Title	KHOVANOV HOMOLOGY AND WORDS
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Citation	Hokkaido University Preprint Series in Mathematics, 943, 1-52
Issue Date	2009-7-1
DOI	10.14943/84090
Doc URL	http://hdl.handle.net/2115/69750
Type	bulletin (article)
File Information	pre943.pdf

KHOVANOV HOMOLOGY AND WORDS

FUKUNAGA TOMONORI AND ITO NOBORU

Abstract. In word and phrase theory of Turaev, we interpret links or virtual links as equivalences of phrases over an alphabet consisting four letters. V. Turaev constructed a version of the Jones polynomial for phrases. We study the well-definedness of the Jones polynomial for phrases in word theory. On the other hand, M. Khovanov introduced a collection of homology groups which is a strictly stronger link invariant than the Jones polynomial and O. Viro reconstructed these Khovanov homology groups. We construct phrase invariants as the homology groups of certain chain complexes for phrases where the coefficients of the Jones polynomial are the Euler characteristics of these complexes using the Viro's method of Khovanov theory. The invariance of these homology groups is showed in only terminology of Turaev's theory of phrases. Moreover, we apply the homology groups to getting invariants for an other type of phrases over an alphabet consisting any letters.

Keywords Turaev's homotopy theory of phrases, categorification, Jones polynomial, homotopy invariants of phrases.

1. Introduction.

V. Turaev introduced a *phrase* over a set α_* consisting four letters, which is one to one corresponding to a stable equivalence class of a knot diagram on surfaces or a virtual link [7, 8, 9]. In this study, we construct cohomology groups $KH^{i,j}(P)$ satisfying a Poincaré series (1) for a phrase P, so-called a pseudolink, a projection image of a nanophrase over α_*. $\hat{J}(P)$ is a version of the Jones polynomial for phrases defined by Turaev's homotopy theory of phrases, we define in Section 3.

$$\hat{J}(P) = \sum_{j=-\infty}^{\infty} q^j \sum_{i=-\infty}^{\infty} (-1)^i \mathrm{rk} KH^{i,j}(P). \tag{1}$$

α_* is the set composed of 4 distinct elements a_+, a_-, b_+, b_- and α_* has an involution $\tau : a_\pm \mapsto b_\mp$. Let $S_* = \{(a_\pm, a_\pm, a_\pm), (a_\pm, a_\pm, a_\mp), (a_\mp, a_\pm, a_\pm), (b_\pm, b_\pm, b_\pm), (b_\pm, b_\pm, b_\mp), (b_\mp, b_\pm, b_\pm)\}$ where three upper signs or three lower signs should be chosen in the double signs for each triple [8, Subsection 4.2]. α_*-alphabet $\mathcal{A}$ is a set where every element A of $\mathcal{A}$ has a projection $|\ | : A \mapsto |A| \in \alpha_*$. A *word* of length $n \geq 1$ in an alphabet $\mathcal{A}$ is a mapping $w : \hat{n} \to \mathcal{A}$ where $\hat{n} = \{i \in \mathbb{N} \mid 1 \leq i \leq n\}$. Such a word is encoded by the sequence $w(1)w(2)\cdots w(n)$. By definition, there is a unique word $\emptyset$ of length 0. A word $w : \hat{n} \to \mathcal{A}$ is a *Gauss word* if each element of $\mathcal{A}$ is the image of precisely two elements of $\hat{n}$ or w is $\emptyset$. A *nanoword* $(\mathcal{A}, w)$

The authors are Research Fellows of the Japan Society for the Promotion of Science. This work was partly supported by KAKENHI.

over α_* is a pair (an *α_*-alphabet* $\mathcal{A}$, a Gauss word in the alphabet $\mathcal{A}$). For a nanoword $(\mathcal{A}, w = w_1w_2\cdots w_k)$ over α_* consisting of subwords w_i $(1 \le i \le k)$ of w, a *nanophrase* of length $k \ge 0$ over α_* is defined as $(\mathcal{A}, (w_1|w_2|\cdots|w_k))$.

For a nanophrase $(\mathcal{A}, (w_1|w_2|\cdots|w_k))$ over α_*, we associate w_i to a pointed components of links and the order $(1 \le i \le k)$ to a order of components of links. Then, by introducing an appropriate isomorphism and equivalence to nanophrases using a map $|\cdot|$ and an involution $\nu(a_\pm) = b_\pm$, we get the set $\mathcal{P}(\alpha_*, \nu)$ of equivalence classes of nanophrases over α_* corresponding to the set of link diagrams [8, Subsection 6.3]. Considering certain equivalence relations depended on S_*, called S_*-homotopy, for nanophrases over α_* corresponding to Reidemeister moves of links, Turaev gives bijection between the set of stable equivalence classes of knot diagrams on surfaces and $\mathcal{P}(\alpha_*, S_*, \nu)$ that is $\mathcal{P}(\alpha_*, \nu)/S_*$-homotopy. Moreover, for $\mathcal{P}(\alpha_*, S_*, \nu)$, let $\alpha_1 := \{1, -1\}$ and $S_1 := \{(\pm1, \pm1, \pm1), (\pm1, \pm1, \mp1), (\pm1, \mp1, \mp1)\}$ where three upper signs or three lower signs should be chosen in the double signs for each triple. We can consider the set $\mathcal{P}(\alpha_1, S_1, \mathrm{id})$ that is an image of the projection: $\alpha_* \to \alpha_1$; $a_+, b_+ \mapsto 1$ and $a_-, b_- \mapsto -1$ induces $\mathcal{P}(\alpha_*, S_*, \nu) \to \mathcal{P}(\alpha_1, S_1, \mathrm{id})$, Turaev construct the Jones polynomial as S_1-homotopy invariants of elements, called pseudolinks, of $\mathcal{P}(\alpha_1, S_1, \mathrm{id})$. However, Turaev's definition of the Jones pol ynomial $J(P)$ for a pseudolink P is depend on nanophrases over α_* [8, Section 8]. It is obvious the existence of $J(P)$ by using geometrical objects (i.e. links). However, it is not clear that the well-definedness of $J(P)$ in only word theory.

In this paper, we give $J(P)$, and $KH^{i,j}(P)$ and show they are pseudolink invariants by using only P of $\mathcal{P}(\alpha_1, S_1, \mathrm{id})$. $KH^{i,j}(P)$ preserve the property of the Khovanov homology group as follows: $KH^{i,j}(P)$ is a strictly stronger invariant than $J(P)$. Moreover, we apply $KH^{i,j}(P)$ to getting invariants for an other type of phrases over an alphabet consisting any letters.

2. Turaev's theory of words

2.1. **Nanowords and Nanophrase.** For our preliminary, we define *nanophrases* and their *S-homotopy* as the manner in Turaev's original paper [7, Section 2], [8, Section 2], Gibson's paper[3, Section 2], or Fukunaga's paper [2, Section 2.1] that gives the detailed description of their terminology.

An *alphabet* is a finite set and *letters* are its elements. α-alphabet $\mathcal{A}$ is a set where every element A of $\mathcal{A}$ has a projection $|\ | : A \mapsto |A| \in \alpha$. A *word of length* $n \ge 1$ in an alphabet $\mathcal{A}$ is a mapping $w : \hat{n} \to \mathcal{A}$ where $\hat{n} = \{i \in \mathbb{N} \mid 1 \le i \le n\}$. Such a word is encoded by the sequence $w(1)w(2)\cdots w(n)$. By definition, there is a unique word $\emptyset$ of length 0. We define *opposite word* by writing the letters of a word w in the opposite order. For example, if $w = abc$, then $w^- = cba$. A word $w : \hat{n} \to \mathcal{A}$ is a *Gauss word* in an alphabet $\mathcal{A}$ if each element of $\mathcal{A}$ is the image of precisely two elements of $\hat{n}$ or w is $\emptyset$. A *Gauss phrase* in an alphabet $\mathcal{A}$ is a sequence of words x_1, $x_2, \ldots, x_m$ in $\mathcal{A}$ denoted by $(x_1|x_2|\ldots|x_m)$ such that $x_1x_2\cdots x_m$ is a Gauss word in $\mathcal{A}$. We call x_i ith component of the Gauss phrase. In particular, if a Gauss phrase has only one component, that com ponen t is a Gauss word. A *nanoword* $(\mathcal{A}, w)$ over α is a pair (an *α-alphabet* $\mathcal{A}$, a Gauss word in the alphabet $\mathcal{A}$). For a nanoword $(\mathcal{A}, w = w_1w_2\cdots w_k)$ over α consisting of subwords w_i $(1 \le i \le k)$ of w, a *nanophrase* of length $k \geq 0$ over α is defined as $(\mathcal{A}, (w_1|w_2|\cdots|w_k))$. Whenever possible, $(\mathcal{A}, (w_1|w_2|\cdots|w_k))$ is indicated by simple symbols: $(w_1|w_2|\cdots|w_k)$, $(\mathcal{A}, P)$ or P. We call w_i ith component of the nanophrase.

An arbitrary nanoword w over α yields a nanophrase (w) of length 1. However, we distinguish between nanowords and nanophrases of length 1. By definition, there is a unique nanophrase of length 0. Pay attention to the fact that $(\emptyset)$ is not a nanophrase of length 0. (cf. [8, Subsection 6.1]. Turaev makes no difference between nanowords and nanophrases of length 1). We denote the nanophrase of length 0 by $\emptyset$. Note that we distinguish the nanophrase $(\emptyset|\emptyset|\dots|\emptyset)$ of length k from the nanophrase $(\emptyset|\emptyset|\dots|\emptyset)$ of length l if $k \neq l$.

An *isomorphism* of α-alphabets $\mathcal{A}_1$, $\mathcal{A}_2$ is bijection $f : \mathcal{A}_1 \to \mathcal{A}_2$ such that $|A| = |f(A)|$ for an arbitrary $A \in \mathcal{A}_1$. Two nanophrases $(\mathcal{A}_1, p_1 = (w_1|w_2|\cdots|w_k))$ and $(\mathcal{A}_2, p_2 = (w'_1|w'_2|\cdots|w'_{k'}))$ over α are *isomorphic* if $k = k'$ and there is an isomorphism of α-alphabets $f : \mathcal{A}_1 \to \mathcal{A}_2$ such that $w'_i = fw_i$ for every $i \in \{1, 2, \cdots, k\}$.

2.2. **Homotopy of nanophrases.** To define homotopy of nanophrases we fix a finite set α with an involution $\tau : \alpha \to \alpha$ and a subset $S \subset \alpha \times \alpha \times \alpha$. We call the triple (α, τ, S) *homotopy data.* Turaev defines *S-homotopy* as follows (cf. [8, Section 2.2], [2, Section 2.1], [3, Section 2]).

Definition 2.1. Let (α, τ, S) be a homotopy data. Two nanowords $(\mathcal{A}_1, w_1)$ and $(\mathcal{A}_2, w_2)$ are S-homotopic if one nanophrase is changed into the other by the finite sequence of the isomorphisms and the following three type deformations (1)–(3), called *homotopy moves*, and their inverse. The relation S-homotopy is denoted by $\simeq_S$.

(H1) Replace $(\mathcal{A}, (xAAy))$ by $(\mathcal{A}\setminus\{A\}, (xy))$ for $\mathcal{A}$ and x, y are words in $\mathcal{A}\setminus\{A\}$ possibly including the $|$ character such that (xy) is a Gauss phrase.

(H2) Replace $(\mathcal{A}, (xAByBAz))$ by $(\mathcal{A} \setminus \{A, B\}, (xyz))$ if $A, B \in \mathcal{A}$ with $\tau(|A|) = |B|$ where x, y, z are words in $\mathcal{A} \setminus \{A, B\}$ possibly including the $|$ character such that (xyz) is a Gauss phrase.

(H3) Replace $(\mathcal{A}, (xAByACzBCt))$ by $(\mathcal{A}, (xBAyCAzCBt))$ for $(|A|, |B|, |C|) \in S$ where x, y, z, t are words in $\mathcal{A}$ possibly including the $|$ character such that $(xyzt)$ is a Gauss phrase.

Recall the following two lemmas from [8, Lemma 2.1, Lemma 2.2] (cf. [2, Lemma 2.4, Lemma 2.5]).

Lemma 2.1. *Let (α, τ, S) be a homotopy data and $\mathcal{A}$ be an α-alphabet. Let A, B, C be distinct letters in $\mathcal{A}$ and let x, y, z, t be words possibly including the $|$ character in the alphabet $\mathcal{A} \setminus \{A, B, C\}$ such that $(xyzt)$ is a Gauss phrase in this alphabet. Then,*

(i) $(\mathcal{A}, (xAByCAzBCt)) \simeq_S (\mathcal{A}, (xBAyACzCBt))$ *for* $(|A|, \tau(|B|), |C|) \in S$;

(ii) $(\mathcal{A}, (xAByCAzCBt)) \simeq_S (\mathcal{A}, (xBAyACzBCt))$ *for* $(\tau(|A|), \tau(|B|), |C|) \in S$;

(iii) $(\mathcal{A}, (xAByACzCBt)) \simeq_S (\mathcal{A}, (xBAyCAzBCt))$ *for* $(\tau(|A|), |B|, |C|) \in S$.

Lemma 2.2. *Suppose that $S \cap (\alpha\times\{b\}\times\{b\}) \neq \emptyset$ for all $b \in \alpha$. Let $(\mathcal{A}, (xAByABz))$ be a nanophrase over α with $|B| = \tau(|A|)$ where x, y, z words possibly including* *the $|$ character in the alphabet $\mathcal{A}\setminus\{A,B\}$ such that xyz is a Gauss phrase in this alphabet. Then, $(\mathcal{A},(xAByABz))\simeq_S(\mathcal{A}\setminus\{A,B\},(xyz))$.*

Definition 2.2. Let α be a finite set. Fix an involution $\nu\ \alpha\to\alpha$ called the *shift involution*. The ν-shift of a nanoword $(\mathcal{A},w:\hat{n}\to\mathcal{A})$ over α is the nanoword $(\mathcal{A}',w':\hat{n}\to\mathcal{A}')$ obtained by the following steps (1)–(3): (1) Let $\mathcal{A}:=(\mathcal{A}-\{A\})\cup\{A_\nu\}$ where A_ν is a letter not belonging to $\mathcal{A}$.

(2) The projection $\mathcal{A}'\to\alpha$ extends the given projection $\mathcal{A}-\{A\}\to\alpha$ by $|A_\nu|=\nu(|A|)$.

(3) The word w' in the alphabet $\mathcal{A}'$ is defined by $w'=xA_\nu yA_\nu$ for $w=AxAy$.

We define ν-shifts and ν-permutations of words in a nanophrase $P=(\mathcal{A},\ (w_1|w_2|\cdots|w_k))$ over α and define $\mathcal{P}(\alpha,S,\nu)$ in the following manner as in [8, Subsection 6.2].

Fix a homotopy data (α,τ,S) and a shift involution in α.

Definition 2.3. For $i=1,\dots,k$, the ith *ν-shift* of a nanophrase P moves the first letter, say A, of w_i to the end of w_i keeping $|A|\in\alpha$ if A appears in w_i only once and applying ν if A appears in w_i twice. All other words in P are preserved.

Definition 2.4. Given two words u, v on an α-alphabet $\mathcal{A}$, consider the mapping $\mathcal{A}\to\alpha$ sending $A\in\mathcal{A}$ to $\nu(|A|)\in\alpha$ if A appears both in u and v and sending A to $|A|$ otherwise. The set $\mathcal{A}$ with this projection to α is an α-alphabet denoted by $\mathcal{A}_{u\cap v}$. For $i=1,\dots,k-1$, the *ν-permutation* of the ith and $(i+1)$st words transforms a nanophrase $P=(\mathcal{A},(w_1|w_2|\cdots|w_k))$ into the nanophrase $(\mathcal{A},(w_1|w_2|\cdots|w_{i-1}|w_{i+1}|w_i|w_{i+2}|\cdots|w_k))$. The operation is involutive. The ν-permutations define an action of the symmetric group S_k on the set of nanophrases of length k.

Denote by $\mathcal{P}(\alpha,S,\nu)$ the set of nanophrases over α quotiented by the equivalence relation generated by S-homotopy, ν-permutations and ν-shifts on words.

Turaev defines *pseudolinks* in the following manner as in [8, Subsection 7.1].

Definition 2.5. Let $\alpha_1=\{-1,1\}$ with involution τ permuting 1 and -1 and let $S_1\subset\alpha_1\times\alpha_1\times\alpha_1$ consists of the following six triples: $(1,1,1)$, $(1,1,-1)$, $(-1,1,1)$, $(-1,-1,-1)$, $(-1,-1,1)$, $(1,-1,-1)$. Let $\nu=\mathrm{id}$. Nanophrases in $\mathcal{P}(\alpha_1,S_1,\mathrm{id})$ is called *pseudolinks*.

Remark 2.1. Let α_* be the set composed of 4 distinct elements a_+,a_-,b_+,b_- with involution $\tau:a_\pm\mapsto b_\mp$. Let $S_*=\{(a_\pm,a_\pm,a_\pm),\ (a_\pm,a_\pm,a_\mp),\ (a_\mp,a_\pm,a_\pm),\ (b_\pm,b_\pm,b_\pm),\ (b_\pm,b_\pm,b_\mp),\ (b_\mp,b_\pm,\ b_\pm)\}$. A projection $\alpha_*\to\alpha_1:=\{1,-1\}$; $a_+,b_+\mapsto 1$ and $a_-,b_-\mapsto -1$ induces surjective mapping $\mathcal{P}(\alpha_*,S_*,\nu)\to\mathcal{P}(\alpha_1,S_1,\mathrm{id})$.

