ON A RAMIFICATION BOUND OF SEMI-STABLE TORSION REPRESENTATIONS OVER A LOCAL FIELD

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Abstract. Let p be a rational prime, k be a perfect field of characteristic p, $W = W(k)$ be the ring of Witt vectors, K be a finite totally ramified extension of $\mathrm{Frac}(W)$ of degree e and r be a non-negative integer satisfying $r < p-1$. Let V be a semi-stable p-adic G_K-representation with Hodge-Tate weights in $\{0, \ldots, r\}$. In this paper, we prove the upper numbering ramification group $G_K^{(j)}$ for $j > u(K, r, n)$ acts trivially on the mod p^n representations associated to V, where $u(K, 0, n) = 0$, $u(K, 1, n) = 1 + e(n + 1/(p-1))$ and $u(K, r, n) = 1 - p^{-n}e(K(\zeta_p)/K)^{-1} + e(n + r/(p-1))$ for $r > 1$.

1. Introduction

Let p be a rational prime, k be a perfect field of characteristic p, $W = W(k)$ be the ring of Witt vectors and K be a finite totally ramified extension of $K_0 = \mathrm{Frac}(W)$ of degree $e = e(K)$. Let the maximal ideal of K be denoted by m_K, an algebraic closure of K by $\bar{K}$ and the absolute Galois group of K by $G_K = \mathrm{Gal}(\bar{K}/K)$. We normalize the valuation v_K of K as $v_K(p) = e$ and extend this to $\bar{K}$. Let $G_K^{(j)}$ denote the j-th upper numbering ramification group in the sense of [7]. Namely, we put $G_K^{(j)} = G_K^{j-1}$, where the latter is the upper numbering ramification group defined in [15].

Let X_K be a proper smooth scheme over K and put $X_{\bar{K}} = X_K \times_K \bar{K}$. Consider the r-th étale cohomology group $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ and its G_K-stable $\mathbb{Z}_p$-lattices $\mathcal{L} \supseteq \mathcal{L}'$. In [7], Fontaine conjectured the upper numbering ramification group $G_K^{(j)}$ acts trivially on the G_K-module $\mathcal{L}/\mathcal{L}'$ for $j > e(n + r/(p-1))$ if X_K has good reduction and this module is killed by p^n. For $e = 1$ and $r < p-1$, this conjecture was proved independently by himself ([8], for $n = 1$) and Abrashkin ([2], for any n), using the theory of Fontaine-Laffaille ([10]) and the comparison theorem of Fontaine-Messing ([11]) between the p-adic étale cohomology groups of X_K and the crystalline cohomology groups of the reduction of X_K. From this result, Fontaine also showed some rareness of a proper smooth scheme over $\mathbb{Q}$ with everywhere good reduction ([8, Théorème 1]). In fact, they proved this ramification bound for the torsion

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representations of the crystalline p-adic representations of G_K with Hodge-Tate weights in $\{0,\dots,r\}$ in the case where K is absolutely unramified.

On the other hand, for a semi-stable p-adic representation V with Hodge-Tate weights in the same range, a similar ramification bound for $e=1$ and $n=1$ is obtained by Breuil (see [4, Proposition 9.2.2.2]). He showed, assuming the Griffiths transversality which in general does not hold, that if $e=1$ and $r<p-1$, then the ramification group $G_K^{(j)}$ acts trivially on the mod p representations of V for $j>2+1/(p-1)$.

In this paper, we prove a version of the result of Breuil for the case where K is absolutely ramified, under the condition $r<p-1$. Our main theorem is the following.

Theorem 1.1. *Let r be a non-negative integer such that $r<p-1$. Let V be a semi-stable p-adic G_K-representation with Hodge-Tate weights in $\{0,\dots,r\}$ and $\mathcal{L}\supseteq\mathcal{L}'$ be G_K-stable $\mathbb{Z}_p$-lattices in V. Suppose that the quotient $\mathcal{L}/\mathcal{L}'$ is killed by p^n. Then the j-th upper numbering ramification group $G_K^{(j)}$ acts trivially on the G_K-module $\mathcal{L}/\mathcal{L}'$ for $j>u(K,r,n)$, where*

$$u(K,r,n)=\begin{cases} 0 & (r=0),\\ 1+e(n+\frac{1}{p-1}) & (r=1),\\ 1-\frac{1}{p^n e(K(\zeta_p)/K)}+e(n+\frac{r}{p-1}) & (r>1)\end{cases}$$

and $e(K(\zeta_p)/K)$ denotes the relative ramification index of the extension $K(\zeta_p)/K$.

We can check that this bound is sharp for $r\le 1$ (Remark 5.13). From this theorem and [7, Proposition 1.3], we have the following corollary.

Corollary 1.2. *Let the notation be as in the theorem and L be the finite extension of K cut out by the G_K-module $\mathcal{L}/\mathcal{L}'$. Let $\mathfrak{D}_{L/K}$ denote the different of the extension L/K. Then we have the inequality*

$$v_K(\mathfrak{D}_{L/K})<u(K,r,n)$$

for $r>0$ and $v_K(\mathfrak{D}_{L/K})=0$ for $r=0$.

For the proof of Theorem 1.1, we essentially follow a beautiful argument of Abrashkin ([2]). We may assume $p\ge 3$ and $r\ge 1$. Thanks to Liu's theorem ([14]) on the G_K-stable $\mathbb{Z}_p$-lattices in semi-stable p-adic representations, it is enough to bound the ramification of the G_K-module

$$T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M}_n)=\mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r,N}(\mathcal{M}_n,\hat{A}_{\mathrm{st},\infty}),$$

where $\mathcal{M}_n$ is a p^n-torsion object of a category $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ of filtered (ϕ_r,N)-modules over S defined by Breuil ([3]) and $\hat{A}_{\mathrm{st},\infty}$ is a p-adic period ring. We may also assume $\zeta_p\in K$ and consider the finite Galois extension

$$F_n=K(\pi^{1/p^n},\zeta_{p^{n+1}})$$

of K whose upper ramification is bounded by the same value as in the theorem. Let L_n be the finite Galois extension of F_n cut out by $T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M}_n)|_{G_{F_n}}$.

Then we bound the ramification of L_n over K. For this, we show that to study this G_{F_n}-module we can use a variant over a smaller coefficient ring Σ of filtered (ϕ_r, N)-modules over S. In precise, let $E(u)$ be the Eisenstein polynomial of a uniformizer π of K over W and we set

$$\Sigma = W[[u, E(u)^p/p]].$$

This ring Σ is small enough for the method of Abrashkin, in which he uses filtered modules of Fontaine-Laffaille ([10]) whose coefficient ring is W, to work also in the case where K is absolutely ramified.

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2. Filtered (ϕ_r, N)-modules of Breuil

In this section, we recall the theory of filtered (ϕ_r, N)-modules over S of Breuil, which is developed by himself and most recently by Caruso and Liu (see for example [3], [5], [14], [6]). In what follows, we always take the divided power envelope of a W-algebra with the compatibility condition with the natural divided power structure on pW.

Let $p \geq 3$ be a rational prime and σ be the Frobenius endomorphism of W. We fix once and for all a uniformizer π of K and a system $\{\pi_n\}_{n\in\mathbb{Z}_{\geq 0}}$ of p-power roots of π such that $\pi_0 = \pi$ and $\pi_n = \pi_{n+1}^p$ for any n. Let $E(u)$ be the Eisenstein polynomial of π over W and set $S = (W[u]^{\mathrm{PD}})^\wedge$, where PD means the divided power envelope and this is taken with respect to the ideal $(E(u))$, and $\wedge$ means the p-adic completion. The ring S is endowed with the σ-semilinear endomorphism $\phi : u \mapsto u^p$ and a natural filtration $\mathrm{Fil}^t S$ induced by the divided power structure such that $\phi(\mathrm{Fil}^t S) \subseteq p^t S$ for any non-negative integer t. We set $\phi_t = p^{-t}\phi|_{\mathrm{Fil}^t S}$ and $c = \phi_1(E(u)) \in S^\times$. Let N denote the W-linear derivation on S defined by the formula $N(u) = -u$. We also define a filtration, ϕ, ϕ_t, N on $S_n = S/p^n S$ similarly.

Let $r \in \{0, \ldots, p-2\}$ be an integer. Set $'\mathrm{Mod}^{r,\phi,N}_{/S}$ to be the category consisting of the following data:

- an S-module $\mathcal{M}$ and its S-submodule $\mathrm{Fil}^r\mathcal{M}$ containing $\mathrm{Fil}^r S.\mathcal{M}$,
- a ϕ-semilinear map $\phi_r : \mathrm{Fil}^r\mathcal{M} \to \mathcal{M}$ satisfying

$$\phi_r(s_r m) = \phi_r(s_r)\phi(m)$$

  for any $s_r \in \mathrm{Fil}^r S$ and $m \in \mathcal{M}$, where we set $\phi(m) = c^{-r}\phi_r(E(u)^r m)$,
- a W-linear map $N : \mathcal{M} \to \mathcal{M}$ such that
  - $N(sm) = N(s)m + sN(m)$ for any $s \in S$ and $m \in \mathcal{M}$,
  - $E(u)N(\mathrm{Fil}^r\mathcal{M}) \subseteq \mathrm{Fil}^r\mathcal{M}$,

– the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Fil}^r\mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M} \\ {\scriptstyle E(u)N}\downarrow & & \downarrow{\scriptstyle cN} \\ \mathrm{Fil}^r\mathcal{M} & \xrightarrow[\phi_r]{} & \mathcal{M}, \end{array}$$

and the morphisms of ${}'\mathrm{Mod}^{r,\phi,N}_{/S}$ are defined to be the S-linear maps preserving Fil^r and commuting with ϕ_r and N. The category defined in the same way but dropping the data N is denoted by ${}'\mathrm{Mod}^{r,\phi}_{/S}$. These categories have obvious notions of exact sequences. Let $\mathrm{Mod}^{r,\phi,N}_{/S_1}$ denote the full subcategory of ${}'\mathrm{Mod}^{r,\phi,N}_{/S}$ consisting of $\mathcal{M}$ such that $\mathcal{M}$ is free of finite rank over S_1 and generated as an S_1-module by the image of ϕ_r. We write $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ for the smallest full subcategory which contains $\mathrm{Mod}^{r,\phi,N}_{/S_1}$ and is stable under extensions. We let $\mathrm{Mod}^{r,\phi,N}_{/S}$ denote the full subcategory consisting of $\mathcal{M}$ such that

- the S-module $\mathcal{M}$ is free of finite rank and generated by the image of ϕ_r,
- the quotient $\mathcal{M}/\mathrm{Fil}^r\mathcal{M}$ is p-torsion free.

We define full subcategories $\mathrm{Mod}^{r,\phi}_{/S_1}$, $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ and $\mathrm{Mod}^{r,\phi}_{/S}$ of ${}'\mathrm{Mod}^{r,\phi}_{/S}$ in a similar way. For $\hat{\mathcal{M}} \in \mathrm{Mod}^{r,\phi,N}_{/S}$ (*resp.* $\mathrm{Mod}^{r,\phi}_{/S}$), the quotient $\hat{\mathcal{M}}/p^n\hat{\mathcal{M}}$ has a natural structure as an object of $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ (*resp.* $\mathrm{Mod}^{r,\phi}_{/S_\infty}$).

For p-torsion objects, we also have the following categories. Consider the k-algebra $k[u]/(u^{ep}) \cong S_1/\mathrm{Fil}^p S_1$ and let this algebra be denoted by $\tilde{S}_1$. The algebra $\tilde{S}_1$ is equipped with the natural filtration, ϕ and N induced by those of S. Namely, $\mathrm{Fil}^t\tilde{S}_1 = u^{et}\tilde{S}_1$, $\phi(u) = u^p$ and $N(u) = -u$. Let ${}'\mathrm{Mod}^{r,\phi,N}_{/\tilde{S}_1}$ denote the category consisting of the following data:

- an $\tilde{S}_1$-module $\tilde{\mathcal{M}}$ and its $\tilde{S}_1$-submodule $\mathrm{Fil}^r\tilde{\mathcal{M}}$ containing $u^{er}\tilde{\mathcal{M}}$,
- a ϕ-semilinear map $\phi_r : \mathrm{Fil}^r\tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$,
- a k-linear map $N : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ such that
  - $N(sm) = N(s)m + sN(m)$ for any $s \in \tilde{S}_1$ and $m \in \tilde{\mathcal{M}}$,
  - $u^eN(\mathrm{Fil}^r\tilde{\mathcal{M}}) \subseteq \mathrm{Fil}^r\tilde{\mathcal{M}}$,
  - the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Fil}^r\tilde{\mathcal{M}} & \xrightarrow{\phi_r} & \tilde{\mathcal{M}} \\ {\scriptstyle u^eN}\downarrow & & \downarrow{\scriptstyle cN} \\ \mathrm{Fil}^r\tilde{\mathcal{M}} & \xrightarrow[\phi_r]{} & \tilde{\mathcal{M}}, \end{array}$$

and whose morphisms are defined as before. Its full subcategory $\mathrm{Mod}^{r,\phi,N}_{/\tilde{S}_1}$ is defined by the following condition:

- As an $\tilde{S}_1$-module, $\tilde{\mathcal{M}}$ is free of finite rank and generated by the image of ϕ_r.

