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Author(s)	Hattori, Shin
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ON A RAMIFICATION BOUND OF SEMI-STABLE TORSION REPRESENTATIONS OVER A LOCAL FIELD

SHIN HATTORI

ABSTRACT. Let p be a rational prime, k be a perfect field of characteristic p , $W = W(k)$ be the ring of Witt vectors, K be a finite totally ramified extension of $\text{Frac}(W)$ of degree e and r be a non-negative integer satisfying $r < p - 1$. Let V be a semi-stable p -adic G_K -representation with Hodge-Tate weights in $\{0, \dots, r\}$. In this paper, we prove the upper numbering ramification group $G_K^{(j)}$ for $j > u(K, r, n)$ acts trivially on the mod p^n representations associated to V , where $u(K, 0, n) = 0$, $u(K, 1, n) = 1 + e(n + 1/(p - 1))$ and $u(K, r, n) = 1 - p^{-n}e(K(\zeta_p)/K)^{-1} + e(n + r/(p - 1))$ for $r > 1$.

1. INTRODUCTION

Let p be a rational prime, k be a perfect field of characteristic p , $W = W(k)$ be the ring of Witt vectors and K be a finite totally ramified extension of $K_0 = \text{Frac}(W)$ of degree $e = e(K)$. Let the maximal ideal of K be denoted by m_K , an algebraic closure of K by \bar{K} and the absolute Galois group of K by $G_K = \text{Gal}(\bar{K}/K)$. We normalize the valuation v_K of K as $v_K(p) = e$ and extend this to \bar{K} . Let $G_K^{(j)}$ denote the j -th upper numbering ramification group in the sense of [7]. Namely, we put $G_K^{(j)} = G_K^{j-1}$, where the latter is the upper numbering ramification group defined in [15].

Let X_K be a proper smooth scheme over K and put $X_{\bar{K}} = X_K \times_K \bar{K}$. Consider the r -th étale cohomology group $H_{\text{ét}}^r(X_{\bar{K}}, \mathbb{Q}_p)$ and its G_K -stable \mathbb{Z}_p -lattices $\mathcal{L} \supseteq \mathcal{L}'$. In [7], Fontaine conjectured the upper numbering ramification group $G_K^{(j)}$ acts trivially on the G_K -module \mathcal{L}/\mathcal{L}' for $j > e(n + r/(p-1))$ if X_K has good reduction and this module is killed by p^n . For $e = 1$ and $r < p - 1$, this conjecture was proved independently by himself ([8], for $n = 1$) and Abrashkin ([2], for any n), using the theory of Fontaine-Laffaille ([10]) and the comparison theorem of Fontaine-Messing ([11]) between the p -adic étale cohomology groups of X_K and the crystalline cohomology groups of the reduction of X_K . From this result, Fontaine also showed some rareness of a proper smooth scheme over \mathbb{Q} with everywhere good reduction ([8, Théorème 1]). In fact, they proved this ramification bound for the torsion

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representations of the crystalline p -adic representations of G_K with Hodge-Tate weights in $\{0, \dots, r\}$ in the case where K is absolutely unramified.

On the other hand, for a semi-stable p -adic representation V with Hodge-Tate weights in the same range, a similar ramification bound for $e = 1$ and $n = 1$ is obtained by Breuil (see [4, Proposition 9.2.2.2]). He showed, assuming the Griffiths transversality which in general does not hold, that if $e = 1$ and $r < p - 1$, then the ramification group $G_K^{(j)}$ acts trivially on the mod p representations of V for $j > 2 + 1/(p - 1)$.

In this paper, we prove a version of the result of Breuil for the case where K is absolutely ramified, under the condition $r < p - 1$. Our main theorem is the following.

Theorem 1.1. *Let r be a non-negative integer such that $r < p - 1$. Let V be a semi-stable p -adic G_K -representation with Hodge-Tate weights in $\{0, \dots, r\}$ and $\mathcal{L} \supseteq \mathcal{L}'$ be G_K -stable \mathbb{Z}_p -lattices in V . Suppose that the quotient \mathcal{L}/\mathcal{L}' is killed by p^n . Then the j -th upper numbering ramification group $G_K^{(j)}$ acts trivially on the G_K -module \mathcal{L}/\mathcal{L}' for $j > u(K, r, n)$, where*

$$u(K, r, n) = \begin{cases} 0 & (r = 0), \\ 1 + e(n + \frac{1}{p-1}) & (r = 1), \\ 1 - \frac{1}{p^n e(K(\zeta_p)/K)} + e(n + \frac{r}{p-1}) & (r > 1) \end{cases}$$

and $e(K(\zeta_p)/K)$ denotes the relative ramification index of the extension $K(\zeta_p)/K$.

We can check that this bound is sharp for $r \leq 1$ (Remark 5.13). From this theorem and [7, Proposition 1.3], we have the following corollary.

Corollary 1.2. *Let the notation be as in the theorem and L be the finite extension of K cut out by the G_K -module \mathcal{L}/\mathcal{L}' . Let $\mathfrak{D}_{L/K}$ denote the different of the extension L/K . Then we have the inequality*

$$v_K(\mathfrak{D}_{L/K}) < u(K, r, n)$$

for $r > 0$ and $v_K(\mathfrak{D}_{L/K}) = 0$ for $r = 0$.

For the proof of Theorem 1.1, we essentially follow a beautiful argument of Abrashkin ([2]). We may assume $p \geq 3$ and $r \geq 1$. Thanks to Liu's theorem ([14]) on the G_K -stable \mathbb{Z}_p -lattices in semi-stable p -adic representations, it is enough to bound the ramification of the G_K -module

$$T_{\text{st}, \overline{\pi}}^*(\mathcal{M}_n) = \text{Hom}_{S, \text{Fil}^r, \phi_r, N}(\mathcal{M}_n, \hat{A}_{\text{st}, \infty}),$$

where \mathcal{M}_n is a p^n -torsion object of a category $\text{Mod}_{S_\infty}^{r, \phi_r, N}$ of filtered (ϕ_r, N) -modules over S defined by Breuil ([3]) and $\hat{A}_{\text{st}, \infty}$ is a p -adic period ring. We may also assume $\zeta_p \in K$ and consider the finite Galois extension

$$F_n = K(\pi^{1/p^n}, \zeta_{p^{n+1}})$$

of K whose upper ramification is bounded by the same value as in the theorem. Let L_n be the finite Galois extension of F_n cut out by $T_{\text{st}, \overline{\pi}}^*(\mathcal{M}_n)|_{G_{F_n}}$.

Then we bound the ramification of L_n over K . For this, we show that to study this G_{F_n} -module we can use a variant over a smaller coefficient ring Σ of filtered (ϕ_r, N) -modules over S . In precise, let $E(u)$ be the Eisenstein polynomial of a uniformizer π of K over W and we set

$$\Sigma = W[[u, E(u)^p/p]].$$

This ring Σ is small enough for the method of Abrashkin, in which he uses filtered modules of Fontaine-Laffaille ([10]) whose coefficient ring is W , to work also in the case where K is absolutely ramified.

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2. FILTERED (ϕ_r, N) -MODULES OF BREUIL

In this section, we recall the theory of filtered (ϕ_r, N) -modules over S of Breuil, which is developed by himself and most recently by Caruso and Liu (see for example [3], [5], [14], [6]). In what follows, we always take the divided power envelope of a W -algebra with the compatibility condition with the natural divided power structure on pW .

Let $p \geq 3$ be a rational prime and σ be the Frobenius endomorphism of W . We fix once and for all a uniformizer π of K and a system $\{\pi_n\}_{n \in \mathbb{Z}_{\geq 0}}$ of p -power roots of π such that $\pi_0 = \pi$ and $\pi_n = \pi_{n+1}^p$ for any n . Let $E(u)$ be the Eisenstein polynomial of π over W and set $S = (W[u]^{\text{PD}})^\wedge$, where PD means the divided power envelope and this is taken with respect to the ideal $(E(u))$, and \wedge means the p -adic completion. The ring S is endowed with the σ -semilinear endomorphism $\phi : u \mapsto u^p$ and a natural filtration $\text{Fil}^t S$ induced by the divided power structure such that $\phi(\text{Fil}^t S) \subseteq p^t S$ for any non-negative integer t . We set $\phi_t = p^{-t} \phi|_{\text{Fil}^t S}$ and $c = \phi_1(E(u)) \in S^\times$. Let N denote the W -linear derivation on S defined by the formula $N(u) = -u$. We also define a filtration, ϕ, ϕ_t, N on $S_n = S/p^n S$ similarly.

Let $r \in \{0, \dots, p-2\}$ be an integer. Set $'\text{Mod}_{/S}^{r, \phi, N}$ to be the category consisting of the following data:

- an S -module \mathcal{M} and its S -submodule $\text{Fil}^r \mathcal{M}$ containing $\text{Fil}^r S \cdot \mathcal{M}$,
- a ϕ -semilinear map $\phi_r : \text{Fil}^r \mathcal{M} \rightarrow \mathcal{M}$ satisfying

$$\phi_r(s_r m) = \phi_r(s_r) \phi(m)$$

for any $s_r \in \text{Fil}^r S$ and $m \in \mathcal{M}$, where we set $\phi(m) = c^{-r} \phi_r(E(u)^r m)$,

- a W -linear map $N : \mathcal{M} \rightarrow \mathcal{M}$ such that
 - $N(sm) = N(s)m + sN(m)$ for any $s \in S$ and $m \in \mathcal{M}$,
 - $E(u)N(\text{Fil}^r \mathcal{M}) \subseteq \text{Fil}^r \mathcal{M}$,

– the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Fil}^r \mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M} \\ E(u)N \downarrow & & \downarrow cN \\ \mathrm{Fil}^r \mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M}, \end{array}$$

and the morphisms of $'\mathrm{Mod}_{/S}^{r,\phi,N}$ are defined to be the S -linear maps preserving Fil^r and commuting with ϕ_r and N . The category defined in the same way but dropping the data N is denoted by $'\mathrm{Mod}_{/S}^{r,\phi}$. These categories have obvious notions of exact sequences. Let $\mathrm{Mod}_{/S_1}^{r,\phi,N}$ denote the full subcategory of $'\mathrm{Mod}_{/S}^{r,\phi,N}$ consisting of \mathcal{M} such that \mathcal{M} is free of finite rank over S_1 and generated as an S_1 -module by the image of ϕ_r . We write $\mathrm{Mod}_{/S_\infty}^{r,\phi,N}$ for the smallest full subcategory which contains $\mathrm{Mod}_{/S_1}^{r,\phi,N}$ and is stable under extensions. We let $\mathrm{Mod}_{/S}^{r,\phi,N}$ denote the full subcategory consisting of \mathcal{M} such that

- the S -module \mathcal{M} is free of finite rank and generated by the image of ϕ_r ,
- the quotient $\mathcal{M}/\mathrm{Fil}^r \mathcal{M}$ is p -torsion free.

We define full subcategories $\mathrm{Mod}_{/S_1}^{r,\phi}$, $\mathrm{Mod}_{/S_\infty}^{r,\phi}$ and $\mathrm{Mod}_{/S}^{r,\phi}$ of $'\mathrm{Mod}_{/S}^{r,\phi}$ in a similar way. For $\hat{\mathcal{M}} \in \mathrm{Mod}_{/S}^{r,\phi,N}$ (*resp.* $\mathrm{Mod}_{/S}^{r,\phi}$), the quotient $\hat{\mathcal{M}}/p^n \hat{\mathcal{M}}$ has a natural structure as an object of $\mathrm{Mod}_{/S_\infty}^{r,\phi,N}$ (*resp.* $\mathrm{Mod}_{/S_\infty}^{r,\phi}$).

For p -torsion objects, we also have the following categories. Consider the k -algebra $k[u]/(u^{ep}) \cong S_1/\mathrm{Fil}^p S_1$ and let this algebra be denoted by \tilde{S}_1 . The algebra \tilde{S}_1 is equipped with the natural filtration, ϕ and N induced by those of S . Namely, $\mathrm{Fil}^t \tilde{S}_1 = u^{et} \tilde{S}_1$, $\phi(u) = u^p$ and $N(u) = -u$. Let $'\mathrm{Mod}_{/\tilde{S}_1}^{r,\phi,N}$ denote the category consisting of the following data:

- an \tilde{S}_1 -module $\tilde{\mathcal{M}}$ and its \tilde{S}_1 -submodule $\mathrm{Fil}^r \tilde{\mathcal{M}}$ containing $u^{er} \tilde{\mathcal{M}}$,
- a ϕ -semilinear map $\phi_r : \mathrm{Fil}^r \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}$,
- a k -linear map $N : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}$ such that
 - $N(sm) = N(s)m + sN(m)$ for any $s \in \tilde{S}_1$ and $m \in \tilde{\mathcal{M}}$,
 - $u^e N(\mathrm{Fil}^r \tilde{\mathcal{M}}) \subseteq \mathrm{Fil}^r \tilde{\mathcal{M}}$,
 - the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Fil}^r \tilde{\mathcal{M}} & \xrightarrow{\phi_r} & \tilde{\mathcal{M}} \\ u^e N \downarrow & & \downarrow cN \\ \mathrm{Fil}^r \tilde{\mathcal{M}} & \xrightarrow{\phi_r} & \tilde{\mathcal{M}}, \end{array}$$

and whose morphisms are defined as before. Its full subcategory $\text{Mod}_{/\tilde{S}_1}^{r,\phi,N}$ is defined by the following condition:

- As an \tilde{S}_1 -module, $\tilde{\mathcal{M}}$ is free of finite rank and generated by the image of ϕ_r .