In the last of this section, we prepare the notation $\mathcal{A}_w$ as in [8, Subsection 6.2] and also prepare the notation P_w as in [8, Subsection 8.2].

Notation 2.1. For a word w, denote by $\mathcal{A}_w$ the same alphabet $\mathcal{A}$ with new projection $|\cdots|_w$ to α defined as follows: for $A\in\mathcal{A}$ set $|A|_w=\tau(|A|)$ if A occurs once, $|A|_w=\nu(|A|)$ if A occurs in twice, and $|A|_w=|A|$ otherwise. For a phrase P in an α_1-alphabet $\mathcal{A}$ and a word w on $\mathcal{A}$, denote by P_w the same phrase on the α_1-alphabet $\mathcal{A}_w$.

3. The Jones polynomial for pseudolinks.

Turaev define the Jones polynomial for pseudolinks by using recursive relations for the bracket polynomial of nanophrases over α_* [8, Section 8]. In this section, we give a state sum representation of the Jones polynomial for pseudolinks.

Definition 3.1. For every pseudolink $P = (\mathcal{A}, (w_1|w_2|\cdots|w_k))$, we assign A with the sign $= -1$ or 1 and call the sign the *marker* of A, denoted by $mark(A)$. Let a *state* s of P be P with their markers for all the elements of $\mathcal{A}$.

For an arbitrary pseudolink P assigned state s, we consider the following deformation $(*)$:

(2)

$$(*) \begin{cases} (w_1|\cdots|AxAy|\cdots|w_k) \to \begin{cases} (w_1|\cdots|x|y|\cdots|w_k) \text{ if mark}(A) = |A| \\ (w_1|\cdots|x^- y|\cdots|w_k)_x \text{ if mark}(A) = -|A| \end{cases} \\ (w_1|\cdots|Ax|Ay|\cdots|w_k) \to \begin{cases} (w_1|\cdots|xy|\cdots|w_k) \text{ if mark}(A) = |A| \\ (w_1|\cdots|x^- y|\cdots|w_k)_x \text{ if mark}(A) = -|A|. \end{cases} \end{cases}$$

A pseudolink $(\emptyset|\cdots|\emptyset)$ is obtained by repeating these deformations from P. We denote the length of this pseudolink $(\emptyset|\cdots|\emptyset)$ by $|s|$.

Notation 3.1. We denote a letter A with $|A| = 1$ and $\text{mark}(A) = +1$ (respectively $\text{mark}(A) = -1$) by A_+ (respectively A_-), and we denote a letter A with $|A| = -1$ and $\text{mark}(A) = +1$ (respectively $\text{mark}(A) = -1$) by $\overline{A}_+$ (respectively $\overline{A}_-$).

Example 3.1. Consider $P = (ABAB)$ with $|A| = |B| = 1$. If $\text{mark}(A) = 1$ and $\text{mark}(B) = -1$, P is represented as $(A_+B_-A_+B_-)$ and $(A_+B_-A_+B_-) \overset{(*)}{\to} (B_-|B_-) \overset{(*)}{\to} (\emptyset)$. If P has $\text{mark}(A) = 1$ and $\text{mark}(B) = -1$, $(A_-B_+A_-B_+) \overset{(*)}{\to} (\overline{B}_+\overline{B}_+) \overset{(*)}{\to} (\emptyset)$.

Example 3.2. Let us add two more examples. $(A_+\overline{B}_+A_+C_+\overline{B}_+C_+) \overset{(*)}{\to} (\overline{B}_+|C_+\overline{B}_+C_+) \overset{(*)}{\to} (C_+C_+) \overset{(*)}{\to} (\emptyset|\emptyset)$. $(A_-\overline{B}_-A_-C_+\overline{B}_-C_+) \overset{(*)}{\to} (B_-C_+B_-C_+) \overset{(*)}{\to} (\overline{C}_+\overline{C}_+) \overset{(*)}{\to} (\emptyset)$.

Lemma 3.1. *The $|s|$ is well-defined. In other words $|s|$ does not depend on the order of deleting letters.*

Proof. • On the case $A_+xA_+yB_+zB_+t$

If we delete A first, then

$$\begin{aligned} A_+xA_+yB_+zB_+t &\longrightarrow x|yB_+B_+t \\ &\longrightarrow x|B_+B_+ty \\ &\longrightarrow x|y|ty \end{aligned}$$

If we delete B first, then

$$\begin{array}{rcl} A_+xA_+yB_+zB_+t & \longrightarrow & B_+zB_+tA_+xA_+y \\ & \longrightarrow & z|tA_+xA_+y \\ & \longrightarrow & z|A_+xA_+yt \\ & \longrightarrow & z|x|yt \end{array}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_-xA_-yB_+zB_+t$
If we delete A first, then

$$\begin{array}{rcl} A_-xA_-yB_+zB_+t & \longrightarrow & x^-yB_+zB_+t \\ & \longrightarrow & B_+zB_+tx^-y \\ & \longrightarrow & z|tx^-y \end{array}$$

If we delete B first, then

$$\begin{array}{rcl} A_-xA_-yB_+zB_+t & \longrightarrow & B_+zB_+tA_-xA_-y \\ & \longrightarrow & z|tA_-xA_-y \\ & \longrightarrow & z|A_+xA_+yt \\ & \longrightarrow & z|x^-yt \end{array}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On $A_+xA_+yB_-zB_-t$.
In this case we can prove similar as the case of $A_-xA_-yB_+zB_+t$.

• On the case $A_-xA_-yB_-zB_-t$
If we delete A first, then

$$\begin{array}{rcl} A_-xA_-yB_-zB_-t & \longrightarrow & B_-zB_-tx^-y \\ & \longrightarrow & z^-tx^-y \end{array}$$

If we delete B first, then

$$\begin{array}{rcl} A_-xA_-yB_-zB_-t & \longrightarrow & z^-tA_-xA_-y \\ & \longrightarrow & A_-xA_yz^-t \\ & \longrightarrow & x^-yz^-t \end{array}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the cases $\overline{A}_{\epsilon_1}x\overline{A}_{\epsilon_1}yB_{\epsilon_2}zB_{\epsilon_2}t$, where $\epsilon_1, \epsilon_2 \in \{+, -\}$.

In this case we can prove similarly as the cases of $A_{-\epsilon_1}xA_{-\epsilon_1}yB_{\epsilon_2}zB_{\epsilon_2}t$.

• On the cases $\overline{A}_{\epsilon_1}x\overline{A}_{\epsilon_1}y\overline{B}_{\epsilon_2}z\overline{B}_{\epsilon_2}t$, where $\epsilon_1, \epsilon_2 \in \{+,-\}$.
In this case we can prove similarly as the case of $A_{-\epsilon_1}xA_{-\epsilon_1}yB_{-\epsilon_2}zB_{-\epsilon_2}t$.

• On the case $A_+xB_+yA_+zB_+$.
In this case, If we delete A first, then

$$\begin{array}{rcl} A_+xB_+yA_+zB_+t & \longrightarrow & xB_+y|zB_+t \\ & \longrightarrow & B_+yx|B_+tz \\ & \longrightarrow & yxtz \end{array}$$

If we delete B first, then

$$\begin{array}{rcl} A_+xB_+yA_+zB_+t & \longrightarrow & B_+yA_+zB_+tA_+x \\ & \longrightarrow & yA_+z|tA_+x \\ & \longrightarrow & A_+zy|A_+xt \\ & \longrightarrow & zyxt \end{array}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_-xB_+yA_-zB_+$.
In this case, If we delete A first, then

$$\begin{array}{rcl} A_-xB_+yA_-zB_+t & \longrightarrow & y^-\overline{B}_+x^-z\overline{B}_+t \\ & \longrightarrow & \overline{B}_+x^-z\overline{B}_+ty^- \\ & \longrightarrow & z^-xty^- \end{array}$$

If we delete B first, then

$$\begin{array}{rcl} A_-xB_+yA_-zB_+t & \longrightarrow & B_+yA_-zB_+tA_-x \\ & \longrightarrow & yA_-z|tA_-x \\ & \longrightarrow & A_-zy|A_-xt \\ & \longrightarrow & y^-z^-xt \end{array}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_+xB_-yA_+zB_-$.
In this case we can prove similarly as the case of $A_-xB_+yA_-zB_+$.

• On the case $A_-xB_-yA_-zB_-$.

In this case, If we delete A first, then

$$\begin{aligned} A_-xB_-yA_-zB_-t &\longrightarrow y^-\overline{B}_-x^-z\overline{B}_-t \\ &\longrightarrow \overline{B}_-x^-z\overline{B}_-ty^- \\ &\longrightarrow x^-z|ty^- \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_-xB_+yA_-zB_+t &\longrightarrow B_-yA_-zB_-tA_-x \\ &\longrightarrow z^-\overline{A}_-y^-t\overline{A}_-x \\ &\longrightarrow \overline{A}_-y^-t\overline{A}_-xz^- \\ &\longrightarrow yt|xz^- \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the cases $\overline{A}_{\epsilon_1}xB_{\epsilon_2}y\overline{A}_{\epsilon_1}zB_{\epsilon_2}$, where $\epsilon_1, \epsilon_2 \in \{+,-\}$.
In this cases we can prove similarly as the cases of $A_{-\epsilon_1}xB_{\epsilon_2}yA_{-\epsilon_1}zB_{\epsilon_2}t$.

• On the cases $\overline{A}_{\epsilon_1}x\overline{B}_{\epsilon_2}y\overline{A}_{\epsilon_1}z\overline{B}_{\epsilon_2}$, where $\epsilon_1, \epsilon_2 \in \{+,-\}$.
In this cases we can prove similarly as the cases of $A_{-\epsilon_1}xB_{-\epsilon_2}yA_{-\epsilon_1}zB_{-\epsilon_2}t$.

• On the case $A_+xA_+y|B_+zB_+t$
If we delete A first, then

$$\begin{aligned} A_+xA_+y|B_+zB_+t &\longrightarrow x|y|B_+zB_+t \\ &\longrightarrow x|y|z|t. \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_+xA_+y|B_+zB_+t &\longrightarrow A_+xA_+y|z|t \\ &\longrightarrow x|y|z|t. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_-xA_-y|B_+zB_+t$
If we delete A first, then

$$\begin{aligned} A_-xA_-y|B_+zB_+t &\longrightarrow x^-y|B_+zB_+t \\ &\longrightarrow x^-y|z|t. \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_-xA_-y|B_+zB_+t &\longrightarrow A_-xA_-y|z|t \\ &\longrightarrow x^-y|z|t. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.
• On the case $A_-xA_-y|B_+zB_+t$
In this case we can prove similarly as the case of $A_-xA_-y|B_+zB_+t$.

• On the case $A_-xA_-y|B_-zB_-t$
If we delete A first, then

$$\begin{aligned} A_-xA_-y|B_-zB_-t &\longrightarrow x^-y|B_-zB_-t \\ &\longrightarrow x^-y|z^-t. \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_-xA_-y|B_-zB_-t &\longrightarrow A_-xA_-y|z^-t \\ &\longrightarrow x^-y|z^-t. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the cases $\overline{A}_{\epsilon_1}x\overline{A}_{\epsilon_1}y|B_{\epsilon_2}zB_{\epsilon_2}t$ and $\overline{A}_{\epsilon_1}x\overline{A}_{\epsilon_1}y|\overline{B}_{\epsilon_2}z\overline{B}_{\epsilon_2}t$ where $\epsilon_1, \epsilon_2 \in \{+,-\}$.
We can prove similarly as the cases of $A_{-\epsilon_1}xA_{-\epsilon_1}y|B_{\epsilon_2}zB_{\epsilon_2}t$ and $A_{-\epsilon_1}xA_{-\epsilon_1}y|B_{-\epsilon_2}zB_{-\epsilon_2}t$ respectively.

• On the case $A_+xB_+y|A_+zB_+t$
If we delete A first, then

$$\begin{aligned} A_+xB_+y|A_+zB_+t &\longrightarrow xB_+yzB_+t \\ &\longrightarrow B_+yzB_+tx \\ &\longrightarrow yz|tx. \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_+xB_+y|A_+zB_+t &\longrightarrow B_+yA_+x|B_+tA_+z \\ &\longrightarrow yA_+xtA_+z \\ &\longrightarrow A_+xtA_+zy \\ &\longrightarrow xt|zy. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_-xB_+y|A_-zB_+t$
If we delete A first, then

$$\begin{aligned} A_-xB_+y|A_-zB_+t &\longrightarrow y^-\overline{B}_+x^-z\overline{B}_+t \\ &\longrightarrow \overline{B}_+x^-z\overline{B}_+ty^- \\ &\longrightarrow z^-xty^-. \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_-xB_+y|A_-zB_+t &\longrightarrow B_+yA_-x|B_+tA_-z \\ &\longrightarrow yA_-xtA_-z \\ &\longrightarrow A_-xtA_-zy \\ &\longrightarrow t^-x^-zy. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_+xB_-|A_+B_-$.
In this case we can prove similarly as the case of $A_-xB_+|A_-B_+$.

• On the case $A_-xB_-|A_-B_-$.

$$\begin{aligned} A_-xB_-y|A_-zB_-t &\longrightarrow y^-\overline{B}_-x^-z\overline{B}_-t \\ &\longrightarrow \overline{B}_-x^-z\overline{B}_-ty^- \\ &\longrightarrow x^-z|ty^-. \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_-xB_-y|A_-zB_-t &\longrightarrow B_-yA_-x|B_-tA_-z \\ &\longrightarrow x^-\overline{A}_-y^-t\overline{A}_-z \\ &\longrightarrow \overline{A}_-y^-t\overline{A}_-zx^- \\ &\longrightarrow y^-t|zx^-. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the cases $\overline{A}_{\epsilon_1}xB_{\epsilon_2}y|\overline{A}_{\epsilon_1}zB_{\epsilon_2}t$ and $\overline{A}_{\epsilon_1}x\overline{B}_{\epsilon_2}y|\overline{A}_{\epsilon_1}z\overline{B}_{\epsilon_2}t$ where $\epsilon_1, \epsilon_2 \in \{+, -\}$.
We can prove similarly as the cases of $A_{-\epsilon_1}xB_{\epsilon_2}y|A_{-\epsilon_1}zB_{\epsilon_2}t$ and $A_{-\epsilon_1}xB_{-\epsilon_2}y|A_{-\epsilon_1}zB_{-\epsilon_2}t$ respectively.

• On the case $A_+x|A_+yB_+zB_+t$
If we delete A first, then

$$\begin{aligned} A_+x|A_+yB_+zB_+t &\longrightarrow xyB_+zB_+t \\ &\longrightarrow B_+zB_+txy \\ &\longrightarrow z|txy. \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_+x|A_+yB_+zB_+t &\longrightarrow A_+x|B_+zB_+tA_+y \\ &\longrightarrow A_+x|z|tA_+y \\ &\longrightarrow A_+x|A_+yt|z \\ &\longrightarrow xyt|z. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_+x|A_+yB_+zB_+t$
If we delete A first, then

$$\begin{aligned} A_-x|A_-yB_+zB_+t &\longrightarrow x^-yB_+zB_+t \\ &\longrightarrow B_+zB_+tx^-y \\ &\longrightarrow z|tx^-y. \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_-x|A_-yB_+zB_+t &\longrightarrow A_-x|B_+zB_+tA_-y \\ &\longrightarrow A_-x|z|tA_-y \\ &\longrightarrow A_-x|A_-yt|z \\ &\longrightarrow x^-yt|z. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_+x|A_+yB_-zB_-t$
If we delete A first, then

$$\begin{aligned} A_+x|A_+yB_-zB_-t &\longrightarrow xyB_-zB_-t \\ &\longrightarrow B_-zB_-txy \\ &\longrightarrow z^-txy. \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_+x|A_+yB_-zB_-t &\longrightarrow A_+x|B_-zB_-tA_+y \\ &\longrightarrow A_+x|z^-tA_+y \\ &\longrightarrow A_+x|A_+yz^-t \\ &\longrightarrow xyz^-t. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_-x|A_-yB_-zB_-t$
If we delete A first, then

$$\begin{aligned} A_-x|A_-yB_-zB_-t &\longrightarrow x^-yB_-zB_-t \\ &\longrightarrow B_-zB_-tx^-y \\ &\longrightarrow z^-tx^-y. \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_-x|A_-yB_-zB_-t &\longrightarrow A_-x|B_-zB_-tA_-y \\ &\longrightarrow A_-x|z^-tA_-y \\ &\longrightarrow A_-x|A_-yz^-t \\ &\longrightarrow x^-yz^-t. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $\overline{A}_{\epsilon_1}x|\overline{A}_{\epsilon_1}yB_{\epsilon_2}zB_{\epsilon_2}t$, $A_{\epsilon_1}x|A_{\epsilon_1}y\overline{B}_{\epsilon_2}z\overline{B}_{\epsilon_2}t$ and $\overline{A}_{\epsilon_1}x|\overline{A}_{\epsilon_1}y\overline{B}_{\epsilon_2}z\overline{B}_{\epsilon_2}t$ is proved similarly as the cases of above.