We define categories ${}'\mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ and $\mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ similarly.

Let D be a weakly admissible filtered (ϕ, N)-module over K satisfying $\mathrm{Fil}^0 D_K = D_K$ and $\mathrm{Fil}^{r+1} D_K = 0$. Set $S_{K_0} = S \otimes_W K_0$ and $\mathcal{D} = D \otimes_{K_0} S_{K_0}$. Then the S_{K_0}-module $\mathcal{D}$ is equipped with the natural ϕ-semilinear map $\phi \otimes \sigma$ and K_0-linear derivation $N \otimes 1 + 1 \otimes N$, which are denoted by ϕ and N, respectively. We define a filtration on $\mathcal{D}$ inductively by $\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ and

$$\mathrm{Fil}^{i+1}\mathcal{D} = \{x \in \mathcal{D} \mid N(x) \in \mathrm{Fil}^i \mathcal{D} \text{ and } f_\pi(x) \in \mathrm{Fil}^{i+1} D_K\},$$

where $f_\pi : \mathcal{D} \to D_K$ is induced by the map $S \to \mathcal{O}_K$ sending u to π. An S-submodule $\hat{\mathcal{M}}$ of $\mathcal{D}$ is said to be a strongly divisible lattice of $\mathcal{D}$ if the following conditions are satisfied:

- the S-module $\hat{\mathcal{M}}$ is free of finite rank,
- $\hat{\mathcal{M}} \otimes_W K_0 = \mathcal{D}$,
- $\hat{\mathcal{M}}$ is stable under ϕ and N,
- $\phi(\mathrm{Fil}^r \hat{\mathcal{M}}) \subseteq p^r \hat{\mathcal{M}}$, where we set $\mathrm{Fil}^r \hat{\mathcal{M}} = \hat{\mathcal{M}} \cap \mathrm{Fil}^r \mathcal{D}$.

We put $\phi_r = p^{-r}\phi|_{\mathrm{Fil}^r \hat{\mathcal{M}}}$. Then the S-module $\hat{\mathcal{M}}$ is generated by $\phi_r(\mathrm{Fil}^r \hat{\mathcal{M}})$ ([3, Proposition 2.1.3]) and we can consider $\hat{\mathcal{M}}$ as an object of $\mathrm{Mod}^{r,\phi,N}_{/S}$.

Let A_{crys} and $\hat{A}_{\mathrm{st}}$ be p-adic period rings. These are constructed as follows. Set R to be the ring

$$R = \varprojlim (\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}} \leftarrow \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}} \leftarrow \cdots),$$

where every arrow is the p-power map. For an element $x = (x_i)_{i \in \mathbb{Z}_{\geq 0}} \in R$ and an integer $n \geq 0$, we set

$$x^{(n)} = \lim_{m \to \infty} \hat{x}_{n+m}^{p^m} \in \mathcal{O}_{\mathbb{C}},$$

where $\hat{x}_i$ is a lift of x_i in $\mathcal{O}_{\bar{K}}$ and $\mathcal{O}_{\mathbb{C}}$ is the p-adic completion of $\mathcal{O}_{\bar{K}}$. Let v_p denote the valuation of $\mathcal{O}_{\mathbb{C}}$ normalized as $v_p(p) = 1$. Then the ring R is a complete valuation ring whose valuation of an element $x \in R$ is given by $v_R(x) = v_p(x^{(0)})$. We define a natural ring homomorphism θ by

$$\begin{aligned} \theta : W(R) &\to \mathcal{O}_{\mathbb{C}} \\ (x_0, x_1, \ldots) &\mapsto \sum_{n \geq 0} p^n x_n^{(n)}. \end{aligned}$$

Then A_{crys} is the p-adic completion of the divided power envelope of $W(R)$ with respect to the principal ideal $\mathrm{Ker}(\theta)$ and $\hat{A}_{\mathrm{st}}$ is the p-adic completion of the divided power polynomial ring $A_{\mathrm{crys}}\langle X \rangle$ over A_{crys}. We set $A_{\mathrm{crys},\infty} = A_{\mathrm{crys}} \otimes_W K_0/W$ and $\hat{A}_{\mathrm{st},\infty} = \hat{A}_{\mathrm{st}} \otimes_W K_0/W$. Put $\underline{\pi} = (\pi_n)_{n \in \mathbb{Z}_{\geq 0}} \in R$, where we abusively let π_n denote the image of $\pi_n \in \mathcal{O}_{\bar{K}}$ in $\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$. These rings are considered as S-algebras by the ring homomorphisms $S \to \hat{A}_{\mathrm{st}}$ and $\hat{A}_{\mathrm{st}} \to A_{\mathrm{crys}}$ which are defined by $u \mapsto [\underline{\pi}]/(1+X)$ and $X \mapsto 0$, respectively. The ring A_{crys} is endowed with a natural filtration induced by the divided power structure, a natural Frobenius endomorphism ϕ and the ϕ-semilinear map $\phi_t = p^{-t}\phi|_{\mathrm{Fil}^t A_{\mathrm{crys}}}$. With these structures, A_{crys} and $A_{\mathrm{crys},\infty}$ are considered as objects of $'\mathrm{Mod}^{r,\phi}_{/S}$. Moreover, the absolute Galois group G_K acts naturally on these two rings. As for $\hat{A}_{\mathrm{st}}$, its filtration is defined by

$$\mathrm{Fil}^t \hat{A}_{\mathrm{st}} = \left\{ \sum_{i\geq 0} a_i \frac{X^i}{i!} \;\middle|\; a_i \in \mathrm{Fil}^{t-i} A_{\mathrm{crys}},\ \lim_{i\to\infty} a_i = 0 \right\}$$

and the Frobenius structure of A_{crys} extends to $\hat{A}_{\mathrm{st}}$ by

$$\begin{aligned} \phi(X) &= (1+X)^p - 1, \\ \phi_t &= p^{-t}\phi|_{\mathrm{Fil}^t \hat{A}_{\mathrm{st}}}. \end{aligned}$$

We write N also for the A_{crys}-linear derivation on $\hat{A}_{\mathrm{st}}$ defined by $N(X) = 1+X$. The rings $\hat{A}_{\mathrm{st}}$ and $\hat{A}_{\mathrm{st},\infty}$ are objects of $'\mathrm{Mod}^{r,\phi,N}_{/S}$. The G_K-action on A_{crys} naturally extends to an action on $\hat{A}_{\mathrm{st}}$. Indeed, the action of $g \in G_K$ on $\hat{A}_{\mathrm{st}}$ is defined by the formula

$$g(X) = [\underline{\varepsilon}(g)](1+X) - 1,$$

where $g(\pi_n) = \varepsilon_n(g)\pi_n$ and $\underline{\varepsilon}(g) = (\varepsilon_n(g))_{n\in\mathbb{Z}_{\geq 0}} \in R$ with the abusive notation as above.

These rings have other descriptions, as follows. For an integer $n \geq 1$, put $W_n = W/p^nW$ and let $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ be the ring of Witt vectors of length n associated to $\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$. We define a W_n-algebra structure on $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ by twisting the natural W_n-algebra structure by σ^{-n}. Then the natural ring homomorphism

$$\begin{aligned} \theta_n : W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) &\to \mathcal{O}_{\bar{K}}/p^n\mathcal{O}_{\bar{K}} \\ (a_0, \ldots, a_{n-1}) &\mapsto \sum_{i=0}^{n-1} p^i \hat{a}_i^{p^{n-i}}, \end{aligned}$$

where $\hat{a}_i$ is a lift of a_i in $\mathcal{O}_{\bar{K}}$, is W_n-linear. Let us denote

$$W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$$

the divided power envelope of $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ with respect to the ideal $\mathrm{Ker}(\theta_n)$. This ring is considered as an S-algebra by $u \mapsto [\pi_n]$. This ring also has a natural filtration defined by the divided power structure, and a natural G_K-module structure. The Frobenius endomorphism of the ring of Witt vectors induces on this ring a ϕ-semilinear Frobenius endomorphism, which is denoted also by ϕ. Then, by the S-linear transition maps

$$W_{n+1}^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \to W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$$
$$(a_0, \dots, a_n) \mapsto (a_0^p, \dots, a_{n-1}^p),$$

these S-algebras form a projective system compatible with all structures. Using this transition map, a ϕ-semilinear map

$$\phi_r : \mathrm{Fil}^r W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \to W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$$

is defined by setting $\phi_r(x)$ to be the image of $p^{-r}\phi(\hat{x})$, where $\hat{x}$ is a lift of x in $\mathrm{Fil}^r W_{n+r}^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$. By definition, the maps ϕ_r are also compatible with the transition maps. The S-algebra $W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ is considered as an object of $'\mathrm{Mod}_{/S}^{r,\phi}$. Then we have a natural isomorphism in $'\mathrm{Mod}_{/S}^{r,\phi}$

$$A_{\mathrm{crys}}/p^n A_{\mathrm{crys}} \to W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$$
$$(x_0, \dots, x_{n-1}) \mapsto (x_{0,n}, \dots, x_{n-1,n}),$$

where we set $x_i = (x_{i,k})_{k \in \mathbb{Z}_{\geq 0}}$.

Similarly, the divided power polynomial ring

$$W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})\langle X \rangle$$

over $W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ is considered as an S-algebra by $u \mapsto [\pi_n]/(1+X)$. This ring has a natural filtration coming from the divided power structure. We define a G_K-action on this ring by

$$g(X) = [\varepsilon_n(g)](1+X) - 1.$$

We also define a ϕ-semilinear Frobenius endomorphism, which we also write as ϕ, by $\phi(X) = (1+X)^p - 1$ and a $W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$-linear derivation N by $N(X) = 1 + X$. These rings form a projective system of S-algebras compatible with all structures by the transition maps defined by the maps above and $X \mapsto X$. We define ϕ-semilinear maps

$$\phi_r : \mathrm{Fil}^r W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})\langle X \rangle \to W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})\langle X \rangle$$

compatible with the transition maps as before. The S-algebra $W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})\langle X \rangle$ is considered as an object of $'\mathrm{Mod}_{/S}^{r,\phi,N}$ and there exists a natural isomorphism in $'\mathrm{Mod}_{/S}^{r,\phi,N}$

$$\hat{A}_{\mathrm{st}}/p^n \hat{A}_{\mathrm{st}} \to W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})\langle X \rangle$$
$$(x_0, \dots, x_{n-1}) \mapsto (x_{0,n}, \dots, x_{n-1,n})$$
$$X \mapsto X$$

which is G_K-linear.

Put $K_n = K(\pi_n)$ and $K_\infty = \cup_n K_n$. For $\mathcal{M} \in \mathrm{Mod}_{/S_\infty}^{r,\phi,N}$, we define a G_K-module $T_{\mathrm{st},\underline{\pi}}^*(\mathcal{M})$ to be

$$T_{\mathrm{st},\underline{\pi}}^*(\mathcal{M}) = \mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r,N}(\mathcal{M}, \hat{A}_{\mathrm{st},\infty}).$$

When $\mathcal{M}$ is killed by p^n, we have a natural identification of G_K-modules

$$T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M}) = \mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r,N}(\mathcal{M}, W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})\langle X\rangle).$$

Note that the G_K-module on the right-hand side is independent of the choice of π_k for $k > n$. Since the natural map

$$\begin{gathered} W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})\langle X\rangle \to W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \\ X \mapsto 0 \end{gathered}$$

is by definition G_{K_n}-linear, we also have a G_{K_n}-linear isomorphism ([3, Lemme 2.3.1.1])

$$T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M})|_{G_{K_n}} \to \mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r}(\mathcal{M}, W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})).$$

A variant of filtered (ϕ_r, N)-modules over S is also introduced by Breuil and Kisin, and developed also by Caruso and Liu (see for example [12], [13], [14], [6]). Put $\mathfrak{S} = W[[u]]$ and let $\phi : \mathfrak{S} \to \mathfrak{S}$ be the σ-semilinear Frobenius endomorphism defined by $\phi(u) = u^p$. Let $'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ denote the category consisting of the following data:

- an $\mathfrak{S}$-module $\mathfrak{M}$,
- a ϕ-semilinear map $\mathfrak{M} \to \mathfrak{M}$, which is denoted also by ϕ, such that the cokernel of the map $1 \otimes \phi : \phi^*\mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$,

and whose morphisms are defined as before. The full subcategory of $'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ consisting of $\mathfrak{M}$ such that $\mathfrak{M}$ is free of finite rank over $\mathfrak{S}/p\mathfrak{S}$ (*resp.* over $\mathfrak{S}$) is denoted by $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ (*resp.* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$). We let $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ denote the smallest full subcategory which contains $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ and is stable under extensions, as before. Then we have an exact functor ([6, Proposition 2.1.2], see also [12, Proposition 1.1.11])

$$\mathcal{M}_{\mathfrak{S}_\infty} : \mathrm{Mod}^{r,\phi}_{\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\phi}_{S_\infty}.$$

For $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{\mathfrak{S}_\infty}$, the filtered ϕ_r-module $\mathcal{M} = \mathcal{M}_{\mathfrak{S}_\infty}(\mathfrak{M})$ over S is defined as follows:

- $\mathcal{M} = S \otimes_{\phi,\mathfrak{S}} \mathfrak{M}$,
- $\mathrm{Fil}^r\mathcal{M} = \mathrm{Ker}(\mathcal{M} \overset{1\otimes\phi}{\to} S \otimes_{\mathfrak{S}} \mathfrak{M} \to (S/\mathrm{Fil}^r S) \otimes_{\mathfrak{S}} \mathfrak{M})$,
- $\phi_r : \mathrm{Fil}^r\mathcal{M} \overset{1\otimes\phi}{\to} \mathrm{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \overset{\phi_r\otimes 1}{\to} S \otimes_{\phi,\mathfrak{S}} \mathfrak{M} = \mathcal{M}$.

