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# ON THE PROLONGATION OF 2-JET SPACE OF 2 INDEPENDENT AND 1 DEPENDENT VARIABLES

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## 1. INTRODUCTION

In this paper, we will consider the extension of "Monster Goursat manifold" in [MZ] to multi independent variables case.

Let  $(R, D)$  be a pair of a manifold  $R$  and a distribution  $D$  on  $R$ . Let  $(J^k(M, n), C^k)$  be a  $k$ -jet space and its canonical system ( $\dim M = m + n$ ). We define a set  $\Sigma(J^k(M, n))$  of integral elements of higher order jet spaces as follows:

$$\Sigma(J^k(M, n)) := \bigcup_{x \in J^k} \Sigma_x,$$

where  $\Sigma_x = \{\text{n-dim integral elements of } (J^k(M, n), C^k)\}$ .

$\Sigma(J^k(M, n))$  are candidates for the extension of "Monster Goursat manifold" in [MZ] to multi independent variables case. However  $\Sigma(J^k(M, n))$  may not be manifolds. This situation is quite different from the case of the 1 independent variable. So one of the main purpose of this paper is to check when a set of integral elements of higher order jet spaces becomes a manifold or not. If a set of integral elements of some higher order jet space is a manifold, then we can define the canonical differential system on this manifold. In this case, we think that the manifold endowed with the canonical differential system is the extension of "Monster Goursat manifold" or the "prolongation" of the jet space. And we will prove the first main result;

**Theorem 3.2.**  $\Sigma(J^k(M^{n+m}, n))$  is a manifold if and only if  $\Sigma(J^2(M^{1+2}, 2))$  or trivial cases.

Here, the trivial cases are;

The case  $n = 1$ . In this case,  $\Sigma(J^k(M^{m+1}, 1))$  is nothing but a rank 1 prolongation (remark 3.1.). The case  $k = m = 1$ .  $\Sigma(J^1(M^{1+n}, n))$  is a Lagrange-Grassmann bundle  $L(J)$ , by definition. hence  $\Sigma(J^1(M^{1+n}, n))$  are also manifolds. we call these cases *trivial cases*.

Next, we will be concerned with the local equivalence problem of  $\Sigma(J^2(M^{1+2}, 2))$  ( $\Sigma(J^2)$  for short). The equivalence problem of "Monster Goursat manifold" of 1 independent variable cases is studied in [M1] and [M3].  $\Sigma(J^2)$  has essentially two types of singularities in the sense of Tanaka theory (proposition 5.2.).

$$\begin{aligned} \Sigma_1 &= \{w \in \Sigma(J^2) \mid \dim(w \cap \text{fiber}) = 1\} \\ \Sigma_2 &= \{w \in \Sigma(J^2) \mid \dim(w \cap \text{fiber}) = 2\} \end{aligned}$$

where  $w \in \Sigma(J^2(M^1, 2))$ , the fiber means that of  $\pi_* : T(J^2) \supset C^2 \rightarrow T(J^1)$ . The first one has normal form:

**Theorem 5.7.(normal form)**

For  $w \in \Sigma_1$ .  $((\Sigma(J^2(M^1, 2)), D), w)$  is locally isomorphic to a germ at the origin in  $(\mathbb{R}^{12}, \hat{D})$ ; where  $(\mathbb{R}^{12}; x, y, z, p, q, r, s, t, a, B, c, e)$  is coordinate and  $\hat{D}$  is expressed by  $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_y = \varpi_r = \varpi_s = 0\}$ , where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_y = dy - adx - Bdt \\ \varpi_1 = dp - rdx - sdy & \varpi_r = dr - cdx - (a^2 + eB)dt \\ \varpi_2 = dq - sdx - tdy & \varpi_s = ds - edx + adt. \end{cases}$$

The other is more complicated one and divided into 3 types (hyperbolic type, elliptic type, parabolic type).

**Theorem 5.12.**

$$\Sigma_2 = \Sigma_h \cup \Sigma_e \cup \Sigma_p$$

where

$$\begin{aligned} \Sigma_h &= \Sigma_2 \cap \{w : \text{hyperbolic point}\} \\ \Sigma_e &= \Sigma_2 \cap \{w : \text{elliptic point}\} \\ \Sigma_p &= \Sigma_2 \cap \{w : \text{parabolic point}\} \end{aligned}$$

For  $w \in \Sigma_2$ ,  $\mathfrak{m}(w)$  is isomorphic to  $\mathfrak{m}(1, 0)$ ,  $\mathfrak{m}(-1, 0)$  or  $\mathfrak{m}(0, 0)$  according to whether  $w \in \Sigma_h$ ,  $w \in \Sigma_e$  or  $w \in \Sigma_p$ , respectively. Moreover,  $w \in \Sigma_h$  is locally isomorphic to a germ at  $(0, \dots, 0, 1, 0)$  in  $(\mathbb{R}^{12}, \bar{D})$ .  $w \in \Sigma_e$  is locally isomorphic to a germ at  $(0, \dots, 0, -1, 0)$  in  $(\mathbb{R}^{12}, \bar{D})$ ,  $w \in \Sigma_p$  is locally isomorphic to a germ at  $(0, \dots, 0, 0, 0)$  in  $(\mathbb{R}^{12}, \bar{D})$ . where  $(\mathbb{R}^{12}; x, y, z, p, q, r, s, t, B, D, E, F)$  is coordinate and  $\bar{D}$  is expressed by  $\bar{D} = \{\varpi^0 = \varpi^1 = \varpi^2 = \varpi_x = \varpi_y = \varpi_t = 0\}$  where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_x = dx - (DE - BF)dr - Bds \\ \varpi_1 = dp - rdx - sdy & \varpi_y = dy - Bdr - Dds \\ \varpi_2 = dq - sdx - tdy & \varpi_t = dt - Edr - Fds. \end{cases}$$

Finally, we summarise the above theorems and will give a complete classification of  $\Sigma(J^2)$ (**Corollary 5.13.**).

## 2. GEOMETRIC CONSTRUCTION OF JET SPACES

In this section, we will briefly recall the geometric construction of jet bundles in general, following [Y1] and [Y2], which is our basis for the later considerations.

Let  $M$  be a manifold of dimension  $m + n$ . Fixing the number  $n$ , we form the space of  $n$ -dimensional *contact elements* to  $M$ , i.e., the Grassmann bundle  $J(M, n)$  over  $M$  consisting of  $n$ -dimensional subspaces of tangent spaces to  $M$ . Namely,  $J(M, n)$  is defined by

$$J(M, n) = \bigcup_{x \in M} J_x, \quad J_x = \text{Gr}(T_x(M), n),$$

where  $\text{Gr}(T_x(M), n)$  denotes the Grassmann manifold of  $n$ -dimensional subspaces in  $T_x(M)$ . Let  $\pi : J(M, n) \rightarrow M$  be the bundle projection. The *canonical system*  $C$  on

$J(M, n)$  is, by definition, the differential system of codimension  $m$  on  $J(M, n)$  defined by

$$C(u) = \pi_*^{-1}(u) = \{v \in T_u(J(M, n)) \mid \pi_*(v) \in u\} \subset T_u(J(M, n)) \xrightarrow{\pi_*} T_x(M),$$

where  $\pi(u) = x$  for  $u \in J(M, n)$ .

Let us describe  $C$  in terms of a canonical coordinate system in  $J(M, n)$ . Let  $u_o \in J(M, n)$ . Let  $(x_1, \dots, x_n, z^1, \dots, z^m)$  be a coordinate system on a neighborhood  $U'$  of  $x_o = \pi(u_o)$  such that  $dx_1, \dots, dx_n$  are linearly independent when restricted to  $u_o \subset T_{x_o}(M)$ . We put  $U = \{u \in \pi^{-1}(U') \mid dx_1|_u, \dots, dx_n|_u \text{ are linearly independent}\}$ . Then  $U$  is a neighborhood of  $u_o$  in  $J(M, n)$ . Here  $dz^\alpha|_u$  is a linear combination of  $dx_i|_u$ 's, i.e.,  $dz^\alpha|_u = \sum_{i=1}^n p_i^\alpha(u) dx_i|_u$ . Thus, there exist unique functions  $p_i^\alpha$  on  $U$  such that  $C$  is defined on  $U$  by the following 1-forms;

$$\varpi^\alpha = dz^\alpha - \sum_{i=1}^n p_i^\alpha dx_i \quad (\alpha = 1, \dots, m),$$

where we identify  $z^\alpha$  and  $x_i$  on  $U'$  with their lifts on  $U$ . The system of functions  $(x_i, z^\alpha, p_i^\alpha)$  ( $\alpha = 1, \dots, m, i = 1, \dots, n$ ) on  $U$  is called a *canonical coordinate system* of  $J(M, n)$  subordinate to  $(x_i, z^\alpha)$ .

$(J(M, n), C)$  is the (geometric) 1-jet space and especially, in case  $m = 1$ , is the so-called contact manifold. Let  $M, \hat{M}$  be manifolds of dimension  $m+n$  and  $\varphi : M \rightarrow \hat{M}$  be a diffeomorphism. Then  $\varphi$  induces the isomorphism  $\varphi_* : (J(M, n), C) \rightarrow (J(\hat{M}, n), \hat{C})$ , i.e., the differential map  $\varphi_* : J(M, n) \rightarrow J(\hat{M}, n)$  is a diffeomorphism sending  $C$  onto  $\hat{C}$ . The reason why the case  $m = 1$  is special is explained by the following theorem of Bäcklund.

**Theorem(Bäcklund)** *Let  $M$  and  $\hat{M}$  be manifolds of dimension  $m+n$ . Assume  $m \geq 2$ . Then, for an isomorphism  $\Phi : (J(M, n), C) \rightarrow (J(\hat{M}, n), \hat{C})$ , there exists a diffeomorphism  $\varphi : M \rightarrow \hat{M}$  such that  $\Phi = \varphi_*$ .*

The essential part of this theorem is to show that  $F = \text{Ker } \pi_*$  is the covariant system of  $(J(M, n), C)$  for  $m \geq 2$ . Namely an isomorphism  $\Phi$  sends  $F$  onto  $\hat{F} = \text{Ker } \hat{\pi}_*$  for  $m \geq 2$ . For the proof, we refer the reader to Theorem 1.4 in [Y2].

In case  $m = 1$ , it is a well known fact that the group of isomorphisms of  $(J(M, n), C)$ , i.e., the group of contact transformations, is larger than the group of diffeomorphisms of  $M$ . Therefore, when we consider the geometric 2-jet spaces, the situation differs according to whether the number  $m$  of dependent variables is 1 or greater.

