# On the general transformation of the Wirtinger integral

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## §1 Introduction.

The Wirtinger integral is the uniformization to the upper half plane $H$ of the hypergeometric function defined on the complex projective line $\mathbf{P}^1$. In [5] we established the transformation formulas of the Wirtinger integral for the linear fractional transformations $\tau \to \tau + 2$ and $\tau \to \dfrac{-1}{\tau}$ with the aide of the theory of theta functions. As a corollary we obtain the transformation formulas of the Wirtinger integral for the linear fractional transformations $\tau \to \tau + 2$ and $\tau \to \dfrac{\tau}{-2\tau+1}$ which are identified with generators of of the principal congruence subgroup $\varGamma(2)$ modulo center. These formulas correspond to the monodromy matrices of the hypergeometric function for generators of the fundamental group of $\mathbf{P}^1$ minus three points. The purpose of this paper is to generalize this result, namely, to establish the transformation formula of the Wirtinger integral for the general element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\varGamma(2)$, which corresponds to a general monodromy matrix of the hypergeometric function. It seems to be quite difficult to obtain such a formula directly from the hypergeometric function because we have no standard way of expressing a general element of the fundamental group corresponding to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \varGamma(2)$.

Following the notation of Chandrasekharan [1], we introduce the four theta functions $\theta(v,\tau), \theta_i(v,\tau)$ $(i = 1,2,3)$ by

$$\theta(v,\tau) = \frac{1}{i}\sum_{n=-\infty}^{+\infty}(-1)^n e^{(n+\frac{1}{2})^2\pi i\tau}e^{(2n+1)\pi iv}, \quad \theta_1(v,\tau) = \sum_{n=-\infty}^{+\infty} e^{(n+\frac{1}{2})^2\pi i\tau}e^{(2n+1)\pi iv}, \quad \theta_2(v,\tau) = \sum_{n=-\infty}^{+\infty}(-1)^n e^{n^2\pi i\tau}e^{2n\pi iv}, \quad \theta_3(v,\tau) = \sum_{n=-\infty}^{+\infty} e^{n^2\pi i\tau}e^{2n\pi iv},$$

which are defined for all $(v,\tau) \in \mathbf{C} \times H$, where $\mathbf{C}$ denote the complex plane. Mumford [2] adopts the symbols $\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}$ to denote the theta functions above. The relations between the two notations are as follows: $\theta(v,\tau) = -\theta_{11}(v,\tau),\ \theta_1(v,\tau) = \theta_{10}(v,\tau),\ \theta_2(v,\tau) = \theta_{01}(v,\tau),\ \theta_3(v,\tau) = \theta_{00}(v,\tau)$. In this paper we also use the following abbreviations: $\theta_i(v) = \theta_i(v,\tau),\ \theta_i = \theta_i(0,\tau)$, etc.

We define two functions $z_1(\tau),\ z_2(\tau)$, which we have been proposing to call *Wirtinger integrals* in our papers [5], [6] (see also [7]), by

$$z_1(\tau) = \theta_3^2 \int_0^{\frac{1}{2}} \theta(v,\tau)^{2\alpha-1}\theta_1(v,\tau)^{2\gamma-2\alpha-1}\theta_2(v,\tau)^{2\beta-2\gamma+3}\theta_3(v,\tau)^{-2\beta-1}dv,$$
$$z_2(\tau) = \theta_3^2 \int_0^{\frac{1}{2}} \theta(v,\tau)^{2\beta-2\gamma+3}\theta_1(v,\tau)^{-2\beta-1}\theta_2(v,\tau)^{2\alpha-1}\theta_3(v,\tau)^{2\gamma-2\alpha-1}dv,$$

where we assume that $\alpha, \beta, \gamma$ satisfy certain conditions such that these integrals are convergent, and that $\arg v = 0$ on the interval of integral $0 < v < 1/2$. They are the lifts of Gauss' hypergeometric functions of $SL$ type to the upper half plane, and form a fundamental system of solutions for the lift of Gauss' hypergeometric differential equation of $SL$ type to the upper half plane. Let $\varGamma(2)$ be the principal congruence subgroup of level 2, and let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\Gamma(2)$. Without loss of generality we may assume $0<c<d$. The problem which we will study in this paper is as follows:

**Problem 1.** Determine the constants $A$ and $B$ with respect to $\tau$ such that $z_1\left(\frac{a\tau+b}{c\tau+d}\right)=Az_1(\tau)+Bz_2(\tau)$.

One can pose a similar problem for $z_2\left(\frac{a\tau+b}{c\tau+d}\right)$, of which we omit the detail in this paper. Since the group $\Gamma(2)$ modulo center is isomorphic to the fundamental group of the Riemann sphere minus three points (which is the defining region of Gauss' hypergeometric differential equation), the constants $A,\ B$ are identified with components of the monodromy matrix for the element of the fundamental group corresponding to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Substitution $\tau\to\frac{a\tau+b}{c\tau+d}$ and $v\to\frac{v}{c\tau+d}$ makes the integral representation above for $z_1(\tau)$ into

$$z_1\left(\frac{a\tau+b}{c\tau+d}\right)=\theta_3\left(0,\frac{a\tau+b}{c\tau+d}\right)^2\int_0^{(c\tau+d)/2}\theta\left(\frac{v}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)^{2\alpha-1}\theta_1\left(\frac{v}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)^{2\gamma-2\alpha-1}\theta_2\left(\frac{v}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)^{2\beta-2\gamma+3}\theta_3\left(\frac{v}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)^{-2\beta-1}\frac{dv}{c\tau+d}.$$

The following formulas are well-known (e.g. [3], [4]):

$$\begin{aligned}
\theta\left(\frac{v}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)&=\left(\frac{c}{d}\right)e^{\pi i(3d-2+bd-cd)/4}\sqrt{\frac{c\tau+d}{i}}e^{c\pi iv^2/(c\tau+d)}\theta(v,\tau),\\
\theta_1\left(\frac{v}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)&=\left(\frac{c}{d}\right)e^{\pi i(d-4+2c+bd-2cd)/4}\sqrt{\frac{c\tau+d}{i}}e^{c\pi iv^2/(c\tau+d)}\theta_1(v,\tau),\\
\theta_2\left(\frac{v}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)&=\left(\frac{c}{d}\right)e^{\pi i(3d-4+2a-ab+bd-cd)/4}\sqrt{\frac{c\tau+d}{i}}e^{c\pi iv^2/(c\tau+d)}\theta_2(v,\tau),\\
\theta_3\left(\frac{v}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)&=\left(\frac{c}{d}\right)e^{\pi i(3d-3+2a+2c-ab-ad-bc+bd-2cd)/4}\sqrt{\frac{c\tau+d}{i}}e^{c\pi iv^2/(c\tau+d)}\theta_3(v,\tau)
\end{aligned}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma(2)$ with $c>0$, where $\left(\frac{c}{d}\right)$ denotes Jacobi's symbol. Substituting these formulas into the integral representation for $z_1\left(\frac{a\tau+b}{c\tau+d}\right)$, we have

$$z_1\left(\frac{a\tau+b}{c\tau+d}\right)=e^{\pi id(b+c)/2}e^{\pi i\alpha}e^{\pi id(\alpha-\gamma)}e^{\pi ic(2-d)(\gamma-\alpha-\beta)/2}e^{\pi ia(b-2)\gamma/2}\theta_3^2\int_0^{(c\tau+d)/2}\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}dv \tag{1.1}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma(2)$ with $c>0$. Noticing that the integers $c,\ d$ are relatively prime to each other, we define integers $k_\nu$ by

$$k_0=0,\qquad k_\nu=\left[\frac{d\nu}{c}\right]+1\qquad(1\le\nu\le c-1),\qquad k_c=d,$$

where for a real number $x$ the symbol $[x]$ denotes the maximal integer not exceeding to $x$.
The branch cuts $L_\nu$ $(\nu=0,\dots,c)$ of the integrand $\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}$ of the integral in (1.1) are given as lines on $\mathbf{C}$. Namely, $L_0$

is defined by the equation $v = s$ for the real paremeter $s$ such that $s \leq 0, \frac{1}{2} \leq s$; if $\nu \geq 1$, $L_\nu$ is defined by $v = s + \dfrac{\nu}{2}\tau$ for the real paremeter $s$ such that $s \leq \dfrac{k_\nu - 1}{2}$, $\dfrac{k_\nu}{2} \leq s$. The integral in (1.1) is decomposed as follows:

$$\int_0^{(c\tau+d)/2} \theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}dv = \sum_{\nu=0}^{c-1}(I_\nu + J_\nu), \tag{1.2}$$

where

$$I_\nu = \int_{(k_\nu+\nu\tau)/2}^{(k_{\nu+1}+\nu\tau)/2} \theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}dv \tag{1.3}$$

and

$$J_\nu = \int_{(k_{\nu+1}+\nu\tau)/2}^{\{k_{\nu+1}+(\nu+1)\tau\}/2} \theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}dv. \tag{1.4}$$

Furthermore, the integral $I_\nu$ has the following decomposition:

$$I_\nu = \sum_{\mu=0}^{l_{\nu+1}-1} \int_0^{\frac{1}{2}} \theta\left(v + \frac{k_\nu+\mu}{2} + \frac{\nu}{2}\tau\right)^{2\alpha-1} \theta_1\left(v + \frac{k_\nu+\mu}{2} + \frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1} \theta_2\left(v + \frac{k_\nu+\mu}{2} + \frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3} \theta_3\left(v + \frac{k_\nu+\mu}{2} + \frac{\nu}{2}\tau\right)^{-2\beta-1} dv, \tag{1.5}$$

where $l_{\nu+1} = k_{\nu+1} - k_\nu$. Thus, in order to solve Problem 1, it suffices to solve the following:

**Problem 2.** Determine the constants $P_\nu$, $Q_\nu$, $R_\nu$, $S_\nu$ with respect to $\tau$ such that $I_\nu = P_\nu z_1(\tau)/\theta_3^2 + Q_\nu z_2(\tau)/\theta_3^2$ and $J_\nu = R_\nu z_1(\tau)/\theta_3^2 + S_\nu z_2(\tau)/\theta_3^2$.