We write $\mathcal{M}_{\mathfrak{S}}$ for the functor $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}} \to \mathrm{Mod}^{r,\phi}_{/S}$ defined similarly.

Finally, let D and $\mathcal{D}$ be as above and $\hat{\mathcal{M}}$ be a strongly divisible lattice in $\mathcal{D}$. The S-module $\mathcal{M}_n = \hat{\mathcal{M}}/p^n\hat{\mathcal{M}}$ has a natural structure as an object of $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$. We set a G_K-module $\hat{T}^*_{\mathrm{st},\underline{\pi}}(\hat{\mathcal{M}})$ to be

$$\hat{T}^*_{\mathrm{st},\underline{\pi}}(\hat{\mathcal{M}}) = \mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r,N}(\hat{\mathcal{M}}, \hat{A}_{\mathrm{st}}).$$

Then we have an exact sequence of G_K-modules

$$0 \longrightarrow \hat{T}^*_{\mathrm{st},\underline{\pi}}(\hat{\mathcal{M}}) \xrightarrow{p^n} \hat{T}^*_{\mathrm{st},\underline{\pi}}(\hat{\mathcal{M}}) \longrightarrow T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M}_n) \longrightarrow 0.$$

The G_K-module $\hat{T}^*_{\mathrm{st},\underline{\pi}}(\hat{\mathcal{M}})$ is naturally considered as a G_K-stable $\mathbb{Z}_p$-lattice in $V^*_{\mathrm{st}}(D)$. By Liu's theorem ([14, Theorem 2.3.5]), the functor $\hat{T}^*_{\mathrm{st},\underline{\pi}}$ gives an anti-equivalence of categories between the category of strongly divisible lattices in $\mathcal{D}$ and the category of G_K-stable $\mathbb{Z}_p$-lattices in $V^*_{\mathrm{st}}(D)$. Moreover, for such a lattice $\mathcal{L}$, its corresponding strongly divisible lattice $\hat{\mathcal{M}}$ in $\mathcal{D}$ is in the essential image of the functor $\mathcal{M}_{\mathfrak{S}}$ ([14, Subsection 3.5]).

3. Filtered ϕ_r-modules over Σ

In this section, we define another variant of filtered ϕ_r-modules over S and prove its properties.

Let $p \geq 3$ be a rational prime and r be an integer such that $0 \leq r < p-1$. Consider the W-algebra $\Sigma = W[[u,Y]]/(E(u)^p - pY)$ as in [3, Subsection 3.2]. We regard Σ as a subring of S by the map sending Y to $E(u)^p/p$. Then the element $c = \phi_1(E(u)) \in S^\times$ is contained in $\Sigma^\times$. We define on Σ a σ-semilinear Frobenius endomorphism ϕ by $\phi(u) = u^p$ and $\phi(Y) = p^{p-1}c^p$. Put $\mathrm{Fil}^t\Sigma = (E(u)^t, Y)$ for $0 \leq t \leq p-1$ and $\mathrm{Fil}^p\Sigma = (Y)$. Then we have $\phi(\mathrm{Fil}^t\Sigma) \subseteq p^t\Sigma$ for $0 \leq t \leq p-1$. We put $\phi_t = p^{-t}\phi|_{\mathrm{Fil}^t\Sigma}$. We also set $\Sigma_n = \Sigma/p^n\Sigma$ and put on this ring the natural structures induced by those of Σ.

We define a category $'\mathrm{Mod}^{r,\phi}_{/\Sigma}$ of filtered ϕ_r-modules over Σ to be the category consisting of the following data:

- a Σ-module M and its Σ-submodule $\mathrm{Fil}^r M$ containing $\mathrm{Fil}^r\Sigma.M$,
- a ϕ-semilinear map $\mathrm{Fil}^r M \to M$ satisfying $\phi_r(s_r m) = \phi_r(s_r)\phi(m)$ for any $s_r \in \mathrm{Fil}^r\Sigma$ and $m \in M$, where we set $\phi(m) = c^{-r}\phi_r(E(u)^r m)$.

and the morphisms are defined in the same manner as $'\mathrm{Mod}^{r,\phi}_{/S}$. This category has a natural notion of exact sequences. We define its full subcategory $\mathrm{Mod}^{r,\phi}_{/\Sigma_1}$ to be the category consisting of M which is free of finite rank and generated by the image of ϕ_r as a Σ_1-module. We also let $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ denote the smallest full subcategory of $'\mathrm{Mod}^{r,\phi}_{/\Sigma}$ which contains $\mathrm{Mod}^{r,\phi}_{/\Sigma_1}$ and is stable under extensions. Moreover, we define a full subcategory $\mathrm{Mod}^{r,\phi}_{/\Sigma}$ of $'\mathrm{Mod}^{r,\phi}_{/\Sigma}$ to be the category consisting of M such that

- the Σ-module M is free of finite rank and generated by the image of ϕ_r,
- the quotient $M/\mathrm{Fil}^r M$ is p-torsion free.

Then we see that for $\hat{M} \in \mathrm{Mod}^{r,\phi}_{/\Sigma}$, the quotient $\hat{M}/p^n\hat{M}$ is naturally considered as an object of $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$.

The natural ring isomorphism $\Sigma_1/\mathrm{Fil}^p\Sigma_1 \cong \tilde{S}_1$ defines a functor $T_0 : \mathrm{Mod}^{r,\phi}_{/\Sigma_1} \to \mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ by $M \mapsto M/\mathrm{Fil}^p\Sigma_1.M$. Then [3, Proposition 2.2.1.3] and Nakayama's lemma shows the following.

Lemma 3.1. *Let M be an object of $\mathrm{Mod}^{r,\phi}_{/\Sigma_1}$ of rank d over Σ_1. Then there exists a basis $\{e_1,\dots,e_d\}$ of M such that $\mathrm{Fil}^r M=\Sigma_1 u^{r_1}e_1\oplus\cdots\oplus\Sigma_1 u^{r_d}e_d+\mathrm{Fil}^p\Sigma_1.M$ for some integers $r_1,\dots,r_d$ with $0\leq r_i\leq er$ for any i.*

Then we can show the following lemma just as in the proof of [3, Lemme 2.3.1.3].

Lemma 3.2. *The functor*

$$M\mapsto \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M,A_{\mathrm{crys},\infty})$$

from $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ to the category of G_{K_∞}-modules is exact.

For $M\in\mathrm{Mod}^{r,\phi}_{/\Sigma_1}$, we can show as in the case of the category $\mathrm{Mod}^{r,\phi}_{/S_1}$ that there is an isomorphism of G_{K_1}-modules

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M,(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})^{\mathrm{PD}})\to \mathrm{Hom}_{\tilde{S}_1,\mathrm{Fil}^r,\phi_r}(T_0(M),\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}),$$

where $\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$ is considered as an object of $'\mathrm{Mod}^{r,\phi}_{\tilde{S}_1}$ by the natural isomorphism

$$(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})^{\mathrm{PD}}/\mathrm{Fil}^p(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})^{\mathrm{PD}}\to\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}.$$

Thus [3, Lemme 2.3.1.2] implies that, for such a Σ_1-module M, we have

$$\#\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M,(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})^{\mathrm{PD}})=p^d,$$

where $d=\dim_{\Sigma_1}M$.

For the category $\mathrm{Mod}^{r,\phi}_{/\Sigma}$, we can show the following lemma just as in the proof of [14, Proposition 4.1.2].

Lemma 3.3. *Let $\hat{M}$ be in $\mathrm{Mod}^{r,\phi}_{/\Sigma}$. Then there exists $\alpha_1,\dots,\alpha_d\in\hat{M}$ such that $\mathrm{Fil}^r\hat{M}=\Sigma\alpha_1\oplus\cdots\oplus\Sigma\alpha_d+\mathrm{Fil}^p\Sigma.\hat{M}$, $E(u)^r\hat{M}\subseteq\Sigma\alpha_1\oplus\cdots\oplus\Sigma\alpha_d$ and the elements $e_1=\phi_r(\alpha_1),\dots,e_d=\phi_r(\alpha_d)$ form a basis of $\hat{M}$.*

Corollary 3.4. *Let $\hat{M}$ be in $\mathrm{Mod}^{r,\phi}_{/\Sigma}$ and A be a Σ-algebra which has a structure as an object of $'\mathrm{Mod}^{r,\phi}_{/\Sigma}$. Let $C\in M_d(\Sigma)$ be the matrix such that*

$$(\alpha_1,\dots,\alpha_d)=(e_1,\dots,e_d)C$$

with the notation of the previous lemma. Then a Σ-linear homomorphism $f:\hat{M}\to A$ preserving Fil^r also commutes with ϕ_r if and only if

$$\phi_r(f(e_1,\dots,e_d)C)=(f(e_1),\dots,f(e_d)).$$

Proof. Suppose that the latter condition holds. By assumption, we have

$$E(u)^r(e_1,\dots,e_d)=(\alpha_1,\dots,\alpha_d)C'$$

for some $C' \in M_d(\Sigma)$. We claim that f commutes with ϕ. Indeed, we have

$$\begin{aligned}\phi(f(e_1,\dots,e_d)) &= c^{-r}\phi_r(E(u)^r(f(e_1),\dots,f(e_d)))\\ &= c^{-r}\phi_r((f(\alpha_1),\dots,f(\alpha_d))C')\\ &= c^{-r}f(e_1,\dots,e_d)\phi(C')\\ &= c^{-r}f(\phi_r(\alpha_1,\dots,\alpha_d))\phi(C')\\ &= c^{-r}f(\phi_r(E(u)^r(e_1,\dots,e_d))) = f(\phi(e_1,\dots,e_d)).\end{aligned}$$

This implies $\phi_r \circ f = f \circ \phi_r$ also on $\mathrm{Fil}^p\Sigma.\hat{M}$.

□

Corollary 3.5. *Let $\hat{M}$ and A be as above and $J \subseteq \mathrm{Fil}^r A$ be an ideal of A such that $\phi_r(J) \subseteq J$. We can consider the Σ-algebra A/J naturally as an object of ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$. Suppose that for any $x \in J$, there exists $t \in \mathbb{Z}_{\geq 0}$ such that $\phi_r^t(x) = 0$. Then we have an isomorphism*

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(\hat{M}, A) \to \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(\hat{M}, A/J).$$

Proof. Let $f : \hat{M} \to A/J$ be an element of the abelian group on the right-hand side and $\hat{x}_i$ be an lift of $f(e_i)$ in A. By the previous corollary, it is enough to show that for any $(\hat{c}_1,\dots,\hat{c}_d) \in J^d$, there is a unique solution $(\hat{y}_1,\dots,\hat{y}_d) \in J^d$ of the equation

$$(\hat{c}_1,\dots,\hat{c}_d) + (\phi_r(\hat{y}_1),\dots,\phi_r(\hat{y}_d))\phi(C) = (\hat{y}_1,\dots,\hat{y}_d).$$

By assumption, the d-tuple

$$\sum_{i=0}^{t}(\phi_r^i(\hat{c}_1),\dots,\phi_r^i(\hat{c}_d))\phi^i(C)\phi^{i-1}(C)\cdots\phi(C)$$

is stable for sufficiently large t and we see that this limit gives a unique solution of the equation. □

For an $\mathfrak{S}$-module $\mathfrak{M}$ in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ (*resp.* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$), we associate to it a Σ-module $M \in {}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$ as follows:

- $M = \Sigma \otimes_{\phi,\mathfrak{S}} \mathfrak{M}$,
- $\mathrm{Fil}^r M = \mathrm{Ker}(M \overset{1\otimes\phi}{\to} \Sigma \otimes_{\mathfrak{S}} \mathfrak{M} \to (\Sigma/\mathrm{Fil}^r\Sigma) \otimes_{\mathfrak{S}} \mathfrak{M})$,
- $\phi_r : \mathrm{Fil}^r M \overset{1\otimes\phi}{\to} \mathrm{Fil}^r\Sigma \otimes_{\mathfrak{S}} \mathfrak{M} \overset{\phi_r\otimes 1}{\to} \Sigma \otimes_{\phi,\mathfrak{S}} \mathfrak{M} = M$.

We can check that this defines an exact functor $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ (*resp.* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}} \to \mathrm{Mod}^{r,\phi}_{/\Sigma}$) as in the proof of [12, Proposition 1.1.11]. We let this functor be denoted by $M_{\mathfrak{S}_\infty}$ (*resp.* $M_{\mathfrak{S}}$).

