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Uniqueness of positive solutions to semilinear elliptic equations with double power nonlinearities

Shinji Kawano

Abstract
We consider the problem
\[
\begin{aligned}
\Delta u + f(u) &= 0 \quad \text{in } \mathbb{R}^n, \\
\lim_{|x| \to \infty} u(x) &= 0,
\end{aligned}
\]
where
\[f(u) = -\omega u + u^p - u^q, \quad \omega > 0, \quad q > p > 1.\]

It is known that a positive solution to (1) exists if and only if \(F(u) := \int_0^u f(s)ds > 0\) for some \(u > 0\). Moreover, Ouyang and Shi in 1998 found that the solution is unique if \(f\) satisfies furthermore the condition that \(\tilde{f}(u) := (uf'(u))' f(u) - uf'(u)^2 < 0\) for any \(u > 0\). In the present paper we remark that this additional condition is unnecessary.

1 Introduction

We shall consider a boundary value problem
\[
\begin{aligned}
u_{rr} + \frac{n-1}{r} u_r + f(u) &= 0 \quad \text{for } r > 0, \\
u_r(0) &= 0, \\
\lim_{r \to \infty} u(r) &= 0,
\end{aligned}
\]
where \(n \in \mathbb{N}\) and
\[f(u) = -\omega u + u^p - u^q, \quad \omega > 0, \quad q > p > 1.\]

The above problem arises in the study of
\[
\begin{aligned}
\Delta u + f(u) &= 0 \quad \text{in } \mathbb{R}^n, \\
\lim_{|x| \to \infty} u(x) &= 0.
\end{aligned}
\]

Indeed, the classical work of Gidas, Ni and Nirenberg [4, 5] tells us that any positive solution to (3) is radially symmetric. On the other hand, for a solution \(u(r)\) of (2), \(v(x) := u(|x|)\) is a solution to (3).

The condition to assure the existence of positive solutions to (3) (and so (2)) was given by Berestycki and Lions [1] and Berestycki, Lions and Peletier [2].
**Proposition 1.** A positive solution to (2) exists if and only if

\[ F(u) := \int_0^u f(s)ds > 0, \quad \text{for some } u > 0. \]  

Uniqueness of positive solutions to (2) had long remained unknown. Finally in 1998 Ouyang and Shi [9] proved uniqueness for (2) with \( f \) satisfying the additional condition (See also Kwong and Zhang [7]):

**Proposition 2.** If \( f \) satisfies furthermore the following condition, then the positive solution is unique;

\[ \tilde{f}(u) := (uf'(u))'f(u) - uf'(u)^2 < 0, \quad \text{for any } u > 0. \]  

Following is the main result of the present paper:

**Theorem 1.** If the nonlinearity \( f \) satisfies the existence condition (4), then the uniqueness condition (5) is automatically fulfilled.

This paper is organized as follows. In section 2 we give a straightforward proof of the theorem. In section 3 we give an alternative proof of the theorem, in which an interesting technical lemma is used. In section 4 we explain the technical lemma.

### 2 Proof of Theorem 1.

**Lemma 1.** The existence condition (4) is equivalent to

\[ \omega < \omega_{p,q}, \]

where

\[ \omega_{p,q} = \frac{2(q-p)}{(p+1)(q-1)} \left[ \frac{p-1}{q-1} \right]^{\frac{q-1}{p-q}}. \]

(See Ouyang and Shi [9] and the appendix of Fukuizumi [3].)

**Lemma 2.** The uniqueness condition (5) is equivalent to

\[ \omega < \eta_{p,q}, \]

where

\[ \eta_{p,q} = \frac{q-p}{q-1} \left[ \frac{p-1}{q-1} \right]^{\frac{q-2}{q-p}}. \]

The proofs of these Lemmas are nothing but straightforward calculation and shall be omitted.

**Proof of Theorem 1.** It is apparent that

\[ 0 < \omega_{p,q} < \eta_{p,q}, \]

which asserts the theorem.
3 Alternative Proof of Theorem 1.

The following is the key lemma. The proof of this lemma is given in the next section as a corollary of more general statement.

Lemma 3. The existence condition (4) is equivalent to the following condition;

\[ \tilde{F}(u) = (uf(u))'F(u) - uf(u)^2 < 0, \quad \text{for any} \quad u > 0. \]  

The uniqueness condition (5) is equivalent to the following condition;

\[ f(u) > 0, \quad \text{for some} \quad u > 0. \]  

This Lemma means that the relation between \( f \) and \( \tilde{f} \) is parallel to the relation between \( F \) and \( \tilde{F} \).

Alternative Proof of Theorem 1. From Lemma 3 it is enough to show that if \( F \) has positive parts (i.e. the existence condition (4)) then \( f \) has positive parts (i.e. the condition (7)). The contraposition of this statement is clear by the monotonicity of the integral. \( \square \)

4 Classification of double power nonlinear functions

To state the main result of this section, from now on we mean a more general function

\[ f(u) = -au^p + bu^q - cu^r, \quad \text{for} \quad u > 0, \]  

where \( a, b, c > 0 \) and \( p < q < r \), by the same notation \( f \). We also steal the former notation

\[ \tilde{f}(u) = (uf(u))'f(u) - uf'(u)^2. \]  

Following is the statement.

Theorem 2. There only can occur the following three cases;

\begin{align*}
(a) & \quad a < b \frac{r - q}{r - p} \left[ \frac{b(q - p)}{c(r - p)} \right]^{\frac{q - p}{r - q}} \iff f \text{ has positive parts} \iff \tilde{f} \text{ remains negative.} \\
(b) & \quad a = b \frac{r - q}{r - p} \left[ \frac{b(q - p)}{c(r - p)} \right]^{\frac{q - p}{r - q}} \iff f \text{ has just one zero} \iff \tilde{f} \text{ has just one zero.} \\
(c) & \quad a > b \frac{r - q}{r - p} \left[ \frac{b(q - p)}{c(r - p)} \right]^{\frac{q - p}{r - q}} \iff f \text{ remains negative} \iff \tilde{f} \text{ has positive parts.}
\end{align*}
Proof. The statement with respect to \( f \) is trivial. We obtain from the definition (9) that
\[
\tilde{f} = -ab(q-p)^2u^{q+p-1} + ca(r-p)^2u^{r+p-1} - bc(r-q)^2u^{r+q-1}.
\]
This is in the form of (8) and use the result with respect to \( f \).

At last we give the proof of Lemma 3, which concludes this paper.

**Proof of Lemma 3.** For the first part, we consider
\[
F(u) = -\frac{\omega}{2}u^2 + \frac{u^{p+1}}{p+1} - \frac{u^{q+1}}{q+1}, \quad \omega > 0, \quad q > p > 1,
\]
which is in the form of (8) and use the result (a) of Theorem 4.
For the second part, we consider
\[
f(u) = -\omega u + u^p - u^q, \quad \omega > 0, \quad q > p > 1,
\]
which is in the form of (8) and use the result (a) of Theorem 4.

**References**