We define categories $'\text{Mod}_{/\tilde{S}_1}^{r,\phi}$ and $\text{Mod}_{/\tilde{S}_1}^{r,\phi}$ similarly.

Let D be a weakly admissible filtered (ϕ, N) -module over K satisfying $\text{Fil}^0 D_K = D_K$ and $\text{Fil}^{r+1} D_K = 0$. Set $S_{K_0} = S \otimes_W K_0$ and $\mathcal{D} = D \otimes_{K_0} S_{K_0}$. Then the S_{K_0} -module \mathcal{D} is equipped with the natural ϕ -semilinear map $\phi \otimes \sigma$ and K_0 -linear derivation $N \otimes 1 + 1 \otimes N$, which are denoted by ϕ and N , respectively. We define a filtration on \mathcal{D} inductively by $\text{Fil}^0 \mathcal{D} = \mathcal{D}$ and

$$\text{Fil}^{i+1} \mathcal{D} = \{x \in \mathcal{D} \mid N(x) \in \text{Fil}^i \mathcal{D} \text{ and } f_\pi(x) \in \text{Fil}^{i+1} D_K\},$$

where $f_\pi : \mathcal{D} \rightarrow D_K$ is induced by the map $S \rightarrow \mathcal{O}_K$ sending u to π . An S -submodule $\hat{\mathcal{M}}$ of \mathcal{D} is said to be a strongly divisible lattice of \mathcal{D} if the following conditions are satisfied:

- the S -module $\hat{\mathcal{M}}$ is free of finite rank,
- $\hat{\mathcal{M}} \otimes_W K_0 = \mathcal{D}$,
- $\hat{\mathcal{M}}$ is stable under ϕ and N ,
- $\phi(\text{Fil}^r \hat{\mathcal{M}}) \subseteq p^r \hat{\mathcal{M}}$, where we set $\text{Fil}^r \hat{\mathcal{M}} = \hat{\mathcal{M}} \cap \text{Fil}^r \mathcal{D}$.

We put $\phi_r = p^{-r} \phi|_{\text{Fil}^r \hat{\mathcal{M}}}$. Then the S -module $\hat{\mathcal{M}}$ is generated by $\phi_r(\text{Fil}^r \hat{\mathcal{M}})$ ([3, Proposition 2.1.3]) and we can consider $\hat{\mathcal{M}}$ as an object of $\text{Mod}_{/S}^{r,\phi,N}$.

Let A_{crys} and \hat{A}_{st} be p -adic period rings. These are constructed as follows. Set R to be the ring

$$R = \varprojlim (\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}} \leftarrow \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}} \leftarrow \cdots),$$

where every arrow is the p -power map. For an element $x = (x_i)_{i \in \mathbb{Z}_{\geq 0}} \in R$ and an integer $n \geq 0$, we set

$$x^{(n)} = \lim_{m \rightarrow \infty} \hat{x}_{n+m}^{p^m} \in \mathcal{O}_{\mathbb{C}},$$

where \hat{x}_i is a lift of x_i in $\mathcal{O}_{\bar{K}}$ and $\mathcal{O}_{\mathbb{C}}$ is the p -adic completion of $\mathcal{O}_{\bar{K}}$. Let v_p denote the valuation of $\mathcal{O}_{\mathbb{C}}$ normalized as $v_p(p) = 1$. Then the ring R is a complete valuation ring whose valuation of an element $x \in R$ is given by $v_R(x) = v_p(x^{(0)})$. We define a natural ring homomorphism θ by

$$\begin{aligned} \theta : W(R) &\rightarrow \mathcal{O}_{\mathbb{C}} \\ (x_0, x_1, \dots) &\mapsto \sum_{n \geq 0} p^n x_n^{(n)}. \end{aligned}$$

Then A_{crys} is the p -adic completion of the divided power envelope of $W(R)$ with respect to the principal ideal $\text{Ker}(\theta)$ and \hat{A}_{st} is the p -adic completion of the divided power polynomial ring $A_{\text{crys}}\langle X \rangle$ over A_{crys} . We set $A_{\text{crys},\infty} = A_{\text{crys}} \otimes_W K_0/W$ and $\hat{A}_{\text{st},\infty} = \hat{A}_{\text{st}} \otimes_W K_0/W$. Put $\underline{\pi} = (\pi_n)_{n \in \mathbb{Z}_{\geq 0}} \in R$, where we abusively let π_n denote the image of $\pi_n \in \mathcal{O}_{\bar{K}}$ in $\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$. These rings

are considered as S -algebras by the ring homomorphisms $S \rightarrow \hat{A}_{\text{st}}$ and $\hat{A}_{\text{st}} \rightarrow A_{\text{crys}}$ which are defined by $u \mapsto [\pi]/(1+X)$ and $X \mapsto 0$, respectively. The ring A_{crys} is endowed with a natural filtration induced by the divided power structure, a natural Frobenius endomorphism ϕ and the ϕ -semilinear map $\phi_t = p^{-t}\phi|_{\text{Fil}^t A_{\text{crys}}}$. With these structures, A_{crys} and $A_{\text{crys},\infty}$ are considered as objects of $'\text{Mod}_{/S}^{r,\phi}$. Moreover, the absolute Galois group G_K acts naturally on these two rings. As for \hat{A}_{st} , its filtration is defined by

$$\text{Fil}^t \hat{A}_{\text{st}} = \left\{ \sum_{i \geq 0} a_i \frac{X^i}{i!} \mid a_i \in \text{Fil}^{t-i} A_{\text{crys}}, \lim_{i \rightarrow \infty} a_i = 0 \right\}$$

and the Frobenius structure of A_{crys} extends to \hat{A}_{st} by

$$\begin{aligned} \phi(X) &= (1+X)^p - 1, \\ \phi_t &= p^{-t}\phi|_{\text{Fil}^t \hat{A}_{\text{st}}}. \end{aligned}$$

We write N also for the A_{crys} -linear derivation on \hat{A}_{st} defined by $N(X) = 1+X$. The rings \hat{A}_{st} and $\hat{A}_{\text{st},\infty}$ are objects of $'\text{Mod}_{/S}^{r,\phi,N}$. The G_K -action on A_{crys} naturally extends to an action on \hat{A}_{st} . Indeed, the action of $g \in G_K$ on \hat{A}_{st} is defined by the formula

$$g(X) = [\underline{\varepsilon}(g)](1+X) - 1,$$

where $g(\pi_n) = \varepsilon_n(g)\pi_n$ and $\underline{\varepsilon}(g) = (\varepsilon_n(g))_{n \in \mathbb{Z}_{\geq 0}} \in R$ with the abusive notation as above.

These rings have other descriptions, as follows. For an integer $n \geq 1$, put $W_n = W/p^n W$ and let $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ be the ring of Witt vectors of length n associated to $\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$. We define a W_n -algebra structure on $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ by twisting the natural W_n -algebra structure by σ^{-n} . Then the natural ring homomorphism

$$\begin{aligned} \theta_n : W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) &\rightarrow \mathcal{O}_{\bar{K}}/p^n \mathcal{O}_{\bar{K}} \\ (a_0, \dots, a_{n-1}) &\mapsto \sum_{i=0}^{n-1} p^i \hat{a}_i^{p^{n-i}}, \end{aligned}$$

where \hat{a}_i is a lift of a_i in $\mathcal{O}_{\bar{K}}$, is W_n -linear. Let us denote

$$W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$$

the divided power envelope of $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ with respect to the ideal $\text{Ker}(\theta_n)$. This ring is considered as an S -algebra by $u \mapsto [\pi_n]$. This ring also has a natural filtration defined by the divided power structure, and a natural G_K -module structure. The Frobenius endomorphism of the ring of Witt vectors

induces on this ring a ϕ -semilinear Frobenius endomorphism, which is denoted also by ϕ . Then, by the S -linear transition maps

$$\begin{aligned} W_{n+1}^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) &\rightarrow W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \\ (a_0, \dots, a_n) &\mapsto (a_0^p, \dots, a_{n-1}^p), \end{aligned}$$

these S -algebras form a projective system compatible with all structures. Using this transition map, a ϕ -semilinear map

$$\phi_r : \text{Fil}^r W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \rightarrow W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$$

is defined by setting $\phi_r(x)$ to be the image of $p^{-r}\phi(\hat{x})$, where \hat{x} is a lift of x in $\text{Fil}^r W_{n+r}^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$. By definition, the maps ϕ_r are also compatible with the transition maps. The S -algebra $W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ is considered as an object of $'\text{Mod}_{/S}^{r,\phi}$. Then we have a natural isomorphism in $'\text{Mod}_{/S}^{r,\phi}$

$$\begin{aligned} A_{\text{crys}}/p^n A_{\text{crys}} &\rightarrow W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \\ (x_0, \dots, x_{n-1}) &\mapsto (x_{0,n}, \dots, x_{n-1,n}), \end{aligned}$$

where we set $x_i = (x_{i,k})_{k \in \mathbb{Z}_{\geq 0}}$.

Similarly, the divided power polynomial ring

$$W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})\langle X \rangle$$

over $W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ is considered as an S -algebra by $u \mapsto [\pi_n]/(1+X)$. This ring has a natural filtration coming from the divided power structure. We define a G_K -action on this ring by

$$g(X) = [\varepsilon_n(g)](1+X) - 1.$$

We also define a ϕ -semilinear Frobenius endomorphism, which we also write as ϕ , by $\phi(X) = (1+X)^p - 1$ and a $W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ -linear derivation N by $N(X) = 1+X$. These rings form a projective system of S -algebras compatible with all structures by the transition maps defined by the maps above and $X \mapsto X$. We define ϕ -semilinear maps

$$\phi_r : \text{Fil}^r W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})\langle X \rangle \rightarrow W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})\langle X \rangle$$

compatible with the transition maps as before. The S -algebra $W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})\langle X \rangle$ is considered as an object of $'\text{Mod}_{/S}^{r,\phi,N}$ and there exists a natural isomorphism in $'\text{Mod}_{/S}^{r,\phi,N}$

$$\begin{aligned} \hat{A}_{\text{st}}/p^n \hat{A}_{\text{st}} &\rightarrow W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})\langle X \rangle \\ (x_0, \dots, x_{n-1}) &\mapsto (x_{0,n}, \dots, x_{n-1,n}) \\ X &\mapsto X \end{aligned}$$

which is G_K -linear.

Put $K_n = K(\pi_n)$ and $K_\infty = \cup_n K_n$. For $\mathcal{M} \in \text{Mod}_{/S_\infty}^{r,\phi,N}$, we define a G_K -module $T_{\text{st},\pi}^*(\mathcal{M})$ to be

$$T_{\text{st},\pi}^*(\mathcal{M}) = \text{Hom}_{S, \text{Fil}^r, \phi_r, N}(\mathcal{M}, \hat{A}_{\text{st},\infty}).$$

When \mathcal{M} is killed by p^n , we have a natural identification of G_K -modules

$$T_{\text{st}, \overline{\pi}}^*(\mathcal{M}) = \text{Hom}_{S, \text{Fil}^r, \phi_r, N}(\mathcal{M}, W_n^{\text{PD}}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})\langle X \rangle).$$

Note that the G_K -module on the right-hand side is independent of the choice of π_k for $k > n$. Since the natural map

$$\begin{aligned} W_n^{\text{PD}}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})\langle X \rangle &\rightarrow W_n^{\text{PD}}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \\ X &\mapsto 0 \end{aligned}$$

is by definition G_{K_n} -linear, we also have a G_{K_n} -linear isomorphism ([3, Lemme 2.3.1.1])

$$T_{\text{st}, \overline{\pi}}^*(\mathcal{M})|_{G_{K_n}} \rightarrow \text{Hom}_{S, \text{Fil}^r, \phi_r}(\mathcal{M}, W_n^{\text{PD}}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})).$$

A variant of filtered (ϕ_r, N) -modules over S is also introduced by Breuil and Kisin, and developed also by Caruso and Liu (see for example [12], [13], [14], [6]). Put $\mathfrak{S} = W[[u]]$ and let $\phi : \mathfrak{S} \rightarrow \mathfrak{S}$ be the σ -semilinear Frobenius endomorphism defined by $\phi(u) = u^p$. Let $'\text{Mod}_{\mathfrak{S}}^{r, \phi}$ denote the category consisting of the following data:

- an \mathfrak{S} -module \mathfrak{M} ,
- a ϕ -semilinear map $\mathfrak{M} \rightarrow \mathfrak{M}$, which is denoted also by ϕ , such that the cokernel of the map $1 \otimes \phi : \phi^*\mathfrak{M} \rightarrow \mathfrak{M}$ is killed by $E(u)^r$,

and whose morphisms are defined as before. The full subcategory of $'\text{Mod}_{\mathfrak{S}}^{r, \phi}$ consisting of \mathfrak{M} such that \mathfrak{M} is free of finite rank over $\mathfrak{S}/p\mathfrak{S}$ (*resp.* over \mathfrak{S}) is denoted by $\text{Mod}_{\mathfrak{S}_1}^{r, \phi}$ (*resp.* $\text{Mod}_{\mathfrak{S}}^{r, \phi}$). We let $\text{Mod}_{\mathfrak{S}_\infty}^{r, \phi}$ denote the smallest full subcategory which contains $\text{Mod}_{\mathfrak{S}_1}^{r, \phi}$ and is stable under extensions, as before. Then we have an exact functor ([6, Proposition 2.1.2], see also [12, Proposition 1.1.11])

$$\mathcal{M}_{\mathfrak{S}_\infty} : \text{Mod}_{\mathfrak{S}_\infty}^{r, \phi} \rightarrow \text{Mod}_{S_\infty}^{r, \phi}.$$

For $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}_\infty}^{r, \phi}$, the filtered ϕ_r -module $\mathcal{M} = \mathcal{M}_{\mathfrak{S}_\infty}(\mathfrak{M})$ over S is defined as follows:

- $\mathcal{M} = S \otimes_{\phi, \mathfrak{S}} \mathfrak{M}$,
- $\text{Fil}^r \mathcal{M} = \text{Ker}(\mathcal{M} \xrightarrow{1 \otimes \phi} S \otimes_{\mathfrak{S}} \mathfrak{M} \rightarrow (S/\text{Fil}^r S) \otimes_{\mathfrak{S}} \mathfrak{M})$,
- $\phi_r : \text{Fil}^r \mathcal{M} \xrightarrow{1 \otimes \phi} \text{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\phi_r \otimes 1} S \otimes_{\phi, \mathfrak{S}} \mathfrak{M} = \mathcal{M}$.