Proposition 3.6. *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ which is killed by p^n. Set $M = M_{\mathfrak{S}_\infty}(\mathfrak{M})$ and $\mathcal{M} = \mathcal{M}_{\mathfrak{S}_\infty}(\mathfrak{M})$. Then there exists a natural isomorphism of G_{K_n}-modules*

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})) \to \mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r}(\mathcal{M}, W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})).$$

Proof. By definition, $\mathcal{M} = S \otimes_\Sigma M$ and we have a natural isomorphism

$$\mathrm{Hom}_\Sigma(M, W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})) \to \mathrm{Hom}_S(\mathcal{M}, W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})).$$

Let f be an element of $\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}))$ and f' be the image of f in the right-hand side of the above isomorphism. Let us check that f' preserves Fil^r and commutes with ϕ_r. Since f' is S-linear, it maps $\mathrm{Fil}^p S.\mathcal{M}$ into $\mathrm{Fil}^p W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$. For $x \in \mathrm{Fil}^r\mathcal{M} \cap \mathrm{Im}(M \to \mathcal{M})$, the commutative diagram whose right vertical arrow is an isomorphism

$$\begin{tikzcd}
M = \Sigma \otimes_{\phi,\mathfrak{S}} \mathfrak{M} \arrow[r, "1\otimes\phi"] \arrow[d] & \Sigma \otimes_{\mathfrak{S}} \mathfrak{M} \arrow[r] \arrow[d] & \Sigma/\mathrm{Fil}^r\Sigma \otimes_{\mathfrak{S}} \mathfrak{M} \arrow[d] \\
\mathcal{M} = S \otimes_{\phi,\mathfrak{S}} \mathfrak{M} \arrow[r, "1\otimes\phi"] & S \otimes_{\mathfrak{S}} \mathfrak{M} \arrow[r] & S/\mathrm{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M}
\end{tikzcd}$$

implies $x \in \mathrm{Im}(\mathrm{Fil}^r M \to \mathrm{Fil}^r\mathcal{M})$ and thus $f'(x) \in \mathrm{Fil}^r W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$.

As for the compatibility with ϕ_r, again by the S-linearity of f' it suffices to show $f'(\phi_r(x)) = \phi_r(f'(x))$ for $x \in \mathrm{Fil}^r\mathcal{M} \cap \mathrm{Im}(M \to \mathcal{M}) = \mathrm{Im}(\mathrm{Fil}^r M \to \mathrm{Fil}^r\mathcal{M})$. This follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{Fil}^r M \arrow[r, "\phi_r"] \arrow[d] & M \arrow[r, "f"] \arrow[d] & W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \arrow[d, equal] \\
\mathrm{Fil}^r \mathcal{M} \arrow[r, "\phi_r"] & \mathcal{M} \arrow[r, "f'"] & W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}).
\end{tikzcd}$$

Hence the map in the proposition is well-defined and injective. To prove the bijectivity, by devissage we may assume that $p\mathfrak{M} = 0$. Then both sides of this injection have the same cardinality by the above remark. Thus the proposition follows. □

4. A method of Abrashkin

In this section, we study the G_{K_n}-module $\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}))$ following Abrashkin ([2]).

Let $p \geq 3$ and $0 \leq r < p-1$ be as before. Consider the Lubin-Tate logarithm

$$l(X) = X + \frac{X^p}{p} + \cdots + \frac{X^{p^n}}{p^n} + \cdots$$

and put $\psi(X) = l^{-1}(\log(1+X))$. Then ψ gives a strict isomorphism of formal groups between the formal group associated to the logarithm $l(X)$ and the multiplicative group $\hat{\mathbb{G}}_m$ over $\mathbb{Z}_p$. We fix a system of p-power roots of unity $\{\zeta_{p^n}\}_{n\in\mathbb{Z}_{\geq 0}}$ such that $\zeta_p \neq 1$ and $\zeta_{p^n} = \zeta_{p^{n+1}}^p$ for any n, and set an element $\underline{\varepsilon}$ of R to be $(\zeta_{p^n})_{n\in\mathbb{Z}_{\geq 0}}$. Then the elements $[\underline{\varepsilon}] - 1$ and $[\underline{\varepsilon}^{1/p}] - 1$ are topologically nilpotent in $W(R)$ and the element of $W(R)$

$$t = \psi([\underline{\varepsilon}] - 1)/\psi([\underline{\varepsilon}^{1/p}] - 1)$$

is a generator of the principal ideal $\mathrm{Ker}(\theta)$. The element $Z = \psi([\underline{\varepsilon}]-1)^{p-1}/p$ of A_{crys} is topologically nilpotent and $\phi(t)$ is contained in the subset

$$p(1 + ZW(R)[[Z]])$$

of A_{crys} ([2, Subsection 1.8]). We set

$$\hat{A} = W(R)[[Z]] \subseteq A_{\mathrm{crys}}.$$

Lemma 4.1. *The element t^p/p of A_{crys} is contained in the subring $\hat{A}$ and topologically nilpotent in this subring.*

Proof. Put $t' = ([\underline{\varepsilon}] - 1)/([\underline{\varepsilon}^{1/p}] - 1)$. This is another generator of $\mathrm{Ker}(\theta)$. We have

$$\frac{([\underline{\varepsilon}] - 1)^{p-1}}{p} = \frac{(t')^{p-1}}{p} \cdot ([\underline{\varepsilon}^{1/p}] - 1)^{p-1}$$

and $\theta([\underline{\varepsilon}^{1/p}] - 1) = \zeta_p - 1$. Take an element $a \in W(R)^\times$ such that $\theta(a) = (\zeta_p - 1)^{p-1}/p$. Then we have

$$\frac{([\underline{\varepsilon}] - 1)^{p-1}}{p} = a(t')^{p-1} + b(t')^p/p$$

for some $b \in W(R)^\times$. Indeed, to show $b \in W(R)^\times$, it suffices to check that the element $([\underline{\varepsilon}^{1/p}]-1)^{p-1} - pa$ of $\mathrm{Ker}(\theta)$ also generates this ideal. This follows from the fact that the 0-th entry $(\underline{\varepsilon}^{1/p}-1)^{p-1}$ of this element satisfies $v_R((\underline{\varepsilon}^{1/p} - 1)^{p-1}) = 1$. Then we see that $(t')^p/p$ is topologically nilpotent because so is t' in $W(R)$. $\square$

In the following, we set the element a in the proof of the lemma to be

$$a = \sum_{k=1}^{p-2} p^{-1}((-1)^{p-1-k}{}_{p-1}C_k - 1)[\underline{\varepsilon}^{1/p}]^k,$$

where ${}_{p-1}C_k = (p-1)!/(k!(p-1-k)!)$ is the binomial coefficient. Note that the coefficient of $[\underline{\varepsilon}^{1/p}]^k$ in each term is an integer.

From this lemma, we can consider the ring $\hat{A}$ as a Σ-algebra by $u \mapsto [\underline{\pi}]$. Put $\mathrm{Fil}^i\hat{A} = (t^i, Z)$ for $0 \leq i \leq p-1$. The Frobenius endomorphism ϕ of A_{crys} preserves $\hat{A}$ and satisfies $\phi(\mathrm{Fil}^i\hat{A}) \subseteq p^i\hat{A}$ for $0 \leq i \leq p-1$. Set $\phi_r = p^{-r}\phi|_{\mathrm{Fil}^r\hat{A}}$. Then we can consider the ring $\hat{A}$ also as an object of the category ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$, and similarly for $\hat{A}_n = \hat{A}/p^n\hat{A}$ and $\hat{A}_\infty = \hat{A} \otimes_W K_0/W$. The absolute Galois group G_{K_∞} acts naturally on these Σ-algebras. The following lemma is used implicitly in [2].

Lemma 4.2. *We have a natural decomposition*

$$\hat{A}_1 = R/(t^p) \oplus (Z).$$

Proof. Consider the natural inclusion $W(R) \to \hat{A}$. First we claim that this induces an injection $R/(t^p) \to \hat{A}_1$. Let x be in the ring R. If the element

$[x] \in W(R)$ is contained in $p\hat{A}$, then its image in $A_{\mathrm{crys}}/pA_{\mathrm{crys}}$ is zero. We have an isomorphism of R-algebras

$$R[Y_1, Y_2, \ldots]/(t^p, Y_1^p, Y_2^p, \ldots) \to A_{\mathrm{crys}}/pA_{\mathrm{crys}}$$

which sends Y_i to the image of $t^{p^i}/(p^i)!$. Thus the inequality $v_R(x) \geq p$ holds. Conversely, if $v_R(x) \geq p$, then we have

$$[x] = w(\psi([\underline{\varepsilon}] - 1)^{p-1}) + pw'$$

for some $w, w' \in W(R)$ and this implies $[x] \in p\hat{A}$.

Let us consider the commutative diagram of R-algebras

$$\begin{tikzcd} R/(t^p) \arrow[r] \arrow[dr] & \hat{A}_1 \arrow[d] \\ & \hat{A}_1/(Z). \end{tikzcd}$$

By definition, the left downward arrow is surjective. We claim that this arrow is an isomorphism. Indeed, let x be in the kernel of this surjection. From the proof of Lemma 4.1, we see that the image of Z in the ring on the left-hand side of the above isomorphism can be written as $a't^{p-1} + b'Y_1$ for some $a', b' \in R^\times$. By assumption, in this ring, we have

$$x = c_1(a't^{p-1} + b'Y_1) + c_2(a't^{p-1} + b'Y_1)^2 + \cdots + c_{p-1}(a't^{p-1} + b'Y_1)^{p-1}$$

for some elements $c_1, \ldots, c_{p-1}$ of R. Then we see that $c_i = 0$ for any i and $v_R(x) \geq p$. This concludes the proof. $\square$

Since $r < p - 1$, from this lemma we can show the following lemma as in the proof of [3, Lemme 2.3.1.3].

Lemma 4.3. *The functor*

$$M \mapsto \mathrm{Hom}_{\Sigma, \mathrm{Fil}^r, \phi_r}(M, \hat{A}_\infty)$$

from $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ *to the category of* G_{K_∞}*-modules is exact.*

Corollary 4.4. *For any* $M \in \mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$*, the natural map*

$$\mathrm{Hom}_{\Sigma, \mathrm{Fil}^r, \phi_r}(M, \hat{A}_\infty) \to \mathrm{Hom}_{\Sigma, \mathrm{Fil}^r, \phi_r}(M, A_{\mathrm{crys}, \infty})$$

is an isomorphism of G_{K_∞}*-modules.*

Proof. By Lemma 3.2 and Lemma 4.3, we may assume $pM = 0$. Consider the commutative diagram of rings

$$\begin{tikzcd} \hat{A}_1 \arrow[r] \arrow[dr] & A_{\mathrm{crys}}/pA_{\mathrm{crys}} \arrow[d] \\ & R/(t^{p-1}) \end{tikzcd}$$

whose downward arrows are defined by modulo Fil^{p-1} of the rings $\hat{A}_1$ and $A_{\mathrm{crys}}/pA_{\mathrm{crys}}$, respectively. Since $r < p-1$, we have $\phi_r(\mathrm{Fil}^{p-1}\hat{A}_1) = 0$ and similarly for the ring $A_{\mathrm{crys}}/pA_{\mathrm{crys}}$. Thus these two surjections induce on the ring $R/(t^{p-1})$ the same structure of a filtered ϕ_r-module over Σ. Hence, as in the proof of Corollary 3.5, we see from Lemma 3.1 that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M,\hat{A}_1) \arrow[r] \arrow[dr] & \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M,A_{\mathrm{crys}}/pA_{\mathrm{crys}}) \arrow[d] \\
 & \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M,R/(t^{p-1}))
\end{tikzcd}$$

whose downward arrows are isomorphisms. This concludes the proof. □

We sketch the proof of the following lemma stated in [2, Subsection 3.2].

Lemma 4.5. *The natural inclusion $W(R) \to \hat{A}$ induces an isomorphism of $W(R)$-algebras $W_n(R)/(\psi([\underline{\varepsilon}]-1)^{p-1}) \to \hat{A}_n/(Z)$.*

Proof. For a subring B of A_{crys}, put

$$I^{[s]}B = \{x \in B \mid \phi^i(x) \in \mathrm{Fil}^s A_{\mathrm{crys}} \text{ for any } i\ \}$$

as in [9, Subsection 5.3]. Then we have $I^{[s]}W(R) = ([\underline{\varepsilon}]-1)^s W(R)$ and the natural ring homomorphism

$$W(R)/I^{[s]}W(R) \to A_{\mathrm{crys}}/I^{[s]}A_{\mathrm{crys}}$$

is an injection ([9, Proposition 5.1.3, Proposition 5.3.5]). Since the element Z is contained in the ideal $I^{[p-1]}A_{\mathrm{crys}}$, this injection factors as

$$W(R)/I^{[p-1]}W(R) \to \hat{A}/(Z) \to A_{\mathrm{crys}}/I^{[p-1]}A_{\mathrm{crys}}.$$

Hence the former arrow is an isomorphism and the lemma follows. □

Since the ideal (Z) of $\hat{A}_n$ satisfies the condition of Corollary 3.5, the Σ-algebra $\hat{A}_n/(Z)$ is naturally considered as an object of $'\mathrm{Mod}^{r,\phi}_{/\Sigma}$. We also give the ring $W_n(R)/(\psi([\underline{\varepsilon}]-1)^{p-1})$ the structures of a Σ-algebra and a filtered ϕ_r-module over Σ induced by those of $\hat{A}_n/(Z)$. The map

$$\Sigma \to W_n(R)/(\psi([\underline{\varepsilon}]-1)^{p-1})$$

sends the element $u \in \Sigma$ to the image of $[\underline{\pi}]$ in the ring on the right-hand side. Put $v = t'/E([\underline{\pi}]) \in W(R)^\times$ with the notation of Lemma 4.1. As for the element $Y \in \Sigma$, the equality

$$Y = -ab^{-1}v^{-1}E([\underline{\pi}])^{p-1} + wb^{-1}v^{-p}Z$$

holds in $\hat{A}$, where a and b are the elements in $W(R)^\times$ as in the proof of Lemma 4.1 and the remark after this lemma, and $w \in W(R)^\times$ is a power series of $[\underline{\varepsilon}]-1$. Hence the above homomorphism sends the element Y to the image of $-ab^{-1}v^{-1}E([\underline{\pi}])^{p-1}$.

