Nonlocal spatially inhomogeneous Hamilton-Jacobi equation with unusual free boundary

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Abstract

We consider the weighted mean curvature flow with a driving term in the plane. For anisotropy functions this evolution problem degenerates to a first order Hamilton-Jacobi equation with a free boundary. The resulting problem may be written as a Hamilton-Jacobi equation with a spatially non-local and discontinuous Hamiltonian. We prove existence and uniqueness of solutions. On the way we show a comparison principle and a stability theorem for viscosity solutions.

Key words: driven curvature flow, singular energies, Hamilton-Jacobi equation, free boundary, discontinuous Hamiltonian, comparison principle

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1 Introduction

Our goal is to study the theory of viscosity solutions to Hamilton-Jacobi equation with discontinuous Hamiltonians and an unusual free boundary. Here we have in mind problems arising when we try to solve the weighted mean curvature (wmc) flow

$$\beta V = \kappa_\gamma + \sigma \quad \text{on } \Gamma(t), \tag{1.1}$$

for a graph of a Lipschitz function over $\mathbb{R}$. Formally, (1.1) is a parabolic equation of the second order, but for the interesting anisotropy function it degenerates to a first order system. In order to explain it, we recall that the curvature, κ_γ, appearing in (1.1) is defined by

$$\kappa_\gamma = -\mathrm{div}_S \left(\nabla_\zeta \gamma(\zeta)|_{\zeta=\mathbf{n}} \right), \tag{1.2}$$

where $\mathbf{n}$ is the normal to the curve. In our case vector $\mathbf{n}$ is defined only $\mathcal{H}^1$-a.e. Moreover, γ is an isotropy function (the surface energy density). The physical examples we have in mind, see [GR1], give us the motivation to consider

$$\gamma(p_1, p_2) = \gamma_\Lambda |p_1| + \gamma_T |p_2|. \tag{1.3}$$

Thus, if there are parts of $\Gamma(t)$ with positive $\mathcal{H}^1$ measure, where $\mathbf{n}$ equals $\mathbf{n}_R = (0, 1)$, then (1.2) makes no sense. At the same time, due to the above definition of γ, the curvature κ_γ is zero on curved parts of $\Gamma(t)$, where $\mathbf{n} \neq \mathbf{n}_R$. The driving term σ is given here. Thus, we claim that (1.1) degenerates to a first order equation (except four directions of orientions including $\mathbf{n}_R$), which may be studied with the methods of viscosity solutions.

In [GR4] we have defined bent facets and we constructed a facet bending solution to (1.1) for a special choice of β, so that the evolution in the direction of $\mathbf{n}_R$ is, rougly speaking, just the upward tranlation. We would like to extend our results so that quite general β will be allowed. The goal of this paper is to develop a theory for Hamilton-Jacobi equations with an unusual free boundary which is useful for this purpose.

In a companion paper, [GGR], we fully explain the process of deriving from (1.1) a tractable system in a local coordinate system. Here, we give only a necessary sketch. We notice that the formula (1.2) becomes meaningful again, once we replace the gradient $\nabla_\zeta \gamma$ with a selection ξ of the subdifferential $\partial_\zeta \gamma$. We need a rule to select $\xi(x) \in \partial_\zeta \gamma(\mathbf{n}(x))$. In [GR4] we considered a variational principle which which goes back to [FG], [GGM], [GPR] and it was developed by Bellettini, Caselles, Chambolle, Moll, Novaga and Paolini, see [BNP1], [BNP2], [CMN].

We note that if β is properly chosen so that equation (1.1) can be written in the form of a differential inclusion $u_t \in -\partial\varphi$ for the graph $\{(x, y) : y = u(t, x)\}$, then a unique solution for initial value problem is constructed in [GG] even if σ depends on x. Also in [GR3] we considered a variational principle, similar to the one studied in [GR4], to show persitency of facets in a free boundary problem involing (1.1). There, the kinetic coefficient β was special. In the present paper, however, we consider quite general β.

With the help of the variational principle mentioned above, we defined [GR4] a variational solution to (1.1) as a couple (d, ξ), where $\Gamma(t)$ is the graph of $d(t, \cdot)$ and $\xi(t)$ is a minimizer of a variational functional.

What is now interesting for us, is that equation (1.1) may be conveniently written for a variational solution if we assume that $\Gamma(t)$ has just one facet, *i.e.*, it has just one line segment with a normal equal to $\mathbf{n}$. Namely, one has (see [GR4, GGR] for a derivation),

$$
\begin{aligned}
\beta_R \dot{L}_0 = \fint_0^{r_0} \sigma(t,s,L_0)\,ds + \frac{\gamma_\Lambda}{r_0} \qquad &\text{on} \quad [0, r_0(t)],\\
d_t = \sigma(t,x,d) m(d_x) \qquad &\text{on} \quad [r_0(t), +\infty),
\end{aligned}
\tag{1.4}
$$

(where γ_Λ equals γ evaluated at $(1,0)$) and it is augmented with the following initial conditions,

$$
r_0(0) = r_{00}, \quad L_0(0) = L_{00}, \quad d(0,x) = d_0(x).
$$

For the sake of simplicity, we assume that the data d_0, σ (see (2.2)) and the solution d are even in x, thus it suffices to consider only $x > 0$.

An important observation is that in order to close this system we need information about the evolution of $r_0(\cdot)$. These points form a zero-dimensional free boundary, whose evolution is not determined by (1.4). Apparently, as a result we have a Hamilton-Jacobi equation with a free boundary which is coupled to an ODE with a nonlocally defined nonlinearity. However, we would like to develop a unifying framework based upon the viscosity theory of Hamilton-Jacobi equations. For this purpose we set

$$
\bar{d}(t,x) = d(t,x) \quad \text{for } |x| > r_0(t), \qquad \bar{d}(t,x) = L_0(t) \quad \text{for } |x| \le r_0(t)
$$

and we require continuity of $\bar{d}$. In this way we may rewrite (1.4) as a single Hamilton-Jacobi equation,

$$
\bar{d}_t + \bar{H}(t,x,\bar{d},\bar{d}_x) = 0,
$$

where we expect to be able to write $\bar{H}$ as

$$
\tilde{H}(t,x,u,p) = \begin{cases} -\sigma(t,x,u)m(p) & \text{for } |x| > r_0(t), \\ -\sigma(t,r_0^*(t),u)m(0) & \text{for } |x| \le r_0(t), \end{cases}
$$

where $r_0^*(t)$ is properly chosen, see §4.3. In general, the Hamiltonian $\bar{H}$ is not continuous along the free boundary $r_0(\cdot)$. This is why we must study the relationship between the right-hand-sides of (1.4_1) and (1.4_2). This requires understanding the behavior of r_0. We shall concentrate on those cases which lead to new problems of the theory of Hamilton-Jacobi equations with discontinuous Hamiltonians. We shall see that a particularly interesting $\bar{H}$ arises when $\dot{r}_0 > 0$. Our main geometric results Theorems 4.1 and 4.2may be stated as follows.

Theorem 1.1 *Let us suppose that σ satisfies the symmetry relation (2.2) and Berg's effect (2.1) and d_0 is admissible (see §2.1 a definition), $d_0|_{\mathbb{R}\setminus(-r_{00},r_{00})} \in C^1 \cap L^\infty$ and $\Xi_0^R > 0$ (this quantity is defined in (2.11). (a) If $d_{0,x}(r_{00}) > 0$, then there exist $T_1 > 0$ and a unique solution to problem (1.4), i.e., there is a unique matching curve $r_0(\cdot)$, L_0 a unique solution to ODE (1.4_1) and d a unique solution to the Hamilton-Jacobi equation (1.4_2). Moreover, $L_0(t) = d(t, r_0(t))$.*
(b)If $d_{0,x}(r_{00}) = 0$, then there exists a unique proper matching curve r_0 and a unique solution L_0, d to problem (1.4_1)–(1.4_2), in addition $L_0(t) = d(t, r_0(t))$.

The notion of the proper matching curve is introduced at the end of §4.2.

The key point is the construction of the interfacial curve r_0. In case (a) it is done with the help of the Banach contraction principle which yields a unique solution. The story gets complicated in case (b) because the Banach theorem is no longer applicable. While is it possible to construct the matching curves by Schauder theorem the uniqueness is not obvious. This case requires the power of the viscosity theory for Hamilton-Jacobi equation with discontinuous Hamiltonians. The starting point is the observation that a small perturbation in the C^0-topology of data in part (b) reduces them to case (a). The passage to the limit is independent of the perturbation and thus selects the mentioned above proper matching curve. This process requires re-writing our problem (1.4) as as single Hamilton-Jacobi equation. A stability result is shown, see Proposition 4.6, to ensure that the uniform limit of viscosity solutions is a solution to the limit problem. We also develop a Comparison Principle, in Theorem 4.3, to establish uniqueness of solutions. It is valid for a restricted class of super-/subsolutions, but it is sufficient for our purposes.

For the sake of consistency we left out many geometric questions, for example a complete catalogue of possible configurations of initial conditions for (1.1) is studied in the companion paper [GGR]. We stress that we permit a general driving term σ in (1.1) conforming to (2.1) and (2.2) and which are of class C^1. We do not discuss here any possible relaxation of this regularity assumption here.

2 Setting up the problem

2.1 Definitions

Here we recall the known facts and introduce the notions necessary to study equation (1.1).

The physical examples we have in mind, see [GR1], give us the motivation to consider

$$\gamma(p_1, p_2) = \gamma_\Lambda |p_1| + \gamma_T |p_2|.$$

It is also natural to consider σ for which the following Berg's effect (see [GR2] and references therein) and symmetry conditions, hold *i.e.* for all $t \in [0, T)$

$$x_i \frac{\partial \sigma}{\partial x_i}(t, x_1, x_2) > 0 \quad \text{for } x_i \neq 0, \quad i = 1, 2, \tag{2.1}$$

and

$$\sigma(t, x_1, x_2) = \sigma(t, -x_1, x_2), \quad \sigma(t, x_1, x_2) = \sigma(t, x_1, -x_2). \tag{2.2}$$

However, conditions (2.2) are just for the sake of simplicity of the presentation.

In this paper, in order to avoid unnecessary technical complications related to the boundary conditions we consider only curves, which are graphs of functions defined over $\mathbb{R}$ with values in $\mathbb{R}_+$, *i.e.*, $\Gamma(t) = \{(x, d(t, x) : \ x \in \mathbb{R}\}$. We assume that for each t the function $d(t, \cdot)$ in the above definition is Lipschitz and *admissible* in the following sense: (a) for all $x \in \mathbb{R}$ we have $d(t, -x) = d(t, x)$, (b) 0 belongs to an open set, where d_x vanishes and (c) $d|_{[0,+\infty)}$ is non-decreasing. Later, we shall impose further restrictions of the class of considered curves.

We will denote the normal vector to Γ by $\mathbf{n}$. Parts of $\Gamma(t)$, where $\mathbf{n}$ belongs to set of normals to the Wulff shape of γ, which coincides here with the set of non-differentiability points of γ on the sphere, are special. If further conditions are satisfied they are called *faceted regions* and in our case their normal is vector $\mathbf{n}_R = (0, 1)$. We refer an interested reader to [GR4, GR5, GGR] for more details on that matter.

2.2 The reduction to the local coordinate system

In this subsection we take for granted that (1.1) leads to (1.4) in the local coordinate system for variational solutions, (see [GR4, GGR] for details). However, we have to specify the properties of m, (also called a mobility coefficient), which are inferred from the general properties of β, (see [GGR] for more comments). Namely, they are:

$$\frac{1}{\beta_R} = m(0) \leq m(p), \tag{2.3}$$

$$m(p) = m(-p), \tag{2.4}$$

$$m \text{ is Lipschitz continuous and } m \in C^2(\mathbb{R} \setminus \{0\}), \tag{2.5}$$

$$m \text{ is convex for } |p| \leq 1, \tag{2.6}$$

$$m(p) \leq C(1 + |p|). \tag{2.7}$$

Here we use the shorthand $\beta_R = \beta(\mathbf{n}_R)$.

