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A remark on the uniqueness of positive solutions to semilinear elliptic equations with double power nonlinearities

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Abstract
We consider the uniqueness of positive solutions to

$$\begin{aligned}
\triangle u - \omega u + u^p - u^{2p-1} &= 0 \quad \text{in } \mathbb{R}^n, \\
\lim_{|x| \to \infty} u(x) &= 0.
\end{aligned}$$

It is known that for fixed $p > 1$, a positive solution to (1) exists if and only if $\omega \in (0, \omega_p)$, where $\omega_p := \frac{p}{(p+1)^2}$. We deduce the uniqueness in the case where $\omega$ is close to $\omega_p$, from the argument in the classical paper by Peletier and Serrin [9], thereby recovering a part of the uniqueness result of Ouyang and Shi [8] for all $\omega \in (0, \omega_p)$.

1 Introduction

We shall consider a boundary value problem

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r} \frac{\partial u}{\partial r} - \omega u + u^p - u^{2p-1} &= 0 \quad \text{for } r > 0, \\
u_r(0) &= 0, \\
\lim_{r \to \infty} u(r) &= 0,
\end{aligned}$$

where $n \in \mathbb{N}$, $p > 1$ and $\omega > 0$. The above problem arises in the study of

$$\begin{aligned}
\triangle u - \omega u + u^p - u^{2p-1} &= 0 \quad \text{in } \mathbb{R}^n, \\
\lim_{|x| \to \infty} u(x) &= 0.
\end{aligned}$$

Indeed, the classical work of Gidas, Ni and Nirenberg [4, 5] tells us that any positive solution to (3) is radially symmetric. On the other hand, for a solution $u(r)$ of (2), $v(x) := u(|x|)$ is a solution to (3).
The condition to assure the existence of positive solutions to (3) (and so (2)) was given by Berestycki and Lions [1] and Berestycki, Lions and Peletier [2]: A solution to (2) with fixed $p > 1$ exists if and only if $\omega \in (0, \omega_p)$, where

$$\omega_p = \frac{p}{(p+1)^2}.$$ 

We shall review what this $\omega_p$ is for in Section 2. Throughout this paper, a solution means a classical solution.

Uniqueness of positive solutions to (2) had long remained unknown. Finally in 1998 Ouyang and Shi [8] proved uniqueness for (2) with all $\omega \in (0, \omega_p)$, $p > 1$. See also Kwong and Zhang [6].

In this present paper, we prove that for $\omega$ close to $\omega_p$, the uniqueness result is obtained directly from the classic result given by Peletier and Serrin [9] in 1983. For another attempt to obtain the uniqueness when $\omega$ is close to $\omega_p$, see Mizumachi [7]. Our result of the present paper is the following:

**Theorem 1.** Let $n \in \mathbb{N}$, $p > 1$ and $\omega \in [a_p, \omega_p)$, where $a_p := \frac{p(7p - 5)}{4(p+1)(2p-1)^2}$. Then (2) has exactly one positive solution.

**Remark 1.** Note that

$$0 < a_p < \omega_p = \frac{p}{(p+1)^2}, \quad p > 1.$$ 

In the next section we clarify the definitions of $\omega_p$ and $a_p$ from the point of view from [9].

## 2 Study of the nonlinearity as a function

In this section, we study the properties of the function $f(u) := -\omega u + u^p - u^{2p-1}$ in $(0, \infty)$, where $\omega > 0$ and $p > 1$ are given constants.

First we define $F(u) := \int_0^u f_{\omega,p}(s)ds$, and by a direct calculation we have

$$F(u) = -\frac{\omega}{2}u^2 + \frac{u^{p+1}}{p+1} - \frac{u^{2p}}{2p}$$

$$= \frac{u^2}{2p(p+1)} \left[ -\omega p(p+1) + 2pu^{p-1} - (p+1)u^{2(p-1)} \right].$$ 

(4)

There are two cases of concern:

(a) $\omega < \omega_p \iff F$ has two zeros in $(0, \infty)$.

(b) $\omega \geq \omega_p \iff F$ has at most one zero in $(0, \infty)$.

The condition to assure the existence of positive solutions of (2) given in [1, 2] is the following:
Lemma 1. The problem (2) has a positive solution if and only if both of the following hypotheses are fulfilled:

(H1) \( \lim_{u \to +0} \frac{f(u)}{u} \) exists and is negative,

(H2) \( F(\delta) > 0 \) for some positive constant \( \delta \).

Lemma 2. The problem (2) has a positive solution if and only if
\[
\omega \in (0, \omega_p)
\]
for \( p > 1 \).

Proof. (H1) is equivalent to the condition \( \omega > 0 \). (H2) is equivalent to the condition (a) above.