Consider the surjective ring homomorphism

$$R \to \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$$
$$x = (x_0, x_1, \ldots) \mapsto x_n$$

and the induced surjection $W_n(R) \to W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$. Let

$$J = \{(x_0, \ldots, x_{n-1}) \in W_n(R) \mid v_R(x_i) \geq p^n \text{ for any } i\ \}$$

be the kernel of the latter surjection.

Lemma 4.6. *The ideal J is contained in the ideal $(\psi([\underline{\varepsilon}]-1)^{p-1})$ of the ring $W_n(R)$.*

Proof. Write $([\underline{\varepsilon}]-1)^{p-1}$ also as $x = (x_0, \ldots, x_{n-1}) \in W_n(R)$ with $v_R(x_0) = p$. Take an element $z = (z_0, \ldots, z_{n-1})$ in the ideal J. We construct $y \in W_n(R)$ such that $xy = z$. By induction, it is enough to show that if $z_0 = \cdots = z_{i-1} = 0$ for some $0 \leq i \leq n-1$ and $(x_0, \ldots, x_i)(0, \ldots, 0, y_i) = (0, \ldots, 0, z_i)$ in $W_{i+1}(R)$, then $x(0, \ldots, 0, y_i, 0, \ldots, 0) \in J$. Let us write this element as $(0, \ldots, 0, w_i, \ldots, w_{n-1})$ with $w_i = z_i$. We have $v_R(y_i) \geq p^n - p^{i+1}$. In the ring of Witt vectors $W_n(\mathbb{F}_p[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}])$, the k-th entry of the vector

$$(X_0, \ldots, X_{n-1})(0, \ldots, 0, Y_i, 0, \ldots, 0)$$

is $X_{k-i}^{p^i} Y_i^{p^{k-i}}$ for any $k \geq i$. Thus we have $v_R(w_k) \geq p^n$. □

Note that the elements $[\zeta_{p^n}]-1$ and $[\zeta_{p^{n+1}}]-1$ is nilpotent in $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$. By the above lemma, we have an isomorphism of rings

$$W_n(R)/(\psi([\underline{\varepsilon}]-1)^{p-1}) \to W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})/(\psi([\zeta_{p^n}]-1)^{p-1}).$$

We give the ring on the right-hand side the structure of a filtered ϕ_r-module over Σ induced by this isomorphism.

Put $F_n = K_n(\zeta_{p^{n+1}})$. For an algebraic extension F of F_n, let us consider the ideals

$$\bar{m}_{n,F} = \{(x_0, \ldots, x_{n-1}) \in W_n(\mathcal{O}_F/p\mathcal{O}_F) \mid x_i \in m_F/p\mathcal{O}_F \text{ for any } i\ \}$$
$$m_{n,F} = \{(x_0, \ldots, x_{n-1}) \in W_n(\mathcal{O}_F) \mid x_i \in m_F \text{ for any } i\ \}$$

of $W_n(\mathcal{O}_F/p\mathcal{O}_F)$ and $W_n(\mathcal{O}_F)$, respectively. The elements $[\zeta_{p^n}]-1$ and $[\zeta_{p^{n+1}}]-1$ are topologically nilpotent in $W_n(\mathcal{O}_F)$ and we define an element $\hat{t} \in W_n(\mathcal{O}_F)$ to be

$$\hat{t} = \psi([\zeta_{p^n}]-1)/\psi([\zeta_{p^{n+1}}]-1).$$

Note that these elements are non-zero divisors of $W_n(\mathcal{O}_F)$. Let the ring

$$W_n(\mathcal{O}_F/p\mathcal{O}_F)/\psi([\zeta_{p^n}]-1)^r \bar{m}_{n,F}$$

be denoted by $\bar{A}_{n,F,r+}$. We also put $\bar{m}_n = \bar{m}_{n,\bar{K}}$, $m_n = m_{n,\bar{K}}$ and $\bar{A}_{n,r+} = \bar{A}_{n,\bar{K},r+}$.

For an algebraic extension F of K, we put

$$\mathfrak{b}_F = \{x \in \mathcal{O}_F \mid v_K(x) > er/(p-1)\}.$$

Note that the ring $\mathcal{O}_F/\mathfrak{b}_F$ is killed by p. When F contains F_n, we also put

$$\bar{A}'_{n,F,r+} = W_n(\mathcal{O}_F/\mathfrak{b}_F)/\psi([\zeta_{p^n}]-1)^r \bar{m}_{n,F}.$$

Then, for $0 \leq r < p-1$, we have natural isomorphisms of rings

$$W_n(\mathcal{O}_F)/\psi([\zeta_{p^n}]-1)^r m_{n,F} \to \bar{A}_{n,F,r+} \to \bar{A}'_{n,F,r+}.$$

Indeed, as in the proof of Lemma 4.6, we can show that both of the kernels of the maps $W_n(\mathcal{O}_F) \to W_n(\mathcal{O}_F/p\mathcal{O}_F)$ and $W_n(\mathcal{O}_F) \to W_n(\mathcal{O}_F/\mathfrak{b}_F)$ are contained in the ideal $\psi([\zeta_{p^n}]-1)^r m_{n,F}$ of the ring $W_n(\mathcal{O}_F)$. We often identify these rings. We also put $\bar{A}'_{n,r+} = \bar{A}'_{n,\bar{K},r+}$.

Write Z_n for the image of the element Z of A_{crys} in $W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$. Then we have a commutative diagram of Σ-algebras

$$\begin{tikzcd}
 & \hat{A}_n \arrow[r] \arrow[dd] & A_{\mathrm{crys}}/p^n A_{\mathrm{crys}} \arrow[d, "\wr"] \\
 & & W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \arrow[dd] \\
W_n(R)/(\psi([\zeta_{p^n}]-1)^{p-1}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \hat{A}_n/(Z) \arrow[dr] & \\
W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})/(\psi([\zeta_{p^n}]-1)^{p-1}) \arrow[rr] \arrow[d] & & W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})/(Z_n), \\
\bar{A}_{n,r+} \arrow[d, "\wr"'] & & \\
\bar{A}'_{n,r+} & &
\end{tikzcd}$$

where all vertical arrows are surjections satisfying the condition of Corollary 3.5. Thus we see that this is also a commutative diagram in $'\mathrm{Mod}^{r,\phi}_{/\Sigma}$. Note that these rings and homomorphisms are independent of the choice of a system $\{\zeta_{p^n}\}_{n\in\mathbb{Z}_{\geq 0}}$. Moreover, let $\hat{M}$ be in $\mathrm{Mod}^{r,\phi}_{/\Sigma}$ and put $M_n = \hat{M}/p^n\hat{M}$. Then, by Corollary 3.5 and Corollary 4.4, we have a natural isomorphism of abelian groups

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M_n, W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})) \simeq \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M_n, \bar{A}_{n,r+}).$$

To study G_{F_n}-actions on both sides of this isomorphism, we need the following proposition.

Proposition 4.7. *The image of the element $Y \in \Sigma$ in the ring $\bar{A}_{n,r+}$ is contained in its subring $\bar{A}_{n,F_n,r+}$.*

Proof. We have the equality

$$([\underline{\varepsilon}^{1/p}]-1)^{p-1} = pa - E([\underline{\pi}])bv$$

in $W(R)$. By definition, we see that the images of the elements a, $[\underline{\varepsilon}^{1/p}]$ and $E([\underline{\pi}])$ in $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ are contained in the subring $W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$. Write β for the product $bv \in W(R)$. Let a_n, β_n and α_n denote the images of the elements a, β and $pa-([\underline{\varepsilon}^{1/p}]-1)^{p-1}$ in the ring $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$, respectively. Then the element α_n is also contained in the subring $W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$. Now we have the equality

$$E([\pi_n])\beta_n = \alpha_n.$$

Note that any element $\beta'_n \in W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ satisfying the same equality is invertible and thus the elements $(\beta'_n)^{-1}E([\pi_n])$ are equal to each other. Since $Y = -a_n\beta_n^{-1}E([\pi_n])^{p-1}$ in $\bar{A}_{n,r+}$, it suffices to construct an element β'_n in the ring $W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$ such that the equality $E([\pi_n])\beta'_n = \alpha_n$ holds. Take a lift $\hat{\alpha}_n$ of α_n in $W_n(\mathcal{O}_{F_n})$. Since $E([\pi_n]) \in W_n(\mathcal{O}_{\bar{K}})$ divides every element in the kernel of the surjection $W_n(\mathcal{O}_{\bar{K}}) \to W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$, we have $E([\pi_n])\hat{\beta}'_n = \hat{\alpha}_n$ for some $\hat{\beta}'_n \in W_n(\mathcal{O}_{\bar{K}})$. Then the element $\hat{\beta}'_n$ is contained in the subring $W_n(\mathcal{O}_{F_n})$ and we set β'_n to be the image of $\hat{\beta}'_n$.

By a similar argument, we can also check that the ring $\bar{A}_{n,F,r+}$ is a subring of $\bar{A}_{n,r+}$ and coincides with the image of $W_n(\mathcal{O}_F)$ in $\bar{A}_{n,r+}$. This concludes the proof. □

Let t_n and $\bar{t}_n$ be the images of $t \in W(R)$ in $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ and $\bar{A}_{n,r+}$ (or $\bar{A}'_{n,r+}$), respectively. Then $\hat{t}$ is a lift of t_n and $\bar{t}_n$ to $W_n(\mathcal{O}_{\bar{K}})$ by the natural surjections

$$W_n(\mathcal{O}_{\bar{K}}) \to W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \to \bar{A}_{n,r+}.$$

Note that we defined the filtration of $\bar{A}_{n,r+}$ as $\mathrm{Fil}^r \bar{A}_{n,r+} = \bar{t}_n^r \bar{A}_{n,r+}$.

Lemma 4.8. *Let $\bar{x}$ be in $\mathrm{Fil}^r \bar{A}_{n,r+}$. Then we have*

$$\phi_r(\bar{x}) = \phi(y) \bmod \psi([\zeta_{p^n}]-1)^r \bar{m}_n,$$

where y is any element of $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ such that the element $t_n^r y$ is a lift of $\bar{x}$. In particular, the right-hand side of the above equality is independent of the choice of a system $\{\zeta_{p^n}\}_{n\in\mathbb{Z}_{\geq 0}}$. Similar assertions also hold for the ring

$$W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})/(\psi([\zeta_{p^n}]-1)^{p-1}).$$

Proof. Since the filtered ϕ_r-module structure of $\bar{A}_{n,r+}$ is induced from that of $\hat{A}$ and $\phi_r(t^r) \equiv 1 \bmod Z$ in $\hat{A}$, we see that there exists an element y as in the lemma.

To prove the independence of the choice of a lift, let $z = (z_0, \ldots, z_{n-1})$ be an element of $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ killed by t_n. The element z is also killed by

$p-[p_n]$, where $p_k=p^{1/p^k}$. This implies

$$\begin{cases} p_n z_0 \in p\mathcal{O}_{\bar{K}} \\ z_0^p + p_{n-1}z_1 \in p\mathcal{O}_{\bar{K}} \\ \vdots \\ z_{n-2}^p + p_1 z_{n-1} \in p\mathcal{O}_{\bar{K}} \end{cases}$$

and $v_p(z_k)\geq 1-1/p^{n-k}$ for $0\leq k\leq n-1$. Repeating this, we see that if z is killed by t_n^r, then $v_p(z_k)\geq 1-r/p^{n-k}$. For such an element z, we have $\phi(z)=0$ in the ring $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$.

Let y_1 and y_2 be elements as in the lemma. Then we have

$$y_1-y_2=\psi([\zeta_{p^{n+1}}]-1)^r w+z,$$

where $w\in\bar{m}_n$ and z is an element as above. The Frobenius endomorphism ϕ sends the element on the right-hand side to an element which is contained in the ideal $\psi([\zeta_{p^n}]-1)^r\bar{m}_n$. Thus the assertions for the ring $\bar{A}_{n,r+}$ follows. We can show the assertion for the ring $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})/(\psi([\zeta_{p^n}]-1)^{p-1})$ similarly. □

From this lemma and Proposition 4.7, we see that the natural G_{F_n}-actions on the rings $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})/(\psi([\zeta_{p^n}]-1)^{p-1})$, $\bar{A}_{n,r+}$ and $\bar{A}'_{n,r+}$ are compatible with the filtered ϕ_r-module structures over Σ. In the commutative diagram above, the lowest horizontal arrow and lower right vertical arrow are G_{F_n}-linear by definition. Hence we have shown the following proposition.