2.3 The interfacial curves

We showed in [GR5] that the interfacial curve r_0 may be of two types: either a tangency curve or a matching curve. We shall say here that the *tangency condition* is satisfied at $(t, r_0(t))$, provided that (see [GR4, Proposition 2.1] and [GR4, (3.10)]),

$$\sigma(t, r_0(t), L_0(t)) = \fint_0^{r_0(t)} \sigma(t, s, L_0(t))\, ds + \frac{\gamma_\Lambda}{r_0(t)}. \tag{2.8}$$

This is a slight modification of the notion, because here we explicitly include time t as opposed to [GR4].

If the tangency condition is satisfied at $(t, r_0(t))$ for all $t \in [0, T)$, then we call the curve $r_0(\cdot)$ a *tangency curve*.

We always demand that $\Gamma(t)$ is a Lipschitz curve, thus the solutions to (1.4) must satisfy

$$d(t, r_0(t)) = L_0(t). \tag{2.9}$$

We call (2.9) the *matching condition*, because L_0 must match d. The matching condition is automatically satisfied for the tangency curves. However, there are curves which do not satisfy the tangency condition for $t > 0$. They are defined just by (2.9). Let us note the following result.

Proposition 2.1 *(cf. [GR5, Proposition 3.3]) Let us suppose that σ of class $C^1([0,T) \times \mathbb{R}^2)$ is given, it satisfies the Berg's effect (2.1) and the symmetry relations (2.2). We assume that we have a variational solution to (1.1), in particular if $\Gamma(t)$ is a family of graphs of admissible Lipschitz functions evolving according to (1.4), and $r_0(\cdot)$ is a C^1 curve in addition $d(\cdot, x)$ is a piecewise C^1 function. If the tangency as well as matching conditions are satisfied at $(t, r_0(t))$ for all $t \in [0, \epsilon)$, then $r_0(\cdot)$ is decreasing.*

For the proof we refer the reader to [GR5, GGR]. $\square$

In order not to distract the reader with unnecessary technical considerations we do not present the notion of a variational solution to (1.1), for it is not needed here. We refer an interested reader for more details to [GR5, GGR].

We need a device which helps us deciding the sign of $\dot r_0(0)$ as well as the type of the curve from the data. In [GR4]–[GR5] we introduced the following quantity,

$$\begin{aligned}\Sigma_0^R &= \int_0^{r_{00}} \sigma_t(0,y,L_{00})\,dy - \sigma_t(0,r_{00},L_{00}) \\ &\quad + \sigma(0,r_{00},L_{00}) \left(\int_0^{r_{00}} \sigma_{x_2}(0,y,L_{00})\,dy - \sigma_{x_2}(0,r_{00},L_{00}) \right). \end{aligned} \tag{2.10}$$

In [GGR] we needed

$$\Xi_0^R = \frac{\Sigma_0^R}{\beta_R} + \sigma(0,r_{00},L_{00})\sigma_{x_1}(0,r_{00},L_{00}) m_p(0^+). \tag{2.11}$$

The role of Ξ_0^R is explained in the following Proposition, which is an adjustment of [GR5, Proposition 3.4] to the present situation.

Proposition 2.2 *([GGR, §2.3], cf. also [GR4, Proposition 3.4]) Let us suppose* (Γ,ξ) *is a variational solution on* $(0,T)$ *and* σ_{x_1}, σ_{x_2}, σ_t *are continuous on* $[0,T)\times\mathbb{R}^2$. *We also assume that* $d(t,\cdot)$ *is of class* $C^{1,1}$ *in the complement of the interior of the faceted regions,* $r_0(\cdot)$ *is a matching curve and* r_0 *is strictly monotone. Moreover, the tangency condition is satisfied at* $r_{00} = r_0(0)$. *Then,*
(a) If, $d_x^+(t,r_0(t)) > 0$, *i.e., the right derivative of* $d(t,\cdot)$ *at* $r_0(t)$ *is positive, then* $r_0(\cdot)$ *is differentiable for* $t\in(0,T)$ *and*

$$\dot r_0(t) = \frac{1}{d_x^+(t,r_0(t))}\left(\dot L_0(t)/\beta_R - \sigma(t,r_0(t),L_0(t)) m(d_x^+(t,r_0(t)))\right), \tag{2.12}$$

moreover $\dot r_0(0) = 0$.
(b) If $d^+_{0,x}(r_{00}) = 0$ *and the right second derivative of* d_0 *vanishes at* $x = r_{00}$, *i.e.,* $d^+_{0,xx}(r_{00}) = 0$, *then the derivative of* r_0 *at* $t = 0$ *exists and it is positive as long as* $\Xi_0^R > 0$ *and it is given by* $\dot r_0(0) = \frac{1}{2}\Xi_0^R/\sigma_{x_1}(0,r_{00},L_{00})$. *In particular, this derivative is positive, provided that* $\Xi_0^R > 0$.
(c) If $d^+_{0,x}(r_{00}) = 0$ *and* $d^+_{0,xx}(r_{00}) > 0$, *then*

$$\dot r_0(0) = -\frac{\sigma_{x_1}(0,r_{00},L_{00})}{d^+_{0,xx}(r_{00})}\left(1 - \sqrt{1 + \frac{\Xi_0^R d^+_{0,xx}(r_{00})}{(\sigma_{x_1}(0,r_{00},L_{00}))^2}}\right).$$

In particular, $\dot r_0(0)$ *is positive if* $\Xi_0^R > 0$.
(d) If $\Xi_0^R = 0$, *then* $\dot r_0(0) = 0$.

After familiarizing with this proposition, a reader may ask, e.g. if there are decreasing matching curves different from tangency curves. It turns out that this is not possible. A discussion of this and related problems is beyond the scope of the present article. It is included in the companion paper [GGR], where a complete catalogue of initial configurations is presented. □

3 Evolution of graphs by a Hamilton-Jacobi equation

We want to solve the equation written in the local coordinate system, (1.4). We notice its several components: a Hamilton-Jacobi equation with a free boundary $r_0(\cdot)$ coupled with a nonlocal ODE. We want to present first the facts implied by the classical theory of viscosity solutions to Hamilton-Jacobi equations.

Proposition 3.1 *Let us suppose that σ belonging to $C^1([0,T)\times\mathbb{R}^2)$ is given and it satisfies the Berg's effect (2.1) and (2.2). The mobility coefficient m fulfills (2.3)–(2.7), d_0 is an admissible Lipschitz function on $\mathbb{R}$, in particular it is increasing on $[r_{00},\infty)$ and $d_0(x) = L_{00}$ for $|x| \le r_{00}$. If we set*

$$H_a(t,x,d,p) = -\sigma(t,x,d)m(p), \tag{3.1}$$

then, there exists a unique viscosity solution to

$$\begin{array}{ll} d_t + H_a(t,x,d,d_x) = 0 & \text{in } (0,T)\times\mathbb{R}, \\ d(0,x) = d_0(x) & \text{for } x\in\mathbb{R}. \end{array} \tag{3.2}$$

Moreover, the modulus of continuity of d is defined by the continuity moduli of m, σ and d_0.

Proof. This is in fact a corollary to the classical Perron's method, see [I]. □

In order to extract more information about smoothness of (3.2) we will study its regularization. By this method we will discover further properties of solutions in a neighborhood of r_0. We hope to be able to localize (3.2). This is indeed true, see Proposition 3.3.

Now, we proceed with the regularization. We apply the standard mollification to m, d_0 and σ. We consider $m^\epsilon = m*\rho_\epsilon$, $d_0^\epsilon = d_0*\rho_\epsilon$ and $\sigma^\epsilon = \sigma*\rho_\epsilon$, where the standard mollifier ρ_ϵ has a support in $B(0,\epsilon)$. We note that in the first two cases the mollification is on the real line, in the last one in $\mathbb{R}^3$.

As a result of regularization we have $m^\epsilon(0) > 1$ and $m^\epsilon_p(0) = 0$. We end up with the problem

$$\begin{array}{l} d_t - \sigma^\epsilon(t,x,d)m^\epsilon(d_x) = 0 \quad \text{in } x\in(0,T)\times\mathbb{R}, \\ d(x,0) = d_0^\epsilon(x) \quad \text{for } x\in\mathbb{R}, \end{array} \tag{3.3}$$

where we suppressed ϵ over d.

We begin with writing the characteristic system, which after simplifications takes the form

$$\begin{array}{l} \dot{x} = -\sigma^\epsilon(t,x,d)m^\epsilon_p(p), \\ \dot{p} = \left(\sigma^\epsilon_{x_1}(t,x,d) + \sigma^\epsilon_{x_2}(t,x,d)p\right) m^\epsilon(p), \\ \dot{d} = \sigma^\epsilon(t,x,d)\left(m^\epsilon(p) - m^\epsilon_p(p)p\right), \\ x(0,\xi) = \xi, \quad d(0,\xi) = d_0(\xi), \quad p(0,\xi) = d_{0,x}(\xi), \quad \xi\in\mathbb{R}. \end{array} \tag{3.4}$$

Let us make the first observation about (3.4).

Proposition 3.2 *Let us suppose that $\sigma^\epsilon, m^\epsilon$ and d_0^ϵ are the mentioned above mollifications of σ, m and d_0 which satisfy the assumption of Proposition 3.1. Then, for all $\epsilon > 0$ there exists $T_\epsilon \in (0,T)$ and a unique smooth solution to (3.4). Moreover, $T_\epsilon \ge T_0 > 0$, where T_0 is independent of ϵ.*

Proof. Existence and uniqueness part is a conclusion from the classical theory of characteristic systems.

For the second part of the proposition, let us now recall a fact from the theory of ODE's,

$$y' = f(y,t), \quad y(t_0) = y_0,$$

where $f : \Omega \times (t_1, t_2) \to \mathbb{R}$, Ω is an open subset of $\mathbb{R}^k$ and $y_0 \in \Omega$, $t_0 \in (t_1, t_2)$. We know that T_{max}, the length of maximal existence interval can be estimated from below only in the following terms: (a) the distance r from (y_0, t_0) to the boundary of $\Omega \times (t_1, t_2)$; (b) the maximum of f over $\Omega \times (t_1, t_2)$. In our case f, depending upon (x, d, p, t), is given by the RHS of (3.4) and $\Omega \times (t_1, t_2)$ is equal to $U := \mathbb{R} \times \mathbb{R} \times [-1, 1] \times \mathbb{R}$. The RHS of (3.4) is of course bounded on U. Thus, we will succeed in proving a uniform bound on T_{max} provided that we can show a bound on p. This fact is the content of the Lemma below. $\square$

Lemma 3.1 *Let us suppose that* $m(p) \leq C(1 + |p|)$, (x, d, p) *is the unique solution to system (3.4) and* $0 \leq p(0, \cdot) \leq p_0 < 1$, *then there exits* $T_0 > 0$ *independent of* ϵ *such that* $|p(t, \xi)| \leq 1$ *for all* $t \in [0, T_0]$.

