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<td>Citation</td>
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<tr>
<td>Hokkaido University Preprint Series in Mathematics, 920, 1-21</td>
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<tr>
<td>Issue Date</td>
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<td>DOI</td>
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<td>10.14943/84069</td>
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Pairs of foliations on timelike surfaces in the de Sitter space $S^3_1$

Shyuichi Izumiya and Farid Tari

June 18, 2008

Abstract

We define in this paper the asymptotic, characteristic and principal directions associated to the de Sitter Gauss map on a smooth timelike surface $M$ in the de Sitter space $S^3_1$. We study their properties and determine the local topological configurations of their integral curves. These curves form pairs of foliations on some regions of $M$ and are defined in an analogous way to their classical counterpart on surfaces in the Euclidean 3-space. However, we show that their behaviour is distinct from that of their analogue on surface in the Euclidean 3-space.

1 Introduction

Let $M$ be a smooth and oriented surface in the Euclidean space $\mathbb{R}^3$. The shape operator $S$ (or the Weingarten map) is a self-adjoint operator defined on each tangent plane of $M$ and describes the shape of $M$ in $\mathbb{R}^3$. It also determines on $M$ three pairs of foliations as follows. At each point $p \in M$, the shape operator $S_p$ is a symmetric operator on $T_p M$. As this vector space inherits the Euclidean scalar product $\cdot$, it has a basis of orthonormal vectors given by the eigenvectors of $S_p$. The directions of these vectors are called principal directions and their integral curves the lines of principal curvature. These form a pair of foliation on $M$ away from umbilic points. A direction $u \in T_p M$ is asymptotic if $S_p(u).u = 0$. There are two asymptotic directions at each hyperbolic point and their integral curves are called the asymptotic curves. At an elliptic point there is a unique pair of conjugate directions for which the included
angle is extremal ([10]). These directions are called the characteristic directions and their integral curves the characteristic curves.

The above three pairs of foliations are given, in a local chart, by binary differential equations (BDEs), also known as quadratic differential equations. These are equations in the form

\[ a(u, v)dv^2 + 2b(u, v)dvdv + c(u, v)du^2 = 0, \]

where the coefficients \( a, b, c \) are smooth functions on some open set \( U \subset \mathbb{R}^2 \) (here smooth means \( C^\infty \)). A BDE defines no directions at points where \( \delta(u, v) = (b^2 - ac)(u, v) < 0 \), two directions at points in the region where \( \delta(u, v) > 0 \), and a double direction at points on the set \( \Delta = \{(u, v) \in U : \delta(u, v) = 0\} \) when the coefficients of the equation do not all vanish at a given point. At such points, every direction is a solution of the BDE. The set \( \Delta \) is called the discriminant of the equation and the function \( \delta \) the discriminant function. BDEs are well studied; see for example [9, 21] for references.

It is shown in [3] that the equations of the asymptotic, characteristic and principal curves are related. A BDE can be viewed as a quadratic form and represented at each point in the plane by a point in the projective plane. If \( \Gamma \) denotes the set of degenerate quadratic forms, then the asymptotic, characteristic and principal BDEs represent a self-polar triangle with respect to \( \Gamma \) ([3]). In particular, any two of them determine the third one. (See [20] for a generalisation of these results.)

We consider in this paper timelike surfaces \( M \) in the de Sitter space \( S^3_1 \). Given \( p \in M \), there is a well defined unit normal vector \( e(p) \) to \( M \) at \( p \) ([12]), see §2. (If \( M \) is orientable, then \( e(p) \) is globally defined. However, it is always locally defined and our investigation here is local in nature.) The vector \( e(p) \) is in the de Sitter space \( S^3_1 \) and we have the de Sitter Gauss map

\[ \mathbb{E} : M \to S^3_1 \]

\[ p \to e(p) \]

The Weingarten map \(-d\mathbb{E}_p\) is a self-adjoint operator on \( T_pM \) ([12]). However, as \( M \) is timelike, the restriction of the pseudo-scalar product in the Minkowski space to \( T_pM \) is also a pseudo scalar product. Therefore \(-d\mathbb{E}_p\) does not always have real eigenvalues. We can still define the concepts of principal/asymptotic/characteristic directions at \( p \) using \(-d\mathbb{E}_p\) and the pseudo-scalar product on \( T_pM \). The difference here to the Euclidean case is that we may not have principal directions (which are along the eigenvectors of \(-d\mathbb{E}_p\)) at every point on \( M \).

The principal/asymptotic/characteristic directions determine pairs of foliations on some regions of the surface. We study in this paper the local behaviour of these foliations. We deal with the lines of principal curvature in §3, the asymptotic curves in §4 and the characteristic curves in §5. We carry out in the Appendix (§6) a topological classification of BDEs with 1-jet equivalent to \((ax + y)dy^2 \pm 2xdxdy\) and with discriminant having an \( A_3\)-singularity. The models in §6 are used to determine the
configuration of the characteristic curves at timelike umbilic points. For the figures in this paper, we draw one foliation determined by a given BDE in blue and the other in red. The discriminant is drawn in thick black.

The results in this paper are true for self-adjoint operators defined on a two-dimensional Laurentzian manifold (compare with [20]). This is the case, for example, for 2-dimensional timelike surfaces in the Minkowski 3-space ([16]) and in the Anti de Sitter 4-space ([8]). In each of these cases a self-adjoint operator is derived from the differential of a well defined Gauss map.

2 Preliminaries

The Minkowski space \((\mathbb{R}^4, \langle, \rangle)\) is the 4-dimensional vector space \(\mathbb{R}^4\) endowed with the pseudo scalar product \(\langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^{3} x_i y_i\), where \(x = (x_0, x_1, x_2, x_3)\) and \(y = (y_0, y_1, y_2, y_3)\) in \(\mathbb{R}^4\). We say that a vector \(x\) in \(\mathbb{R}^4 \setminus \{0\}\) is spacelike if \(\langle x, x \rangle > 0\), lightlike if \(\langle x, x \rangle = 0\), timelike if \(\langle x, x \rangle < 0\).

The norm of a vector \(x \in \mathbb{R}^4\) is defined by \(\|x\| = \sqrt{\langle x, x \rangle}\). Given a vector \(v \in \mathbb{R}^4\) and a real number \(c\), the hyperplane with pseudo normal \(v\) is defined by

\[ HP(v, c) = \{ x \in \mathbb{R}^4 \mid \langle x, v \rangle = c \}. \]

We say that \(HP(v, c)\) is a spacelike, timelike or lightlike hyperplane if \(v\) is timelike, spacelike or lightlike respectively. We also say that a two dimensional vector space is spacelike if all its vectors are spacelike, timelike if it has a spacelike and a timelike vector and lightlike otherwise.