Proposition 4.9. *Let $\hat{M}$ be in $\mathrm{Mod}^{r,\phi}_{/\Sigma}$ and put $M_n=\hat{M}/p^n\hat{M}$. Then we have an isomorphism of G_{F_n}-modules*

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M_n,\bar{A}_{n,r+})\simeq \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M_n,W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})).$$

Let $e_1,\ldots,e_d$ be a basis of $\hat{M}$ as in Lemma 3.3 and $C=(c_{i,j})\in M_d(\Sigma)$ be the associated matrix representing ϕ_r as in Corollary 3.4. Then the underlying G_{F_n}-set of the G_{F_n}-module

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M_n,\bar{A}_{n,r+})$$

is identified with the set of d-tuples $(\bar{x}_1,\ldots,\bar{x}_d)$ in $\bar{A}_{n,r+}$ such that $c_{1,i}\bar{x}_1+\cdots+c_{d,i}\bar{x}_d\in\mathrm{Fil}^r\bar{A}_{n,r+}$ for any i and the following equality holds:

$$(1)\qquad \begin{cases} \phi_r(c_{1,1}\bar{x}_1+\cdots+c_{d,1}\bar{x}_d)=\bar{x}_1 \\ \vdots \\ \phi_r(c_{1,d}\bar{x}_1+\cdots+c_{d,d}\bar{x}_d)=\bar{x}_d. \end{cases}$$

We choose a lift $(\hat{c}_{i,j})\in M_d(W_n(\mathcal{O}_{F_n}))$ of the image of C in $M_d(\bar{A}_{n,r+})$ by the natural ring homomorphism

$$W_n(\mathcal{O}_{\bar{K}})\to W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})\to\bar{A}_{n,r+}.$$

Fix a polynomial $\Phi_i\in\mathbb{Z}[X_0,\ldots,X_{n-1}]$ such that $\Phi_i\equiv X_i^p \bmod p$. This induces for any commutative ring B a map $\Phi=(\Phi_0,\ldots,\Phi_{n-1}):W_n(B)\to W_n(B)$ which is a lift of the Frobenius endomorphism on $W_n(B/pB)$. In particular, set B to be the polynomial ring $\mathbb{Z}[X_0,\dots,X_{n-1},Y_0,\dots,Y_{n-1}]$. Put $X=(X_0,\dots,X_{n-1})$ and $Y=(Y_0,\dots,Y_{n-1})$ in the ring $W_n(B)$. Then we see that there exists elements $U_0,\dots,U_{n-1}$ and $U'_0,\dots,U'_{n-1}$ of the polynomial ring B such that

$$\Phi(X+Y)=\Phi(X)+\Phi(Y)+(pU_0,\dots,pU_{n-1}),$$
$$\Phi(XY)=\Phi(X)\Phi(Y)+(pU'_0,\dots,pU'_{n-1})$$

in the ring $W_n(B)$.

Proposition 4.10. *Every solution $(\bar{x}_1,\dots,\bar{x}_d)$ in $\bar{A}_{n,r+}$ of the equation* (1) *such that $c_{1,i}\bar{x}_1+\cdots+c_{d,i}\bar{x}_d\in \mathrm{Fil}^r\bar{A}_{n,r+}$ for any i uniquely lifts to a d-tuple $(\hat{x}_1,\dots,\hat{x}_d)$ in $W_n(\mathcal{O}_{\bar{K}})$ such that $\hat{c}_{1,i}\hat{x}_1+\cdots+\hat{c}_{d,i}\hat{x}_d\in\hat{t}^rW_n(\mathcal{O}_{\bar{K}})$ for any i and the following equality holds:*

$$\text{(2)}\qquad \begin{cases}\Phi((\hat{c}_{1,1}\hat{x}_1+\cdots+\hat{c}_{d,1}\hat{x}_d)/\hat{t}^r)=\hat{x}_1\\ \qquad\vdots\\ \Phi((\hat{c}_{1,d}\hat{x}_1+\cdots+\hat{c}_{d,d}\hat{x}_d)/\hat{t}^r)=\hat{x}_d.\end{cases}$$

Proof. Fix a lift $\hat{x}_i$ of $\bar{x}_i$ to $W_n(\mathcal{O}_{\bar{K}})$. Then we have

$$\begin{cases}\Phi((\hat{c}_{1,1}\hat{x}_1+\cdots+\hat{c}_{d,1}\hat{x}_d)/\hat{t}^r)=\hat{x}_1+([\zeta_{p^n}]-1)^r\hat{\delta}_1\\ \qquad\vdots\\ \Phi((\hat{c}_{1,d}\hat{x}_1+\cdots+\hat{c}_{d,d}\hat{x}_d)/\hat{t}^r)=\hat{x}_d+([\zeta_{p^n}]-1)^r\hat{\delta}_d\end{cases}$$

for some $\hat{\delta}_1,\dots,\hat{\delta}_d\in m_n$. It suffices to show that there exists a unique d-tuple $(\hat{y}_1,\dots,\hat{y}_d)$ in m_n such that

$$\begin{aligned}\Phi((\hat{c}_{1,i}(\hat{x}_1+([\zeta_{p^n}]-1)^r\hat{y}_1)+\cdots+\hat{c}_{d,i}(\hat{x}_d+([\zeta_{p^n}]-1)^r\hat{y}_d))/\hat{t}^r)&\\ =\hat{x}_i+([\zeta_{p^n}]-1)^r\hat{y}_i&\end{aligned}$$

for any i. For this, we need the following lemma.

Lemma 4.11. *Let N be a complete discrete valuation field and m_N be the maximal ideal of N. Let $\epsilon_1,\dots,\epsilon_d$ be in m_N. Let $P_1,\dots,P_d$ and $P'_1\dots,P'_d$ be elements of $\mathcal{O}_N[[Y_1,\dots,Y_d]]$ such that $P_i\in(Y_1,\dots,Y_d)^2$. Then the equation*

$$\begin{cases}Y_1-P_1(Y_1,\dots,Y_d)-\epsilon_1P'_1(Y_1,\dots,Y_d))=0\\ \qquad\vdots\\ Y_d-P_d(Y_1,\dots,Y_d)-\epsilon_dP'_d(Y_1,\dots,Y_d))=0\end{cases}$$

has a unique solution in m_N.

Proof. By assumption, we see that for any integer $l\geq 1$, a d-tuple $y=(y_1,\dots,y_d)$ in m_N/m_N^l satisfying the above equation lifts uniquely to a d-tuple in m_N/m_N^{l+1} satisfying the same equation. Thus the lemma follows. □

Let us write as $\hat{y}_i = (\hat{y}_{i,0}, \dots, \hat{y}_{i,n-1})$. Since the image of $\Phi(([\zeta_{p^{n+1}}] - 1)^r)$ in $\bar{A}_{n,r+}$ is divisible by $([\zeta_{p^n}] - 1)^r$, we can find $\hat{b} \in W_n(\mathcal{O}_{\bar{K}})$ such that

$$\Phi(([\zeta_{p^n}] - 1)^r/\hat{t}^r) = ([\zeta_{p^n}] - 1)^r \hat{b}.$$

Then there exists polynomials $U_{i,m}$ over $\mathcal{O}_{\bar{K}}$ of the indeterminates $\underline{Y} = (Y_{i,m})_{1 \le i \le d, 0 \le m \le n-1}$ such that the equation we have to solve is

$$\begin{aligned} \hat{x}_i + ([\zeta_{p^n}] - 1)^r \hat{y}_i = \hat{x}_i &+ ([\zeta_{p^n}] - 1)^r \hat{\delta}_i \\ &+ ([\zeta_{p^n}] - 1)^r \hat{b}(\Phi(\hat{c}_{i,1})\Phi(\hat{y}_1) + \cdots + \Phi(\hat{c}_{i,d})\Phi(\hat{y}_d)) \\ &+ (pU_{i,0}(\underline{\hat{y}}), \dots, pU_{i,n-1}(\underline{\hat{y}})) \end{aligned}$$

for any i, where we put $\underline{\hat{y}} = (\hat{y}_{i,m})_{1 \le i \le d, 0 \le m \le n-1}$. Note that, for any elements $P_0, \dots, P_{n-1}$ of the polynomial ring $\mathcal{O}_{\bar{K}}[\underline{Y}]$, we can uniquely find elements $Q_0, \dots, Q_{n-1}$ of this ring such that the coefficients of these polynomials are in the maximal ideal $m_{\bar{K}}$ and the equality

$$(pP_0, \dots, pP_{n-1}) = ([\zeta_{p^n}] - 1)^r (Q_0, \dots, Q_{n-1})$$

holds in the ring of Witt vectors $W_n(\mathcal{O}_{\bar{K}}[\underline{Y}])$. Therefore, this equation is equivalent to the equation

$$\begin{aligned} \hat{y}_i = \hat{\delta}_i &+ \hat{b}(\Phi(\hat{c}_{i,1})\Phi(\hat{y}_1) + \cdots + \Phi(\hat{c}_{i,d})\Phi(\hat{y}_d)) \\ &+ (V_{i,0}(\underline{\hat{y}}), \dots, V_{i,n-1}(\underline{\hat{y}})), \end{aligned}$$

where $V_{i,m}$ is a polynomial of $\underline{Y}$ over $\mathcal{O}_{\bar{K}}$ whose coefficients are in the maximal ideal $m_{\bar{K}}$. From the definition of Φ, we see that the elements $\hat{y}_{i,m}$ is a solution of a system of equations

$$Y_{i,m} - P_{i,m}(\underline{Y}) - \epsilon_{i,m} P'_{i,m}(\underline{Y}) = 0$$

satisfying the condition of Lemma 4.11 for a sufficiently large finite extension N of K. Then, by this lemma, we can solve the equation uniquely in $m_{\bar{K}}$. $\square$

Let F be an algebraic extension of $F_n = K_n(\zeta_{p^{n+1}})$ and consider the ring $\bar{A}_{n,F,r+}$. By Lemma 4.7, we can consider this ring as a Σ-algebra and also as an object of ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$ by putting $\mathrm{Fil}^r \bar{A}_{n,F,r+} = \bar{t}^r_n \bar{A}_{n,F,r+}$ and for $\bar{x} \in \mathrm{Fil}^r \bar{A}_{n,F,r+}$,

$$\phi_r(\bar{x}) = \phi(y) \bmod \psi([\zeta_{p^n}] - 1)^r \bar{m}_{n,F},$$

where y is any element of $W_n(\mathcal{O}_F/p\mathcal{O}_F)$ such that the element $t^r_n y$ is a lift of $\bar{x}$. For $M_n = \hat{M}/p^n\hat{M} \in \mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ as before, let us set

$$T^*_{\mathrm{crys},\pi_n,F}(M_n) = \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M_n, \bar{A}_{n,F,r+}).$$

We see that

$$\bar{A}_{n,r+} = \bar{A}_{n,\bar{K},r+} = \bigcup_{F/F_n} \bar{A}_{n,F,r+}$$

in $'\mathrm{Mod}^{r,\phi}_{/\Sigma}$ and thus we have a natural identification of abelian groups

$$T^*_{\mathrm{crys},\pi_n,\bar{K}}(M_n) = \bigcup_{F/F_n} T^*_{\mathrm{crys},\pi_n,F}(M_n).$$

The absolute Galois group G_{F_n} acts on the abelian group on the left-hand side.