We write $\mathcal{M}_{\mathfrak{S}}$ for the functor $\text{Mod}_{\mathfrak{S}}^{r, \phi} \rightarrow \text{Mod}_{S}^{r, \phi}$ defined similarly.

Finally, let D and \mathcal{D} be as above and $\hat{\mathcal{M}}$ be a strongly divisible lattice in \mathcal{D} . The S -module $\mathcal{M}_n = \hat{\mathcal{M}}/p^n \hat{\mathcal{M}}$ has a natural structure as an object of $\text{Mod}_{S_\infty}^{r, \phi, N}$. We set a G_K -module $\hat{T}_{\text{st}, \overline{\pi}}^*(\hat{\mathcal{M}})$ to be

$$\hat{T}_{\text{st}, \overline{\pi}}^*(\hat{\mathcal{M}}) = \text{Hom}_{S, \text{Fil}^r, \phi_r, N}(\hat{\mathcal{M}}, \hat{A}_{\text{st}}).$$

Then we have an exact sequence of G_K -modules

$$0 \longrightarrow \hat{T}_{\text{st}, \overline{\pi}}^*(\hat{\mathcal{M}}) \xrightarrow{p^n} \hat{T}_{\text{st}, \overline{\pi}}^*(\hat{\mathcal{M}}) \longrightarrow T_{\text{st}, \overline{\pi}}^*(\mathcal{M}_n) \longrightarrow 0.$$

The G_K -module $\hat{T}_{\text{st},\pi}^*(\hat{\mathcal{M}})$ is naturally considered as a G_K -stable \mathbb{Z}_p -lattice in $V_{\text{st}}^*(D)$. By Liu's theorem ([14, Theorem 2.3.5]), the functor $\hat{T}_{\text{st},\pi}^*$ gives an anti-equivalence of categories between the category of strongly divisible lattices in \mathcal{D} and the category of G_K -stable \mathbb{Z}_p -lattices in $V_{\text{st}}^*(D)$. Moreover, for such a lattice \mathcal{L} , its corresponding strongly divisible lattice $\hat{\mathcal{M}}$ in \mathcal{D} is in the essential image of the functor $\mathcal{M}_{\mathfrak{S}}$ ([14, Subsection 3.5]).

3. FILTERED ϕ_r -MODULES OVER Σ

In this section, we define another variant of filtered ϕ_r -modules over S and prove its properties.

Let $p \geq 3$ be a rational prime and r be an integer such that $0 \leq r < p-1$. Consider the W -algebra $\Sigma = W[[u, Y]]/(E(u)^p - pY)$ as in [3, Subsection 3.2]. We regard Σ as a subring of S by the map sending Y to $E(u)^p/p$. Then the element $c = \phi_1(E(u)) \in S^\times$ is contained in Σ^\times . We define on Σ a σ -semilinear Frobenius endomorphism ϕ by $\phi(u) = u^p$ and $\phi(Y) = p^{p-1}c^p$. Put $\text{Fil}^t\Sigma = (E(u)^t, Y)$ for $0 \leq t \leq p-1$ and $\text{Fil}^p\Sigma = (Y)$. Then we have $\phi(\text{Fil}^t\Sigma) \subseteq p^t\Sigma$ for $0 \leq t \leq p-1$. We put $\phi_t = p^{-t}\phi|_{\text{Fil}^t\Sigma}$. We also set $\Sigma_n = \Sigma/p^n\Sigma$ and put on this ring the natural structures induced by those of Σ .

We define a category $'\text{Mod}_{\Sigma}^{r,\phi}$ of filtered ϕ_r -modules over Σ to be the category consisting of the following data:

- a Σ -module M and its Σ -submodule $\text{Fil}^r M$ containing $\text{Fil}^r\Sigma.M$,
- a ϕ -semilinear map $\text{Fil}^r M \rightarrow M$ satisfying $\phi_r(s_r m) = \phi_r(s_r)\phi(m)$ for any $s_r \in \text{Fil}^r\Sigma$ and $m \in M$, where we set $\phi(m) = c^{-r}\phi_r(E(u)^r m)$.

and the morphisms are defined in the same manner as $'\text{Mod}_{S}^{r,\phi}$. This category has a natural notion of exact sequences. We define its full subcategory $\text{Mod}_{\Sigma_1}^{r,\phi}$ to be the category consisting of M which is free of finite rank and generated by the image of ϕ_r as a Σ_1 -module. We also let $\text{Mod}_{\Sigma_\infty}^{r,\phi}$ denote the smallest full subcategory of $'\text{Mod}_{\Sigma}^{r,\phi}$ which contains $\text{Mod}_{\Sigma_1}^{r,\phi}$ and is stable under extensions. Moreover, we define a full subcategory $\text{Mod}_{\Sigma}^{r,\phi}$ of $'\text{Mod}_{\Sigma}^{r,\phi}$ to be the category consisting of M such that

- the Σ -module M is free of finite rank and generated by the image of ϕ_r ,
- the quotient $M/\text{Fil}^r M$ is p -torsion free.

Then we see that for $\hat{M} \in \text{Mod}_{\Sigma}^{r,\phi}$, the quotient $\hat{M}/p^n \hat{M}$ is naturally considered as an object of $\text{Mod}_{\Sigma_\infty}^{r,\phi}$.

The natural ring isomorphism $\Sigma_1/\text{Fil}^p\Sigma_1 \cong \tilde{S}_1$ defines a functor $T_0 : \text{Mod}_{\Sigma_1}^{r,\phi} \rightarrow \text{Mod}_{\tilde{S}_1}^{r,\phi}$ by $M \mapsto M/\text{Fil}^p\Sigma_1.M$. Then [3, Proposition 2.2.1.3] and Nakayama's lemma shows the following.

Lemma 3.1. *Let M be an object of $\text{Mod}_{/\Sigma_1}^{r,\phi}$ of rank d over Σ_1 . Then there exists a basis $\{e_1, \dots, e_d\}$ of M such that $\text{Fil}^r M = \Sigma_1 u^{r_1} e_1 \oplus \dots \oplus \Sigma_1 u^{r_d} e_d + \text{Fil}^p \Sigma_1.M$ for some integers r_1, \dots, r_d with $0 \leq r_i \leq er$ for any i .*

Then we can show the following lemma just as in the proof of [3, Lemme 2.3.1.3].

Lemma 3.2. *The functor*

$$M \mapsto \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, A_{\text{crys}, \infty})$$

from $\text{Mod}_{/\Sigma_\infty}^{r,\phi}$ to the category of G_{K_∞} -modules is exact.

For $M \in \text{Mod}_{/\Sigma_1}^{r,\phi}$, we can show as in the case of the category $\text{Mod}_{/S_1}^{r,\phi}$ that there is an isomorphism of G_{K_1} -modules

$$\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, (\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})^{\text{PD}}) \rightarrow \text{Hom}_{\tilde{S}_1, \text{Fil}^r, \phi_r}(T_0(M), \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}),$$

where $\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$ is considered as an object of $'\text{Mod}_{/S_1}^{r,\phi}$ by the natural isomorphism

$$(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})^{\text{PD}}/\text{Fil}^p(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})^{\text{PD}} \rightarrow \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}.$$

Thus [3, Lemme 2.3.1.2] implies that, for such a Σ_1 -module M , we have

$$\#\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, (\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})^{\text{PD}}) = p^d,$$

where $d = \dim_{\Sigma_1} M$.

For the category $\text{Mod}_{/\Sigma}^{r,\phi}$, we can show the following lemma just as in the proof of [14, Proposition 4.1.2].

Lemma 3.3. *Let \hat{M} be in $\text{Mod}_{/\Sigma}^{r,\phi}$. Then there exists $\alpha_1, \dots, \alpha_d \in \hat{M}$ such that $\text{Fil}^r \hat{M} = \Sigma \alpha_1 \oplus \dots \oplus \Sigma \alpha_d + \text{Fil}^p \Sigma.\hat{M}$, $E(u)^r \hat{M} \subseteq \Sigma \alpha_1 \oplus \dots \oplus \Sigma \alpha_d$ and the elements $e_1 = \phi_r(\alpha_1), \dots, e_d = \phi_r(\alpha_d)$ form a basis of \hat{M} .*

Corollary 3.4. *Let \hat{M} be in $\text{Mod}_{/\Sigma}^{r,\phi}$ and A be a Σ -algebra which has a structure as an object of $'\text{Mod}_{/\Sigma}^{r,\phi}$. Let $C \in M_d(\Sigma)$ be the matrix such that*

$$(\alpha_1, \dots, \alpha_d) = (e_1, \dots, e_d)C$$

with the notation of the previous lemma. Then a Σ -linear homomorphism $f : \hat{M} \rightarrow A$ preserving Fil^r also commutes with ϕ_r if and only if

$$\phi_r(f(e_1, \dots, e_d)C) = (f(e_1), \dots, f(e_d)).$$

Proof. Suppose that the latter condition holds. By assumption, we have

$$E(u)^r(e_1, \dots, e_d) = (\alpha_1, \dots, \alpha_d)C'$$

for some $C' \in M_d(\Sigma)$. We claim that f commutes with ϕ . Indeed, we have

$$\begin{aligned} \phi(f(e_1, \dots, e_d)) &= c^{-r} \phi_r(E(u)^r(f(e_1), \dots, f(e_d))) \\ &= c^{-r} \phi_r((f(\alpha_1), \dots, f(\alpha_d))C') \\ &= c^{-r} f(e_1, \dots, e_d)\phi(C') \\ &= c^{-r} f(\phi_r(\alpha_1, \dots, \alpha_d))\phi(C') \\ &= c^{-r} f(\phi_r(E(u)^r(e_1, \dots, e_d))) = f(\phi(e_1, \dots, e_d)). \end{aligned}$$

This implies $\phi_r \circ f = f \circ \phi_r$ also on $\text{Fil}^p \Sigma \cdot \hat{M}$. □

Corollary 3.5. *Let \hat{M} and A be as above and $J \subseteq \text{Fil}^r A$ be an ideal of A such that $\phi_r(J) \subseteq J$. We can consider the Σ -algebra A/J naturally as an object of $'\text{Mod}_{\Sigma}^{r, \phi}$. Suppose that for any $x \in J$, there exists $t \in \mathbb{Z}_{\geq 0}$ such that $\phi_r^t(x) = 0$. Then we have an isomorphism*

$$\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(\hat{M}, A) \rightarrow \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(\hat{M}, A/J).$$

Proof. Let $f : \hat{M} \rightarrow A/J$ be an element of the abelian group on the right-hand side and \hat{x}_i be an lift of $f(e_i)$ in A . By the previous corollary, it is enough to show that for any $(\hat{c}_1, \dots, \hat{c}_d) \in J^d$, there is a unique solution $(\hat{y}_1, \dots, \hat{y}_d) \in J^d$ of the equation

$$(\hat{c}_1, \dots, \hat{c}_d) + (\phi_r(\hat{y}_1), \dots, \phi_r(\hat{y}_d))\phi(C) = (\hat{y}_1, \dots, \hat{y}_d).$$

By assumption, the d -tuple

$$\sum_{i=0}^t (\phi_r^i(\hat{c}_1), \dots, \phi_r^i(\hat{c}_d))\phi^i(C)\phi^{i-1}(C) \cdots \phi(C)$$

is stable for sufficiently large t and we see that this limit gives a unique solution of the equation. □

For an \mathfrak{S} -module \mathfrak{M} in $\text{Mod}_{\mathfrak{S}}^{r, \phi}$ (resp. $\text{Mod}_{\mathfrak{S}}^{r, \phi}$), we associate to it a Σ -module $M \in '\text{Mod}_{\Sigma}^{r, \phi}$ as follows:

- $M = \Sigma \otimes_{\phi, \mathfrak{S}} \mathfrak{M}$,
- $\text{Fil}^r M = \text{Ker}(M \xrightarrow{1 \otimes \phi} \Sigma \otimes_{\mathfrak{S}} \mathfrak{M} \rightarrow (\Sigma/\text{Fil}^r \Sigma) \otimes_{\mathfrak{S}} \mathfrak{M})$,
- $\phi_r : \text{Fil}^r M \xrightarrow{1 \otimes \phi} \text{Fil}^r \Sigma \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\phi_r \otimes 1} \Sigma \otimes_{\phi, \mathfrak{S}} \mathfrak{M} = M$.