(1) Case  $m = 1$ . We should start from a contact manifold  $(J, C)$  of dimension  $2n+1$ , which is locally a space of 1-jet for one dependent variable by Darboux's theorem. Then we can construct the geometric second order jet space  $(L(J), E)$  as follows: We consider the Lagrange-Grassmann bundle  $L(J)$  over  $J$  consisting of all  $n$ -dimensional integral elements of  $(J, C)$ ;

$$L(J) = \bigcup_{u \in J} L_u \subset J(J, n),$$

where  $L_u$  is the Grassmann manifolds of all Lagrangian (or Legendrian) subspaces of the symplectic vector space  $(C(u), d\varpi)$ . Here  $\varpi$  is a local contact form on  $J$ . Namely,

$v \in J(J, n)$  is an integral element if and only if  $v \subset C(u)$  and  $d\varpi|_v = 0$ , where  $u = \pi(v)$ . Let  $\pi : L(J) \rightarrow J$  be the projection. Then the canonical system  $E$  on  $L(J)$  is defined by

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \xrightarrow{\pi_*} T_u(J),$$

where  $\pi(v) = u$  for  $v \in L(J)$ . We have  $\partial E = \pi_*^{-1}(C)$  and  $\text{Ch}(C) = \{0\}$  (cf.[Y1]). Hence we get  $\text{Ch}(\partial E) = \text{Ker } \pi_*$ , which implies the Bäcklund theorem for  $(L(J), E)$  (cf. [Y1]).

Now we put

$$(J^2(M, n), C^2) = (L(J(M, n)), E),$$

where  $M$  is a manifold of dimension  $n + 1$ .

(2) Case  $m \geq 2$ . Since  $F = \text{Ker } \pi_*$  is a covariant system of  $(J(M, n), C)$ , we define  $J^2(M, n) \subset J(J(M, n), n)$  by

$$J^2(M, n) = \{n\text{-dim. integral elements of } (J(M, n), C), \text{ transversal to } F\},$$

$C^2$  is defined as the restriction to  $J^2(M, n)$  of the canonical system on  $J(J(M, n), n)$ .

Now the higher order (geometric) jet spaces  $(J^{k+1}(M, n), C^{k+1})$  for  $k \geq 2$  are defined (simultaneously for all  $m$ ) by induction on  $k$ . Namely, for  $k \geq 2$ , we define  $J^{k+1}(M, n) \subset J(J^k(M, n), n)$  and  $C^{k+1}$  inductively as follows:

$$J^{k+1}(M, n) = \{n\text{-dim. integral elements of } (J^k(M, n), C^k), \text{ transversal to } \text{Ker } (\pi_{k-1}^k)_*\},$$

where  $\pi_{k-1}^k : J^k(M, n) \rightarrow J^{k-1}(M, n)$  is the projection. Here we have

$$\text{Ker } (\pi_{k-1}^k)_* = \text{Ch}(\partial C^k),$$

and  $C^{k+1}$  is defined as the restriction to  $J^{k+1}(M, n)$  of the canonical system on  $J(J^k(M, n), n)$ .

where the derived system and Cauchy characteristic system of  $(R, D)$  are generally defined as follows;

The derived system  $\partial D$  of  $D$  is defined, in terms of sections, by

$$\partial \mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

where  $\mathcal{D} = \Gamma(D)$  denotes the space of sections of  $D$ . In general  $\partial D$  is obtained as a subsheaf of the tangent sheaf of  $R$  (for the precise argument, see e.g.[Y1], [BCG3]). Moreover higher derived systems  $\partial^i D$  are defined successively by

$$\partial^i D = \partial(\partial^{i-1} D),$$

where we put  $\partial^0 D = D$  by convention.  $D$  is called regular, if  $\partial^i D$  is subbundle for all  $i$ .

the Cauchy characteristic system  $\text{Ch}(D)$  of a differential system  $(R, D)$  is defined by

$$\text{Ch}(D)(x) = \{X \in D(x) \mid X \lrcorner d\omega_i \equiv 0 \pmod{\omega_1, \dots, \omega_s} \text{ for } i = 1, \dots, s\},$$

where  $D = \{\omega_1 = \dots = \omega_s = 0\}$  is defined locally by defining 1-forms  $\{\omega_1, \dots, \omega_s\}$ .

Here we observe that, if we drop the transversality condition in our definition of  $J^k(M, n)$  and collect all  $n$ -dimensional integral elements, we may have some singularities in  $J^k(M, n)$  in general. Namely, a set of all  $n$ -dimensional integral elements of  $(J^k(M, n), C^k)$  may be a variety.

In the next section, we will introduce that the geometric construction of the jet spaces without the transversality condition are candidates for the generalization of "Monster

Goursat manifold". We will give a criteria for the generalization of "Monster Goursat manifold" to be a manifold.

### 3. MAIN THEOREM

In this section, we extend "Monster Goursat manifolds" to multi independent variables cases. Namely we consider geometric construction of the jet spaces without transversality conditions.

Let  $(J^k(M, n), C^k)$  be a  $k$ -jet space and its canonical system( $\dim M = m + n$ ). We define  $\Sigma(J^k(M, n))$  as follows:

$$\Sigma(J^k(M, n)) := \bigcup_{x \in J^k} \Sigma_x$$

where  $\Sigma_x = \{\text{n-dim integral elements of } (J^k(M, n), C^k)\}$ . By definitin,

$$J^{k+1}(M, n) \subset \Sigma(J^k(M, n)) \subset J(C^k, n) \subset J(J^k(M, n), n),$$

where  $J(C^k, n)$  is defined by

$$J(C^k, n) = \bigcup_{x \in J^k} C_x, \quad C_x = \text{Gr}(C^k(x), n),$$

*i.e.*  $C_x$  is the Grassmann manifold consisting of  $n$ -dimensional subspaces in  $C^k(x)$ .

*Remark 3.1.* When  $n = 1$ ,  $\Sigma(J^k(M, 1))$  are called "rank 1 prolongation" of  $J^k(M, 1)$  in [SY]. Note that

$$\Sigma(J^k(M, 1)) = J(C^k, 1).$$

We can repeat the procedure of "rank 1 prolongation", starting from any differential system. We can define "k-th rank 1 prolongation" inductively. Moreover, when  $n = m = 1$ , "k-th rank 1 prolongation" of  $(J(M, 1), C)$  are called "Monster Goursat manifold" in [MZ].

Generally  $\Sigma(J^k(M, n))$  is a variety and is not a submanifold in  $J(C^k, n)$ . But we have the one of the main theorem in this paper;

**Theorem 3.2.**  $\Sigma(J^k(M^{n+m}, n))$  is a manifold if and only if  $\Sigma(J^2(M^{1+2}, 2))$  or trivial cases.

*Proof.* We showed the following theorem in [S];

**Theorem 3.3.** ([S])  $\Sigma(J^k(M^{n+m}, n))$  are not manifolds except for  $\Sigma(J^2(M^{1+2}, 2))$  and trivial cases.

Therefore, we only prove that  $\Sigma(J^2(M^{1+2}, 2))$  is a manifold. □

**Theorem 3.4.** A set of integral elements  $\Sigma(J^2(M^{1+2}, 2))$  of  $(J^2(M^{1+2}, 2), C^2)$  is a manifold.

*Proof.* For  $w \in \Sigma(J^2(M^{1+2}, 2))$ , let  $p(w) = v \in J^2(M^{1+2}, 2)$ , where  $p : \Sigma(J^2(M^{1+2}, 2)) \rightarrow J^2(M^{1+2}, 2)$  is the projection. And let  $(U, (x, y, z, p, q, r, s, t))$  be a canonical coordinate in  $J^2(M^{1+2}, 2)$  around  $v$ . Then the canonical system  $C^2$  is expressed by  $\{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ , where  $\varpi_0 = dz - pdx - qdy$ ,  $\varpi_1 = dp - rdx - sdy$ ,  $\varpi_2 = dq - sdx - tdy$ .

Let  $\pi : J(C^2, 2) \rightarrow J^2(M^{1+2}, 2)$  be the projection. Then  $\pi^{-1}(U)$  is covered by 10 open sets in  $J(C^2, 2)$ :

$$\pi^{-1}(U) = U_{xy} \cup U_{xr} \cup U_{xs} \cup U_{xt} \cup U_{yr} \cup U_{ys} \cup U_{yt} \cup U_{rs} \cup U_{rt} \cup U_{st},$$

where

$$U_{xy} := \{w \in \pi^{-1}(U) \mid dx \wedge dy|_w \neq 0\}$$

$$U_{xr} := \{w \in \pi^{-1}(U) \mid dx \wedge dr|_w \neq 0\}$$

$\vdots$

$$U_{st} := \{w \in \pi^{-1}(U) \mid ds \wedge dt|_w \neq 0\}.$$

In the following, we will explicitly describes the defining equation of  $\Sigma(J^2(M^{1+2}, 2))$  in terms of the inhomogeneous Grassmann coordinate of  $U_{xy}, \dots, U_{st}$ .

(0) On  $U_{xy}$ ;

In this case, note that  $dx \wedge dy|_w \neq 0$  is the transversality condition of the geometric construction of the jet spaces (§2). So the defining equation of  $\Sigma(J^2(M^{1+2}, 2))$  in  $U_{xy}$  will be that of the third order jet space  $J^3(M^{1+2}, 2)$ .

For  $w \in U_{xy}$ ,  $w$  is a 2-dim subspace of  $C^2(v)$ ,  $p(w) = v$ . Hence, restricting  $dr, ds, dt$  to  $w$ , we can introduce the inhomogeneous coordinate in  $U_{xy}$  of  $J(C^2, 2)$  around  $w$  as follows;

$$\begin{cases} dr|_w = p_{111}(w)dx|_w + p_{112}(w)dy|_w \\ ds|_w = p_{121}(w)dx|_w + p_{122}(w)dy|_w \\ dt|_w = p_{221}(w)dx|_w + p_{222}(w)dy|_w. \end{cases}$$

Moreover 2-dim integral element  $w$  satisfies  $d\varpi^1|_w = d\varpi^2|_w = 0$ ;

$$d\varpi^1|_w = -dr \wedge dx|_w - ds \wedge dy|_w = (p_{112}(w) - p_{121}(w))dx \wedge dy|_w$$

$$d\varpi^2|_w = -ds \wedge dx|_w - dt \wedge dy|_w = (p_{122}(w) - p_{221}(w))dx \wedge dy|_w.$$

In this way, we obtain the defining equations  $f_1 = f_2 = 0$  of  $\Sigma(J^2(M^{1+2}, 2))$  in the inhomogeneous coordinate  $U_{xr}$  of  $J(C^2, 2)$ , where  $f_1 = p_{112} - p_{121}$ ,  $f_2 = p_{122} - p_{221}$ ;

$$\{f_1 = f_2 = 0\} \subset U_{xy}.$$

Then  $df_1, df_2$  are independent on  $\{f_1 = f_2 = 0\}$ . Thus, we have

$$\begin{cases} dr|_w = p_{111}(w)dx|_w + p_{112}(w)dy|_w \\ ds|_w = p_{112}(w)dx|_w + p_{122}(w)dy|_w \\ dt|_w = p_{122}(w)dx|_w + p_{222}(w)dy|_w. \end{cases}$$

$(x, y, z, p, q, r, s, t, p_{111}, p_{112}, p_{122}, p_{222})$  is a coordinate of  $\Sigma(J^2(M^{1+2}, 2))$  in  $U_{xy}$ . This coordinate is called the canonical coordinate of the  $J^3(M^{1+2}, 2)$  (§2, §4).

