \hat{A}_n/(Z) \).

**Proof.** For a subring \( B \) of \( A_{\text{crys}} \), put

\[
I^s[B] = \{ x \in B \mid \phi^i(x) \in \text{Fil}^s A_{\text{crys}} \text{ for any } i \}
\]

as in [9, Subsection 5.3]. Then we have \( I^s[W(R)] = (\varepsilon - 1)^s W(R) \) and the natural ring homomorphism

\[
W(R)/I^s[W(R)] \rightarrow A_{\text{crys}}/I^s A_{\text{crys}}
\]

is an injection ([9, Proposition 5.1.3, Proposition 5.3.5]). Since the element \( Z \) is contained in the ideal \( I^{p-1} A_{\text{crys}} \), this injection factors as

\[
W(R)/I^{p-1}[W(R)] \rightarrow \hat{A}/(Z) \rightarrow A_{\text{crys}}/I^{p-1}[A_{\text{crys}}].
\]

Hence the former arrow is an isomorphism and the lemma follows. \( \square \)

Since the ideal \( (Z) \) of \( \hat{A}_n \) satisfies the condition of Corollary 3.5, the \( \Sigma \)-algebra \( \hat{A}_n/(Z) \) is naturally considered as an object of \( \text{Mod}^\phi_{\Sigma} \). We also give the ring \( W_n(R)/(\psi(\varepsilon) - 1)^{p-1} \) the structures of a \( \Sigma \)-algebra and a filtered \( \phi_r \)-module over \( \Sigma \) induced by those of \( \hat{A}_n/(Z) \). The map

\[
\Sigma \rightarrow W_n(R)/(\psi(\varepsilon) - 1)^{p-1}
\]

sends the element \( u \in \Sigma \) to the image of \( \varepsilon/t \) in the ring on the right-hand side. Put \( v = \varepsilon/t \in W(R)^\times \) with the notation of Lemma 4.1. As for the element \( Y \in \Sigma \), the equality

\[
Y = -ab^{-1}v^{-1}E(\varepsilon/t)^{p-1} + wv^{-1}v^{-p}Z
\]

holds in \( \hat{A} \), where \( a \) and \( b \) are the elements in \( W(R)^\times \) as in the proof of Lemma 4.1 and the remark after this lemma, and \( w \in W(R)^\times \) is a power series of \( \varepsilon - 1 \). Hence the above homomorphism sends the element \( Y \) to the image of \( -ab^{-1}v^{-1}E(\varepsilon/t)^{p-1} \).
Consider the surjective ring homomorphism
\[ R \to \mathcal{O}_K/p\mathcal{O}_K \]
\[ x = (x_0, x_1, \ldots) \mapsto x_n \]
and the induced surjection \( W_n(R) \to W_n(\mathcal{O}_K/p\mathcal{O}_K) \). Let
\[ J = \{ (x_0, \ldots, x_{n-1}) \in W_n(R) \mid v_R(x_i) \geq p^n \text{ for any } i \} \]
be the kernel of the latter surjection.

**Lemma 4.6.** The ideal \( J \) is contained in the ideal \( (\psi([x] - 1)^{p-1}) \) of the ring \( W_n(R) \).

**Proof.** Write \( ([x] - 1)^{p-1} \) also as \( x = (x_0, \ldots, x_{n-1}) \in W_n(R) \) with \( v_R(x_0) = p \). Take an element \( z = (z_0, \ldots, z_{n-1}) \) in the ideal \( J \). We construct \( y \in W_n(R) \) such that \( xy = z \). By induction, it is enough to show that if \( z_0 = \cdots = z_{i-1} = 0 \) for some \( 0 \leq i \leq n - 1 \) and \( (x_0, \ldots, x_i)(0, \ldots, 0, y_i) = (0, \ldots, 0, z_i) \) in \( W_{i+1}(R) \), then \( x(0, \ldots, 0, y_i, 0, \ldots, 0) \in J \). Let us write this element as \( (0, \ldots, 0, w_i, \ldots, w_{n-1}) \) with \( w_i = z_i \). We have \( v_R(y_i) \geq p^n - p^{i+1} \).

In the ring of Witt vectors \( W_n(\mathbb{F}_p[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}]) \), the \( k \)-th entry of the vector
\[ (X_0, \ldots, X_{n-1})(0, \ldots, 0, Y_i, 0, \ldots, 0) \]
is \( X_{k-i}^{p^i}Y_i^{p^{k-i}} \) for any \( k \geq i \). Thus we have \( v_R(w_k) \geq p^n \). \( \Box \)

Note that the elements \( [\zeta^{p^n}] - 1 \) and \( [\zeta^{p^{n+1}}] - 1 \) is nilpotent in \( W_n(\mathcal{O}_K/p\mathcal{O}_K) \).

By the above lemma, we have an isomorphism of rings
\[ W_n(R)/(\psi([x] - 1)^{p-1}) \cong W_n(\mathcal{O}_K/p\mathcal{O}_K)/(\psi([\zeta^{p^n}] - 1)^{p-1}). \]

We give the ring on the right-hand side the structure of a filtered \( \phi_r \)-module over \( \Sigma \) induced by this isomorphism.

Put \( F_n = K_n(\zeta^{p^{n+1}}) \). For an algebraic extension \( F \) of \( F_n \), let us consider the ideals
\[ m_{n,F} = \{(x_0, \ldots, x_{n-1}) \in W_n(\mathcal{O}_F/p\mathcal{O}_F) \mid x_i \in m_F/p\mathcal{O}_F \text{ for any } i \} \]
\[ m_{n,F} = \{(x_0, \ldots, x_{n-1}) \in W_n(\mathcal{O}_F) \mid x_i \in m_F \text{ for any } i \} \]
of \( W_n(\mathcal{O}_F/p\mathcal{O}_F) \) and \( W_n(\mathcal{O}_F) \), respectively. The elements \( [\zeta^{p^n}] - 1 \) and \( [\zeta^{p^{n+1}}] - 1 \) are topologically nilpotent in \( W_n(\mathcal{O}_F) \) and we define an element \( \hat{\ell} \in W_n(\mathcal{O}_F) \) to be
\[ \hat{\ell} = \psi([\zeta^{p^n}] - 1)/\psi([\zeta^{p^{n+1}}] - 1). \]

Note that these elements are non-zero divisors of \( W_n(\mathcal{O}_F) \). Let the ring
\[ W_n(\mathcal{O}_F/p\mathcal{O}_F)/(\psi([\zeta^{p^n}] - 1))m_{n,F} \]
be denoted by \( \tilde{A}_{n,F,r+} \). We also put \( \tilde{m}_n = \hat{\tilde{m}}_{n,F} \), \( m_n = m_{n,K} \) and \( \tilde{A}_{n,r+} = \tilde{A}_{n,K,r+} \).