The de Sitter space is the pseudo-sphere \(S^3_1 = \{ x \in \mathbb{R}^4 \mid \langle x, x \rangle = 1 \}\).

The wedge product of three vectors \(a_1, a_2, a_3 \in \mathbb{R}^4\) is given by

\[ a_1 \wedge a_2 \wedge a_3 = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ a_0^1 & a_1^1 & a_2^1 & a_3^1 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 \end{vmatrix}, \]

where \(\{e_0, e_1, e_2, e_3\}\) is the canonical basis of \(\mathbb{R}^4\) and \(a_i = (a_i^0, a_i^1, a_i^2, a_i^3), i = 1, 2, 3\). We have \(\langle a, a_1 \wedge a_2 \wedge a_3 \rangle = \det(a, a_1, a_2, a_3)\), so the vector \(a_1 \wedge a_2 \wedge a_3\) is pseudo orthogonal to all the vectors \(a_i, i = 1, 2, 3\), that is, \(\langle a_i, a_1 \wedge a_2 \wedge a_3 \rangle = 0\) for \(i = 1, 2, 3\).

Definition 2.1 A smooth surface \(M\) embedded in \(S^3_1\) is timelike if its tangent space \(T_p M\) at any point \(p \in M\) is a timelike vector space.

Given a local chart \(i : U \to M\), where \(U\) is an open subset of \(\mathbb{R}^2\), we denote by \(x : U \to S^3_1\) such embedding, identify \(x(U)\) with \(U\) through the embedding \(x\) and write \(M = x(U)\). The first fundamental form of \(M\) at a point \(p\) is the quadratic
form $I_p : T_p M \to \mathbb{R}$ given by $I_p(v) = \langle v, v \rangle$. If $v = ax_u + bx_v \in T_p M$, then $I_p(v) = Ea^2 + 2Fab +Gb^2$, where

$$E = \langle x_u, x_u \rangle, \quad F = \langle x_u, x_v \rangle, \quad G = \langle x_v, x_v \rangle$$

are the coefficients of the first fundamental form at $p$. Because $M$ is timelike, we have $EG - F^2 < 0$, so at any point $p \in M$ there are two lightlike directions in $T_p M$. These are the solutions of $I_p(v) = 0$.

The lightlike directions can be given locally on $x(U)$ by two smooth direction fields. One can show the following (the proof is identical to the Euclidean case; see for example [7] page 216).

**Theorem 2.2** At any point $p \in M$, there is a local parametrisation of a neighbourhood $V$ of $p$, such that for any $p' \in V$, the coordinate curves through $p'$ are tangent to the lightlike directions. Equivalently, there exist a local parametrisation $x : U \to V \subset M$ with $E = G = 0$ in $U$.

Since $\langle x, x \rangle \equiv 1$, we have $\langle x_u, x \rangle \equiv 0$ and $\langle x_v, x \rangle \equiv 0$. Let $q = (u, v) \in U$. We define the spacelike unit normal vector $e(q)$ to $M$ at $p = x(q)$ by

$$e(q) = \frac{x(q) \wedge x_u(q) \wedge x_v(q)}{\|x(q) \wedge x_u(q) \wedge x_v(q)\|}.$$ 

This de Sitter Gauss map of $M$ is the map

$$E : U \to S^3_1$$

$q \mapsto e(q)$

(see [12]). For any $p = x(q) \in M$ and $v \in T_p M$, one can show that $D_v E \in T_p M$, where $D_v$ denotes the covariant derivative with respect to the tangent vector $v$.

The linear transformation $A_p = -dE(q)$ is called the de Sitter shape operator. One can show that this is a self-adjoint operator, that is, a linear and symmetric operator on $T_p M$.

**Remark 2.3** There is a crucial fact here. The surface $M$ is timelike, so the restriction of the pseudo scalar product in $\mathbb{R}_1^4$ to $M$ is a pseudo scalar product. Therefore $A_p$ does not always have real eigenvalues.

The second fundamental form of $M$ at a point $p$ is the quadratic form $\Pi_p : T_p M \to \mathbb{R}$ given by $\Pi_p(v) = \langle A_p(v), v \rangle$. If $v = ax_u + bx_v \in T_p M$, then $\Pi_p(v) = la^2 + 2mab + nb^2$, where

$$l = -\langle e_u, x_u \rangle = \langle e, x_{uu} \rangle$$
$$m = -\langle e_u, x_v \rangle = \langle e, x_{uv} \rangle = \langle e, x_{vu} \rangle = -\langle e_v, x_u \rangle$$
$$n = -\langle e_v, x_v \rangle = \langle e, x_{vv} \rangle$$
are the coefficients of the second fundamental form at \( p \). The matrix of the de Sitter shape operator \( A_p \) with respect to the basis \( \{ x_u, x_v \} \) is given by

\[
A_p = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} l \\ m \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}
\]

We call

\[
K(p) = \det(A_p) = \frac{ln - m^2}{EG - F^2}
\]

the (de Sitter) Gauss-Kronecker curvature of \( M \) at \( p \). The set of points where this curvature vanishes is labeled the (de Sitter) parabolic set of \( M \). We also call

\[
H(p) = \frac{lG - 2mF + nE}{2(EG - F^2)}
\]

the (de Sitter) mean curvature of \( M \) at \( p \).

We say that a property is generic if it holds for a residual set of embeddings of \( M \) in \( S^3 \).

We consider here topological equivalence between BDEs and say that two BDEs are topologically equivalent if there is a local homeomorphism in the plane taking the integral curves of one to those of the other.

### 3 Lines of principal curvature

Following Remark 2.3, the de Sitter shape operator \( A_p \) does not always have real eigenvalues. When it does, we denote by \( \kappa_i, i = 1, 2 \), these eigenvalues and call them the (de Sitter) principal curvature. Then \( K(p) = \kappa_1(p)\kappa_2(p) \) and \( H(p) = \frac{1}{2}(\kappa_1(p) + \kappa_2(p)) \). The eigenvectors of \( A_p \) are called the (de Sitter) principal directions.