Lemma 4.12. *For an algebraic extension F of F_n, the fixed part $T^*_{\mathrm{crys},\pi_n,\bar{K}}(M_n)^{G_F}$ is equal to $T^*_{\mathrm{crys},\pi_n,F}(M_n)$.*

Proof. From Proposition 4.10, we see that the elements of $T^*_{\mathrm{crys},\pi_n,\bar{K}}(M_n)$ correspond bijectively to the solutions of the equation (2) in $W_n(\mathcal{O}_{\bar{K}})$ satisfying the condition on Fil^r. The uniqueness assertion of this proposition shows that $g \in G_F$ fixes a solution in $W_n(\mathcal{O}_{\bar{K}})$ if and only if g fixes its image in $\bar{A}_{n,r+}$. Hence a solution is fixed by G_F if and only if this solution is contained in the image of $W_n(\mathcal{O}_F)$. Thus the lemma follows. $\square$

Corollary 4.13. *Let L_n be the finite Galois extension of F_n corresponding to the kernel of the map $G_{F_n} \to \mathrm{Aut}(T^*_{\mathrm{crys},\pi_n,\bar{K}}(M_n))$. Then an algebraic extension F of F_n contains L_n if and only if $\#T^*_{\mathrm{crys},\pi_n,F}(M_n) = \#T^*_{\mathrm{crys},\pi_n,\bar{K}}(M_n)$.*

Proof. An algebraic extension F of F_n contains L_n if and only if the action of G_F on $T^*_{\mathrm{crys},\pi_n,\bar{K}}(M_n)$ is trivial. By Lemma 4.12, this is equivalent to $T^*_{\mathrm{crys},\pi_n,F}(M_n) = T^*_{\mathrm{crys},\pi_n,\bar{K}}(M_n)$. $\square$

5. Ramification bound

In this section, we prove Theorem 1.1. Take G_K-stable $\mathbb{Z}_p$-lattices $\mathcal{L} \supseteq \mathcal{L}'$ in V such that $\mathcal{L}' \supseteq p^n\mathcal{L}$. Since the G_K-module $\mathcal{L}/\mathcal{L}'$ is a quotient of $\mathcal{L}/p^n\mathcal{L}$, we may assume $\mathcal{L}' = p^n\mathcal{L}$. If $r = 0$, then the G_K-module V is unramified and the theorem is trivial. Thus we may assume $r \geq 1$ and $p \geq 3$. Let L be the finite Galois extensions of K corresponding to the kernel of the map

$$G_K \to \mathrm{Aut}(\mathcal{L}/p^n\mathcal{L}).$$

It is enough to show that, for the greatest upper ramification break $u_{L(\zeta_p)/K}$ of the Galois extension $L(\zeta_p)/K$, the inequality

$$u_{L(\zeta_p)/K} \leq u(K,r,n)$$

holds. Since the Herbrand function is transitive and the finite Galois extension $K(\zeta_p)$ is tamely ramified over K, we may assume $\zeta_p \in K$. We fix a uniformizer π of K and a system $\{\pi_n\}_{n\in\mathbb{Z}_{\geq 0}}$ as before. Then, by Liu's theorem ([14, Theorem 2.3.5]), it suffices to show the following.

Theorem 5.1. *Let r be an integer such that $1 \leq r < p-1$ and $\hat{\mathcal{M}}$ be the strongly divisible lattice corresponding to $\mathcal{L}$. Put $\mathcal{M}_n = \hat{\mathcal{M}}/p^n\hat{\mathcal{M}} \in \mathrm{Mod}^{r,\phi,N}_{/S_\infty}$. Then $G_K^{(j)}$ acts trivially on the G_K-module $T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M}_n)$ for $j > u(K,r,n)$.*

Let L_n be the finite Galois extension of $F_n = K_n(\zeta_{p^{n+1}})$ corresponding to the kernel of the map

$$G_{F_n} \to \mathrm{Aut}(T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M}_n)).$$

Since F_n is Galois over K, the extension L_n is also a Galois extension of K. Let $\hat{\mathfrak{M}}$ be the object of the category $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ such that $\mathcal{M}_{\mathfrak{S}}(\hat{\mathfrak{M}}) \simeq \hat{\mathcal{M}}$. From Proposition 3.6 and Proposition 4.9, we see that L_n is also the finite extension of F_n cut out by the G_{F_n}-module $T^*_{\mathrm{crys},\pi_n,\bar{K}}(M_n)$ for $M_n = M_{\mathfrak{S}}(\hat{\mathfrak{M}})/p^n M_{\mathfrak{S}}(\hat{\mathfrak{M}})$. It is enough to prove the inequality

$$u_{L_n/K} \leq u(K,r,n) = \begin{cases} 1 + e(n + \frac{1}{p-1}) & (r = 1), \\ \frac{p^n-1}{p^n} + e(n + \frac{r}{p-1}) & (r > 1). \end{cases}$$

Before proving this, we state some general lemmas to calculate the ramification bound. Let N be a complete discrete valuation field of positive residue characteristic, v_N be its valuation normalized as $v_N(N^\times) = \mathbb{Z}$ and N^{sep} be its separable closure.

Lemma 5.2. *Let $f(T) \in \mathcal{O}_N[T]$ be a separable monic polynomial and $z_1, \ldots, z_d$ be the zeros of f in $\mathcal{O}_{N^{\mathrm{sep}}}$. Suppose that the set $\{v_N(z_k - z_i) \mid k = 1, \ldots, d, k \neq i\}$ is independent of i. Put*

$$s = \sum_{\substack{k=1,\ldots,d \\ k \neq i}} v_N(z_k - z_i) \text{ and } \alpha = \sup_{\substack{k=1,\ldots,d \\ k \neq i}} v_N(z_k - z_i),$$

which are independent of i by assumption. If $j > s + \alpha$, then we have the decomposition

$$\{x \in \mathcal{O}_{N^{\mathrm{sep}}} \mid v_N(f(x)) \geq j\} = \coprod_{i=1,\ldots,d} \{x \in \mathcal{O}_{N^{\mathrm{sep}}} \mid v_N(x - z_i) \geq j - s\}.$$

Otherwise, the set on the left-hand side contains

$$\{x \in \mathcal{O}_{N^{\mathrm{sep}}} \mid v_N(x - z_i) \geq \alpha\},$$

which contains at least two zeros of f.

Proof. A verbatim argument in the proof of [1, Lemma 6.6] shows the claim. □

Corollary 5.3. *Let $f(T)$ be as above and put $B = \mathcal{O}_N[T]/(f(T))$. Suppose that the algebra B is finite flat and of relative complete intersection over $\mathcal{O}_N$. Let us write the N-algebra $N' = B \otimes_{\mathcal{O}_N} N$ as the product $N_1 \times \cdots \times N_t$ of finite separable extensions $N_1, \ldots, N_t$ of N. If $j > s + \alpha$, then the j-th upper numbering ramification group ([1]), which we let be denoted by $G_N^{(j)}$, is contained in G_{N_i} for any i. Moreover, if N' is a field and B coincides with $\mathcal{O}_{N'}$, then $j > s + \alpha$ if and only if $G_N^{(j)} \subseteq G_{N'}$.*

Proof. From the previous lemma, the conductor $c(B)$ of the $\mathcal{O}_N$-algebra B ([1, Proposition 6.4]) is equal to $s+\alpha$. Thus we have the inequality

$$c(\mathcal{O}_{N_1}\times\cdots\times\mathcal{O}_{N_t})\leq c(B)=s+\alpha$$

by the definition of the conductor and a functoriality of the functor $\mathcal{F}^j$ defined in [1]. This implies the corollary. $\square$

Corollary 5.4. *Consider the finite Galois extension $F_n=K_n(\zeta_{p^{n+1}})$ of K and let $u_{F_n/K}$ denote the greatest upper ramification break of F_n/K. Then we have the equality*

$$u_{F_n/K}=1+e(n+\frac{1}{p-1}).$$

Proof. Note that we are assuming that ζ_p is contained in K. Applying the previous corollary to the Eisenstein polynomial $f(T)=T^{p^n}-\pi$ shows that $j>1+e(n+1/(p-1))$ if and only if $G_K^{(j)}\subseteq G_{K_n}$. Similarly, putting $f(T)=T^{p^n}-\zeta_p$, we see that if $j>e(n+1/(p-1))$, then $G_K^{(j)}\subseteq G_{K(\zeta_{p^{n+1}})}$. Since $G_{F_n}=G_{K_n}\cap G_{K(\zeta_{p^{n+1}})}$, we conclude that $j>1+e(n+1/(p-1))$ if and only if $G_K^{(j)}\subseteq G_{F_n}$. $\square$

Remark 5.5. Note that this argument also shows the equality

$$u_{K_n(\zeta_{p^n})/K}=1+e(n+\frac{1}{p-1})$$

without assuming $\zeta_p\in K$.

Next we assume that the residue field of N is perfect. For an algebraic extension F of N, we put

$$\mathfrak{a}^j_{F/N}=\{x\in\mathcal{O}_F\mid v_N(x)\geq j\}.$$

For a finite Galois extension Q of N, we write $u_{Q/N}$ for the greatest upper ramification break ([7]) of Q/N. Let us consider the property

$$(P_j)\begin{cases}\text{for any algebraic extension } F \text{ of } N\text{, if there exists}\\ \text{an } \mathcal{O}_N\text{-algebra homomorphism } \mathcal{O}_Q\to\mathcal{O}_F/\mathfrak{a}^j_{F/N},\\ \text{then there exists an } N\text{-algebra injection } Q\to F\end{cases}$$

for $j\in\mathbb{R}_{\geq 0}$, as in [7, Proposition 1.5]. Then we have the following proposition, which is due to Yoshida. Here we reproduce his proof for the convenience of the reader.

Proposition 5.6 ([16])**.**

$$u_{Q/N}=\inf\{j\in\mathbb{R}_{\geq 0}\mid \textit{the property } (P_j) \textit{ holds }\}.$$

Proof. By [7, Proposition 1.5 (i)], it is enough to show that the property (P_j) does not hold for $j=u_{Q/N}-(e')^{-1}$ with an arbitrarily large $e'>0$. As in the proof of [7, Proposition 1.5 (ii)], we may assume that Q is totally and wildly ramified over N. Let N' be a finite tamely ramified Galois extension of N such that $Q \cap N' = N$ and put $Q' = QN'$. Note that we have $u_{Q'/N} = u_{Q/N}$ by assumption. From this proposition in [7], we see that for some algebraic extension F of N, there exists an $\mathcal{O}_N$-algebra homomorphism $\mathcal{O}_{Q'} \to \mathcal{O}_F/\mathfrak{a}^j_{F/N}$ for $j = u_{Q/N} - e(Q'/N)^{-1}$ but no N-algebra injection $Q' \to F$. Since Q/N is wildly ramified, we see that $e(Q/N)u_{Q/N} - 1 > e(Q/N)$. Hence we have $u_{Q/N} - e(Q'/N)^{-1} > 1 = u_{N'/N}$ and there exists an N-algebra injection $N' \to F$ also by this proposition. Thus there exists no N-algebra injection $Q \to F$ and the property (P_j) for Q/N does not hold. Since we can choose an arbitrarily large N' as above, the proposition follows. □

We see from Proposition 5.6 that to bound the greatest upper ramification break $u_{L_n/K}$, it is enough to show the following proposition.

Proposition 5.7. *Let F be an algebraic extension of K. If $j > u(K, r, n)$ and there exists an $\mathcal{O}_K$-algebra homomorphism*

$$\eta : \mathcal{O}_{L_n} \to \mathcal{O}_F/\mathfrak{a}^j_{F/K},$$

then there exists a K-algebra injection $L_n \to F$.

Proof. By assumption, we have $j > er/(p-1)$ and $\mathfrak{b}_F \supseteq \mathfrak{a}^j_{F/K}$. Thus η induces an $\mathcal{O}_K$-algebra homomorphism

$$\mathcal{O}_{L_n} \to \mathcal{O}_F/\mathfrak{b}_F.$$

Since η also induces an $\mathcal{O}_K$-algebra homomorphism $\mathcal{O}_{F_n} \to \mathcal{O}_F/\mathfrak{a}^j_{F/K}$ and $r \geq 1$, from Corollary 5.4 and [7, Proposition 1.5] we get a K-linear injection $F_n \to F$. Thus we see that F contains π_n and $\zeta_{p^{n+1}}$. More precisely, we have the following lemma.

Lemma 5.8. *For some integers i and i' such that $i' \equiv 1 \bmod p$, we have $\eta(\pi_n) \equiv \pi_n \zeta^i_{p^n} \bmod \mathfrak{b}_F$ and $\eta(\zeta_{p^{n+1}}) \equiv \zeta^{i'}_{p^{n+1}} \bmod \mathfrak{b}_F$. Moreover, there exists $g \in G_K$ such that $g(\pi_n) = \pi_n \zeta^i_{p^n}$ and $g(\zeta_{p^{n+1}}) = \zeta^{i'}_{p^{n+1}}$.*

Proof. Since the map η is $\mathcal{O}_K$-linear, the equality $\eta(\pi_n)^{p^n} = \pi$ holds in $\mathcal{O}_F/\mathfrak{a}^j_{F/K}$. Set $\hat{x}$ to be a lift of $\eta(\pi_n)$ in $\mathcal{O}_F$. Then we have

$$v_K(\hat{x}^{p^n} - \pi) = \sum_{i=0}^{p^n-1} v_K(\hat{x} - \pi_n \zeta^i_{p^n}) \geq j.$$

Let us apply Lemma 5.2 to $f(T) = T^{p^n} - \pi$. Then, with the notation of the lemma, we have

$$s = ne + \frac{p^n - 1}{p^n} \text{ and } \alpha = \frac{1}{p^n} + \frac{e}{p-1}.$$

Since $j - s > er/(p-1)$ by assumption, we have

$$\hat{x} \equiv \pi_n \zeta^i_{p^n} \bmod \mathfrak{b}_F$$

for some i.