We can check that this defines an exact functor $\text{Mod}_{\mathfrak{S}}^{r, \phi} \rightarrow \text{Mod}_{\Sigma}^{r, \phi}$ (resp. $\text{Mod}_{\mathfrak{S}}^{r, \phi} \rightarrow \text{Mod}_{\Sigma}^{r, \phi}$) as in the proof of [12, Proposition 1.1.11]. We let this functor be denoted by $M_{\mathfrak{S}_{\infty}}$ (resp. $M_{\mathfrak{S}}$).

Proposition 3.6. *Let \mathfrak{M} be an object of $\text{Mod}_{\mathfrak{S}_{\infty}}^{r, \phi}$ which is killed by p^n . Set $M = M_{\mathfrak{S}_{\infty}}(\mathfrak{M})$ and $\mathcal{M} = M_{\mathfrak{S}_{\infty}}(\mathfrak{M})$. Then there exists a natural isomorphism of G_{K_n} -modules*

$$\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})) \rightarrow \text{Hom}_{\mathfrak{S}, \text{Fil}^r, \phi_r}(\mathcal{M}, W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})).$$

Proof. By definition, $\mathcal{M} = S \otimes_{\Sigma} M$ and we have a natural isomorphism

$$\mathrm{Hom}_{\Sigma}(M, W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})) \rightarrow \mathrm{Hom}_S(\mathcal{M}, W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})).$$

Let f be an element of $\mathrm{Hom}_{\Sigma, \mathrm{Fil}^r, \phi_r}(M, W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}))$ and f' be the image of f in the right-hand side of the above isomorphism. Let us check that f' preserves Fil^r and commutes with ϕ_r . Since f' is S -linear, it maps $\mathrm{Fil}^p S \cdot \mathcal{M}$ into $\mathrm{Fil}^p W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$. For $x \in \mathrm{Fil}^r \mathcal{M} \cap \mathrm{Im}(M \rightarrow \mathcal{M})$, the commutative diagram whose right vertical arrow is an isomorphism

$$\begin{array}{ccccc} M = \Sigma \otimes_{\phi, \mathfrak{S}} \mathfrak{M} & \xrightarrow{1 \otimes \phi} & \Sigma \otimes_{\mathfrak{S}} \mathfrak{M} & \longrightarrow & \Sigma / \mathrm{Fil}^r \Sigma \otimes_{\mathfrak{S}} \mathfrak{M} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M} = S \otimes_{\phi, \mathfrak{S}} \mathfrak{M} & \xrightarrow{1 \otimes \phi} & S \otimes_{\mathfrak{S}} \mathfrak{M} & \longrightarrow & S / \mathrm{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \end{array}$$

implies $x \in \mathrm{Im}(\mathrm{Fil}^r M \rightarrow \mathrm{Fil}^r \mathcal{M})$ and thus $f'(x) \in \mathrm{Fil}^r W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$.

As for the compatibility with ϕ_r , again by the S -linearity of f' it suffices to show $f'(\phi_r(x)) = \phi_r(f'(x))$ for $x \in \mathrm{Fil}^r \mathcal{M} \cap \mathrm{Im}(M \rightarrow \mathcal{M}) = \mathrm{Im}(\mathrm{Fil}^r M \rightarrow \mathrm{Fil}^r \mathcal{M})$. This follows from the commutative diagram

$$\begin{array}{ccccc} \mathrm{Fil}^r M & \xrightarrow{\phi_r} & M & \xrightarrow{f} & W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \\ \downarrow & & \downarrow & & \parallel \\ \mathrm{Fil}^r \mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M} & \xrightarrow{f'} & W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}). \end{array}$$

Hence the map in the proposition is well-defined and injective. To prove the bijectivity, by devissage we may assume that $p\mathfrak{M} = 0$. Then both sides of this injection have the same cardinality by the above remark. Thus the proposition follows. \square

4. A METHOD OF ABRASHKIN

In this section, we study the G_{K_n} -module $\mathrm{Hom}_{\Sigma, \mathrm{Fil}^r, \phi_r}(M, W_n^{\mathrm{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}))$ following Abrashkin ([2]).

Let $p \geq 3$ and $0 \leq r < p - 1$ be as before. Consider the Lubin-Tate logarithm

$$l(X) = X + \frac{X^p}{p} + \cdots + \frac{X^{p^n}}{p^n} + \cdots$$

and put $\psi(X) = l^{-1}(\log(1 + X))$. Then ψ gives a strict isomorphism of formal groups between the formal group associated to the logarithm $l(X)$ and the multiplicative group \hat{G}_m over \mathbb{Z}_p . We fix a system of p -power roots of unity $\{\zeta_{p^n}\}_{n \in \mathbb{Z}_{\geq 0}}$ such that $\zeta_p \neq 1$ and $\zeta_{p^n} = \zeta_{p^{n+1}}^p$ for any n , and set an element $\underline{\varepsilon}$ of R to be $(\zeta_{p^n})_{n \in \mathbb{Z}_{\geq 0}}$. Then the elements $[\underline{\varepsilon}] - 1$ and $[\underline{\varepsilon}^{1/p}] - 1$ are topologically nilpotent in $\bar{W}(R)$ and the element of $W(R)$

$$t = \psi([\underline{\varepsilon}] - 1) / \psi([\underline{\varepsilon}^{1/p}] - 1)$$

is a generator of the principal ideal $\text{Ker}(\theta)$. The element $Z = \psi([\varepsilon] - 1)^{p-1}/p$ of A_{crys} is topologically nilpotent and $\phi(t)$ is contained in the subset

$$p(1 + ZW(R)[[Z]])$$

of A_{crys} ([2, Subsection 1.8]). We set

$$\hat{A} = W(R)[[Z]] \subseteq A_{\text{crys}}.$$

Lemma 4.1. *The element t^p/p of A_{crys} is contained in the subring \hat{A} and topologically nilpotent in this subring.*

Proof. Put $t' = ([\varepsilon] - 1)/([\varepsilon^{1/p}] - 1)$. This is another generator of $\text{Ker}(\theta)$. We have

$$\frac{([\varepsilon] - 1)^{p-1}}{p} = \frac{(t')^{p-1}}{p} \cdot ([\varepsilon^{1/p}] - 1)^{p-1}$$

and $\theta([\varepsilon^{1/p}] - 1) = \zeta_p - 1$. Take an element $a \in W(R)^\times$ such that $\theta(a) = (\zeta_p - 1)^{p-1}/p$. Then we have

$$\frac{([\varepsilon] - 1)^{p-1}}{p} = a(t')^{p-1} + b(t')^p/p$$

for some $b \in W(R)^\times$. Indeed, to show $b \in W(R)^\times$, it suffices to check that the element $([\varepsilon^{1/p}] - 1)^{p-1} - pa$ of $\text{Ker}(\theta)$ also generates this ideal. This follows from the fact that the 0-th entry $(\varepsilon^{1/p} - 1)^{p-1}$ of this element satisfies $v_R((\varepsilon^{1/p} - 1)^{p-1}) = 1$. Then we see that $(t')^p/p$ is topologically nilpotent because so is t' in $W(R)$. \square

In the following, we set the element a in the proof of the lemma to be

$$a = \sum_{k=1}^{p-2} p^{-1}((-1)^{p-1-k} {}_{p-1}C_k - 1)[\varepsilon^{1/p}]^k,$$

where ${}_{p-1}C_k = (p-1)!/(k!(p-1-k)!)$ is the binomial coefficient. Note that the coefficient of $[\varepsilon^{1/p}]^k$ in each term is an integer.

From this lemma, we can consider the ring \hat{A} as a Σ -algebra by $u \mapsto [\pi]$. Put $\text{Fil}^i \hat{A} = (t^i, Z)$ for $0 \leq i \leq p-1$. The Frobenius endomorphism ϕ of A_{crys} preserves \hat{A} and satisfies $\phi(\text{Fil}^i \hat{A}) \subseteq p^i \hat{A}$ for $0 \leq i \leq p-1$. Set $\phi_r = p^{-r} \phi|_{\text{Fil}^r \hat{A}}$. Then we can consider the ring \hat{A} also as an object of the category $'\text{Mod}'_{\Sigma}^{r, \phi}$, and similarly for $\hat{A}_n = \hat{A}/p^n \hat{A}$ and $\hat{A}_\infty = \hat{A} \otimes_W K_0/W$. The absolute Galois group G_{K_∞} acts naturally on these Σ -algebras. The following lemma is used implicitly in [2].

Lemma 4.2. *We have a natural decomposition*

$$\hat{A}_1 = R/(t^p) \oplus (Z).$$

Proof. Consider the natural inclusion $W(R) \rightarrow \hat{A}$. First we claim that this induces an injection $R/(t^p) \rightarrow \hat{A}_1$. Let x be in the ring R . If the element

$[x] \in W(R)$ is contained in $p\hat{A}$, then its image in $A_{\text{crys}}/pA_{\text{crys}}$ is zero. We have an isomorphism of R -algebras

$$R[Y_1, Y_2, \dots]/(t^p, Y_1^p, Y_2^p, \dots) \rightarrow A_{\text{crys}}/pA_{\text{crys}}$$

which sends Y_i to the image of $t^{p^i}/(p^i)!$. Thus the inequality $v_R(x) \geq p$ holds. Conversely, if $v_R(x) \geq p$, then we have

$$[x] = w(\psi([\varepsilon] - 1)^{p-1}) + pw'$$

for some $w, w' \in W(R)$ and this implies $[x] \in p\hat{A}$.

Let us consider the commutative diagram of R -algebras

$$\begin{array}{ccc} R/(t^p) & \longrightarrow & \hat{A}_1 \\ & \searrow & \downarrow \\ & & \hat{A}_1/(Z). \end{array}$$

By definition, the left downward arrow is surjective. We claim that this arrow is an isomorphism. Indeed, let x be in the kernel of this surjection. From the proof of Lemma 4.1, we see that the image of Z in the ring on the left-hand side of the above isomorphism can be written as $a't^{p-1} + b'Y_1$ for some $a', b' \in R^\times$. By assumption, in this ring, we have

$$x = c_1(a't^{p-1} + b'Y_1) + c_2(a't^{p-1} + b'Y_1)^2 + \dots + c_{p-1}(a't^{p-1} + b'Y_1)^{p-1}$$

for some elements c_1, \dots, c_{p-1} of R . Then we see that $c_i = 0$ for any i and $v_R(x) \geq p$. This concludes the proof. \square

Since $r < p - 1$, from this lemma we can show the following lemma as in the proof of [3, Lemme 2.3.1.3].

Lemma 4.3. *The functor*

$$M \mapsto \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, \hat{A}_\infty)$$

from $\text{Mod}_{/\Sigma_\infty}^{r, \phi}$ to the category of G_{K_∞} -modules is exact.

Corollary 4.4. *For any $M \in \text{Mod}_{/\Sigma_\infty}^{r, \phi}$, the natural map*

$$\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, \hat{A}_\infty) \rightarrow \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, A_{\text{crys}, \infty})$$

is an isomorphism of G_{K_∞} -modules.

Proof. By Lemma 3.2 and Lemma 4.3, we may assume $pM = 0$. Consider the commutative diagram of rings

$$\begin{array}{ccc} \hat{A}_1 & \longrightarrow & A_{\text{crys}}/pA_{\text{crys}} \\ & \searrow & \downarrow \\ & & R/(t^{p-1}) \end{array}$$

whose downward arrows are defined by modulo Fil^{p-1} of the rings \hat{A}_1 and $A_{\text{crys}}/pA_{\text{crys}}$, respectively. Since $r < p - 1$, we have $\phi_r(\text{Fil}^{p-1}\hat{A}_1) = 0$ and similarly for the ring $A_{\text{crys}}/pA_{\text{crys}}$. Thus these two surjections induce on the ring $R/(t^{p-1})$ the same structure of a filtered ϕ_r -module over Σ . Hence, as in the proof of Corollary 3.5, we see from Lemma 3.1 that we have a commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, \hat{A}_1) & \longrightarrow & \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, A_{\text{crys}}/pA_{\text{crys}}) \\ & \searrow & \downarrow \\ & & \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, R/(t^{p-1})) \end{array}$$

whose downward arrows are isomorphisms. This concludes the proof. \square

We sketch the proof of the following lemma stated in [2, Subsection 3.2].