The equation for the lines of principal curvature (which are the integral curves of the line fields determined by the principal directions) is given by the same formula of their contrepart for surfaces in the Euclidean 3-space, namely by

\[
\begin{vmatrix}
    dv^2 & -dvdu & du^2 \\
    E & F & G \\
    l & m & n
\end{vmatrix} = 0,
\]

equivalently,

\[
(Gm - Fn)dv^2 + (Gl - En)dvdu + (Fl - Em)du^2 = 0.
\] (1)

The discriminant function of this equation is

\[
\delta(u, v) = ((Gl - En)^2 - 4(Gm - Fn)(Fl - Em))(u, v).
\]
When $\delta(u, v) > 0$, there are two distinct principal directions at $\mathbf{x}(u, v)$. These coincide at points where $\delta(u, v) = 0$. There are no principal directions at points where $\mathbf{x}(u, v) < 0$. We labeled in [14] the locus of points where $\delta(u, v) = 0$ the *Lightlike Principal Locus* (LPL). We have the following from [14].

**Proposition 3.1** (1) For a generic timelike surface $M$ in the de Sitter space, the LPL is a curve on $M$. It can be characterised as the set of points on $M$ where the two principal directions coincide and become a lightlike direction.

(2) The LPL divides the surface into two regions. In one of them there are no principal directions and in the other there are two distinct principal directions at each point. In the later case, the principal directions are orthogonal and one is spacelike while the other is timelike.

We have an extra information about the LPL.

**Proposition 3.2** For a generic timelike surface $M$ in the de Sitter space, the LPL curve is a smooth curve with possible Morse singularities of type node. The singular points are the points where the de Sitter shape operator is a multiple of the identity. For this reason, they are labeled “timelike umbilic points”.

**Proof.** We take a local parametrisation $\mathbf{x} : U \to V \subset M$ with $E = G = 0$ in $U$, that is, the images of $u = \text{constant}$ and $v = \text{constant}$ have lightlike tangent directions at all points (Theorem 2.2). Then the equation of the lines of principal curvature becomes

$$ndv^2 - ldu^2 = 0.$$

The discriminant function of this equation is $\delta(u, v) = (ln)(u, v)$. The LPL is the zero set of $\delta$. It is clear that this is a curve when $M$ is a generic surface. In general, this curve is smooth unless $l(q) = n(q) = 0$ at some point $q \in U$. These points can occur generically at isolated points on the LPL (we have two independent conditions). At such points the singularity of the LPL is generically a Morse $A_1^-$-singularity, i.e., it is equivalent by smooth changes of coordinates in the source to $\pm(u^2 - v^2)$.

The matrix of $A_p$, with $p = \mathbf{x}(q)$, with respect to the above parametrisation is

$$-\frac{1}{F^2} \begin{pmatrix} 0 & -F \\ -F & 0 \end{pmatrix} \begin{pmatrix} l & m \\ m & n \end{pmatrix}.$$

It is a multiple of the identity at $q$ if and only if $l(q) = n(q) = 0$ \hfill $\Box$

We are interested in the local topological configurations of the lines of principal curvature. Away from the LPL, we either have locally a pair of transverse foliations or no lines of principal curvature. So we analyse the configurations at points on the LPL. The lines of principal curvature are given by a BDE. These equations are well studied (see for example [9, 21] for references). The terminology used in the following result is clarified in its proof and in Figures 1 and 2.
Figure 1: Folded singularities: saddle (left), node (centre), focus (right).

Figure 2: Topological configurations of the solutions of a BDE at a Morse Type 2 singularity when the discriminant has an $A_1^-$-singularity, [6].

**Theorem 3.3**

1. At smooth points on the LPL, the lines of principal curvature form a family of cusps with the cusps tracing the LPL, except maybe at some isolated points on this curve. At such points, the equation of the lines of principal curvature has generically a folded singularity of type saddle, node or focus (Figure 1).

2. At a timelike umbilic point, the equation of the lines of principal curvature has generically a Morse Type 2 singularity of type $A_1^-$. All the five generic cases of such singularities can occur (Figure 2).

**Proof.** We take a local parametrisation $\mathbf{x} : U \to V \subset M$ with $E = G = 0$ in $U$ (Theorem 2.2), so that the equation of the lines of principal curvature is given by $ndv^2 - ldu^2 = 0$.

1. At a smooth point $p_0 = \mathbf{x}(q_0)$ on the LPL, we have either $l(q_0)$ or $n(q_0)$ non zero. Assume that $n(q_0) \neq 0$ and write the equation locally at $q_0$ in the form

$$dv^2 - \frac{l}{n}du^2 = 0.$$ 

We can now use the criteria for recognition of the singularities of a BDE (see for example [4] and [9] for terminology). The equation is locally smoothly equivalent to $dv^2 - udu^2 = 0$ if and only if $\partial l/\partial u(q_0) \neq 0$. In this case the configuration is a family
of cusps. When \( \partial l/\partial u(q_0) = 0 \), the BDE has a singularity. Well folded singularities are topologically equivalent to

\[
dv^2 + (v - \lambda u^2)du^2 = 0,
\]

with \( \lambda \) any fixed value in the open intervals determined by the exceptional values 0 and 1/16. The singularity is of type folded saddle if \( \lambda < 0 \), folded node if \( 0 < \lambda < 1/16 \) and folded focus if \( 1/16 < \lambda \) (Figure 1). For the equation of the lines of principal curvature, a necessary condition to have a folded singularity is \((\partial l/\partial v)(q_0). (\partial^2 l/\partial u^2)(q_0) \neq 0\). Then

\[
\lambda = n(q_0) \frac{\partial^2 l}{\partial u^2}(q_0) \frac{\partial l}{\partial v}(q_0),
\]

so all the folded singularities can occur.

(2) If \( p_0 \) is a timelike umbilic point, \( l(q_0) = n(q_0) = 0 \). So the coefficients of the BDE of the lines of principal curvature vanish at \( q_0 \). As the \( LPL \), which is the discriminant of the equation of the lines of principal curvature, has a Morse singularity of type \( A_1 \), the equation has generically a Morse Type 2 singularity of type \( A_1 \) ([6]).