• On the case $A_+x|A_+y|B_+zB_+t$
If we delete A first, then

$$\begin{aligned} A_+x|A_+y|B_+zB_+t &\longrightarrow xy|B_+zB_+t \\ &\longrightarrow xy|z|t. \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_+x|A_+y|B_+zB_+t &\longrightarrow A_+x|A_+y|z|t \\ &\longrightarrow xy|z|t. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_-x|A_-y|B_+zB_+t$
If we delete A first, then

$$\begin{array}{rcl} A_-x|A_-y|B_+zB_+t & \longrightarrow & x^-y|B_+zB_+t \\ & \longrightarrow & x^-y|z|t. \end{array}$$

If we delete B first, then

$$\begin{array}{rcl} A_-x|A_-y|B_+zB_+t & \longrightarrow & A_-x|A_-|z|t \\ & \longrightarrow & x^-y|z|t. \end{array}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_+x|A_+y|B_-zB_-t$
If we delete A first, then

$$\begin{array}{rcl} A_+x|A_+y|B_-zB_-t & \longrightarrow & xy|B_-zB_-t \\ & \longrightarrow & xy|z^-t. \end{array}$$

If we delete B first, then

$$\begin{array}{rcl} A_+x|A_+y|B_-zB_-t & \longrightarrow & A_+x|A_+y|z^-t \\ & \longrightarrow & xy|z^-t. \end{array}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_-x|A_-y|B_-zB_-t$
If we delete A first, then

$$\begin{array}{rcl} A_-x|A_-y|B_-zB_-t & \longrightarrow & x^-y|B_-zB_-t \\ & \longrightarrow & x^-y|z^-t. \end{array}$$

If we delete B first, then

$$\begin{array}{rcl} A_-x|A_-y|B_-zB_-t & \longrightarrow & A_-x|A_-y|z^-t \\ & \longrightarrow & x^-y|z^-t. \end{array}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $\overline{A}_{\epsilon_1}x|\overline{A}_{\epsilon_1}y|B_{\epsilon_2}zB_{\epsilon_2}t$, $A_{\epsilon_1}x|A_{\epsilon_1}y|\overline{B}_{\epsilon_2}z\overline{B}_{\epsilon_2}t$ and $\overline{A}_{\epsilon_1}x|\overline{A}_{\epsilon_1}y|\overline{B}_{\epsilon_2}z\overline{B}_{\epsilon_2}t$ is proved similarly as the cases of above.

• On the case $A_+x|B_+y|A_+zB_+t$
If we delete A first, then

$$\begin{array}{rcl} A_+x|B_+y|A_+zB_+t & \longrightarrow & A_+x|A_+zB_+t|B_+y \\ & \longrightarrow & xzB_+t|B_+y \\ & \longrightarrow & B_+txz|B_+y \\ & \longrightarrow & txzy. \end{array}$$

If we delete B first, then

$$\begin{array}{rcl} A_+x|B_+y|A_+zB_+t & \longrightarrow & A_+x|B_+y|B_+tA_+z \\ & \longrightarrow & A_+x|ytA_+z \\ & \longrightarrow & A_+x|A_+zyt \\ & \longrightarrow & xzyt. \end{array}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_-x|B_+y|A_-zB_+t$
If we delete A first, then

$$\begin{array}{rcl} A_-x|B_+y|A_-zB_+t & \longrightarrow & A_-x|A_-zB_+t|B_+y \\ & \longrightarrow & x^-zB_+t|B_+y \\ & \longrightarrow & B_+tx^-z|B_+y \\ & \longrightarrow & tx^-zy. \end{array}$$

If we delete B first, then

$$\begin{array}{rcl} A_-x|B_+y|A_-zB_+t & \longrightarrow & A_-x|B_+y|B_+tA_-z \\ & \longrightarrow & A_-x|ytA_-z \\ & \longrightarrow & A_-x|A_-zyt \\ & \longrightarrow & x^-zyt. \end{array}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_-x|B_+y|A_-zB_+t$

This case is proved similarly as the case of $A_{-}x|B_{+}y|A_{-}zB_{+}t$.
• On the case $A_{-}x|B_{-}y|A_{-}zB_{-}t$
If we delete A first, then

$$\begin{array}{rcl} A_{-}x|B_{-}y|A_{-}zB_{-}t & \longrightarrow & A_{-}x|A_{-}zB_{-}t|B_{+}y \\ & \longrightarrow & x^{-}zB_{-}t|B_{-}y \\ & \longrightarrow & B_{-}tx^{-}z|B_{-}y \\ & \longrightarrow & z^{-}xt^{-}y. \end{array}$$

If we delete B first, then

$$\begin{array}{rcl} A_{-}x|B_{-}y|A_{-}zB_{-}t & \longrightarrow & A_{-}x|B_{-}y|B_{-}tA_{-}z \\ & \longrightarrow & A_{-}x|y^{-}tA_{-}z \\ & \longrightarrow & A_{-}x|A_{-}zy^{-}t \\ & \longrightarrow & t^{-}yz^{-}x. \end{array}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $\overline{A}_{\epsilon_1}x|B_{\epsilon_2}y|\overline{A}_{\epsilon_1}zB_{\epsilon_2}t$ and $\overline{A}_{\epsilon_1}x|\overline{B}_{\epsilon_2}y|\overline{A}_{\epsilon_1}z\overline{B}_{\epsilon_2}t$ is proved similarly as the cases of above.

• On the case $A_{+}x|A_{+}y|B_{+}zB_{+}t$
If we delete A first, then

$$\begin{array}{rcl} A_{+}x|A_{+}y|B_{+}z|B_{+}t & \longrightarrow & xy|B_{+}z|B_{+}t \\ & \longrightarrow & xy|zt. \end{array}$$

If we delete B first, then

$$\begin{array}{rcl} A_{+}x|A_{+}y|B_{+}z|B_{+}t & \longrightarrow & A_{+}x|A_{+}y|zt \\ & \longrightarrow & xy|zt. \end{array}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_{-}x|A_{-}y|B_{+}z|B_{+}t$
If we delete A first, then

$$\begin{array}{rcl} A_{-}x|A_{-}y|B_{+}z|B_{+}t & \longrightarrow & x^{-}y|B_{+}z|B_{+}t \\ & \longrightarrow & x^{-}y|z|t. \end{array}$$

If we delete B first, then

$$\begin{aligned} A_-x|A_-y|B_+z|B_+t &\longrightarrow A_-x|A_-y|zt \\ &\longrightarrow x^-y|zt. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_+x|A_+y|B_-z|B_-t$
If we delete A first, then

$$\begin{aligned} A_+x|A_+y|B_-z|B_-t &\longrightarrow x^-y|B_-z|B_-t \\ &\longrightarrow xy|z^-t. \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_+x|A_+y|B_-z|B_-t &\longrightarrow A_+x|A_+y|z^-t \\ &\longrightarrow xy|z^-t. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_-x|A_-y|B_-z|B_-t$
If we delete A first, then

$$\begin{aligned} A_-x|A_-y|B_-z|B_-t &\longrightarrow x^-y|B_-z|B_-t \\ &\longrightarrow x^-y|z^-t. \end{aligned}$$

If we delete B first, then

$$\begin{aligned} A_-x|A_-y|B_-z|B_-t &\longrightarrow A_-x|A_-y|z^-t \\ &\longrightarrow x^-y|z^-t. \end{aligned}$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $\overline{A}_{\epsilon_1}x|\overline{A}_{\epsilon_1}y|B_{\epsilon_2}z|B_{\epsilon_2}t$ and $\overline{A}_{\epsilon_1}x|\overline{A}_{\epsilon_1}y|\overline{B}_{\epsilon_2}z|\overline{B}_{\epsilon_2}t$ is proved similarly as the cases of above.

Now we have completed the proof. □

Remark 3.1. The deformation $(*)$ corresponds to smoothing crossings of a link diagrams in the following figures (cf. [11, Page 320, Figure 1]).

FIGURE 1. Smoothing of a diagram according to thick segments corresponding to markers.

Definition 3.2. For an arbitrary pseudolink P and state s of P, we define $[P]$, $[P|s]$ $\in \mathbb{Z}[t,u,d]$ by

$$[P|s] := t^{\sharp\{\text{positive marker}\}} u^{\sharp\{\text{negative marker}\}} d^{|s|-1}, \tag{3}$$

$$[P] := \sum_s [P|s]. \tag{4}$$

Proposition 3.1. *The polynomial $[P]$ is invariant under S_1-homotopy moves (H2) for an arbitrary pseudolink P if and only if $u = t^{-1}$ and $d = -t^2 - t^{-2}$.*

Proof. Consider a nanophrase $P = (P_1|ABxBAy|P_2)$ with $|A| = +$ and $|B| = -$, where x and y are words not including "|" character. Then

$$\begin{aligned}
[(P_1|ABxBAy|P_2)] &= t[(P_1|BxB|y|P_2)] + s[(P_1|Bx^-By|P_2)_x] \\
&= (t^2 + tsd + s^2)[(P_1|x^-|y|P_2)_x] + st[(P_1|xy|P_2)]
\end{aligned}$$

So if $[P]$ does not change by the second homotopy move, then $t^2 + tsd + s^2 = 0$ and $st = 1$. In other words $s = t^{-1}$ and $d = -t^2 - t^{-2}$.

Converse is checked easily by the above equation. $\square$

Remark 3.2. Substituting t^{-1} for u and $-t^2 - t^{-2}$ for d, we have

$$[P] = \sum_s t^{\sigma(s)}(-t^2 - t^{-2})^{|s|-1}$$

where $\sigma(s) := \sharp\{$ positive marker $\} - \sharp\{$ negative marker $\}$.

Proposition 3.2. *$[P]$ is invariant under S_1-homotopy move (H3) for an arbitrary pseudolink P.*

Proof. First we consider the case of $(\epsilon(A), \epsilon(B), \epsilon(C)) = (\pm, \pm, \pm)$. Consider the 3rd homotopy move

$$(P_1|ABxACyBCz|P_2) \longrightarrow (P_1|BAxCAyCBz|P_2).$$

Then

$$\begin{aligned}
[(P_1|ABxACyBCz|P_2)] &= t^{3\epsilon(A)}[(P_1|xy|z|P_2)] \\
&+ (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))[(P_1|zx^-y^-|P_2)] \\
&+ t^{\epsilon(A)}[(P_1|x^-y|z|P_2)] \\
&+ t^{-\epsilon(A)}[(P_1|xy^-z|P_2)] \\
&+ t^{-\epsilon(A)}[(P_1|x^-yz|P_2)]
\end{aligned}$$

and

$$\begin{array}{rcl}[(P_1|BAxCAyCBz|P_2)] & = & t^{3\epsilon(A)}[(P_1|xy|z|P_2)] \\ & + & (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))[(P_1|x^- y|z|P_2)] \\ & + & t^{\epsilon(A)}[(P_1|x^- y^- z|P_2)] \\ & + & t^{-\epsilon(A)}[(P_1|xy^- z|P_2)] \\ & + & t^{-\epsilon(A)}[(P_1|z^- y^- x|P_2)].\end{array}$$

Note that

$$2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}) = t^{\epsilon(A)}$$

So $[(P_1|ABxACyBCz|P_2)]$ is equal to $[(P_1|BAxCAyCBz|P_2)]$.

Consider the 3rd homotopy move

$$(P_1|ABx|ACyBCz|P_2) \longrightarrow (P_1|BAx|CAyCBz|P_2).$$

Then

$$\begin{array}{rcl}[(P_1|ABx|ACyBCz|P_2)] & = & t^{3\epsilon(A)}[(P_1|xzy|P_2)] \\ & + & (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))[(P_1|zx^- y^-|P_2)] \\ & + & t^{-\epsilon(A)}[(P_1|x^-|y^- z|P_2)] \\ & + & t^{\epsilon(A)}[(P_1|xzy^-|P_2)] \\ & + & t^{-\epsilon(A)}[(P_1|x^- yz|P_2)]\end{array}$$

and

$$\begin{array}{rcl}[(P_1|BAx|CAyCBz|P_2)] & = & t^{3\epsilon(A)}[(P_1|xyz|P_2)] \\ & + & (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))[(P_1|z^- x^- y|P_2)] \\ & + & t^{-\epsilon(A)}[(P_1|x^- yz|P_2)] \\ & + & t^{-\epsilon(A)}[(P_1|x^-|y^- z|P_2)] \\ & + & t^{\epsilon(A)}[(P_1|x^- y^- z|P_2)].\end{array}$$

So $[(P_1|ABx|ACyBCz|P_2)]$ is equal to $[(P_1|BAx|CAyCBz|P_2)]$.

Consider the 3rd homotopy move

$$(P_1|ABxACy|BCz|P_2) \longrightarrow (P_1|BAxCAy|CBz|P_2).$$

Then

$$\begin{array}{rcl}[(P_1|ABxACy|BCz|P_2)] & = & t^{3\epsilon(A)}[(P_1|xzy|P_2)] \\ & + & (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))[(P_1|xy^- z|P_2)] \\ & + & t^{-\epsilon(A)}[(P_1|x^- y^- z|P_2)] \\ & + & t^{\epsilon(A)}[(P_1|x^- yz|P_2)] \\ & + & t^{-\epsilon(A)}[(P_1|yx^-|z|P_2)]\end{array}$$

and

$$
\begin{aligned}
[(P_1|BAx|CAyCBz|P_2)] &= t^{3\epsilon(A)}[(P_1|xyz|P_2)] \\
&+ (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))[(P_1|x^- yz|P_2)] \\
&+ t^{-\epsilon(A)}[(P_1|x^- y^- z|P_2)] \\
&+ t^{-\epsilon(A)}[(P_1|y^- x|z|P_2)] \\
&+ t^{\epsilon(A)}[(P_1|xy^- z|P_2)].
\end{aligned}
$$

So $[(P_1|ABxACy|BCz|P_2)]$ is equal to $[(P_1|BAxCAy|CBz|P_2)]$.

Consider the 3rd homotopy move

$$(P_1|ABx|ACy|BCz|P_2) \longrightarrow (P_1|BAx|CAy|CBz|P_2).$$

Then

$$
\begin{aligned}
[(P_1|ABx|ACy|BCz|P_2)] &= t^{3\epsilon(A)}[(P_1|y|zx|P_2)] \\
&+ (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))[(P_1|y^- zx|P_2)] \\
&+ t^{-\epsilon(A)}[(P_1|x^-|zy^-|P_2)] \\
&+ t^{\epsilon(A)}[(P_1|xzy^-|P_2)] \\
&+ t^{-\epsilon(A)}[(P_1|yx^-|z|P_2)]
\end{aligned}
$$

and

$$
\begin{aligned}
[(P_1|BAx|CAy|CBz|P_2)] &= t^{3\epsilon(A)}[(P_1|yz|x|P_2)] \\
&+ (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))[(P_1|y^- xz|P_2)] \\
&+ t^{-\epsilon(A)}[(P_1|zy^-|x^-|P_2)] \\
&+ t^{-\epsilon(A)}[(P_1|x^- y|z|P_2)] \\
&+ t^{\epsilon(A)}[(P_1|z^- yx^-|P_2)].
\end{aligned}
$$

So $[(P_1|ABx|ACy|BCz|P_2)]$ is equal to $[(P_1|BAx|CAy|CBz|P_2)]$.

The cases of $(\epsilon(A), \epsilon(B), \epsilon(C)) = (\mp, \pm, \pm)$ and $(\epsilon(A), \epsilon(B), \epsilon(C)) = (\pm, \pm, \mp)$ are proved similarly as the above case. □

Proposition 3.3. *For an arbitrary pseudolink P, the Jones polynomial $J(P)$ for pseudolinks,*

$$J(P) = (-t)^{-3w(P)} \sum_{s:states} t^{\sigma(s)}(-t^2 - t^{-2})^{|s|-1} \tag{5}$$

where $w(P) = \sum_{letters\ A\ in\ P} |A|$

Remark 3.3. The Jones polynomial $J(P)$ of a pseudolink P is given by using recursive relations for the bracket polynomial of nanophrases over α_* [8, Section 8]. It is obvious the existence of $J(P)$ by using geometrical objects (i.e. links). However, it is not clear that the well-definedness of $J(P)$ is given in only word theory of Turaev. Then, we give the well-definedness by Lemma 3.1 and (5) using only P of $\mathcal{P}(\alpha_1, S_1, \mathrm{id})$.

Definition 3.3 of *enhanced states* is given in the manner as in [11, Page 326, Subsection 4.3].

Definition 3.3. By an *enhanced state* S of pseudolink P we mean a collection of markers constituting a state s of P enhanced by an assignment of a plus or minus sign to each of the components $(\emptyset|\cdots|\emptyset)$. (Recall that $(\emptyset|\cdots|\emptyset)$ is obtained by deformations $(*)$.) We denote $\emptyset$ with a positive marker $+$ by $\emptyset_+$ and $\emptyset$ with a negative marker $-$ by $\emptyset_-$.

