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A remark on the uniqueness of positive solutions to semilinear elliptic equations with double power nonlinearities

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Abstract

We consider the uniqueness of positive solutions to

$$\begin{cases} \Delta u - \omega u + u^p - u^{2p-1} = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} u(x) = 0. \end{cases} \quad (1)$$

It is known that for fixed $p > 1$, a positive solution to (1) exists if and only if $\omega \in (0, \omega_p)$, where $\omega_p := \frac{p}{(p+1)^2}$. We deduce the uniqueness in the case where ω is close to ω_p , from the argument in the classical paper by Peletier and Serrin [9], thereby recovering a part of the uniqueness result of Ouyang and Shi [8] for all $\omega \in (0, \omega_p)$.

1 Introduction

We shall consider a boundary value problem

$$\begin{cases} u_{rr} + \frac{n-1}{r}u_r - \omega u + u^p - u^{2p-1} = 0 & \text{for } r > 0, \\ u_r(0) = 0, \\ \lim_{r \rightarrow \infty} u(r) = 0, \end{cases} \quad (2)$$

where $n \in \mathbb{N}$, $p > 1$ and $\omega > 0$. The above problem arises in the study of

$$\begin{cases} \Delta u - \omega u + u^p - u^{2p-1} = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} u(x) = 0. \end{cases} \quad (3)$$

Indeed, the classical work of Gidas, Ni and Nirenberg [4, 5] tells us that any positive solution to (3) is radially symmetric. On the other hand, for a solution $u(r)$ of (2), $v(x) := u(|x|)$ is a solution to (3).

The condition to assure the existence of positive solutions to (3) (and so (2)) was given by Berestycki and Lions [1] and Berestycki, Lions and Peletier [2]: A solution to (2) with fixed $p > 1$ exists if and only if $\omega \in (0, \omega_p)$, where

$$\omega_p = \frac{p}{(p+1)^2}.$$

We shall review what this ω_p is for in Section 2. Throughout this paper, a *solution* means a classical solution.

Uniqueness of positive solutions to (2) had long remained unknown. Finally in 1998 Ouyang and Shi [8] proved uniqueness for (2) with all $\omega \in (0, \omega_p)$, $p > 1$. See also Kwong and Zhang [6].

In this present paper, we prove that for ω close to ω_p , the uniqueness result is obtained directly from the classical result given by Peletier and Serrin [9] in 1983. For another attempt to obtain the uniqueness when ω is close to ω_p , see Mizumachi [7]. Our result of the present paper is the following:

Theorem 1. *Let $n \in \mathbb{N}$, $p > 1$ and $\omega \in [a_p, \omega_p)$, where $a_p := \frac{p(7p-5)}{4(p+1)(2p-1)^2}$. Then (2) has exactly one positive solution.*

Remark 1. Note that

$$0 < a_p < \omega_p = \frac{p}{(p+1)^2}, \quad p > 1.$$

In the next section we clarify the definitions of ω_p and a_p from the point of view from [9].

2 Study of the nonlinearity as a function

In this section, we study the properties of the function $f(u) := -\omega u + u^p - u^{2p-1}$ in $(0, \infty)$, where $\omega > 0$ and $p > 1$ are given constants.

First we define $F(u) := \int_0^u f_{\omega,p}(s) ds$, and by a direct calculation we have

$$\begin{aligned} F(u) &= -\frac{\omega}{2}u^2 + \frac{u^{p+1}}{p+1} - \frac{u^{2p}}{2p} \\ &= \frac{u^2}{2p(p+1)} \left[-\omega p(p+1) + 2pu^{p-1} - (p+1)u^{2(p-1)} \right]. \end{aligned} \quad (4)$$

There are two cases of concern:

- (a) $\omega < \omega_p \iff F$ has two zeros in $(0, \infty)$.
- (b) $\omega \geq \omega_p \iff F$ has at most one zero in $(0, \infty)$.

The condition to assure the existence of positive solutions of (2) given in [1, 2] is the following;

Lemma 1. *The problem (2) has a positive solution if and only if both of the following hypotheses are fulfilled:*

(H1) $\lim_{u \rightarrow +0} \frac{f(u)}{u}$ exists and is negative,

(H2) $F(\delta) > 0$ for some positive constant δ .

Lemma 2. *The problem (2) has a positive solution if and only if*

$$\omega \in (0, \omega_p)$$

for $p > 1$.

Proof. (H1) is equivalent to the condition $\omega > 0$. (H2) is equivalent to the condition (a) above. \square

This is the origin of ω_p . Next we turn to the exponent a_p .