(1) On  $U_{xr}$ ;

For  $w \in U_{xr}$ ,  $w$  is a 2-dim subspace of  $C^2(v)$ ,  $p(w) = v$ . Hence, restricting  $dy, ds, dt$  to  $w$ , we can introduce the inhomogeneous coordinate in  $U_{xr}$  of  $J(C^2, 2)$  around  $w$  as follows;

$$\begin{cases} dy|_w = a(w)dx|_w + B(w)dr|_w \\ ds|_w = c(w)dx|_w + D(w)dr|_w \\ dt|_w = e(w)dx|_w + F(w)dr|_w. \end{cases}$$

Moreover 2-dim integral element  $w$  satisfies  $d\varpi^1|_w = d\varpi^2|_w = 0$ ;

$$d\varpi^1|_w = -dr \wedge dx|_w - ds \wedge dy|_w = (-1 - a(w)D(w) + B(w)c(w))dr \wedge dx|_w$$

$$d\varpi^2|_w = -ds \wedge dx|_w - dt \wedge dy|_w = (-D(w) + B(w)e(w) - a(w)F(w))dr \wedge dx|_w.$$

In this way, we obtain the defining equations  $f_1 = f_2 = 0$  of  $\Sigma(J^2(M^{1+2}, 2))$  in the inhomogeneous coordinate  $U_{xr}$  of  $J(C^2, 2)$ , where  $f_1 = -1 - aD + Bc$ ,  $f_2 = -D + Be - aF$ ;

$$\{f_1 = f_2 = 0\} \subset U_{xr}.$$

Then  $df_1, df_2$  are independent on  $\{f_1 = f_2 = 0\}$ .

(2) On  $U_{xs}$ ;

For  $w \in U_{xs}$ ,  $w$  is a 2-dim subspace of  $C^2(v)$ ,  $p(w) = v$ . Hence, restricting  $dy, dr, dt$  to  $w$ , we have

$$\begin{cases} dy|_w = a(w)dx|_w + B(w)ds|_w \\ dr|_w = c(w)dx|_w + D(w)ds|_w \\ dt|_w = e(w)dx|_w + F(w)ds|_w. \end{cases}$$

Moreover 2-dim integral element  $w$  satisfies  $d\varpi^1|_w = d\varpi^2|_w = 0$ ;

$$d\varpi^1|_w = -dr \wedge dx|_w - ds \wedge dy|_w = (-D(w) - a(w))ds \wedge dx|_w$$

$$d\varpi^2|_w = -ds \wedge dx|_w - dt \wedge dy|_w = (-1 - a(w)F(w) + e(w)B(w))ds \wedge dx|_w.$$

Then the defining functions of  $\Sigma(J^2(M^{1+2}, 2))$  are independent by the same reasoning as in (1).

(3) On  $U_{xt}$ ;

For  $w \in U_{xt}$ ,  $w$  is a 2-dim subspace of  $C^2(v)$ ,  $p(w) = v$ . Hence, restricting  $dy, dr, ds$  to  $w$ , we have

$$\begin{cases} dy|_w = a(w)dx|_w + B(w)dt|_w \\ dr|_w = c(w)dx|_w + D(w)dt|_w \\ ds|_w = e(w)dx|_w + F(w)dt|_w. \end{cases}$$

Moreover 2-dim integral element  $w$  satisfies  $d\varpi^1|_w = d\varpi^2|_w = 0$ ;

$$d\varpi^1|_w = (-D(w) + e(w)B(w) - a(w)F(w))dt \wedge dx|_w$$

$$d\varpi^2|_w = (-F(w) - a(w))dt \wedge dx|_w.$$

Hence, we have

$$\begin{cases} dy|_w = a(w)dx|_w + B(w)dt|_w \\ dr|_w = c(w)dx|_w + (a^2(w) + e(w)B(w))dt|_w \\ ds|_w = e(w)dx|_w - a(w)dt|_w. \end{cases}$$



$(x, y, z, p, q, r, s, t, a, B, c, e)$  is a coordinate of  $\Sigma(J^2(M^{1+2}, 2))$  in  $U_{xt}$ .

(4) On  $U_{yr}$ ;

For  $w \in U_{yr}$ ,  $w$  is a 2-dim subspace of  $C^2(v)$ ,  $p(w) = v$ . Hence, restricting  $dx, ds, dt$  to  $w$ , we have

$$\begin{cases} dx|_w = a(w)dy|_w + B(w)dr|_w \\ ds|_w = c(w)dy|_w + D(w)dr|_w \\ dt|_w = e(w)dy|_w + F(w)dr|_w. \end{cases}$$

Moreover 2-dim integral element  $w$  satisfies  $d\varpi^1|_w = d\varpi^2|_w = 0$

$$\begin{aligned} d\varpi^1|_w &= (-a(w) - D(w))dr \wedge dy|_w \\ d\varpi^2|_w &= (c(w)B(w) - a(w)D(w) - F(w))dr \wedge dy|_w. \end{aligned}$$

(5) On  $U_{ys}$ ;

For  $w \in U_{ys}$ ,  $w$  is a 2-dim subspace of  $C^2(v)$ ,  $p(w) = v$ . Hence, restricting  $dx, dr, dt$  to  $w$ , we have

$$\begin{cases} dx|_w = a(w)dy|_w + B(w)ds|_w \\ dr|_w = c(w)dy|_w + D(w)ds|_w \\ dt|_w = e(w)dy|_w + F(w)ds|_w. \end{cases}$$

Moreover 2-dim integral element  $w$  satisfies  $d\varpi^1|_w = d\varpi^2|_w = 0$ ;

$$\begin{aligned} d\varpi^1|_w &= (c(w)B(w) - a(w)D(w) - 1)ds \wedge dy|_w \\ d\varpi^2|_w &= (-a(w) - F(w))ds \wedge dy|_w. \end{aligned}$$

(6) On  $U_{yt}$ ;

For  $w \in U_{yt}$ ,  $w$  is a 2-dim subspace of  $C^2(v)$ ,  $p(w) = v$ . Hence, restricting  $dx, dr, ds$  to  $w$ , we have

$$\begin{cases} dx|_w = a(w)dy|_w + B(w)dt|_w \\ dr|_w = c(w)dy|_w + D(w)dt|_w \\ ds|_w = e(w)dy|_w + F(w)dt|_w. \end{cases}$$

Moreover 2-dim integral element  $w$  satisfies  $d\varpi^1|_w = d\varpi^2|_w = 0$ ;

$$\begin{aligned} d\varpi^1|_w &= (c(w)B(w) - a(w)D(w) - F(w))dt \wedge dy|_w \\ d\varpi^2|_w &= (e(w)B(w) - a(w)F(w) - 1)dt \wedge dy|_w. \end{aligned}$$

(7) On  $U_{rs}$ ;

For  $w \in U_{rs}$ ,  $w$  is a 2-dim subspace of  $C^2(v)$ ,  $p(w) = v$ . Hence, restricting  $dx, dy, dt$  to  $w$ , we have

$$\begin{cases} dx|_w = A(w)dr|_w + B(w)ds|_w \\ dy|_w = C(w)dr|_w + D(w)ds|_w \\ dt|_w = E(w)dr|_w + F(w)ds|_w. \end{cases}$$

Moreover 2-dim integral element  $w$  satisfies  $d\varpi^1|_w = d\varpi^2|_w = 0$ ;

$$\begin{aligned} d\varpi^1|_w &= -dr \wedge dx|_w - ds \wedge dy|_w = (-B(w) + C(w))dr \wedge ds|_w \\ d\varpi^2|_w &= -ds \wedge dx|_w - dt \wedge dy|_w = (-A(w) + D(w)E(w) - C(w)F(w))ds \wedge dr|_w. \end{aligned}$$

Hence, we have

$$\begin{cases} dx|_w &= (D(w)E(w) - B(w)F(w))dr|_w + B(w)ds|_w \\ dy|_w &= B(w)dr|_w + D(w)ds|_w \\ dt|_w &= E(w)dr|_w + F(w)ds|_w. \end{cases}$$

$(x, y, z, p, q, r, s, t, B, D, E, F)$  is a coordinate of  $\Sigma(J^2(M^{1+2}, 2))$  in  $U_{rs}$ .

(8) On  $U_{rt}$ ;

For  $w \in U_{rt}$ ,  $w$  is a 2-dim subspace of  $C^2(v), p(w) = v$ . Hence, restricting  $dx, dy, ds$  to  $w$ , we have

$$\begin{cases} dx|_w &= A(w)dr|_w + B(w)dt|_w \\ dy|_w &= C(w)dr|_w + D(w)dt|_w \\ ds|_w &= E(w)dr|_w + F(w)dt|_w. \end{cases}$$

Moreover 2-dim integral element  $w$  satisfies  $d\varpi^1|_w = d\varpi^2|_w = 0$ ;

$$\begin{aligned} d\varpi^1|_w &= (-B(w) - D(w)E(w) + C(w)F(w))dr \wedge dt|_w \\ d\varpi^2|_w &= (-C(w) - A(w)F(w) + B(w)E(w))dt \wedge dr|_w. \end{aligned}$$

(9) On  $U_{st}$ ;

For  $w \in U_{st}$ ,  $w$  is a 2-dim subspace of  $C^2(v), p(w) = v$ . Hence, restricting  $dx, dy, dr$  to  $w$ , we have

$$\begin{cases} dx|_w &= A(w)ds|_w + B(w)dt|_w \\ dy|_w &= C(w)ds|_w + D(w)dt|_w \\ dr|_w &= E(w)ds|_w + F(w)dt|_w. \end{cases}$$

Moreover 2-dim integral element  $w$  satisfies  $d\varpi^1|_w = d\varpi^2|_w = 0$ ;

$$\begin{aligned} d\varpi^1|_w &= (-B(w)E(w) + A(w)F(w) - D(w))ds \wedge dt|_w \\ d\varpi^2|_w &= (-B(w) + C(w))ds \wedge dt|_w. \end{aligned}$$

From (0),  $\dots$ , (9), we conclude  $\Sigma(J^2(M^{1+2}, 2))$  is a submanifold in  $J(C^2, 2)$ . □

#### 4. REVIEW OF TANAKA THEORY

Next we will consider the local equivalence problem of  $(\Sigma(J^2(M^{1+2}, 2)), D)$ , where  $D$  is a canonical system on  $\Sigma(J^2(M^{1+2}, 2))$  (see §5). To this purpose, we first recall Tanaka theory of weakly regular differential systems in this section(see [T],[Y1]).

**4.1. Weak derived system.** Let  $D$  be a differential system, then we define  $k$ -th weak higher derived system  $\partial^{(k)}\mathcal{D}$  by ;

$$\partial^{(1)}\mathcal{D} = \partial\mathcal{D} , \quad \partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [D, \partial^{(k-1)}\mathcal{D}]$$

where  $\mathcal{D} = \Gamma(D)$ .  $D$  is called weakly regular, if  $\partial^{(i)}D$  is subbundle for all  $i$ .

For weakly regular differential system  $D$ , the followings hold ;

$$(S1) \quad \exists \mu \text{ s.t. } D^{-1} \subset D^{-2} \subset \dots \subset D^{-k} \subset \dots \\ \dots \subset D^{-(\mu-1)} \subset D^{-\mu} = D^{-(\mu+1)} = \dots$$

$$(S2) \quad [\mathcal{D}^p, \mathcal{D}^q] \subset D^{p+q} \quad \forall p, q < 0 \\ \text{i.e. } [X, Y] \in \mathcal{D}^{p+q} , \quad X \in \mathcal{D}^p, Y \in \mathcal{D}^q \quad \forall p, q < 0$$

where  $D^{-1} := D$ ,  $D^{-k} := \partial^{(k-1)}D$  ( $k \geq 2$ ).

**4.2. Symbol algebra of weakly regular differential system.** Let  $(R, D)$  be a weakly regular differential system such that

$$T(R) = D^{-\mu} \supset D^{-(\mu-1)} \supset \dots \supset D^{-1} = D.$$

For all  $x \in R$ , we put  $\mathfrak{g}_{-1}(x) := D^{-1}(x) = D(x)$ ,  $\mathfrak{g}_p(x) := D^p(x)/D^{p+1}(x)$ .

$$\mathfrak{m}(x) := \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

Then  $\dim \mathfrak{m}(x) = \dim M$ . For  $X \in \mathfrak{g}_p(x)$ ,  $Y \in \mathfrak{g}_q(x)$ , Lie bracket  $[X, Y] \in \mathfrak{g}_{p+q}(x)$  is defined by ;

Let  $\tilde{X} \in \mathcal{D}^p$ ,  $\tilde{Y} \in \mathcal{D}^q$  be extensions ( $\tilde{X}_x = X$ ,  $\tilde{Y}_x = Y$ ), then  $[\tilde{X}, \tilde{Y}] \in \mathcal{D}^{p+q}$   $[X, Y] := [\tilde{X}, \tilde{Y}]_x \in \mathfrak{g}_{p+q}(x)$  is not depend on the extensions.

We call  $(\mathfrak{m}(x), [ \ ])$  Symbol Algebra of  $(R, D)$  at  $x$ .

### 4.3. Example.

*Example 4.1.* Let  $J^3(M^{1+2}, 2); (x_1, x_2, y, p_1, p_2, p_{11}, p_{12}, p_{22}, p_{111}, p_{112}, p_{122}, p_{222})$  be a canonical coordinate, then  $C^3 = \{\varpi = \varpi_1 = \varpi_2 = \varpi_{11} = \varpi_{12} = \varpi_{22} = 0\}$ , where

$$\begin{cases} \varpi &= dy - p_1 dx_1 - p_2 dx_2 \\ \varpi_i &= dp_i - p_{i1} dx_1 - p_{i2} dx_2 \\ \varpi_{ij} &= dp_{ij} - p_{ij1} dx_1 - p_{ij2} dx_2. \end{cases}$$

And the structure equation is

$$\begin{cases} d\varpi &\equiv 0 & (\text{mod } C^3) \\ d\varpi_i &\equiv 0 & (\text{mod } C^3) \\ d\varpi_{ij} &= -dp_{ij1} \wedge dx_1 - dp_{ij2} \wedge dx_2 & (\text{mod } C^3). \end{cases}$$

Therefore  $\partial^{(1)}C^3 = \partial C^3 = \{\varpi = \varpi_1 = \varpi_2 = 0\}$ . The structure equations for  $\partial C^3$  and  $\partial^{(1)}C^3$  are

$$\begin{cases} d\varpi &\equiv 0 & (\text{mod } \partial C^3) \\ d\varpi_i &\equiv -dp_{i1} \wedge dx_1 - dp_{i2} \wedge dx_2 & (\text{mod } \partial C^3) \end{cases}$$

$$\begin{cases} d\varpi \equiv 0 & (\text{mod } \partial^{(1)}C^3, \varpi_{ij} \wedge \varpi_{kl}) \\ d\varpi_i \equiv -dp_{i1} \wedge dx_1 - dp_{i2} \wedge dx_2 & (\text{mod } \partial^{(1)}C^3, \varpi_{ij} \wedge \varpi_{kl}). \end{cases}$$

Thus  $\partial^{(2)}C^3 = \partial^2C^3 = \{\varpi = 0\}$ . The structure equations for  $\partial^2C^3, \partial^{(2)}C^3$  are

$$\begin{cases} d\varpi \equiv -dp_1 \wedge dx_1 - dp_2 \wedge dx_2 & (\text{mod } \partial^2C^3) \\ \begin{cases} d\varpi \equiv -dp_1 \wedge dx_1 - dp_2 \wedge dx_2 \\ (\text{mod } \partial^{(2)}C^3, \varpi_{ij} \wedge \varpi_{kl}, \varpi_i \wedge \varpi_{jk}, \varpi_i \wedge \varpi_j). \end{cases} \end{cases}$$

Therefore  $\partial^{(3)}C^3 = \partial^3C^3 = T(J^3)$ . Especially,  $(J^3(M, 1), C^3)$  is regular and weakly regular.

Symbol algebra of  $J^3(M^{1+2}, 2)$ ;

We take coframe:  $\{\varpi, \varpi_1, \varpi_2, \varpi_{11}, \varpi_{12}, \varpi_{22}, dp_{111}, dp_{112}, dp_{122}, dp_{222}, dx_1, dx_2, \}$  and its dual frame  $\{\frac{\partial}{\partial y}, \frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_{ij}}, \frac{\partial}{\partial p_{ijk}}, \frac{d}{dx_i}\}$ ,

where  $\frac{d}{dx_i} := \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial y} + p_{1i} \frac{\partial}{\partial p_1} + p_{2i} \frac{\partial}{\partial p_2} + p_{11i} \frac{\partial}{\partial p_{11}} + p_{12i} \frac{\partial}{\partial p_{12}} + p_{22i} \frac{\partial}{\partial p_{22}}$ .

Then, at  $x \in J^3$ ,

$$\mathfrak{g}_{-1}(x) := C^3 = \left\langle \frac{\partial}{\partial p_{ijk}}, \frac{d}{dx_i} \right\rangle, \quad \mathfrak{g}_{-2}(x) := \left\langle \frac{\partial}{\partial p_{ij}} \right\rangle,$$

$$\mathfrak{g}_{-3}(x) := \left\langle \frac{\partial}{\partial p_i} \right\rangle, \quad \mathfrak{g}_{-4}(x) := \left\langle \frac{\partial}{\partial y} \right\rangle,$$

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-4}.$$

The bracket relations are;

$$\left[ \frac{\partial}{\partial p_{jkl}}, \frac{d}{dx_i} \right] = \delta_{il} \frac{\partial}{\partial p_{jk}}, \quad \left[ \frac{\partial}{\partial p_{jk}}, \frac{d}{dx_i} \right] = \delta_{ik} \frac{\partial}{\partial p_j}, \quad \left[ \frac{\partial}{\partial p_j}, \frac{d}{dx_i} \right] = \delta_{ij} \frac{\partial}{\partial y}, \quad \text{the other is 0.}$$

## 5. EQUIVALENCE PROBLEM OF $(\Sigma(J^2(M^{1+2}, 2)), D)$

From now on, we denote  $(\Sigma(J^2(M^{1+2}, 2)), D)$  simply by  $(\Sigma(J^2), D)$ . Because  $\Sigma(J^2)$  is manifold, we can define the canonical system  $D$  on  $\Sigma(J^2)$  as follows;  
 $\forall u \in \Sigma(J^2), p(u) = x$

$$D(u) = p_*^{-1}(u) \subset T_u(\Sigma(J^2)) \xrightarrow{p_*} T_x(J^2)$$

where  $p : \Sigma(J^2) \rightarrow J^2(M^{1+2}, 2)$  is the projection.

In this section, we will consider the equivalence problem of  $(\Sigma(J^2), D)$ . Namely we will give the orbit decomposition under the  $\text{Aut}(\Sigma(J^2), D)$ , where

$$\text{Aut}(\Sigma(J^2), D) = \{\varphi : \Sigma(J^2) \rightarrow \Sigma(J^2) \mid \varphi: \text{local diffeomorphism such that } \varphi_*(D) = D\}.$$

*Remark 5.1.* Let  $\varphi : J^2(M, 2) \rightarrow J^2(M, 2)$  be an isomorphism, i.e.,  $\varphi$  is a diffeomorphism such that  $\varphi_*(C^2) = C^2$ . Then  $\varphi$  induces the isomorphism  $\varphi_* : (\Sigma(J^2), D) \rightarrow (\Sigma(J^2), D)$ , namely, the differential map  $\varphi_* : \Sigma(J^2) \rightarrow \Sigma(J^2)$  is a diffeomorphism sending  $D$  onto  $D$ .

First, we explain geometric meaning of the decomposition in the proof of theorem 3.4.  $\Sigma(J^2)$  has another decomposition;

$$\Sigma(J^2) = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \text{ (disjoint union),}$$

where  $\Sigma_i = \{w \in \Sigma(J^2) \mid \dim(w \cap \text{fiber}) = i\}$  ( $i = 0, 1, 2$ ), the fiber means that of  $\pi_* : T(J^2) \supset C^2 \rightarrow T(J^1)$ . Then,

$$\begin{aligned} \Sigma_0|_{p^{-1}(U)} &= U_{xy} \\ \Sigma_1|_{p^{-1}(U)} &= (U_{xr} \cup U_{xs} \cup U_{xt} \cup U_{yr} \cup U_{ys} \cup U_{yt}) \setminus U_{xy} \\ \Sigma_2|_{p^{-1}(U)} &= (U_{rs} \cup U_{rt} \cup U_{st}) \setminus (U_{xy} \cup U_{xr} \cup U_{xs} \cup U_{xt} \cup U_{yr} \cup U_{ys} \cup U_{yt}). \end{aligned}$$

$\Sigma_0 = J^3$  is an open set in  $\Sigma(J^2)$ .  $\Sigma_1$  is a codimension 1 submanifold in  $\Sigma(J^2)$ .  $\Sigma_2$  is a codimension 2 submanifold in  $\Sigma(J^2)$  and is a  $\mathbb{P}^2$ -bundle over  $J^2$ .

**Proposition 5.2.**  $(\Sigma(J^2), D)$  is regular, but is not weakly regular, precisely

$$D \subset \partial D \subset \partial^2 D \subset \partial^3 D = T\Sigma(J^2(M^1, 2)).$$

Moreover  $\partial^2 D = \partial^{(2)} D$  and

$$\begin{cases} \partial^{(3)} D = T\Sigma(J^2) & \text{on } \Sigma_0 \cup \Sigma_1 \\ \partial^{(3)} D = \partial^{(2)} D & \text{on } \Sigma_2 \end{cases}$$

*Remark 5.3.* Note that  $\Sigma_0 = J^3$  by definition. So the derived system, weak derived system around  $w \in \Sigma_0$  and the symbol algebra at  $w \in \Sigma_0$  are given in the example 4.1.

*Proof.* We take canonical coordinate on  $J^2$  and consider the decomposition of  $\Sigma(J^2): U_{xy} \cup U_{xr} \cup U_{xs} \cup U_{xt} \cup U_{yr} \cup U_{ys} \cup U_{yt} \cup U_{rs} \cup U_{rt} \cup U_{st}$  (see proof of theorem 4.3.). First of all, we show that it is enough to work on three open sets  $U_{xr}, U_{rs}, U_{rt}$ .

**Lemma 5.4.**  $p^{-1}(U) = U_{xy} \cup U_{xr} \cup U_{xs} \cup U_{xt} \cup U_{yr} \cup U_{ys} \cup U_{yt} \cup U_{rs} \cup U_{rt} \cup U_{st} = U_{xy} \cup U_{xt} \cup U_{yr} \cup U_{rs} \cup U_{rt} \cup U_{st}$ , under the notation of the proof of theorem 3.4.

*Proof.* First, we prove  $U_{xr} \subset U_{xt} \cup U_{xy}$ .

For  $w \in U_{xr}$ ,

$$\begin{cases} dy|_w = a(w)dx|_w + B(w)dr|_w \\ ds|_w = c(w)dx|_w + D(w)dr|_w \\ dt|_w = e(w)dx|_w + F(w)dr|_w, \end{cases}$$

and the relations are  $f_1 = -1 - aD + Bc = 0$ ,  $f_2 = -D + Be - aF = 0$ . Note that  $w \in U_{xy}$  if and only if  $dx$  and  $dy$  are independent at  $w$ , i.e.,  $B(w) \neq 0$ . Assume that  $B(w) = 0$ , then  $F(w) \neq 0$  from  $f_1 = 0$ . So

$$dx \wedge dt|_w = dx \wedge (e(w)dx + F(w)dr)|_w = F(w)dx \wedge dr|_w \neq 0.$$

Therefore  $w \in U_{xt} \cup U_{xy}$ .

The same argument yeilds  $U_{xs} \subset U_{xt} \cup U_{xy}$ ,  $U_{ys} \subset U_{yr} \cup U_{xy}$  and  $U_{yt} \subset U_{yr} \cup U_{xy}$ .  $\square$

From above remark, lemma and natural symmetry, where natural symmetry means the isomorphism induced by  $\bar{x} = y, \bar{y} = x, \bar{p} = q, \bar{q} = p, \bar{r} = t, \bar{t} = r$ , it is enough to work on  $U_{xr}, U_{rs}, U_{rt}$ , because every germ in  $U_{yr}$  appears in  $U_{xt}$  and that of  $U_{st}$  appears in  $U_{rs}$ . On  $U_{xr}$ ;

We take a coordinate  $(x, y, z, p, q, r, s, t, a, B, c, e)$  on  $U_{xt}$  (see proof of theorem 4.3.), then  $D = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_y = \varpi_r = \varpi_s = 0\}$ , where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_y = dy - adx - Bdt \\ \varpi_1 = dp - rdx - sdy & \varpi_r = dr - cdx - (a^2 + eB)dt \\ \varpi_2 = dq - sdx - tdy & \varpi_s = ds - edx + adt. \end{cases}$$

Recall that, for  $w = (x, y, z, p, q, r, s, t, a, B, c, e)$ ,  $B \neq 0$  if and only if  $w \in \Sigma_0$ , therefore it is enough to consider at  $w$  in the hypersurface  $\{B = 0\} \subset \Sigma(J^2)$ . The structure equation at a point in  $\{B = 0\}$  is

$$\begin{cases} d\varpi_i \equiv 0 & (i = 0, 1, 2) \\ d\varpi_y = -da \wedge dx - dB \wedge dt \neq 0 \\ d\varpi_r = -dc \wedge dx - (edB + 2ada) \wedge dt \neq 0 \\ d\varpi_s = -de \wedge dx + da \wedge dt \neq 0 & (\text{mod } D). \end{cases}$$

Hence  $\partial D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ . The structure equation of  $\partial D$  at a point in  $\{B = 0\}$  is

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv (-\varpi_r - a\varpi_s + e\varpi_y) \wedge dx - a\varpi_y \wedge dt \neq 0 \\ d\varpi_2 = -\varpi_s \wedge dx - dt \wedge \varpi_y \neq 0 \\ & (\text{mod } \partial D, \varpi_\alpha \wedge \varpi_\beta (\alpha, \beta = y, r, s)). \end{cases}$$

Hence  $\partial^{(2)}D = \partial^2D = \{\varpi_0 = 0\}$ . The structure equation of  $\partial^2D$  at a point in  $\{B = 0\}$  is

$$\begin{cases} d\varpi_0 \equiv -(\varpi_1 + a\varpi_2) \wedge dx \neq 0 \\ & (\text{mod } \partial^2D, \varpi_\alpha \wedge \varpi_\beta (\alpha, \beta \in \{y, r, s, 1, 2\})). \end{cases}$$

Hence  $\partial^{(3)}D = \partial^3D = T(\Sigma(J^2))$ . We conclude

$$\partial^{(3)}D = T\Sigma(J^2) \text{ on } \Sigma_0 \cup \Sigma_1.$$

On  $U_{rs}$ ;

Let  $(x, y, z, p, q, r, s, t, B, D, E, F)$  be a coordinate on  $U_{rs}$ . Then  $D = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_y = \varpi_t = 0\}$ , where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_x = dx - (DE - BF)dr - Bds \\ \varpi_1 = dp - rdx - sdy & \varpi_y = dy - Bdr - Dds \\ \varpi_2 = dq - sdx - tdy & \varpi_t = dt - Edr - Fds. \end{cases}$$

$w \in \Sigma_2$  if and only if  $dx|_w = dy|_w = 0$ . So, in this coordinate,  $\Sigma_2$  is a  $\{B = D = 0\}$ : codimension 2 submanifold in  $\Sigma(J^2)$ .

The structure equation at a point in  $\{B = D = 0\}$  is

$$\begin{cases} d\varpi_i \equiv 0 & (i = 0, 1, 2) \\ d\varpi_x = -(EdD - FdB) \wedge dr - dB \wedge ds \neq 0 \\ d\varpi_y = -dB \wedge dr - dD \wedge ds \neq 0 \\ d\varpi_t = -dE \wedge dr - dF \wedge ds \neq 0 & (\text{mod } D). \end{cases}$$

Hence  $\partial D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ . The structure equation of  $\partial D$  at a point in  $\{B = D = 0\}$  is

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv -dr \wedge \varpi_x - ds \wedge \varpi_y \neq 0 \\ d\varpi_2 \equiv -ds \wedge \varpi_x - (E dr + F ds) \wedge \varpi_y \neq 0 \\ \quad \quad \quad (\text{mod } \partial D, \varpi_\alpha \wedge \varpi_\beta \ (\alpha, \beta = x, y, t)). \end{cases}$$

Hence  $\partial^{(2)} D = \partial^2 D = \{\varpi_0 = 0\}$ . The structure equation of  $\partial^2 D$  at a point in  $\{B = D = 0\}$  is

$$\begin{cases} d\varpi_0 \equiv -\varpi_1 \wedge \varpi_x - \varpi_2 \wedge \varpi_y \neq 0 \\ \quad \quad \quad (\text{mod } \partial^2 D) \\ d\varpi_0 \equiv 0 \\ \quad \quad \quad (\text{mod } \partial^2 D, \varpi_\alpha \wedge \varpi_\beta \ (\alpha, \beta \in \{x, y, t, 1, 2\})). \end{cases}$$

Therefore  $\partial^3 D = T(\Sigma(J^2))$  on  $U_{rs}$  and  $\partial^{(3)} D = \{\varpi_0 = 0\}$  on  $\Sigma_2 \cap U_{rs}$ .

On  $U_{rt}$ ;

$$\begin{cases} dx|_w = A(w)dr|_w + B(w)dt|_w \\ dy|_w = C(w)dr|_w + D(w)dt|_w \\ ds|_w = E(w)dr|_w + F(w)dt|_w, \end{cases}$$

where defining equations are  $-B - DE + CF = 0$ ,  $-C - AF + BE = 0$ .  $w \in \Sigma_2$  if and only if  $A = B = C = D = 0$ . Moreover, if  $(E, F) \neq (0, 0)$  then the point is in  $U_{rs}$  or  $U_{st}$ . So we consider a point  $(E, F) = (0, 0)$  and take a coordinate  $(x, y, z, p, q, r, s, t, A, D, E, F)$  around  $(E, F) = (0, 0)$ . Then  $D = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_y = \varpi_s = 0\}$  where

$$\begin{cases} \varpi_0 = dz - p dx - q dy & \varpi_x = dx - A dr - \frac{DE + AF^2}{EF - 1} dt \\ \varpi_1 = dp - r dx - s dy & \varpi_y = dy - \frac{DE^2 + AF}{EF - 1} dr - D dt \\ \varpi_2 = dq - s dx - t dy & \varpi_s = ds - E dr - F dt. \end{cases}$$

The structure equation at a point  $(A, D, E, F) = 0$  is

$$\begin{cases} d\varpi_i \equiv 0 & (i = 0, 1, 2) \\ d\varpi_x = -dA \wedge dr \neq 0 \\ d\varpi_y = -dD \wedge dt \neq 0 \\ d\varpi_s = -dE \wedge dr - dF \wedge dt \neq 0 \quad (\text{mod } D). \end{cases}$$

Hence  $\partial D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ . The structure equation of  $\partial D$  at a point in  $(A, D, E, F) = 0$  is

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv -dr \wedge \varpi_x \neq 0 \\ d\varpi_2 \equiv -dt \wedge \varpi_y \neq 0 \\ \quad \quad \quad (\text{mod } \partial D, \varpi_\alpha \wedge \varpi_\beta \ (\alpha, \beta = x, y, s)). \end{cases}$$





$$\begin{cases} d\varpi_0 \equiv -\hat{\varpi}_1 \wedge dx \neq 0 \\ \quad \quad \quad (\text{mod } \partial^2 D, \varpi_\alpha \wedge \varpi_i, \varpi_i \wedge \varpi_j \ (\alpha \in \{y, r, s\}, i, j \in \{1, 2\})) \end{cases}$$

Hence its dual frame satisfies the relation of this proposition.  $\square$

**Theorem 5.7. (normal form)**

For  $w \in \Sigma_1$ .  $((\Sigma(J^2(M^1, 2)), D), w)$  is locally isomorphic to a germ at the origin in  $(\mathbb{R}^{12}, \hat{D})$ ; where  $(\mathbb{R}^{12}; x, y, z, p, q, r, s, t, a, B, c, e)$  is coordinate and  $\hat{D}$  is expressed by  $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_y = \varpi_r = \varpi_s = 0\}$ , where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_y = dy - adx - Bdt \\ \varpi_1 = dp - rdx - sdy & \varpi_r = dr - cdx - (a^2 + eB)dt \\ \varpi_2 = dq - sdx - tdy & \varpi_s = ds - edx + adt. \end{cases}$$

*Proof.* We construct the paths from any points to the origin, directly. For  $w_0 \in \Sigma_1$ , we may assume  $w_0$  is expressed by a germ at  $w_0 = (0, \dots, 0, a_0, 0, c_0, e_0)$ ,

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_y = dy - adx - Bdt \\ \varpi_1 = dp - rdx - sdy & \varpi_r = dr - cdx - (a^2 + eB)dt \\ \varpi_2 = dq - sdx - tdy & \varpi_s = ds - edx + adt, \end{cases}$$

because of normal form of  $J^2$  and  $w_0 \in \Sigma_1$  if and only if  $B = 0$ . Hence what we have to do is to construct  $\varphi \in \text{Aut}(\Sigma(J^2), D)$  sending  $(0, \dots, 0, a_0, 0, c_0, e_0)$  to  $(0, \dots, 0)$ . Let  $\varphi_e$  be a

$$\begin{aligned} \varphi_e : (x, y, z, p, q, r, s, t, a, B, c, e) \mapsto \\ (x, y, z - \frac{e_0}{2}x^2y, p - e_0xy, q - \frac{e_0}{2}x^2, r - e_0y, s - e_0x, t, a, B, c - e_0a, e - e_0) \end{aligned}$$

Then

$$\begin{cases} \varphi_e^* \varpi_0 = \varpi_0 & \varphi_e^* \varpi_y = \varpi_y \\ \varphi_e^* \varpi_1 = \varpi_1 & \varphi_e^* \varpi_r = \varpi_r - e_0 \varpi_y \\ \varphi_e^* \varpi_2 = \varpi_2 & \varphi_e^* \varpi_s = \varpi_s. \end{cases}$$

Therefore,  $\varphi_e$  leaves  $D$  invariant and sends a germ  $(0, \dots, 0, a_0, 0, c_0, e_0)$  to a germ  $(0, \dots, 0, a_0, 0, c'_0, 0)$  where  $c'_0 = c_0 - e_0a_0$ .

Similarly, Let  $\varphi_a, \varphi_c$  be

$$\begin{aligned} \varphi_a : (x, y, z, p, q, r, s, t, a, B, c, e) \mapsto \\ (x, y, z, p + a_0q, q, r + 2a_0s + a_0^2t, s + a_0t, t, a - a_0, B, c + 2ea_0, e) \\ \varphi_c : (x, y, z, p, q, r, s, t, a, B, c, e) \mapsto \\ (x, y, z - \frac{c'_0}{6}x^3, p - \frac{c'_0}{2}x^2, q, r - c'_0x, s, t, a, B, c - c'_0, e) \end{aligned}$$

Then these maps preserve  $D$  and the composition  $\varphi_c \circ \varphi_a$  sends a germ  $(0, \dots, 0, a_0, 0, c'_0, 0)$  to a germ  $(0, \dots, 0, 0, 0, 0, 0)$ , where above isomorphisms are obtained by focusing on the form  $\varpi_c = dc + 2ade - eda$  in the proof of proposition 5.5. and leaving the form invariant to keep the symbol algebras.  $\square$

## 5.2. Classification of $\Sigma_2$ .

Finally, we will classify points in  $\Sigma_2$ . From the proposition 5.2.,  $w \in \Sigma_2$ , we can not define the symbol algebra at  $w$ . But  $\partial^{(1)}D$  and  $\partial^{(2)}D$  are subbundle, so we can define pseudo symbol algebra at  $w$  as follows;

For  $w \in \Sigma_2$ , we put  $\mathfrak{g}_{-1}(w) := D^{-1}(w) = D(w)$ ,  $\mathfrak{g}_{-2}(w) := D^{-2}(w)/D^{-1}(w)$ ,  $\mathfrak{g}_{-3}(w) := D^{-3}(w)/D^{-2}(w)$ ,  $\mathfrak{g}_{-4}(w) := T_w(\Sigma(J^2))/D^{-3}(w)$ .

$$\mathfrak{m}(w) = \mathfrak{g}_{-1}(w) \oplus \mathfrak{g}_{-2}(w) \oplus \mathfrak{g}_{-3}(w) \oplus \mathfrak{g}_{-4}(w).$$

We define Lie bracket by the same way of the usual symbol algebra except for  $[\mathfrak{g}_{-1}, \mathfrak{g}_{-3}]$ . For  $[\mathfrak{g}_{-1}, \mathfrak{g}_{-3}]$ , we define  $[\mathfrak{g}_{-1}, \mathfrak{g}_{-3}] = 0$ .

Note that this pseudo symbol algebra does not satisfy the generating condition  $[\mathfrak{g}_{-1}, \mathfrak{g}_{-3}] = \mathfrak{g}_{-4}$ .

**Lemma 5.8.** *For  $w_0 \in \Sigma_2$ , there exists  $w \in U_{rs}$  such that  $w$  is locally isomorphic to  $w_0$*

*Proof.* Note that  $\Sigma_2$  is covered by  $U_{rs} \cup U_{rt} \cup U_{st}$ . From the symmetry  $x$  and  $y$ ,  $U_{st}$  is isomorphic to  $U_{rs}$ . So it is enough to consider the points in  $U_{rt} \setminus (U_{rs} \cup U_{st})$ .  $U_{rt} \setminus (U_{rs} \cup U_{st})$  is a set consisting of a point  $w_0$ .  $w_0$  is the origin in the coordinate  $U_{rt}$ , i.e.,  $w_0$  is the integral element given by;

$$w_0 = \{dx = dy = dz = dp = dq = ds = 0\} = \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial t} \right\rangle.$$

Then we consider the isomorphism  $\varphi : (J^2, C^2) \rightarrow (J^2, C^2)$ ;

$$\varphi : (x, y, z, p, q, r, s, t) \mapsto (\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{t}) = (x - y, y, z, p, q + p, r, s + r, t + 2s + r).$$

$\varphi$  sends the integral element  $w_0$  to  $\bar{w}_0$  where the  $\bar{w}_0$  is expressed by

$$\bar{w}_0 = \{d\bar{x} = d\bar{y} = d\bar{z} = d\bar{p} = d\bar{q} = d(\bar{s} - \bar{r}) = 0\} = \left\langle \frac{\partial}{\partial \bar{t}}, \frac{\partial}{\partial \bar{r}} + \frac{\partial}{\partial \bar{s}} \right\rangle,$$

in the new coordinate system. Thus  $\bar{w}_0 \in U_{\bar{r}\bar{s}}$  in this new coordinate system.  $\square$

From above lemma, it is enough to classify the points in  $U_{rs}$ .

**Proposition 5.9.** *For  $w \in \Sigma_2$ , pseudo symbol algebra  $\mathfrak{m}(w)$  is isomorphic to  $\mathfrak{m}(E, F)$ ,  $\mathfrak{m}(E, F) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-4}$  and  $[\cdot, \cdot]$  is given by;*

$$[X_B, X_r] = X_y - FX_x, \quad [X_D, X_s] = X_y, \quad [X_B, X_s] = X_x, \quad [X_D, X_r] = EX_x$$

$$[X_E, X_r] = X_t, \quad [X_F, X_s] = X_t$$

$$[X_r, X_x] = X_p, \quad [X_s, X_y] = X_p + FX_q, \quad [X_s, X_x] = X_q, \quad [X_r, X_y] = EX_q$$

the other is trivial,

where  $\{X_z, X_p, X_q, X_x, X_y, X_t, X_r, X_s, X_B, X_D, X_E, X_F\}$  are basis, and

$$\mathfrak{g}_{-1} = \langle \{X_r, X_s, X_B, X_D, X_E, X_F\} \rangle$$

$$\mathfrak{g}_{-2} = \langle \{X_x, X_y, X_t\} \rangle$$

$$\mathfrak{g}_{-3} = \langle \{X_p, X_q\} \rangle$$

$$\mathfrak{g}_{-4} = \langle \{X_z\} \rangle$$

where  $E, F \in \mathbb{R}$  are parameters.

*Proof.* We may assume  $w \in U_{rs}$  by the above lemma. From the proof of theorem 3.4., in  $U_{rs}$ ,  $D$  is expressed by  $D = \{\varpi^0 = \varpi^1 = \varpi^2 = \varpi_x = \varpi_y = \varpi_t = 0\}$  where  $(x, y, z, p, q, r, s, t, B, D, E, F)$  is the coordinate and

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_x = dx - (DE - BF)dr - Bds \\ \varpi_1 = dp - rdx - sdy & \varpi_y = dy - Bdr - Dds \\ \varpi_2 = dq - sdx - tdy & \varpi_t = dt - Edr - Fds. \end{cases}$$

Recall that  $w \in \Sigma_2$  if and only if  $B = D = 0$  in this coordinate. Let  $\{X_z, X_p, X_q, X_x, X_y, X_t, X_r, X_s, X_B, X_D, X_E, X_F\}$  be the dual frame of the coframe  $\{\varpi^0, \varpi^1, \varpi^2, \varpi_x, \varpi_y, \varpi_t, dr, ds, dB, dD, dE, dF\}$ . From the proof of proposition 5.2., the structure equations are;

$$\begin{cases} d\varpi_i \equiv 0 & (i = 0, 1, 2) \\ d\varpi_x = -(EdD - FdB) \wedge dr - dB \wedge ds \neq 0 \\ d\varpi_y = -dB \wedge dr - dD \wedge ds \neq 0 \\ d\varpi_t = -dE \wedge dr - dF \wedge ds \neq 0 & (\text{mod } D). \end{cases}$$

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv -dr \wedge \varpi_x - ds \wedge \varpi_y \neq 0 \\ d\varpi_2 \equiv -ds \wedge \varpi_x - (Edr + Fds) \wedge \varpi_y \neq 0 \\ & (\text{mod } \partial D, \varpi_\alpha \wedge \varpi_\beta (\alpha, \beta = x, y, t)). \end{cases}$$

$$\begin{cases} d\varpi_0 \equiv -\varpi_1 \wedge \varpi_x - \varpi_2 \wedge \varpi_y \neq 0 \\ & (\text{mod } \partial^2 D) \\ d\varpi_0 \equiv 0 \\ & (\text{mod } \partial^2 D, \varpi_\alpha \wedge \varpi_\beta (\alpha, \beta \in \{x, y, t, 1, 2\})). \end{cases}$$

Thus we obtain the result. □

For the pseudo symbol algebra  $\mathfrak{m}(E, F)$ , the followings are intrinsic;

$$\begin{aligned} \mathfrak{g}_{-1}^V &= \{X \in \mathfrak{g}_{-1} \mid ad(X)|_{\mathfrak{g}_{-2}} = 0\} = \langle X_B, X_D, X_E, X_F \rangle \\ \mathfrak{g}_{-2}^V &= \{X \in \mathfrak{g}_{-2} \mid ad(X)|_{\mathfrak{g}_{-1}} = 0\} = \langle X_t \rangle \\ \tilde{\mathfrak{g}}_{-1} &= \{X \in \mathfrak{g}_{-1} \mid \text{Im } ad(X)|_{\mathfrak{g}_{-1}} \in \mathfrak{g}_{-2}^V\} = \langle X_E, X_F \rangle. \end{aligned}$$

**Lemma 5.10.** *For the pseudo symbol algebra  $\mathfrak{m}(E, F)$ , let  $Ch(\mathfrak{m}(E, F))$  be a set of the characteristic directions, that is,*

$$Ch(\mathfrak{m}(E, F)) = \{V \subset \mathfrak{g}_{-1} : 1 - \dim \text{subspace} \mid X \in V, X \neq 0, \text{rank } ad(X)|_{\mathfrak{g}_{-2}} = 1\}.$$

Then

$$\#Ch(\mathfrak{m}(E, F)) = \begin{cases} 2 & (F^2 + 4E > 0) \\ 1 & (F^2 + 4E = 0) \\ 0 & (F^2 + 4E < 0). \end{cases}$$

*Remark 5.11.* For  $w \in \Sigma_2$ ,  $w$  is said to be *hyperbolic*, *elliptic* or *parabolic* according to whether  $F^2 + 4E$  is positive, negative or zero, respectively.

*Proof.* For  $X \in \mathfrak{g}_{-1}$ ,

$$X = \xi X_r + \eta X_s + X^V \quad (\xi, \eta \in \mathbb{R}, X^V \in \mathfrak{g}_{-1}^V)$$

Then

$$\begin{cases} ad(X)(X_x) &= \xi X_p + \eta X_q \\ ad(X)(X_y) &= \xi(EX_q) + \eta(X_p + FX_q) = \eta X_p + (\xi E + \eta F)X_q \\ ad(X)(X_t) &= 0. \end{cases}$$

Hence  $X$  is a characteristic direction if and only if  $X$  is a null direction for the quadratic form

$$\xi(\xi E + \eta F) - \eta^2 = E\xi^2 + F\xi\eta - \eta^2.$$

Therefore the determinant of this quadratic form classifies the number of the characteristic directions.  $\square$

From above lemma,  $\Sigma_2$  has at least 3 components. But this classification is sufficient by the following theorem.

**Theorem 5.12.**

$$\Sigma_2 = \Sigma_h \cup \Sigma_e \cup \Sigma_p$$

where

$$\begin{aligned} \Sigma_h &= \Sigma_2 \cap \{w : \text{hyperbolic point}\} \\ \Sigma_e &= \Sigma_2 \cap \{w : \text{elliptic point}\} \\ \Sigma_p &= \Sigma_2 \cap \{w : \text{parabolic point}\} \end{aligned}$$

For  $w \in \Sigma_2$ ,  $\mathfrak{m}(w)$  is isomorphic to  $\mathfrak{m}(1, 0)$ ,  $\mathfrak{m}(-1, 0)$  or  $\mathfrak{m}(0, 0)$  according to whether  $w \in \Sigma_h$ ,  $w \in \Sigma_e$  or  $w \in \Sigma_p$ , respectively. Moreover,

$w \in \Sigma_h$  is locally isomorphic to a germ at  $(0, \dots, 0, 1, 0)$  in  $(\mathbb{R}^{12}, \bar{D})$ .

$w \in \Sigma_e$  is locally isomorphic to a germ at  $(0, \dots, 0, -1, 0)$  in  $(\mathbb{R}^{12}, \bar{D})$ ,

$w \in \Sigma_p$  is locally isomorphic to a germ at  $(0, \dots, 0, 0, 0)$  in  $(\mathbb{R}^{12}, \bar{D})$ .

where  $(\mathbb{R}^{12}; x, y, z, p, q, r, s, t, B, D, E, F)$  is coordinate and  $\bar{D}$  is expressed by  $\bar{D} = \{\varpi^0 = \varpi^1 = \varpi^2 = \varpi_x = \varpi_y = \varpi_t = 0\}$  where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_x = dx - (DE - BF)dr - Bds \\ \varpi_1 = dp - rdx - sdy & \varpi_y = dy - Bdr - Dds \\ \varpi_2 = dq - sdx - tdy & \varpi_t = dt - Edr - Fds. \end{cases}$$

*Proof.* First, we introduce the isomorphisms  $\varphi_a (a \in \mathbb{R}) : U_{rs} \rightarrow U_{rs}$  and  $\psi : U_{rs} \rightarrow U_{rs}$ . these isomorphisms will preserve the determinant of the quadratic form;

$$F^2 + 4E = \bar{F}^2 + 4\bar{E}.$$

For nonzero  $a \in \mathbb{R}$ , we define  $\varphi_a$  by;

$$\begin{aligned} \varphi_a : (x, y, z, p, q, r, s, t, B, D, E, F) &\mapsto (\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{t}, \bar{B}, \bar{D}, \bar{E}, \bar{F}) \\ &= \left( \frac{x}{a^2}, \frac{y}{a}, \frac{z}{a^4}, \frac{p}{a^2}, \frac{q}{a^3}, r, \frac{s}{a}, \frac{t}{a^2}, \frac{B}{a}, D, \frac{E}{a^2}, \frac{F}{a} \right). \end{aligned}$$

$\psi$  is defined by;

$$\begin{aligned} \psi : (x, y, z, p, q, r, s, t, B, D, E, F) &\mapsto (\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{t}, \bar{B}, \bar{D}, \bar{E}, \bar{F}) \\ &= \left( x - \frac{y}{2}, y, z, p, q + \frac{p}{2}, r, s + \frac{r}{2}, t + \frac{r}{4} + s, B - \frac{D}{2}, D, E - \frac{F}{2} - \frac{1}{4}, F + 1 \right). \end{aligned}$$

Then

$$\begin{cases} \psi^* \bar{\omega}_0 = \bar{\omega}_0 & \psi^* \bar{\omega}_x = \bar{\omega}_x - \frac{1}{2} \bar{\omega}_y \\ \psi^* \bar{\omega}_1 = \bar{\omega}_1 & \psi^* \bar{\omega}_y = \bar{\omega}_y \\ \psi^* \bar{\omega}_2 = \bar{\omega}_2 + \frac{1}{2} \bar{\omega}_1 & \psi^* \bar{\omega}_t = \bar{\omega}_t. \end{cases}$$

(1) For  $w \in \Sigma_h$ , we may assume  $w = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, E_0, F_0) \in U_{rs}$ . Then  $F_0^2 + 4E_0 > 0$ . If  $F_0 \neq 0$ ,  $\varphi_{-F_0}$  sends  $w$  to  $w' = (0, \dots, 0, \frac{E_0}{F_0^2}, -1)$ .  $\psi$  sends  $w'$  to  $w'' = (0, \dots, 0, \frac{E_0}{F_0^2} + \frac{1}{2} - \frac{1}{4}, 0)$ . Furthermore  $\varphi_{\sqrt{E'_0}}$  sends  $w''$  to  $(0, \dots, 0, 1, 0)$ , where  $E'_0 = \frac{E_0}{F_0^2} + \frac{1}{2} - \frac{1}{4}$ .  
If  $F_0 = 0$ ,  $\varphi_{\sqrt{E_0}}$  sends  $w$  to  $(0, \dots, 0, 1, 0)$ .

(2) For  $w \in \Sigma_e$ , we may assume  $w = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, E_0, F_0) \in U_{rs}$ . Then  $F_0^2 + 4E_0 < 0$ . If  $F_0 \neq 0$ ,  $\varphi_{-F_0}$  sends  $w$  to  $w' = (0, \dots, 0, \frac{E_0}{F_0^2}, -1)$ .  $\psi$  sends  $w'$  to  $w'' = (0, \dots, 0, \frac{E_0}{F_0^2} + \frac{1}{2} - \frac{1}{4}, 0)$ . Furthermore  $\varphi_{\sqrt{-E'_0}}$  sends  $w''$  to  $(0, \dots, 0, -1, 0)$ , where  $E'_0 = \frac{E_0}{F_0^2} + \frac{1}{2} - \frac{1}{4}$ .  
If  $F_0 = 0$ ,  $\varphi_{\sqrt{-E_0}}$  sends  $w$  to  $(0, \dots, 0, -1, 0)$ .

(3) For  $w \in \Sigma_p$ , we may assume  $w = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, E_0, F_0) \in U_{rs}$ . Then  $F_0^2 + 4E_0 = 0$ . If  $F_0 = 0$ ,  $E_0 = 0$ .  
If  $F_0 \neq 0$ ,  $\varphi_{-F_0}$  sends  $w$  to  $w' = (0, \dots, 0, -\frac{1}{4}, -1)$ .  $\psi$  sends  $w'$  to  $(0, \dots, 0, 0, 0)$ .

The normal forms of the pseudo symbol algebras are obtained by the above local normal forms. □

We summarize

**Corollary 5.13.**

$$\Sigma(J^2) = \Sigma_0 \cup \Sigma_1 \cup (\Sigma_h \cup \Sigma_e \cup \Sigma_p)$$

where

$$\begin{aligned} \Sigma_0 &= \{w \in \Sigma(J^2) \mid \dim(w \cap \text{fiber}) = 0\} = J^3 \\ \Sigma_1 &= \{w \in \Sigma(J^2) \mid \dim(w \cap \text{fiber}) = 1\} \\ \Sigma_2 &= \{w \in \Sigma(J^2) \mid \dim(w \cap \text{fiber}) = 2\} \\ \Sigma_2 &= \Sigma_h \cup \Sigma_p \cup \Sigma_e \\ \Sigma_h &= \Sigma_2 \cap \{w : \text{hyperbolic point}\} \\ \Sigma_e &= \Sigma_2 \cap \{w : \text{elliptic point}\} \\ \Sigma_p &= \Sigma_2 \cap \{w : \text{parabolic point}\} \end{aligned}$$

$\Sigma_0$  is an open set in  $\Sigma(J^2)$ .  $\Sigma_1$  is an codimension 1 submanifold in  $\Sigma(J^2)$   $\Sigma_2$  is an codimension 2 submanifold in  $\Sigma(J^2)$  and  $P^2$ -bundle over  $J^2$ .  $\Sigma_h, \Sigma_e$  are also

codimension 2 submanifolds in  $\Sigma(J^2)$ .  $\Sigma_p$  is an codimension 3 submanifold in  $\Sigma(J^2)$ .

Moreover, the each component have the following normal forms;

(0)  $\Sigma_0$  has jet type normal form.

(1)  $w \in \Sigma_1$  is locally isomorphic to a germ at the origin in  $(\mathbb{R}^{12}, \hat{D})$  where  $(\mathbb{R}^{12}; x, y, z, p, q, r, s, t, a, B, c, e)$  is coordinate and  $\hat{D}$  is expressed by  $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_y = \varpi_r = \varpi_s = 0\}$ , where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_y = dy - adx - Bdt \\ \varpi_1 = dp - rdx - sdy & \varpi_r = dr - cdx - (a^2 + eB)dt \\ \varpi_2 = dq - sdx - tdy & \varpi_s = ds - edx + adt. \end{cases}$$

(2)

$w \in \Sigma_h$  is locally isomorphic to a germ at  $(0, \dots, 0, 1, 0)$  in  $(\mathbb{R}^{12}, \bar{D})$ .

$w \in \Sigma_e$  is locally isomorphic to a germ at  $(0, \dots, 0, -1, 0)$  in  $(\mathbb{R}^{12}, \bar{D})$ ,

$w \in \Sigma_p$  is locally isomorphic to a germ at  $(0, \dots, 0, 0, 0)$  in  $(\mathbb{R}^{12}, \bar{D})$ .

where  $(\mathbb{R}^{12}; x, y, z, p, q, r, s, t, B, D, E, F)$  is coordinate and  $\bar{D}$  is expressed by  $\bar{D} = \{\varpi^0 = \varpi^1 = \varpi^2 = \varpi_x = \varpi_y = \varpi_t = 0\}$  where

$$\begin{cases} \varpi_0 = dz - pdx - qdy & \varpi_x = dx - (DE - BF)dr - Bds \\ \varpi_1 = dp - rdx - sdy & \varpi_y = dy - Bdr - Dds \\ \varpi_2 = dq - sdx - tdy & \varpi_t = dt - Edr - Fds. \end{cases}$$

## Appendix

### A description of integral manifolds of $(\Sigma(J^2), D)$

In this appendix, we will consider integral manifolds of  $(\Sigma(J^2), D)$ . If  $S$  is a 2-dim submanifold of  $\Sigma(J^2)$  and satisfies  $TS \subset D$ , then  $S$  is called a 2-dim integral manifold of  $(\Sigma(J^2), D)$ . And If  $S$  is a 2-dim integral manifold of  $(\Sigma(J^2), D)$  with  $\Omega|_S \neq 0$ , then  $S$  is called an integral manifold of  $(\Sigma(J^2), D)$  with independence condition  $\Omega$ , where  $\Omega$  is a 2-form on  $(\Sigma(J^2), D)$  independent modulo  $D$ .

We describe the relation between the integral manifolds of  $(\Sigma(J^2), D)$  and singular solutions of partial differential equations of second order.

For  $(J^k(M^{m+n}, n), C^k)$ , the integral manifolds  $S$  with independence condition  $dx_1 \wedge \dots \wedge dx_n$  correspond to the graph of the  $k$ -jet extension of  $m$  functions of  $n$  variables. Hence, the integral manifolds with independence condition  $dx_1 \wedge \dots \wedge dx_n$  depend on  $m$  functions of  $n$  variables.

*Example 5.14.* Let  $(x, y, z, p, q, r, s, t, p_{111}, p_{112}, p_{122}, p_{222})$  be a normal coordinate on  $J^3(M^{1+2}, 2)$ . the solution  $S$  with independence condition  $dx \wedge dy$  is expressed by;

$$S = (x, y, z(x, y), z_x(x, y), z_y(x, y), z_{xx}(x, y), z_{xy}(x, y), z_{yy}(x, y), z_{xxx}(x, y), z_{xxy}(x, y), z_{xyy}(x, y), z_{yyy}(x, y)).$$

And the integral manifolds depend on 1 function of 2 variables  $z(x, y)$ .

Now, we consider the integral manifolds of  $(\Sigma(J^2), D)$  passing through  $\Sigma_1 = \{\dim(w \cap \text{fiber}) = 1\}$  with some independence condition. Note that the integral manifolds  $S \subset \{\dim(w \cap \text{fiber}) = 0\}$  is in the above example.

**Proposition 5.15.** *Let  $(x, y, z, p, q, r, s, t, a, B, c, e)$  be the normal coordinate around  $\Sigma_1$  on  $(\Sigma(J^2), D)$  (theorem 5.7.).*

*If  $S$  is an integral manifold of  $(\Sigma(J^2), D)$  passing through  $\Sigma_1$  with independence condition  $dx \wedge dt$ , then  $S$  is written by;*

$$(1) \quad S = (x, y(x, t), \int qy_t dt + z_0(x), z_x - qy_x, \int ty_t dt + q_0(x), p_x - sy_x, q_x - ty_x, t, y_x, y_t, r_x, s_x).$$

*In other words, the integral manifolds depend on 1 function of 2 variables  $y(x, t)$  and 2 functions of 1 variable  $q_0(x), z_0(x)$ .*

*Conversely, for any  $y(x, t)$  with  $y_t(0, 0) = 0$  and  $q_0(x), z_0(x)$ . We can construct the integral manifold by (1).*

*Proof.* Let  $S$  be an integral manifold, then  $S = (x, y(x, t), z(x, t), \dots, t, \dots, e(x, t))$  from independence condition. And  $S^* \varpi_i = 0$  ( $i = 0, 1, 2, y, r, s$ );

$$(2) \quad S^* \varpi_0 = (z_x - qy_x - p)dx + (z_t - qy_t)dt = 0$$

$$(3) \quad S^* \varpi_1 = (p_x - r - sy_x)dx + (p_t - sy_t)dt = 0$$

$$(4) \quad S^* \varpi_2 = (q_x - s - ty_x)dx + (q_t - ty_t)dt = 0$$

$$(5) \quad S^* \varpi_y = (y_x - a)dx + (y_t - B)dt = 0$$

$$(6) \quad S^* \varpi_r = (r_x - c)dx + (r_t - (a^2 + eB))dt = 0$$

$$(7) \quad S^* \varpi_s = (s_x - e)dx + (s_t + a)dt = 0.$$

$a = y_x$  and  $B = y_t$  is determined by (5), and note that the condition passing through  $\Sigma_1$  is  $B = y_t = 0$ . From (4),  $q = \int ty_t dt + q_0(x)$  where  $q_0(x)$  is a function on  $S$  depending only on  $x$ , and  $s = q_x - ty_x$ . From (7),  $e = s_x$ . From (2),  $z = \int qy_t dt + z_0(x)$  where  $z_0(x)$  is a function on  $S$  depending only on  $x$ , and  $p = z_x - qy_x$ . From (3),  $r = p_x - sy_x$ . From (6)  $c = r_x$ . Therefore

$$(8) \quad S = (x, y(x, t), \int qy_t dt + z_0(x), z_x - qy_x, \int ty_t dt + q_0(x), p_x - sy_x, q_x - ty_x, t, y_x, y_t, r_x, s_x).$$

Conversely, for any  $y(x, t)$  with  $y_t(0, 0) = 0$  and  $q_0(x), z_0(x)$ . We define the 2-dim submanifold  $S$  by (1), then  $y_t(0, 0) = 0$  ensure that passing through  $\Sigma_1$  and the rest 3 conditions in (2),  $\dots$ , (7) are satisfied by definition, automatically.  $\square$

Let  $p : \Sigma(J^2) \rightarrow J^1$  be a natural projection,  $(x, y, z, p, q, r, s, t, \dots) \mapsto (x, y, z, p, q)$  ( $J^1$  is a 5-dim contact manifold).

**Corollary 5.16.** *The projection of the integral manifolds passing through  $\Sigma_1$  with independence condition  $dx \wedge dt$  have singularities at the origin.*

**Corollary 5.17.** *Let  $S$  be an integral manifolds with independence condition  $dx \wedge dt$ . Assume  $S \subset \Sigma_1$ , then the projection of  $S$  is a regular curve.*

*Proof.* The condition is  $B = y_t \equiv 0$ . Hence  $y(x, t) = y(x)$  depends only on  $x$ . From the above theorem, we have

$$\begin{aligned}
x &= x \\
y &= y(x) \\
z &= z_0(x) \\
p &= z'_0 - q'_0 y' \\
q &= q_0(x) \\
r &= (q''_0 + 3)t^3 - 3q''_0 x t + z''_0 \\
s &= q'_0 - t y' \\
t &= t \\
a &= y' \\
B &= 0 \\
c &= z'''_0 - (q'''_0 y' + q''_0 y'') - (q''_0 y'' + q'_0 y''') - (q''_0 - t y'') y' - (q'_0 - t y') y'' \\
e &= q''_0 - t y''.
\end{aligned}$$

□

*Example 5.18.* (cusp) Let  $y(x, t) = t^3 - 3xt$ ,  $z_0(x)$  and  $q_0(x)$ . Then the integral manifold  $S(x, t)$  is

$$\begin{aligned}
x &= x \\
y &= t^3 - 3xt \\
z &= \frac{9}{28}t^7 - \frac{27}{20}xt^5 + (q_0 + \frac{3}{2}x^2)t^3 - 3q_0xt + z_0 \\
p &= \frac{9}{10}t^5 + (q'_0 - \frac{3}{2}x)t^3 - 3q'_0xt + z'_0 \\
q &= \frac{3}{4}t^4 - \frac{3}{2}xt^2 + q_0 \\
r &= (q''_0 + 3)t^3 - 3q''_0xt + z''_0 \\
s &= \frac{3}{2}t^2 + q'_0 \\
t &= t \\
a &= -3t \\
B &= 3t^2 - 3x \\
c &= q'''_0 t^3 - 3(q''_0 + xq'''_0)t + z'''_0 \\
e &= q''_0
\end{aligned}$$

from direct calculation.

*Example 5.19.* (Cartan's overdetermined system ) We consider the Cartan's overdetermined system

$$r = \frac{1}{3}t^3, \quad s = \frac{1}{2}t^2.$$



The Lie algebra of infinitesimal contact transformations of the system is isomorphic to the 14-dim exceptional simple Lie algebra  $G_2$ .

Let  $y(x, t) = -xt$ ,  $z_0(x) = 0$  and  $q_0(x) = 0$ . Then the integral manifold  $S(x, t)$  is

$$\begin{array}{ll} x = x & s = \frac{1}{2}t^2 \\ y = -xt & t = t \\ z = \frac{1}{6}x^2t^3 & a = -t \\ p = -\frac{1}{6}xt^3 & B = -x \\ q = -\frac{1}{2}xt^2 & c = 0 \\ r = \frac{1}{3}t^3 & e = 0. \end{array}$$

Therefore the projection of the integral manifold  $S(x, t)$  is a singular solution of the Cartan's overdetermined system, where the projection is  $\Sigma(J^2) \rightarrow J^2$ .

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