Proof. Indeed, let us have a look at (3.4$_2$). The assumptions on d and m imply existence of $K > 0$ such that

$$\dot{p} \leq K(1 + p + p^2), \qquad p(0) = d_{0,x}(\xi). \tag{3.5}$$

Thus, as long as $p(t) \leq 1$ holds, then (3.5) implies

$$\dot{p} \leq K(1 + 2p), \qquad p(0) = d_{0x}(\xi) \leq p_0 < 1.$$

This differential inequality combined with Gronwall inequality lead us to

$$(1 + 2p_0)e^{2Kt} \leq 3.$$

Hence, we conclude that if t belongs to the interval $[0, T_0]$, where $T_0 = \frac{1}{2K} \ln \frac{3}{1+2p_0}$, which is independent from ϵ, then

$$p(t) \leq 1$$

and the claim follows. $\square$

Let us draw some qualitative conclusion about the projected characteristics. If $d_{0,x}(\xi) = 0$, then $p(0, \xi) = 0$ and $\dot{x}(0, \xi) = -\sigma(t, x, d)m_p^\epsilon(0) = 0$. Hence, the projected characteristic is perpendicular to the x-axis. On the other hand, by (2.6) for any $a > 0$, we have $m_p^\epsilon(a) \geq 0$. Thus, since d is nondecreasing we notice that always $\dot{x}(0) \leq 0$, see Fig. 1.

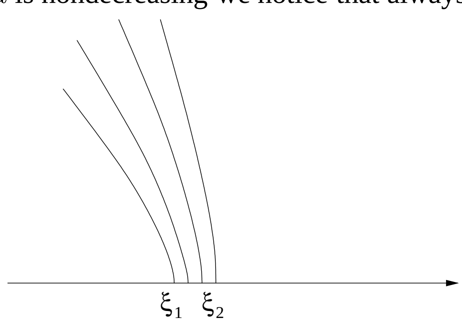

Fig. 1

Remark. This property of the characteristic curves implies that they emanate from any tangency curve if $d_x(t, r_0(t)) = 0$. This will be indeed shown in Proposition 3.4. Thus, these facts justify the statement that a matching condition is automatically fulfilled on the tangency curve. We may also say that a tangency curve behaves like a rarefaction wave. On the other hand the same property of the characteristics implies that the matching curve terminate on them, thus the matching curve are like shock waves.

Now, we will state the first of our qualitative results. It explains our interest in the regularized system. It is also useful in the process of constructing a solution to the free boundary problem (1.4). Of course we have to localize it in a neighborhood of r_0.

Theorem 3.1 *Let us assume that m fulfills (2.3)–(2.7), σ satisfies Berg's effect, $\sigma \in C^1([0,T)\times\mathbb{R}^2)$, d_0 is a bounded admissible Lipschitz function and $d_0|_{[r_{00},+\infty)} \in C^1\cap L^\infty$ and d is the corresponding viscosity solution to (3.2). If Lip$(d_0) = p_0 < 1$, then:*
(a) for all $t \in (0,T_0]$, where T_0 is provided by Lemma 3.1, we have Lip$(d(t,\cdot)) \leq 1$ and Lip$(d(\cdot,x)) \leq M := m(1)\sup\sigma$ on $[0,T_0]$ for all $x \in \mathbb{R}$;
(b) if $d^+_{0,x}(r_{00}) \geq \delta > 0$ and $\lambda_1 > r_{00}$, then there is $\delta_0 = \delta_0(\lambda_1) > 0$ such that for all $t \in [0,T_0]$ and all x, y satisfying $r_{00} \leq x \leq y \leq \lambda_1$ we have $d(t,y) - d(t,x) \geq \delta_0(y-x)$;
(c) if $d_{0,x}(r_{00}) = 0$, then there exists a continuous function $\delta_0 : (0,T_0] \to (0,1)$, such that for all $t \in (0,T_0]$ we have $d(t,y) - d(t,x) \geq \delta_0(t)(y-x)$ for $x, y \in [r_0,\lambda_1]$.

Proof. Existence of solutions to the regularized characteristic system (3.4) was established in Proposition 3.2. Lemma 3.1 implies existence of a uniform bound, equal to 1, (with respect to ϵ) on d^ϵ_x for $t \in [0,T_0]$. Since d^ϵ_0 converges uniformly to d_0, then the theory of viscosity solutions implies the uniform convergence of d^ϵ to the unique viscosity solution of (3.2). As a result the Lipschitz constant of $d(t,\cdot)$, the limit of d^ϵ is bounded by one on $[0,T_0]$.

By the same token

$$\text{Lip}\,(d(\cdot,x)) \leq \max_{t,x} d^\epsilon_t(t,x) \leq m(1)\sup_{t,x,u}\sigma(t,x,u).$$

Thus, (a) is established.

In order to prove (b) we use the convexity of m. This property and (3.4$_2$) imply that $d^\epsilon_x(t,x) \geq \delta := \min\{d_{0,x}(x) : x \in [r_{00},\lambda_1]\}$ as long as $d^\epsilon_x \leq 1$. Subsequently, we proceed by approximation,

$$d(t,y) - d(t,x) = \lim_{\epsilon\to 0^+}(d^\epsilon(t,y) - d^\epsilon(t,x)) = \lim_{\epsilon\to 0^+}\int_x^y d^\epsilon_x(t,\tau)\,d\tau \geq \delta(y-x).$$

We prove (c) in similar manner. We fix $t > 0$, then we have

$$d(t,y) - d(t,x) = \lim_{\epsilon\to 0^+}\int_x^y d^\epsilon_x(t,\tau)\,d\tau.$$

By (3.4$_2$), we deduce that at $t > 0$ the derivative $d_x(t,x)$ must be at least

$$t\sigma_{x_2}(t, r_{00}/2, d_0(r_{00}))m(0) =: \delta(t) > 0.$$

Thus,

$$d(t,y) - d(t,x) \geq \delta(t)(y-x).$$

□

Sometimes we have to localize solutions, this is of course possible because of the finite speed of propagation. In fact this is permitted by our next result.

Proposition 3.3 *Let us suppose that d_{01}, d_{02} are two pieces of initial data required by Proposition 3.1, satisfying $(d_{01}-d_{02})|_{[\lambda_0,\lambda_1]} \equiv 0$ and d_1, d_2 are the corresponding solutions to (3.4). Then, for any $t < T_0$, such that $\lambda_1 - \mu t > \lambda_0$ we have*

$$(d_1 - d_2)|_{[0,t]\times[\lambda_0,\lambda_1-\mu t]} \equiv 0,$$

where $\mu = \sup\sigma\cdot\sup m_p$.

Proof. Let us look at the system of characteristics (3.4) augmented with d^ϵ_{0i}, $i=1,2$ the mollification of d_{0i}. Since the mollification kernel ρ_ϵ is supported in $(-\epsilon,\epsilon)$, then the initial data d^ϵ_{01} d^ϵ_{02} agree on $[\lambda_0+\epsilon,\lambda_1-\epsilon]$. By Theorem 3.1, the solutions (x_i,d_i,p_i), $i=1,2$ exist for $t\le T_0$, where T_0 is introduced in Lemma 3.1. We notice that due to the structure of the RHS of (3.4) we have $|\dot x_i|\le\mu$, $i=1,2$. Thus, if $y\in[\lambda_0+\epsilon,\lambda_1-\epsilon-\mu t]$, then $d^\epsilon_1(t,y)=d^\epsilon_2(t,y)$, as long as $\lambda_0+\epsilon<\lambda_1-\epsilon-\mu t$. The equality holds after the passage to the limit with ϵ. Thus $d_1=d_2$ on $[0,t]\times(\lambda_0,\lambda_1-\mu t)$. Our claim follows. □

3.1 Regularity of solutions

Here, we shall study the regularity of viscosity solutions in a non-cylindrical domain. If r_0 is strictly decreasing and positive and L_0 is a unique solution to

$$\dot L_0 = \fint_0^{r_0}\sigma(t,y,L_0)\,dy+\frac{\gamma_\Lambda}{r_0},$$

then we define a region $G_T=\{(t,x)\in(0,T_0)\times\mathbb{R}:\ |x|>r_0(t)\}$.

Proposition 3.4 *Let us suppose that $T>0$, H_a is given by (3.1), $r_0(\cdot)$ is a tangency curve and d is a viscosity solution to*

$$\begin{array}{lll} d_t+H_a(t,x,d,d_x)=0 & \text{in } G_T:=\{(t,x):t\ge0,x\ge r_0(t)\}, & \\ d(0,x)=d_0(x) & \text{for } |x|\ge r_{00}, & d(t,r_0(t))=L_0(t)\quad \text{for } t>0. \end{array} \tag{3.6}$$

If $d_{0,x}(r_{00})=0$ and d_x, d_t are continuous in $U\cap G_T$, where U is an open set containing all points $(t,r_0(t))$, $t\ge0$, then $d_x(t,r_0(t))=0$ for all $t\ge0$.

Proof. By the general theory a sufficiently smooth viscosity solution satisfies the equation pointwise in G_T. The classical theory stipulates the compatibility conditions for data g given on a curve. In our case, this curve is a graph of function r_0, $\Gamma(r_0)$, and g equals L_0, but it is convenient to express it in terms of the the arc-length parameter s. We notice,

$$\frac{dg}{ds}=\frac{dL_0}{ds}=\frac{dL_0}{dt}\frac{dt}{ds}=\frac{\dot L_0}{\sqrt{(\dot r_0)^2+1}}=\frac{\sigma(t,r_0,L_0)}{\beta_R\sqrt{(\dot r_0)^2+1}}.$$

The unit tangent $\vec\tau$ and normal vectors $\mathbf{n}$ are

$$\vec\tau=(1,\dot r_0)/\sqrt{(\dot r_0)^2+1},\qquad \mathbf{n}=(-\dot r_0,1)/\sqrt{(\dot r_0)^2+1}.$$

Once we write $u_{\vec\tau}=\frac{\partial u}{\partial\vec\tau}$ and $u_{\mathbf{n}}=\frac{\partial u}{\partial\mathbf{n}}$ then we obviously have,

$$\frac{\partial u}{\partial t}=e_t\cdot\vec\tau u_{\vec\tau}+e_t\cdot\mathbf{n}u_{\mathbf{n}},\quad \frac{\partial u}{\partial x}=e_x\cdot\vec\tau u_{\vec\tau}+e_x\cdot\mathbf{n}u_{\mathbf{n}}.$$

Subsequently,

$$\frac{\partial u}{\partial t}=\frac{1}{\sqrt{(\dot r_0)^2+1}}u_{\vec\tau}-\frac{\dot r_0}{\sqrt{(\dot r_0)^2+1}}u_{\mathbf{n}},\quad \frac{\partial u}{\partial x}=\frac{\dot r_0}{\sqrt{(\dot r_0)^2+1}}u_{\vec\tau}+\frac{1}{\sqrt{(\dot r_0)^2+1}}u_{\mathbf{n}}.$$

If d is a differentiable solution to (3.6), then we are able to calculate $d_{\vec\tau}$ from the data, because $d_{\vec\tau}=\frac{dg}{ds}$. The compatibility conditions require existence of a solution $p_{\mathbf{n}}$, which equals $d_{\mathbf{n}}$, to the following equation

$$\frac{\sigma(t,r_0,L_0)}{\beta_\Lambda((\dot r_0)^2+1)}-\frac{p_{\mathbf{n}}\dot r_0}{\sqrt{(\dot r_0)^2+1}}-\sigma(t,r_0,L_0)m\left(\frac{p_{\mathbf{n}}}{\sqrt{(\dot r_0)^2+1}}+\frac{\dot r_0\sigma(t,r_0,L_0)}{\beta_R((\dot r_0)^2+1)}\right)=0. \tag{3.7}$$

It is easy to see that

$$p_{\mathbf{n}} = -\frac{\sigma(t, r_0, L_0)\dot{r}_0}{\beta_R\sqrt{(\dot{r}_0)^2 + 1}}$$

is a solution to (3.7) yielding $d_x(t, r_0(t)) = 0$. It is a separate question if there are other solutions. Even if they exist, then the structure of equation (3.7) implies that they must be separated from $p_{\mathbf{n}}$ given above. As a result we would construct d with discontinuous space derivative at $t = 0$. Our claim follows. $\square$

We stress that in the above result we do not use any further regularity properties of m, because we know that in our case $d_x(t, r_0(t)) \geq 0$. In particular, the actual value of m_p^+ at $p = 0$ is unimportant for us. However, formally we extend m smoothly to the negative argument and use Proposition 3.4 anyway.

4 A Hamilton-Jacobi equation with a free boundary

Now, we begin our construction of the free boundary problem (1.4). We noticed that we have only two types of monotone interfacial curves r_0, if $r_0(\cdot)$ is decreasing, then we have tangency curves. In virtue of Proposition 3.4 we expect that d_x is continuous along such curves, provided that $d_{0,x}(r_{00}) = 0$. Moreover, since the characteristics start from tangency curves we may impose boundary data on curve r_0. By the Remark in §4.3 they do not require any advanced theory of viscosity solutions to the Hamilton-Jacobi equations hence their construction is performed in the companion paper [GGR].

If $r_0(\cdot)$ is increasing, then this is a matching curve determined by the conditions

$$d(t, r_0(t)) = L_0(t).$$

In this case the characteristics terminate on the matching curve. Such curves are constructed in subsection 4.1.

We also seek a unifying framework for our free boundary problem (1.4). A natural choice is the theory of viscosity solutions to Hamilton-Jacobi equations. The situation is quite interesting in the case of matching curves, because a natural choice of Hamiltonian $\bar{H}$ transforms (1.4) into an equation with a spatially nonlocal and discontinuous Hamiltonian. Moreover, the discontinuity is with respect to $p = d_x$. This is is why we have to develop theory of solutions to such problems. It is however restricted to deal with sufficiently regular, *i.e.*, Lipschitz continuous solutions. We shall establish a comparison principle and stability in this restricted setting. These results are presented in subsections 4.3 and 4.2 respectively.

We have already learned what are the component of these systems: Proposition 2.2 gives us a tool to detect the type of curve emanating from the interfacial point r_0. Due to Theorem 3.1 we know how to solve the Hamilton-Jacobi equations (1.4_2), *i.e.*, (3.2).

Let us now concentrate on the interfacial curves. We have the following possibilities at the interfacial point r_{00}:
(Ξ) we have two possibilities of a sign of Ξ^R, ($\Xi^R = 0$ does not permit us to decide what is the type of the interfacial curve);
(τ) the tangency condition holds or fails;
(D) the derivative $d_{0,x}(r_{00})$ either vanishes or it is different from zero.

Thus, in order to solve the geometric problem of the evolution of a graph we would have to consider the eight cases. However, only those related to the matching curve lead to non-trivial problems of the viscosity theory of Hamilton-Jacobi equations. The remaining cases will be considered in detail in the companion paper [GGR]. Here are our main geometric results.

Theorem 4.1 *Let us suppose that $\sigma \in C^1([0,T)\times\mathbb{R}^2)$ satisfies the symmetry relation (2.2) and Berg's effect (2.1) and d_0 is a bounded admissible Lipschitz function with Lip$(d_0) = p_0 < 1$ and $d_0|_{[r_{00},+\infty)} \in C^1 \cap L^\infty$. If $\Xi_0^R > 0$ and $d_{0,x}(r_{00}) > 0$, then there exist $T_1 \le T$ and a unique solution to problem (1.4), i.e., there is a unique matching curve $r_0(\cdot)$, L_0 a unique solution to ODE (1.4$_1$) and d a unique solution to the Hamilton-Jacobi equation (1.4$_2$). Moreover, $L_0(t) = d(t, r_0(t))$.*

The proof is given in subsection 4.1, where the appropriate interfacial curves are constructed. If $d_{0,x}(r_{00}) = 0$ and $\Xi_0^R > 0$, then further considerations are required. This is so, because the method of construction of the matching curve developed in §4.1 does not yield uniqueness. The necessary tools are created in subsections 4.2 and 4.3. They are based on a selection principle leading us to the notion of a proper matching curve. We also need a stability result and a comparison principle for Hamilton-Jacobi equations with non-local discontinuous Hamiltonians. They lead us to the proof of the following statement, where the notion of a proper matching curve is introduced at the end of subsection 4.2.

Theorem 4.2 *Let us suppose that $\sigma \in C^1([0,T)\times\mathbb{R}^2)$ satisfies the symmetry relation (2.2) and Berg's effect (2.1). We assume that d_0 is a bounded admissible Lipschitz function with Lip$(d_0) = p_0 < 1$ and $d_0|_{[r_{00},+\infty)} \in C^1 \cap L^\infty$. If $\Xi_0^R > 0$ and $d_{0,x}(r_{00}) = 0$, then there exist $T_1 \le T$ and a unique proper matching curve r_0 and a unique solution L_0, d to problem (1.4$_1$)–(1.4$_2$).*

4.1 A matching curve emanating from r_{00}

We shall prove here Theorem 4.1. For this purpose we turn our attention to a construction of a matching curve emanating from r_{00}. Here, we do not care if the tangency condition is satisfied and we admit vanishing derivative of d_0 at r_{00}. We strive to present a general existence result covering as many cases as possible.

Proposition 4.1 *Let us assume that d_0 is an admissible initial condition and $\Xi_0^R > 0$.*

(a) Then, there exists $T_1 \le T_0$ and $(r_0, L_0) \in C^1([0,T_1];\mathbb{R}^2)$ a solution to the following problem,

$$\begin{aligned}
&\beta_R \dot{L}_0(t) = \frac{1}{r_0(t)}\int_0^{r_0(t)} \sigma(t,s,L_0(t))\,ds + \frac{\gamma_\Lambda}{r_0(t)},\\
&L_0(t) = d(t, r_0(t)),\\
&L_0(0) = L_{00}, \quad r_0(0) = r_{00}.
\end{aligned} \tag{4.1}$$

Here, d is a given solution to the Hamilton-Jacobi equation (3.2) in $(0,T_1)\times\mathbb{R}$ with the initial data d_0.

(b) If in addition, $d_{0,x}^+(r_{00}) > 0$, then the solution constructed in part (a) is unique.

Remark. It turns out that $d^+_{0,x}(r_{00}) > 0$ leads to a simple uniqueness proof. On the other hand, if $d_{0,x}(r_{00}) = 0$, then we have to apply a selection procedure. This will be treated separately in subsections 4.2 and 4.3.

Proof. Let us notice that for a given d_0 Proposition 3.1 yields the existence of a unique solution to the Hamilton-Jacobi equation (3.2). Subsequently, we notice that if we are interested in d for $x \geq r_{00}$, $t \in [0, T_1]$, then due to Theorem 3.1 $d(t, \cdot)$ is increasing.

For any $\tau \in (0, T_0)$, let us define the set

$$\begin{aligned} X_{\tau,\eta} &= \left\{(r, L) \in C([0, \tau]; \mathbb{R}^2) : \ (r, L)(0) = (r_{00}, L_{00}), \ \|r - r_{00}\|_{C[0,\tau]} \leq r_{00}/2, \right. \\ &\quad \left. \|L - L_{00}\|_{C[0,\tau]} \leq \eta \right\}, \end{aligned}$$

where $C([0, \tau]; \mathbb{R}^k)$, $k \in \mathbb{N}$, is the space of all continuous functions from $[0, \tau]$ with values in $\mathbb{R}^k$ equipped with the sup-norm defined as follows

$$\|u\|_{C[0,\tau]} = \max_{s\in[0,\tau]} |u(s)| \text{ and } \|\vec{v}\|_{C([0,\tau];\mathbb{R}^k)} = \left(\sum_{i=1}^{k} \|v_i\|^2_{C[0,\tau]}\right)^{1/2}, \quad (\vec{v} = (v_1, \ldots, v_k)).$$

Subsequently, we define a map $\mathcal{L} : X_{\tau,\eta} \to C([0, \tau]; \mathbb{R}^2)$ by

$$\mathcal{L}(r, L) = (\hat{r}, \hat{L}),$$

where $\hat{L}(t)$ is given by formula

$$\hat{L}(t) = L_{00} + \int_0^t \left(\frac{1}{r(z)} \int_0^{r(z)} \sigma(z, s, L_0(z))\, ds + \frac{\gamma_\Lambda}{r(z)}\right) dz, \tag{4.2}$$

and

$$\hat{r}(t) = d^{-1}(t, .) \left(\hat{L}(t)\right). \tag{4.3}$$

Taking the inverse function is justified by the strict monotonicity of d, see Theorem 3.1 (b), (c).

For an appropriate choice of η and τ we establish that $\mathcal{L} : X_{\tau,\eta} \to X_{\tau,\eta}$. Indeed,

$$\begin{aligned} \left|\hat{L}(t) - L_{00}\right| &= \left|\int_0^t \left(\frac{1}{r(z)} \int_0^{r(z)} \sigma(z, s, L_0(z))\, ds + \frac{\gamma_\Lambda}{r(z)}\right) dz\right| \\ &\leq \frac{\tau}{\inf_{s\in[0,\tau]} |r(s)|} \left(\sup_{s\in[0,\tau]} |r(s)| \sup_{\mathbb{R}_+\times\mathbb{R}^2} \sigma(t, x, y) + \gamma_\Lambda\right) \\ &\leq \frac{2\tau}{r_{00}} \left(\frac{3}{2} r_{00} \sup_{\mathbb{R}_+\times\mathbb{R}^2} \sigma(t, x, y) + \gamma_\Lambda\right) =: \tau K_0. \end{aligned}$$

Hence, taking $\tau = T_1 \leq T_0$ sufficiently small we obtain

$$\|\hat{L} - L_{00}\|_{C([0,\tau])} \leq \eta.$$

Next, we estimate

$$|\hat{r}(t) - r_{00}| = |d^{-1}(t, .)(\hat{L}(t)) - r_{00}| \leq \sup_{t\in[0,\tau], |y-L_{00}|\leq\eta} |d^{-1}(t, .)(y) - r_{00}|.$$

Since, the map $(t,y) \to d^{-1}(t,.)(y) - r_{00}$ is uniformly continuous on the compact set $d([0,\tau]\times[L_{00}-\eta, L_{00}+\eta])$, we can choose τ (by taking smaller T_1, if necessary) and η such that

$$\|\hat{r} - r_{00}\|_{C([0,\tau])} \leq \frac{r_{00}}{2}.$$

Now, we shall show that mapping $\mathcal{L}$ is compact. For this purpose we take a bounded sequence $(r_n, L_n) \in X_{\tau,\eta}$. Thus, the sequence $\hat{L}_n$ given by (4.2) is bounded not only in $C^0([0,\tau])$ but also in the space $C^1([0,\tau])$. Indeed, since $\mathcal{L}: X_{\tau,\eta} \to X_{\tau,\eta}$, then we have

$$\|\hat{L} - L_{00}\|_{C([0,\tau])} \leq \eta.$$

Moreover, $\dot{\hat{L}}_n$ is bounded in $C^0([0,\tau])$. Indeed, due to (4.2) we have

$$\left|\dot{\hat{L}}_n\right| \leq K_0.$$

Consequently,

$$\|\hat{L}\|_{C^1([0,\tau])} \leq \|L_{00}\|_{C^0([0,\tau])} + \eta + K_0.$$

This is why by Arzela-Ascoli Theorem we can extract a subsequence $\hat{L}_{n_k}$ of $\hat{L}_n$ converging uniformly $\hat{L}$. Subsequently, we deduce that $\hat{r}_{n_k} \to \hat{r}$ in $C[0,\tau]$, where $\hat{r}(t) = d^{-1}(t,.)(\hat{L}(t))$. This finishes proof of the compactness of map $\mathcal{L}$.