For an algebraic extension \( F \) of \( K \), we put
\[ b_F = \{ x \in \mathcal{O}_F \mid v_K(x) > cr/(p-1) \}. \]
Note that the ring \( \mathcal{O}_F/b_F \) is killed by \( p \). When \( F \) contains \( F_n \), we also put

\[
\mathring{A}_{n,F,r+} = W_n(\mathcal{O}_F/b_F)/\psi([\zeta_p^n] - 1)^r \mathfrak{m}_{n,F}.
\]

Then, for \( 0 \leq r < p - 1 \), we have natural isomorphisms of rings

\[
W_n(\mathcal{O}_F)/\psi([\zeta_p^n] - 1)^r m_{n,F} \rightarrow \mathring{A}_{n,F,r+} \rightarrow \mathring{A}_{n,F,r+}.
\]

Indeed, as in the proof of Lemma 4.6, we can show that both of the kernels of the maps \( W_n(\mathcal{O}_F) \rightarrow W_n(\mathcal{O}_F/p\mathcal{O}_F) \) and \( W_n(\mathcal{O}_F) \rightarrow W_n(\mathcal{O}_F/b_F) \) are contained in the ideal \( \psi([\zeta_p^n] - 1)^r m_{n,F} \) of the ring \( W_n(\mathcal{O}_F) \). We often identify these rings. We also put \( \overline{A}_{n,r} = \overline{A}_{n,K,r} \).

Write \( Z_n \) for the image of the element \( Z \) of \( A_{\text{crys}} \) in \( W^{\text{PD}}_n(\mathcal{O}_K/p\mathcal{O}_K) \). Then we have a commutative diagram of \( \Sigma \)-algebras

\[
\begin{array}{ccc}
\hat{A}_n & \rightarrow & A_{\text{crys}}/p^n A_{\text{crys}} \\
\downarrow & & \downarrow \\
W^n_{\text{PD}}(\mathcal{O}_F/p\mathcal{O}_F) & \rightarrow & W^n_{\text{PD}}(\mathcal{O}_K/p\mathcal{O}_K)
\end{array}
\]

\[
\begin{array}{ccc}
W_n(R)/(\psi([\zeta_p^n] - 1)^{p-1}) & \sim & \hat{A}_n/(Z) \\
\downarrow & & \downarrow \\
W_n(\mathcal{O}_K/p\mathcal{O}_K)/(\psi([\zeta_p^n] - 1)^{p-1}) & \rightarrow & W^n_{\text{PD}}(\mathcal{O}_K/p\mathcal{O}_K)/(Z_n),
\end{array}
\]

\[
\begin{array}{ccc}
\hat{A}_{n,r+} & \rightarrow & \overline{A}_{n,r+} \\
\downarrow & & \downarrow \\
\overline{A}_{n,F,r+} & \rightarrow & \overline{A}_{n,F,r+}
\end{array}
\]

where all vertical arrows are surjections satisfying the condition of Corollary 3.5. Thus we see that this is also a commutative diagram in \( \text{Mod}_{r,\phi}^{\Sigma} \). Note that these rings and homomorphisms are independent of the choice of a system \( \{\zeta_p^n\}_{n \in \mathbb{Z}_{\geq 0}} \). Moreover, let \( M \) be in \( \text{Mod}_{r,\phi}^{\Sigma} \) and put \( M_n = M/p^n M \). Then, by Corollary 3.5 and Corollary 4.4, we have a natural isomorphism of abelian groups

\[
\text{Hom}_{\Sigma,\text{Fil}_r,\phi_r}(M_n, W^n_{\text{PD}}(\mathcal{O}_K/p\mathcal{O}_K)) \simeq \text{Hom}_{\Sigma,\text{Fil}_r,\phi_r}(M_n, \hat{A}_{n,r+}).
\]

To study \( G_{F_n} \)-actions on both sides of this isomorphism, we need the following proposition.

**Proposition 4.7.** The image of the element \( Y \in \Sigma \) in the ring \( \mathring{A}_{n,r+} \) is contained in its subring \( \mathring{A}_{n,F_n,r+} \).
Proof. We have the equality

\[
([x^{1/p}] - 1)^{p - 1} = pa - E([x])bv
\]

in W(R). By definition, we see that the images of the elements \(a, [x^{1/p}]\) and \(E([x])\) in \(W_n(O_K/pO_K)\) are contained in the subring \(W_n(O_{F_n}/pO_{F_n})\). Write \(\beta\) for the product \(bv \in W(R)\). Let \(a_n, \beta_n\) and \(\alpha_n\) denote the images of the elements \(a, \beta\) and \(pa - ([x^{1/p}] - 1)^{p - 1}\) in the ring \(W_n(O_K/pO_K)\), respectively. Then the element \(\alpha_n\) is also contained in the subring \(W_n(O_{F_n}/pO_{F_n})\). Now we have the equality

\[
E([\pi_n])\beta_n = \alpha_n.
\]

Note that any element \(\beta'_n \in W_n(O_K/pO_K)\) satisfying the same equality is invertible and thus the elements \((\beta'_n)^{-1}E([\pi_n])\) are equal to each other. Since \(Y = -a_n\beta_n^{-1}E([\pi_n])^{p - 1}\) in \(A_{n,r+}\), it suffices to construct an element \(\beta'_n\) in the ring \(W_n(O_{F_n}/pO_{F_n})\) such that the equality \(E([\pi_n])\beta'_n = \alpha_n\) holds. Take a lift \(\hat{\alpha}_n\) of \(\alpha_n\) in \(W_n(O_{F_n})\). Since \(E([\pi_n]) \in W_n(O_K)\) divides every element in the kernel of the surjection \(W_n(O_K) \to W_n(O_{F_n}/pO_{F_n})\), we have \(E([\pi_n])\hat{\beta}'_n = \hat{\alpha}_n\) for some \(\hat{\beta}'_n \in W_n(O_K)\). Then the element \(\hat{\beta}'_n\) is contained in the subring \(W_n(O_{F_n})\) and we set \(\beta'_n\) to be the image of \(\hat{\beta}'_n\).

By a similar argument, we can also check that the ring \(\tilde{A}_{n, r+}\) is a subring of \(\tilde{A}_{n, r+}\) and coincides with the image of \(W_n(O_{\tilde{F}})\) in \(\tilde{A}_{n, r+}\). This concludes the proof. 

Let \(t_n\) and \(l_n\) be the images of \(t \in W(R)\) in \(W_n(O_K/pO_K)\) and \(\tilde{A}_{n, r+}\) (or \(\tilde{A}'_{n, r+}\)), respectively. Then \(t\) is a lift of \(t_n\) and \(l_n\) to \(W_n(O_{\tilde{K}})\) by the natural surjections

\[
W_n(O_K) \to W_n(O_K/pO_K) \to \tilde{A}_{n, r+}.
\]

Note that we defined the filtration of \(A_{n, r+}\) as \(\text{Fil}^r\tilde{A}_{n, r+} = \tilde{t}_n\tilde{A}_{n, r+}\).

**Lemma 4.8.** Let \(\bar{x}\) be in \(\text{Fil}^r\tilde{A}_{n, r+}\). Then we have

\[
\phi_r(\bar{x}) = \phi(y) \mod \psi([\zeta^{p^n}] - 1)^r\tilde{m}_n,
\]

where \(y\) is any element of \(W_n(O_K/pO_K)\) such that the element \(t^n y\) is a lift of \(\bar{x}\). In particular, the right-hand side of the above equality is independent of the choice of a system \(\{\zeta^{p^n}\}_{n \in \mathbb{Z}_{\geq 0}}\). Similar assertions also hold for the ring

\[
W_n(O_K/pO_K)/(\psi([\zeta^{p^n}] - 1)^{p - 1}).
\]

**Proof.** Since the filtered \(\phi_r\)-module structure of \(\tilde{A}_{n, r+}\) is induced from that of \(\tilde{A}\) and \(\phi_r(t^n) \equiv 1 \mod Z\) in \(\tilde{A}\), we see that there exists an element \(y\) as in the lemma.