This is the origin of \( \omega_p \). Next we turn to the exponent \( a_p \).
As a preparation, we calculate the derivatives of \( f(u) = -\omega u + u^p - u^2 \):
\[
\begin{align*}
f'(u) &= -\omega + pu^{p-1} - (2p-1)u^{2(p-1)}, \\
f''(u) &= 2(p-1)(2p-1)u^{p-2} \left[ \frac{p}{2(2p-1)} - u^{p-1} \right].
\end{align*}
\]
We shall introduce four positive constants \( \alpha \), \( b \), \( c \) and \( \beta \).

- Let \( \alpha \) denote the unique zero of \( f'' \) in \( (0, \infty) \):
\[
\alpha = \left[ \frac{p}{2(2p-1)} \right]^{\frac{1}{p-1}}.
\]
- Let \( b \) denote the first zero of \( f \) in \( (0, \infty) \):
\[
b = \left[ \frac{1 - \sqrt{1 - 4\omega}}{2} \right]^{\frac{1}{p-1}}.
\]
- Let \( c \) denote the last zero of \( f \) in \( (0, \infty) \):
\[
c = \left[ \frac{1 + \sqrt{1 - 4\omega}}{2} \right]^{\frac{1}{p-1}}.
\]
- Let \( \beta \) denote the first zero of \( F \) in \( (0, \infty) \):
\[
\beta = \left[ \frac{p}{p+1} \left( 1 - \sqrt{1 - \frac{(p+1)^2}{p^2} \omega} \right) \right]^{\frac{1}{p-1}}.
\]

It is easy to check that
\[
\beta \in (b, c) \quad (5)
\]
either by observing the graphs or by a straightforward calculation. From (5) we deduce
\[
f(\beta) > 0, \quad (6)
\]
which will be used later.

We are not able to give a clear explanation on the relation between \( \alpha \) and \( \beta \).
Lemma 3. The condition $\alpha \leq \beta$ is equivalent to $\omega \geq a_p = \frac{p(7p - 5)}{4(p + 1)(2p - 1)^2}$.

Proof. A simple calculation.

This is where our $a_p$ comes into play. In the next section, we see what this condition stands for.

3 Proof of Theorem 1.

First we state the result by Peletier and Serrin [9], which assures the uniqueness of solutions of (2).

Lemma 4. Let $f$ satisfy (H1-3), where (H1), (H2) are in Lemma 1., and (H3) is the following:

(H3) $G(u) := \frac{f(u)}{u - \beta}$ is nonincreasing in $(\beta, c)$.

Then (2) has exactly one positive solution.

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. We shall see that for $\omega \in [a_p, \omega_p)$, (H1-3) are satisfied. It is enough to show that if $\omega \geq a_p$, then

$k(u) := f'(u)(u - \beta) - f(u) \leq 0 \quad \text{in} \quad (\beta, c).$  \hfill (7)

To prove (7) we calculate the derivative of $k(u)$

$k'(u) = f''(u)(u - \beta),$

and note that

$f''(u) > 0 \quad \text{in} \quad (0, \alpha);$

$f''(u) < 0 \quad \text{in} \quad (\alpha, \infty).$

So if $\alpha \leq \beta$ (i.e. $\omega \geq a_p$, see Lemma 3), then $k'(u) < 0$ in $(\beta, c)$, i.e. $k$ is decreasing in the interval. Therefore

$k(u) < k(\beta) = -f(\beta) < 0 \quad \text{in} \quad (\beta, c),$

where the last inequality follows by (6).

This proves (7) and completes the proof.

If $\alpha > \beta$, we need to check that $k(\alpha) \leq 0$, i.e.

$\alpha - \frac{f(\alpha)}{f'(\alpha)} \leq \beta.$ \hfill (8)

This condition provides an implicit relation between $\omega$ and $p$. Besides,
Remark 2. The condition (8) does not cover all $\omega \in (0, \omega_p)$. That is for $\omega$ close to zero, $\alpha - \frac{f(\alpha)}{f'(\alpha)} > \beta$.

Proof. The left hand side of (8) is estimated from below as

$$\alpha - \frac{f(\alpha)}{f'(\alpha)} = \frac{(p - 1)\alpha^p(1 - 2\alpha^{p-1})}{-\omega + p\alpha^{p-1} - (2p - 1)\alpha^{2(p-1)}}$$

$$> \frac{(p - 1)\alpha^p(1 - 2\alpha^{p-1})}{p\alpha^{p-1} - (2p - 1)\alpha^{2(p-1)}} > 0,$$

for all $\omega \in (0, \omega_p)$, whereas the right hand side $\beta$ decreases to zero as $\omega$ decreases to zero.

When $\omega$ is close to zero, a very delicate observation is needed. See Ouyang and Shi [8] for details.

References