We assume that \( q_0 \) is the origin. The bi-valued field determined by the BDE can be lifted to a single direction field \( \xi \) on a surface

\[
N = \{(u, v, [\alpha : \beta]) \in \mathbb{R}^2, 0 \times \mathbb{R}P^1 : a\beta^2 + 2b\alpha\beta + c\alpha^2 = 0\},
\]

where \( a, b, c \) are the coefficients of the lines of principal curvature BDE. If we consider the affine chart \( p = \beta/\alpha \) and set \( F(x, y, p) = a(x, y)p^2 + 2b(x, y)p + c(x, y) \), then the lifted direction field is parallel to the vector field \( \xi = F_p \partial/\partial u + pF_p \partial/\partial v - (F_u + pF_v) \partial/\partial p \). The topological type of the BDE is determined by the number and type of the singularities of the vector field \( \xi \) on the exceptional fibre \((0, 0) \times \mathbb{R}P^1\) (see [6] for the analytic case and Remark 2 in [21] for the smooth case). If we write \( j^1l = l_1u + l_2v \) and \( j^1n = n_1u + n_2v \), then the singularities of \( \xi \) are at the roots of the cubic \( \phi = (F_u + pF_v)(0, 0, p) = n_2p^3 + n_1p^2 - l_2p - l_1 \). We need \( \phi \) to have simple roots, that is

\[
4n_1^2l_1 + n_1^2l_2^2 + 18n_1n_2l_1l_2 - 27n_2^2l_1^2 + 4n_2l_2^3 \neq 0.
\]

The eigenvalues of the linear part of \( \xi \) at the roots of \( \phi \) are \(-\phi'(p)\) and \( \alpha_1(p) \), where \( \alpha_1(p) = 2p(n_2p + n_1) \) (see [6]). So we also need \( \alpha_1 \) not to vanish at a root of \( \phi \), that is,

\[
l_1(n_2l_1 - n_1l_2) \neq 0.
\]

When the above conditions are satisfied, which is the case at generic timelike umbilic points, \( \xi \) has either saddle or node singularities at the roots of \( \phi \). The number and type of these singularities, which is determined by the sign of \(-\phi'(p)\alpha_1(p)\), determines the topological configuration of the BDE. It is clear that we can have all the five possible generic cases in [6], see Figure 2 where the labeling refers to the number and type of the zeros of \( \xi \) on the exceptional fibre.
Figure 3: Topological configurations of the solutions of a BDE at a Morse Type 2 singularity when the discriminant has an $A_1^+$-singularity.

**Remark 3.4** (1) The principal directions can be detected via singularity theory. It is shown in [14] that, away from the LPL, the folding map of the surface with respect to a hyperplane has a degenerate singularity if and only if the normal to the hyperplane is parallel to a principal direction. The LPL and the folded singularities of the lines of principal curvature can also be detected via the singularities of projections along lightlike geodesics [15].

(2) For surfaces in the Euclidean 3-space (or two dimensional Riemannian surfaces with a self-adjoint operator [20]), the lines of principal curvature have Morse type 2 singularities of type $A_1^+$ at umbilic points ([17, 2], see Figure 3). Their configurations are distinct from those of timelike surfaces at timelike umbilic points.

### 4 Asymptotic curves

We define an asymptotic direction via the de Sitter shape operator.

**Definition 4.1** A direction $v \in T_p M$ is called asymptotic if $\Pi_p(v) = \langle A_p(v), v \rangle = 0$.

It follows from the above definition that the asymptotic directions are given by

$$ndv^2 + 2mdudv + ldu^2 = 0.$$  \hfill (2)

The discriminant of equation (2) is the set of points where $m^2 = nl$ vanishes. This is exactly the de Sitter parabolic set given by $K = (nl - m^2)/(EG - F^2) = 0$. As $EG - F^2 < 0$, there are two distinct asymptotic directions in the region $K > 0$ and no asymptotic directions in the region $K < 0$ (the opposite happens in the Euclidean case). On the de Sitter parabolic set there is a unique double asymptotic direction. The de Sitter parabolic set of a generic surface, when not empty, is a smooth curve.

**Proposition 4.2** The de Sitter parabolic set and the LPL are tangential at their points of intersection. At such points the unique asymptotic direction on the de Sitter parabolic set is lightlike; on one side of such points it is spacelike and on the other it is timelike.
Proof. We take a local parametrisation \( x : U \to V \subset M \) with \( E = G = 0 \) in \( U \) (Theorem 2.2). Then the LPL is given by \( nl = 0 \) and the de Sitter parabolic set by \( m^2 - nl = 0 \). We observe first that the singular points of the LPL are not generically de Sitter parabolic points (otherwise \( n = l = m = 0 \)). Suppose, without loss of generality, that \( n(q) = 0 \) and \( l(q) \neq 0 \) at the point in consideration. So the LPL is given locally by \( n = 0 \). Then the point \( p = x(q) \) is also a parabolic point if and only if \( m(q) = 0 \). Now the gradient of \( m^2 - nl \) at \( q \) is \((-nu, -nv)(q)) \) and is parallel to the gradient of \( n \) at \( q \). So the two curves are tangential at their meeting point.

At \( q \), equation (2) becomes \( l(q)du^2 = 0 \), so the unique asymptotic direction on the surface is along \( x_u(q) \) which is lightlike. Near the point \( q \) on the de Sitter parabolic set, the double asymptotic direction is along \((du, dv) = (m, -n) \) in the parameter space, so is along \( mx_u - nx_v \) on the surface. We have \( \langle mx_u - nx_v, mx_u - nx_v \rangle = -2nmF \), and this changes sign generically at \( q \).

**Proposition 4.3** An asymptotic direction at a point \( p \in M \) is also a principal direction at \( p \) if and only if \( p \) is a de Sitter parabolic point or a point on the LPL. On the LPL, the asymptotic direction is lightlike.

**Proof.** We take a local parametrisation as in the proof of Proposition 4.2. Then the equation of the lines of principal curvature is given by \( ndv^2 - ldu^2 = 0 \). Subtracting this from equation (2) yields \( du(du + pdv) = 0 \).

The lightlike direction \( du = 0 \) is both asymptotic and principal if and only if \( n(q) = 0 \), if and only if \( p = x(q) \) is on the LPL. The direction \( ldu + pdv = 0 \) is both asymptotic and principal if and only if \((m^2 - nl)(q) = 0 \), if and only if \( p \) is a de Sitter parabolic point.

**Theorem 4.4** On the de Sitter parabolic curve, the asymptotic curves form a family of cusps with the cusps tracing the de Sitter parabolic curve, except may be at some isolated points on this curve. At such points, the equation of the asymptotic curves has generically a folded singularity of type saddle, node or focus (Figure 1).