Let $h(T)$ be the minimal polynomial of $\zeta_{p^{n+1}}$ over $\mathcal{O}_K$. Since h divides $T^{p^n}-\zeta_p$, the $\mathcal{O}_K$-algebra $B'=\mathcal{O}_K[T]/(h(T))$ is also finite flat of relative complete intersection and the K-algebra $B'\otimes_{\mathcal{O}_K}K$ is étale. The Galois group $\mathrm{Gal}(K(\zeta_{p^{n+1}})/K)$ acts transitively on the set of zeros of h. Hence h also satisfies the conditions of Lemma 5.2. Let s' and α' be as in this lemma for h. Then we have $s'\leq ne$ and $\alpha'\leq e/(p-1)$. This implies $j-s'>er/(p-1)$. By this lemma, there exists an element $g'\in\mathrm{Gal}(K(\zeta_{p^{n+1}})/K)$ such that the element $g'(\zeta_{p^{n+1}})=\zeta_{p^{n+1}}^{i'}$ satisfies

$$\eta(\zeta_{p^{n+1}})\equiv g'(\zeta_{p^{n+1}}) \bmod \mathfrak{b}_F.$$

Since $K_n\cap K(\zeta_{p^{n+1}})=K$ (see for example [14, Lemma 5.1.2]), we can find an element $g\in G_K$ such that $g(\pi_n)=\pi_n\zeta_{p^n}^i$ and $g(\zeta_{p^{n+1}})=g'(\zeta_{p^{n+1}})$. This concludes the proof. □

Lemma 5.9. *The $\mathcal{O}_K$-algebra homomorphism η induces an $\mathcal{O}_K$-algebra injection*

$$\eta_{\mathfrak{b}}:\mathcal{O}_{L_n}/\mathfrak{b}_{L_n}\to\mathcal{O}_F/\mathfrak{b}_F.$$

Proof. We write the Eisenstein polynomial of a uniformizer π_{L_n} of L_n over $\mathcal{O}_K$ as

$$P(T)=T^{e'}+c_1T^{e'-1}+\cdots+c_{e'-1}T+c_{e'},$$

where $e'=e(L_n/K)$. Then $z=\eta(\pi_{L_n})$ satisfies $P(z)=0$ in $\mathcal{O}_F/\mathfrak{a}_{F/K}^j$. Let $\hat{z}$ be a lift of z in $\mathcal{O}_F$. Since $j>1$, we have $v_F(\hat{z})=e(F/K)/e'$. The condition $i>e(L_n)r/(p-1)$ is equivalent to the condition

$$v_F(\hat{z}^i)>\frac{e(L_n)r}{p-1}\cdot\frac{e(F/K)}{e'}=\frac{e(F)r}{p-1}.$$

Thus the claim follows. □

Since L_n contains F_n, we can consider the ring

$$\bar{A}'_{n,L_n,r+}=W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n})/\psi([\zeta_{p^n}]-1)^r\bar{m}_{n,L_n}$$

and similarly $\bar{A}'_{n,F,r+}$ for F. We give these rings structures of Σ-algebras as follows. The ring $\bar{A}'_{n,L_n,r+}$ is considered as a Σ-algebra by using the system $\{\pi_n\}_{n\in\mathbb{Z}_{\geq 0}}$ we chose of p-power roots of π, as in the previous section. On the other hand, using i and i' in Lemma 5.8, put $\tilde{\pi}_n=\pi_n\zeta_{p^n}^i$ and $\tilde{\zeta}_{p^{n+1}}=\zeta_{p^{n+1}}^{i'}$. Then we consider the ring $\bar{A}'_{n,F,r+}$ as a Σ-algebra by using a system of p-power roots of π containing $\tilde{\pi}_n$. We define Fil^r and ϕ_r of these rings in the same way as before.

Lemma 5.10. *The induced ring homomorphism*

$$\bar{\eta}:\bar{A}'_{n,L_n,r+}\to\bar{A}'_{n,F,r+}$$

is a morphism of the category ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$.

Proof. Firstly, we check that $\bar{\eta}$ is Σ-linear. By definition, this homomorphism commutes with the action of the element $u \in \Sigma$. To show the compatibility with the element $Y \in \Sigma$, let us consider the commutative diagram

$$\begin{array}{ccc}
W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n}) & \longrightarrow & W_n(\mathcal{O}_F/p\mathcal{O}_F) \\
\downarrow & & \downarrow \\
W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n}) & \xrightarrow{\eta_{\mathfrak{b}}} & W_n(\mathcal{O}_F/\mathfrak{b}_F) \\
\downarrow & & \downarrow \\
\bar{A}'_{n,L_n,r+} & \xrightarrow{\bar{\eta}} & \bar{A}'_{n,F,r+},
\end{array}$$

where the horizontal arrows are induced by η. Note that we have $\eta_{\mathfrak{b}}(\pi_n) = \tilde{\pi}_n$ and $\eta_{\mathfrak{b}}(\zeta_{p^{n+1}}) = \tilde{\zeta}_{p^{n+1}}$. Put $\beta \in W(R)^\times$ as in the proof of Proposition 4.7. Namely, the element β is the solution in $W(R)$ of the equation

$$E([\underline{\pi}])\beta = pa - ([\underline{\varepsilon}^{1/p}] - 1)^{p-1},$$

where the element $a \in W(R)$ is as in the remark after Lemma 4.1. Let a_n and β_n denote the images of a and β in $W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n})$, respectively. Then the element β_n is a solution of the equation

$$E([\pi_n])\beta_n = pa_n - ([\zeta_{p^{n+1}}] - 1)^{p-1}.$$

Similarly, we define elements $\tilde{a}_n$ and $\tilde{\beta}_n$ of $W_n(\mathcal{O}_F/p\mathcal{O}_F)$ using $\tilde{\pi}_n$ and $\tilde{\zeta}_{p^{n+1}}$. By definition, the element $\tilde{\beta}_n$ is a solution of the equation

$$E([\tilde{\pi}_n])\tilde{\beta}_n = p\tilde{a}_n - ([\tilde{\zeta}_{p^{n+1}}] - 1)^{p-1}.$$

Now what we have to show is the equality

$$\bar{\eta}(a_n\beta_n^{-1}E([\pi_n])^{p-1}) = \tilde{a}_n\tilde{\beta}_n^{-1}E([\tilde{\pi}_n])^{p-1}$$

in the ring $\bar{A}'_{n,F,r+}$. Since the element a_n of $W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n})$ is a linear combination of the elements $1, [\zeta_{p^{n+1}}], \ldots, [\zeta_{p^{n+1}}]^{p-1}$ over $\mathbb{Z}$, we have $\bar{\eta}(a_n) = \tilde{a}_n$ in $\bar{A}'_{n,F,r+}$. Thus the elements $\tilde{\beta}_n$ and $\bar{\eta}(\beta_n)$ satisfy the same equation in $\bar{A}'_{n,F,r+}$. Since these two elements are invertible, we see that $\bar{\eta}(\beta_n)^{-1}E([\tilde{\pi}_n]) = \tilde{\beta}_n^{-1}E([\tilde{\pi}_n])$ and the equality holds. Since the diagram above is compatible with the Frobenius endomorphisms, we see from the definition that $\bar{\eta}$ also preserves Fil^r and commutes with ϕ_r of both sides. □

Thus the homomorphism $\bar{\eta}$ induces a homomorphism of abelian groups

$$T^*_{\mathrm{crys},L_n,\pi_n}(M_n) \to T^*_{\mathrm{crys},F,\tilde{\pi}_n}(M_n).$$

Then the following lemma, whose proof is omitted in [2, Subsection 3.13], implies that this homomorphism is an injection. We insert here a proof of this lemma for the convenience of the reader.

Lemma 5.11. *The ring homomorphism $\bar{\eta} : \bar{A}'_{n,L_n,r+} \to \bar{A}'_{n,F,r+}$ is an injection.*

Proof. For an algebraic extension N of F_n, let us write $\bar{A}'_N$ for the ring $\bar{A}'_{n,N,r+}$. Note that the ring $\bar{A}'_N/p\bar{A}'_N$ is isomorphic to the ring

$$\mathcal{O}_N/\{x \in \mathcal{O}_N \mid v_p(x) > \frac{r}{p^{n-1}(p-1)}\}.$$

As in the proof of Lemma 5.9, we see that the homomorphism $\bar{\eta}$ induces an injection

$$\bar{A}'_{L_n}/p\bar{A}'_{L_n} \to \bar{A}'_F/p\bar{A}'_F.$$

Thus it is enough to show the exactness of the sequence

$$0 \to \bar{A}'_N/p^m\bar{A}'_N \xrightarrow{\times p} \bar{A}'_N/p^{m+1}\bar{A}'_N \to \bar{A}'_N/p\bar{A}'_N \to 0.$$

Let $\bar{x}$ and $\bar{y}$ be in $\bar{A}'_N$ such that $p\bar{x} = p^{m+1}\bar{y}$. Let $\hat{x} = (\hat{x}_0, \dots, \hat{x}_{n-1})$ and $\hat{y} = (\hat{y}_0, \dots, \hat{y}_{n-1})$ be lifts of $\bar{x}$ and $\bar{y}$ in the ring $W_n(\mathcal{O}_N)$, respectively. In this ring, we have

$$(0, \hat{x}_0^p, \hat{x}_1^p, \dots,) = (0, \dots, 0, \hat{y}_0^{p^{m+1}}, \hat{y}_1^{p^{m+1}}, \dots) + ([\zeta_{p^n}] - 1)^r \hat{z},$$

where $\hat{z}$ is in the ideal $m_{n,N}$. From this equality we see that $v_p(\hat{x}_0) > r/(p^{n-1}(p-1))$ and

$$\hat{x} = ([\zeta_{p^n}] - 1)^r \hat{w} + (0, \hat{x}'_1, \hat{x}'_2, \dots)$$

for some $\hat{w} \in m_{n,N}$ and $\hat{x}'_i \in \mathcal{O}_N$. The image of the first term on the right-hand side in $\bar{A}'_N$ is zero. Hence we may assume $\hat{x}_0 = 0$. Repeating this, we can see that $\bar{x} \in p^m\bar{A}'_N$ and the above sequence is exact. □

Now Corollary 4.13 shows that the abelian group $T^*_{\mathrm{crys},L_n,\pi_n}(M_n)$ is of order p^{nd}, where $d = \dim_{\mathbb{Q}_p} V$. This implies that the the abelian group $T^*_{\mathrm{crys},F,\tilde{\pi}_n}(M_n)$ is also of order p^{nd}. Let $g \in G_K$ be as in Lemma 5.8. Then we have the following lemma.

Lemma 5.12. *The G_{F_n}-action on $T^*_{\mathrm{crys},\bar{K},\tilde{\pi}_n}(M_n)$ is the conjugate of the action on $T^*_{\mathrm{crys},\bar{K},\pi_n}(M_n)$ by the element g.*

Proof. Let $a_n, \tilde{a}_n$ and $\beta_n, \tilde{\beta}_n$ be the elements of $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ as in the proof of Lemma 5.10. Let us consider the composite

$$\Sigma \to \bar{A}'_{n,r+} \xrightarrow{g} \bar{A}'_{n,r+}$$

of the ring homomorphism defined by $u \mapsto [\pi_n]$ and $Y \mapsto -a_n\beta_n^{-1}E([\pi_n])^{p-1}$, and the map induced by g. We claim that this is the natural ring homomorphism defined by $\tilde{\pi}_n$. For this, we only have to check that this composite sends the element $Y \in \Sigma$ to $-\tilde{a}_n\tilde{\beta}_n^{-1}E([\tilde{\pi}_n])$. Since the equality

$$E([\pi_n])\beta_n = pa_n - ([\zeta_{p^{n+1}}] - 1)^{p-1}$$

holds in the ring $\bar{A}'_{n,r+}$ on the source of the above map g, we have

$$E([\tilde{\pi}_n])g(\beta_n) = p\tilde{a}_n - ([\tilde{\zeta}_{p^{n+1}}] - 1)^{p-1}$$

in the ring $\bar{A}'_{n,r+}$ on the target. Since the elements $g(\beta_n)$ and $\tilde{\beta}_n$ are invertible, we have $g(\beta_n)^{-1}E([\tilde{\pi}_n]) = \tilde{\beta}_n^{-1}E([\tilde{\pi}_n])$ and the claim follows. Thus we have an isomorphism of abelian groups

$$\begin{aligned} \mathrm{Hom}_\Sigma(M_n, \bar{A}'_{n,r+}) &\to \mathrm{Hom}_\Sigma(M_n, \bar{A}'_{n,r+}) \\ f &\mapsto g \circ f, \end{aligned}$$

where we consider on the ring $\bar{A}'_{n,r+}$ on the right-hand side the filtered ϕ_r-module structure over Σ defined by $\tilde{\pi}_n$. Since $g(t_n) = \tilde{t}_n$, we can check that this isomorphism induces an injection

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M_n, \bar{A}'_{n,r+}) \to \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M_n, \bar{A}'_{n,r+}).$$

Since these abelian groups have the same cardinality, this is also an isomorphism. $\square$

Since L_n is Galois over K, the above lemma shows that the finite Galois extension of F_n cut out by the action on $T^*_{\mathrm{crys},\bar{K},\tilde{\pi}_n}(M_n)$ is also L_n. Hence we see from Corollary 4.13 that F also contains L_n and Proposition 5.7 follows. This concludes the proof of Theorem 1.1. $\square$

Remark 5.13. The ramification bound in Theorem 1.1 is sharp for $r \leq 1$. Indeed, the greatest upper ramification break $1 + e(n + 1/(p-1))$ for $r = 1$ is obtained by the p^n-torsion of the Tate curve $\bar{K}^\times/\pi^{\mathbb{Z}}$ (see Remark 5.5). The author does not know whether this bound is sharp also for $r > 1$.

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