Lemma 4.5. *The natural inclusion $W(R) \rightarrow \hat{A}$ induces an isomorphism of $W(R)$ -algebras $W_n(R)/(\psi([\underline{\varepsilon}] - 1)^{p-1}) \rightarrow \hat{A}_n/(Z)$.*

Proof. For a subring B of A_{crys} , put

$$I^{[s]}B = \{x \in B \mid \phi^i(x) \in \text{Fil}^s A_{\text{crys}} \text{ for any } i\}$$

as in [9, Subsection 5.3]. Then we have $I^{[s]}W(R) = ([\underline{\varepsilon}] - 1)^s W(R)$ and the natural ring homomorphism

$$W(R)/I^{[s]}W(R) \rightarrow A_{\text{crys}}/I^{[s]}A_{\text{crys}}$$

is an injection ([9, Proposition 5.1.3, Proposition 5.3.5]). Since the element Z is contained in the ideal $I^{[p-1]}A_{\text{crys}}$, this injection factors as

$$W(R)/I^{[p-1]}W(R) \rightarrow \hat{A}/(Z) \rightarrow A_{\text{crys}}/I^{[p-1]}A_{\text{crys}}.$$

Hence the former arrow is an isomorphism and the lemma follows. \square

Since the ideal (Z) of \hat{A}_n satisfies the condition of Corollary 3.5, the Σ -algebra $\hat{A}_n/(Z)$ is naturally considered as an object of $'\text{Mod}_{\Sigma}^{r, \phi}$. We also give the ring $W_n(R)/(\psi([\underline{\varepsilon}] - 1)^{p-1})$ the structures of a Σ -algebra and a filtered ϕ_r -module over Σ induced by those of $\hat{A}_n/(Z)$. The map

$$\Sigma \rightarrow W_n(R)/(\psi([\underline{\varepsilon}] - 1)^{p-1})$$

sends the element $u \in \Sigma$ to the image of $[\underline{\pi}]$ in the ring on the right-hand side. Put $v = t'/E([\underline{\pi}]) \in W(R)^\times$ with the notation of Lemma 4.1. As for the element $Y \in \Sigma$, the equality

$$Y = -ab^{-1}v^{-1}E([\underline{\pi}])^{p-1} + wb^{-1}v^{-p}Z$$

holds in \hat{A} , where a and b are the elements in $W(R)^\times$ as in the proof of Lemma 4.1 and the remark after this lemma, and $w \in W(R)^\times$ is a power series of $[\underline{\varepsilon}] - 1$. Hence the above homomorphism sends the element Y to the image of $-ab^{-1}v^{-1}E([\underline{\pi}])^{p-1}$.

Consider the surjective ring homomorphism

$$\begin{aligned} R &\rightarrow \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}} \\ x = (x_0, x_1, \dots) &\mapsto x_n \end{aligned}$$

and the induced surjection $W_n(R) \rightarrow W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$. Let

$$J = \{(x_0, \dots, x_{n-1}) \in W_n(R) \mid v_R(x_i) \geq p^n \text{ for any } i\}$$

be the kernel of the latter surjection.

Lemma 4.6. *The ideal J is contained in the ideal $(\psi([\underline{\varepsilon}] - 1)^{p-1})$ of the ring $W_n(R)$.*

Proof. Write $([\underline{\varepsilon}] - 1)^{p-1}$ also as $x = (x_0, \dots, x_{n-1}) \in W_n(R)$ with $v_R(x_0) = p$. Take an element $z = (z_0, \dots, z_{n-1})$ in the ideal J . We construct $y \in W_n(R)$ such that $xy = z$. By induction, it is enough to show that if $z_0 = \dots = z_{i-1} = 0$ for some $0 \leq i \leq n-1$ and $(x_0, \dots, x_i)(0, \dots, 0, y_i) = (0, \dots, 0, z_i)$ in $W_{i+1}(R)$, then $x(0, \dots, 0, y_i, 0, \dots, 0) \in J$. Let us write this element as $(0, \dots, 0, w_i, \dots, w_{n-1})$ with $w_i = z_i$. We have $v_R(y_i) \geq p^n - p^{i+1}$. In the ring of Witt vectors $W_n(\mathbb{F}_p[X_0, \dots, X_{n-1}, Y_0, \dots, Y_{n-1}])$, the k -th entry of the vector

$$(X_0, \dots, X_{n-1})(0, \dots, 0, Y_i, 0, \dots, 0)$$

is $X_{k-i}^{p^i} Y_i^{p^{k-i}}$ for any $k \geq i$. Thus we have $v_R(w_k) \geq p^n$. \square

Note that the elements $[\zeta_{p^n}] - 1$ and $[\zeta_{p^{n+1}}] - 1$ is nilpotent in $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$. By the above lemma, we have an isomorphism of rings

$$W_n(R)/(\psi([\underline{\varepsilon}] - 1)^{p-1}) \rightarrow W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})/(\psi([\zeta_{p^n}] - 1)^{p-1}).$$

We give the ring on the right-hand side the structure of a filtered ϕ_r -module over Σ induced by this isomorphism.

Put $F_n = K_n(\zeta_{p^{n+1}})$. For an algebraic extension F of F_n , let us consider the ideals

$$\begin{aligned} \bar{m}_{n,F} &= \{(x_0, \dots, x_{n-1}) \in W_n(\mathcal{O}_F/p\mathcal{O}_F) \mid x_i \in m_F/p\mathcal{O}_F \text{ for any } i\} \\ m_{n,F} &= \{(x_0, \dots, x_{n-1}) \in W_n(\mathcal{O}_F) \mid x_i \in m_F \text{ for any } i\} \end{aligned}$$

of $W_n(\mathcal{O}_F/p\mathcal{O}_F)$ and $W_n(\mathcal{O}_F)$, respectively. The elements $[\zeta_{p^n}] - 1$ and $[\zeta_{p^{n+1}}] - 1$ are topologically nilpotent in $W_n(\mathcal{O}_F)$ and we define an element $\hat{t} \in W_n(\mathcal{O}_F)$ to be

$$\hat{t} = \psi([\zeta_{p^n}] - 1)/\psi([\zeta_{p^{n+1}}] - 1).$$

Note that these elements are non-zero divisors of $W_n(\mathcal{O}_F)$. Let the ring

$$W_n(\mathcal{O}_F/p\mathcal{O}_F)/\psi([\zeta_{p^n}] - 1)^r \bar{m}_{n,F}$$

be denoted by $\bar{A}_{n,F,r+}$. We also put $\bar{m}_n = \bar{m}_{n,\bar{K}}$, $m_n = m_{n,\bar{K}}$ and $\bar{A}_{n,r+} = \bar{A}_{n,\bar{K},r+}$.

For an algebraic extension F of K , we put

$$\mathfrak{b}_F = \{x \in \mathcal{O}_F \mid v_K(x) > er/(p-1)\}.$$

Note that the ring $\mathcal{O}_F/\mathfrak{b}_F$ is killed by p . When F contains F_n , we also put

$$\bar{A}'_{n,F,r+} = W_n(\mathcal{O}_F/\mathfrak{b}_F)/\psi([\zeta_{p^n}] - 1)^r \bar{m}_{n,F}.$$

Then, for $0 \leq r < p - 1$, we have natural isomorphisms of rings

$$W_n(\mathcal{O}_F)/\psi([\zeta_{p^n}] - 1)^r m_{n,F} \rightarrow \bar{A}_{n,F,r+} \rightarrow \bar{A}'_{n,F,r+}.$$

Indeed, as in the proof of Lemma 4.6, we can show that both of the kernels of the maps $W_n(\mathcal{O}_F) \rightarrow W_n(\mathcal{O}_F/p\mathcal{O}_F)$ and $W_n(\mathcal{O}_F) \rightarrow W_n(\mathcal{O}_F/\mathfrak{b}_F)$ are contained in the ideal $\psi([\zeta_{p^n}] - 1)^r m_{n,F}$ of the ring $W_n(\mathcal{O}_F)$. We often identify these rings. We also put $\bar{A}'_{n,r+} = \bar{A}'_{n,\bar{K},r+}$.

Write Z_n for the image of the element Z of A_{crys} in $W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$. Then we have a commutative diagram of Σ -algebras

$$\begin{array}{ccc} & \hat{A}_n & \longrightarrow & A_{\text{crys}}/p^n A_{\text{crys}} \\ & \downarrow & & \downarrow \wr \\ & & & W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \\ W_n(R)/(\psi([\zeta_{p^n}] - 1)^{p-1}) & \xrightarrow{\sim} & \hat{A}_n/(Z) & \longrightarrow & W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})/(Z_n), \\ \downarrow \wr & & & & \downarrow \\ W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})/(\psi([\zeta_{p^n}] - 1)^{p-1}) & \longrightarrow & & & \\ \downarrow & & & & \\ \bar{A}_{n,r+} & & & & \\ \downarrow \wr & & & & \\ \bar{A}'_{n,r+} & & & & \end{array}$$

where all vertical arrows are surjections satisfying the condition of Corollary 3.5. Thus we see that this is also a commutative diagram in $'\text{Mod}_{/\Sigma}^{r,\phi}$. Note that these rings and homomorphisms are independent of the choice of a system $\{\zeta_{p^n}\}_{n \in \mathbb{Z}_{\geq 0}}$. Moreover, let \hat{M} be in $\text{Mod}_{/\Sigma}^{r,\phi}$ and put $M_n = \hat{M}/p^n \hat{M}$. Then, by Corollary 3.5 and Corollary 4.4, we have a natural isomorphism of abelian groups

$$\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M_n, W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})) \simeq \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M_n, \bar{A}_{n,r+}).$$

To study G_{F_n} -actions on both sides of this isomorphism, we need the following proposition.

Proposition 4.7. *The image of the element $Y \in \Sigma$ in the ring $\bar{A}_{n,r+}$ is contained in its subring $\bar{A}_{n,F_n,r+}$.*

Proof. We have the equality

$$([\varepsilon^{1/p}] - 1)^{p-1} = pa - E([\pi])bv$$

in $W(R)$. By definition, we see that the images of the elements a , $[\varepsilon^{1/p}]$ and $E([\pi])$ in $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ are contained in the subring $W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$. Write β for the product $bv \in W(R)$. Let a_n , β_n and α_n denote the images of the elements a , β and $pa - ([\varepsilon^{1/p}] - 1)^{p-1}$ in the ring $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$, respectively. Then the element α_n is also contained in the subring $W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$. Now we have the equality

$$E([\pi_n])\beta_n = \alpha_n.$$

Note that any element $\beta'_n \in W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ satisfying the same equality is invertible and thus the elements $(\beta'_n)^{-1}E([\pi_n])$ are equal to each other. Since $Y = -a_n\beta_n^{-1}E([\pi_n])^{p-1}$ in $\bar{A}_{n,r+}$, it suffices to construct an element β'_n in the ring $W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$ such that the equality $E([\pi_n])\beta'_n = \alpha_n$ holds. Take a lift $\hat{\alpha}_n$ of α_n in $W_n(\mathcal{O}_{F_n})$. Since $E([\pi_n]) \in W_n(\mathcal{O}_{\bar{K}})$ divides every element in the kernel of the surjection $W_n(\mathcal{O}_{\bar{K}}) \rightarrow W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$, we have $E([\pi_n])\hat{\beta}'_n = \hat{\alpha}_n$ for some $\hat{\beta}'_n \in W_n(\mathcal{O}_{\bar{K}})$. Then the element $\hat{\beta}'_n$ is contained in the subring $W_n(\mathcal{O}_{F_n})$ and we set β'_n to be the image of $\hat{\beta}'_n$.

By a similar argument, we can also check that the ring $\bar{A}_{n,F,r+}$ is a subring of $\bar{A}_{n,r+}$ and coincides with the image of $W_n(\mathcal{O}_F)$ in $\bar{A}_{n,r+}$. This concludes the proof. \square

Let t_n and \bar{t}_n be the images of $t \in W(R)$ in $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ and $\bar{A}_{n,r+}$ (or $\bar{A}'_{n,r+}$), respectively. Then \hat{t} is a lift of t_n and \bar{t}_n to $W_n(\mathcal{O}_{\bar{K}})$ by the natural surjections

$$W_n(\mathcal{O}_{\bar{K}}) \rightarrow W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \rightarrow \bar{A}_{n,r+}.$$

Note that we defined the filtration of $\bar{A}_{n,r+}$ as $\text{Fil}^r \bar{A}_{n,r+} = \bar{t}_n^r \bar{A}_{n,r+}$.

Lemma 4.8. *Let \bar{x} be in $\text{Fil}^r \bar{A}_{n,r+}$. Then we have*

$$\phi_r(\bar{x}) = \phi(y) \bmod \psi([\zeta_{p^n}] - 1)^r \bar{m}_n,$$

where y is any element of $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ such that the element $t_n^r y$ is a lift of \bar{x} . In particular, the right-hand side of the above equality is independent of the choice of a system $\{\zeta_{p^n}\}_{n \in \mathbb{Z}_{\geq 0}}$. Similar assertions also hold for the ring

$$W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})/(\psi([\zeta_{p^n}] - 1)^{p-1}).$$

Proof. Since the filtered ϕ_r -module structure of $\bar{A}_{n,r+}$ is induced from that of \hat{A} and $\phi_r(t^r) \equiv 1 \bmod Z$ in \hat{A} , we see that there exists an element y as in the lemma.

To prove the independence of the choice of a lift, let $z = (z_0, \dots, z_{n-1})$ be an element of $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ killed by t_n . The element z is also killed by

$p - [p_n]$, where $p_k = p^{1/p^k}$. This implies

$$\begin{cases} p_n z_0 \in p\mathcal{O}_{\bar{K}} \\ z_0^p + p_{n-1} z_1 \in p\mathcal{O}_{\bar{K}} \\ \vdots \\ z_{n-2}^p + p_1 z_{n-1} \in p\mathcal{O}_{\bar{K}} \end{cases}$$

and $v_p(z_k) \geq 1 - 1/p^{n-k}$ for $0 \leq k \leq n-1$. Repeating this, we see that z is killed by t_n^r , then $v_p(z_k) \geq 1 - r/p^{n-k}$. For such an element z , we have $\phi(z) = 0$ in the ring $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$.