**Proof.** We can suppose, without loss of generality, that the point in consideration is \( p_0 = (0, 1, 0, 0) \) and take the surface in de Sitter Monge form

\[
\phi(x, y) = (f(x, y), \sqrt{1 + f^2(x, y) - x^2 - y^2}, x, y),
\]

with

\[
f(x, y) = a_{10}x + a_{11}y + a_{20}x^2 + a_{21}xy + a_{22}y^2 + a_{30}x^3 + a_{31}x^2y + a_{32}xy^2 + a_{33}y^3 + h.o.t.
\]

and \((x, y)\) in some neighbourhood of the origin. We require \(a_{10}^2 + a_{11}^2 > 1\) for the surface to be timelike near the \( p_0 \). We calculate the 2-jet of the coefficients of the asymptotic BDE and find
j^2l = 2a_{20} + 6a_{30}x + 2a_{31}y + 12a_{40}x^2 + (-2a_{20}a_{10}a_{11} + 6a_{41} + a_{21}(a_{10}^2 - 1))xy + 
(2a_{12} + a_{20}(1 - a_{11}^2) + a_{22}(a_{10}^2 - 1))y^2,

j^2m = -a_{21} - 2a_{31}x - 2a_{32}y + (-3a_{41} + \frac{1}{2}a_{21}(a_{10}^2 - 1) + a_{20}a_{10}a_{11})x^2 + \n2(a_{21}a_{10}a_{11} - 2a_{42})xy + (-3a_{43} + \frac{1}{2}a_{21}(-1 + a_{11}^2) + a_{11}a_{10}a_{22})y^2,

j^2n = 2a_{22} + 2a_{32}x + 6a_{33}y + (2a_{42} - a_{20} + a_{22}(1 + a_{11}^2 - a_{10}^2))x^2 + \n(2a_{11}a_{10}a_{22} + 6a_{43} + a_{21}(a_{11}^2 - 1))xy + 12a_{44}y^2.

The origin is a de Sitter parabolic point (i.e., a point on the discriminant of the asymptotic curves BDE) if and only if \(a_{21}^2 - 4a_{20}a_{22} = 0\). We can suppose, without loss of generality, that \(\varphi_x(0, 0)\) is an asymptotic direction so that \(a_{20} = 0\). Then the origin is a parabolic point if \(a_{21} = 0\). We have generically \(a_{22} \neq 0\), otherwise the parabolic set is singular. The origin is a singularity of the asymptotic BDE if \(a_{30} = 0\). The singularity is in general a folded one and its type is determined by

\[ \lambda_{as} = \frac{1}{2} \frac{(12a_{22}a_{40} - 6a_{30}a_{32} - a_{31}^2)a_{22}^2}{2(3a_{22}a_{41} - a_{31}a_{32} - 9a_{30}a_{33})^2} \]

(see the criteria in \([4]\)). We have a folded saddle if \(\lambda_{as} < 0\), a folded node if \(0 < \lambda_{as} < \frac{1}{16}\), and a folded focus \(\lambda_{as} > \frac{1}{16}\). It is clear that the three cases can occur for generic surfaces. \(\square\)

**Remark 4.5** (1) At the point of tangency of the de Sitter parabolic curve and the LPL the equation of the asymptotic curves is in general not singular. So in a neighbourhood of such points the asymptotic curves form a family of cusps.

(2) Asymptotic directions can be detected via singularity theory. It is shown in \([15]\) that a projection along parallel geodesics to an orthogonal quadric has a singularity of type cusp or worse at \(p\) if and only if the tangent direction to the geodesic at \(p\) is along an asymptotic direction.

## 5 Characteristic curves

We use the results in \([3]\) to define the characteristic directions. A BDE can be viewed as a quadratic form and represented at each point in the plane by a point in the projective plane. If \(\Gamma\) denotes the set of degenerate quadratic forms, then the asymptotic and principal BDEs of a timelike surface determine a unique BDE, which we call the **characteristic curves BDE**, in such a way that the three equations represent at each point on \(M\) a self-polar triangle in the projective plane with respect to \(\Gamma\) \(([3])\). The characteristic curves BDE is given as the jacobian of the asymptotic and principal curves BDE and has the following expression

\[
(2m(mG - nF) - n(lG - nE))dv^2 + 2(m(lG + nE) - 2lnF)dvdv + 
(l(lG - nE) - 2m(lF - mE))du^2 = 0. \quad (3)
\]
If we take a parametrisation of the surface with $E = G = 0$ (Theorem 2.2), then equation (3) becomes

$$mndv^2 + 2lndvdu + mldu^2 = 0.$$  

The discriminant of this equation is $ln(ln - m^2) = 0$, which is the union of the de Sitter parabolic set and the LPL.

We say that two tangent directions $u, v \in T_pM$ are conjugate, and write $v = \bar{u}$, if $\langle A_p(u), v \rangle = 0$. For instance, a direction $u \in T_pM$ is asymptotic if and only if $\bar{u} = u$. If we take a local parametrisation as in Theorem 2.2, then $u = ax_u + bx_v$ and $v = cx_u + dx_v$ are conjugate if and only if

$$acl + (ad + bc)m + bdn = 0.$$  

We are interested in directions whose conjugate is a lightlike direction. Suppose that $u$ is a lightlike direction at $p = x(q)$, and take $u = x_u(q)$ with $x$ as in Theorem 2.2. Then $\bar{u} = (mx_u - lx_v)(q)$. So $\bar{u}$ is lightlike if and only if $l(q) = 0$ or $m(q) = 0$. The condition $l(q) = 0$ means that $p$ is on the LPL and $\bar{u} = u$. The condition $m(q) = 0$ is equivalent to $H(p) = 0$. Then $\bar{u}$ is the other lightlike direction. Observe that when $m(q) = 0$ characteristic directions at $p$ are both lightlike and are along $x_u$ and $x_v$. So we have the following result.

**Proposition 5.1** Let $M$ be a timelike surface and $p \in M$.

(1) A lightlike direction $u$ at $p$ is self-conjugate if and only if $p$ is on the LPL and $u$ is the unique principal direction at $p$ (which is also an asymptotic direction, see Proposition 4.3).

(2) The conjugate of a lightlike direction $u$ at $p$ is the other lightlike direction at $p$ if and only if $p$ is on the curve $H = 0$. At such points both lightlike directions are characteristic directions.

We analyse now to the configuration of the characteristic curves.

**Theorem 5.2** (1) On the de Sitter parabolic curve and away from its points of tangency with the LPL, the characteristic curves form a family of cusps with the cusps tracing the de Sitter parabolic curve, except maybe at some isolated points on this curve. At such points, the equation of the characteristic curves has generically a folded singularity of type saddle, node or focus (Figure 1). The folded singularities of the characteristic and asymptotic BDEs coincide. The indices of the two equations at such points are not related in general.