To prove the independence of the choice of a lift, let \(z = (z_0, \ldots, z_{n-1})\) be an element of \(W_n(O_K/pO_K)\) killed by \(t_n\). The element \(z\) is also killed by
Let \( p - [p_n] \), where \( p_k = p^{1/p^k} \). This implies
\[
\begin{align*}
  p_n z_0 & \in p \mathcal{O}_K \\
  z_0^p + p_n^{-1} z_1 & \in p \mathcal{O}_K \\
  & \vdots \\
  z_{n-2}^p + p_{n-1} z_{n-1} & \in p \mathcal{O}_K
\end{align*}
\]
and \( v_p(z_k) \geq 1 - 1/p^{n-k} \) for \( 0 \leq k \leq n - 1 \). Repeating this, we see that if \( z \) is killed by \( t'_r \), then \( v_p(z_k) \geq 1 - r/p^{n-k} \). For such an element \( z \), we have \( \phi(z) = 0 \) in the ring \( W_n(\mathcal{O}_K/p \mathcal{O}_K) \).

Let \( y_1 \) and \( y_2 \) be elements as in the lemma. Then we have
\[
y_1 - y_2 = \psi([\zeta_{p^k}^n - 1]^r w + z,
\]
where \( w \in \bar{m}_n \) and \( z \) is an element as above. The Frobenius endomorphism \( \phi \) sends the element on the right-hand side to an element which is contained in the ideal \( \psi([\zeta_{p^k}^n - 1]^r \bar{m}_n \). Thus the assertions for the ring \( \bar{A}_{n,r^+} \) follows.

We can show the assertion for the ring \( W_n(\mathcal{O}_K/p \mathcal{O}_K)/(\psi([\zeta_{p^k}^n - 1]^r)^{-1}) \) similarly.

From this lemma and Proposition 4.7, we see that the natural \( G_{F_n} \)-actions on the rings \( W_n(\mathcal{O}_K/p \mathcal{O}_K)/(\psi([\zeta_{p^k}^n - 1]^r)^{-1}) \), \( \bar{A}_{n,r^+} \) and \( A'_{n,r^+} \) are compatible with the \( \phi_r \)-module structures over \( \Sigma \). In the commutative diagram above, the lowest horizontal arrow and lower right vertical arrow are \( G_{F_n} \)-linear by definition. Hence we have shown the following proposition.

**Proposition 4.9.** Let \( \tilde{M} \) be in \( \text{Mod}^{\mathcal{T}/\Sigma} \) and put \( M_n = \tilde{M}/p^n \tilde{M} \). Then we have an isomorphism of \( G_{F_n} \)-modules
\[
\text{Hom}_{\Sigma, \text{Fil}^{r,w}}(M_n, \tilde{A}_{n,r^+}) \simeq \text{Hom}_{\Sigma, \text{Fil}^{r,w}}(M_n, W_{n, \text{PD}}(\mathcal{O}_K/p \mathcal{O}_K)).
\]

Let \( e_1, \ldots, e_d \) be a basis of \( \tilde{M} \) as in Lemma 3.3 and \( C = (c_{i,j}) \in M_d(\Sigma) \) be the associated matrix representing \( \phi_r \) as in Corollary 3.4. Then the underlying \( G_{F_n} \)-set of the \( G_{F_n} \)-module
\[
\text{Hom}_{\Sigma, \text{Fil}^{r,w}}(M_n, \tilde{A}_{n,r^+})
\]
is identified with the set of \( d \)-tuples \( (\bar{x}_1, \ldots, \bar{x}_d) \) in \( \tilde{A}_{n,r^+} \) such that \( c_{1,i} \bar{x}_1 + \cdots + c_{d,i} \bar{x}_d \in \text{Fil}^1 \tilde{A}_{r,n^+} \) for any \( i \) and the following equality holds:
\[
\begin{align*}
  \phi_r(c_{1,1} \bar{x}_1 + \cdots + c_{d,1} \bar{x}_d) &= \bar{x}_1 \\
  \vdots \\
  \phi_r(c_{1,d} \bar{x}_1 + \cdots + c_{d,d} \bar{x}_d) &= \bar{x}_d.
\end{align*}
\]

We choose a lift \( (\bar{e}_{i,j}) \in M_d(W_n(\mathcal{O}_{F_n})) \) of the image of \( C \) in \( M_d(\tilde{A}_{n,r^+}) \) by the natural ring homomorphism
\[
W_n(\mathcal{O}_K) \to W_n(\mathcal{O}_K/p \mathcal{O}_K) \to \tilde{A}_{n,r^+}.
\]

Fix a polynomial \( \Phi_i \in \mathbb{Z}[X_0, \ldots, X_{n-1}] \) such that \( \Phi_i \equiv X_i^p \mod p \). This induces for any commutative ring \( B \) a map \( \Phi = (\Phi_0, \ldots, \Phi_{n-1}) : W_n(B) \to W_n(B) \) which is a lift of the Frobenius endomorphism on \( W_n(B/pB) \). In
particular, set $B$ to be the polynomial ring $\mathbb{Z}[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}]$. Put $X = (X_0, \ldots, X_{n-1})$ and $Y = (Y_0, \ldots, Y_{n-1})$ in the ring $W_n(B)$. Then we see that there exists elements $U_0, \ldots, U_{n-1}$ and $U'_0, \ldots, U'_{n-1}$ of the polynomial ring $B$ such that

$$
\Phi(X + Y) = \Phi(X) + \Phi(Y) + (pU_0, \ldots, pU_{n-1}),
$$
$$
\Phi(XY) = \Phi(X)\Phi(Y) + (pU'_0, \ldots, pU'_{n-1})
$$
in the ring $W_n(B)$.

**Proposition 4.10.** Every solution $(\bar{x}_1, \ldots, \bar{x}_d)$ in $\tilde{A}_{n,r+}$ of the equation (1) such that $c_{1,i}\bar{x}_1 + \cdots + c_{d,i}\bar{x}_d \in \text{Fil}^r \tilde{A}_{n,r+}$ for any $i$ uniquely lifts to a $d$-tuple $(\hat{x}_1, \ldots, \hat{x}_d)$ in $W_n(\mathcal{O}_K)$ such that $\hat{c}_{1,i}\hat{x}_1 + \cdots + \hat{c}_{d,i}\hat{x}_d \in \tilde{t}W_n(\mathcal{O}_K)$ for any $i$ and the following equality holds:

$$
\begin{cases}
\Phi((\hat{c}_{1,1}\hat{x}_1 + \cdots + \hat{c}_{d,1}\hat{x}_d)/\tilde{t}') = \hat{x}_1 \\
\vdots \\
\Phi((\hat{c}_{1,d}\hat{x}_1 + \cdots + \hat{c}_{d,d}\hat{x}_d)/\tilde{t}') = \hat{x}_d
\end{cases}
$$

(2)