As a preparation, we calculate the derivatives of $f(u) = -\omega u + u^p - u^{2p-1}$:

$$\begin{aligned} f'(u) &= -\omega + pu^{p-1} - (2p-1)u^{2(p-1)}, \\ f''(u) &= 2(p-1)(2p-1)u^{p-2} \left[\frac{p}{2(2p-1)} - u^{p-1} \right]. \end{aligned}$$

We shall introduce four positive constants α , b , c and β .

- Let α denote the unique zero of f'' in $(0, \infty)$: $\alpha = \left[\frac{p}{2(2p-1)} \right]^{\frac{1}{p-1}}$.
- Let b denote the first zero of f in $(0, \infty)$: $b = \left[\frac{1 - \sqrt{1 - 4\omega}}{2} \right]^{\frac{1}{p-1}}$.
- Let c denote the last zero of f in $(0, \infty)$: $c = \left[\frac{1 + \sqrt{1 - 4\omega}}{2} \right]^{\frac{1}{p-1}}$.
- Let β denote the first zero of F in $(0, \infty)$: $\beta = \left[\frac{p}{p+1} \left(1 - \sqrt{1 - \frac{(p+1)^2}{p}\omega} \right) \right]^{\frac{1}{p-1}}$.

It is easy to check that

$$\beta \in (b, c) \tag{5}$$

either by observing the graphs or by a straightforward calculation. From (5) we deduce

$$f(\beta) > 0, \tag{6}$$

which will be used later.

We are not able to give a clear explanation on the relation between α and β .

Lemma 3. *The condition $\alpha \leq \beta$ is equivalent to $\omega \geq a_p = \frac{p(7p-5)}{4(p+1)(2p-1)^2}$.*

Proof. A simple calculation. \square

This is where our a_p comes into play. In the next section, we see what this condition stands for.

3 Proof of Theorem 1.

First we state the result by Peletier and Serrin [9], which assures the uniqueness of solutions of (2).

Lemma 4. *Let f satisfy (H1-3), where (H1), (H2) are in Lemma 1., and (H3) is the following:*

(H3) $G(u) := \frac{f(u)}{u-\beta}$ is nonincreasing in (β, c) .

Then (2) has exactly one positive solution.

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. We shall see that for $\omega \in [a_p, \omega_p)$, (H1-3) are satisfied. It is enough to show that if $\omega \geq a_p$, then

$$k(u) := f'(u)(u-\beta) - f(u) \leq 0 \quad \text{in } (\beta, c). \quad (7)$$

To prove (7) we calculate the derivative of $k(u)$

$$k'(u) = f''(u)(u-\beta),$$

and note that

$$\begin{aligned} f''(u) &> 0 && \text{in } (0, \alpha); \\ f''(u) &< 0 && \text{in } (\alpha, \infty). \end{aligned}$$

So if $\alpha \leq \beta$ (i.e. $\omega \geq a_p$, see Lemma 3), then $k'(u) < 0$ in (β, c) , i.e. k is decreasing in the interval. Therefore

$$k(u) < k(\beta) = -f(\beta) < 0 \quad \text{in } (\beta, c),$$

where the last inequality follows by (6).

This proves (7) and completes the proof. \square

If $\alpha > \beta$, we need to check that $k(\alpha) \leq 0$, i.e.

$$\alpha - \frac{f(\alpha)}{f'(\alpha)} \leq \beta. \quad (8)$$

This condition provides an implicit relation between ω and p . Besides,

Remark 2. The condition (8) does not cover all $\omega \in (0, \omega_p)$. That is for ω close to zero, $\alpha - \frac{f(\alpha)}{f'(\alpha)} > \beta$.

Proof. The left hand side of (8) is estimated from below as

$$\begin{aligned} \alpha - \frac{f(\alpha)}{f'(\alpha)} &= \frac{(p-1)\alpha^p(1-2\alpha^{p-1})}{-\omega + p\alpha^{p-1} - (2p-1)\alpha^{2(p-1)}} \\ &> \frac{(p-1)\alpha^p(1-2\alpha^{p-1})}{p\alpha^{p-1} - (2p-1)\alpha^{2(p-1)}} > 0, \end{aligned}$$

for all $\omega \in (0, \omega_p)$, whereas the right hand side β decreases to zero as ω decreases to zero. \square

When ω is close to zero, a very delicate observation is needed. See Ouyang and Shi [8] for details.

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