(2) At a point of tangency of the de Sitter parabolic curve and LPL, the characteristic BDE is topologically equivalent to

$$vdu^2 + 2udvdu + v^3du^2 = 0 \quad \text{Figure 4 left, or to}$$
$$vdu^2 - 2udvdu + v^3du^2 = 0 \quad \text{Figure 4 right.}$$
Figure 4: Configurations of the characteristic curves at the point of tangency of the de Sitter parabolic set and the LPL.

(3) At a timelike umbilic point the characteristic BDE has generically a Morse Type 2 singularity with discriminant of type $A_1^-$. All the generic five topological models of such singularities can occur (Figure 2).

Proof. (1) We follow the same setting as in the proof of Theorem 4.4. The 2-jets the origin of the coefficients of the characteristic BDE are too lengthy to reproduce here. The calculations show that if $p_0$ is not on the LPL, then it is a point on the discriminant (i.e., the de Sitter parabolic set) if and only if $a_{21}^2 - 4a_{20}a_{22} = 0$. We can take without loss of generality $a_{20} = 0$. Then $p_0$ is a parabolic point if $a_{21} = 0$ and in this case $\phi_x(0,0)$ is a characteristic direction. (It is also an asymptotic direction. In fact on the parabolic set, the characteristic and asymptotic directions coincide and become a principal direction.) We have generically $a_{22} \neq 0$, otherwise the de Sitter parabolic set is singular, and $a_{10}^2 - 1 \neq 0$ otherwise $p_0$ is also on the LPL. Then the origin is a singularity of the characteristic BDE if $a_{30} = 0$. Using the criteria in [4], we find that the singularity is in general a folded one and its type is determined by

$$\lambda_{\text{char}} = \frac{1}{2} \frac{a_{22}^2(a_{10}^2 - 1)^2(a_{31}^2 - 12a_{22}a_{40})}{((5a_{31}a_{32} - 3a_{22}a_{41})(a_{10}^2 - 1) + 4a_{10}a_{11}a_{31})^2}.$$

We have a folded saddle if $\lambda_{\text{char}} < 0$, a folded node if $0 < \lambda_{\text{char}} < \frac{1}{16}$ and a folded focus $\lambda_{\text{char}} > \frac{1}{16}$. In general $\lambda_{\text{char}} \neq \lambda_{\text{as}}$ (see the proof of Theorem 4.4 for $\lambda_{\text{as}}$), so the type of the folded singularities of the asymptotic and characteristic BDEs are not related in general. (We remark that for surfaces in the Euclidean 3-space the two BDEs have opposite indices at their folded singularities, [3].)

(2) We take the surface in de Sitter Monge form (see the proof of Theorem 4.4) and let $a_{10} = a_{11} = 1$. Then $\phi_x(0,0)$ and $\phi_y(0,0)$ are lightlike directions. The point $p_0$ is on both the LPL and the de Sitter parabolic set if and only if $l(0) = m(0) = 0$, that is $a_{20} = a_{21} = 0$, or $m(0) = n(0) = 0$, that is $a_{21} = a_{22} = 0$. We assume that $a_{20} = a_{21} = 0$ and $a_{22} \neq 0$. Then the discriminant of the BDE of the characteristic curves has an $A_3^-$ singularity (i.e., equivalent to $\pm(x^2 - y^4)$) provided $3a_{30}a_{32} - a_{31}^2 \neq 0$, which is satisfied generically.
Let \((a, b, c)\) denote the coefficients of the BDE the characteristic curves. Then we have at the origin, after scaling, \(j^1a = -a_{31}x - a_{32}y\), \(j^1b = 3a_{30}x + a_{31}y\) and \(j^1c = 0\). As \(3a_{30}a_{32} - a_{31}^2 \neq 0\), we can make change of coordinates in the source and multiply by non-zero constant so that \(j^1a = \alpha x + y\), \(j^1b = \pm x\) and \(j^1c = 0\), with \(\alpha = 1/(8|a_{22}| \sqrt{3|a_{30}a_{32} - a_{31}^2|})\). The result now follows from Theorem 6.1 in the Appendix and the fact that the discriminant has an \(A_3^\pm\)-singularity.

(3) We return here to the parametrisation in Theorem 2.2. If \(p_0 = x(q_0)\) is a timelike umbilic point, then \(l(q_0) = n(q_0) = 0\) (we have generically \(m(q_0) \neq 0\)). So the coefficients of the BDE of the characteristic curves vanish at \(q_0\). As the \(LPL\), which is locally the discriminant of the equation, has a Morse singularity of type \(A_1^\pm\), the equation has generically a Morse Type 2 singularity ([6]) with a discriminant having an \(A_1^\pm\)-singularity. We assume that \(q_0\) is the origin. The 1-jet of the BDE is \(j^1ndv^2 + j^1ldu^2\). Following the calculation and notation in the proof of Theorem 3.3, the singularities of the lifted field \(\xi\) are the roots of the cubic \(\phi = n_2p^3 + n_1p^2 + l_2p + l_1\). We need \(\phi\) to have simple roots, that is

\[4n_3^3l_1 - n_1^2l_2^2 - 18n_1n_2l_1l_2 + 27n_2^2l_1^2 + 4n_2 l_2^3 \neq 0.\]

The eigenvalues of the linear part of \(\xi\) at the roots of \(\phi\) are \(-\phi'(p)\) and \(\alpha_1(p)\), where \(\alpha_1(p) = 2p(n_2p + n_1)\) (see [6]). So we also need \(\alpha_1\) not to vanish at the roots of \(\phi\), that is

\[l_1(n_2l_1 - n_1l_2) \neq 0.\]

Then \(\xi\) has either saddle or node singularities at the roots of \(\phi\). The number and type of these singularities (which is determined by the sign of \(-\phi'(p)\alpha_1(p)\)) determine the topological configuration of the BDE. It is clear that we can have all the five possible generic cases in [6], see Figure 2. We observe here that, in general, there is no relation between the type of the singularity of the BDE of the lines of principal curvature and that of the characteristic curves.

\[\square\]

**Remark 5.3** Let \(C(M, S^3)\) denotes the space of embeddings of an orientable timelike surface \(M\) in the de Sitter space, endowed with the Whitney topology. It follows by Thom’s transversality theorem that the set of embeddings with the properties (a)-(f) below form a residual subset of \(C(M, S^3)\).

(a) The de Sitter parabolic set, when not empty, is a smooth curve.

(b) The \(LPL\), when not empty, is a smooth curve or has isolated singularities of type \(A_1^\pm\).

(c) The de Sitter parabolic set and the \(LPL\) have ordinary tangency at their points of intersection.