Let y_1 and y_2 be elements as in the lemma. Then we have

$$y_1 - y_2 = \psi([\zeta_{p^{n+1}}] - 1)^r w + z,$$

where $w \in \bar{m}_n$ and z is an element as above. The Frobenius endomorphism ϕ sends the element on the right-hand side to an element which is contained in the ideal $\psi([\zeta_{p^n}] - 1)^r \bar{m}_n$. Thus the assertions for the ring $\bar{A}_{n,r+}$ follows. We can show the assertion for the ring $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})/(\psi([\zeta_{p^n}] - 1)^{p-1})$ similarly. \square

From this lemma and Proposition 4.7, we see that the natural G_{F_n} -actions on the rings $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})/(\psi([\zeta_{p^n}] - 1)^{p-1})$, $\bar{A}_{n,r+}$ and $\bar{A}'_{n,r+}$ are compatible with the filtered ϕ_r -module structures over Σ . In the commutative diagram above, the lowest horizontal arrow and lower right vertical arrow are G_{F_n} -linear by definition. Hence we have shown the following proposition.

Proposition 4.9. *Let \hat{M} be in $\text{Mod}_{\Sigma}^{r,\phi}$ and put $M_n = \hat{M}/p^n \hat{M}$. Then we have an isomorphism of G_{F_n} -modules*

$$\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M_n, \bar{A}_{n,r+}) \simeq \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M_n, W_n^{\text{PD}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})).$$

Let e_1, \dots, e_d be a basis of \hat{M} as in Lemma 3.3 and $C = (c_{i,j}) \in M_d(\Sigma)$ be the associated matrix representing ϕ_r as in Corollary 3.4. Then the underlying G_{F_n} -set of the G_{F_n} -module

$$\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M_n, \bar{A}_{n,r+})$$

is identified with the set of d -tuples $(\bar{x}_1, \dots, \bar{x}_d)$ in $\bar{A}_{n,r+}$ such that $c_{1,i}\bar{x}_1 + \dots + c_{d,i}\bar{x}_d \in \text{Fil}^r \bar{A}_{n,r+}$ for any i and the following equality holds:

$$(1) \quad \begin{cases} \phi_r(c_{1,1}\bar{x}_1 + \dots + c_{d,1}\bar{x}_d) = \bar{x}_1 \\ \vdots \\ \phi_r(c_{1,d}\bar{x}_1 + \dots + c_{d,d}\bar{x}_d) = \bar{x}_d. \end{cases}$$

We choose a lift $(\hat{c}_{i,j}) \in M_d(W_n(\mathcal{O}_{F_n}))$ of the image of C in $M_d(\bar{A}_{n,r+})$ by the natural ring homomorphism

$$W_n(\mathcal{O}_{\bar{K}}) \rightarrow W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) \rightarrow \bar{A}_{n,r+}.$$

Fix a polynomial $\Phi_i \in \mathbb{Z}[X_0, \dots, X_{n-1}]$ such that $\Phi_i \equiv X_i^p \pmod{p}$. This induces for any commutative ring B a map $\Phi = (\Phi_0, \dots, \Phi_{n-1}) : W_n(B) \rightarrow W_n(B)$ which is a lift of the Frobenius endomorphism on $W_n(B/pB)$. In

particular, set B to be the polynomial ring $\mathbb{Z}[X_0, \dots, X_{n-1}, Y_0, \dots, Y_{n-1}]$. Put $X = (X_0, \dots, X_{n-1})$ and $Y = (Y_0, \dots, Y_{n-1})$ in the ring $W_n(B)$. Then we see that there exists elements U_0, \dots, U_{n-1} and U'_0, \dots, U'_{n-1} of the polynomial ring B such that

$$\begin{aligned}\Phi(X + Y) &= \Phi(X) + \Phi(Y) + (pU_0, \dots, pU_{n-1}), \\ \Phi(XY) &= \Phi(X)\Phi(Y) + (pU'_0, \dots, pU'_{n-1})\end{aligned}$$

in the ring $W_n(B)$.

Proposition 4.10. *Every solution $(\bar{x}_1, \dots, \bar{x}_d)$ in $\bar{A}_{n,r+}$ of the equation (1) such that $c_{1,i}\bar{x}_1 + \dots + c_{d,i}\bar{x}_d \in \text{Fil}^r \bar{A}_{n,r+}$ for any i uniquely lifts to a d -tuple $(\hat{x}_1, \dots, \hat{x}_d)$ in $W_n(\mathcal{O}_{\bar{K}})$ such that $\hat{c}_{1,i}\hat{x}_1 + \dots + \hat{c}_{d,i}\hat{x}_d \in \hat{t}^r W_n(\mathcal{O}_{\bar{K}})$ for any i and the following equality holds:*

$$(2) \quad \begin{cases} \Phi((\hat{c}_{1,1}\hat{x}_1 + \dots + \hat{c}_{d,1}\hat{x}_d)/\hat{t}^r) = \hat{x}_1 \\ \vdots \\ \Phi((\hat{c}_{1,d}\hat{x}_1 + \dots + \hat{c}_{d,d}\hat{x}_d)/\hat{t}^r) = \hat{x}_d. \end{cases}$$

Proof. Fix a lift \hat{x}_i of \bar{x}_i to $W_n(\mathcal{O}_{\bar{K}})$. Then we have

$$\begin{cases} \Phi((\hat{c}_{1,1}\hat{x}_1 + \dots + \hat{c}_{d,1}\hat{x}_d)/\hat{t}^r) = \hat{x}_1 + ([\zeta_{p^n}] - 1)^r \hat{\delta}_1 \\ \vdots \\ \Phi((\hat{c}_{1,d}\hat{x}_1 + \dots + \hat{c}_{d,d}\hat{x}_d)/\hat{t}^r) = \hat{x}_d + ([\zeta_{p^n}] - 1)^r \hat{\delta}_d \end{cases}$$

for some $\hat{\delta}_1, \dots, \hat{\delta}_d \in m_n$. It suffices to show that there exists a unique d -tuple $(\hat{y}_1, \dots, \hat{y}_d)$ in m_n such that

$$\begin{aligned}\Phi((\hat{c}_{1,i}(\hat{x}_1 + ([\zeta_{p^n}] - 1)^r \hat{y}_1) + \dots + \hat{c}_{d,i}(\hat{x}_d + ([\zeta_{p^n}] - 1)^r \hat{y}_d))/\hat{t}^r) \\ = \hat{x}_i + ([\zeta_{p^n}] - 1)^r \hat{y}_i\end{aligned}$$

for any i . For this, we need the following lemma.

Lemma 4.11. *Let N be a complete discrete valuation field and m_N be the maximal ideal of N . Let $\epsilon_1, \dots, \epsilon_d$ be in m_N . Let P_1, \dots, P_d and P'_1, \dots, P'_d be elements of $\mathcal{O}_N[[Y_1, \dots, Y_d]]$ such that $P_i \in (Y_1, \dots, Y_d)^2$. Then the equation*

$$\begin{cases} Y_1 - P_1(Y_1, \dots, Y_d) - \epsilon_1 P'_1(Y_1, \dots, Y_d) = 0 \\ \vdots \\ Y_d - P_d(Y_1, \dots, Y_d) - \epsilon_d P'_d(Y_1, \dots, Y_d) = 0 \end{cases}$$

has a unique solution in m_N .

Proof. By assumption, we see that for any integer $l \geq 1$, a d -tuple $y = (y_1, \dots, y_d)$ in m_N/m_N^l satisfying the above equation lifts uniquely to a d -tuple in m_N/m_N^{l+1} satisfying the same equation. Thus the lemma follows. \square

Let us write as $\hat{y}_i = (\hat{y}_{i,0}, \dots, \hat{y}_{i,n-1})$. Since the image of $\Phi(([\zeta_{p^{n+1}}] - 1)^r)$ in $\bar{A}_{n,r+}$ is divisible by $([\zeta_{p^n}] - 1)^r$, we can find $\hat{b} \in W_n(\mathcal{O}_{\bar{K}})$ such that

$$\Phi(([\zeta_{p^n}] - 1)^r / \hat{t}^r) = ([\zeta_{p^n}] - 1)^r \hat{b}.$$

Then there exists polynomials $U_{i,m}$ over $\mathcal{O}_{\bar{K}}$ of the indeterminates $\underline{Y} = (Y_{i,m})_{1 \leq i \leq d, 0 \leq m \leq n-1}$ such that the equation we have to solve is

$$\begin{aligned} \hat{x}_i + ([\zeta_{p^n}] - 1)^r \hat{y}_i &= \hat{x}_i + ([\zeta_{p^n}] - 1)^r \hat{\delta}_i \\ &+ ([\zeta_{p^n}] - 1)^r \hat{b} (\Phi(\hat{c}_{i,1})\Phi(\hat{y}_1) + \dots + \Phi(\hat{c}_{i,d})\Phi(\hat{y}_d)) \\ &+ (pU_{i,0}(\hat{y}), \dots, pU_{i,n-1}(\hat{y})) \end{aligned}$$

for any i , where we put $\hat{y} = (\hat{y}_{i,m})_{1 \leq i \leq d, 0 \leq m \leq n-1}$. Note that, for any elements P_0, \dots, P_{n-1} of the polynomial ring $\mathcal{O}_{\bar{K}}[\underline{Y}]$, we can uniquely find elements Q_0, \dots, Q_{n-1} of this ring such that the coefficients of these polynomials are in the maximal ideal $m_{\bar{K}}$ and the equality

$$(pP_0, \dots, pP_{n-1}) = ([\zeta_{p^n}] - 1)^r (Q_0, \dots, Q_{n-1})$$

holds in the ring of Witt vectors $W_n(\mathcal{O}_{\bar{K}}[\underline{Y}])$. Therefore, this equation is equivalent to the equation

$$\begin{aligned} \hat{y}_i &= \hat{\delta}_i + \hat{b} (\Phi(\hat{c}_{i,1})\Phi(\hat{y}_1) + \dots + \Phi(\hat{c}_{i,d})\Phi(\hat{y}_d)) \\ &+ (V_{i,0}(\hat{y}), \dots, V_{i,n-1}(\hat{y})), \end{aligned}$$

where $V_{i,m}$ is a polynomial of \underline{Y} over $\mathcal{O}_{\bar{K}}$ whose coefficients are in the maximal ideal $m_{\bar{K}}$. From the definition of Φ , we see that the elements $\hat{y}_{i,m}$ is a solution of a system of equations

$$Y_{i,m} - P_{i,m}(\underline{Y}) - \epsilon_{i,m} P'_{i,m}(\underline{Y}) = 0$$

satisfying the condition of Lemma 4.11 for a sufficiently large finite extension N of K . Then, by this lemma, we can solve the equation uniquely in $m_{\bar{K}}$. \square

Let F be an algebraic extension of $F_n = K_n(\zeta_{p^{n+1}})$ and consider the ring $\bar{A}_{n,F,r+}$. By Lemma 4.7, we can consider this ring as a Σ -algebra and also as an object of ${}^r\text{Mod}_{\Sigma}^{\phi}$ by putting $\text{Fil}^r \bar{A}_{n,F,r+} = \bar{t}_n^r \bar{A}_{n,F,r+}$ and for $\bar{x} \in \text{Fil}^r \bar{A}_{n,F,r+}$,

$$\phi_r(\bar{x}) = \phi(y) \bmod \psi([\zeta_{p^n}] - 1)^r \bar{m}_{n,F},$$

where y is any element of $W_n(\mathcal{O}_F/p\mathcal{O}_F)$ such that the element $t_n^r y$ is a lift of \bar{x} . For $M_n = \hat{M}/p^n \hat{M} \in \text{Mod}_{\Sigma_{\infty}}^{\phi}$ as before, let us set

$$T_{\text{crys}, \pi_n, F}^*(M_n) = \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M_n, \bar{A}_{n,F,r+}).$$

We see that

$$\bar{A}_{n,r+} = \bar{A}_{n,\bar{K},r+} = \bigcup_{F/F_n} \bar{A}_{n,F,r+}$$

in $'\text{Mod}'_{/\Sigma}^{r,\phi}$ and thus we have a natural identification of abelian groups

$$T_{\text{crys},\pi_n,\bar{K}}^*(M_n) = \bigcup_{F/F_n} T_{\text{crys},\pi_n,F}^*(M_n).$$

The absolute Galois group G_{F_n} acts on the abelian group on the left-hand side.