We may now invoke the Schauder Fixed Point Theorem, because $\mathcal{L}$ is compact and it maps a closed convex set $X_{\tau,\delta}$ into itself. Thus, we conclude that mapping $\mathcal{L}$ has a fixed point.

(b) In order to show uniqueness of the fixed point we shall show that $\mathcal{L}$ is a contraction on $X_{T,\eta}$, provided that $d^+_{0,x}(r_{00}) > 0$.

We recall that due to Theorem 3.1, part (b) $d(t,\cdot)$ is strictly increasing, more precisely for any $\lambda_1 \geq r_{00}$ there is a positive η such that for all $t \in (0, T_1]$ and all $r_{00} \leq x < y \leq \lambda_1$ we have $d(t,y) - d(t,x) \geq \eta(y-x)$. Thus, we can write:

$$\left|\hat{L}_1(t) - \hat{L}_2(t)\right| = |d(t,\hat{r}_1(t)) - d(t,\hat{r}_2(t))| \geq \eta\,|\hat{r}_1(t) - \hat{r}_2(t)|\,.$$

Hence

$$\|\hat{r}_1 - \hat{r}_2\|_{C([0,\tau])} \leq \frac{1}{\eta}\|\hat{L}_1 - \hat{L}_2\|_{C([0,\tau])}.$$

Now, employing (4.2) we come to the following estimate

$$\begin{aligned}
\left|\hat{L}_1(t) - \hat{L}_2(t)\right| \;\leq\;& \left|\int_0^t \left(\frac{1}{r_1(z)} - \frac{1}{r_2(z)}\right)\left(\gamma_\Lambda + \int_0^{r_1(z)} \sigma(z, L_1(z), s)\,ds\right)dz\right| \\
&+ \left|\int_0^t \frac{1}{r_2(z)} \int_0^{r_1(z)} \left(\sigma(z,L_1(z),s) - \sigma(z,L_2(z),s)\right)ds\,dz\right| \\
&+ \left|\int_0^t \left(\frac{1}{r_2(z)}\int_{r_2(z)}^{r_1(z)} \sigma(z,L_2(z),s)ds\right)dz\right| \qquad (4.4) \\
\leq\;& \tau\left(\frac{2}{r_{00}}\left(K_0 + \sup_{s\in[0,\tau]}\sigma\right)\|r_1 - r_2\|_{C[0,\tau]} + 3\sup_{s\in[0,\tau]}|\sigma_{x_2}|\|L_1 - L_2\|_{C[0,\tau]}\right).
\end{aligned}$$

Finally, restricting further T_1 for any $\tau \leq T_1$ we obtain that $\mathcal{L}$ is a contraction on $X_{\tau,\eta}$. $\square$

We notice, that this proposition due to the used method does not yield uniqueness of the interfacial curve if $d_{0,x}(r_{00}) = 0$. We shall find a separate argument later.

Let us now turn to the *proof of Theorem 4.1*. In fact, the existence of a matching curve follows from Proposition 4.1 (a). Its uniqueness is guaranteed by Proposition 4.1 (b). Existence of d and L is given also in Proposition 4.1. It is based on Proposition 3.1 and the theory of ODE's, respectively. $\square$

4.2 Stability

We will establish a stability result and in next section a comparison principle. They are interesting for its own sake because they deal with discontinuous Hamiltonians. They are also needed to select a proper solution r_0 when Theorem 4.1 fails to guarantee uniqueness if $d_{0,x}(r_{00}) = 0$. We stress that we want to solve simultaneously two cases: the tangency condition may or may not hold at the start of a matching curve.

Our restricted stability result is based on a comparison principle which guarantees uniqueness of viscosity solutions for Hamilton-Jacobi equations with discontinuous Hamiltonians. Apparently, there is not much literature on this subject, see however [CS], [St], [CR].

Here is our approach, we will first regularize data, d_0, in a canonical way (see below), so that the uniqueness part of Theorem 4.1 will apply to the regularization d_0^ϵ. Then, we will set up our problem as a Hamilton-Jacobi equation, on the real line, with a discontinuous Hamiltonian $\bar{H}^\epsilon$ involving parameter ϵ. Its viscosity solution $\bar{d}^\epsilon$ depends upon the interfacial curve r_0^ϵ. Subsequently, we show that the family of solutions $\bar{d}^\epsilon$ converges uniformly. From this fact we deduce the uniform convergence of d^ϵ as well as r_0^ϵ. The last statement requires some effort. Next, we claim that the limit $r_0^0 = \lim_{\epsilon\to 0^+} r_0^\epsilon$ is a solution to (4.1). Finally, we show that the limit does not depend upon a particular way to regularize data.

We begin with regularization, which will be called *canonical*, we set

$$d_0^\epsilon(x) = d_0(x) + \eta^\epsilon(x), \quad \text{where } \eta^\epsilon(x) = \begin{cases} 0 & \text{if } x \leq r_{00}, \\ \text{if } \epsilon(x - r_{00}) & r_{00} \leq x < \rho_0, \\ \text{if } \epsilon(\rho_0 - r_{00}) & x \geq \rho_0. \end{cases} \tag{4.5}$$

where $\rho_0 > r_{00}$ is a fixed number. Automatically, by Proposition 4.1 we will obtain existence and uniqueness of d^ϵ and r_0^ϵ. Furthermore, we have the following observation.

Lemma 4.1 *Let us suppose that $d_{0,x}(r_{00}) = 0$ and the assumptions of Proposition 4.1 hold (except $d_{0,x}(r_{00}) > 0$). Let us consider d_0^ϵ defined by (4.5). If $0 < \epsilon_1 < \epsilon_2$, then*

$$d^{\epsilon_1}(t,x) \leq d^{\epsilon_2}(t,x), \quad L_0^{\epsilon_1}(t) \leq L_0^{\epsilon_2}(t).$$

Before commencing the proof we will introduce a convenient notation,

$$\bar{d}^\epsilon(t,x) = \begin{cases} L_0^\epsilon(t) & \text{if } |x| < r_0^\epsilon(t), \\ d^\epsilon(t,x) & \text{if } |x| \geq r_0^\epsilon(t). \end{cases} \tag{4.6}$$

Thus, the Lemma states that

$$\bar{d}^{\epsilon_1}(t,x) \leq \bar{d}^{\epsilon_2}(t,x). \tag{4.7}$$

Proof. By Proposition 4.1 for all $\epsilon > 0$ there exists a unique solution to system (4.1). For the sake of simplicity, we will use the following shorthands: $d^i := d^{\epsilon_i}$, $L_0^i := L_0^{\epsilon_i}$, $r_0^i := r_0^{\epsilon_i}$. We will divide the domain $(0, T_1) \times \mathbb{R}$ into three open sets, G_I, G_{II} and G_{III}, (see Fig. 2 below),

$$G_I = \left\{(t,x): \ |x| \geq \max\{r_0^1(t), r_0^2(t)\}\right\}, \quad G_{III} = \left\{(t,x): \ |x| \leq \min\{r_0^1(t), r_0^2(t)\}\right\},$$

$$G_{II} = (0, T_1) \times \mathbb{R} \setminus (G_I \cup G_{III}). \tag{4.8}$$

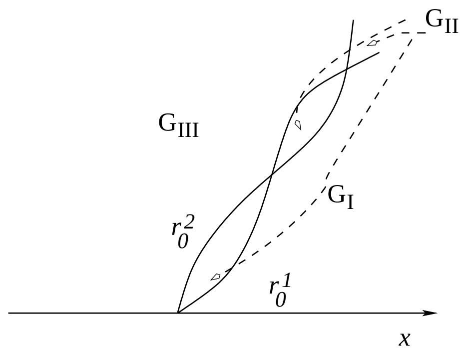

Fig. 2

Let us define a set S by the formula,

$$S = \{s \in [0, T_1) : (4.7) \text{ holds for all } t \leq s\}.$$

Due to the definition of the regularization of d_0 zero belongs to S, hence this set is not empty.

The set S is closed. Indeed, if s_n converges to s, then for all $t < s$ inequality (4.7) holds. Hence, due to continuity of $\bar{d}^i$, $i = 1, 2$ it does hold for $t = s$ as well.

Now, we claim that S is open. In order to prove this we take $s \in S$, we have to show that there exists $\delta > 0$ such that $(s - \delta, s + \delta) \subset S$. It will suffice to prove that $s + \delta \in S$ for some δ.

If $s = T_1$, then there is nothing to prove. Before studying the general situation we make the following observation. It is true that

$$d^{\epsilon_1}(t,x) \leq d^{\epsilon_2}(t,x) \quad \text{for all } (t,x) \in G_I. \tag{4.9}$$

Indeed, d^{ϵ_1}, d^{ϵ_2} satisfy equation (3.1). The classical comparison principle applies, see [I], hence the claim follows.

Due to this observation, if $s \in S$ is such that for all x we have $\bar{d}^{\epsilon_1}(s,x) < \bar{d}^{\epsilon_2}(s,x)$, then automatically there exists $\delta > 0$ such that $s + \delta \in S$. Indeed, in G_{II} and G_{III} we use continuity of $\bar{d}^i$, $i = 1, 2$, while in G_I we use the observation above, (4.9).

It remains to consider the case when there is x_0 such that $\bar{d}^{\epsilon_1}(s, x_0) = \bar{d}^{\epsilon_2}(s, x_0)$. Due to the presence of flat regions we have the following possibilities (cf. Fig. 2): (a) $\bar{d}^1$ and $\bar{d}^2$ touch along a facet, *i.e.*, $\bar{d}^1(s,x) = \bar{d}^2(s,x)$ for $x \leq r_0^2(s)$, and the facet of $\bar{d}^1$ is longer, *i.e.*, $r_0^2(s) < r_0^1(s)$; (b) same as (a) but $r_0^2(s) = r_0^1(s)$; (c) $\bar{d}^1$ and $\bar{d}^2$ touch only in the interior of G_I; (d) $r_0^1(s) < r_0^2(s)$ and $\bar{d}^1(s, r_0^2(s)) = \bar{d}^2(s, r_0^2(s))$. We should keep in mind that some combinations of the above case may occur simultaneously.

In fact, we have already noticed that (c) follows from (4.9).

Now, we will treat case (a). We notice that the mapping

$$r_0 \mapsto \Psi(r_0) := \fint_0^{r_0} \sigma(t, y, L_0)\, dy + \frac{\gamma_\Lambda}{r_0} \tag{4.10}$$

has a negative derivative as long as $\dot{L}_0 > \sigma(t, r_0, L_0)$. Thus, due to $L_0^2(s) = L_0^1(s)$ and $r_0^2(s) < r_0^1(s)$ we conclude that $\dot{L}_0^2(s) > \dot{L}_0^1(s)$. As a result, $L_0^2(t) > L_0^1(t)$ for $t \in (s, s+\eta)$, for a positive η.

We turn our attention to (b). We first notice that for all $s < t < s + t_1$, where t_1 is sufficiently small we have that $r_0^1(t) < r_0^2(t)$. Indeed, formula (2.12) for $\dot{r}_0$,

$$\dot{r}_0(s) = \frac{\dot{L}_0(s) - d_t(s, r_0(s))}{d_x^+(s, r_0(s))} = \frac{\dot{L}_0(s) - \sigma(s, L_0(s), r_0(s)) m(d_x^+(s, r_0(s)))}{d_x^+(s, r_0(s))}$$

and $L_0^1(0) = L_0^2(0)$ combined with $(d^{\epsilon_1})_x^+(s, r_0(s))) < (d^{\epsilon_2})_x^+(s, r_0(s)))$, where $r_0(s) = r_0^1(s) = r_0^2(s)$ yield

$$\dot{r}_0^1(s) > \dot{r}_0^2(s).$$

We note that it is permitted to take one side derivatives of Lipschitz continuous viscosity solutions, which are monotone in x.

By continuity of $\dot{r}_0(\cdot)$ the inequality above holds on an interval, so that $r_0^1(t) < r_0^2(t)$ holds for $t \in [s, s + t_1)$, where t_1 is maximal with this property.

Now, we are in a position to compare $L_0^1(t)$ and $L_0^2(t)$ for $t \in [s, s + t_1)$. We note that

$$\begin{aligned} \ddot{L}_0(t) &= -\frac{\dot{r}_0(t)}{r_0(t)}(\dot{L}_0(t) - \sigma(t, r_0(t), L_0(t))) \\ &\quad + \frac{1}{r_0(t)} \int_0^{r_0(t)} (\sigma_t(t, y, L_0(t)) + \dot{L}_0(t)\sigma_{x_1}(t, y, L_0(t))\, dy. \end{aligned}$$

Since $\dot{L}_0(0) > \sigma(0, r_{00}, L_{00})$ we conclude that

$$\ddot{L}_0^2(0) > \ddot{L}_0^1(0).$$

This implies that

$$L_0^2(t) > L_0^1(t) \qquad \text{for } t < t_*,$$

where $t_* > s$.

We now turn our attention to (d). In this case

$$r_0^1(s) < r_0^2(s) \tag{4.11}$$

and for an $x_0 \in [r_0^1(s), r_0^2(s)]$ we have $\bar{d}^1(s, x_0) = \bar{d}^2(s, x_0)$. Let us suppose that $x_0 = r_0^1(s)$, then due to monotonicity of d^1 we conclude that $\bar{d}^1(s, x) = \bar{d}^2(s, x)$ for all $x \in [r_0^1(s), r_0^2(s)]$. Furthermore, $L_0^2(s) = L^1(s)$, hence $r_0^1(s) \geq r_0^2(s)$ contrary to (4.11).

As a result x_0 must be bigger than $r_0^1(s)$. Let us suppose further that $x_0 < r_0^2(s)$. Since the interfacial curves r_0^i, $i = 1, 2$ are the matching curves, then

$$\dot{L}_0^2 > \sigma(s, r_0^2(s), L_0^2(s)) m(0).$$

Due to Berg's effect we know that

$$\sigma(s, r_0^2(s), L_0^2(s)) > \sigma(s, x, d^1(s, x))$$

for $x \in [x_0, r_0^2(s)]$. Once we recall that $d_x^1(s,x) = 0$ on this set, we conclude that

$$\dot{L}_0^2 > d_t^1(s,x) \quad \text{for } x \in [x_0, r_0^2(s)].$$

Thus, there is a positive number η such that $L_0^2(t) > d_t^1(t,x)$ for $(t,x) \in G_{II}$ and $t \in (s, s+\eta)$.

The final sub-case is $x_0 = r_0^2(s)$, but in fact this is a special case of (c) which has been already treated.

Summarizing, we have thus proved that set S is open and closed, hence it equals $[0, T_1)$. $\square$

In principle there is more than one solution to (4.1). We have to choose one, *i.e.*, we need a tool to select r_0. Let us suppose that $\bar{d}_0^\epsilon$ is the canonical regularization, but in fact it may be any admissible one (see the definition below). Then, for a fixed (t,x) the family $\bar{d}^\epsilon(t,x)$ is decreasing, hence converging,

$$\lim_{\epsilon \to 0^+} \bar{d}^\epsilon(t,x) = \bar{d}(t,x). \tag{4.12}$$

Of course, the pointwise convergence is not sufficient.

Proposition 4.2 *The family $\bar{d}^\epsilon$ converges locally uniformly and its limit $\bar{d}$ is a Lipschitz continuous function.*

Proof. By Theorem 3.1 the family $\bar{d}^\epsilon$ has a common bound on the Lipschitz constant, hence for any compact set $\mathcal{C} \subset \mathbb{R}$ by the Arzela-Ascoli Theorem there exists a subsequence converging uniformly to $\bar{d}$ on $\mathcal{C}$. Since the convergence is monotone decreasing the whole family converges locally uniformly to $\bar{d}$. Moreover, the uniform convergence preserves the Lipschitz constant. $\square$

A more important question is whether the limit $\bar{d}$ depends upon the regularization $\bar{d}^\epsilon$. For this reason we restrict our attention to regularizations of initial data, which we call *admissible*. They are such that:
(i) d_0^ϵ is Lipschitz continuous and $\mathrm{Lip}(d_0^\epsilon - d_0) \le M(\epsilon)$, where $M(\epsilon) \to 0$;
(ii) $d_0^\epsilon(x) = L_{00}$ for $|x| \le r_{00}$, $d_0^\epsilon(x) > d_0(x)$ for $|x| > r_{00}$ and there is $\rho_0 > r_{00}$ such that $d_0^\epsilon(x) = d_0^\epsilon(\rho_0)$ for $|x| > r_{00}$;
(iii) $(d_0^\epsilon)_x^+(r_{00}) = \epsilon$;
(iv) for each fixed x the family $d_0^\epsilon(x)$ is decreasing.

We notice that Lemma 4.1 and its proof are valid for any admissible regularization. We shall see that $\bar{d}$ does not depend on the choice of d_0^ϵ. Indeed, we have

Proposition 4.3 *Let us assume that d_0^ϵ is the canonical regularization of data and δ_0^η is any admissible one. Then*

$$\lim_{\epsilon \to 0^+} \bar{d}^\epsilon(t,x) = \bar{d}(t,x) = \lim_{\eta \to 0^+} \bar{\delta}^\eta(t,x),$$

where $\bar{\delta}^\eta$ is defined as in (4.6).

Proof. Let us set $\epsilon(\eta) := \max\{M(\eta), \eta\}$. We notice that $d_0^{\epsilon(\eta)} \ge \delta_0^\eta$. Indeed, by definition $|\delta_0^\eta(x) - d_0|/|x - r_{00}| \le M(\eta)$ for all $x \in [r_{00}, +\infty)$. On the other hand, see (4.5), $|d_0^{\epsilon(\eta)}(x) - d_0|/|x - r_{00}| = |\eta^\epsilon(x) - \eta^\epsilon(r_{00})|/|x - r_{00}| \le \epsilon(\eta)$ for all $x \in [r_{00}, +\infty)$. Thus, due to the definition of $\epsilon(\eta)$ and part (a) of the defintion of an admissible regularization we conclude that $\bar\delta^\eta(t,x) \le \bar d^{\epsilon(\eta)}(t,x)$, hence

$$\bar d(t,x) \ge \lim_{\eta\to 0^+} \bar\delta^\eta(t,x).$$

It is now sufficient to prove that the converse inequality. Let us look at

$$\min_{x\in[r_{00},\rho_0]} \frac{|\delta_0^\eta(x) - d_0(r_{00})|}{|x-r_{00}|} =: \mu(\eta).$$

This number is positive and the minimum is attained at $x_0 \in (r_{00}, \rho_0)$. By the definition we have $\delta_0^\eta(x) > d_0^{\mu(\eta)}(x)$ for all $x_0 \in (r_{00}, \infty)$. Thus, $\delta^\eta(t,x) > d^{\mu(\eta)}(t,x)$. In this way we conclude that

$$d(t,x) \le \lim_{\eta\to 0^+} \bar\delta^\eta(t,x).$$

Our claim follows. $\square$

Once we defined a unique$\bar d$ we shall identify r_0. Let us notice that if $d^\epsilon_{0,x} = 0$ only on $[-r_{00}, r_{00}]$, then due to Theorem 3.1, for all $t>0$ the derivative $d^\epsilon_x(t,\cdot)$ vanishes only on an interval containing zero, more precisely, $d^\epsilon_x(t,x) = 0$ iff $x \in [-r_0^\epsilon(t), r_0^\epsilon(t)]$. Thus, we set

$$r_0(t) = \sup\{x : \ \bar d_x(t,x) = 0\}. \tag{4.13}$$

We are now able to show the desired result.

Proposition 4.4 *The family of functions $r_0^\epsilon(\cdot)$ converges uniformly to $r_0(\cdot)$, and $r_0(\cdot)$ is Lipschitz continuous and increasing. Moreover, $r_0(\cdot)$ is a solution to (4.1).*

Proof. We will first show the pointwise convergence. We fix $t>0$. We recall that due to Proposition 4.2 $\bar d^\epsilon$ converges locally uniformly to $\bar d$. This implies that $d^\epsilon(t, r_0^\epsilon(t)) = \bar d^\epsilon(t,0) = L_0^\epsilon(t)$, goes to $d(t, r_0(t)) = \bar d(t,0) = L_0(t)$, where the last two equalities follow from (4.13). Let us now look at the difference

$$D = d(t, r_0(t)) - d(t, r_0^\epsilon(t)).$$

On one side, due to strict monotonicity of d guaranteed by Theorem 3.1, we have

$$|D| \ge \eta(t)|r_0(t) - r_0^\epsilon(t)|.$$

On the other hand we can see that

$$\begin{aligned} |D| &\le |d(t,r_0(t)) - d^\epsilon(t, r_0^\epsilon(t))| + |d^\epsilon(t, r_0^\epsilon(t)) - d(t, r_0^\epsilon(t))| \\ &\le |L_0^\epsilon(t) - L_0(t)| + \sup_x |d^\epsilon(t,x) - d(t,x)| \\ &\le 2\|\bar d^\epsilon - \bar d\|_{C([0,T_1]\times\mathbb{R})}. \end{aligned}$$

After combining these two estimates we obtain

$$\eta(t)|r_0(t) - r_0^\epsilon(t)| \le 2\|d^\epsilon - d\|_{C([0,T_1]\times\mathbb{R})},$$

which implies the pointwise convergence of r_0^ϵ. However, due to $\eta(0)=0$ we cannot deduce the uniform convergence at this stage.

We cannot claim that the family r_0^ϵ, $\epsilon > 0$ has a common bound on the Lipschitz constant, because this is not true. However, by Lemma 4.2 the family r_0^ϵ is equicontinuous, it is also uniformly bounded, hence by Arzela-Ascoli Theorem we can select a uniformly convergent subsequence

$$r_0^{\epsilon_k} \to r_0^0 \qquad \text{in } C[0, T_1].$$

But the limit r_0^0 must be equal to r_0 and it satisfies the Lipschitz condition with same constant. We now see that the whole family r_0^ϵ must converge uniformly to r_0.

Now, we shall show that r_0 is a solution to (4.1). Since r_0^ϵ and L_0^ϵ converge uniformly, it follows from $(4.1)_1$ that $\dot{L}_0^\epsilon$ converges uniformly too. Thus, we may let ϵ to zero and conclude that (4.1) holds after the passage to the limit, because we have already seen that $d(t, r_0(t)) = L_0(t)$. Our claim follows. $\square$

We chose an interfacial curve among possibly many. The selection is natural, thus we shall call its result as a *proper* matching curve $r_0(\cdot)$. Subsequently we shall deal only with the proper interfacial curves even without mentioning this explicitly.

We close this section with a technical result.

Lemma 4.2 *The family of functions r_0^ϵ, $\epsilon > 0$, is equicontinuous.*

Proof. *Step i), preparations.* In order to show our claim we have to recall from the proof of Proposition 4.1 that the couple L_0^ϵ, r_0^ϵ is a solution to equation (4.1), but this time we may not use the fact that $\mathcal{L}$ is a contraction. Furthermore, due to (4.3) we have

$$r_0^\epsilon(t) = (d^\epsilon(t, \cdot))^{-1}(L_0^\epsilon(t)) =: f_\epsilon(t, L_0^\epsilon(t)),$$

where function f_ϵ is introduced for the sake of easy notation.