(d) The singularities of the BDE of the lines of principal curvature are those described in Theorem 3.3.
(e) The singularities of the BDE of the asymptotic curves are those described in Theorem 4.4.
(f) The singularities of the BDE of the characteristic curves are those described in Theorem 5.2.

One can always construct a patch of a timelike surface in $S_3^1$ with one of the pairs
of foliations described in this paper having one of its possible stable local singularities.
In the following example, we find explicit expressions for the above foliations on a
special surface. Consider the timelike surface $M$ given by $x : \mathbb{R}^2 \to S_3^1$ with

$$x(u, v) = \frac{1}{\sqrt{2}}(\sinh(u), \cosh(u), \cos(v), \sin(v)).$$

We have $x_u = \frac{1}{\sqrt{2}}(\cosh(u), \sinh(u), 0, 0)$ and $x_v = \frac{1}{\sqrt{2}}(0, 0, -\sin(v), \cos(v))$, so the
coefficients of the first fundamental form are given by

$$E = -\frac{1}{2}, \quad F = 0, \quad G = \frac{1}{2}.$$ 

The de Sitter Gauss map is given by

$$e(u, v) = \frac{1}{\sqrt{2}}(\sinh(u), \cosh(u), -\cos(v), -\sin(v)).$$

Therefore the coefficients of the second fundamental form are given by

$$l = \frac{1}{2}, \quad m = 0, \quad n = \frac{1}{2}.$$ 

It follows that the equations for the pairs of foliations described in this paper are as follows.

- Lines of principal curvature: $dvdu = 0$.
- Asymptotic curves: $dv^2 + du^2 = 0$.
- Characteristic curves: $dv^2 - du^2 = 0$.

So the lines of principal curvature are given $u = constant$ and $v = constant$, there are
no asymptotic curves and the characteristic curves are given by $v \pm u = constant$.

6 Appendix

We are interested in BDEs with coefficients with 1-jets $(\alpha x + y, \pm x, 0)$. By equivalent
we mean taking one BDE to another by smooth changes of coordinates in the source
and multiplication by nowhere zero functions.
Theorem 6.1 A BDE with 1-jet equivalent to \((\alpha x + y, \pm x, 0)\) and with a discriminant with \(A_3\)-singularity is topologically equivalent to one of the following cases.

(i) Discriminant has an \(A_3^-\)-singularity:

\[
\begin{align*}
ydy^2 + 2xdydx + y^3dx^2 &= 0 \quad \text{Figure 5 A, or to} \\
ydy^2 - 2xdydx + y^3dx^2 &= 0 \quad \text{Figure 5 B}
\end{align*}
\]

(ii) Discriminant has an \(A_3^+\)-singularity

\[
\begin{align*}
ydy^2 + 2xdydx - y^3dx^2 &= 0 \quad \text{Figure 5 C, or to} \\
ydy^2 - 2xdydx - y^3dx^2 &= 0 \quad \text{Figure 5 D}
\end{align*}
\]

Proof. We take the coefficients of the BDE \(\omega = (a, b, c)\) in the form

\[
\begin{align*}
a &= \alpha x + y + M_1(x, y) \\
b &= \epsilon x + M_2(x, y) \\
a &= \lambda y^3 + M_3(x, y)
\end{align*}
\]

where \(M_i\) are germs, at the origin, of smooth functions with \(j^1M_1 = j^1M_2 = 0, j^3M_3 = 0, \epsilon = \pm 1\) and \(\lambda = \pm 1\).

We consider a blowing up of the BDE following the method introduced in [17, 11] for BDEs whose discriminants are isolated points, and extended in [19, 21] for general BDEs.

Following the notation in [11], let \(f_i(w), i = 1, 2\) denote the foliation associated to the BDE \(\omega = (a, b, c)\), which is tangent to the vector field \(a \frac{\partial}{\partial x} + (-b + (-1)^i \sqrt{b^2 - ac}) \frac{\partial}{\partial y}\). If \(\psi\) is a diffeomorphism and \(\lambda(x, y)\) is a non-vanishing real valued function, then ([11]) for \(k = 1, 2\)

1. \(\psi(f_k(w)) = f_k(\psi^*(\omega))\), if \(\psi\) is orientation preserving;
2. \(\psi(f_k(w)) = f_{3-k}(\psi^*(\omega))\), if \(\psi\) is orientation reversing;
3. \(f_k(\lambda w) = f_k(\omega)\), if \(\lambda(x, y)\) is positive;
4. \(f_k(\lambda w) = f_{3-k}(\omega)\), if \(\lambda(x, y)\) is negative.

We consider the directional blowing-up

\[
\begin{align*}
(1) : \begin{cases} x = v^2 \\ y = uv \end{cases} & \quad \text{(2) : \begin{cases} x = -v^2 \\ y = uv \end{cases}} & \quad \text{(3) : \begin{cases} x = uv^2 \\ y = v \end{cases}}
\end{align*}
\]

The blowing up (1) (resp. (2)) is a diffeomorphism from the region \(v > 0\) (resp. \(v < 0\)) in the \((u, v)\)-plane to the region \(x > 0\) (resp. \(x < 0\)) in the \((x, y)\)-plane and is orientation reversing (resp. preserving) in this region. Applying the blowing up (2) gives similar results to applying (1). So we deal in more details with blowing up (1). We also consider the blowing up (3) but this does not reveal any extra information.

Consider the blowing-up (1). Then the new BDE \(\omega_0 = (u, v)^*\omega\) has coefficients
\[ \tilde{a} = v(u^2 + auv + 4ev^2) + N_1(u, v), \]
\[ \tilde{b} = v^2((u^2 + auv + 2ev^2) + N_2(u, v)), \]
\[ \tilde{c} = v^3(u + av + N_3(u, v)), \]

where \( N_i, i = 1, 2, 3, \) are smooth functions with \( j^3N_1 = 0, j^2N_2 = 0 \) and \( j^1N_3 = 0. \)

We can write \((\bar{a}, \bar{b}, \bar{c}) = v(A_1, vB_1, v^2C_1)\) with
\[
A_1 = u(u^2 + auv + 4ev^2) + N_1(u, v),
B_1 = u^2 + auv + 2ev^2 + N_2(u, v),
C_1 = u + av + N_3(u, v).
\]

and consider the quadratic form \( \omega_1 = (A_1, vB_1, v^2C_1). \)

The discriminant of \( \omega_1 \), which is the blowing up of the discriminant of \( \omega \), is the exceptional fibre \( v = 0 \) when \( \lambda = -1 \) and the union of the exceptional fibre with two smooth curves \( C_i, i = 1, 2, \) meeting transversally the exceptional fibre at \((u_i, 0)\), where \( u_i \) are solutions of \((b_2^2 - 1)u^4 + 2eb_2u^2 + 1 = 0. \)

We decompose \( \omega_1 \) into two 1-forms, and to these 1-forms are associated the vector fields
\[
X_i = A_1 \frac{\partial}{\partial u} + (-vB_1 + (-1)^i \sqrt{v^2(B_1^2 - A_1C_1)}) \frac{\partial}{\partial v}, \quad i = 1, 2.
\]

These vector fields are tangent to the foliations defined by \( \omega_1 \) and have the exceptional fibre \( v = 0 \) (or part of it in the case \( \lambda = +1 \)) as an integral curve. We deal now with the cases \( A_3^+ \) and \( A_3^- \) separately.