**Proof.** Fix a lift $\hat{x}_i$ of $\bar{x}_i$ to $W_n(\mathcal{O}_K)$. Then we have

$$
\begin{cases}
\Phi((\hat{c}_{1,1}\hat{x}_1 + \cdots + \hat{c}_{d,1}\hat{x}_d)/\tilde{t}') = \hat{x}_1 + ((\zeta_p^r) - 1)^r \hat{\delta}_1 \\
\vdots \\
\Phi((\hat{c}_{1,d}\hat{x}_1 + \cdots + \hat{c}_{d,d}\hat{x}_d)/\tilde{t}') = \hat{x}_d + ((\zeta_p^r) - 1)^r \hat{\delta}_d
\end{cases}
$$

for some $\hat{\delta}_1, \ldots, \hat{\delta}_d \in m_n$. It suffices to show that there exists a unique $d$-tuple $(\hat{y}_1, \ldots, \hat{y}_d)$ in $m_n$ such that

$$
\Phi((\hat{c}_{1,i}(\hat{x}_1 + ((\zeta_p^r) - 1)^r \hat{y}_1) + \cdots + \hat{c}_{d,i}(\hat{x}_d + ((\zeta_p^r) - 1)^r \hat{y}_d))/\tilde{t}')
$$

$$
= \hat{x}_i + ((\zeta_p^r) - 1)^r \hat{y}_i
$$

for any $i$. For this, we need the following lemma.

**Lemma 4.11.** Let $N$ be a complete discrete valuation field and $m_N$ be the maximal ideal of $N$. Let $\epsilon_1, \ldots, \epsilon_d$ be in $m_N$. Let $P_1, \ldots, P_d$ and $P'_1, \ldots, P'_d$ be elements of $\mathcal{O}_N[[Y_1, \ldots, Y_d]]$ such that $P_i \in (Y_1, \ldots, Y_d)^2$. Then the equation

$$
\begin{cases}
Y_1 - P_1(Y_1, \ldots, Y_d) - \epsilon_1 P'_1(Y_1, \ldots, Y_d) = 0 \\
\vdots \\
Y_d - P_d(Y_1, \ldots, Y_d) - \epsilon_d P'_d(Y_1, \ldots, Y_d) = 0
\end{cases}
$$

has a unique solution in $m_N$.

**Proof.** By assumption, we see that for any integer $l \geq 1$, a $d$-tuple $y = (y_1, \ldots, y_d)$ in $m_N/m_N^{l+1}$ satisfying the above equation lifts uniquely to a $d$-tuple in $m_N/m_N^{l+1}$ satisfying the same equation. Thus the lemma follows. \qed
Let us write as \( \hat{\eta}_i = (\hat{\eta}_{i,0}, \ldots, \hat{\eta}_{i,n-1}) \). Since the image of \( \Phi((\zeta_p 1) - 1)^r \) in \( \bar{A}_{n,r+} \) is divisible by \( ((\zeta_p^n) - 1)^r \), we can find \( \hat{b} \in W_n(O_K) \) such that
\[
\Phi((\zeta_p^n) - 1)^r/\hat{b} = ((\zeta_p^n) - 1)^r \hat{b}.
\]
Then there exists polynomials \( U_{i,m} \) over \( O_K \) of the indeterminates \( Y = (Y_{i,m})_{1 \leq i \leq d, 0 \leq m \leq n-1} \) such that the equation we have to solve is
\[
\hat{x}_i + ((\zeta_p^n) - 1)^r \hat{y}_i = \hat{x}_i + ((\zeta_p^n) - 1)^r \hat{\delta}_i
\]
\[
+ ((\zeta_p^n) - 1)^r \hat{b}(\Phi(\hat{c}_{i,1})\Phi(\hat{y}_i) + \cdots + \Phi(\hat{c}_{i,d})\Phi(\hat{y}_d))
\]
\[
+ (pU_{i,0}(\hat{y}), \ldots, pU_{i,n-1}(\hat{y}))
\]
for any \( i \), where we put \( \hat{y} = (\hat{y}_{i,m})_{1 \leq i \leq d, 0 \leq m \leq n-1} \). Note that, for any elements \( P_0, \ldots, P_{n-1} \) of the polynomial ring \( O_K[Y] \), we can uniquely find elements \( Q_0, \ldots, Q_{n-1} \) of this ring such that the coefficients of these polynomials are in the maximal ideal \( m_{K} \) and the equality
\[
(pP_0, \ldots, pP_{n-1}) = ((\zeta_p^n) - 1)^r(Q_0, \ldots, Q_{n-1})
\]
holds in the ring of Witt vectors \( W_n(O_K[Y]) \). Therefore, this equation is equivalent to the equation
\[
\hat{y}_i = \hat{\delta}_i + \hat{b}(\Phi(\hat{c}_{i,1})\Phi(\hat{y}_i) + \cdots + \Phi(\hat{c}_{i,d})\Phi(\hat{y}_d))
\]
\[
+ (V_{i,0}(\hat{y}), \ldots, V_{i,n-1}(\hat{y}))
\]
where \( V_{i,m} \) is a polynomial of \( Y \) over \( O_K \) whose coefficients are in the maximal ideal \( m_{K} \). From the definition of \( \Phi \), we see that the elements \( \hat{y}_{i,m} \) is a solution of a system of equations
\[
Y_{i,m} - P_{i,m}(Y) - \epsilon_{i,m}P'_{i,m}(Y) = 0
\]
satisfying the condition of Lemma 4.11 for a sufficiently large finite extension \( N \) of \( K \). Then, by this lemma, we can solve the equation uniquely in \( m_{K} \). \( \Box \)

Let \( F \) be an algebraic extension of \( F_n = K_n(\zeta_p 1) \) and consider the ring \( \bar{A}_{n,F,r+} \). By Lemma 4.7, we can consider this ring as a \( \Sigma \)-algebra and also as an object of \( \text{Mod}^{\rho,\omega}_{\Sigma} \) by putting \( \text{Fil}^r \hat{A}_{n,F,r+} = \bar{r}_n \bar{A}_{n,F,r+} \) and for \( \bar{x} \in \text{Fil}^r \bar{A}_{n,F,r+} \),
\[
\phi_r(\bar{x}) = \phi(y) \mod \psi((\zeta_p^n) - 1)^r \bar{m}_{n,F},
\]
where \( y \) is any element of \( W_n(O_F/p\bar{O}_F) \) such that the element \( t^r_n y \) is a lift of \( \bar{x} \). For \( M_n = \bar{M}/p^n \bar{M} \in \text{Mod}^{\rho,\omega}_{\Sigma_\infty} \) as before, let us set
\[
T_{\text{cryst},n,F}^r(M_n) = \text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M_n, \bar{A}_{n,F,r+}).
\]
We see that
\[
\bar{A}_{n,r+} = \bar{A}_{n,K,r+} = \bigcup_{F/F_n} \bar{A}_{n,F,r+}.
\]
From Proposition 4.10, we see that the elements of
\[ T^*_{\text{crys}, \pi_n, \mathbb{K}}(M_n) = \bigcup_{F/F_n} T^*_{\text{crys}, \pi_n, F}(M_n). \]
The absolute Galois group \( G_{F_n} \) acts on the abelian group on the left-hand side.

**Lemma 4.12.** For an algebraic extension \( F \) of \( F_n \), the fixed part \( T^*_{\text{crys}, \pi_n, \mathbb{K}}(M_n)^{G_F} \) is equal to \( T^*_{\text{crys}, \pi_n, F}(M_n) \).