We use the method employed in the course of proof of Proposition 4.1. In particular we take d, a viscosity solution to (3.2) with initial data $\bar{d}_0$ coinciding with d_0 for $|x| \geq r_{00}$. We notice that $d^\epsilon(t, x) \geq d(t, x)$, see Lemma 4.1. This implies that

$$(d(t, \cdot))^{-1}(y) \geq (d^\epsilon(t, y))^{-1}.$$

We will denote by ω the modulus of continuity of the function $y \mapsto (d(t, \cdot))^{-1}(y)$. Thus, if $\delta > 0$ is given, then there is $\tau > 0$ such that for all t, $y - L_{00}$ smaller than τ we have

$$|(d(t, \cdot))^{-1}(y) - r_{00}| \leq \omega(2\tau) \leq \frac{\delta}{2}. \tag{4.14}$$

Step ii) We have to show that for a given $\delta > 0$ there exists $\rho > 0$ independent of ϵ, such that

$$\text{if } |t - s| < \rho, \quad \text{then } |r_0^\epsilon(t) - r_0^\epsilon(s)| < \delta.$$

We first take $s < t < \tau$, where τ is defined by (4.14). We notice due to $r_0^\epsilon(s) \geq r_{00}$

$$r_0^\epsilon(t) - r_0^\epsilon(s) \leq f_\epsilon(t, L_0^\epsilon(t)) - r_{00} \leq \omega(\tau) \leq \frac{\delta}{2}.$$

Now, we take $s < \tau \leq t$, then we calculate using the above estimate

$$r_0^\epsilon(t) - r_0^\epsilon(s) \leq r_0^\epsilon(t) - r_0^\epsilon(\tau) + \omega(\tau).$$

We shall look closer at the first term on the RHS,

$$\begin{aligned} r_0^\epsilon(t) - r_0^\epsilon(\tau) &\leq |f_\epsilon(t, L^\epsilon(t)) - f_\epsilon(t, L^\epsilon(\tau))| + |f_\epsilon(t, L^\epsilon(\tau)) - f_\epsilon(\tau, L^\epsilon(\tau))| \\ &\leq I_1 + I_2. \end{aligned}$$

Now, t is bigger or equal to τ, which is defined independently of ϵ, by Theorem 3.1 (c) we have

$$d^\epsilon(t,y) - d^\epsilon(t,x) \geq \eta(t)(y-x), \tag{4.15}$$

where

$$\eta(t) = t \min_{u\in[\tau,T_1]} \sigma_{x_2}(t, r_{00}/2, d_0(r_{00}))m(0) > 0$$

for $t \geq \tau$. This implies that

$$\begin{aligned} I_1 &\leq \frac{1}{\eta(\tau)}|L^\epsilon(t) - L^\epsilon(\tau)| \\ &\leq \frac{\max_{[0,T_1]} \dot{L}_0^\epsilon}{\eta(\tau)}(t-\tau) \leq K_1(t-s), \end{aligned} \tag{4.16}$$

where

$$K_1 = \frac{\max_{[0,T_1]} \dot{L}_0^\epsilon}{\eta(\tau)}$$

is independent of ϵ due to (4.1).

In order to estimate I_2 we make the following observation,

$$0 = y - y = d^\epsilon(\tau, f_\epsilon(\tau,y)) - d^\epsilon(t, f_\epsilon(t,y)) + d^\epsilon(t, f_\epsilon(\tau,y)) - d^\epsilon(t, f_\epsilon(\tau,y)). \tag{4.17}$$

Since $t \geq \tau$, we notice that $f_\epsilon(\tau,y) > f_\epsilon(t,y)$. Furthermore, by (4.17) and (4.15) we can see that

$$d^\epsilon(t, f_\epsilon(\tau,y)) - d^\epsilon(\tau, f_\epsilon(\tau,y)) = d^\epsilon(\tau, f_\epsilon(\tau,y)) - d^\epsilon(t, f_\epsilon(t,y)) \geq \eta(\tau)(f_\epsilon(\tau,y) - f_\epsilon(t,y)).$$

The LHS of the above inequality may be estimated due to Theorem 3.1 (a) as follows,

$$d^\epsilon(t, f_\epsilon(\tau,y)) - d^\epsilon(\tau, f_\epsilon(\tau,y)) \leq M(t-\tau).$$

Combing these estimates yields a bound on I_2,

$$I_2 \leq \frac{M}{\eta(\tau)}(t-\tau) \leq \frac{M}{\eta(\tau)}(t-s).$$

Finally, we come to the inequality

$$r_0^\epsilon(t) - r_0^\epsilon(s) \leq \left(\frac{M}{\eta(\tau)} + K_1\right)(t-s),$$

thus ρ must be smaller than $\delta/(\frac{M}{\eta(\tau)} + K_1)$.

In fact, the remaining case $t, s \geq \tau$ has been already considered in the course of the above analysis. We conclude that indeed, the family is equicontinuous. $\square$

4.3 A Comparison Principle

Once we selected a proper matching curve as a result of a limiting process we wish to apply the same approach to construction of L, d a solution to (1.4_1)–(1.4_2). It is convenient to introduce a unified framework. For this purpose we will use the theory of viscosity solutions to Hamilton-Jacobi equations with discontinuous Hamiltonians.

We will use a natural modification of the standard definition of the (sub-, super-) solution u. This modification is required due to the discontinuity of the Hamiltonian. But first we introduce the Hamiltonian itself.

Let us assume that r_0^ϵ is a tangency curve yielded by Theorem 4.1 (a) for d_0^ϵ which is an admissible regularization of d_0. We introduce the notation, for all $\epsilon \geq 0$ we set

$$F^\epsilon = \{(t,x) \in (0,T_1) \times \mathbb{R} : |x| \leq r_0^\epsilon(t)\} \tag{4.18}$$

In order to define the Hamiltonian we will need another observation. Namely, let us suppose that

$$\frac{1}{\beta}_R \dot{L}_0(t) > \sigma(t, r_0(t), L_0(t)) \tag{4.19}$$

The mapping $r_0 \mapsto ⨍_0^{r_0} \sigma(t,y,L_0)\,dy + \frac{\gamma_\Lambda}{r_0}$ is decreasing. Thus, for a given $r_0(t)$ there is a unique $r^*(t)$ such that $\frac{1}{\beta}_R \dot{L}_0(t) = \sigma(t, r^*(t), L_0(t))$, moreover since σ is increasing in the second and third variable we deduce that $r^*(t) > r_0(t)$.

Finally, we set

$$\bar{H}^\epsilon(t,x,d,p) = -\sigma\left(t, (r_0^\epsilon)^*(t)\chi_{F^\epsilon} + x(1-\chi_{F^\epsilon}), d\right) m(p). \tag{4.20}$$

Here, $\epsilon \geq 0$ with the understanding that if $\epsilon = 0$, then we take r_0 the proper interfacial curve in place of r_0^0 in $\bar{H}^0$ and in F^0. We emphasize that our Hamiltonian is not only discontinuous but also nonlocal due to the definition of r^*.

Remark. Let us notice that the Hamiltonian $\bar{H}^\epsilon$, $\epsilon \geq 0$, is in fact Lipschitz continuous provided that r^* coincides with a tangency curve r_0.

Let us now suppose that $\bar{H}$ equals $\bar{H}^\epsilon$ for some $\epsilon \geq 0$. We need some preparations prior to formulation the definition of viscosity solution to

$$\bar{d}_t + \bar{H}(t,x,\bar{d},\bar{d}_x) = 0, \quad \bar{d}(0,x) = \bar{d}_0(x). \tag{4.21}$$

We recall that for a locally bounded function $u : (0,T_1) \times \mathbb{R}^n \to \mathbb{R}$ we set

$$u_*(x) = \liminf_{y \to x} u(y), \qquad u^*(x) = \limsup_{y \to x} u(y).$$

The function u_* (resp. u^*) is lower (resp. upper) semicontinuous.

Definition 4.1 (a) We shall say that a bounded, uniformly continuous function $u : (0,T_1) \times \mathbb{R} \to \mathbb{R}$ is a *viscosity subsolution* of (4.21) provided that for all C^1 functions $\varphi : (0,T_1) \times \mathbb{R} \to \mathbb{R}$ such that $u - \varphi$ has a local maximum at (t_0, x_0), then

$$\varphi_t(t_0,x_0) + (\bar{H})_*(t_0,x_0,u(t_0,x_0),\varphi_x(t_0,x_0)) \leq 0.$$

(b) We shall say that a bounded, uniformly continuous function $v : (0,T_1) \times \mathbb{R} \to \mathbb{R}$ is a *viscosity supersolution* of (4.21) if it for all C^1 functions $\varphi : (0,T_1) \times \mathbb{R} \to \mathbb{R}$ such that $v - \varphi$ has a local minimum at (t_0,x_0), then

$$\varphi_t(t_0,x_0) + (\bar{H})^*(t_0,x_0,v(t_0,x_0),\varphi_x(t_0,x_0)) \geq 0.$$

(c) We shall say that a bounded, uniformly continuous function $d : (0,T_1) \times \mathbb{R} \to \mathbb{R}$ is a *viscosity solution* of (4.21) provided that it is a viscosity subsolution as well as a viscosity supersolution of (4.21).

We notice that this definition is in the line of notion of sub-(super-)solution introduced by [BP], [I] and more recently by [CR] for discontinuous Hamiltonians.

We state now the basic observation.

Proposition 4.5 *Let us suppose that $\bar{H} = \bar{H}^\epsilon$, where ϵ is positive, the function $\bar{d} = \bar{d}^\epsilon$ is defined by (4.6). Then, $\bar{d}$ is a viscosity solution to (4.21).*

Proof. We first check that $\bar{d}$ is a subsolution of (4.21). We notice that $(\bar{H})_* = \bar{H}$ in $F^c = (0,T) \times \mathbb{R} \setminus F$, where set F is defined in (4.18).

Let us take a test function φ such that the difference $\bar{d} - \varphi$ attains a local maximum at (t_0, x_0). We have three cases to consider: (i) $x_0 > r_0(t_0)$, (ii) $x_0 < r_0(t_0)$, (iii) $x_0 = r_0(t_0)$.

In the first case, we notice that $\bar{d} = d$ and d a viscosity solution to a Hamilton-Jacobi equation with Lipschitz continuous Hamiltonian

$$d_t - \sigma(t,x,d)m(d_x) = 0 \quad \text{in } F^c.$$

In this case our claim follows from the fact that d is a regular viscosity subsolution, even a viscosity solution.

If $x_0 < r_0(t_0)$, then $\bar{d}(t_0, x_0) = L_0(t_0)$ on $(-r_0(t_0), r_0(t_0))$ and (t_0, x_0) is a point of differentiability of $\bar{d}$. Hence, $\varphi_x(t_0, x_0) = 0$ and $\varphi_t(t_0, x_0) = \dot{L}_0(t_0)$. Moreover, since $(\bar{H})_* = \bar{H}$, then

$$\varphi_t(t_0, x_0) + (\bar{H})_* \left(t_0, x_0, \bar{d}(t_0, x_0), \varphi_x(t_0, x_0)\right) = \dot{L}_0(t_0) - \sigma(t_0, x_0, L_0(t_0))m(0) = 0.$$

In order to finish checking that $\bar{d}$ is a subsolution we consider (iii). Let us notice that always the left derivative of $\bar{d}^-$ at (t_0, x_0) is zero, *i.e.*, $\bar{d}_x^-(t_0, x_0) = 0$ and $\bar{d}_x^+(t_0, x_0) > 0$. Hence, there is no C^1 function φ such that $\varphi(t_0, x_0) = \bar{d}(t_0, x_0)$ and $\varphi \geq \bar{d}$ in a neighborhood of (t_0, x_0). We conclude that $\bar{d}$ is a viscosity subsolution of (4.21).

We shall check now that $\bar{d}$ is a viscosity supersolution of (4.21). As before we have to consider the three above cases. The first two can be handled as in the case of subsolution. The remaining case (iii) is the most difficult. Let us suppose that φ is a test function such that $\bar{d} - \varphi$ attains a local minimum at $(t_0, r_0(t_0))$. We have already noticed that

$$0 \leq \varphi_x(t_0, r_0(t_0)) \leq \bar{d}_x^+(t_0, r_0(t_0)). \tag{4.22}$$

We have to discover the restrictions on $\varphi_t(t_0, r_0(t_0))$. Due to the matching condition we see that

$$\dot{L}_0(t_0) = \bar{d}_t^-(t_0, r_0(t_0)) + \bar{d}_x^+(t_0, r_0(t_0))\dot{r}_0(t_0).$$

Since $\dot{r}_0(t_0) > 0$ we conclude that $\dot{L}_0(t_0) > \dot{r}_0(t_0)$. Moreover we infer

$$\bar{d}_t^-(t_0, r_0(t_0)) \leq \varphi_t(t_0, r_0(t_0)) \leq \dot{L}_0(t_0).$$

We can see that $(\bar{H})^*(t_0, r_0(t_0), L_0(t_0), p) = -\sigma(t_0, r_0(t_0), L_0(t_0))m(p)$. Thus, we have to check that

$$\varphi_t(t_0, r_0(t_0)) - \sigma(t_0, r_0(t_0), L_0(t_0))m(\varphi_x(t_0, r_0(t_0))) \geq 0.$$

Once we denote its left-hand-side by LHS, then we have

$$LHS \geq \bar{d}_t^-(t_0, r_0(t_0)) - \sigma(t_0, r_0(t_0), L_0(t_0))m(\varphi_x(t_0, r_0(t_0))).$$

In set F^c the characteristics of the smoothed out system do not cross, also after passage to the limit. Thus, $(t_0, r_0(t_0))$ is a differentiability point of d considered as a viscosity of solution of (3.2) over $(0, T_1) \times \mathbb{R}$. As as result,

$$LHS \geq \sigma\,(t_0, r_0(t_0), L_0(t_0))\, m(\bar{d}_x(t_0, r_0(t_0)) - \sigma\,(t_0, r_0(t_0), L_0(t_0))\, m(\varphi_x(t_0, r_0(t_0))).$$

Finally, due to (4.22) and monotonicity of m we conclude that $LHS \geq 0$. In other words, $\bar{d}$ is a viscosity supersolution as well as a viscosity subsolution. Our claim follows. $\square$

Here is one of our main results, a Comparison Principle, which is proved in a restrictive setting.

Theorem 4.3 *(A Comparison Principle) For each $\epsilon \geq 0$ we take $\bar{H} = \bar{H}^\epsilon$, where $\bar{H}^\epsilon$ is defined in (4.20). We assume that u, v are even, Lipschitz continuous viscosity sub- (resp. super-) solution to (4.21). In addition, if $(t, x) \in F$, then $u_x(t, x) = 0$ and v in non-decreasing for $x \geq 0$. If $u(0, x) \leq v(0, x)$, then for all t, x we have*

$$u(t, x) \leq v(t, x).$$

Proof. We know by the classical argument, in the complement of F the classical comparison principle holds, see (4.9) and Fig. 2. Thus $u(t, x) \leq v(t, x)$ in the complement of F. As a result we conclude that

$$u(t, r_0(t)) \leq v(t, r_0(t)).$$

We turn our attention to set F, we first use the fact that a differentiable sub-/supersolution satisfies the equation in a classical sense at that point. Thus, we have to worry separately about the null set of non-differentiability points of u and v.

By assumption, at any common differentiability point of u and v in F we have

$$u_t - \sigma(t, r^*(t), u)/\beta_R \leq 0, \qquad v_t - \sigma(t, r^*(t), v) m(v_x) \geq 0.$$

Subsequently we conclude that

$$(u - v)_t \leq \frac{1}{\beta_R}(\sigma(t, r^*(t), u) - \sigma(t, r^*(t), v)), \tag{4.23}$$

because $m(0) \leq m(p)$.

In order to proceed we will make an observation about the structure of $E \subset (0, T_1) \times (-\lambda, \lambda)$ the set points, where u and v are differentiable. Of course, $\lambda_2(E) = 2\lambda T_1$, on the other hand

$$\lambda_2(E) = \int_{-\lambda}^{\lambda} \lambda_1(E_x)\, dx, \tag{4.24}$$

where $E_x = E \cap \{x\} \times (0, T_1)$ and λ^k stands for the k-dimensional Lebesgue measure. Let us consider $Z = \{x \in (-\lambda, \lambda) : \ \lambda_1(E_x) < T_1\}$. Of course, $\lambda_1(Z) = 0$ for otherwise (4.24) would be violated. As a result for almost all x_0 such that (t_0, x_0) is in E, almost all points of the interval (t_0, x_0), belong to E.

Since $u \mapsto \sigma(t, r^*(t), u)$ is strictly increasing, the above observation permits us to integrate (4.23) over $[0, t]$,

$$u(x, t) - v(x, t) \leq u(0, x) - v(0, x) \leq 0$$

or

$$u(x,t) - v(x,t) \leq u(t, r_0(t)) - v(t, r_0(t)) \leq 0$$

as desired.

We have to deal with the non-differentiability points of u and v, say (t_0, x_0) is one of them . Then in any neighborhood of (t_0, x_0) we can find points of differentiability (t_n, x_n), where the desired inequality is satisfied, $u(x_n, t_n) - v(x_n, t_n) \leq 0$. Due to continuity of u, v and convergence $(x_n, t_n) \to (t_0, x_0)$ we conclude that $u(x_0, t_0) - v(x_0, t_0) \leq 0$ as well. □

An immediate conclusion is that $\bar{d}^\epsilon$, $\epsilon > 0$, are unique viscosity solutions to (4.21) with $\bar{H} = \bar{H}^\epsilon$ in the restricted (Lipschitz continuous) class of functions. Moreover, if $\bar{H} = \bar{H}^0$, then there is at most one solution to (4.21). However, below we shall prove that there is at least one viscosity solution to (4.21) with $\bar{H} = \bar{H}^0$. We have to define convergence of discontinuous Hamiltonians. For this purpose we will use the standard notions (see [G, §2.1.2]) of *upper relaxed limit* and *lower relaxed limit* to study convergence of sequences of discontinuous functions. If u^ϵ, $\epsilon > 0$ is a sequence of locally bounded measurable functions, then we set

$$\limsup_{\epsilon\to 0^+}{}^* u^\epsilon(x) = \limsup_{z\to x, \epsilon\to 0^+} u^\epsilon(z) = \lim_{\delta\to 0^+} \sup\{u^\delta(z) : \ z \in B(x,\epsilon), \ 0 < \delta < \epsilon\},$$

$$\liminf_{\epsilon\to 0^+}{}_* u^\epsilon(x) = \liminf_{z\to x, \epsilon\to 0^+} u^\epsilon(z) = \lim_{\delta\to 0^+} \inf\left\{u^\delta(z) : \ z \in B(x,\epsilon), \ 0 < \delta < \epsilon\right\}.$$

We shall say that a sequence of discontinuous Hamiltonians $\bar{H}^\epsilon$, $\epsilon > 0$, *converges* to $\bar{H}$ provided that:

$$\limsup_{\epsilon\to 0^+}{}^* \bar{H}^\epsilon = \bar{H}^* \quad \text{and} \quad \liminf_{\epsilon\to 0^+}{}_* \bar{H}^\epsilon = \bar{H}_*.$$

This definition allows for some indefiniteness of $\bar{H}$, because we are really interested in the upper and lower envelopes.

Proposition 4.6 *Let us assume that $\bar{d}_0^\epsilon$, $\epsilon > 0$ is a sequence of admissible regularization of initial data d_0 and $\bar{d}^\epsilon$ are the corresponding viscosity solutions to (4.21). Then, $\bar{H}^\epsilon$ converges to $\bar{H}^0$ and the limit $\bar{d}$ of $\bar{d}^\epsilon$ is a unique viscosity solution to (4.21) with $\bar{H} = \bar{H}^0$ and with initial data $\bar{d}_0$, i.e.,*

$$\bar{d}_t + \bar{H}(t, x, \bar{d}, \bar{d}_x) = 0, \quad \bar{d}(0,x) = \bar{d}_0(x). \tag{4.25}$$

Proof. We have already proved the uniform convergence of $r_0^\epsilon(\cdot)$ to $r_0(\cdot)$. It is now easy to see from the definition of the upper/lower relaxed limits that

$$\liminf_{\epsilon\to 0^+}{}_* \bar{H}^\epsilon(t,x,d,p) = (\bar{H}^0)_*(t,x,d,p), \qquad \limsup_{\epsilon\to 0^+}{}_* \bar{H}^\epsilon(t,x,d,p) = (\bar{H}^0)^*(t,x,d,p).$$

That is, the Hamiltonians $\bar{H}^\epsilon$ converge to $\bar{H}^0$.

Now we shall show that the limit $\bar{d}$ of viscosity subsolutions $\bar{d}^\epsilon$ is a viscosity subsolution to (4.25) with $\bar{H}$ equal to $\bar{H}^0$. Let us suppose that φ is a C^1-test function and the difference $\bar{d} - \varphi$ attains its local maximum at (t_0, x_0). We know, that we may assume that this maximum is strict in a ball $B((t_0, x_0), r)$ (see, e.g. [G, Proposition 2.2.2]). By [BP, Lemma A.3] (see also [G, Lemma 2.2.5]) we conclude that if (t_n, x_n) is a sequence of maxima of functions $\bar{d}^{\epsilon} - \varphi$ in $B((t_0, x_0), r)$, then (t_n, x_n) converges to (t_0, x_0) as n tends to infinity. By assumption,

$$\varphi_t(t_n, x_n) + (\bar{H}^{\epsilon})_* \left(t_n, x_n, \bar{d}^{\epsilon}(t_n, x_n), \varphi_x(t_n, x_n)\right) \leq 0.$$

Due to the definition of the lower relaxed limit we have,

$$\begin{aligned}
&\varphi_t(t_0, x_0) + \liminf_{\epsilon \to 0^+}{}_* \bar{H}^0 \left(t_0, x_0, \bar{d}(t_0, x_0), \varphi_x(t_0, x_0)\right) \\
&\leq \liminf_{\epsilon \to 0^+} \left(\varphi_t(t_n, x_n) + (\bar{H}^{\epsilon})_* \left(t_n, x_n, \bar{d}^{\epsilon}(t_n, x_n), \varphi_x(t_n, x_n)\right)\right) \\
&\leq 0.
\end{aligned}$$

This implies that $\bar{d}$ a subsolution to (4.25). A similar argument shows that $\bar{d}$ is also a supersolution to (4.25). Thus, it is a viscosity solution. Once we showed that, an application of the comparison principle, Theorem 4.3 yields uniqueness of $\bar{d}$. $\square$

Having at hand the results of this and previous subsection we may provide the *proof of Theorem 4.2*. Namely, due to Proposition 4.4 we have existence of a unique proper interfacial curve r_0. On the other hand, Proposition 4.6 yields existence of a unique viscosity solution $\bar{d}$, *i.e.*, L_0 and d. $\square$

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