Notation 3.2. We rewrite the deformation $(*)$ as follows:

(6)
$$(**)\begin{cases} (w_1|\cdots|AxAy|\cdots|w_k) \to \begin{cases} (w_1|\cdots|ax|ay|\cdots|w_k) \text{ if mark}(A)=|A| \\ (w_1|\cdots|ax^{-}ay|\cdots|w_k)_x \text{ if mark}(A)=-|A| \end{cases} \\ (w_1|\cdots|Ax|Ay|\cdots|w_k) \to \begin{cases} (w_1|\cdots|axay|\cdots|w_k) \text{ if mark}(A)=|A| \\ (w_1|\cdots|ax^{-}ay|\cdots|w_k)_x \text{ if mark}(A)=-|A|. \end{cases} \end{cases}$$

A pseudolink $(a^1_1\cdots a^1_{n_1}|a^1_2\cdots a^2_{n_2}|\cdots|a^{k'}_1\cdots a^{k'}_{n_{k'}})$ given by repeating these deformations $(**)$ from P represents an enhanced state S and the pseudolink is denoted by $(\overset{B^1_1\cdots B^1_{m_1}}{\emptyset_{\epsilon_1}}|\overset{B^2_1\cdots B^2_{m_1}}{\emptyset_{\epsilon_2}}|\cdots|\overset{B^{k'}_1\cdots B^{k'}_{m_{k'}}}{\emptyset_{\epsilon_{k'}}})$ where $B^i_1\cdots B^i_{m_i}$ is a word obtained by arranging all the distinct letters in $\{A^i_1,\dots,A^i_{n_i}\}$ in any desired order. Note that $\{A^i_1,\dots,A^i_{n_i}\}$ corresponds to $\{a^i_1,\dots,a^i_{n_i}\}$.

Example 3.3. Example 3.2 is rewritten by using Notation 3.2. $(A_+\overline{B}_+A_+C_+\overline{B}_+C_+)$ $\overset{(**)}{\to}$ $(a\overline{B}_+|aC_+\overline{B}_+C_+) = (\overline{B}_+a|\overline{B}_+C_+aC_+)$ $\overset{(**)}{\to}$ $(babC_+aC_+) = (C_+aC_+bab)$ $\overset{(**)}{\to}$ $(ca|cbab) = (\overset{AC}{\emptyset}|\overset{ABC}{\emptyset})$. $(A_-\overline{B}_-A_-C_+\overline{B}_-C_+)$ $\overset{(**)}{\to}$ $(aB_-aC_+B_-C_+) = (B_-aC_+B_-C_+a)$ $\overset{(**)}{\to}$ $(b\overline{C}_+ab\overline{C}_+a) = (\overline{C}_+ab\overline{C}_+ab)$ $\overset{(**)}{\to}$ $(cbacab) = (\overset{ABC}{\emptyset})$.

Notation 3.3. For an arbitrary enhanced state S of pseudolink P, let

$$i(S) := \frac{w(P)-\sigma(S)}{2}, \tag{7}$$

$$\tau(S) := \sharp\{\emptyset_+ \text{ in } P_S\} - \sharp\{\emptyset_- \text{ in } P_S\}, \tag{8}$$

$$j(S) := -\frac{\sigma(S)+2\tau(S)-3w(P)}{2} \in \mathbb{Z}. \tag{9}$$

Let s be a state of a pseudolink P, S be an enhanced state of P and $\hat{J}(P) = (-t^2 - t^{-2})J(P)$. By using these notations above we have

$$\hat{J}(P) = (-t)^{-3w(P)} \sum_{\text{states } s} t^{\sigma(s)}(-t^2 - t^{-2})^{|s|} \tag{10}$$

$$= (-t)^{-3w(P)} \sum_{\text{enhanced states } S} t^{\sigma(S)}(-t^2)^{\tau(S)} \tag{11}$$

$$= \sum_{\text{enhanced states } S} (-1)^{w(P)+\tau(S)}\ t^{-2j(S)} \tag{12}$$

$$= \sum_{\text{enhanced states } S} (-1)^{\frac{w(P)-\sigma(S)}{2}} q^{j(S)} \quad (q = -t^{-2}) \tag{13}$$

$$= \sum_{\text{enhanced states } S} (-1)^{i(S)} q^{j(S)}. \tag{14}$$

Remark 3.4. Let α_0 be the set $\{-1, 1\}$ with the involution $\tau_0 : \pm 1 \mapsto \mp 1$ and S_0 be $\{(-1, -1, -1), (1, 1, 1)\}$. Note that every S_1-homotopy invariant of pseudolinks is S_0-homotopy invariant of nanophrases over α_0 because $S_0 \subset S_1$.

Corollary 3.1. *$J(P)$ and $\hat{J}(P)$ are S_0-homotopy invariants for nanophrases P over α_0.*

4. Khovanov Homology for pseudolinks.

Definition 4.1. For an arbitrary pseudolink P, let $C(P)$ be a free abelian group generated by enhanced states of P. We define the subgroup $C^{i,j}(P)$ of $C(P)$ by

$$C^{i,j}(P) := \langle S : \text{enhanced states} \mid j(S) = j, i(S) = i\rangle \ \ (i,\ j \in \mathbb{Z}).$$

Remark 4.1. The Jones polynomial $\hat{J}(P) = \sum_{j=-\infty}^{\infty} q^j \sum_{i=-\infty}^{\infty} (-1)^i \mathrm{rk} C^{i,j}(P)$.

Let us define the differential d of bidegree $(1, 0)$ as follows:

$$d(S) = \sum_{\text{enhanced states } T} (S : T)T.$$

In other words, for two arbitrary enhanced states S and T, we define incidence numbers $(S : T)$. We give the definition of differential in the manner as in [11, Section 5]. Assume that the order of letters in the alphabet of a pseudolink P is given.

Definition 4.2. The incidence number $(S : T)$ is zero unless the markers of S and T differ at only one letter of P and this letter is called *the different part between S and T*. The marker of S is positive, and that of T is negative at this different part. If $(S : T) \neq 0$, the different part between S and T satisfy one of the six cases (15)–(20) in the following:

(15)

$$S:(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\ \overset{\cdots A\cdots}{\emptyset_-}\ |\ \overset{\cdots A\cdots}{\emptyset_-}\ |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})$$

$$\rightsquigarrow T:(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\ \overset{\cdots A\cdots}{\emptyset_-}\ |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k});$$

(16)

$$S:(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\ \overset{\cdots A\cdots}{\emptyset_-}\ |\ \overset{\cdots A\cdots}{\emptyset_+}\ |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})$$

$$\rightsquigarrow T:(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\ \overset{\cdots A\cdots}{\emptyset_+}\ |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k});$$

(17)

$$S:(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\ \overset{\cdots A\cdots}{\emptyset_+}\ |\ \overset{\cdots A\cdots}{\emptyset_-}\ |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})$$

$$\rightsquigarrow T:(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\ \overset{\cdots A\cdots}{\emptyset_+}\ |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k});$$

(18)

$$S:(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\ \overset{\cdots A\cdots}{\emptyset_+}\ |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})$$

$$\rightsquigarrow T:(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\ \overset{\cdots A\cdots}{\emptyset_+}\ |\ \overset{\cdots A\cdots}{\emptyset_+}\ |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k});$$

(19)

$$S:(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\ \overset{\cdots A\cdots}{\emptyset_-}\ |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})$$

$$\rightsquigarrow T:(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\ \overset{\cdots A\cdots}{\emptyset_+}\ |\ \overset{\cdots A\cdots}{\emptyset_-}\ |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k});$$

(20)

$$S:(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\ \overset{\cdots A\cdots}{\emptyset_-}\ |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})$$

$$\rightsquigarrow T:(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\ \overset{\cdots A\cdots}{\emptyset_-}\ |\ \overset{\cdots A\cdots}{\emptyset_+}\ |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k});$$

For (15)–(20), $(S:T)$ is defined as

$$(S:T):=1. \tag{21}$$

Theorem 4.1. *$d\circ d=0$ modulo 2.*

Proof. Let ϵ_i be ith marker of ith letter and so ϵ_i is an element of $\{+,-\}$. Consider the tuple $(\epsilon_1, \epsilon_2, \dots, \epsilon_k)$ consisting of all the markers of a phrase. If card $\{j \mid \epsilon_j = +\} \leq 1$, $d^2(S) = 0$. So we can assume that card $\{j \mid \epsilon_j = +\} \geq 2$ now.

To prove

$$d \circ d(S) = \sum_{\text{enhanced states } T,U} (S:T)(T:U)U = 0,$$

we show $\sum_{\text{enhanced states } T}(S:T)(T:U) = 0$.

Let A and B be different parts between S and U. We can assume that the other letters in the phrase are already delated by the deformation $(**)$. We denote phrases which were consisted of letter replaced by the deformation $(**)$ by α_j ($j \in \{1, \cdots k\}$), x,y,z and t. We denote a state S by $S =$ (a phrase P with markers, a pseudolink given by repeating deformation $(**)$ from P to the end). We check following 26 cases:

(1)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|A_+xA_+yB_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\qquad \cdots A \cdots \ \cdots AB \cdots \ \cdots A \cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad | \quad \emptyset_{\varepsilon_{13}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(2)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{A}_+yB_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\qquad \cdots AB \cdots \ \cdots B \cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(3)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{A}_+y\overline{B}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\qquad \cdots AB \cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(4)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|A_+xB_+yA_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\qquad \cdots AB \cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(5)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+xB_+y\overline{A}_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\qquad \cdots AB \cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(6)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{B}_+y\overline{A}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\qquad \cdots AB \cdots \ \cdots AB \cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(7)

$$\begin{aligned} S &= ((\alpha_1|\cdots|\alpha_{l-1}|A_+xA_+y|B_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k),\\ &\qquad\qquad \cdots A\cdots \ \ \cdots A\cdots \ \ \cdots B\cdots \ \ \cdots B\cdots\\ &\quad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad | \quad \emptyset_{\varepsilon_{13}} \quad | \quad \emptyset_{\varepsilon_{14}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{aligned}$$

(8)

$$\begin{aligned} S &= ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{A}_+y|B_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k),\\ &\qquad\qquad \cdots A\cdots \ \ \cdots B\cdots \ \ \cdots B\cdots\\ &\quad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad | \quad \emptyset_{\varepsilon_{13}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{aligned}$$

(9)

$$\begin{aligned} S &= ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{A}_+y|\overline{B}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k),\\ &\qquad\qquad \cdots A\cdots \ \ \cdots B\cdots\\ &\quad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{aligned}$$

(10)

$$\begin{aligned} S &= ((\alpha_1|\cdots|\alpha_{l-1}|A_+xB_+y|A_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k),\\ &\qquad\qquad \cdots AB\cdots \ \ \cdots AB\cdots\\ &\quad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad | \quad \emptyset_{\varepsilon_{12}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{aligned}$$

(11)

$$\begin{aligned} S &= ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+xB_+y|\overline{A}_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k),\\ &\qquad\qquad \cdots AB\cdots\\ &\quad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{aligned}$$

(12)

$$\begin{aligned} S &= ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{B}_+y|\overline{A}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k),\\ &\qquad\qquad \cdots AB\cdots \ \ \cdots AB\cdots\\ &\quad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad | \quad \emptyset_{\varepsilon_{12}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{aligned}$$

(13)

$$\begin{aligned} S &= ((\alpha_1|\cdots|\alpha_{l-1}|A_+|xA_+yB_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k),\\ &\qquad\qquad \cdots AB\cdots \ \ \cdots AB\cdots\\ &\quad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad | \quad \emptyset_{\varepsilon_{12}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{aligned}$$

(14)

$$\begin{aligned} S &= ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x|\overline{A}_+yB_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k),\\ &\qquad\qquad \cdots AB\cdots \ \ \cdots AB\cdots\\ &\quad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad | \quad \emptyset_{\varepsilon_{12}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{aligned}$$

(15)

$$\begin{aligned} S &= ((\alpha_1|\cdots|\alpha_{l-1}|A_+x|A_+y\overline{B}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k),\\ &\qquad\qquad \cdots AB\cdots\\ &\quad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{aligned}$$

(16)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x|\overline{A}_+y\overline{B}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(17)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_+x|A_+y|B_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots A\cdots \ \cdots B\cdots \ \cdots B\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad | \quad \emptyset_{\varepsilon_{13}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(18)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x|\overline{A}_+y|B_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots A\cdots \ \cdots B\cdots \ \cdots B\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad | \quad \emptyset_{\varepsilon_{13}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(19)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_+x|A_+y|\overline{B}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots A\cdots \ \cdots B\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(20)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x|\overline{A}_+y|\overline{B}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots A\cdots \ \cdots B\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(21)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_+x|B_+y|A_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(22)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x|B_+y|\overline{A}_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(23)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x|\overline{B}_+y|\overline{A}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(24)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_+x|A_+y|B_+z|B_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots A\cdots \ \cdots B\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

(25)

$$
\begin{array}{rcl}
S & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x|\overline{A}_+y|B_+z|B_+t|\alpha_{l+1}|\cdots|\alpha_k), \\
& & \qquad\qquad\quad \cdots A\cdots \ \ \cdots B\cdots \\
& & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k}))
\end{array}
$$

(26)

$$
\begin{array}{rcl}
S & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x|\overline{A}_+y|\overline{B}_+z|\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k), \\
& & \qquad\qquad\quad \cdots A\cdots \ \ \cdots B\cdots \\
& & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})).
\end{array}
$$

- Consider the case (1)
  Let

$$
\begin{array}{rcl}
U & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_-xA_-yB_-zB_-t|\alpha_{l+1}|\cdots|\alpha_k), \\
& & \qquad\qquad\qquad \cdots AB\cdots \\
& & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{41}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})).
\end{array}
$$

It is sufficient to show that for each $(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}) \in \{+,-\} \times \{+,-\} \times \{+,-\}$, the coefficient of U in $d^2(S)$ is even for all $\varepsilon_{41} \in \{+,-\}$. Hence for S and U, we have to check the total number of ways to get U from S is even (we denote the condition by $(\sharp)$). Let us localize the problem of the difference parts of S, A_+ and B_+. Two routes (i) and (ii) can be found to change A_+ (respectively B_+) into A_- (respectively B_+) as follows:

(i)

$$
\begin{array}{rcl}
S & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_+xA_+yB_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\
& & \qquad\qquad\quad \cdots A\cdots \ \ \cdots AB\cdots \ \ \cdots A\cdots \\
& & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \qquad | \quad \emptyset_{\varepsilon_{13}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \\
\\
\to T & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_-xA_-yB_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\
& & \qquad\qquad\quad \cdots AB\cdots \ \ \cdots B\cdots \\
& & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{21}} \qquad | \quad \emptyset_{\varepsilon_{22}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U,
\end{array}
$$

(ii)

$$
\begin{array}{rcl}
S & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_+xA_+yB_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\
& & \qquad\qquad\quad \cdots A\cdots \ \ \cdots AB\cdots \ \ \cdots A\cdots \\
& & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \qquad | \quad \emptyset_{\varepsilon_{13}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \\
\\
\to T & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_+xA_+yB_-zB_-t|\alpha_{l+1}|\cdots|\alpha_k), \\
& & \qquad\qquad\quad \cdots A\cdots \ \ \cdots AB\cdots \\
& & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{31}} \quad | \quad \emptyset_{\varepsilon_{32}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U.
\end{array}
$$

Then the condition $(\sharp)$ can be also state that the sum of the contribution of (i) to the coefficient of U and the contribution of (ii) to the coefficient of U is even.

The case $(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}) = (+,+,+)$.

In this case $(S,T) = 0$ for all $(\varepsilon_{21}, \varepsilon_{22})$ and $(\varepsilon_{31}, \varepsilon_{32})$. So the condition $(\sharp)$ holds.

The case $(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}) = (-,+,+)$.

Consider the route (i), (S,T) is not equal to 0 if and only if $(\varepsilon_{21},\varepsilon_{22}) = (+,+)$. Then for this T, $(T,U) = 0$ for all $\varepsilon_{41} \in \{\pm\}$. On the other hand, in route (ii), $(S,T) = 0$ for all $\varepsilon_{31},\varepsilon_{32} \in \{\pm\}$. So the condition ($\sharp$) holds.
The case $(\varepsilon_{11},\varepsilon_{12},\varepsilon_{13}) = (+,-,+)$.

Consider the route (i) (respectively the route (ii)), (S,T) is not equal to 0 if and only if $(\varepsilon_{21},\varepsilon_{22}) = (+,+)$ (respectively $(\varepsilon_{21},\varepsilon_{22}) = (+,+)$). Then for this T, $(T,U) = 0$ for all ε_{41}. So the condition ($\sharp$) holds.
The case $(\varepsilon_{11},\varepsilon_{12},\varepsilon_{13}) = (+,+,-)$.

Consider the route (i), in this route $(S,T) = 0$ for all $\varepsilon_{21},\varepsilon_{22} \in \{\pm\}$. On the other hand, in route (ii), (S,T) is not equal to 0 if and only if $(\varepsilon_{31},\varepsilon_{32}) = (+,+)$. Then for this T, $(T,U) = 0$ for all $\varepsilon_{41} \in \{\pm\}$. So the condition ($\sharp$) holds.
The case $(\varepsilon_{11},\varepsilon_{12},\varepsilon_{13}) = (+,-,-)$.

