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Uniqueness of positive solutions to semilinear elliptic equations with double power nonlinearities

Shinji Kawano

Abstract
We consider the problem
\[ \begin{cases} \Delta u + f(u) = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \to \infty} u(x) = 0, \end{cases} \] (1)

where
\[ f(u) = -\omega u + u^p - u^q, \quad \omega > 0, \quad q > p > 1. \]

It is known that a positive solution to (1) exists if and only if \( F(u) := \int_0^u f(s)ds > 0 \) for some \( u > 0 \). Moreover, Ouyang and Shi in 1998 found that the solution is unique if \( f \) satisfies furthermore the condition that
\[ \tilde{f}(u) := \left( uf'(u)\right)'f(u) - uf'(u)^2 < 0 \]
for any \( u > 0 \). In the present paper we remark that this additional condition is unnecessary.

1 Introduction
We shall consider a boundary value problem
\[ \begin{cases} u_{rr} + \frac{n-1}{r}u_r + f(u) = 0 & \text{for } r > 0, \\ u_r(0) = 0, \\ \lim_{r \to \infty} u(r) = 0, \end{cases} \] (2)

where \( n \in \mathbb{N} \) and
\[ f(u) = -\omega u + u^p - u^q, \quad \omega > 0, \quad q > p > 1. \]

The above problem arises in the study of
\[ \begin{cases} \Delta u + f(u) = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases} \] (3)

Indeed, the classical work of Gidas, Ni and Nirenberg [4, 5] tells us that any positive solution to (3) is radially symmetric. On the other hand, for a solution \( u(r) \) of (2), \( v(x) := u(|x|) \) is a solution to (3).

The condition to assure the existence of positive solutions to (3) (and so (2)) was given by Berestycki and Lions [1] and Berestycki, Lions and Peletier [2]:
Proposition 1. A positive solution to (2) exists if and only if
\[ F(u) := \int_0^u f(s) ds > 0, \quad \text{for some } u > 0. \quad (4) \]

Uniqueness of positive solutions to (2) had long remained unknown. Finally in 1998 Ouyang and Shi [9] proved uniqueness for (2) with \( f \) satisfying the additional condition (See also Kwong and Zhang [7]):

Proposition 2. If \( f \) satisfies furthermore the following condition, then the positive solution is unique;
\[ \tilde{f}(u) := (uf'(u))'f(u) - uf'(u)^2 < 0, \quad \text{for any } u > 0. \quad (5) \]

Following is the main result of the present paper:

Theorem 1. If the nonlinearity \( f \) satisfies the existence condition (4), then the uniqueness condition (5) is automatically fulfilled.

This paper is organized as follows. In section 2 we give a straightforward proof of the theorem. In section 3 we give an alternative proof of the theorem, in which an interesting technical lemma is used. In section 4 we explain the technical lemma.

2 Proof of Theorem 1.

Lemma 1. The existence condition (4) is equivalent to
\[ \omega < \omega_{p,q}, \]
where
\[ \omega_{p,q} = \frac{2(q - p)}{(p + 1)(q - 1)} \left[ \frac{(p - 1)(q + 1)}{(p + 1)(q - 1)} \right] \frac{q - 1}{p - 1}. \]
(See Ouyang and Shi [9] and the appendix of Fukuizumi [3].)

Lemma 2. The uniqueness condition (5) is equivalent to
\[ \omega < \eta_{p,q}, \]
where
\[ \eta_{p,q} = \frac{q - p}{q - 1} \left[ \frac{p - 1}{q - 1} \right] \frac{q - 1}{p - 1}. \]

The proofs of these Lemmas are nothing but straightforward calculation and shall be omitted.

Proof of Theorem 1. It is apparent that
\[ 0 < \omega_{p,q} < \eta_{p,q}, \]
which asserts the theorem. \(\square\)
3 Alternative Proof of Theorem 1.

The following is the key lemma. The proof of this lemma is given in the next section as a corollary of more general statement.

**Lemma 3.** The existence condition (4) is equivalent to the following condition;

\[ \tilde{F}(u) = (uf(u))'F(u) - uf(u)^2 < 0, \quad \text{for any } u > 0. \]  

The uniqueness condition (5) is equivalent to the following condition;

\[ f(u) > 0, \quad \text{for some } u > 0. \]  

This Lemma means that the relation between \( f \) and \( \tilde{f} \) is parallel to the relation between \( F \) and \( \tilde{F} \).

**Alternative Proof of Theorem 1.** From Lemma 3 it is enough to show that if \( F \) has positive parts (i.e. the existence condition (4)) then \( f \) has positive parts (i.e. the condition (7)). The contraposition of this statement is clear by the monotonicity of the integral. \( \square \)

4 Classification of double power nonlinear functions

To state the main result of this section, from now on we mean a more general function

\[ f(u) = -au^p + bu^q - cu^r, \quad \text{for } u > 0, \]  

where \( a, b, c > 0 \) and \( p < q < r \), by the same notation \( f \). We also steal the former notation

\[ \tilde{f}(u) = (uf'(u))'f(u) - uf'(u)^2. \]

Following is the statement.

**Theorem 2.** There only can occur the following three cases;

\begin{align*}
(a) & \quad a < b \frac{r - q}{r - p} \left[ \frac{b(q - p)}{c(r - p)} \right]^{\frac{q - p}{r - q}} \iff f \text{ has positive parts} \iff \tilde{f} \text{ remains negative.} \\
(b) & \quad a = b \frac{r - q}{r - p} \left[ \frac{b(q - p)}{c(r - p)} \right]^{\frac{q - p}{r - q}} \iff f \text{ has just one zero} \iff \tilde{f} \text{ has just one zero.} \\
(c) & \quad a > b \frac{r - q}{r - p} \left[ \frac{b(q - p)}{c(r - p)} \right]^{\frac{q - p}{r - q}} \iff f \text{ remains negative} \iff \tilde{f} \text{ has positive parts.}
\end{align*}
Proof. The statement with respect to \(f\) is trivial. We obtain from the definition (9) that
\[
\tilde{f} = -ab(q-p)^2u^{q+p-1} + ca(r-p)^2u^{r+p-1} - bc(r-q)^2u^{r+q-1}.
\]
This is in the form of (8) and use the result with respect to \(f\).

At last we give the proof of Lemma 3, which concludes this paper.

Proof of Lemma 3. For the first part, we consider
\[
F(u) = -\frac{\omega}{2}u^2 + \frac{u^{p+1}}{p+1} - \frac{u^{q+1}}{q+1}, \quad \omega > 0, \quad q > p > 1,
\]
which is in the form of (8) and use the result (a) of Theorem 4.

For the second part, we consider
\[
f(u) = -\omega u^p - u^q, \quad \omega > 0, \quad q > p > 1,
\]
which is in the form of (8) and use the result (a) of Theorem 4.

References