**Proof.** From Proposition 4.10, we see that the elements of \( T^*_{\text{crys}, \pi_n, \mathbb{K}}(M_n) \) correspond bijectively to the solutions of the equation (2) in \( W_n(\mathcal{O}_\mathbb{K}) \) satisfying the condition on \( \text{Fil} \). The uniqueness assertion of this proposition shows that \( g \in G_F \) fixes a solution in \( W_n(\mathcal{O}_\mathbb{K}) \) if and only if \( g \) fixes its image in \( \mathbb{A}_{n, r+} \). Hence a solution is fixed by \( G_F \) if and only if this solution is contained in the image of \( W_n(\mathcal{O}_F) \). Thus the lemma follows. \( \square \)

**Corollary 4.13.** Let \( L_n \) be the finite Galois extension of \( F_n \) corresponding to the kernel of the map \( G_{F_n} \to \text{Aut}(T^*_{\text{crys}, \pi_n, \mathbb{K}}(M_n)) \). Then an algebraic extension \( F \) of \( F_n \) contains \( L_n \) if and only if \#\( T^*_{\text{crys}, \pi_n, F}(M_n) = \#T^*_{\text{crys}, \pi_n, \mathbb{K}}(M_n) \).

**Proof.** An algebraic extension \( F \) of \( F_n \) contains \( L_n \) if and only if the action of \( G_F \) on \( T^*_{\text{crys}, \pi_n, \mathbb{K}}(M_n) \) is trivial. By Lemma 4.12, this is equivalent to \( T^*_{\text{crys}, \pi_n, F}(M_n) = T^*_{\text{crys}, \pi_n, \mathbb{K}}(M_n) \). \( \square \)

### 5. Ramification Bound

In this section, we prove Theorem 1.1. Take \( G_K \)-stable \( \mathbb{Z}_p \)-lattices \( \mathcal{L} \supseteq \mathcal{L}' \) in \( V \) such that \( \mathcal{L}' \supseteq p^n \mathcal{L} \). Since the \( G_K \)-module \( \mathcal{L}/\mathcal{L}' \) is a quotient of \( \mathcal{L}/p^n \mathcal{L} \), we may assume \( \mathcal{L}' = p^n \mathcal{L} \). If \( r = 0 \), then the \( G_K \)-module \( V \) is unramified and the theorem is trivial. Thus we may assume \( r \geq 1 \) and \( p \geq 3 \). Let \( L \) be the finite Galois extensions of \( K \) corresponding to the kernel of the map \( G_K \to \text{Aut}(\mathcal{L}/p^n \mathcal{L}) \).

It is enough to show that, for the greatest upper ramification break \( u_{L(\zeta_p)/K} \) of the Galois extension \( L(\zeta_p)/K \), the inequality
\[ u_{L(\zeta_p)/K} \leq u(K, r, n) \]
holds. Since the Herbrand function is transitive and the finite Galois extension \( K(\zeta_p) \) is tamely ramified over \( K \), we may assume \( \zeta_p \in K \). We fix a uniformizer \( \pi \) of \( K \) and a system \( \{\pi_n\}_{n \in \mathbb{Z}_{\geq 0}} \) as before. Then, by Liu’s theorem ([14, Theorem 2.3.5]), it suffices to show the following.

**Theorem 5.1.** Let \( r \) be an integer such that \( 1 \leq r < p - 1 \) and \( \mathcal{M} \) be the strongly divisible lattice corresponding to \( \mathcal{L} \). Put \( \mathcal{M}_n = \hat{\mathcal{M}}/p^n \mathcal{M} \in \text{Mod}_{r, \phi, \mathcal{N}} \). Then \( G_K^{(j)} \) acts trivially on the \( G_K \)-module \( T^*_{\text{st}, \mathcal{L}}(\mathcal{M}_n) \) for \( j > u(K, r, n) \).
Let $L_n$ be the finite Galois extension of $F_n = K_n(\zeta_{p^n+1})$ corresponding to the kernel of the map

$$G_{F_n} \to \text{Aut}(T^\ast_{\text{st, } \mathcal{M}_n}).$$

Since $F_n$ is Galois over $K$, the extension $L_n$ is also a Galois extension of $K$. Let $\hat{M}$ be the object of the category $\text{Mod}_{/\phi}^{/S}$ such that $\mathcal{M}_n = M_0(\hat{M})$. From Proposition 3.6 and Proposition 4.9, we see that $L_n$ is also the finite extension of $F_n$ cut out by the $G_{F_n}$-module $T^\ast_{\text{crys}, \pi_n, K}(M_n)$ for $M_n = M_0(\mathfrak{M})/p^n M_0(\mathfrak{M})$. It is enough to prove the inequality

$$u_{L_n/K} \leq u(K, r, n) = \begin{cases} 1 + e(n + \frac{1}{p-1}) & (r = 1), \\ \frac{p^n-1}{p^r-1} + e(n + \frac{r}{p-1}) & (r > 1). \end{cases}$$

Before proving this, we state some general lemmas to calculate the ramification bound. Let $N$ be a complete discrete valuation field of positive residue characteristic, $v_N$ be its valuation normalized as $v_N(N') = \mathbb{Z}$ and $N_{\text{sep}}$ be its separable closure.

**Lemma 5.2.** Let $f(T) \in O_N[T]$ be a separable monic polynomial and $z_1, \ldots, z_d$ be the zeros of $f$ in $O_{N_{\text{sep}}}$. Suppose that the set $\{v_N(z_k - z_i) \mid k = 1, \ldots, d, k \neq i\}$ is independent of $i$. Put

$$s = \sum_{k=1, \ldots, d, k \neq i} v_N(z_k - z_i) \text{ and } \alpha = \sup_{k=1, \ldots, d, k \neq i} v_N(z_k - z_i),$$

which are independent of $i$ by assumption. If $j > s + \alpha$, then we have the decomposition

$$\{x \in O_{N_{\text{sep}}} \mid v_N(f(x)) \geq j\} = \prod_{i=1, \ldots, d} \{x \in O_{N_{\text{sep}}} \mid v_N(x - z_i) \geq j - s\}.$$  

Otherwise, the set on the left-hand side contains

$$\{x \in O_{N_{\text{sep}}} \mid v_N(x - z_i) \geq \alpha\},$$

which contains at least two zeros of $f$.

**Proof.** A verbatim argument in the proof of [1, Lemma 6.6] shows the claim. \hfill \square

**Corollary 5.3.** Let $f(T)$ be as above and put $B = O_N[T]/(f(T))$. Suppose that the algebra $B$ is finite flat and of relative complete intersection over $O_N$. Let us write the $N$-algebra $N' = B \otimes_{O_N} N$ as the product $N_1 \times \cdots \times N_t$ of finite separable extensions $N_1, \ldots, N_t$ of $N$. If $j > s + \alpha$, then the $j$-th upper numbering ramification group $\{G^{(j)}_N\}$, which we let be denoted by $G^{(j)}_N$, is contained in $G^{(j)}_{N_i}$ for any $i$. Moreover, if $N'$ is a field and $B$ coincides with $O_N$, then $j > s + \alpha$ if and only if $G^{(j)}_N \subseteq G^{(j)}_{N'}$. 
Proof. From the previous lemma, the conductor $c(B)$ of the $\mathcal{O}_N$-algebra $B$ ([1, Proposition 6.4]) is equal to $s + \alpha$. Thus we have the inequality

$$c(\mathcal{O}_N \times \cdots \times \mathcal{O}_N) \leq c(B) = s + \alpha$$

by the definition of the conductor and a functoriality of the functor $\mathcal{F}^j$ defined in [1]. This implies the corollary. \hfill $\square$

**Corollary 5.4.** Consider the finite Galois extension $F_n = K_n(\zeta_{p^n+1})$ of $K$ and let $u_{F_n/K}$ denote the greatest upper ramification break of $F_n/K$. Then we have the equality

$$u_{F_n/K} = 1 + e(n + \frac{1}{p-1}).$$

**Proof.** Note that we are assuming that $\zeta_p$ is contained in $K$. Applying the previous corollary to the Eisenstein polynomial $f(T) = T^{p^n} - \pi$ shows that $j > 1 + e(n + 1/(p-1))$ if and only if $G_K^{(j)} \subseteq G_{K_n}$. Similarly, putting $f(T) = T^{p^n} - \zeta_p$, we see that if $j > e(n + 1/(p-1))$, then $G_K^{(j)} \subseteq G_K(\zeta_{p^n+1})$. Since $G_{F_n} = G_{K_n} \cap G_K(\zeta_{p^n+1})$, we conclude that $j > 1 + e(n + 1/(p-1))$ if and only if $G_K^{(j)} \subseteq G_{F_n}$. \hfill $\square$

**Remark 5.5.** Note that this argument also shows the equality

$$u_{K_n(\zeta_{p^n})/K} = 1 + e(n + \frac{1}{p-1})$$

without assuming $\zeta_p \in K$.

Next we assume that the residue field of $N$ is perfect. For an algebraic extension $F$ of $N$, we put

$$\mathfrak{a}_F^j = \{ x \in \mathcal{O}_F \mid v_N(x) \geq j \}.$$

For a finite Galois extension $Q$ of $N$, we write $u_{Q/N}$ for the greatest upper ramification break ([7]) of $Q/N$. Let us consider the property

$$(P_j) \begin{cases} \text{for any algebraic extension } F \text{ of } N, \text{ if there exists} \\ \text{an } \mathcal{O}_N\text{-algebra homomorphism } \mathcal{O}_Q \to \mathcal{O}_F/\mathfrak{a}_F^j, \\ \text{then there exists an } N\text{-algebra injection } Q \to F \end{cases}$$

for $j \in \mathbb{R}_{\geq 0}$, as in [7, Proposition 1.5]. Then we have the following proposition, which is due to Yoshida. Here we reproduce his proof for the convenience of the reader.

**Proposition 5.6** ([16]).

$$u_{Q/N} = \inf \{ j \in \mathbb{R}_{\geq 0} \mid \text{the property } (P_j) \text{ holds } \}.$$

**Proof.** By [7, Proposition 1.5 (i)], it is enough to show that the property $(P_j)$ does not hold for $j = u_{Q/N} - (e')^{-1}$ with an arbitrarily large $e' > 0$. As in the proof of [7, Proposition 1.5 (ii)], we may assume that $Q$ is totally and wildly ramified over $N$. Let $N'$ be a finite tamely ramified Galois
Let \( Q \cap N' = N \) and put \( Q' = QN' \). Note that we have \( u_{Q'/N} = u_{Q/N} \) by assumption. From this proposition in [7], we see that for some algebraic extension \( F \) of \( N' \), there exists an \( \mathcal{O}_N \)-algebra homomorphism \( \mathcal{O}_{Q'} \to \mathcal{O}_F/\mathfrak{a}_{F/K}^j \) for \( j = u_{Q/N} - e(Q'/N)^{-1} \) but no \( N \)-algebra injection \( Q' \to F \). Since \( Q'/N \) is wildly ramified, we see that \( e(Q/N)u_{Q/N} - 1 > e(Q/N) \). Hence we have \( u_{Q/N} - e(Q'/N)^{-1} > 1 = u_{N'/N} \) and there exists an \( N \)-algebra injection \( N' \to F \) also by this proposition. Thus there exists no \( N \)-algebra injection \( Q \to F \) and the property \((P_j)\) for \( Q/N \) does not hold. Since we can choose an arbitrarily large \( N' \) as above, the proposition follows. \( \square \)

We see from Proposition 5.6 that to bound the greatest upper ramification break \( u_{L_n/K} \), it is enough to show the following proposition.

**Proposition 5.7.** Let \( F \) be an algebraic extension of \( K \). If \( j > u(K,r,n) \) and there exists an \( \mathcal{O}_K \)-algebra homomorphism

\[
\eta : \mathcal{O}_{L_n} \to \mathcal{O}_F/\mathfrak{a}_{F/K}^j,
\]

then there exists a \( K \)-algebra injection \( L_n \to F \).

**Proof.** By assumption, we have \( j > er/(p-1) \) and \( \mathfrak{b}_F \supseteq \mathfrak{a}_{F/K}^j \). Thus \( \eta \) induces an \( \mathcal{O}_K \)-algebra homomorphism

\[
\mathcal{O}_{L_n} \to \mathcal{O}_F/\mathfrak{b}_F.
\]

Since \( \eta \) also induces an \( \mathcal{O}_K \)-algebra homomorphism \( \mathcal{O}_{F_n} \to \mathcal{O}_F/\mathfrak{a}_{F/K}^j \) and \( r \geq 1 \), from Corollary 5.4 and [7, Proposition 1.5] we get a \( K \)-linear injection \( F_n \to F \). Thus we see that \( F \) contains \( \pi_n \) and \( \zeta_{p^{n+1}} \). More precisely, we have the following lemma.

**Lemma 5.8.** For some integers \( i \) and \( i' \) such that \( i' \equiv 1 \mod p \), we have \( \eta(\pi_n) \equiv \pi_n \zeta_{p^n}^i \mod \mathfrak{b}_F \) and \( \eta(\zeta_{p^{n+1}}) \equiv \zeta_{p^{n+1}}^{i'} \mod \mathfrak{b}_F \). Moreover, there exists \( g \in G_K \) such that \( g(\pi_n) = \pi_n \zeta_{p^n}^i \) and \( g(\zeta_{p^{n+1}}) = \zeta_{p^{n+1}}^{i'} \).

**Proof.** Since the map \( \eta \) is \( \mathcal{O}_K \)-linear, the equality \( \eta(\pi_n)^{p^n} = \pi \) holds in \( \mathcal{O}_F/\mathfrak{a}_{F/K}^j \). Set \( \hat{x} \) to be a lift of \( \eta(\pi_n) \) in \( \mathcal{O}_F \). Then we have

\[
v_K(\hat{x}^{p^n} - \pi) = \sum_{i=0}^{p^n-1} v_K(\hat{x} - \pi_n \zeta_{p^n}^i) \geq j.
\]

Let us apply Lemma 5.2 to \( f(T) = T^{p^n} - \pi \). Then, with the notation of the lemma, we have

\[
s = ne + \frac{p^n - 1}{p^n} \quad \text{and} \quad \alpha = \frac{1}{p^n} + \frac{e}{p - 1}.
\]

Since \( j - s > er/(p-1) \) by assumption, we have

\[
\hat{x} \equiv \pi_n \zeta_{p^n}^i \mod \mathfrak{b}_F
\]

for some \( i \).
Let \( h(T) \) be the minimal polynomial of \( \zeta_{p^{n+1}} \) over \( \mathcal{O}_K \). Since \( h \) divides \( T^{p^n} - \zeta_p \), the \( \mathcal{O}_K \)-algebra \( B' = \mathcal{O}_K[T]/(h(T)) \) is also finite flat of relative complete intersection and the \( \mathcal{K} \)-algebra \( B' \otimes_{\mathcal{O}_K} K \) is étale. The Galois group \( \text{Gal}(K(\zeta_{p^{n+1}})/K) \) acts transitively on the set of zeros of \( h \). Hence \( h \) also satisfies the conditions of Lemma 5.2. Let \( s' \) and \( \alpha' \) be as in this lemma for \( h \). Then we have \( s' \leq ne \) and \( \alpha' \leq e/(p-1) \). This implies \( j - s' > er/(p-1) \). By this lemma, there exists an element \( g' \in \text{Gal}(K(\zeta_{p^{n+1}})/K) \) such that the element \( g'(\zeta_{p^{n+1}}) = \zeta_{p^{n+1}}' \) satisfies

\[
\eta(\zeta_{p^{n+1}}) = g'(\zeta_{p^{n+1}}) \mod b_F.
\]

Since \( K \cap K(\zeta_{p^{n+1}}) = K \) (see for example [14, Lemma 5.1.2]), we can find an element \( g \in G_K \) such that \( g(\pi_n) = \pi_n \zeta_{p^n} \) and \( g(\zeta_{p^{n+1}}) = g'(\zeta_{p^{n+1}}) \). This concludes the proof.

**Lemma 5.9.** The \( \mathcal{O}_K \)-algebra homomorphism \( \eta \) induces an \( \mathcal{O}_K \)-algebra injection

\[
\eta_b : \mathcal{O}_{L_n}/b_{L_n} \to \mathcal{O}_F/b_F.
\]

**Proof.** We write the Eisenstein polynomial of a uniformizer \( \pi_{L_n} \) of \( L_n \) over \( \mathcal{O}_K \) as

\[
P(T) = T^{e'} + c_1 T^{e'-1} + \cdots + c_{e'-1} T + c_{e'},
\]

where \( e' = e(L_n/K) \). Then \( z = \eta(\pi_{L_n}) \) satisfies \( P(z) = 0 \) in \( \mathcal{O}_F/\mathcal{a}_F^2/K \).

Let \( \hat{z} \) be a lift of \( z \) in \( \mathcal{O}_F \). Since \( j > 1 \), we have \( v_F(\hat{z}) = e(F/K)/e' \). The condition \( i > e(L_n)r/(p-1) \) is equivalent to the condition

\[
v_F(\hat{z}^i) > \frac{e(L_n)r}{p-1} \cdot \frac{e(F/K)}{e'} = \frac{e(F)r}{p-1}.
\]

Thus the claim follows. \( \square \)

Since \( L_n \) contains \( F_n \), we can consider the ring

\[
\mathcal{A}'_{n,L_n,r+} = W_n(\mathcal{O}_{L_n}/b_{L_n})/\psi([\zeta_{p^n}]-1)^r \bar{m}_{n,L_n}
\]

and similarly \( \mathcal{A}'_{n,F,r+} \) for \( F \). We give these rings structures of \( \Sigma \)-algebras as follows. The ring \( \mathcal{A}'_{n,L_n,r+} \) is considered as a \( \Sigma \)-algebra by using the system \( \{\pi_n\}_{n \in \mathbb{Z}_{\geq 0}} \) we chose of \( p \)-power roots of \( \pi \), as in the previous section. On the other hand, using \( i \) and \( i' \) in Lemma 5.8, put \( \bar{\pi}_n = \pi_n \zeta_{p^n} \) and \( \bar{\zeta}_{p^{n+1}} = \zeta_{p^{n+1}}^{i'} \).

Then we consider the ring \( \mathcal{A}'_{n,F,r+} \) as a \( \Sigma \)-algebra by using a system of \( p \)-power roots of \( \pi \) containing \( \bar{\pi}_n \). We define \( \text{Fil}^r \) and \( \phi_r \) of these rings in the same way as before.

**Lemma 5.10.** The induced ring homomorphism

\[
\bar{\eta} : \mathcal{A}'_{n,L_n,r+} \to \mathcal{A}'_{n,F,r+}
\]

is a morphism of the category \( ^r\text{Mod}^{\phi_r}_\Sigma \).
Proof. Firstly, we check that $\tilde{\eta}$ is $\Sigma$-linear. By definition, this homomorphism commutes with the action of the element $u \in \Sigma$. To show the compatibility with the element $Y \in \Sigma$, let us consider the commutative diagram

$$
\begin{array}{ccc}
W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n}) & \longrightarrow & W_n(\mathcal{O}_F/p\mathcal{O}_F) \\
\downarrow & & \downarrow \\
W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n}) & \xrightarrow{\eta_b} & W_n(\mathcal{O}_F/\mathfrak{b}_F) \\
\downarrow & & \downarrow \\
\bar{A}'_{n,L_n,r+} & \xrightarrow{\bar{\eta}} & \bar{A}'_{n,F,r+},
\end{array}
$$

where the horizontal arrows are induced by $\eta$. Note that we have $\eta_b(\pi_n) = \tilde{\pi}_n$ and $\eta_b(\zeta_{p^n+1}) = \tilde{\zeta}_{p^n+1}$. Put $\beta \in W(R)^{\times}$ as in the proof of Proposition 4.7. Namely, the element $\beta$ is the solution in $W(R)$ of the equation

$$E([\mathbb{Z}]) \beta = p a - ([\mathbb{Z}^{1/p}] - 1)^{p-1},$$

where the element $a \in W(R)$ is as in the remark after Lemma 4.1. Let $a_n$ and $\beta_n$ denote the images of $a$ and $\beta$ in $W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n})$, respectively. Then the element $\beta_n$ is a solution of the equation

$$E([\pi_n]) \beta_n = p a_n - ([\zeta_{p^n+1}] - 1)^{p-1}.$$

Similarly, we define elements $\tilde{a}_n$ and $\tilde{\beta}_n$ of $W_n(\mathcal{O}_F/p\mathcal{O}_F)$ using $\tilde{\pi}_n$ and $\tilde{\zeta}_{p^n+1}$. By definition, the element $\tilde{\beta}_n$ is a solution of the equation

$$E([\tilde{\pi}_n]) \tilde{\beta}_n = p \tilde{a}_n - ([\tilde{\zeta}_{p^n+1}] - 1)^{p-1}.$$

Now what we have to show is the equality

$$\bar{\eta}(a_n \beta_n E([\pi_n])^{p-1}) = \tilde{a}_n \tilde{\beta}_n E([\tilde{\pi}_n])^{p-1}$$

in the ring $\bar{A}'_{n,F,r+}$. Since the element $a_n$ of $W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n})$ is a linear combination of the elements $1, [\zeta_{p^n+1}], \ldots, [\zeta_{p^n+1}]^{p-1}$ over $\mathbb{Z}$, we have $\bar{\eta}(a_n) = \tilde{a}_n$ in $\bar{A}'_{n,F,r+}$. Thus the elements $\tilde{\beta}_n$ and $\bar{\eta}(\beta_n)$ satisfy the same equation in $\bar{A}'_{n,F,r+}$. Since these two elements are invertible, we see that $\bar{\eta}(\beta_n)^{-1} E([\tilde{\pi}_n]) = \tilde{\beta}_n^{-1} E([\tilde{\pi}_n])$ and the equality holds. Since the diagram above is compatible with the Frobenius endomorphisms, we see from the definition that $\bar{\eta}$ also preserves Fil$^r$ and commutes with $\phi_r$ of both sides. $\square$

Thus the homomorphism $\bar{\eta}$ induces a homomorphism of abelian groups

$$T^r_{\text{crys},L_n,\pi_n}(M_n) \to T^r_{\text{crys},F,\tilde{\pi}_n}(M_n).$$

Then the following lemma, whose proof is omitted in [2, Subsection 3.13], implies that this homomorphism is an injection. We insert here a proof of this lemma for the convenience of the reader.

Lemma 5.11. The ring homomorphism $\bar{\eta} : \bar{A}'_{n,L_n,r+} \to \bar{A}'_{n,F,r+}$ is an injection.
Proof. For an algebraic extension $N$ of $F$, let us write $A'_{n,N,r+}$ for the ring $A'_{n,N,r+}$. Note that the ring $A'_{n,N}/pA'_{n,N}$ is isomorphic to the ring

$$\mathcal{O}_N/\{x \in \mathcal{O}_N \mid v_p(x) > \frac{r}{p^{n-1}(p-1)}\}.$$ 

As in the proof of Lemma 5.9, we see that the homomorphism $\bar{\eta}$ induces an injection $A'_{L_n}/pA'_{L_n} \rightarrow A'_{F}/pA'_{F}$. Thus it is enough to show the exactness of the sequence

$$0 \rightarrow A'_{n,N}/p^n A'_{n,N} \xrightarrow{\times p} A'_{n,N}/p^{n+1} A'_{n,N} \rightarrow A'_{n,N}/p A'_{n,N} \rightarrow 0.$$ 

Let $\bar{x}$ and $\bar{y}$ be in $A'_{n,N}$ such that $px = p^{n+1} \bar{y}$. Let $\bar{x} = (\hat{x}_0, \ldots, \hat{x}_{n-1})$ and $\bar{y} = (\hat{y}_0, \ldots, \hat{y}_{n-1})$ be lifts of $\bar{x}$ and $\bar{y}$ in the ring $W_n(\mathcal{O}_N)$, respectively. In this ring, we have

$$(0, \hat{x}_0^p, \hat{x}_1^p, \ldots, ) = (0, \ldots, 0, \hat{y}_0^{p^{m+1}}, \hat{y}_1^{p^{m+1}}, \ldots) + ([\zeta_{p^n}] - 1)^r \hat{z},$$

where $\hat{z}$ is in the ideal $m_{n,N}$. From this equality we see that $v_p(\hat{x}_0) > r/(p^{n-1}(p-1))$ and $\hat{x} = ([\zeta_{p^n}] - 1)^r \hat{w} + (0, \hat{x}_1^p, \hat{x}_2^p, \ldots)$ for some $\hat{w} \in m_{n,N}$ and $\hat{x}_i^p \in \mathcal{O}_N$. The image of the first term on the right-hand side in $A'_{n,N}$ is zero. Hence we may assume $\hat{x}_0 = 0$. Repeating this, we can see that $\bar{x} \in p^n A'_{n,N}$ and the above sequence is exact. 

Now Corollary 4.13 shows that the abelian group $T^*_{\text{crys},L_n,\pi_n}(M_n)$ is of order $p^n d$, where $d = \dim_{\mathbb{Q}_p} V$. This implies that the the abelian group $T^*_{\text{crys},F,\pi_n}(M_n)$ is also of order $p^n d$. Let $g \in G_K$ be as in Lemma 5.8. Then we have the following lemma.

**Lemma 5.12.** The $G_{F_n}$-action on $T^*_{\text{crys},K,\pi_n}(M_n)$ is the conjugate of the action on $T^*_{\text{crys},K,\pi_n}(M_n)$ by the element $g$.

**Proof.** Let $\alpha_n, \tilde{\alpha}_n$ and $\beta_n, \tilde{\beta}_n$ be the elements of $W_n(\mathcal{O}_K/p\mathcal{O}_K)$ as in the proof of Lemma 5.10. Let us consider the composite

$$\Sigma \rightarrow A'_{n,r+} \xrightarrow{g} A'_{n,r+}$$

of the ring homomorphism defined by $u \mapsto [\pi_n]$ and $Y \mapsto -a_n^0 \beta_n^{-1} E([\pi_n])^{p-1}$, and the map induced by $g$. We claim that this is the natural ring homomorphism defined by $\pi_n$. For this, we only have to check that this composite sends the element $Y \in \Sigma$ to $-\tilde{a}_n \beta_n^{-1} E([\pi_n])$. Since the equality

$$E([\pi_n]) \beta_n = p a_n - ([\zeta_{p^n+1}] - 1)^{p-1}$$

holds in the ring $A'_{n,r+}$ on the source of the above map $g$, we have

$$E([\pi_n]) g(\beta_n) = p \tilde{a}_n - ([\zeta_{p^n+1}] - 1)^{p-1}$$

for all $\gamma_n \in \Sigma$. Therefore, the above composite is the natural ring homomorphism.
in the ring $\mathcal{A}_{n,r+}'$ on the target. Since the elements $g(\beta_n)$ and $\tilde{\beta}_n$ are invertible, we have $g(\beta_n)^{-1}E([\pi_n]) = \tilde{\beta}_n^{-1}E([\pi_n])$ and the claim follows. Thus we have an isomorphism of abelian groups

$$\text{Hom}_\Sigma(M_n, \mathcal{A}_{n,r+}') \rightarrow \text{Hom}_\Sigma(M_n, \mathcal{A}_{n,r+}')$$

where we consider on the ring $\mathcal{A}_{n,r+}'$ on the right-hand side the filtered $\phi_r$-module structure over $\Sigma$ defined by $\tilde{\pi}_n$. Since $g(t_n) = \tilde{t}_n$, we can check that this isomorphism induces an injection

$$\text{Hom}_{\Sigma, \text{Fil}_r, \phi_r}(M_n, \mathcal{A}_{n,r+}') \rightarrow \text{Hom}_{\Sigma, \text{Fil}_r, \phi_r}(M_n, \mathcal{A}_{n,r+}')$$

Since these abelian groups have the same cardinality, this is also an isomorphism.

Since $L_n$ is Galois over $K$, the above lemma shows that the finite Galois extension of $F_n$ cut out by the action on $T_{\text{crys},K,t_n}(M_n)$ is also $L_n$. Hence we see from Corollary 4.13 that $F$ also contains $L_n$ and Proposition 5.7 follows. This concludes the proof of Theorem 1.1.

**Remark 5.13.** The ramification bound in Theorem 1.1 is sharp for $r \leq 1$. Indeed, the greatest upper ramification break $1 + e(n + 1/(p - 1))$ for $r = 1$ is obtained by the $p^n$-torsion of the Tate curve $\mathcal{K}^\times / \pi\mathbb{Z}$ (see Remark 5.5). The author does not know whether this bound is sharp also for $r > 1$.

**References**


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