Consider the route (i), (S,T) is not equal to 0 if and only if $(\varepsilon_{21},\varepsilon_{22}) = (+,-)$. Then for this T, (T,U) is not equal to 0 if and only if $\varepsilon_{41} = +$. Similarly, in route (ii), (S,T) is not equal to 0 for all $(\varepsilon_{31},\varepsilon_{32}) = (+,-)$. Then for this T, (T,U) is not equal to 0 if and only if $\varepsilon_{41} = +$. So the condition ($\sharp$) holds.
The case $(\varepsilon_{11},\varepsilon_{12},\varepsilon_{13}) = (-,-,+)$.

Consider the route (i), (S,T) is not equal to 0 if and only if $(\varepsilon_{21},\varepsilon_{22}) = (-,+)$. Then for this T, (T,U) is not equal to 0 if and only if $\varepsilon_{41} = +$. Similarly, in route (ii), (S,T) is not equal to 0 for all $(\varepsilon_{31},\varepsilon_{32}) = (-,+)$. Then for this T, (T,U) is not equal to 0 if and only if $\varepsilon_{41} = +$. So the condition ($\sharp$) holds.
The case $(\varepsilon_{11},\varepsilon_{12},\varepsilon_{13}) = (-,-,-)$.

Consider the route (i), (S,T) is not equal to 0 if and only if $(\varepsilon_{21},\varepsilon_{22}) = (-,-)$. Then for this T, (T,U) is not equal to 0 if and only if $\varepsilon_{41} = -$. Similarly, in route (ii), (S,T) is not equal to 0 for all $(\varepsilon_{31},\varepsilon_{32}) = (-,-)$. Then for this T, (T,U) is not equal to 0 if and only if $\varepsilon_{41} = -$. So the condition ($\sharp$) holds.

• Consider the case (2)

Let

$$\begin{array}{rcl} U & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_{-}x\overline{A}_{-}yB_{-}zB_{-}t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots A\cdots \quad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{41}} \quad | \quad \emptyset_{\varepsilon_{42}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})). \end{array}$$

It is sufficient to show that for each $(\varepsilon_{11},\varepsilon_{12}) \in \{(\pm,\pm)\}$ where double signs are arbitrary, the coefficient of U in $d^2(S)$ is even for all $\varepsilon_{41},\varepsilon_{42} \in \{\pm\}$. Hence for S and U, we have to check the total number of ways to get U from S is even (we denote the condition by ($\sharp$)). Let us localize the problem of the difference parts of S, A_+ and B_+. Two routes (i) and (ii) can be found to change A_+ (respectively B_+) into A_- (respectively B_+) as follows:
(i)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_{+}x\overline{A}_{+}yB_{+}zB_{+}t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \quad \cdots B\cdots \quad \cdots A\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad | \quad \emptyset_{\varepsilon_{13}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

$$\begin{array}{rcl}\to T & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_- x\overline{A}_- yB_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots A\cdots \ \ \cdots AB\cdots \ \ \cdots B\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{21}} \quad | \quad \emptyset_{\varepsilon_{22}} \quad | \quad \emptyset_{\varepsilon_{23}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U,\end{array}$$

(ii)

$$\begin{array}{rcl}S & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+ x\overline{A}_+ yB_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \ \ \cdots B\cdots \ \ \cdots A\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad | \quad \emptyset_{\varepsilon_{13}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k}))\end{array}$$

$$\begin{array}{rcl}\to T & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+ x\overline{A}_+ yB_-zB_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{31}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U.\end{array}$$

Then the condition ($\sharp$) can be also state that the sum of the contribution of (i) to the coefficient of U and the contribution of (ii) to the coefficient of U is even.
The case $(\varepsilon_{11}, \varepsilon_{12}) = (+,+)$.

Consider the route (i). Then (S,T) is not equal to 0 if and only if $(\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (+,+,+)$. For this T, $(T,U) = 0$ for all $\epsilon_{41}, \varepsilon_{42} \in \{\pm\}$. On the other hand, in route (ii), we obtain $(S,T) = 0$ for all $\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23} \in \{\pm\}$. So the condition ($\sharp$) holds.
The case $(\varepsilon_{11}, \varepsilon_{12}) = (+,-)$.

Consider the route (i). Then (S,T) is not equal to 0 if and only if $(\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (+,+,+)$. For this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+,+)$. Consider the route (ii). Then (S,T) is not equal to 0 if and only if $\varepsilon_{31} = +$. For this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+,+)$. So the condition ($\sharp$) holds.
The case $(\varepsilon_{11}, \varepsilon_{12}) = (-,+)$.

Consider the route (i). Then (S,T) is not equal to 0 if and only if $(\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (-,+,+)$ or $(\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (+,-,+)$. Put $(\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (-,+,+)$. For this T, we obtain $(T,U) = 0$ for all $\varepsilon_{41}, \varepsilon_{42} \in \{\pm\}$. Put $(\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (+,-,+)$. For this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+,+)$. Consider the route (ii). Then (S,T) is not equal to 0 if and only if $\varepsilon_{31} = +$. For this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+,+)$. So the condition ($\sharp$) holds. The case $(\varepsilon_{11}, \varepsilon_{12}) = (-,-)$.

Consider the route (i). Then (S,T) is not equal to 0 if and only if $(\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (-,+,-)$ or $(\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (+,-,-)$. Put $(\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (-,+,-)$. For this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (-,+)$. Put $(\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (+,-,-)$. For this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+,-)$. Consider the route (ii). Then (S,T) is not equal to 0 if and only if $\varepsilon_{31} = -$. For this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+,-)$ or $(\varepsilon_{41}, \varepsilon_{42}) = (-,+)$. So the condition ($\sharp$) holds.

- Consider the case (3)

Let

$$\begin{array}{rcl}U & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_- x\overline{A}_- y\overline{B}_- z\overline{B}_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots A\cdots \ \ \cdots AB\cdots \ \ \cdots B\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{41}} \quad | \quad \emptyset_{\varepsilon_{42}} \quad | \quad \emptyset_{\varepsilon_{43}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})).\end{array}$$

It is sufficient to show that for each $\varepsilon_{11} \in \{\pm\}$ the coefficient of U in $d^2(S)$ is even for all $\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43} \in \{\pm\}$. Hence for S and U, we have to check the total number of ways to get U from S is even (we denote the condition by ($\sharp$)). Let us localize the problem of the difference parts of S, A_+ and B_+. Two routes (i) and (ii) can be found to change A_+ (respectively B_+) into A_- (respectively B_+) as follows:
(i)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{A}_+y\overline{B}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

$$\begin{array}{rcl} \to T & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_-x\overline{A}_-y\overline{B}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots A\cdots \ \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{21}} \quad | \quad \emptyset_{\varepsilon_{22}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U, \end{array}$$

(ii)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{A}_+y\overline{B}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

$$\begin{array}{rcl} \to T & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{A}_+y\overline{B}_-z\overline{B}_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \ \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{31}} \quad | \quad \emptyset_{\varepsilon_{32}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U. \end{array}$$

Then the condition ($\sharp$) can be also state that the sum of the contribution of (i) to the coefficient of U and the contribution of (ii) to the coefficient of U is even.
The case $\varepsilon_{11} = +$.

Consider the route (i). Then (S,T) is not equal to 0 if and only if $(\varepsilon_{21}, \varepsilon_{22}) = (+,+)$. For this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (+,+,+)$. Consider the route (ii). Then (S,T) is not equal to 0 if and only if $(\varepsilon_{31}, \varepsilon_{32}) = (+,+)$. For this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (+,+,+)$. So in this case the condition ($\sharp$) holds.
The case $\varepsilon_{11} = -$.

Consider the route (i). In this case (S,T) is not equal to 0 if and only if $(\varepsilon_{21}, \varepsilon_{22}) = (+,-)$ or $(\varepsilon_{21}, \varepsilon_{22}) = (-,+)$. Put $(\varepsilon_{21}, \varepsilon_{22}) = (+,-)$, then for this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (+,+,-)$. Put $(\varepsilon_{21}, \varepsilon_{22}) = (-,+)$, then for this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (-,+,+)$ or $(\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (+,-,+)$. On the other hand, in route (ii) (S,T) is not equal to 0 if and only if $(\varepsilon_{31}, \varepsilon_{32}) = (+,-)$ or $(\varepsilon_{31}, \varepsilon_{32}) = (-,+)$. Put $(\varepsilon_{31}, \varepsilon_{32}) = (+,-)$, then for this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (+,+,-)$. Put $(\varepsilon_{31}, \varepsilon_{32}) = (-,+)$, then for this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (-,+,+)$ or $(\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (+,-,+)$. So in this case the condition ($\sharp$) holds.
• Consider the case (4)

Let

$$\begin{array}{rcl} U & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_-xB_-yA_-zB_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{41}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})). \end{array}$$

It is sufficient to show that for each $\varepsilon_{11} \in \{\pm\}$ the coefficient of U in $d^2(S)$ is even for all $\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43} \in \{\pm\}$. Hence for S and U, we have to check the total number of ways to get U from S is even (we denote the condition by ($\sharp$)). Let us localize the problem of the difference parts of S, A_+ and B_+. Two routes (i) and (ii) can be found to change A_+ (respectively B_+) into A_- (respectively B_+) as follows:
(i)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_+xB_+yA_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \\ & & \\ \to T & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_-xB_+yA_-zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{21}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U, \end{array}$$

(ii)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_+xB_+yA_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \\ & & \\ \to T & = & ((\alpha_1|\cdots|\alpha_{l-1}|A_+xB_-yA_+zB_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{31}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U. \end{array}$$

Then the condition ($\sharp$) can be also state that the sum of the contribution of (i) to the coefficient of U and the contribution of (ii) to the coefficient of U is even.
In this case, it is clear that $(S,T)(T,U) = 0$ for all T by the definition of d.

- Consider the case (5)

Let

$$\begin{array}{rcl} U & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_-xB_-y\overline{A}_-zB_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB\cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{41}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})). \end{array}$$

It is sufficient to show that for each $\varepsilon_{11} \in \{\pm\}$ the coefficient of U in $d^2(S)$ is even for all $\varepsilon_{41} \in \{\pm\}$. Hence for S and U, we have to check the total number of ways to get U from S is even (we denote the condition by ($\sharp$)). Let us localize the problem of the difference parts of S, A_+ and B_+. Two routes (i) and (ii) can be found to change A_+ (respectively B_+) into A_- (respectively B_+) as follows:

(i)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+xB_+y\overline{A}_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\qquad \cdots AB\cdots \\ && \;(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

$$\begin{array}{rcl} \to T &=& ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_-xB_+y\overline{A}_-zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\qquad \cdots AB\cdots \\ && \;(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{21}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U, \end{array}$$

(ii)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+xB_+y\overline{A}_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\qquad \cdots AB\cdots \\ && \;(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

$$\begin{array}{rcl} \to T &=& ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+xB_-y\overline{A}_+zB_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\qquad \cdots AB\cdots \quad \cdots AB\cdots \\ && \;(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{31}} \quad | \quad \emptyset_{\varepsilon_{32}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U. \end{array}$$

Then the condition ($\sharp$) can be also state that the sum of the contribution of (i) to the coefficient of U and the contribution of (ii) to the coefficient of U is even.
The case $\varepsilon_{11} = +$

On the route (i), we obtain $(S,T) = 0$ for all T by the definition of d. Consider the route (ii). In this case (S,T) is not equal to 0 if and only if $(\varepsilon_{31}, \varepsilon_{32}) = (+,+)$. For this T, $(T,U) = 0$ for all $\varepsilon_{41} \in \{\pm\}$. So in this case the condition ($\sharp$) holds.
The case $\varepsilon_{11} = -$

Consider the route (i). Then $(S,T) = 0$ for all T. Consider the route (ii). Then (S,T) is not equal to 0 if and only if $(\varepsilon_{31}, \varepsilon_{32}) = (+,-)$ or $(\varepsilon_{31}, \varepsilon_{32}) = (-,+)$. Put $(\varepsilon_{31}, \varepsilon_{32}) = (+,-)$. Then for this T, we obtain (T,U) is not equal to 0 if and only if $\varepsilon_{41} = +$. Put $(\varepsilon_{31}, \varepsilon_{32}) = (-,+)$. Then for this T, we obtain (T,U) is not equal to 0 if and only if $\varepsilon_{41} = +$. So the condition ($\sharp$) holds.

• Consider the case (6)

Let

$$\begin{array}{rcl} U &=& ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_-x\overline{B}_-y\overline{A}_-z\overline{B}_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\qquad \cdots AB\cdots \\ && \;(\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{41}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})). \end{array}$$

It is sufficient to show that for each $(\varepsilon_{11}, \varepsilon_{12}) \in \{(\pm,\pm)\}$ where double signs are arbitrary, the coefficient of U in $d^2(S)$ is even for all $\varepsilon_{41} \in \{\pm\}$. Hence for S and U, we have to check the total number of ways to get U from S is even (we denote the condition by ($\sharp$)). Let us localize the problem of the difference parts of S, A_+ and B_+. Two routes (i) and (ii) can be found to change A_+ (respectively B_+) into A_- (respectively B_+) as follows:

(i)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{B}_+y\overline{A}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\quad \cdots AB\cdots \ \cdots AB\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

$$\begin{array}{rcl} \to T &=& ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_-x\overline{B}_+y\overline{A}_-z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\quad \cdots AB\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{21}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U, \end{array}$$

(ii)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{B}_+y\overline{A}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\quad \cdots AB\cdots \ \cdots AB\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

$$\begin{array}{rcl} \to T &=& ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{B}_-y\overline{A}_+z\overline{B}_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\quad \cdots AB\cdots \ \cdots AB\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{31}} \quad | \quad \emptyset_{\varepsilon_{32}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U. \end{array}$$

Then the condition ($\sharp$) can be also state that the sum of the contribution of (i) to the coefficient of U and the contribution of (ii) to the coefficient of U is even.
In this case, we can easily check that $(S,T)(T,U) = 0$ for all T by the definition of T.

• Consider the case (7)
Let

$$\begin{array}{rcl} U &=& ((\alpha_1|\cdots|\alpha_{l-1}|A_-xA_-y|B_-zB_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\quad \cdots A\cdots \ \cdots AB\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{41}} \quad | \quad \emptyset_{\varepsilon_{42}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})). \end{array}$$

It is sufficient to show that for each $(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{14}) \in \{(\pm,\pm,\pm,\pm)\}$ where double signs are arbitrary, the coefficient of U in $d^2(S)$ is even for all $\varepsilon_{41}, \varepsilon_{42} \in \{\pm\}$. Hence for S and U, we have to check the total number of ways to get U from S is even (we denote the condition by ($\sharp$)). Let us localize the problem of the difference parts of S, A_+ and B_+. Two routes (i) and (ii) can be found to change A_+ (respectively B_+) into A_- (respectively B_+) as follows:
(i)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|A_+xA_+y|B_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\quad \cdots A\cdots \ \cdots A\cdots \ \cdots B\cdots \ \cdots B\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad | \quad \emptyset_{\varepsilon_{13}} \quad | \quad \emptyset_{\varepsilon_{14}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

$$\begin{array}{rcl} \to T &=& ((\alpha_1|\cdots|\alpha_{l-1}|A_-xA_-y|B_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \qquad\quad \cdots A\cdots \ \cdots B\cdots \ \cdots B\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{21}} \quad | \quad \emptyset_{\varepsilon_{22}} \quad | \quad \emptyset_{\varepsilon_{23}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U, \end{array}$$

(ii)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|A_+xA_+y|B_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \quad\quad\quad\quad \cdots A\cdots \quad \cdots A\cdots \quad \cdots B\cdots \quad \cdots B\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad | \quad \emptyset_{\varepsilon_{13}} \quad | \quad \emptyset_{\varepsilon_{14}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \\ \\ \to T &=& ((\alpha_1|\cdots|\alpha_{l-1}|A_+xA_+y|B_-zB_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \quad\quad\quad\quad \cdots A\cdots \quad \cdots A\cdots \quad \cdots B\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{31}} \quad | \quad \emptyset_{\varepsilon_{32}} \quad | \quad \emptyset_{\varepsilon_{33}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U. \end{array}$$

Then the condition ($\sharp$) can be also state that the sum of the contribution of (i) to the coefficient of U and the contribution of (ii) to the coefficient of U is even.
In this case we can easily check that the condition ($\sharp$) holds since empty words which relates A and empty words which relates B are independent.

- Consider the cases (8) and (9).
  In this cases the condition ($\sharp$) holds similarly as the case (7).
- Consider the case (10)
  Let

$$\begin{array}{rcl} U &=& ((\alpha_1|\cdots|\alpha_{l-1}|A_-xB_-y|A_-zB_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \quad\quad\quad\quad \cdots AB\cdots \quad \cdots AB\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{41}} \quad | \quad \emptyset_{\varepsilon_{42}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})). \end{array}$$

It is sufficient to show that for each $(\varepsilon_{11}, \varepsilon_{12}) \in \{(\pm, \pm)\}$ where double signs are arbitrary, the coefficient of U in $d^2(S)$ is even for all $\varepsilon_{41}, \varepsilon_{42} \in \{\pm\}$. Hence for S and U, we have to check the total number of ways to get U from S is even (we denote the condition by ($\sharp$)). Let us localize the problem of the difference parts of S, A_+ and B_+. Two routes (i) and (ii) can be found to change A_+ (respectively B_+) into A_- (respectively B_+) as follows:
(i)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|A_+xB_+y|A_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \quad\quad\quad\quad \cdots AB\cdots \quad \cdots AB\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad |\emptyset_{\epsilon_{l-1}}|\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \\ \\ \to T &=& ((\alpha_1|\cdots|\alpha_{l-1}|A_-xB_+y|A_-zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \quad\quad\quad\quad \cdots AB\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{21}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U, \end{array}$$

(ii)

$$\begin{array}{rcl} S &=& ((\alpha_1|\cdots|\alpha_{l-1}|A_+xB_+y|A_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \quad\quad\quad\quad \cdots AB\cdots \quad \cdots AB\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \quad | \quad \emptyset_{\varepsilon_{12}} \quad |\emptyset_{\epsilon_{l-1}}|\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \\ \\ \to T &=& ((\alpha_1|\cdots|\alpha_{l-1}|A_+xB_-y|A_+zB_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ && \quad\quad\quad\quad \cdots AB\cdots \\ && (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{31}} \quad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U. \end{array}$$

Then the condition ($\sharp$) can be also state that the sum of the contribution of (i) to the coefficient of U and the contribution of (ii) to the coefficient of U is even.
The case $(\varepsilon_{11}, \varepsilon_{12}) = (+,+)$.
In this case, both in the route (i) and in the route (ii), $(S,T) = 0$ for all T. So the condition ($\sharp$) holds.
The case $(\varepsilon_{11}, \varepsilon_{12}) = (+,-)$.
Consider the route (i). In this route (S,T) is not equal to 0 if and only if $\varepsilon_{21} = +$. Then for this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+,+)$. Consider the route (ii). In this route (S,T) is not equal to 0 if and only if $\varepsilon_{31} = +$. Then for this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+,+)$. So the condition ($\sharp$) holds.
The case $(\varepsilon_{11}, \varepsilon_{12}) = (-,+)$.
Consider the route (i). Then (S,T) is not equal to 0 if and only if $\varepsilon_{21} = +$. Moreover for this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+,+)$. Consider the route (ii). Then (S,T) is not equal to 0 if and only if $\varepsilon_{31} = +$. Moreover for this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+,+)$. So the condition ($\sharp$) holds.
The case $(\varepsilon_{11}, \varepsilon_{12}) = (-,-)$.
Consider the route (i). Then (S,T) is not equal to 0 if and only if $\varepsilon_{21} = -$. Moreover for this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+,-)$ or $(\varepsilon_{41}, \varepsilon_{42}) = (-,+)$. Consider the route (ii). Then (S,T) is not equal to 0 if and only if $\varepsilon_{31} = -$. Moreover for this T, (T,U) is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+,-)$. $(\varepsilon_{41}, \varepsilon_{42}) = (-,+)$. So the condition ($\sharp$) holds.
• Consider the case (11)
Let

$$\begin{array}{rcl} U & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_-xB_-y|\overline{A}_-zB_-t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB \cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{41}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})). \end{array}$$

It is sufficient to show that for each $\varepsilon_{11} \in \{\pm\}$. the coefficient of U in $d^2(S)$ is even for all $\varepsilon_{41} \in \{\pm\}$. Hence for S and U, we have to check the total number of ways to get U from S is even (we denote the condition by ($\sharp$)). Let us localize the problem of the difference parts of S, A_+ and B_+. Two routes (i) and (ii) can be found to change A_+ (respectively B_+) into A_- (respectively B_+) as follows:
(i)

$$\begin{array}{rcl} S & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+xB_+y|\overline{A}_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB \cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l-1}}|\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \end{array}$$

$$\begin{array}{rcl} \to T & = & ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_-xB_+y|\overline{A}_-zB_+t|\alpha_{l+1}|\cdots|\alpha_k), \\ & & \qquad\qquad \cdots AB \cdots \quad \cdots AB \cdots \\ & & (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}| \quad \emptyset_{\varepsilon_{21}} \quad | \quad \emptyset_{\varepsilon_{22}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U, \end{array}$$

(ii)

$$\begin{aligned}
S &= ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+xB_+y|\overline{A}_+zB_+t|\alpha_{l+1}|\cdots|\alpha_k),\\
&\qquad\qquad\qquad \cdots AB\cdots\\
&\qquad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\quad \emptyset_{\varepsilon_{11}} \qquad |\emptyset_{\epsilon_{l-1}}|\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k}))\\
\to T &= ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+xB_-y|\overline{A}_+zB_-t|\alpha_{l+1}|\cdots|\alpha_k),\\
&\qquad\qquad\qquad \cdots AB\cdots \ \cdots AB\cdots\\
&\qquad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\quad \emptyset_{\varepsilon_{31}} \qquad | \quad \emptyset_{\varepsilon_{32}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U.
\end{aligned}$$

In this case we can chose ε_{21}, ε_{22}, ε_{31} ,$\varepsilon_{32} \in \{\pm\}$ so that $(\varepsilon_{21}, \varepsilon_{22}) = (\varepsilon_{31}, \varepsilon_{32})$ and (S, T) is not equal to 0. Moreover Ts in the route (i) and in the route (ii) have same form. So the condition ($\sharp$) holds.

• Consider the case (12)

Let

$$\begin{aligned}
U &= ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_-x\overline{B}_-y|\overline{A}_-z\overline{B}_-t|\alpha_{l+1}|\cdots|\alpha_k),\\
&\qquad\qquad\qquad \cdots AB\cdots \ \cdots AB\cdots\\
&\qquad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\quad \emptyset_{\varepsilon_{41}} \qquad | \quad \emptyset_{\varepsilon_{42}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})).
\end{aligned}$$

Then the condition ($\sharp$) can be also state that the sum of the contribution of (i) to the coefficient of U and the contribution of (ii) to the coefficient of U is even.

It is sufficient to show that for each $(\varepsilon_{11}, \varepsilon_{12}) \in \{\pm, \pm\}$ where double signs are arbitrary, the coefficient of U in $d^2(S)$ is even for all $\varepsilon_{41}, \varepsilon_{42} \in \{\pm\}$. Hence for S and U, we have to check the total number of ways to get U from S is even (we denote the condition by ($\sharp$)). Let us localize the problem of the difference parts of S, A_+ and B_+. Two routes (i) and (ii) can be found to change A_+ (respectively B_+) into A_- (respectively B_+) as follows:

(i)

$$\begin{aligned}
S &= ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{B}_+y|\overline{A}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k),\\
&\qquad\qquad\qquad \cdots AB\cdots \ \cdots AB\cdots\\
&\qquad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\quad \emptyset_{\varepsilon_{11}} \qquad | \quad \emptyset_{\varepsilon_{12}} \qquad |\emptyset_{\epsilon_{l-1}}|\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k}))\\
\to T &= ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_-x\overline{B}_+y|\overline{A}_-z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k),\\
&\qquad\qquad\qquad \cdots AB\cdots\\
&\qquad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\quad \emptyset_{\varepsilon_{21}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U,
\end{aligned}$$

(ii)

$$\begin{aligned}
S &= ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{B}_+y|\overline{A}_+z\overline{B}_+t|\alpha_{l+1}|\cdots|\alpha_k),\\
&\qquad\qquad\qquad \cdots AB\cdots \ \cdots AB\cdots\\
&\qquad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\quad \emptyset_{\varepsilon_{11}} \qquad | \quad \emptyset_{\varepsilon_{12}} \qquad |\emptyset_{\epsilon_{l-1}}|\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k}))\\
\to T &= ((\alpha_1|\cdots|\alpha_{l-1}|\overline{A}_+x\overline{B}_-y|\overline{A}_+z\overline{B}_-t|\alpha_{l+1}|\cdots|\alpha_k),\\
&\qquad\qquad\qquad \cdots AB\cdots\\
&\qquad (\emptyset_{\epsilon_1}|\cdots|\emptyset_{\epsilon_{l-1}}|\quad \emptyset_{\varepsilon_{31}} \qquad |\emptyset_{\epsilon_{l+1}}|\cdots|\emptyset_{\epsilon_k})) \to U.
\end{aligned}$$

Then the condition ($\sharp$) can be also state that the sum of the contribution of (i) to the coefficient of U and the contribution of (ii) to the coefficient of U is even.

This case is completely same as the case (ii).
•The cases (13) - (23)
We can easily check the condition (♯) by the definition of d.
•The cases (24) - (26)
In this case we can prove the condition (♯) holds same as the case (7).
Now we have completed the proof. □

Definition 4.3. We denote the mapping d modulo $2 : C^{i,j}(P;\mathbb{Z}_2) \to C^{i+1,j}(P;\mathbb{Z}_2)$ by d_2^i for i and j. The Khovanov homology group $KH^{i,j}(P)$ for a pseudolink P is defined as

$$KH^{i,j}(P) := \mathrm{Ker} d_2^i / \mathrm{Im} d_2^{i-1}. \tag{22}$$

Remark 4.2. $KH^{i,j}(P)$ is independent of the order of the letters of P because the incidence number $(S : T)$ is always either 0 or 1 modulo 2 for enhanced states S and T.

5. Invariance under S_1-homotopy moves.

Theorem 5.1. *$KH^{i,j}(P)$ is S_1-homotopy invariants for pseudolinks.*

Proof. By construction of $KH^{i,j}(P)$, $KH^{i,j}(P)$ is not depend on an arbitrary isomorphism of P. Then $KH^{i,j}$ is invariant under isomorphisms. It remains to prove that if a nanophrase P is obtained from a nanophrase P' by a homotopy move then $KH^{i,j}(P') \simeq KH^{i,j}(P)$.

(I) Consider the first homotopy move $(xAAy) \to (xy)$ and its inverse move where $|A| = 1$. For P' and P, denote by $S_+(\epsilon, \eta)$ the state $(u|\overset{Aw}{\emptyset_\epsilon}|\overset{A}{\emptyset_\eta}|v)$ of P' with mark(A) $= 1$ and denote by $S_-(\epsilon)$ the state $(u|\overset{Aw}{\emptyset_\epsilon}|v)$ of P' with mark$(A) = -1$ where ϵ, η $\in \{+,-\}$. The subcomplex C' of $C(P')$ is defined by $C' := C(S_+(+,+),\ S_+(+,-)$ $-\ S_+(-,+))$.

First, the retraction

$$\rho : C(P') \to C(S_+(+,+),\ S_+(+,-) - S_+(-,+))$$

is defined by the formulas

$$\begin{aligned}
&S_+(+,+) \mapsto S_+(+,+),\\
&S_+(-,+) \mapsto S_+(-,+) - S_+(+,-),\\
&\text{otherwise} \mapsto 0.
\end{aligned}$$

Second, the isomorphism

$$C(S_+(+,+), S_+(+,-) - S_+(-,+)) \to C(P) = C((u|\overset{w}{\emptyset_+}|v), (u|\overset{w}{\emptyset_-}|v))$$

is defined by the formulas

$$S_+(+,+) \mapsto (u|\overset{w}{\emptyset_+}|v),$$

$$S_+(+,-) - S_+(-,+) \mapsto (u|\overset{w}{\emptyset_-}|v).$$

Third, consider the following composition of this isomorphism with ρ

$$C(P') \xrightarrow{\rho} C' \overset{\text{isom}}{\to} C(P).$$

The map $h : C(P') \to C(P')$ such that $d \circ h + h \circ d = \mathrm{id} - \mathrm{in} \circ \rho$, is defined by the formulas

$$\begin{aligned} S_-(+) &\mapsto S_+(+,-),\\ S_-(-) &\mapsto S_+(-,-),\\ \text{otherwise} &\mapsto 0. \end{aligned}$$

(II) Consider the second homotopy move $P' = (xAByBAz) \to (xyz) = P$ and its inverse move where $(|A|, |B|) = (1, -1)$. It is necessary to consider two distinct cases (II-1), (II-2) as follows.

(II–1) Consider the case where the state of P' with (mark(A), mark(B)) $= (1, 1)$ is represented as $(u|\ \overset{ABw}{\emptyset_\epsilon}\ |v)$.

Denote by $S_{+-}(\epsilon, \eta)$ the state $(u|\ \overset{Aw}{\emptyset_\epsilon} \mid \overset{AB}{\emptyset_\eta}\ |v)$ of P' with (mark(A), mark(B)) $= (1, -1)$, denote by $S_{-+}(\epsilon, \eta)$ the state $(u|\ \overset{ABw}{\emptyset_\epsilon} \mid \overset{ABt}{\emptyset_\eta}\ |v)$ of P' with (mark(A), mark(B)) $= (-1,\ 1)$, denote by $S_{++}(\epsilon)$ the state $(u|\ \overset{ABw}{\emptyset_\epsilon}\ |v)$ of P' with (mark(A), mark(B)) $= (1,\ 1)$ and denote by $S_{--}(\epsilon)$ the state $(u|\ \overset{ABw}{\emptyset_\epsilon}\ |v)$ of P' with (mark(A), mark(B)) $= (-1,\ -1)$ where $\epsilon,\ \eta \in \{+, -\}$. The subcomplex C' of $C(P')$ is defined by $C' := C\big(S_{-+}(+,+),\ S_{-+}(+,-) + S_{+-}(+,-),\ S_{-+}(-,+) + S_{+-}(-,-),\ S_{-+}(-,-) + S_{+-}(-,-)\big)$.

First, the retraction $\rho : C(P') \to C'$ is defined by the formulas

$$\begin{aligned} S_{-+}(+,+) &\mapsto S_{-+}(+,+),\\ S_{-+}(+,-) &\mapsto S_{-+}(+,-) + S_{+-}(+),\\ S_{-+}(-,+) &\mapsto S_{-+}(-,+) + S_{+-}(+),\\ S_{-+}(-,-) &\mapsto S_{-+}(-,-) + S_{+-}(-),\\ S_{+-}(+,+) &\mapsto S_{-+}(+,+),\\ S_{+-}(-,+) &\mapsto S_{-+}(+,-) + S_{-+}(-,+),\\ \text{otherwise} &\mapsto 0. \end{aligned}$$

Second, the isomorphism

$$C' \to C(P) = C((u|\overset{w}{\emptyset_\epsilon}|\overset{t}{\emptyset_\eta}|v))$$

is defined by the formulas

$$
\begin{aligned}
S_{-+}(+,+) &\mapsto (u|\overset{w}{\emptyset_+}|\overset{t}{\emptyset_+}|v),\\
S_{-+}(+,-)+S_{+-}(+) &\mapsto (u|\overset{w}{\emptyset_+}|\overset{t}{\emptyset_-}|v),\\
S_{-+}(-,+)+S_{+-}(+) &\mapsto (u|\overset{w}{\emptyset_-}|\overset{t}{\emptyset_+}|v),\\
S_{-+}(-,-)+S_{+-}(-) &\mapsto (u|\overset{w}{\emptyset_-}|\overset{t}{\emptyset_-}|v).
\end{aligned}
$$

Third, consider the following composition of this isomorphism with ρ

$$C(P') \xrightarrow{\rho} C' \overset{\text{isom}}{\to} C(P).$$

The map $h : C(P') \to C(P')$ such that $d \circ h + h \circ d = \mathrm{id} - \mathrm{in} \circ \rho$, is defined by the formulas

$$
\begin{aligned}
S_{--}(\epsilon) &\mapsto S_{+-}(\epsilon,-),\\
S_{+-}(\epsilon,+) &\mapsto S_{++}(\epsilon),\\
\text{otherwise} &\mapsto 0.
\end{aligned}
$$

(II–2) Consider the case where the state of P' with (mark(A), mark(B)) $= (1,\,1)$ is represented as $(u|\overset{Aw}{\emptyset_\epsilon}|\overset{ABt}{\emptyset_\eta}|v)$.

Denote by $S_{+-}(\epsilon,\zeta,\eta)$ the state $(u|\overset{Aw}{\emptyset_\epsilon}|\overset{AB}{\emptyset_\zeta}|\overset{Bt}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B)) $= (1,\,-1)$, denote by $S_{-+}(\epsilon)$ the state $(u|\overset{ABwt'}{\emptyset_\epsilon}|v)$ of P' with (mark(A), mark(B)) $=$ $(-1,\,1)$, denote by $S_{++}(\epsilon)$ the state $(u|\overset{Aw}{\emptyset_\epsilon}|\overset{ABt}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B)) $= (1,\,1)$ and denote by $S_{--}(\epsilon,\eta)$ the state $(u|\overset{ABw}{\emptyset_\epsilon}|\overset{Bt}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B)) $= (-1,\,-1)$ where $\epsilon,\,\eta \in \{+,-\}$ and the word t' is obtained by deleting from t all letters which appear in w. The subcomplex C' of $C(P')$ is defined by $C' := C\big(\, S_{-+}(+) + S_{+-}(+,-,+),\, S_{-+}(-) + S_{+-}(+,-,-) + S_{+-}(-,-,+)\,\big)$.

First, the retraction $\rho : C(P') \to C'$ is defined by the formulas

$$
\begin{aligned}
S_{-+}(+) &\mapsto S_{-+}(+) + S_{+-}(+,-,+),\\
S_{-+}(-) &\mapsto S_{-+}(-) + S_{+-}(+,-,-) + S_{+-}(-,-,+),\\
S_{+-}(+,+,-) &\mapsto S_{-+}(+) + S_{+-}(+,-,+),\\
S_{+-}(-,+,+) &\mapsto S_{-+}(+) + S_{+-}(+,-,+),\\
S_{+-}(-,+,-) &\mapsto S_{-+}(-) + S_{+-}(+,-,-) + S_{+-}(-,-,+),\\
\text{otherwise} &\mapsto 0.
\end{aligned}
$$

Second, the isomorphism

$$C' \to C(P) = C((u|\overset{wt'}{\emptyset_\epsilon}|v))$$

is defined by the formulas

$$S_{-+}(+) + S_{+-}(+,-,+) \mapsto (u|\overset{wt'}{\emptyset_+}|v),$$
$$S_{-+}(-) + S_{+-}(+,-,-) + S_{+-}(-,-,+) \mapsto (u|\overset{wt'}{\emptyset_-}|v).$$

Third, consider the following composition of this isomorphism with ρ

$$C(P') \xrightarrow{\rho} C' \xrightarrow{\text{isom}} C(P).$$

The map $h : C(P') \to C(P')$ such that $d \circ h + h \circ d = \mathrm{id} - \mathrm{in} \circ \rho$, is defined by the formulas

$$S_{--}(\epsilon, \eta) \mapsto S_{+-}(\epsilon, -, \eta),$$
$$S_{+-}(\epsilon, +, \eta) \mapsto S_{++}(\epsilon, \eta),$$
$$\text{otherwise} \mapsto 0.$$

By using (II-1) and (II-2), we proved $KH^{i,j}((xAByBAz)) \simeq KH^{i,j}((xyz))$ if $(|A|, |B|) = (1, -1)$. In addition, the fact is, (II-1) and (II-2) prove that $KH^{i,j}((xAByABz)) \simeq KH^{i,j}((xyz))$ if $(|A|, |B|) = (-1, 1)$. Moreover, by exchanging A, B in the proofs above, (II-1) and (II-2) prove that $KH^{i,j}((xAByBAz)) \simeq KH^{i,j}((xyz))$ if $(|A|, |B|) = (-1, 1)$ and $KH^{i,j}((xAByABz)) \simeq KH^{i,j}((xyz))$ if $(|A|, |B|) = (1, -1)$.

Here, consider

$$xAAy \overset{\mathrm{H1}}{\sim} xABBAy \quad \text{with } |A| = -1, |B| = 1$$
$$\overset{\mathrm{H2}}{\sim} xy.$$

We has already showed that the invariance of $KH^{i,j}$ under these moves above and then $KH^{i,j}$ is preserved under the first homotopy move $xAAy \to xy$ with $|A| = -1$ and its inverse move.

(III) Consider the third homotopy move $P' = (xAByACzBCt) \to (xBAyCAzCBt) = P$ and its inverse move where $(|A|, |B|, |C|) = (-1, -1, -1)$. For the letters A, B and C, we define w_{ABC}, w_{AB}, w_{AC}, w_{BC}, w_A, w_B and w_C in the following. Let w_{ABC} be a word containing A, B and C. Let $(X, Y, Z) = \{(A, B, C), (A, C, B), (B, C, A)\}$. Denote by w_{XY} a word containing X and Y and not containing Z and denote by w_Z a word containing Z and not containing X and Y.

(III–1) Consider the case where the state of P' with (mark(A), mark(B), mark(C)) = (1, 1, 1) is represented as $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|v)$.

Denote by $S_{+++}(\epsilon)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |v)$ of P' with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (1,\ 1,\ 1)$, denote by $S_{-++}(\epsilon,\eta)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |\ \overset{w_{AB}}{\emptyset_\eta}\ |v)$ of P' with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (-1,\ 1,\ 1)$, denote by $S_{+-+}(\epsilon,\eta)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |\ \overset{ABC}{\emptyset_\eta}\ |v)$ of P' with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (1,\ -1,\ 1)$, denote by $S_{--+}(\epsilon)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |v)$ of P' with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (-1,\ -1,\ 1)$, denote by $S_{++-}(\epsilon,\eta)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |\ \overset{w_{BC}}{\emptyset_\eta}\ |v)$ of P' with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (1,\ 1,\ -1)$, denote by $S_{-+-}(\epsilon,\zeta,\eta)$ the state $(u|\ \overset{w_{AC}}{\emptyset_\epsilon}\ |\ \overset{w_{AB}}{\emptyset_\zeta}\ |\ \overset{w_{BC}}{\emptyset_\eta}\ |v)$ of P' with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (-1,\ 1,\ -1)$, denote by $S_{+--}(\epsilon)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |v)$ of P' with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (1,\ -1,\ -1)$, denote by $S_{---}(\epsilon,\eta)$ the state $(u|\ \overset{w_{AC}}{\emptyset_\epsilon}\ |\ \overset{w_{ABC}}{\emptyset_\eta}\ |v)$ of P' with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (-1,\ -1,\ -1)$.

The subcomplex C' of $\mathcal{C}(P')$ is defined by $C' := \mathcal{C}\big(\ S_{-++}(+,+),\ S_{-++}(+,-) + S_{+-+}(+,-),\ S_{-++}(-,+) + S_{+-+}(+,-),\ S_{-++}(-,-) + S_{+-+}(-,-),\ S_{**-}\ \big)$ where S_{**-} denotes every states with $\mathrm{mark}(C) = -1$.

Denote by $T_{+-+}(\epsilon,\eta)$ the state $(u|\ \overset{w_{AB}}{\emptyset_\epsilon}\ |\ \overset{w_{AC}}{\emptyset_\eta}\ |v)$ of P with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (1,\ -1,\ 1)$, denote by $T_{-++}(\epsilon,\zeta,\eta,-)$ the state $(u|\ \overset{w_A}{\emptyset_\epsilon}|\ \overset{w_B}{\emptyset_\zeta}|\ \overset{w_C}{\emptyset_\eta}|\ \overset{ABC}{\emptyset_-}\ |v)$ of P with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (-1, 1, 1)$, denote by $T_{++-}(\epsilon,\eta)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |\ \overset{w_B}{\emptyset_\eta}|v)$ of P with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (1,\ 1,\ -1)$, denote by $T_{-+-}(\epsilon,\zeta,\eta)$ the state $(u|\ \overset{w_A}{\emptyset_\epsilon}|\ \overset{w_{ABC}}{\emptyset_\zeta}\ |\ \overset{w_B}{\emptyset_\eta}|v)$ of P with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (-1,\ 1,\ -1)$, denote by $T_{+--}(\epsilon)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |v)$ of P with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (1,\ -1,\ -1)$, denote by $T_{---}(\epsilon,\eta)$ the state $(u|\ \overset{w_A}{\emptyset_\epsilon}|\ \overset{w_{ABC}}{\emptyset_\eta}\ |v)$ of P with $(\mathrm{mark}(A), \mathrm{mark}(B), \mathrm{mark}(C)) = (-1,\ -1,\ -1)$, denote by T_{**-} every states of P with $\mathrm{mark}(C) = -1$.

The subcomplex C of $\mathcal{C}(P)$ is defined by $C := \mathcal{C}\big(\ T_{+-+}(+,\eta) + T_{-++}(+,+,\eta,-),\ T_{+-+}(-,\eta) + T_{-++}(+,-,\eta,-) + T_{-++}(-,+,\eta,-),\ T_{**-}\ \big)$.

First, the retraction $\rho : C(P') \to C'$ is defined by the formulas

$$\begin{aligned}
S_{-++}(+,+) &\mapsto S_{-++}(+,+),\\
S_{-++}(+,-) &\mapsto S_{-++}(+,-) + S_{+-+}(+,-),\\
S_{-++}(-,+) &\mapsto S_{-++}(-,+) + S_{+-+}(+,-),\\
S_{-++}(-,-) &\mapsto S_{-++}(-,-) + S_{+-+}(-,-),\\
S_{**-} &\mapsto S_{**-},\\
S_{+-+}(+,+) &\mapsto S_{-++}(+,+) + S_{++-}(+,+),\\
S_{+-+}(-,+) &\mapsto S_{-++}(+,-) + S_{-++}(-,+) + S_{++-}(+,-) + S_{++-}(-,+),\\
S_{--+}(\epsilon) &\mapsto S_{+--}(\epsilon),\\
\text{otherwise} &\mapsto 0.
\end{aligned}$$

Second, consider the following composition of the following isomorphism with ρ

$$C(P') \xrightarrow{\rho} C' \xrightarrow{\text{isom}} C \xrightarrow{i} C(P). \tag{23}$$

The isomorphism $C' \to C$ is defined by the formulas

$$\begin{aligned}
S_{-++}(+,+) &\mapsto T_{+-+}(+,+) + T_{-++}(+,+,+,-),\\
S_{-++}(+,-) + S_{+-+}(+,-) &\mapsto T_{+-+}(+,-) + T_{-++}(+,+,-,-),\\
S_{-++}(-,+) + S_{+-+}(+,-) &\mapsto T_{+-+}(-,+) + T_{-++}(+,-,+,-) + T_{-++}(-,+,+,-),\\
S_{-++}(-,-) + S_{+-+}(-,-) &\mapsto T_{+-+}(-,-) + T_{-++}(+,-,-,-) + T_{-++}(-,+,-,-),\\
S_{++-}(\epsilon,\eta) &\mapsto T_{++-}(\epsilon,\eta),\\
S_{-+-}(\epsilon,\zeta,\eta) &\mapsto T_{+-+}(\epsilon,\zeta,\eta),\\
S_{+--}(\epsilon) &\mapsto T_{+--}(\epsilon),\\
S_{---}(\epsilon,\eta) &\mapsto T_{---}(\epsilon,\eta).
\end{aligned}$$

Third, the map $h : C(P') \to C(P')$ such that $d \circ h + h \circ d = \mathrm{id} - \mathrm{in} \circ \rho$, is defined by the formulas

$$\begin{aligned}
S_{--+}(\epsilon) &\mapsto S_{+-+}(\epsilon,-),\\
S_{+-+}(\epsilon,+) &\mapsto S_{+++}(\epsilon),\\
\text{otherwise} &\mapsto 0.
\end{aligned}$$

(III–2) Consider the case where the state of P' with (mark(A), mark(B), mark(C)) $= (1,\ 1,\ 1)$ is represented as $(u|\overset{w_{AB}}{\emptyset_\epsilon}|\overset{w_{ABC}}{\emptyset_\eta}|v)$.

Denote by $S_{+++}(\epsilon,\eta)$ the state $(u|\overset{w_{AB}}{\emptyset_\epsilon}|\overset{w_{ABC}}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) $= (1,\ 1,\ 1)$, denote by $S_{-++}(\epsilon)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|v)$ of P' with (mark(A), mark(B), mark(C)) $= (-1,\ 1,\ 1)$, denote by $S_{+-+}(\epsilon,\zeta,\eta)$ the state $(u|\overset{w_{AC}}{\emptyset_\epsilon}|\overset{ABC}{\emptyset_\zeta}$ $\overset{w_B}{|}\emptyset_\eta|v)$ of P' with (mark(A), mark(B), mark(C)) $= (1,\ -1,\ 1)$, denote by $S_{--+}(\epsilon,\eta)$

the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |\ \overset{w_B}{\emptyset_\eta}\ |v)$ of P' with (mark(A), mark(B), mark(C)) = (−1, −1, 1), denote by $S_{++-}(\epsilon)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |v)$ of P' with (mark(A), mark(B), mark(C)) = (1, 1, −1), denote by $S_{-+-}(\epsilon,\eta)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |\ \overset{w_{AC}}{\emptyset_\eta}\ |v)$ of P' with (mark(A), mark(B), mark(C)) = (−1, 1, −1), denote by $S_{+--}(\epsilon,\eta)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |\ \overset{w_{ABC}}{\emptyset_\eta}\ |v)$ of P' with (mark(A), mark(B), mark(C)) = (1, −1, −1), denote by $S_{---}(\epsilon,\zeta,\eta)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |\ \overset{w_{AC}}{\emptyset_\zeta}\ |\ \overset{w_B}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) = (−1, −1, −1).

The subcomplex C' of $C(P')$ is defined by $C' := C\big(\ S_{-++}(+) + S_{+-+}(+,-,+),\ S_{-++}(-) + S_{+-+}(+,-,-) + S_{+-+}(-,-,+),\ S_{**-}\ \big)$ where S_{**-} denotes every states with mark(C) = −1.

Denote by $T_{+-+}(\epsilon)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |v)$ of P with (mark(A), mark(B), mark(C)) = (1, −1, 1), denote by $T_{-++}(\epsilon,\zeta,-)$ the state $(u|\ \overset{w_{BC}}{\emptyset_\epsilon}\ |\ \overset{w_A}{\emptyset_\zeta}|\ \overset{ABC}{\emptyset_-}\ |v)$ of P with (mark(A), mark(B), mark(C)) = (−1, 1, 1), denote by $T_{++-}(\epsilon)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |v)$ of P with (mark(A), mark(B), mark(C)) = (1, 1, −1), denote by $T_{-+-}(\epsilon,\eta)$ the state $(u|\ \overset{w_{BC}}{\emptyset_\epsilon}\ |\ \overset{w_A}{\emptyset_\eta}|v)$ of P with (mark(A), mark(B), mark(C)) = (−1, 1, −1), denote by $T_{+--}(\epsilon,\eta)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |\ \overset{w_{BC}}{\emptyset_\eta}\ |v)$ of P with (mark(A), mark(B), mark(C)) = (1, −1, −1), denote by $T_{---}(\epsilon,\zeta,\eta)$ the state $(u|\ \overset{w_{BC}}{\emptyset_\epsilon}\ |\ \overset{w_A}{\emptyset_\zeta}|\ \overset{w_{BC}}{\emptyset_\eta}\ |v)$ of P with (mark(A), mark(B), mark(C)) = (−1, −1, −1), denote by T_{**-} every states of P with mark(C) = −1.

The subcomplex C of $C(P)$ is defined by $C := C\big(\ T_{+-+}(+) + T_{-++}(+,+,-),\ T_{+-+}(-) + T_{-++}(+,-,-) + T_{-++}(-,+,-),\ T_{**-}\ \big)$.

First, the retraction $\rho : C(P') \to C'$ is defined by the formulas

$$
\begin{aligned}
S_{-++}(+) &\mapsto S_{-++}(+) + S_{+-+}(+,-,+),\\
S_{-++}(-) &\mapsto S_{-++}(-) + S_{+-+}(+,-,-) + S_{+-+}(-,-,+),\\
S_{**-} &\mapsto S_{**-},\\
S_{+-+}(+,+,-) &\mapsto S_{-++}(+) + S_{+-+}(+,-,+) + S_{++-}(+),\\
S_{+-+}(-,+,+) &\mapsto S_{-++}(+) + S_{+-+}(+,-,+) + S_{++-}(+),\\
S_{+-+}(-,+,-) &\mapsto S_{-++}(-) + S_{+-+}(+,-,-) + S_{+-+}(-,-,-) + S_{++-}(-),\\
S_{--+}(\epsilon,\eta) &\mapsto S_{+--}(\epsilon,\eta),\\
\text{otherwise} &\mapsto 0.
\end{aligned}
$$

Second, consider the following composition (23) of the following isomorphism with ρ. The isomorphism $C' \to C$ is defined by the formulas

$$\begin{aligned}
S_{-++}(+) + S_{+-+}(+,-,+) &\mapsto T_{+-+}(+) + T_{-++}(+,+,-),\\
S_{-++}(-) + S_{+-+}(+,-,-) + S_{+-+}(-,-,+) &\mapsto T_{+-+}(-) + T_{-++}(+,-,-)\\
&\qquad\qquad\qquad + T_{-++}(-,+,-),\\
S_{++-}(\epsilon) &\mapsto T_{++-}(\epsilon),\\
S_{-+-}(\epsilon,\eta) &\mapsto T_{-+-}(\epsilon,\eta),\\
S_{+--}(\epsilon,\eta) &\mapsto T_{+--}(\epsilon,\eta),\\
S_{---}(\epsilon,\zeta,\eta) &\mapsto T_{---}(\epsilon,\zeta,\eta).
\end{aligned}$$

Third, the map $h : C(P') \to C(P')$ such that $d \circ h + h \circ d = \mathrm{id} - \mathrm{in} \circ \rho$, is defined by the formulas

$$\begin{aligned}
S_{--+}(\epsilon,\eta) &\mapsto S_{+-+}(\epsilon,-,\eta),\\
S_{+-+}(\epsilon,+,\eta) &\mapsto S_{+++}(\epsilon,\eta),\\
\text{otherwise} &\mapsto 0.
\end{aligned}$$

(III–3) Consider the case where the state of P' with (mark(A), mark(B), mark(C)) $=$ (1, 1, 1) is represented as $(u|\overset{w_A}{\emptyset_\epsilon}|\overset{w_{ABC}}{\emptyset_\eta}|v)$.

Denote by $S_{+++}(\epsilon,\eta)$ the state $(u|\overset{w_A}{\emptyset_\epsilon}|\overset{w_{ABC}}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) $=$ (1, 1, 1), denote by $S_{-++}(\epsilon)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|v)$ of P' with (mark(A), mark(B), mark(C)) $=$ (-1, 1, 1), denote by $S_{+-+}(\epsilon,\zeta,\eta)$ the state $(u|\overset{w_A}{\emptyset_\epsilon}|\overset{ABC}{\emptyset_\zeta}|\overset{w_{BC}}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) $=$ (1, -1, 1), denote by $S_{--+}(\epsilon,\eta)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{w_{BC}}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) $=$ (-1, -1, 1), denote by $S_{++-}(\epsilon,\zeta,\eta)$ the state $(u|\overset{w_A}{\emptyset_\epsilon}|\overset{w_{BC}}{\emptyset_\zeta}|\overset{w_{ABC}}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) $=$ (1, 1, -1), denote by $S_{-+-}(\epsilon,\eta)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{w_{BC}}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) $=$ (-1, 1, -1), denote by $S_{+--}(\epsilon,\eta)$ the state $(u|\overset{w_A}{\emptyset_\epsilon}|\overset{w_{ABC}}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) $=$ (1, -1, -1), denote by $S_{---}(\epsilon)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|v)$ of P' with (mark(A), mark(B), mark(C)) $=$ (-1, -1, -1).