Lemma 4.12. *For an algebraic extension F of F_n , the fixed part $T_{\text{crys},\pi_n,\bar{K}}^*(M_n)^{G_F}$ is equal to $T_{\text{crys},\pi_n,F}^*(M_n)$.*

Proof. From Proposition 4.10, we see that the elements of $T_{\text{crys},\pi_n,\bar{K}}^*(M_n)$ correspond bijectively to the solutions of the equation (2) in $W_n(\mathcal{O}_{\bar{K}})$ satisfying the condition on Fil^r . The uniqueness assertion of this proposition shows that $g \in G_F$ fixes a solution in $W_n(\mathcal{O}_{\bar{K}})$ if and only if g fixes its image in $\bar{A}_{n,r,+}$. Hence a solution is fixed by G_F if and only if this solution is contained in the image of $W_n(\mathcal{O}_F)$. Thus the lemma follows. \square

Corollary 4.13. *Let L_n be the finite Galois extension of F_n corresponding to the kernel of the map $G_{F_n} \rightarrow \text{Aut}(T_{\text{crys},\pi_n,\bar{K}}^*(M_n))$. Then an algebraic extension F of F_n contains L_n if and only if $\#T_{\text{crys},\pi_n,F}^*(M_n) = \#T_{\text{crys},\pi_n,\bar{K}}^*(M_n)$.*

Proof. An algebraic extension F of F_n contains L_n if and only if the action of G_F on $T_{\text{crys},\pi_n,\bar{K}}^*(M_n)$ is trivial. By Lemma 4.12, this is equivalent to $T_{\text{crys},\pi_n,F}^*(M_n) = T_{\text{crys},\pi_n,\bar{K}}^*(M_n)$. \square

5. RAMIFICATION BOUND

In this section, we prove Theorem 1.1. Take G_K -stable \mathbb{Z}_p -lattices $\mathcal{L} \supseteq \mathcal{L}'$ in V such that $\mathcal{L}' \supseteq p^n \mathcal{L}$. Since the G_K -module \mathcal{L}/\mathcal{L}' is a quotient of $\mathcal{L}/p^n \mathcal{L}$, we may assume $\mathcal{L}' = p^n \mathcal{L}$. If $r = 0$, then the G_K -module V is unramified and the theorem is trivial. Thus we may assume $r \geq 1$ and $p \geq 3$. Let L be the finite Galois extensions of K corresponding to the kernel of the map

$$G_K \rightarrow \text{Aut}(\mathcal{L}/p^n \mathcal{L}).$$

It is enough to show that, for the greatest upper ramification break $u_{L(\zeta_p)/K}$ of the Galois extension $L(\zeta_p)/K$, the inequality

$$u_{L(\zeta_p)/K} \leq u(K, r, n)$$

holds. Since the Herbrand function is transitive and the finite Galois extension $K(\zeta_p)$ is tamely ramified over K , we may assume $\zeta_p \in K$. We fix a uniformizer π of K and a system $\{\pi_n\}_{n \in \mathbb{Z}_{\geq 0}}$ as before. Then, by Liu's theorem ([14, Theorem 2.3.5]), it suffices to show the following.

Theorem 5.1. *Let r be an integer such that $1 \leq r < p - 1$ and $\hat{\mathcal{M}}$ be the strongly divisible lattice corresponding to \mathcal{L} . Put $\mathcal{M}_n = \hat{\mathcal{M}}/p^n \hat{\mathcal{M}} \in \text{Mod}'_{/S_\infty}^{r,\phi,N}$. Then $G_K^{(j)}$ acts trivially on the G_K -module $T_{\text{st},\pi}^*(\mathcal{M}_n)$ for $j > u(K, r, n)$.*

Let L_n be the finite Galois extension of $F_n = K_n(\zeta_{p^{n+1}})$ corresponding to the kernel of the map

$$G_{F_n} \rightarrow \text{Aut}(T_{\text{st}, \pi}^*(\mathcal{M}_n)).$$

Since F_n is Galois over K , the extension L_n is also a Galois extension of K . Let $\hat{\mathfrak{M}}$ be the object of the category $\text{Mod}_{\mathfrak{S}}^{r, \phi}$ such that $\mathcal{M}_{\mathfrak{S}}(\hat{\mathfrak{M}}) \simeq \hat{\mathcal{M}}$. From Proposition 3.6 and Proposition 4.9, we see that L_n is also the finite extension of F_n cut out by the G_{F_n} -module $T_{\text{crys}, \pi_n, \bar{K}}^*(M_n)$ for $M_n = M_{\mathfrak{S}}(\hat{\mathfrak{M}})/p^n M_{\mathfrak{S}}(\hat{\mathfrak{M}})$. It is enough to prove the inequality

$$u_{L_n/K} \leq u(K, r, n) = \begin{cases} 1 + e(n + \frac{1}{p-1}) & (r = 1), \\ \frac{p^n - 1}{p^n} + e(n + \frac{r}{p-1}) & (r > 1). \end{cases}$$

Before proving this, we state some general lemmas to calculate the ramification bound. Let N be a complete discrete valuation field of positive residue characteristic, v_N be its valuation normalized as $v_N(N^\times) = \mathbb{Z}$ and N^{sep} be its separable closure.

Lemma 5.2. *Let $f(T) \in \mathcal{O}_N[T]$ be a separable monic polynomial and z_1, \dots, z_d be the zeros of f in $\mathcal{O}_{N^{\text{sep}}}$. Suppose that the set $\{v_N(z_k - z_i) \mid k = 1, \dots, d, k \neq i\}$ is independent of i . Put*

$$s = \sum_{\substack{k=1, \dots, d \\ k \neq i}} v_N(z_k - z_i) \text{ and } \alpha = \sup_{\substack{k=1, \dots, d \\ k \neq i}} v_N(z_k - z_i),$$

which are independent of i by assumption. If $j > s + \alpha$, then we have the decomposition

$$\{x \in \mathcal{O}_{N^{\text{sep}}} \mid v_N(f(x)) \geq j\} = \prod_{i=1, \dots, d} \{x \in \mathcal{O}_{N^{\text{sep}}} \mid v_N(x - z_i) \geq j - s\}.$$

Otherwise, the set on the left-hand side contains

$$\{x \in \mathcal{O}_{N^{\text{sep}}} \mid v_N(x - z_i) \geq \alpha\},$$

which contains at least two zeros of f .

Proof. A verbatim argument in the proof of [1, Lemma 6.6] shows the claim. \square

Corollary 5.3. *Let $f(T)$ be as above and put $B = \mathcal{O}_N[T]/(f(T))$. Suppose that the algebra B is finite flat and of relative complete intersection over \mathcal{O}_N . Let us write the N -algebra $N' = B \otimes_{\mathcal{O}_N} N$ as the product $N_1 \times \dots \times N_t$ of finite separable extensions N_1, \dots, N_t of N . If $j > s + \alpha$, then the j -th upper numbering ramification group ([1]), which we let be denoted by $G_N^{(j)}$, is contained in G_{N_i} for any i . Moreover, if N' is a field and B coincides with $\mathcal{O}_{N'}$, then $j > s + \alpha$ if and only if $G_N^{(j)} \subseteq G_{N'}$.*

Proof. From the previous lemma, the conductor $c(B)$ of the \mathcal{O}_N -algebra B ([1, Proposition 6.4]) is equal to $s + \alpha$. Thus we have the inequality

$$c(\mathcal{O}_{N_1} \times \cdots \times \mathcal{O}_{N_t}) \leq c(B) = s + \alpha$$

by the definition of the conductor and a functoriality of the functor \mathcal{F}^j defined in [1]. This implies the corollary. \square

Corollary 5.4. *Consider the finite Galois extension $F_n = K_n(\zeta_{p^{n+1}})$ of K and let $u_{F_n/K}$ denote the greatest upper ramification break of F_n/K . Then we have the equality*

$$u_{F_n/K} = 1 + e(n + \frac{1}{p-1}).$$

Proof. Note that we are assuming that ζ_p is contained in K . Applying the previous corollary to the Eisenstein polynomial $f(T) = T^{p^n} - \pi$ shows that $j > 1 + e(n + 1/(p-1))$ if and only if $G_K^{(j)} \subseteq G_{K_n}$. Similarly, putting $f(T) = T^{p^n} - \zeta_p$, we see that if $j > e(n + 1/(p-1))$, then $G_K^{(j)} \subseteq G_{K(\zeta_{p^{n+1}})}$. Since $G_{F_n} = G_{K_n} \cap G_{K(\zeta_{p^{n+1}})}$, we conclude that $j > 1 + e(n + 1/(p-1))$ if and only if $G_K^{(j)} \subseteq G_{F_n}$. \square

Remark 5.5. Note that this argument also shows the equality

$$u_{K_n(\zeta_{p^n})/K} = 1 + e(n + \frac{1}{p-1})$$

without assuming $\zeta_p \in K$.

Next we assume that the residue field of N is perfect. For an algebraic extension F of N , we put

$$\mathfrak{a}_{F/N}^j = \{x \in \mathcal{O}_F \mid v_N(x) \geq j\}.$$

For a finite Galois extension Q of N , we write $u_{Q/N}$ for the greatest upper ramification break ([7]) of Q/N . Let us consider the property

$$(P_j) \begin{cases} \text{for any algebraic extension } F \text{ of } N, \text{ if there exists} \\ \text{an } \mathcal{O}_N\text{-algebra homomorphism } \mathcal{O}_Q \rightarrow \mathcal{O}_F/\mathfrak{a}_{F/N}^j, \\ \text{then there exists an } N\text{-algebra injection } Q \rightarrow F \end{cases}$$

for $j \in \mathbb{R}_{\geq 0}$, as in [7, Proposition 1.5]. Then we have the following proposition, which is due to Yoshida. Here we reproduce his proof for the convenience of the reader.

Proposition 5.6 ([16]).

$$u_{Q/N} = \inf\{j \in \mathbb{R}_{\geq 0} \mid \text{the property } (P_j) \text{ holds}\}.$$

Proof. By [7, Proposition 1.5 (i)], it is enough to show that the property (P_j) does not hold for $j = u_{Q/N} - (e')^{-1}$ with an arbitrarily large $e' > 0$. As in the proof of [7, Proposition 1.5 (ii)], we may assume that Q is totally and wildly ramified over N . Let N' be a finite tamely ramified Galois

extension of N such that $Q \cap N' = N$ and put $Q' = QN'$. Note that we have $u_{Q'/N} = u_{Q/N}$ by assumption. From this proposition in [7], we see that for some algebraic extension F of N , there exists an \mathcal{O}_N -algebra homomorphism $\mathcal{O}_{Q'} \rightarrow \mathcal{O}_F/\mathfrak{a}_{F/N}^j$ for $j = u_{Q/N} - e(Q'/N)^{-1}$ but no N -algebra injection $Q' \rightarrow F$. Since Q/N is wildly ramified, we see that $e(Q/N)u_{Q/N} - 1 > e(Q/N)$. Hence we have $u_{Q/N} - e(Q'/N)^{-1} > 1 = u_{N'/N}$ and there exists an N -algebra injection $N' \rightarrow F$ also by this proposition. Thus there exists no N -algebra injection $Q \rightarrow F$ and the property (P_j) for Q/N does not hold. Since we can choose an arbitrarily large N' as above, the proposition follows. \square

We see from Proposition 5.6 that to bound the greatest upper ramification break $u_{L_n/K}$, it is enough to show the following proposition.

Proposition 5.7. *Let F be an algebraic extension of K . If $j > u(K, r, n)$ and there exists an \mathcal{O}_K -algebra homomorphism*

$$\eta : \mathcal{O}_{L_n} \rightarrow \mathcal{O}_F/\mathfrak{a}_{F/K}^j,$$

then there exists a K -algebra injection $L_n \rightarrow F$.

Proof. By assumption, we have $j > er/(p-1)$ and $\mathfrak{b}_F \supseteq \mathfrak{a}_{F/K}^j$. Thus η induces an \mathcal{O}_K -algebra homomorphism

$$\mathcal{O}_{L_n} \rightarrow \mathcal{O}_F/\mathfrak{b}_F.$$

Since η also induces an \mathcal{O}_K -algebra homomorphism $\mathcal{O}_{F_n} \rightarrow \mathcal{O}_F/\mathfrak{a}_{F/K}^j$ and $r \geq 1$, from Corollary 5.4 and [7, Proposition 1.5] we get a K -linear injection $F_n \rightarrow F$. Thus we see that F contains π_n and $\zeta_{p^{n+1}}$. More precisely, we have the following lemma.

Lemma 5.8. *For some integers i and i' such that $i' \equiv 1 \pmod{p}$, we have $\eta(\pi_n) \equiv \pi_n \zeta_{p^n}^i \pmod{\mathfrak{b}_F}$ and $\eta(\zeta_{p^{n+1}}) \equiv \zeta_{p^{n+1}}^{i'} \pmod{\mathfrak{b}_F}$. Moreover, there exists $g \in G_K$ such that $g(\pi_n) = \pi_n \zeta_{p^n}^i$ and $g(\zeta_{p^{n+1}}) = \zeta_{p^{n+1}}^{i'}$.*

Proof. Since the map η is \mathcal{O}_K -linear, the equality $\eta(\pi_n)^{p^n} = \pi$ holds in $\mathcal{O}_F/\mathfrak{a}_{F/K}^j$. Set \hat{x} to be a lift of $\eta(\pi_n)$ in \mathcal{O}_F . Then we have

$$v_K(\hat{x}^{p^n} - \pi) = \sum_{i=0}^{p^n-1} v_K(\hat{x} - \pi_n \zeta_{p^n}^i) \geq j.$$

Let us apply Lemma 5.2 to $f(T) = T^{p^n} - \pi$. Then, with the notation of the lemma, we have

$$s = ne + \frac{p^n - 1}{p^n} \text{ and } \alpha = \frac{1}{p^n} + \frac{e}{p-1}.$$

Since $j - s > er/(p-1)$ by assumption, we have

$$\hat{x} \equiv \pi_n \zeta_{p^n}^i \pmod{\mathfrak{b}_F}$$

for some i .

