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**HOKKAIDO UNIVERSITY**
A remark on the uniqueness of positive solutions to semilinear elliptic equations with double power nonlinearities

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Abstract
We consider the uniqueness of positive solutions to

\[
\begin{align*}
\Delta u - \omega u + u^p - u^{2p-1} &= 0 \quad \text{in } \mathbb{R}^n, \\
\lim_{|x| \to \infty} u(x) &= 0.
\end{align*}
\]

(1)

It is known that for fixed \(p > 1\), a positive solution to (1) exists if and only if \(\omega \in (0, \omega_p)\), where \(\omega_p := \frac{p}{(p+1)^2}\). We deduce the uniqueness in the case where \(\omega\) is close to \(\omega_p\), from the argument in the classical paper by Peletier and Serrin [9], thereby recovering a part of the uniqueness result of Ouyang and Shi [8] for all \(\omega \in (0, \omega_p)\).

1 Introduction
We shall consider a boundary value problem

\[
\begin{align*}
\frac{u_{rr}}{r} + \frac{n-1}{r} u_r - \omega u + u^p - u^{2p-1} &= 0 \quad \text{for } r > 0, \\
u_r(0) &= 0, \\
\lim_{r \to \infty} u(r) &= 0,
\end{align*}
\]

(2)

where \(n \in \mathbb{N}, p > 1\) and \(\omega > 0\). The above problem arises in the study of

\[
\begin{align*}
\Delta u - \omega u + u^p - u^{2p-1} &= 0 \quad \text{in } \mathbb{R}^n, \\
\lim_{|x| \to \infty} u(x) &= 0.
\end{align*}
\]

(3)

Indeed, the classical work of Gidas, Ni and Nirenberg [4, 5] tells us that any positive solution to (3) is radially symmetric. On the other hand, for a solution \(u(r)\) of (2), \(v(x) := u(|x|)\) is a solution to (3).
The condition to assure the existence of positive solutions to (3) (and so (2)) was given by Berestycki and Lions [1] and Berestycki, Lions and Peletier [2]: A solution to (2) with fixed \( p > 1 \) exists if and only if \( \omega \in (0, \omega_p) \), where

\[
\omega_p = \frac{p}{(p+1)^2}.
\]

We shall review what this \( \omega_p \) is for in Section 2. Throughout this paper, a solution means a classical solution.

Uniqueness of positive solutions to (2) had long remained unknown. Finally in 1998 Ouyang and Shi [8] proved uniqueness for (2) with all \( \omega \in (0, \omega_p) \), \( p > 1 \). See also Kwong and Zhang [6].

In this present paper, we prove that for \( \omega \) close to \( \omega_p \), the uniqueness result is obtained directly from the classical result given by Peletier and Serrin [9] in 1983. For another attempt to obtain the uniqueness when \( \omega \) is close to \( \omega_p \), see Mizumachi [7]. Our result of the present paper is the following:

**Theorem 1.** Let \( n \in \mathbb{N} \), \( p > 1 \) and \( \omega \in [a_p, \omega_p) \), where

\[
a_p = \frac{p(7p - 5)}{4(p + 1)(2p - 1)^2}.
\]

Then (2) has exactly one positive solution.

**Remark 1.** Note that

\[
0 < a_p < \omega_p = \frac{p}{(p+1)^2}, \quad p > 1.
\]

In the next section we clarify the definitions of \( \omega_p \) and \( a_p \) from the point of view from [9].

## 2 Study of the nonlinearity as a function

In this section, we study the properties of the function \( f(u) := -\omega u + u^p - u^{2p-1} \) in \((0, \infty)\), where \( \omega > 0 \) and \( p > 1 \) are given constants.

First we define \( F(u) := \int_0^u f_{\omega,p}(s)ds \), and by a direct calculation we have

\[
F(u) = -\frac{\omega}{2} u^2 + \frac{u^{p+1}}{p+1} - \frac{u^{2p}}{2p} = \frac{u^2}{2p(p+1)} \left[-\omega p(p+1) + 2pu^{p-1} - (p+1)u^{2(p-1)}\right].
\]

There are two cases of concern:

(a) \( \omega < \omega_p \iff F \) has two zeros in \((0, \infty)\).

(b) \( \omega \geq \omega_p \iff F \) has at most one zero in \((0, \infty)\).

The condition to assure the existence of positive solutions of (2) given in [1, 2] is the following:
Lemma 1. The problem (2) has a positive solution if and only if both of the following hypotheses are fulfilled:

\((H1)\) \(\lim_{u \to +0} \frac{f(u)}{u}\) exists and is negative,

\((H2)\) \(F(\delta) > 0\) for some positive constant \(\delta\).

Lemma 2. The problem (2) has a positive solution if and only if

\[ \omega \in (0, \omega_p) \]

for \(p > 1\).

Proof. (H1) is equivalent to the condition \(\omega > 0\). (H2) is equivalent to the condition (a) above.

This is the origin of \(\omega_p\). Next we turn to the exponent \(a_p\).

As a preparation, we calculate the derivatives of \(f(u) = -\omega u + u^p - u^{2p-1}\):

\[
\begin{align*}
f'(u) &= -\omega + pu^{p-1} - (2p-1)u^{2(p-1)}, \\
f''(u) &= 2(p-1)(2p-1)u^{p-2} \left[ \frac{p}{2(2p-1)} - u^{p-1} \right].
\end{align*}
\]

We shall introduce four positive constants \(\alpha, b, c\) and \(\beta\).

- Let \(\alpha\) denote the unique zero of \(f''\) in \((0, \infty)\): \(\alpha = \left[ \frac{p}{2(2p-1)} \right]^{\frac{1}{p-1}}\).

- Let \(b\) denote the first zero of \(f\) in \((0, \infty)\): \(b = \left[ \frac{1 + \sqrt{1 - 4\omega}}{2} \right]^{\frac{1}{p-1}}\).

- Let \(c\) denote the last zero of \(f\) in \((0, \infty)\): \(c = \left[ \frac{1 - \sqrt{1 - 4\omega}}{2} \right]^{\frac{1}{p-1}}\).

- Let \(\beta\) denote the first zero of \(F\) in \((0, \infty)\): \(\beta = \left[ \frac{p}{p+1} \left( 1 - \sqrt{1 - \left( \frac{p+1}{p} \right)^2 \omega} \right) \right]^{\frac{1}{p-1}}\).

It is easy to check that

\[
\beta \in (b, c) \quad (5)
\]

either by observing the graphs or by a straightforward calculation. From (5) we deduce

\[
f(\beta) > 0, \quad (6)
\]

which will be used later.

We are not able to give a clear explanation on the relation between \(\alpha\) and \(\beta\).
Lemma 3. The condition \( \alpha \leq \beta \) is equivalent to \( \omega \geq a_p = \frac{p(7p - 5)}{4(p + 1)(2p - 1)^2} \).

Proof. A simple calculation.

This is where our \( a_p \) comes into play. In the next section, we see what this condition stands for.

3 Proof of Theorem 1.

First we state the result by Peletier and Serrin [9], which assures the uniqueness of solutions of (2).

Lemma 4. Let \( f \) satisfy (H1-3), where (H1), (H2) are in Lemma 1., and (H3) is the following:

\[
\text{(H3)} \
G(u) := \frac{f(u)}{u - \beta} \text{ is nonincreasing in } (\beta, c).
\]

Then (2) has exactly one positive solution.

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. We shall see that for \( \omega \in [a_p, \omega_p) \), (H1-3) are satisfied. It is enough to show that if \( \omega \geq a_p \), then

\[
k(u) := f'(u)(u - \beta) - f(u) \leq 0 \text{ in } (\beta, c).
\]

To prove (7) we calculate the derivative of \( k(u) \)

\[k'(u) = f''(u)(u - \beta),\]

and note that

\[
f''(u) > 0 \text{ in } (0, \alpha); \quad f''(u) < 0 \text{ in } (\alpha, \infty).
\]

So if \( \alpha \leq \beta \) (i.e. \( \omega \geq a_p \), see Lemma 3), then \( k'(u) < 0 \) in \( (\beta, c) \), i.e. \( k \) is decreasing in the interval. Therefore

\[k(u) < k(\beta) = -f(\beta) < 0 \text{ in } (\beta, c),\]

where the last inequality follows by (6).

This proves (7) and completes the proof.

If \( \alpha > \beta \), we need to check that \( k(\alpha) \leq 0 \), i.e.

\[
\alpha - \frac{f(\alpha)}{f''(\alpha)} \leq \beta.
\]

This condition provides an implicit relation between \( \omega \) and \( p \). Besides,
Remark 2. The condition (8) does not cover all $\omega \in (0, \omega_p)$. That is for $\omega$ close to zero, $\alpha - \frac{f(\alpha)}{f'(0)} > \beta$.

Proof. The left hand side of (8) is estimated from below as

$$\alpha - \frac{f(\alpha)}{f'(0)} = \frac{(p-1)\alpha^p(1-2\alpha^{p-1})}{\omega + p\alpha^{p-1} - (2p-1)\alpha^{2(p-1)}} \geq \frac{(p-1)\alpha^p(1-2\alpha^{p-1})}{p\alpha^{p-1} - (2p-1)\alpha^{2(p-1)}} > 0,$$

for all $\omega \in (0, \omega_p)$, whereas the right hand side $\beta$ decreases to zero as $\omega$ decreases to zero. When $\omega$ is close to zero, a very delicate observation is needed. See Ouyang and Shi [8] for details.

References