In fact, the constants $A$ and $B$ in Problem 1 are written by

$$\begin{aligned}
A =& e^{\pi i d(b+c)/2}e^{\pi i\alpha}e^{\pi i d(\alpha-\gamma)}e^{\pi i c(2-d)(\gamma-\alpha-\beta)/2}e^{\pi i a(b-2)\gamma/2}\sum_{\nu=0}^{c-1}(P_\nu + R_\nu),\\
B =& e^{\pi i d(b+c)/2}e^{\pi i\alpha}e^{\pi i d(\alpha-\gamma)}e^{\pi i c(2-d)(\gamma-\alpha-\beta)/2}e^{\pi i a(b-2)\gamma/2}\sum_{\nu=0}^{c-1}(Q_\nu + S_\nu).
\end{aligned}$$

**Theorem.** The integrals $I_\nu$'s and $J_\nu$'s in (1.2) are given as follows: $I_0$ is given by (3.2), $I_\nu$ for a positive even integer $\nu$ $(1 \leq \nu \leq c-1)$ by (4.11), $I_\nu$ for a positive odd integer $\nu$ $(1 \leq \nu \leq c-1)$ by (5.10), $J_0$ by (6.3), $J_\nu$ for a positive even integer $\nu$ $(1 \leq \nu \leq c-1)$ by (7.8), and $J_\nu$ for a positive odd integer $\nu$ $(1 \leq \nu \leq c-1)$ by (8.6).

The proof is given in Sections 2–8.

*Remark.* The linear fractional transformation of the general form $\tau \to \dfrac{a\tau+b}{c\tau+d}$ with $a, d$ odd and $b, c$ even is composed of a finite number of the linear fractional transformations each of which is of the form $\tau \to \tau + m$ with $m$ even or $\tau \to \dfrac{\tau}{n\tau+1}$ with $n$ even. According to the results of [5], the matrices $M_0(m)$ and $M_1(n)$

satisfying

$$\begin{pmatrix} z_1(\tau+m) \\ z_2(\tau+m) \end{pmatrix} = M_0(m) \begin{pmatrix} z_1(\tau) \\ z_2(\tau) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} z_1\left(\dfrac{\tau}{n\tau+1}\right) \\ z_2\left(\dfrac{\tau}{n\tau+1}\right) \end{pmatrix} = M_1(n) \begin{pmatrix} z_1(\tau) \\ z_2(\tau) \end{pmatrix}$$

are given by

$$M_0(m) = \begin{bmatrix} e^{m\pi i(\gamma-1)} & 0 \\ 0 & e^{m\pi i(1-\gamma)} \end{bmatrix}$$

and

$$M_1(n) = \begin{bmatrix} \dfrac{\sin\pi(\gamma-\alpha)\sin\pi(\gamma-\beta)e^{n\pi i(\alpha+\beta-\gamma+1)} - \sin\pi\alpha\sin\pi\beta e^{n\pi i(\gamma-\alpha-\beta-1)}}{\sin\pi\gamma\sin\pi(\gamma-\alpha-\beta)} & \dfrac{-2\pi i\Gamma(1-\gamma)\Gamma(2-\gamma)\sin n\pi(\gamma-\alpha-\beta-1)}{\Gamma(-\alpha)\Gamma(-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\beta-\gamma)\sin\pi(\gamma-\alpha-\beta)} \\ \dfrac{-2\pi i\Gamma(\gamma-1)\Gamma(\gamma)\sin n\pi(\gamma-\alpha-\beta-1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)\sin\pi(\gamma-\alpha-\beta)} & \dfrac{\sin\pi(\gamma-\alpha)\sin\pi(\gamma-\beta)e^{n\pi i(\gamma-\alpha-\beta-1)} - \sin\pi\alpha\sin\pi\beta e^{n\pi i(\alpha+\beta-\gamma+1)}}{\sin\pi\gamma\sin\pi(\gamma-\alpha-\beta)} \end{bmatrix}.$$

Another way to obtain the results of Theorem is to calculate the components of a product of the matrices $M_0(m)$ or $M_1(n)$ which represents the monodromy matrix for the general linear fractional transformation $\tau \to \dfrac{a\tau+b}{c\tau+d}$. In this paper, however, we do not choose a way to make such a calculation because of its complicatedness.

## §2 Auxiliary formulas.

In this section we prove some auxiliary formulas which are applied to the study of the integrals $I_\nu$ and $J_\nu$.

**Lemma 2.1.** Let $p, q, r, s$ be complex constants. Then we have

$$\int_0^{\frac{1}{2}} \theta(u)^p\theta_1(u)^q\theta_2(u)^r\theta_3(u)^s du = \int_0^{\frac{1}{2}} \theta(u)^q\theta_1(u)^p\theta_2(u)^s\theta_3(u)^r du.$$

We omit the proof.

**Lemma 2.2.** We have:
(i)

$$\int_0^{\tau/2} \theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1}du = \frac{1-e^{2\pi i(\alpha-\gamma)}}{1-e^{-2\pi i\gamma}}(z_1(\tau)/\theta_3^2) + \frac{e^{\pi i(\alpha-\gamma)}(e^{\pi i\beta}-e^{-\pi i\beta})}{1-e^{-2\pi i\gamma}}(z_2(\tau)/\theta_3^2).$$

(ii)

$$\int_0^{\tau/2} \theta(u)^{2\gamma-2\alpha-1}\theta_1(u)^{2\alpha-1}\theta_2(u)^{-2\beta-1}\theta_3(u)^{2\beta-2\gamma+3}du = \frac{1-e^{-2\pi i\alpha}}{1-e^{-2\pi i\gamma}}(z_1(\tau)/\theta_3^2) + \frac{e^{\pi i(-\alpha-\beta+\gamma)}-e^{\pi i(-\alpha+\beta-\gamma)}}{1-e^{-2\pi i\gamma}}(z_2(\tau)/\theta_3^2).$$

*Proof.* The first formula follows from the equality

$$0=\left(\int_{\tau/2}^{0}+\int_{0}^{1/2}+\int_{1/2}^{(1+\tau)/2}+\int_{(1+\tau)/2}^{\tau/2}\right)\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1}du,$$

and the second one from

$$0=\left(\int_{\tau/2}^{0}+\int_{0}^{1/2}+\int_{1/2}^{(1+\tau)/2}+\int_{(1+\tau)/2}^{\tau/2}\right)\theta(u)^{2\gamma-2\alpha-1}\theta_1(u)^{2\alpha-1}\theta_2(u)^{-2\beta-1}\theta_3(u)^{2\beta-2\gamma+3}du.$$

One can prove the following lemma similarly.

**Lemma 2.3.** We have:
(i)

$$\int_{0}^{\tau/2}\theta(u)^{2\beta-2\gamma+3}\theta_1(u)^{-2\beta-1}\theta_2(u)^{2\alpha-1}\theta_3(u)^{2\gamma-2\alpha-1}du=\frac{e^{\pi i(\alpha+\beta-\gamma)}-e^{\pi i(-\alpha+\beta+\gamma)}}{1-e^{2\pi i\gamma}}(z_1(\tau)/\theta_3^2)+\frac{1-e^{2\pi i\beta}}{1-e^{2\pi i\gamma}}(z_2(\tau)/\theta_3^2).$$

(ii)

$$\int_{0}^{\tau/2}\theta(u)^{-2\beta-1}\theta_1(u)^{2\beta-2\gamma+3}\theta_2(u)^{2\gamma-2\alpha-1}\theta_3(u)^{2\alpha-1}du=\frac{e^{\pi i(-\alpha-\beta+\gamma)}-e^{\pi i(\alpha-\beta+\gamma)}}{1-e^{2\pi i\gamma}}(z_1(\tau)/\theta_3^2)+\frac{1-e^{2\pi i(\gamma-\beta)}}{1-e^{2\pi i\gamma}}(z_2(\tau)/\theta_3^2).$$

We omit the proof.

## §3 The integral $I_0$.

From (1.5) we have

$$I_0=\sum_{\mu=0}^{k_1-1}\int_0^{1/2}\theta\left(v+\frac{\mu}{2}\right)^{2\alpha-1}\theta_1\left(v+\frac{\mu}{2}\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{\mu}{2}\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{\mu}{2}\right)^{-2\beta-1}dv. \tag{3.1}$$

Note that for a non-negative integer $\mu$

$$\begin{aligned}&\theta\left(v+\frac{\mu}{2}\right)^{2\alpha-1}\theta_1\left(v+\frac{\mu}{2}\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{\mu}{2}\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{\mu}{2}\right)^{-2\beta-1}\\&=\frac{1+(-1)^{\mu}}{2}e^{\pi i(1-\gamma)\mu}\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}+\frac{1+(-1)^{\mu+1}}{2}e^{\pi i(1-\gamma)(\mu-1)}e^{\pi i(2\alpha-2\gamma+1)}\theta(v)^{2\gamma-2\alpha-1}\theta_1(v)^{2\alpha-1}\theta_2(v)^{-2\beta-1}\theta_3(v)^{2\beta-2\gamma+3}.\end{aligned}$$

Substituting this expression into (3.1), we have

$$\begin{aligned}I_0&=\sum_{\mu=0}^{k_1-1}\left\{\frac{1+(-1)^{\mu}}{2}e^{\pi i(1-\gamma)\mu}+\frac{1+(-1)^{\mu+1}}{2}e^{\pi i(1-\gamma)\mu}e^{\pi i(2\alpha-\gamma)}\right\}\int_0^{1/2}\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}dv\\&=\left\{\frac{1+e^{\pi i(2\alpha-\gamma)}}{2}\cdot\frac{1-e^{\pi i(1-\gamma)k_1}}{1-e^{\pi i(1-\gamma)}}+\frac{1-e^{\pi i(2\alpha-\gamma)}}{2}\cdot\frac{1-(-1)^{k_1}e^{\pi i(1-\gamma)k_1}}{1+e^{\pi i(1-\gamma)}}\right\}\int_0^{1/2}\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}dv,\end{aligned}$$

from which it follows that

$$I_0 = e^{\pi i(2\alpha-\gamma)/2}\left\{\cos\frac{\pi}{2}(\gamma-2\alpha)\cdot\frac{1-e^{\pi i(1-\gamma)k_1}}{1-e^{\pi i(1-\gamma)}} + i\sin\frac{\pi}{2}(\gamma-2\alpha)\cdot\frac{1-(-1)^{k_1}e^{\pi i(1-\gamma)k_1}}{1+e^{\pi i(1-\gamma)}}\right\} z_1(\tau)/\theta_3^2. \tag{3.2}$$

## §4 The integrals $I_\nu$ for positive even integers $\nu$.

In this section let $\nu$ be a positive even integer. Let $\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_\pm$ be the branches of $\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}$ analytically continued on the real interval $0<v<\frac{1}{2}$ from the upper side for the plus sign and from the lower side for the minus sign, respectively, such that they coincide with each other on the real interval $-(\mu+1)/2<v<-\mu/2$. Then we have

$$\begin{aligned}&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+\\
=&\frac{1+(-1)^\mu}{2}e^{\pi i(1-\gamma)\mu}e^{\pi i(2\alpha-2\gamma+1)}\theta\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_1\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_2\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\theta_3\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\\
+&\frac{1+(-1)^{\mu+1}}{2}e^{\pi i(1-\gamma)(\mu+1)}\theta\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\end{aligned}$$

and

$$\begin{aligned}&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_-\\
=&\frac{1+(-1)^\mu}{2}e^{\pi i(\gamma-1)\mu}e^{\pi i(2\gamma-2\alpha-1)}\theta\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_1\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_2\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\theta_3\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\\
+&\frac{1+(-1)^{\mu+1}}{2}e^{\pi i(\gamma-1)(\mu+1)}\theta\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}.\end{aligned}$$

Combining these relations, we have

$$\begin{aligned}&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+\\
=&\left\{\frac{1+(-1)^\mu}{2}e^{2\pi i(1-\gamma)\mu}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^\mu}{2}e^{2\pi i(1-\gamma)(\mu+1)}\right\}\times\\
&\times\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_-.\end{aligned}$$

Now we have

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_-\\
=&e^{\pi i(\alpha+\beta-\gamma+1)}\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+.
\end{aligned}$$

Combining the preceding two relations, we have

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+\\
=&\left\{\frac{1+(-1)^\mu}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^\mu}{2}e^{2\pi i(1-\gamma)}\right\}e^{2\pi i(1-\gamma)\mu}e^{\pi i(\alpha+\beta-\gamma+1)}\times\\
&\times\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+.
\end{aligned}\tag{4.1}$$

On the other hand, we have

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+\\
=&\frac{1+(-1)^{\mu+l_\nu}}{2}e^{\pi i(\gamma-1)(\mu+l_\nu)}e^{\pi i(2\beta+1)}\theta\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_1\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_2\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_3\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\\
+&\frac{1+(-1)^{\mu+l_\nu+1}}{2}e^{\pi i(\gamma-1)(\mu+l_\nu+1)}\theta\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}
\end{aligned}$$

and

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_-\\
=&\frac{1+(-1)^{\mu+l_\nu}}{2}e^{\pi i(1-\gamma)(\mu+l_\nu)}e^{-\pi i(2\beta+1)}\theta\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_1\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_2\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_3\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\\
+&\frac{1+(-1)^{\mu+l_\nu+1}}{2}e^{\pi i(1-\gamma)(\mu+l_\nu+1)}\theta\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}.
\end{aligned}$$

Combining these two relations, we have

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+\\
&=\left\{\frac{1+(-1)^{\mu+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{\mu+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}e^{2\pi i(\gamma-1)(\mu+l_\nu)}\times\\
&\times\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_- .
\end{aligned}$$

Now we have

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_-\\
&=e^{\pi i(\alpha+\beta-\gamma+1)}\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{-2\beta-1}\right]_+ .
\end{aligned}$$

Combining the preceding two relations, we have

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+\\
&=\left\{\frac{1+(-1)^{\mu+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{\mu+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}e^{2\pi i(\gamma-1)(\mu+l_\nu)}e^{\pi i(\alpha+\beta-\gamma+1)}\times \qquad (4.2)\\
&\times\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{-2\beta-1}\right]_+ .
\end{aligned}$$

From (4.1) and (4.2) we have

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+\\
&=\left\{\frac{1+(-1)^{\mu}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{\mu}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{\mu+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{\mu+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}e^{2\pi i(\gamma-1)l_\nu}e^{2\pi i(\alpha+\beta-\gamma+1)}\times \qquad (4.3)\\
&\times\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{-2\beta-1}\right]_+ .
\end{aligned}$$

Repeating the same procedure, we arrive at the following

**Lemma 4.1.** *For a positive even integer $\nu$ we have*

$$\begin{aligned}&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+\\&=\left\{\frac{1+(-1)^\mu}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^\mu}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{\mu+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{\mu+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}\times\\&\times\left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{2\pi i(\gamma-1)}\right\}\cdots\cdots\left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{2\pi i(1-\gamma)}\right\}\times\\&\times\left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{2\pi i(\gamma-1)}\right\}e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{\nu\pi i(\alpha+\beta-\gamma+1)}\times\\&\times\left[\theta\left(v+\frac{k_\nu+\mu}{2}\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}\right)^{-2\beta-1}\right]_+.\end{aligned}\tag{4.4}$$

We are now in a position to compute $I_\nu$ for a positive even integer $\nu$. Substituting (4.4) into (1.5), we have

$$\begin{aligned}I_\nu=&\sum_{\mu=0}^{l_{\nu+1}-1}\left\{\frac{1+(-1)^\mu}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^\mu}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{\mu+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{\mu+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}\times\\&\times\left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{2\pi i(\gamma-1)}\right\}\cdots\cdots\left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{2\pi i(1-\gamma)}\right\}\times\\&\times\left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{2\pi i(\gamma-1)}\right\}e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{\nu\pi i(\alpha+\beta-\gamma+1)}\times\\&\times\int_0^{\frac12}\theta\left(v+\frac{k_\nu+\mu}{2}\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}\right)^{-2\beta-1}dv.\end{aligned}\tag{4.5}$$

Since

$$\begin{aligned}&\theta\left(v+\frac{k_\nu+\mu}{2}\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}\right)^{-2\beta-1}\\&=\frac{1+(-1)^\mu}{2}e^{\pi i(1-\gamma)\mu}\theta\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1}\\&\quad+\frac{1-(-1)^\mu}{2}e^{\pi i(1-\gamma)(\mu-1)}e^{\pi i(2\alpha-2\gamma+1)}\theta\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1}\theta_1\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1}\theta_2\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1}\theta_3\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3},\end{aligned}$$

(4.5) is turned to

$$
\begin{aligned}
I_\nu = \sum_{\mu=0}^{l_{\nu+1}-1} & \left\{\frac{1+(-1)^\mu}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^\mu}{2} e^{2\pi i(1-\gamma)}\right\} \left\{\frac{1+(-1)^{\mu+l_\nu}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu}}{2} e^{2\pi i(\gamma-1)}\right\} \left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)}\right\} \times \\
& \times \left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{2\pi i(\gamma-1)}\right\} \cdots\cdots \left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{2\pi i(1-\gamma)}\right\} \times \\
& \times \left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{2\pi i(\gamma-1)}\right\} e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \times \\
& \times \left\{\frac{1+(-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} \int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} \theta_1\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} \theta_2\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} \theta_3\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} dv \right.\\
& \left. + \frac{1-(-1)^\mu}{2} e^{\pi i(1-\gamma)(\mu-1)} e^{\pi i(2\alpha-2\gamma+1)} \int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} \theta_1\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} \theta_2\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} \theta_3\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} dv\right\},
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
I_\nu = \sum_{\mu=0}^{l_{\nu+1}-1} & e^{4\pi i(\alpha-\gamma)} \left\{\frac{1+(-1)^{l_\nu}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{l_\nu}}{2} e^{2\pi i(\gamma-1)}\right\} \left\{\frac{1+(-1)^{l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)}\right\} \times \\
& \times \left\{\frac{1+(-1)^{l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{2\pi i(\gamma-1)}\right\} \cdots\cdots \left\{\frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{2\pi i(1-\gamma)}\right\} \times \\
& \times \left\{\frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{2\pi i(\gamma-1)}\right\} e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \times \\
& \times \frac{1+(-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} \int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} \theta_1\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} \theta_2\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} \theta_3\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} dv \\
& + \sum_{\mu=0}^{l_{\nu+1}-1} e^{2\pi i(1-\gamma)} \left\{\frac{1-(-1)^{l_\nu}}{2} e^{4\pi i\beta} + \frac{1+(-1)^{l_\nu}}{2} e^{2\pi i(\gamma-1)}\right\} \left\{\frac{1-(-1)^{l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1+(-1)^{l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)}\right\} \times \\
& \times \left\{\frac{1-(-1)^{l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{4\pi i\beta} + \frac{1+(-1)^{l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{2\pi i(\gamma-1)}\right\} \cdots\cdots \left\{\frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{2\pi i(1-\gamma)}\right\} \times \\
& \times \left\{\frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{4\pi i\beta} + \frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{2\pi i(\gamma-1)}\right\} e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \times \\
& \times \frac{1-(-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} e^{\pi i(2\alpha-\gamma)} \int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} \theta_1\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} \theta_2\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} \theta_3\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} dv.
\end{aligned}
\tag{4.6}
$$

Substituting

$$\sum_{\mu=0}^{l_{\nu+1}-1} \frac{1 \pm (-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} = \frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}} \pm \frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}$$

into (4.6), we have

$$\begin{aligned}
I_\nu = &\left\{\frac{1+(-1)^{l_\nu}}{2}e^{4\pi i\beta} + \frac{1-(-1)^{l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\left\{\frac{1+(-1)^{l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}\times \\
&\times\left\{\frac{1+(-1)^{l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{4\pi i\beta} + \frac{1-(-1)^{l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{2\pi i(\gamma-1)}\right\}\cdots\cdots\left\{\frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{2\pi i(1-\gamma)}\right\}\times \\
&\times\left\{\frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{4\pi i\beta} + \frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{2\pi i(\gamma-1)}\right\}e^{4\pi i(\alpha-\gamma)}e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{\nu\pi i(\alpha+\beta-\gamma+1)}\times \\
&\times\frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}} + \frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}\int_0^{\frac{1}{2}}\theta\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1}dv \\
&+\left\{\frac{1-(-1)^{l_\nu}}{2}e^{4\pi i\beta} + \frac{1+(-1)^{l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\left\{\frac{1-(-1)^{l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)} + \frac{1+(-1)^{l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}\times \\
&\times\left\{\frac{1-(-1)^{l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{4\pi i\beta} + \frac{1+(-1)^{l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{2\pi i(\gamma-1)}\right\}\cdots\cdots\left\{\frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{4\pi i(\alpha-\gamma)} + \frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{2\pi i(1-\gamma)}\right\}\times \\
&\times\left\{\frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{4\pi i\beta} + \frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{2\pi i(\gamma-1)}\right\}e^{2\pi i(1-\gamma)}e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{\nu\pi i(\alpha+\beta-\gamma+1)}e^{\pi i(2\alpha-\gamma)}\times \\
&\times\frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}} - \frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}\int_0^{\frac{1}{2}}\theta\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1}\theta_1\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1}\theta_2\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1}\theta_3\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3}dv,
\end{aligned}$$

which is turned by some elementary calculation to

$$\begin{aligned}
I_\nu = &e^{\pi i\left[\left\{(-1)^{k_\nu-k_{\nu-1}}+(-1)^{k_\nu-k_{\nu-3}}+\cdots+(-1)^{k_\nu-k_1}\right\}(2\beta-\gamma+1)+\left\{(-1)^{k_\nu-k_{\nu-2}}+(-1)^{k_\nu-k_{\nu-4}}+\cdots+(-1)^{k_\nu-k_2}\right\}(2\alpha-\gamma-1)\right]}e^{2\pi i(\alpha-\gamma)}e^{\pi i(\gamma-1)}e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}\times \\
&\times\frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}} + \frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}\int_0^{\frac{1}{2}}\theta\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1}dv \\
&+e^{-\pi i\left[\left\{(-1)^{k_\nu-k_{\nu-1}}+(-1)^{k_\nu-k_{\nu-3}}+\cdots+(-1)^{k_\nu-k_1}\right\}(2\beta-\gamma+1)+\left\{(-1)^{k_\nu-k_{\nu-2}}+(-1)^{k_\nu-k_{\nu-4}}+\cdots+(-1)^{k_\nu-k_2}\right\}(2\alpha-\gamma-1)\right]}e^{2\pi i(\gamma-\alpha)}e^{\pi i(1-\gamma)}e^{\pi i(2\alpha-\gamma)}e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}\times \\
&\times e^{2\nu\pi i(\alpha+\beta-\gamma)}\frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}} - \frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}\int_0^{\frac{1}{2}}\theta\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1}\theta_1\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1}\theta_2\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1}\theta_3\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3}dv.
\end{aligned} \tag{4.7}$$

Now we have

$$\int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} \theta_1\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} \theta_2\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} \theta_3\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} dv$$
$$=\left\{\frac{1+(-1)^{k_\nu}}{2}e^{\pi i(1-\gamma)k_\nu}+\frac{1-(-1)^{k_\nu}}{2}e^{\pi i(1-\gamma)(k_\nu-1)}e^{\pi i(2\alpha-2\gamma+1)}\right\}\int_0^{\frac{1}{2}}\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}dv \tag{4.8}$$
$$=e^{\pi i(1-\gamma)k_\nu}e^{\{1-(-1)^{k_\nu}\}\pi i(2\alpha-\gamma)/2}\int_0^{\frac{1}{2}}\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}dv$$

and

$$\int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} \theta_1\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} \theta_2\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} \theta_3\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} dv$$
$$=\left\{\frac{1+(-1)^{k_\nu}}{2}e^{\pi i(1-\gamma)k_\nu}+\frac{1-(-1)^{k_\nu}}{2}e^{\pi i(1-\gamma)(k_\nu-1)}e^{\pi i(1-2\alpha)}\right\}\int_0^{\frac{1}{2}}\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}dv \tag{4.9}$$
$$=e^{\pi i(1-\gamma)k_\nu}e^{\{1-(-1)^{k_\nu}\}\pi i(\gamma-2\alpha)/2}\int_0^{\frac{1}{2}}\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}dv.$$

Substituting (4.8) and (4.9) into (4.7), we have

$$\begin{aligned}I_\nu =&e^{\pi i\left[\left\{(-1)^{k_\nu-k_{\nu-1}}+(-1)^{k_\nu-k_{\nu-3}}+\cdots+(-1)^{k_\nu-k_1}\right\}(2\beta-\gamma+1)+\left\{(-1)^{k_\nu-k_{\nu-2}}+(-1)^{k_\nu-k_{\nu-4}}+\cdots+(-1)^{k_\nu-k_2}\right\}(2\alpha-\gamma-1)\right]}\times\\ &\times(-1)e^{\pi i(2\alpha-\gamma)/2}e^{\{2-(-1)^{k_\nu}\}\pi i(2\alpha-\gamma)/2}e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(1-\gamma)k_\nu}\frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}}+\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}z_1(\tau)/\theta_3^2\\ &+e^{-\pi i\left[\left\{(-1)^{k_\nu-k_{\nu-1}}+(-1)^{k_\nu-k_{\nu-3}}+\cdots+(-1)^{k_\nu-k_1}\right\}(2\beta-\gamma+1)+\left\{(-1)^{k_\nu-k_{\nu-2}}+(-1)^{k_\nu-k_{\nu-4}}+\cdots+(-1)^{k_\nu-k_2}\right\}(2\alpha-\gamma-1)\right]}\times\\ &\times(-1)e^{\pi i(2\alpha-\gamma)/2}e^{\{2-(-1)^{k_\nu}\}\pi i(\gamma-2\alpha)/2}e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(1-\gamma)k_\nu}\frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}}-\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}z_1(\tau)/\theta_3^2.\end{aligned} \tag{4.10}$$

If we introduce $\Theta_\nu$ by

$$\Theta_\nu=\left\{(-1)^{k_\nu-k_{\nu-1}}+(-1)^{k_\nu-k_{\nu-3}}+\cdots+(-1)^{k_\nu-k_1}\right\}(2\beta-\gamma+1)\pi+\left\{(-1)^{k_\nu-k_{\nu-2}}+(-1)^{k_\nu-k_{\nu-4}}+\cdots+(-1)^{k_\nu-k_2}\right\}(2\alpha-\gamma-1)\pi+\frac{2-(-1)^{k_\nu}}{2}(2\alpha-\gamma)\pi,$$

(4.10) is turned to

$$I_\nu=-e^{\pi i(2\alpha-\gamma)/2}e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(1-\gamma)k_\nu}\left\{\cos\Theta_\nu\cdot\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}}+i\sin\Theta_\nu\cdot\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}z_1(\tau)/\theta_3^2. \tag{4.11}$$

## §5 The integrals $I_\nu$ for positive odd integers $\nu$.

In this section let $\nu$ be a positive odd integer. The formula (4.1) holds also in this case. In case where $\nu \geq 3$, (4.3) holds too. Moreover we have

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-2}{2}\tau\right)^{-2\beta-1}\right]_+\\
&=\left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}e^{2\pi i(1-\gamma)(\mu+l_\nu+l_{\nu-1})}e^{\pi i(\alpha+\beta-\gamma+1)}\times\\
&\times\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-3}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-3}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-3}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-3}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+.
\end{aligned}\tag{5.1}$$

Combining (5.1) with (4.3), we have

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+\\
&=\left\{\frac{1+(-1)^{\mu}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{\mu}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{\mu+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{\mu+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\times\\
&\times\left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}e^{2\pi i(1-\gamma)(\mu+l_{\nu-1})}e^{3\pi i(\alpha+\beta-\gamma+1)}\times\\
&\times\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-3}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-3}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-3}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu-3}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+.
\end{aligned}\tag{5.2}$$

Repeating the same procedure, we arrive at the following

**Lemma 5.1.** *For a positive odd integer $\nu$ we have*

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu+\mu}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+\\
&=\left\{\frac{1+(-1)^{\mu}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{\mu}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{\mu+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{\mu+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}\cdots\\
&\cdots\left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{2\pi i(\gamma-1)}\right\}\left\{\frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{2\pi i(1-\gamma)}\right\}\\
&\times e^{2\pi i(1-\gamma)(\mu+l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)}e^{\nu\pi i(\alpha+\beta-\gamma+1)}\left[\theta\left(v+\frac{k_\nu+\mu}{2}\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_\nu+\mu}{2}\right)^{-2\beta-1}\theta_2\left(v+\frac{k_\nu+\mu}{2}\right)^{2\alpha-1}\theta_3\left(v+\frac{k_\nu+\mu}{2}\right)^{2\gamma-2\alpha-1}\right]_+.
\end{aligned}\tag{5.3}$$

Let us now compute $I_\nu$ for a positive odd integer $\nu$. Substituting (5.3) into (1.5), we have

$$
\begin{aligned}
I_\nu = & \sum_{\mu=0}^{l_{\nu+1}-1} \left\{ \frac{1+(-1)^\mu}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^\mu}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \cdots \\
& \cdots \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{2\pi i(1-\gamma)} \right\} \\
& \times e^{2\pi i(1-\gamma)(\mu+l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu+\mu}{2}\right)^{2\beta-2\gamma+3} \theta_1\left(v+\frac{k_\nu+\mu}{2}\right)^{-2\beta-1} \theta_2\left(v+\frac{k_\nu+\mu}{2}\right)^{2\alpha-1} \theta_3\left(v+\frac{k_\nu+\mu}{2}\right)^{2\gamma-2\alpha-1} dv.
\end{aligned} \tag{5.4}
$$

Since

$$
\begin{aligned}
& \theta\left(v+\frac{k_\nu+\mu}{2}\right)^{2\beta-2\gamma+3} \theta_1\left(v+\frac{k_\nu+\mu}{2}\right)^{-2\beta-1} \theta_2\left(v+\frac{k_\nu+\mu}{2}\right)^{2\alpha-1} \theta_3\left(v+\frac{k_\nu+\mu}{2}\right)^{2\gamma-2\alpha-1} \\
= & \frac{1+(-1)^\mu}{2} e^{\pi i(\gamma-1)\mu} \theta\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} \theta_1\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} \theta_2\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} \theta_3\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} \\
& + \frac{1-(-1)^\mu}{2} e^{\pi i(\gamma-1)(\mu-1)} e^{\pi i(2\beta+1)} \theta\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} \theta_1\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} \theta_2\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} \theta_3\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1},
\end{aligned}
$$

(5.4) is turned to

$$
\begin{aligned}
I_\nu = & \sum_{\mu=0}^{l_{\nu+1}-1} \left\{ \frac{1+(-1)^\mu}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^\mu}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \cdots \\
& \cdots \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{2\pi i(1-\gamma)} \right\} \\
& \times e^{2\pi i(1-\gamma)(\mu+l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \left\{ \frac{1+(-1)^\mu}{2} e^{\pi i(\gamma-1)\mu} \int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} \theta_1\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} \theta_2\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} \theta_3\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} dv \right. \\
& \left. + \frac{1-(-1)^\mu}{2} e^{\pi i(\gamma-1)(\mu-1)} e^{\pi i(2\beta+1)} \int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} \theta_1\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} \theta_2\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} \theta_3\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} dv \right\},
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
I_\nu = &\sum_{\mu=0}^{l_{\nu+1}-1} e^{4\pi i(\alpha-\gamma)} \left\{ \frac{1+(-1)^{l_\nu}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \cdots \\
&\cdots \left\{ \frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{2\pi i(1-\gamma)} \right\} \\
&\times e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \frac{1+(-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} \int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} \theta_1\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} \theta_2\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} \theta_3\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} dv \\
&+ \sum_{\mu=0}^{l_{\nu+1}-1} e^{2\pi i(1-\gamma)} \left\{ \frac{1-(-1)^{l_\nu}}{2} e^{4\pi i\beta} + \frac{1+(-1)^{l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1-(-1)^{l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1+(-1)^{l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \cdots \\
&\cdots \left\{ \frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{4\pi i\beta} + \frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2} e^{2\pi i(1-\gamma)} \right\} \\
&\times e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \frac{1-(-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} e^{\pi i(2\beta-\gamma)} \int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} \theta_1\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} \theta_2\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} \theta_3\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} dv.
\end{aligned}
\tag{5.5}
$$

Substituting

$$
\sum_{\mu=0}^{l_{\nu+1}-1} \frac{1\pm(-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} = \frac{1}{2}\left\{ \frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}} \pm \frac{1-(-1)^{l_{\nu+1}} e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}} \right\}
$$

into (5.5), we have

$$
\begin{aligned}
I_\nu =&\left\{\frac{1+(-1)^{l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\left\{\frac{1+(-1)^{l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}\cdots\\
&\cdots\left\{\frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{2\pi i(\gamma-1)}\right\}\left\{\frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{2\pi i(1-\gamma)}\right\}\\
&\times e^{4\pi i(\alpha-\gamma)}e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)}e^{\nu\pi i(\alpha+\beta-\gamma+1)}\frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}}+\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}\\
&\times\int_0^{\frac{1}{2}}\theta\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1}\theta_2\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1}\theta_3\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1}dv\\
&+\left\{\frac{1-(-1)^{l_\nu}}{2}e^{4\pi i\beta}+\frac{1+(-1)^{l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\left\{\frac{1-(-1)^{l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1+(-1)^{l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}\cdots\\
&\cdots\left\{\frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{4\pi i\beta}+\frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{2\pi i(\gamma-1)}\right\}\left\{\frac{1-(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1+(-1)^{l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{2\pi i(1-\gamma)}\right\}\\
&\times e^{2\pi i(1-\gamma)}e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)}e^{\nu\pi i(\alpha+\beta-\gamma+1)}e^{\pi i(2\beta-\gamma)}\frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}}-\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}\\
&\times\int_0^{\frac{1}{2}}\theta\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1}\theta_1\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3}\theta_2\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1}\theta_3\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1}dv,
\end{aligned}
$$

which is turned by some elementary calculation to

$$
\begin{aligned}
I_\nu =&e^{\pi i\left[\left\{(-1)^{k_\nu-k_{\nu-1}}+(-1)^{k_\nu-k_{\nu-3}}+\cdots+(-1)^{k_\nu-k_2}\right\}(2\beta-\gamma+1)+\left\{(-1)^{k_\nu-k_{\nu-2}}+(-1)^{k_\nu-k_{\nu-4}}+\cdots+(-1)^{k_\nu-k_1}\right\}(2\alpha-\gamma-1)\right]}e^{4\pi i(\alpha-\gamma)}e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)}\\
&\times(-1)e^{(2\nu-1)\pi i(\alpha+\beta-\gamma)}\frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}}+\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}\\
&\times\int_0^{\frac{1}{2}}\theta\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1}\theta_2\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1}\theta_3\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1}dv\\
&+e^{-\pi i\left[\left\{(-1)^{k_\nu-k_{\nu-1}}+(-1)^{k_\nu-k_{\nu-3}}+\cdots+(-1)^{k_\nu-k_2}\right\}(2\beta-\gamma+1)+\left\{(-1)^{k_\nu-k_{\nu-2}}+(-1)^{k_\nu-k_{\nu-4}}+\cdots+(-1)^{k_\nu-k_1}\right\}(2\alpha-\gamma-1)\right]}e^{\pi i(2\beta-3\gamma)}e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)}\\
&\times(-1)e^{(2\nu-1)\pi i(\alpha+\beta-\gamma)}\frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}}-\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}\\
&\times\int_0^{\frac{1}{2}}\theta\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1}\theta_1\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3}\theta_2\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1}\theta_3\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1}dv.
\end{aligned}
\tag{5.6}
$$

Now we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} \theta_1\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} \theta_2\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} \theta_3\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} dv \\
&\quad=\left\{\frac{1+(-1)^{k_\nu}}{2}e^{\pi i(\gamma-1)k_\nu}+\frac{1-(-1)^{k_\nu}}{2}e^{\pi i(\gamma-1)(k_\nu-1)}e^{\pi i(2\beta+1)}\right\}\int_0^{\frac{1}{2}}\theta(v)^{2\beta-2\gamma+3}\theta_1(v)^{-2\beta-1}\theta_2(v)^{2\alpha-1}\theta_3(v)^{2\gamma-2\alpha-1}dv \qquad (5.7)\\
&\quad=e^{\pi i(\gamma-1)k_\nu}e^{\{1-(-1)^{k_\nu}\}\pi i(2\beta-\gamma)/2}\int_0^{\frac{1}{2}}\theta(v)^{2\beta-2\gamma+3}\theta_1(v)^{-2\beta-1}\theta_2(v)^{2\alpha-1}\theta_3(v)^{2\gamma-2\alpha-1}dv
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} \theta_1\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} \theta_2\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} \theta_3\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} dv \\
&\quad=\left\{\frac{1+(-1)^{k_\nu}}{2}e^{\pi i(\gamma-1)k_\nu}+\frac{1-(-1)^{k_\nu}}{2}e^{\pi i(\gamma-1)(k_\nu-1)}e^{\pi i(2\gamma-2\beta-3)}\right\}\int_0^{\frac{1}{2}}\theta(v)^{2\beta-2\gamma+3}\theta_1(v)^{-2\beta-1}\theta_2(v)^{2\alpha-1}\theta_3(v)^{2\gamma-2\alpha-1}dv \qquad (5.8)\\
&\quad=e^{\pi i(\gamma-1)k_\nu}e^{\{1-(-1)^{k_\nu}\}\pi i(\gamma-2\beta)/2}\int_0^{\frac{1}{2}}\theta(v)^{2\beta-2\gamma+3}\theta_1(v)^{-2\beta-1}\theta_2(v)^{2\alpha-1}\theta_3(v)^{2\gamma-2\alpha-1}dv.
\end{aligned}$$

Substituting (5.7) and (5.8) into (5.6), we have

$$\begin{aligned}
I_\nu =&e^{\pi i\left[\left\{(-1)^{k_\nu-k_{\nu-1}}+(-1)^{k_\nu-k_{\nu-3}}+\cdots+(-1)^{k_\nu-k_2}\right\}(2\beta-\gamma+1)+\left\{(-1)^{k_\nu-k_{\nu-2}}+(-1)^{k_\nu-k_{\nu-4}}+\cdots+(-1)^{k_\nu-k_1}\right\}(2\alpha-\gamma-1)\right]}(-1)e^{\pi i(2\alpha-5\gamma)/2}e^{2\pi i(\alpha-\beta)}\\
&\times e^{\{2-(-1)^{k_\nu}\}\pi i(2\beta-\gamma)/2}e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(\gamma-1)k_\nu}\frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}}+\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}z_2(\tau)/\theta_3^2\\
&+e^{-\pi i\left[\left\{(-1)^{k_\nu-k_{\nu-1}}+(-1)^{k_\nu-k_{\nu-3}}+\cdots+(-1)^{k_\nu-k_2}\right\}(2\beta-\gamma+1)+\left\{(-1)^{k_\nu-k_{\nu-2}}+(-1)^{k_\nu-k_{\nu-4}}+\cdots+(-1)^{k_\nu-k_1}\right\}(2\alpha-\gamma-1)\right]}(-1)e^{\pi i(2\alpha-5\gamma)/2}e^{-2\pi i(\alpha-\beta)} \qquad (5.9)\\
&\times e^{\{2-(-1)^{k_\nu}\}\pi i(\gamma-2\beta)/2}e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(\gamma-1)k_\nu}\frac{1}{2}\left\{\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}}-\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}z_2(\tau)/\theta_3^2.
\end{aligned}$$

If we introduce $\Theta_\nu$ by

$$\Theta_\nu=\left\{(-1)^{k_\nu-k_{\nu-1}}+(-1)^{k_\nu-k_{\nu-3}}+\cdots+(-1)^{k_\nu-k_2}\right\}(2\beta-\gamma+1)\pi+\left\{(-1)^{k_\nu-k_{\nu-2}}+(-1)^{k_\nu-k_{\nu-4}}+\cdots+(-1)^{k_\nu-k_1}\right\}(2\alpha-\gamma-1)\pi+\frac{2-(-1)^{k_\nu}}{2}(2\beta-\gamma)\pi+2(\alpha-\beta)\pi,$$

(5.9) is turned to

$$I_\nu=-e^{\pi i(2\alpha-5\gamma)/2}e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(\gamma-1)k_\nu}\left\{\cos\Theta_\nu\cdot\frac{1-e^{\pi i(1-\gamma)l_{\nu+1}}}{1-e^{\pi i(1-\gamma)}}+i\sin\Theta_\nu\cdot\frac{1-(-1)^{l_{\nu+1}}e^{\pi i(1-\gamma)l_{\nu+1}}}{1+e^{\pi i(1-\gamma)}}\right\}z_2(\tau)/\theta_3^2. \qquad (5.10)$$

## §6 The integral $J_0$.

From (1.4) we have

$$J_0=\int_0^{\tau/2}\theta\left(v+\frac{k_1}{2}\right)^{2\alpha-1}\theta_1\left(v+\frac{k_1}{2}\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_1}{2}\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_1}{2}\right)^{-2\beta-1}dv. \tag{6.1}$$

Note that

$$\begin{aligned}&\theta\left(v+\frac{k_1}{2}\right)^{2\alpha-1}\theta_1\left(v+\frac{k_1}{2}\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_1}{2}\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_1}{2}\right)^{-2\beta-1}\\&=\frac{1+(-1)^{k_1}}{2}e^{\pi i(1-\gamma)k_1}\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}+\frac{1-(-1)^{k_1}}{2}e^{\pi i(1-\gamma)(k_1-1)}e^{\pi i(2\alpha-2\gamma+1)}\theta(v)^{2\gamma-2\alpha-1}\theta_1(v)^{2\alpha-1}\theta_2(v)^{-2\beta-1}\theta_3(v)^{2\beta-2\gamma+3}.\end{aligned}$$

Substituting this expression into (6.1), we have

$$\begin{aligned}J_0=&\frac{1+(-1)^{k_1}}{2}e^{\pi i(1-\gamma)k_1}\int_0^{\tau/2}\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}dv\\&+\frac{1-(-1)^{k_1}}{2}e^{\pi i(1-\gamma)(k_1-1)}e^{\pi i(2\alpha-2\gamma+1)}\int_0^{\tau/2}\theta(v)^{2\gamma-2\alpha-1}\theta_1(v)^{2\alpha-1}\theta_2(v)^{-2\beta-1}\theta_3(v)^{2\beta-2\gamma+3}dv.\end{aligned} \tag{6.2}$$

Applying the formulas of Lemma 2.2 to (6.2), we have

$$\begin{aligned}J_0=&\left\{\frac{1+(-1)^{k_1}}{2}e^{\pi i(1-\gamma)k_1}\frac{1-e^{2\pi i(\alpha-\gamma)}}{1-e^{-2\pi i\gamma}}+\frac{1-(-1)^{k_1}}{2}e^{\pi i(1-\gamma)k_1}e^{\pi i(2\alpha-\gamma)}\frac{1-e^{-2\pi i\alpha}}{1-e^{-2\pi i\gamma}}\right\}z_1(\tau)/\theta_3^2\\&+\left\{\frac{1+(-1)^{k_1}}{2}e^{\pi i(1-\gamma)k_1}\frac{e^{\pi i(\alpha-\gamma)}(e^{\pi i\beta}-e^{-\pi i\beta})}{1-e^{-2\pi i\gamma}}+\frac{1-(-1)^{k_1}}{2}e^{\pi i(1-\gamma)k_1}e^{\pi i(2\alpha-\gamma)}\frac{e^{\pi i(\gamma-\alpha-\beta)}-e^{\pi i(-\alpha+\beta-\gamma)}}{1-e^{-2\pi i\gamma}}\right\}z_2(\tau)/\theta_3^2\\=&e^{\pi i(1-\gamma)l_1}e^{\{1+(-1)^{l_1+1}\}\pi i(2\alpha-\gamma)/2}\frac{1-e^{-\pi i\gamma}e^{(-1)^{l_1}\pi i(2\alpha-\gamma)}}{1-e^{-2\pi i\gamma}}z_1(\tau)/\theta_3^2+e^{\pi i(1-\gamma)l_1}e^{\pi i(\alpha-\gamma)}\frac{e^{\pi i\{\gamma+(-1)^{l_1}(2\beta-\gamma)\}/2}-e^{-\pi i\{\gamma+(-1)^{l_1}(2\beta-\gamma)\}/2}}{1-e^{-2\pi i\gamma}}z_2(\tau)/\theta_3^2\\=&e^{\pi i(1-\gamma)l_1}e^{\pi i(\alpha-\gamma)}\frac{e^{\pi i\{\gamma-(-1)^{l_1}(2\alpha-\gamma)\}/2}-e^{-\pi i\{\gamma-(-1)^{l_1}(2\alpha-\gamma)\}/2}}{1-e^{-2\pi i\gamma}}z_1(\tau)/\theta_3^2+e^{\pi i(1-\gamma)l_1}e^{\pi i(\alpha-\gamma)}\frac{e^{\pi i\{\gamma+(-1)^{l_1}(2\beta-\gamma)\}/2}-e^{-\pi i\{\gamma+(-1)^{l_1}(2\beta-\gamma)\}/2}}{1-e^{-2\pi i\gamma}}z_2(\tau)/\theta_3^2,\end{aligned}$$

from which it follows that

$$J_0=e^{\pi i(1-\gamma)l_1}e^{\pi i\alpha}\frac{\sin[\pi\{\gamma-(-1)^{l_1}(2\alpha-\gamma)\}/2]}{\sin\pi\gamma}z_1(\tau)/\theta_3^2+e^{\pi i(1-\gamma)l_1}e^{\pi i\alpha}\frac{\sin[\pi\{\gamma+(-1)^{l_1}(2\beta-\gamma)\}/2]}{\sin\pi\gamma}z_2(\tau)/\theta_3^2. \tag{6.3}$$

## §7 The integrals $J_\nu$ for positive even integers $\nu$.

In this section let $\nu$ be a positive even integer. Then we have

$$\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+$$
$$=\frac{1+(-1)^{l_{\nu+1}}}{2}e^{\pi i(1-\gamma)l_{\nu+1}}e^{\pi i(2\alpha-2\gamma+1)}\theta\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_1\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_2\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\theta_3\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}$$
$$+\frac{1-(-1)^{l_{\nu+1}}}{2}e^{\pi i(1-\gamma)(l_{\nu+1}+1)}\theta\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}$$

and

$$\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_-$$
$$=\frac{1+(-1)^{l_{\nu+1}}}{2}e^{\pi i(\gamma-1)l_{\nu+1}}e^{\pi i(2\gamma-2\alpha-1)}\theta\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_1\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_2\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\theta_3\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}$$
$$+\frac{1-(-1)^{l_{\nu+1}}}{2}e^{\pi i(\gamma-1)(l_{\nu+1}+1)}\theta\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_\nu-1}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}.$$

Combining these relations, we have

$$\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+$$
$$=\left\{\frac{1+(-1)^{l_{\nu+1}}}{2}e^{2\pi i(1-\gamma)l_{\nu+1}}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}}}{2}e^{2\pi i(1-\gamma)(l_{\nu+1}+1)}\right\}\times$$
$$\times\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_-.$$

Now we have

$$\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_-$$
$$=e^{\pi i(\alpha+\beta-\gamma+1)}\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+.$$

Combining the preceding two relations, we have

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+\\
&=\left\{\frac{1+(-1)^{l_{\nu+1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}}}{2}e^{2\pi i(1-\gamma)}\right\}e^{2\pi i(1-\gamma)l_{\nu+1}}e^{\pi i(\alpha+\beta-\gamma+1)}\times\\
&\times\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+.
\end{aligned}\tag{7.1}$$

On the other hand, we have

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+\\
=&\frac{1+(-1)^{l_{\nu+1}+l_\nu}}{2}e^{\pi i(\gamma-1)(l_{\nu+1}+l_\nu)}e^{\pi i(2\beta+1)}\theta\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_1\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_2\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_3\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\\
+&\frac{1-(-1)^{l_{\nu+1}+l_\nu}}{2}e^{\pi i(\gamma-1)(l_{\nu+1}+l_\nu+1)}\theta\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}
\end{aligned}$$

and

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_-\\
=&\frac{1+(-1)^{l_{\nu+1}+l_\nu}}{2}e^{\pi i(1-\gamma)(l_{\nu+1}+l_\nu)}e^{-\pi i(2\beta+1)}\theta\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_1\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_2\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_3\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\\
+&\frac{1-(-1)^{l_{\nu+1}+l_\nu}}{2}e^{\pi i(1-\gamma)(l_{\nu+1}+l_\nu+1)}\theta\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu-1}-1}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}.
\end{aligned}$$