Let $h(T)$ be the minimal polynomial of $\zeta_{p^{n+1}}$ over \mathcal{O}_K . Since h divides $T^{p^n} - \zeta_p$, the \mathcal{O}_K -algebra $B' = \mathcal{O}_K[T]/(h(T))$ is also finite flat of relative complete intersection and the K -algebra $B' \otimes_{\mathcal{O}_K} K$ is étale. The Galois group $\text{Gal}(K(\zeta_{p^{n+1}})/K)$ acts transitively on the set of zeros of h . Hence h also satisfies the conditions of Lemma 5.2. Let s' and α' be as in this lemma for h . Then we have $s' \leq ne$ and $\alpha' \leq e/(p-1)$. This implies $j - s' > er/(p-1)$. By this lemma, there exists an element $g' \in \text{Gal}(K(\zeta_{p^{n+1}})/K)$ such that the element $g'(\zeta_{p^{n+1}}) = \zeta_{p^{n+1}}^{i'}$ satisfies

$$\eta(\zeta_{p^{n+1}}) \equiv g'(\zeta_{p^{n+1}}) \pmod{\mathfrak{b}_F}.$$

Since $K_n \cap K(\zeta_{p^{n+1}}) = K$ (see for example [14, Lemma 5.1.2]), we can find an element $g \in G_K$ such that $g(\pi_n) = \pi_n \zeta_{p^n}^i$ and $g(\zeta_{p^{n+1}}) = g'(\zeta_{p^{n+1}})$. This concludes the proof. \square

Lemma 5.9. *The \mathcal{O}_K -algebra homomorphism η induces an \mathcal{O}_K -algebra injection*

$$\eta_{\mathfrak{b}} : \mathcal{O}_{L_n}/\mathfrak{b}_{L_n} \rightarrow \mathcal{O}_F/\mathfrak{b}_F.$$

Proof. We write the Eisenstein polynomial of a uniformizer π_{L_n} of L_n over \mathcal{O}_K as

$$P(T) = T^{e'} + c_1 T^{e'-1} + \cdots + c_{e'-1} T + c_{e'},$$

where $e' = e(L_n/K)$. Then $z = \eta(\pi_{L_n})$ satisfies $P(z) = 0$ in $\mathcal{O}_F/\mathfrak{a}_{F/K}^j$. Let \hat{z} be a lift of z in \mathcal{O}_F . Since $j > 1$, we have $v_F(\hat{z}) = e(F/K)/e'$. The condition $i > e(L_n)r/(p-1)$ is equivalent to the condition

$$v_F(\hat{z}^i) > \frac{e(L_n)r}{p-1} \cdot \frac{e(F/K)}{e'} = \frac{e(F)r}{p-1}.$$

Thus the claim follows. \square

Since L_n contains F_n , we can consider the ring

$$\bar{A}'_{n,L_n,r+} = W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n})/\psi([\zeta_{p^n}] - 1)^r \bar{m}_{n,L_n}$$

and similarly $\bar{A}'_{n,F,r+}$ for F . We give these rings structures of Σ -algebras as follows. The ring $\bar{A}'_{n,L_n,r+}$ is considered as a Σ -algebra by using the system $\{\pi_n\}_{n \in \mathbb{Z}_{\geq 0}}$ we chose of p -power roots of π , as in the previous section. On the other hand, using i and i' in Lemma 5.8, put $\tilde{\pi}_n = \pi_n \zeta_{p^n}^i$ and $\tilde{\zeta}_{p^{n+1}} = \zeta_{p^{n+1}}^{i'}$. Then we consider the ring $\bar{A}'_{n,F,r+}$ as a Σ -algebra by using a system of p -power roots of π containing $\tilde{\pi}_n$. We define Fil^r and ϕ_r of these rings in the same way as before.

Lemma 5.10. *The induced ring homomorphism*

$$\bar{\eta} : \bar{A}'_{n,L_n,r+} \rightarrow \bar{A}'_{n,F,r+}$$

is a morphism of the category ${}^r\text{Mod}_{\Sigma}^{\phi}$.

Proof. Firstly, we check that $\bar{\eta}$ is Σ -linear. By definition, this homomorphism commutes with the action of the element $u \in \Sigma$. To show the compatibility with the element $Y \in \Sigma$, let us consider the commutative diagram

$$\begin{array}{ccc} W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n}) & \longrightarrow & W_n(\mathcal{O}_F/p\mathcal{O}_F) \\ \downarrow & & \downarrow \\ W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n}) & \xrightarrow{\eta_{\mathfrak{b}}} & W_n(\mathcal{O}_F/\mathfrak{b}_F) \\ \downarrow & & \downarrow \\ \bar{A}'_{n,L_n,r+} & \xrightarrow{\bar{\eta}} & \bar{A}'_{n,F,r+}, \end{array}$$

where the horizontal arrows are induced by η . Note that we have $\eta_{\mathfrak{b}}(\pi_n) = \tilde{\pi}_n$ and $\eta_{\mathfrak{b}}(\zeta_{p^{n+1}}) = \tilde{\zeta}_{p^{n+1}}$. Put $\beta \in W(R)^\times$ as in the proof of Proposition 4.7. Namely, the element β is the solution in $W(R)$ of the equation

$$E([\underline{\pi}])\beta = pa - ([\underline{\pi}^{1/p}] - 1)^{p-1},$$

where the element $a \in W(R)$ is as in the remark after Lemma 4.1. Let a_n and β_n denote the images of a and β in $W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n})$, respectively. Then the element β_n is a solution of the equation

$$E([\pi_n])\beta_n = pa_n - ([\zeta_{p^{n+1}}] - 1)^{p-1}.$$

Similarly, we define elements \tilde{a}_n and $\tilde{\beta}_n$ of $W_n(\mathcal{O}_F/p\mathcal{O}_F)$ using $\tilde{\pi}_n$ and $\tilde{\zeta}_{p^{n+1}}$. By definition, the element $\tilde{\beta}_n$ is a solution of the equation

$$E([\tilde{\pi}_n])\tilde{\beta}_n = p\tilde{a}_n - ([\tilde{\zeta}_{p^{n+1}}] - 1)^{p-1}.$$

Now what we have to show is the equality

$$\bar{\eta}(a_n\beta_n^{-1}E([\pi_n])^{p-1}) = \tilde{a}_n\tilde{\beta}_n^{-1}E([\tilde{\pi}_n])^{p-1}$$

in the ring $\bar{A}'_{n,F,r+}$. Since the element a_n of $W_n(\mathcal{O}_{L_n}/p\mathcal{O}_{L_n})$ is a linear combination of the elements $1, [\zeta_{p^{n+1}}], \dots, [\zeta_{p^{n+1}}]^{p-1}$ over \mathbb{Z} , we have $\bar{\eta}(a_n) = \tilde{a}_n$ in $\bar{A}'_{n,F,r+}$. Thus the elements $\tilde{\beta}_n$ and $\bar{\eta}(\beta_n)$ satisfy the same equation in $\bar{A}'_{n,F,r+}$. Since these two elements are invertible, we see that $\bar{\eta}(\beta_n)^{-1}E([\tilde{\pi}_n]) = \tilde{\beta}_n^{-1}E([\tilde{\pi}_n])$ and the equality holds. Since the diagram above is compatible with the Frobenius endomorphisms, we see from the definition that $\bar{\eta}$ also preserves Fil^r and commutes with ϕ_r of both sides. \square

Thus the homomorphism $\bar{\eta}$ induces a homomorphism of abelian groups

$$T_{\text{crys},L_n,\pi_n}^*(M_n) \rightarrow T_{\text{crys},F,\tilde{\pi}_n}^*(M_n).$$

Then the following lemma, whose proof is omitted in [2, Subsection 3.13], implies that this homomorphism is an injection. We insert here a proof of this lemma for the convenience of the reader.

Lemma 5.11. *The ring homomorphism $\bar{\eta} : \bar{A}'_{n,L_n,r+} \rightarrow \bar{A}'_{n,F,r+}$ is an injection.*

Proof. For an algebraic extension N of F_n , let us write \bar{A}'_N for the ring $\bar{A}'_{n,N,r+}$. Note that the ring $\bar{A}'_N/p\bar{A}'_N$ is isomorphic to the ring

$$\mathcal{O}_N/\{x \in \mathcal{O}_N \mid v_p(x) > \frac{r}{p^{n-1}(p-1)}\}.$$

As in the proof of Lemma 5.9, we see that the homomorphism $\bar{\eta}$ induces an injection

$$\bar{A}'_{L_n}/p\bar{A}'_{L_n} \rightarrow \bar{A}'_F/p\bar{A}'_F.$$

Thus it is enough to show the exactness of the sequence

$$0 \rightarrow \bar{A}'_N/p^m\bar{A}'_N \xrightarrow{\times p} \bar{A}'_N/p^{m+1}\bar{A}'_N \rightarrow \bar{A}'_N/p\bar{A}'_N \rightarrow 0.$$

Let \bar{x} and \bar{y} be in \bar{A}'_N such that $p\bar{x} = p^{m+1}\bar{y}$. Let $\hat{x} = (\hat{x}_0, \dots, \hat{x}_{n-1})$ and $\hat{y} = (\hat{y}_0, \dots, \hat{y}_{n-1})$ be lifts of \bar{x} and \bar{y} in the ring $W_n(\mathcal{O}_N)$, respectively. In this ring, we have

$$(0, \hat{x}_0^p, \hat{x}_1^p, \dots) = (0, \dots, 0, \hat{y}_0^{p^{m+1}}, \hat{y}_1^{p^{m+1}}, \dots) + ([\zeta_{p^n}] - 1)^r \hat{z},$$

where \hat{z} is in the ideal $m_{n,N}$. From this equality we see that $v_p(\hat{x}_0) > r/(p^{n-1}(p-1))$ and

$$\hat{x} = ([\zeta_{p^n}] - 1)^r \hat{w} + (0, \hat{x}'_1, \hat{x}'_2, \dots)$$

for some $\hat{w} \in m_{n,N}$ and $\hat{x}'_i \in \mathcal{O}_N$. The image of the first term on the right-hand side in \bar{A}'_N is zero. Hence we may assume $\hat{x}_0 = 0$. Repeating this, we can see that $\bar{x} \in p^m\bar{A}'_N$ and the above sequence is exact. \square

Now Corollary 4.13 shows that the abelian group $T_{\text{crys}, L_n, \pi_n}^*(M_n)$ is of order p^{nd} , where $d = \dim_{\mathbb{Q}_p} V$. This implies that the the abelian group $T_{\text{crys}, F, \tilde{\pi}_n}^*(M_n)$ is also of order p^{nd} . Let $g \in G_K$ be as in Lemma 5.8. Then we have the following lemma.

Lemma 5.12. *The G_{F_n} -action on $T_{\text{crys}, \bar{K}, \tilde{\pi}_n}^*(M_n)$ is the conjugate of the action on $T_{\text{crys}, \bar{K}, \pi_n}^*(M_n)$ by the element g .*

Proof. Let a_n, \tilde{a}_n and $\beta_n, \tilde{\beta}_n$ be the elements of $W_n(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ as in the proof of Lemma 5.10. Let us consider the composite

$$\Sigma \rightarrow \bar{A}'_{n,r+} \xrightarrow{g} \bar{A}'_{n,r+}$$

of the ring homomorphism defined by $u \mapsto [\pi_n]$ and $Y \mapsto -a_n\beta_n^{-1}E([\pi_n])^{p-1}$, and the map induced by g . We claim that this is the natural ring homomorphism defined by $\tilde{\pi}_n$. For this, we only have to check that this composite sends the element $Y \in \Sigma$ to $-\tilde{a}_n\tilde{\beta}_n^{-1}E([\tilde{\pi}_n])$. Since the equality

$$E([\pi_n])\beta_n = pa_n - ([\zeta_{p^{n+1}}] - 1)^{p-1}$$

holds in the ring $\bar{A}'_{n,r+}$ on the source of the above map g , we have

$$E([\tilde{\pi}_n])g(\beta_n) = p\tilde{a}_n - ([\tilde{\zeta}_{p^{n+1}}] - 1)^{p-1}$$

in the ring $\bar{A}'_{n,r+}$ on the target. Since the elements $g(\beta_n)$ and $\tilde{\beta}_n$ are invertible, we have $g(\beta_n)^{-1}E([\tilde{\pi}_n]) = \tilde{\beta}_n^{-1}E([\tilde{\pi}_n])$ and the claim follows. Thus we have an isomorphism of abelian groups

$$\begin{aligned} \mathrm{Hom}_{\Sigma}(M_n, \bar{A}'_{n,r+}) &\rightarrow \mathrm{Hom}_{\Sigma}(M_n, \bar{A}'_{n,r+}) \\ f &\mapsto g \circ f, \end{aligned}$$

where we consider on the ring $\bar{A}'_{n,r+}$ on the right-hand side the filtered ϕ_r -module structure over Σ defined by $\tilde{\pi}_n$. Since $g(t_n) = \tilde{t}_n$, we can check that this isomorphism induces an injection

$$\mathrm{Hom}_{\Sigma, \mathrm{Fil}^r, \phi_r}(M_n, \bar{A}'_{n,r+}) \rightarrow \mathrm{Hom}_{\Sigma, \mathrm{Fil}^r, \phi_r}(M_n, \bar{A}'_{n,r+}).$$

Since these abelian groups have the same cardinality, this is also an isomorphism. \square

Since L_n is Galois over K , the above lemma shows that the finite Galois extension of F_n cut out by the action on $T_{\mathrm{crys}, \bar{K}, \tilde{\pi}_n}^*(M_n)$ is also L_n . Hence we see from Corollary 4.13 that F also contains L_n and Proposition 5.7 follows. This concludes the proof of Theorem 1.1. \square

Remark 5.13. The ramification bound in Theorem 1.1 is sharp for $r \leq 1$. Indeed, the greatest upper ramification break $1 + e(n + 1/(p - 1))$ for $r = 1$ is obtained by the p^n -torsion of the Tate curve $\bar{K}^{\times}/\pi^{\mathbb{Z}}$ (see Remark 5.5). The author does not know whether this bound is sharp also for $r > 1$.

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E-mail address: `shin-h@math.sci.hokudai.ac.jp`

DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY