A remark on the uniqueness of positive solutions to semilinear elliptic equations with double power nonlinearities

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Abstract

We consider the uniqueness of positive solutions to
\[ \begin{cases} \Delta u - \omega u + u^p - u^{2p-1} = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases} \]

It is known that for fixed \( p > 1 \), a positive solution to (1) exists if and only if \( \omega \in (0, \omega_p) \), where \( \omega_p := \frac{p}{(p+1)^2} \). We deduce the uniqueness in the case where \( \omega \) is close to \( \omega_p \), from the argument in the classical paper by Peletier and Serrin [9], thereby recovering a part of the uniqueness result of Ouyang and Shi [8] for all \( \omega \in (0, \omega_p) \).

1 Introduction

We shall consider a boundary value problem
\[ \begin{cases} u_{rr} + \frac{n-1}{r} u_r - \omega u + u^p - u^{2p-1} = 0 & \text{for } r > 0, \\ u_r(0) = 0, \\ \lim_{r \to \infty} u(r) = 0, \end{cases} \]

where \( n \in \mathbb{N} \), \( p > 1 \) and \( \omega > 0 \). The above problem arises in the study of
\[ \begin{cases} \Delta u - \omega u + u^p - u^{2p-1} = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases} \]

Indeed, the classical work of Gidas, Ni and Nirenberg [4, 5] tells us that any positive solution to (3) is radially symmetric. On the other hand, for a solution \( u(r) \) of (2), \( v(x) := u(|x|) \) is a solution to (3).
The condition to assure the existence of positive solutions to (3) (and so (2)) was given by Berestycki and Lions [1] and Berestycki, Lions and Peletier [2]: A solution to (2) with fixed \( p > 1 \) exists if and only if \( \omega \in (0, \omega_p) \), where

\[
\omega_p = \frac{p}{(p + 1)^2}.
\]

We shall review what this \( \omega_p \) is for in Section 2. Throughout this paper, a solution means a classical solution.

Uniqueness of positive solutions to (2) had long remained unknown. Finally in 1998 Ouyang and Shi [8] proved uniqueness for (2) with all \( \omega \in (0, \omega_p) \), \( p > 1 \). See also Kwong and Zhang [6].

In this present paper, we prove that for \( \omega \) close to \( \omega_p \), the uniqueness result is obtained directly from the classical result given by Peletier and Serrin [9] in 1983. For another attempt to obtain the uniqueness when \( \omega \) is close to \( \omega_p \), see Mizumachi [7]. Our result of the present paper is the following:

**Theorem 1.** Let \( n \in \mathbb{N}, p > 1 \) and \( \omega \in [a_p, \omega_p) \), where \( a_p := \frac{p(7p - 5)}{4(p + 1)(2p - 1)^2} \). Then (2) has exactly one positive solution.

**Remark 1.** Note that

\[
0 < a_p < \omega_p = \frac{p}{(p + 1)^2}, \quad p > 1.
\]

In the next section we clarify the definitions of \( \omega_p \) and \( a_p \) from the point of view from [9].

## 2 Study of the nonlinearity as a function

In this section, we study the properties of the function \( f(u) := -\omega u + u^p - u^{2p-1} \) in \((0, \infty)\), where \( \omega > 0 \) and \( p > 1 \) are given constants.

First we define \( F(u) := \int_0^u f_{\omega,p}(s) \, ds \), and by a direct calculation we have

\[
F(u) = -\frac{\omega}{2} u^2 + \frac{u^{p+1}}{p + 1} - \frac{u^{2p}}{2p} = \frac{u^2}{2p(p+1)} \left[ -\omega p(p + 1) + 2pu^{p-1} - (p + 1)u^{2(p-1)} \right].
\]

There are two cases of concern:

(a) \( \omega < \omega_p \iff F \) has two zeros in \((0, \infty)\).

(b) \( \omega \geq \omega_p \iff F \) has at most one zero in \((0, \infty)\).

The condition to assure the existence of positive solutions of (2) given in [1, 2] is the following:
Lemma 1. The problem (2) has a positive solution if and only if both of the following hypotheses are fulfilled:

(H1) \( \lim_{u \to +0} \frac{f(u)}{u} \) exists and is negative,

(H2) \( F(\delta) > 0 \) for some positive constant \( \delta \).

Lemma 2. The problem (2) has a positive solution if and only if

\[ \omega \in (0, \omega_p) \]

for \( p > 1 \).

Proof. (H1) is equivalent to the condition \( \omega > 0 \). (H2) is equivalent to the condition (a) above.

This is the origin of \( \omega_p \). Next we turn to the exponent \( a_p \). As a preparation, we calculate the derivatives of \( f(u) = -\omega u + u^p - u^{2p-1} \):

\[
\begin{align*}
 f'(u) &= -\omega + pu^{p-1} - (2p-1)u^{2(p-1)}, \\
 f''(u) &= 2(p-1)(2p-1)u^{p-2} \left[ \frac{p}{2(2p-1)} - u^{p-1} \right].
\end{align*}
\]

We shall introduce four positive constants \( \alpha, b, c \) and \( \beta \).

- Let \( \alpha \) denote the unique zero of \( f'' \) in \((0, \infty)\): \( \alpha = \left[ \frac{p}{2(2p-1)} \right]^{\frac{1}{p-1}} \).
- Let \( b \) denote the first zero of \( f \) in \((0, \infty)\): \( b = \left[ \frac{1 - \sqrt{1 - 4\omega}}{2} \right]^{\frac{1}{p-1}} \).
- Let \( c \) denote the last zero of \( f \) in \((0, \infty)\): \( c = \left[ \frac{1 + \sqrt{1 - 4\omega}}{2} \right]^{\frac{1}{p-1}} \).
- Let \( \beta \) denote the first zero of \( F \) in \((0, \infty)\): \( \beta = \left[ \frac{p}{p+1} \left( 1 - \sqrt{1 - \frac{(p+1)^2}{p^2} \omega} \right) \right]^{\frac{1}{p-1}} \).

It is easy to check that

\[ \beta \in (b, c) \] (5)

either by observing the graphs or by a straightforward calculation. From (5) we deduce

\[ f(\beta) > 0, \] (6)

which will be used later.

We are not able to give a clear explanation on the relation between \( \alpha \) and \( \beta \).
Lemma 3. The condition $\alpha \leq \beta$ is equivalent to $\omega \geq a_p = \frac{p(7p - 5)}{4(p + 1)(2p - 1)^2}$.

Proof. A simple calculation.

This is where our $a_p$ comes into play. In the next section, we see what this condition stands for.

3 Proof of Theorem 1.

First we state the result by Peletier and Serrin [9], which assures the uniqueness of solutions of (2).

Lemma 4. Let $f$ satisfy (H1-3), where (H1), (H2) are in Lemma 1., and (H3) is the following:

\[(H3) \quad G(u) := \frac{f(u)}{u - \beta} \text{ is nonincreasing in } (\beta, \varsigma).
\]

Then (2) has exactly one positive solution.

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. We shall see that for $\omega \in [a_p, \omega_p)$, (H1-3) are satisfied. It is enough to show that if $\omega \geq a_p$, then

\[k(u) := f'(u)(u - \beta) - f(u) \leq 0 \quad \text{in } (\beta, \varsigma). \tag{7}\]

To prove (7) we calculate the derivative of $k(u)$

\[k'(u) = f''(u)(u - \beta),\]

and note that

\[
\begin{align*}
    f''(u) > 0 & \quad \text{in } (0, \alpha); \\
    f''(u) < 0 & \quad \text{in } (\alpha, \infty).
\end{align*}
\]

So if $\alpha \leq \beta$ (i.e. $\omega \geq a_p$, see Lemma 3), then $k'(u) < 0$ in $(\beta, \varsigma)$, i.e. $k$ is decreasing in the interval. Therefore

\[k(u) < k(\beta) = -f(\beta) < 0 \quad \text{in } (\beta, \varsigma),\]

where the last inequality follows by (6).

This proves (7) and completes the proof.

If $\alpha > \beta$, we need to check that $k(\alpha) \leq 0$, i.e.

\[\alpha - \frac{f(\alpha)}{f'(\alpha)} \leq \beta. \tag{8}\]

This condition provides an implicit relation between $\omega$ and $p$. Besides,
Remark 2. The condition (8) does not cover all $\omega \in (0, \omega_p)$. That is for $\omega$ close to zero, $\alpha - \frac{f(\alpha)}{f'(\alpha)} > \beta$.

Proof. The left hand side of (8) is estimated from below as

\[
\alpha - \frac{f(\alpha)}{f'(\alpha)} = \frac{(p-1)\alpha^p(1 - 2\alpha^{p-1})}{-\omega + p\alpha^{p-1} - (2p-1)\alpha^{2(p-1)} - (p-1)\alpha^p(1 - 2\alpha^{p-1})} > 0,
\]

for all $\omega \in (0, \omega_p)$, whereas the right hand side $\beta$ decreases to zero as $\omega$ decreases to zero.

When $\omega$ is close to zero, a very delicate observation is needed. See Ouyang and Shi \cite{8} for details.

References


5