Combining these two relations, we have

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+\\
&=\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}e^{2\pi i(\gamma-1)(l_{\nu+1}+l_\nu)}\times\\
&\times\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_-.
\end{aligned}$$

Now we have

$$
\begin{aligned}
&\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_-\\
&=e^{\pi i(\alpha+\beta-\gamma+1)}\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{-2\beta-1}\right]_+.
\end{aligned}
$$

Combining the preceding two relations, we have

$$
\begin{aligned}
&\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-1}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+\\
&=\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}e^{2\pi i(\gamma-1)(l_{\nu+1}+l_\nu)}e^{\pi i(\alpha+\beta-\gamma+1)}\times\\
&\times\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{-2\beta-1}\right]_+.
\end{aligned}
\tag{7.2}
$$

From (7.1) and (7.2) we have

$$
\begin{aligned}
&\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+\\
&=\left\{\frac{1+(-1)^{l_{\nu+1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}e^{2\pi i(\gamma-1)l_\nu}e^{2\pi i(\alpha+\beta-\gamma+1)}\times\\
&\times\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{-2\beta-1}\right]_+.
\end{aligned}
\tag{7.3}
$$

Repeating the same procedure, we arrive at the following

**Lemma 7.1.** *For a positive even integer $\nu$ we have*

$$\begin{aligned}
&\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+\\
&=\left\{\frac{1+(-1)^{l_{\nu+1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\times\\
&\times\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{2\pi i(\gamma-1)}\right\}\cdots\\
&\cdots\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{2\pi i(\gamma-1)}\right\}\times\\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{\nu\pi i(\alpha+\beta-\gamma+1)}\left[\theta\left(v+\frac{k_{\nu+1}}{2}\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}\right)^{-2\beta-1}\right]_+.
\end{aligned}\tag{7.4}$$

Let us now compute $J_\nu$ for a positive even integer $\nu$. From (1.4) we have

$$J_\nu=\int_0^{\tau/2}\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}dv.\tag{7.5}$$

Substituting (7.4) into (7.5), we have

$$\begin{aligned}
J_\nu=&\left\{\frac{1+(-1)^{l_{\nu+1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\times\\
&\times\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{2\pi i(\gamma-1)}\right\}\cdots\\
&\cdots\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{2\pi i(\gamma-1)}\right\}\times\\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{\nu\pi i(\alpha+\beta-\gamma+1)}\int_0^{\tau/2}\theta\left(v+\frac{k_{\nu+1}}{2}\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}\right)^{-2\beta-1}dv.
\end{aligned}\tag{7.6}$$

Since

$$\begin{aligned}
&\theta\left(v+\frac{k_{\nu+1}}{2}\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}\right)^{-2\beta-1}\\
=&\frac{1+(-1)^{k_{\nu+1}}}{2}e^{\pi i(1-\gamma)k_{\nu+1}}\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}+\frac{1-(-1)^{k_{\nu+1}}}{2}e^{\pi i(1-\gamma)(k_{\nu+1}-1)}e^{\pi i(2\alpha-2\gamma+1)}\theta(v)^{2\gamma-2\alpha-1}\theta_1(v)^{2\alpha-1}\theta_2(v)^{-2\beta-1}\theta_3(v)^{2\beta-2\gamma+3},
\end{aligned}$$

(7.6) is turned to

$$
\begin{aligned}
J_\nu =& \left\{\frac{1+(-1)^{l_{\nu+1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\times \\
&\times\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2}e^{2\pi i(\gamma-1)}\right\}\cdots \\
&\cdots\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\cdots+l_3}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\cdots+l_2}}{2}e^{2\pi i(\gamma-1)}\right\}\times \\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_4+l_2)}e^{\nu\pi i(\alpha+\beta-\gamma+1)}\left\{\frac{1+(-1)^{k_{\nu+1}}}{2}e^{\pi i(1-\gamma)k_{\nu+1}}\int_0^{\tau/2}\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}dv\right. \\
&\left.+\frac{1-(-1)^{k_{\nu+1}}}{2}e^{\pi i(1-\gamma)(k_{\nu+1}-1)}e^{\pi i(2\alpha-2\gamma+1)}\int_0^{\tau/2}\theta(v)^{2\gamma-2\alpha-1}\theta_1(v)^{2\alpha-1}\theta_2(v)^{-2\beta-1}\theta_3(v)^{2\beta-2\gamma+3}dv\right\},
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
J_\nu =& e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\beta-\gamma+1)]}e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(1-\gamma)k_{\nu+1}}\times \\
&\times\left\{\frac{1+(-1)^{k_{\nu+1}}}{2}\int_0^{\tau/2}\theta(v)^{2\alpha-1}\theta_1(v)^{2\gamma-2\alpha-1}\theta_2(v)^{2\beta-2\gamma+3}\theta_3(v)^{-2\beta-1}dv+\frac{1-(-1)^{k_{\nu+1}}}{2}e^{\pi i(2\alpha-\gamma)}\int_0^{\tau/2}\theta(v)^{2\gamma-2\alpha-1}\theta_1(v)^{2\alpha-1}\theta_2(v)^{-2\beta-1}\theta_3(v)^{2\beta-2\gamma+3}dv\right\}.
\end{aligned}
\tag{7.7}
$$

Applying the formulas of Lemma 2.2 to (7.7), we have

$$
\begin{aligned}
J_\nu =& e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\beta-\gamma+1)]}\times \\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(1-\gamma)k_{\nu+1}}\left\{\frac{1+(-1)^{k_{\nu+1}}}{2}\frac{1-e^{2\pi i(\alpha-\gamma)}}{1-e^{-2\pi i\gamma}}+\frac{1-(-1)^{k_{\nu+1}}}{2}e^{\pi i(2\alpha-\gamma)}\frac{1-e^{-2\pi i\alpha}}{1-e^{-2\pi i\gamma}}\right\}z_1(\tau)/\theta_3^2 \\
&+e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\beta-\gamma+1)]}\times \\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(1-\gamma)k_{\nu+1}}\left\{\frac{1+(-1)^{k_{\nu+1}}}{2}\frac{e^{\pi i(\alpha-\gamma)}(e^{\pi i\beta}-e^{-\pi i\beta})}{1-e^{-2\pi i\gamma}}+\frac{1-(-1)^{k_{\nu+1}}}{2}e^{\pi i(2\alpha-\gamma)}\frac{e^{\pi i(\gamma-\alpha-\beta)}-e^{-\pi i(-\alpha+\beta-\gamma))}}{1-e^{-2\pi i\gamma}}\right\}z_2(\tau)/\theta_3^2 \\
=& e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\beta-\gamma+1)]}\times \\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(1-\gamma)k_{\nu+1}}\frac{e^{\pi i(\alpha-\gamma)}}{1-e^{-2\pi i\gamma}}\left\{e^{\pi i\gamma/2}e^{(-1)^{k_{\nu+1}}\pi i(\gamma-2\alpha)/2}-e^{-\pi i\gamma/2}e^{(-1)^{k_{\nu+1}}\pi i(2\alpha-\gamma)/2}\right\}z_1(\tau)/\theta_3^2 \\
&+e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\beta-\gamma+1)]}\times \\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(1-\gamma)k_{\nu+1}}\frac{e^{\pi i(\alpha-\gamma)}}{1-e^{-2\pi i\gamma}}\left\{e^{\pi i\gamma/2}e^{(-1)^{k_{\nu+1}}\pi i(2\beta-\gamma)/2}-e^{-\pi i\gamma/2}e^{(-1)^{k_{\nu+1}}\pi i(\gamma-2\beta)/2}\right\}z_2(\tau)/\theta_3^2,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
J_\nu =& e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\beta-\gamma+1)]}\times \\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(1-\gamma)k_{\nu+1}}e^{\pi i\alpha}\frac{\sin\pi\{\gamma+(-1)^{k_{\nu+1}}(\gamma-2\alpha)\}/2}{\sin\pi\gamma}z_1(\tau)/\theta_3^2 \\
&+ e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\beta-\gamma+1)]}\times \\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\cdots+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(1-\gamma)k_{\nu+1}}e^{\pi i\alpha}\frac{\sin\pi\{\gamma+(-1)^{k_{\nu+1}}(2\beta-\gamma)\}/2}{\sin\pi\gamma}z_2(\tau)/\theta_3^2.
\end{aligned}
\tag{7.8}
$$

## §8 The integrals $J_\nu$ for positive odd integers $\nu$.

In this section let $\nu$ be a positive odd integer. The formula (7.1) holds also in this case. In case where $\nu\geq 3$, (7.3) holds too. Moreover we have

$$
\begin{aligned}
&\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-2}{2}\tau\right)^{-2\beta-1}\right]_+ \\
=&\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}e^{2\pi i(1-\gamma)(l_{\nu+1}+l_\nu+l_{\nu-1})}e^{\pi i(\alpha+\beta-\gamma+1)}\times \\
&\times\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-3}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-3}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-3}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-3}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+.
\end{aligned}
\tag{8.1}
$$

Combining (8.1) with (7.3), we have

$$
\begin{aligned}
&\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+ \\
=&\left\{\frac{1+(-1)^{l_{\nu+1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\times \\
&\times\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1})}e^{3\pi i(\alpha+\beta-\gamma+1)}\times \\
&\times\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-3}{2}\tau\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-3}{2}\tau\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-3}{2}\tau\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu-3}{2}\tau\right)^{2\gamma-2\alpha-1}\right]_+.
\end{aligned}
\tag{8.2}
$$

Repeating the same procedure, we arrive at the following

**Lemma 8.1.** *For a positive odd integer $\nu$ we have*

$$
\begin{aligned}
&\left[\theta\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\alpha-1}\theta_1\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3}\theta_3\left(v+\frac{k_{\nu+1}}{2}+\frac{\nu}{2}\tau\right)^{-2\beta-1}\right]_+\\
&=\left\{\frac{1+(-1)^{l_{\nu+1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\\
&\times\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}\cdots\cdots\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3}}{2}e^{2\pi i(\gamma-1)}\right\}\\
&\times\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_2}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+\cdots+l_2}}{2}e^{2\pi i(1-\gamma)}\right\}e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)}e^{\nu\pi i(\alpha+\beta-\gamma+1)}\\
&\times\left[\theta\left(v+\frac{k_{\nu+1}}{2}\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu+1}}{2}\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu+1}}{2}\right)^{2\gamma-2\alpha-1}\right]_+.
\end{aligned}
\tag{8.3}
$$

Let us now compute $J_\nu$ for a positive odd integer $\nu$. Substituting (8.3) into (7.5), we have

$$
\begin{aligned}
J_\nu=&\left\{\frac{1+(-1)^{l_{\nu+1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\times\\
&\times\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}\cdots\cdots\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3}}{2}e^{2\pi i(\gamma-1)}\right\}\times\\
&\times\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_2}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+\cdots+l_2}}{2}e^{2\pi i(1-\gamma)}\right\}e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)}e^{\nu\pi i(\alpha+\beta-\gamma+1)}\\
&\times\int_0^{\tau/2}\theta\left(v+\frac{k_{\nu+1}}{2}\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu+1}}{2}\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu+1}}{2}\right)^{2\gamma-2\alpha-1}dv.
\end{aligned}
\tag{8.4}
$$

Since

$$
\begin{aligned}
&\theta\left(v+\frac{k_{\nu+1}}{2}\right)^{2\beta-2\gamma+3}\theta_1\left(v+\frac{k_{\nu+1}}{2}\right)^{-2\beta-1}\theta_2\left(v+\frac{k_{\nu+1}}{2}\right)^{2\alpha-1}\theta_3\left(v+\frac{k_{\nu+1}}{2}\right)^{2\gamma-2\alpha-1}\\
=&\frac{1+(-1)^{k_{\nu+1}}}{2}e^{\pi i(\gamma-1)k_{\nu+1}}\theta(v)^{2\beta-2\gamma+3}\theta_1(v)^{-2\beta-1}\theta_2(v)^{2\alpha-1}\theta_3(v)^{2\gamma-2\alpha-1}+\frac{1-(-1)^{k_{\nu+1}}}{2}e^{\pi i(\gamma-1)(k_{\nu+1}-1)}e^{\pi i(2\beta+1)}\theta(v)^{-2\beta-1}\theta_1(v)^{2\beta-2\gamma+3}\theta_2(v)^{2\gamma-2\alpha-1}\theta_3(v)^{2\alpha-1},
\end{aligned}
$$

(8.4) is turned to

$$\begin{aligned}J_\nu =&\left\{\frac{1+(-1)^{l_{\nu+1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}}}{2}e^{2\pi i(1-\gamma)}\right\}\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu}}{2}e^{2\pi i(\gamma-1)}\right\}\times\\ &\times\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2}e^{2\pi i(1-\gamma)}\right\}\cdots\cdots\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3}}{2}e^{4\pi i\beta}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+\cdots+l_3}}{2}e^{2\pi i(\gamma-1)}\right\}\times\\ &\times\left\{\frac{1+(-1)^{l_{\nu+1}+l_\nu+\cdots+l_2}}{2}e^{4\pi i(\alpha-\gamma)}+\frac{1-(-1)^{l_{\nu+1}+l_\nu+\cdots+l_2}}{2}e^{2\pi i(1-\gamma)}\right\}e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+l_{\nu-3}+\cdots+l_4+l_2)}e^{\nu\pi i(\alpha+\beta-\gamma+1)}\\ &\times\left\{\frac{1+(-1)^{k_{\nu+1}}}{2}e^{\pi i(\gamma-1)k_{\nu+1}}\int_0^{\tau/2}\theta(v)^{2\beta-2\gamma+3}\theta_1(v)^{-2\beta-1}\theta_2(v)^{2\alpha-1}\theta_3(v)^{2\gamma-2\alpha-1}dv\right.\\ &\left.+\frac{1-(-1)^{k_{\nu+1}}}{2}e^{\pi i(\gamma-1)k_{\nu+1}}e^{\pi i(2\beta-\gamma)}\int_0^{\tau/2}\theta(v)^{-2\beta-1}\theta_1(v)^{2\beta-2\gamma+3}\theta_2(v)^{2\gamma-2\alpha-1}\theta_3(v)^{2\alpha-1}dv\right\},\end{aligned}$$

from which it follows that

$$\begin{aligned}J_\nu =& e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\beta-\gamma+1)]}e^{\pi i(\alpha-\beta-2\gamma)}\\ &\times e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(\gamma-1)k_{\nu+1}}\left\{\frac{1+(-1)^{k_{\nu+1}}}{2}\int_0^{\tau/2}\theta(v)^{2\beta-2\gamma+3}\theta_1(v)^{-2\beta-1}\theta_2(v)^{2\alpha-1}\theta_3(v)^{2\gamma-2\alpha-1}dv\right.\\ &\left.+\frac{1-(-1)^{k_{\nu+1}}}{2}e^{\pi i(2\beta-\gamma)}\int_0^{\tau/2}\theta(v)^{-2\beta-1}\theta_1(v)^{2\beta-2\gamma+3}\theta_2(v)^{2\gamma-2\alpha-1}\theta_3(v)^{2\alpha-1}dv\right\}.\end{aligned}\tag{8.5}$$

Applying the formulas of Lemma 2.3 to (8.5), we have

$$\begin{aligned}J_\nu =& e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\beta-\gamma+1)]}e^{\pi i(\alpha-\beta-2\gamma)}\\ &\times e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(\gamma-1)k_{\nu+1}}\left\{\frac{1+(-1)^{k_{\nu+1}}}{2}\frac{e^{\pi i(\alpha+\beta-\gamma)}-e^{\pi i(\gamma-\alpha+\beta)}}{1-e^{2\pi i\gamma}}+\frac{1-(-1)^{k_{\nu+1}}}{2}\frac{e^{\pi i(-\alpha+\beta)}-e^{\pi i(\alpha+\beta)}}{1-e^{2\pi i\gamma}}\right\}z_1(\tau)/\theta_3^2\\ &+e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\beta-\gamma+1)]}e^{\pi i(\alpha-\beta-2\gamma)}\\ &\times e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(\gamma-1)k_{\nu+1}}\left\{\frac{1+(-1)^{k_{\nu+1}}}{2}\frac{1-e^{2\pi i\beta}}{1-e^{2\pi i\gamma}}+\frac{1-(-1)^{k_{\nu+1}}}{2}\frac{e^{\pi i(2\beta-\gamma)}-e^{\pi i\gamma}}{1-e^{2\pi i\gamma}}\right\}z_2(\tau)/\theta_3^2\\ =& e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\beta-\gamma+1)]}e^{\pi i(\alpha-3\gamma)}\\ &\times e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}\frac{e^{\pi i(\gamma-1)k_{\nu+1}}}{e^{\pi i\gamma}-e^{-\pi i\gamma}}\left\{e^{\pi i\gamma/2}e^{(-1)^{k_{\nu+1}}\pi i(\gamma-2\alpha)/2}-e^{-\pi i\gamma/2}e^{(-1)^{k_{\nu+1}}\pi i(2\alpha-\gamma)/2}\right\}z_1(\tau)/\theta_3^2\\ &+e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\beta-\gamma+1)]}e^{\pi i(\alpha-3\gamma)}\\ &\times e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+\cdots+l_4+l_2)}e^{2\nu\pi i(\alpha+\beta-\gamma)}\frac{e^{\pi i(\gamma-1)k_{\nu+1}}}{e^{\pi i\gamma}-e^{-\pi i\gamma}}\left\{e^{\pi i\gamma/2}e^{(-1)^{k_{\nu+1}}\pi i(2\beta-\gamma)/2}-e^{-\pi i\gamma/2}e^{(-1)^{k_{\nu+1}}\pi i(\gamma-2\beta)/2}\right\}z_2(\tau)/\theta_3^2,\end{aligned}$$

and therefore

$$
\begin{aligned}
J_\nu =& e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\beta-\gamma+1)]}e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+\cdots+l_4+l_2)} \\
&\times e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(\gamma-1)k_{\nu+1}}e^{\pi i(\alpha-3\gamma)}\frac{\sin\pi\{\gamma+(-1)^{k_{\nu+1}}(\gamma-2\alpha)\}/2}{\sin\pi\gamma}z_1(\tau)/\theta_3^2 \\
&+ e^{\pi i[\{(-1)^{k_{\nu+1}-k_\nu}+(-1)^{k_{\nu+1}-k_{\nu-2}}+\cdots+(-1)^{k_{\nu+1}-k_1}\}(2\alpha-\gamma-1)+\{(-1)^{k_{\nu+1}-k_{\nu-1}}+(-1)^{k_{\nu+1}-k_{\nu-3}}+\cdots+(-1)^{k_{\nu+1}-k_2}\}(2\beta-\gamma+1)]}e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+\cdots+l_4+l_2)} \\
&\times e^{2\nu\pi i(\alpha+\beta-\gamma)}e^{\pi i(\gamma-1)k_{\nu+1}}e^{\pi i(\alpha-3\gamma)}\frac{\sin\pi\{\gamma+(-1)^{k_{\nu+1}}(2\beta-\gamma)\}/2}{\sin\pi\gamma}z_2(\tau)/\theta_3^2.
\end{aligned} \tag{8.6}
$$

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