**The case \( A_3^- \) (\( \lambda = +1 \))**

We have \( A_1(u, 0) = u^3 \), so \( A(u_i, 0) \neq 0, i = 1, 2 \) with \( u_i \) as above. This means that near \((u_i, 0), \) \( i = 1, 2, \) the integral curves of \( X_i \) form segments of smooth curves ending transversally on \( C_i, i = 1, 2. \) It also means that both vector fields \( X_1 \) and \( X_2 \) are smooth away from \((0, 0)\) and the points \((u_i, 0), \) \( i = 1, 2. \) So we need to analyse the configurations of these vector fields at \((0, 0)\), with \( v \geq 0 \) (we are dealing with the blowing up \((1)\) which is a diffeomorphism from the region \( v > 0 \) delimited by the exceptional fibre and the curves \( C_1 \) and \( C_2 \) to the region in the \((x, y)\)-plane delimited by the discriminant and with \( x > 0 \)). We have, at the origin,
\[
X_1 = (u(u^2 + auv + 4ev^2) + \text{h.o.t.}) \frac{\partial}{\partial u} + (-uv(u + av) + \text{h.o.t.}) \frac{\partial}{\partial v}.
\]

The first component \( A_1 \) of \( X_1 \) has a \( D_4 \)-singularity provided \( a^2 - 16e \neq 0 \) (i.e., equivalent to \( u(u^2 \pm v^2) \)). So \( A_1 = (u - g_1(u, v))(u^2 + auv + 4ev^2 + g_2(u, v)) \) for some germ of smooth functions \( g_1 \) and \( g_2 \) with \( j^1g_1 = j^2g_2 = 0 \). We observe that when \( u - g_1(u, v) = 0 \) the second component of \( X_1 \) also vanishes. Therefore \( X_1 = (u - g_1(u, v))\tilde{X}_1 \) with
\[
\tilde{X}_1 = (u^2 + auv + 4ev^2 + \text{h.o.t.}) \frac{\partial}{\partial u} + (-v(u + av) + \text{h.o.t.}) \frac{\partial}{\partial v}.
\]
The 2-jet of the vector field $\bar{X}_1$ satisfies the general position condition in Proposition 3.5 in [18]. Therefore, by Takens’ result, $X_1$ is topologically equivalent to $(u^2 + auv + 4\epsilon v^2)\frac{\partial}{\partial u} - v(u + av)\frac{\partial}{\partial v}$ near the origin. Using a polar blowing up, we find that the configuration of $\bar{X}_1$ at the origin and hence of $X_1$ is as in Figure 5. (The singularities of the blown up field are either saddles or nodes and are as shown in Figure 5.)

We now consider the vector field $X_2$. We have, at the origin,

$$X_2 = (u(u^2 + auv + 4\epsilon v^2) + h.o.t.)\frac{\partial}{\partial u} + (-v(u^2 + auv + 4\epsilon v^2) + h.o.t.)\frac{\partial}{\partial v}.$$  

If $a^2 - 16\epsilon > 0$, we can proceed as for $X_1$ above. We can factor out a term $u^2 + auv + 4\epsilon v^2 + h.o.t$ from both component of $X_2$ and consider the vector field $\bar{X}_2 = (u + h.o.t.)\frac{\partial}{\partial u} + (-v + h.o.t.)\frac{\partial}{\partial v}$ which has a saddle-singularity at the origin. So $X_2$ has a saddle singularity at the origin.

If $a^2 - 16\epsilon < 0$, we consider a blowing up of the singularity of $X_2$ and find again that it has a saddle singularity at the origin.

Applying the blowing up (2) gives similar result, with the position of the nodes and saddles of $X_1$ inverted when $a^2 - 8\epsilon > 0$. So we have the configuration of the solution curves of $\omega$ in region $(x, y)$-plane delimited by the discriminant and with $x < 0$ as shown in Figure 5. We also need to invert the colours of the foliations as we factored out a term $v$ in $\omega$ (see comments at the beginning of the proof).

Therefore the configurations of the solutions of the BDE $\omega$ are as in shown in Figure 5.

**The case $A_3^+$ ($\lambda = -1$)**

The situation here is similar to the case $A_3^-$ (the 3-jets of $X_1$ and $X_2$ do not depend on $\lambda$). The difference is that the whole exceptional fibre is an integral curve of both $X_1$ and $X_2$. The configurations are as in Figure 5.

**Remarks 6.2** (1) The configurations in Figure 5 are topologically equivalent to Morse type 2 singularities (Figure 2 and Figure 3).

(2). A BDE with 1-jet $(\alpha x + y, \pm x, 0)$ is equivalent to one with a 1-jet $(y, \pm x + \beta y, 0)$ which is studied in [5]. A result in [5] states that we can reduce formally such BDEs, for almost all values of $\alpha$, by a formal change of coordinates in the source and multiplication by non-zero formal power series, to one in the form

$$(\alpha x + y)dy^2 \pm 2xdxdy + c(y)dx^2 = 0,$$

where $c(y)$ is a formal power series with a zero 2-jet. Theorem 6.1 gives the topological models for the case when $j^3c = \lambda y^3$, with $\lambda \neq 0$.

**Acknowledgement.** Part of the work in this paper was carried out during the visit of FT to Hokkaido University in October 2007. The visit was supported by a Royal Society International Outgoing Short Visit grant 2007/R2.
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Figure 5: Topological models of BDEs with 1-jet equivalent to $(\alpha x + y, \epsilon x, 0)$, $\epsilon = \pm 1$, and with discriminant with an $A_3$-singularity (bottom two rows). The figures in the top row give the configurations of the vector fields $X_1$ and $X_2$ at the origin and those of their blowing up.

**References**


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