The subcomplex C' of $C(P')$ is defined by $C' := C\big(\, S_{-++}(+) + S_{+-+}(+,-,+),\ S_{-++}(-) + S_{+-+}(+,-,-) + S_{+-+}(-,-,+),\ S_{**-} \,\big)$ where S_{**-} denotes every states with mark(C) $= -1$.

Denote by $T_{+-+}(\epsilon)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|v)$ of P with (mark(A), mark(B), mark(C)) $=$ (1, -1, 1), denote by $T_{-++}(\epsilon,-,\eta)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{ABC}{\emptyset_-}|\overset{w_B}{\emptyset_\eta}|v)$ of P with

$(\mathrm{mark}(A),\ \mathrm{mark}(B),\ \mathrm{mark}(C)) = (-1,\ 1,\ 1)$, denote by $T_{++-}(\epsilon, \zeta, \eta)$ the state $(u|\ \overset{w_{AC}}{\emptyset_\epsilon}\ |\ \overset{w_B}{\emptyset_\zeta}|\ \ \overset{w_{ABC}}{\emptyset_\eta}\ |v)$ of P with $(\mathrm{mark}(A),\ \mathrm{mark}(B),\ \mathrm{mark}(C)) = (1,\ 1,\ -1)$, denote by $T_{-+-}(\epsilon, \eta)$ the state $(u|\ \ \overset{w_{ABC}}{\emptyset_\epsilon}\ \ |\ \overset{w_B}{\emptyset_\eta}|v)$ of P with $(\mathrm{mark}(A),\ \mathrm{mark}(B),\ \mathrm{mark}(C)) = (-1,\ 1,\ -1)$, denote by $T_{+--}(\epsilon, \eta)$ the state $(u|\ \overset{w_{AC}}{\emptyset_\epsilon}\ |\ \ \overset{w_{ABC}}{\emptyset_\eta}\ |v)$ of P with $(\mathrm{mark}(A),$ $\mathrm{mark}(B),\ \mathrm{mark}(C)) = (1,\ -1,\ -1)$, denote by $T_{---}(\epsilon)$ the state $(u|\ \ \overset{w_{ABC}}{\emptyset_\epsilon}\ \ |v)$ of P with $(\mathrm{mark}(A),\ \mathrm{mark}(B),\ \mathrm{mark}(C)) = (-1,\ -1,\ -1)$, denote by T_{**-} every states of P with $\mathrm{mark}(C) = -1$.

The subcomplex C of $C(P)$ is defined by $C := C\big(\ T_{+-+}(+) + T_{-++}(+,-,+),$ $T_{+-+}(-) + T_{-++}(+,-,-) + T_{-++}(-,-,+),\ T_{**-}\ \big)$.

First, the retraction $\rho : C(P') \to C'$ is defined by the formulas

$$\begin{aligned}
S_{-++}(+) &\mapsto S_{-++}(+) + S_{+-+}(+,-,+),\\
S_{-++}(-) &\mapsto S_{-++}(-) + S_{+-+}(+,-,-) + S_{+-+}(-,-,+),\\
S_{**-} &\mapsto S_{**-},\\
S_{+-+}(+,+,+) &\mapsto S_{++-}(+,+,+),\\
S_{+-+}(+,+,-) &\mapsto S_{-++}(+) + S_{+-+}(+,-,+) + S_{++-}(+,+,-) + S_{++-}(+,-,+),\\
S_{+-+}(-,+,+) &\mapsto S_{-++}(+) + S_{+-+}(+,-,+) + S_{++-}(-,+,+),\\
S_{+-+}(-,+,-) &\mapsto S_{-++}(-) + S_{+-+}(+,-,-) + S_{+-+}(-,-,+) + S_{++-}(-,+,-)\\
&\qquad + S_{++-}(-,-,+),\\
S_{--+}(\epsilon,\eta) &\mapsto S_{+--}(\epsilon,\eta),\\
\text{otherwise} &\mapsto 0.
\end{aligned}$$

Second, consider the following composition (23) of the following isomorphism with ρ. The isomorphism $C' \to C$ is defined by the formulas

$$\begin{aligned}
S_{-++}(+) + S_{+-+}(+,-,+) &\mapsto T_{+-+}(+) + T_{-++}(+,-,+),\\
S_{-++}(-) + S_{+-+}(+,-,-) + S_{+-+}(-,-,+) &\mapsto T_{+-+}(-) + T_{-++}(+,-,-)\\
&\qquad + T_{-++}(-,-,+),\\
S_{++-}(\epsilon) &\mapsto T_{++-}(\epsilon),\\
S_{-+-}(\epsilon,\eta) &\mapsto T_{-+-}(\epsilon,\eta),\\
S_{+--}(\epsilon,\eta) &\mapsto T_{+--}(\epsilon,\eta),\\
S_{---}(\epsilon,\zeta,\eta) &\mapsto T_{---}(\epsilon,\zeta,\eta).
\end{aligned}$$

Third, the map $h : C(P') \to C(P')$ such that $d \circ h + h \circ d = \mathrm{id} - \mathrm{in} \circ \rho$, is defined by the formulas

$$\begin{aligned}
S_{--+}(\epsilon,\eta) &\mapsto S_{+-+}(\epsilon,-,\eta),\\
S_{+-+}(\epsilon,+,\eta) &\mapsto S_{+++}(\epsilon,\eta),\\
\text{otherwise} &\mapsto 0.
\end{aligned}$$

(III–4) Consider the case where the state of P' with (mark(A), mark(B), mark(C)) $= (1,\ 1,\ 1)$ is represented as $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{w_C}{\emptyset_\eta}|v)$.

Denote by $S_{+++}(\epsilon,\eta)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{w_C}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) $= (1,\ 1,\ 1)$, denote by $S_{-++}(\epsilon,\zeta,\eta)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{w_{AB}}{\emptyset_\epsilon}|\overset{w_C}{\emptyset_\epsilon}|v)$ of P' with (mark(A), mark(B), mark(C)) $= (-1,\ 1,\ 1)$, denote by $S_{+-+}(\epsilon,\zeta,\eta)$ the state $(u|\overset{w_{AB}}{\emptyset_\epsilon}|\overset{ABC}{\emptyset_\zeta}|\overset{w_C}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) $= (1,\ -1,\ 1)$, denote by $S_{--+}(\epsilon,\eta)$ the state $(u|\overset{w_{AB}}{\emptyset_\epsilon}|\overset{w_C}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) $= (-1,\ -1,\ 1)$, denote by $S_{++-}(\epsilon)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|v)$ of P' with (mark(A), mark(B), mark(C)) $= (1,\ 1,\ -1)$, denote by $S_{-+-}(\epsilon,\eta)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{w_{AB}}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) $= (-1,\ 1,\ -1)$, denote by $S_{+--}(\epsilon,\eta)$ the state $(u|\overset{w_{AB}}{\emptyset_\epsilon}|\overset{w_{ABC}}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) $= (1,\ -1,\ -1)$, denote by $S_{---}(\epsilon)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|v)$ of P' with (mark(A), mark(B), mark(C)) $= (-1,\ -1,\ -1)$.

The subcomplex C' of $C(P')$ is defined by $C' := C\big(\ S_{-++}(+,+,\eta),\ S_{-++}(+,-,\eta) + S_{+-+}(+,-,\eta),\ S_{-++}(-,+,\eta) + S_{+-+}(+,-,\eta),\ S_{-++}(-,-,\eta) + S_{+-+}(-,-,\eta),\ S_{**-}\ \big)$ where S_{**-} denotes every states with mark(C) $= -1$.

Denote by $T_{+-+}(\epsilon,\zeta,\eta)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{w_{AB}}{\emptyset_\zeta}|\overset{w_C}{\emptyset_\eta}|\ v)$ of P with (mark(A), mark(B), mark(C)) $= (1,\ -1,\ 1)$, denote by $T_{-++}(\epsilon,-,\eta)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{ABC}{\emptyset_-}|\overset{w_C}{\emptyset_\eta}|\ v)$ of P with (mark(A), mark(B), mark(C)) $= (-1,\ 1,\ 1)$, denote by $T_{++-}(\epsilon)$ the state $(u|\overset{w_{ABC}}{\emptyset_\eta}|v)$ of P with (mark(A), mark(B), mark(C)) $= (1,\ 1,\ -1)$, denote by $T_{-+-}(\epsilon,\eta)$ the state $(u|\overset{w_{AB}}{\emptyset_\epsilon}|\overset{w_{ABC}}{\emptyset_\eta}|v)$ of P with (mark(A), mark(B), mark(C)) $= (-1,\ 1,\ -1)$, denote by $T_{+--}(\epsilon,\eta)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{w_{AC}}{\emptyset_\eta}|v)$ of P with (mark(A), mark(B), mark(C)) $= (1,\ -1,\ -1)$, denote by $T_{---}(\epsilon)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|v)$ of P with (mark(A), mark(B), mark(C)) $= (-1,\ -1,\ -1)$, denote by T_{**-} every states of P with mark(C) $= -1$.

The subcomplex C of $C(P)$ is defined by $C := C\big(\ T_{+-+}(+,+,\eta),\ T_{+-+}(+,-,\eta) + T_{-++}(+,-,\eta),\ T_{+-+}(-,+,\eta) + T_{-++}(+,-,\eta),\ T_{+-+}(-,-,\eta) + T_{-++}(-,-,\eta),\ T_{**-}\ \big)$.

First, the retraction $\rho : C(P') \to C'$ is defined by the formulas

$$\begin{aligned}
S_{-++}(+,+,\eta) &\mapsto S_{-++}(+,+,\eta),\\
S_{-++}(+,-,\eta) &\mapsto S_{-++}(+,-,\eta) + S_{+-+}(+,-,\eta),\\
S_{-++}(-,+,\eta) &\mapsto S_{-++}(-,+,\eta) + S_{+-+}(+,-,\eta),\\
S_{-++}(-,-,\eta) &\mapsto S_{-++}(-,-,\eta) + S_{+-+}(-,-,\eta),\\
S_{**-} &\mapsto S_{**-},\\
S_{+-+}(+,+,+) &\mapsto S_{-++}(+,+,+),\\
S_{+-+}(+,+,-) &\mapsto S_{-++}(+,+,-) + S_{++-}(+),\\
S_{+-+}(-,+,+) &\mapsto S_{-++}(+,-,+) + S_{-++}(-,+,+) + S_{++-}(+),\\
S_{+-+}(-,+,-) &\mapsto S_{-++}(+,-,-) + S_{-++}(-,+,-) + S_{++-}(-),\\
S_{--+}(\epsilon,\eta) &\mapsto S_{+--}(\epsilon,\eta),\\
\text{otherwise} &\mapsto 0.
\end{aligned}$$

Second, consider the following composition (23) of the following isomorphism with ρ. The isomorphism $C' \to C$ is defined by the formulas

$$\begin{aligned}
S_{-++}(+,+,\eta) &\mapsto T_{+-+}(+,+,\eta),\\
S_{-++}(+,-,\eta) + S_{+-+}(+,-,\eta) &\mapsto T_{+-+}(+,-,\eta) + T_{-++}(+,-,\eta),\\
S_{-++}(-,+,\eta) + S_{+-+}(+,-,\eta) &\mapsto T_{+-+}(-,+,\eta) + T_{-++}(+,-,\eta),\\
S_{-++}(-,-,\eta) + S_{+-+}(-,-,\eta) &\mapsto T_{+-+}(-,-,\eta) + T_{-++}(-,-,\eta),\\
S_{++-}(\epsilon) &\mapsto T_{++-}(\epsilon),\\
S_{-+-}(\epsilon,\eta) &\mapsto T_{-+-}(\epsilon,\eta),\\
S_{+--}(\epsilon,\eta) &\mapsto T_{+--}(\epsilon,\eta),\\
S_{---}(\epsilon) &\mapsto T_{---}(\epsilon).
\end{aligned}$$

Third, the map $h : C(P') \to C(P')$ such that $d \circ h + h \circ d = \mathrm{id} - \mathrm{in} \circ \rho$, is defined by the formulas

$$\begin{aligned}
S_{--+}(\epsilon,\eta) &\mapsto S_{+-+}(\epsilon,-,\eta),\\
S_{+-+}(\epsilon,+,\eta) &\mapsto S_{+++}(\epsilon,\eta),\\
\text{otherwise} &\mapsto 0.
\end{aligned}$$

(III–5) Consider the case where the state of P' with (mark(A), mark(B), mark(C)) $= (1,\ 1,\ 1)$ is represented as $(u|\ \overset{w_A}{\emptyset_\epsilon}\ |\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |\ \overset{w_C}{\emptyset_\eta}\ |v)$.

Denote by $S_{+++}(\epsilon,\eta)$ the state $(u|\ \overset{w_A}{\emptyset_\epsilon}|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |\ \overset{w_C}{\emptyset_\eta}\ |v)$ of P' with (mark(A), mark(B), mark(C)) $= (1,\ 1,\ 1)$, denote by $S_{-++}(\epsilon,\zeta,\eta)$ the state $(u|\ \overset{w_{ABC}}{\emptyset_\epsilon}\ |\ \overset{w_C}{\emptyset_\epsilon}|v)$ of P' with (mark(A), mark(B), mark(C)) $= (-1,\ 1,\ 1)$, denote by $S_{+-+}(\epsilon,\zeta,\eta,\theta)$ the state

$(u|\overset{w_A}{\emptyset_\epsilon}|\overset{B}{\emptyset_\zeta}|\overset{w_C}{\emptyset_\eta}|\overset{ABC}{\emptyset_\theta}|v)$ of P' with (mark(A), mark(B), mark(C)) = (1, −1, 1), denote by $S_{--+}(\epsilon,\zeta,\eta)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{w_B}{\emptyset_\zeta}|\overset{w_C}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) = (−1, −1, 1), denote by $S_{++-}(\epsilon,\eta)$ the state $(u|\overset{w_A}{\emptyset_\epsilon}|\overset{w_{ABC}}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) = (1, 1, −1), denote by $S_{-+-}(\epsilon)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|v)$ of P' with (mark(A), mark(B), mark(C)) = (−1, 1, −1), denote by $S_{+--}(\epsilon,\eta)$ the state $(u|\overset{w_A}{\emptyset_\epsilon}|\overset{w_B}{\emptyset_\zeta}|\overset{w_{ABC}}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) = (1, −1, −1), denote by $S_{---}(\epsilon,\eta)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{w_B}{\emptyset_\eta}|v)$ of P' with (mark(A), mark(B), mark(C)) = (−1, −1, −1).

The subcomplex C' of $C(P')$ is defined by $C' := C\big(\, S_{-++}(+,\eta) + S_{+-+}(+,+,\eta,-),$ $S_{-++}(-,\eta) + S_{+-+}(+,-,\eta,-) + S_{+-+}(-,+,\eta,-),\ S_{**-}\,\big)$ where S_{**-} denotes every states with mark(C) = −1.

Denote by $T_{+-+}(\epsilon,\eta)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{w_{AB}}{\emptyset_\eta}|v)$ of P with (mark(A), mark(B), mark(C)) = (1, −1, 1), denote by $T_{-++}(\epsilon,-)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{ABC}{\emptyset_-}|v)$ of P with (mark(A), mark(B), mark(C)) = (−1, 1, 1), denote by $T_{++-}(\epsilon,\eta)$ the state $(u|\overset{w_A}{\emptyset_\eta}|\overset{w_{ABC}}{\emptyset_\eta}|v)$ of P with (mark(A), mark(B), mark(C)) = (1, 1, −1), denote by $T_{-+-}(\epsilon)$ the state $(u|\overset{w_{ABC}}{\emptyset_\eta}|v)$ of P with (mark(A), mark(B), mark(C)) = (−1, 1, −1), denote by $T_{+--}(\epsilon,\zeta,\eta)$ the state $(u|\overset{w_A}{\emptyset_\epsilon}|\overset{w_{BC}}{\emptyset_\eta}|\overset{w_{AB}}{\emptyset_\eta}|v)$ of P with (mark(A), mark(B), mark(C)) = (1, −1, −1), denote by $T_{---}(\epsilon,\eta)$ the state $(u|\overset{w_{ABC}}{\emptyset_\epsilon}|\overset{w_{BC}}{\emptyset_\eta}|v)$ of P with (mark(A), mark(B), mark(C)) = (−1, −1, −1), denote by T_{**-} every states of P with mark(C) = −1.

The subcomplex C of $C(P)$ is defined by $C := C\big(\, T_{+-+}(+,+),\ T_{+-+}(+,-) + T_{-++}(+,-),\ T_{+-+}(-,+) + T_{-++}(+,-),\ T_{+-+}(-,-) + T_{-++}(-,-),\ T_{**-}\,\big)$.

First, the retraction $\rho : C(P') \to C'$ is defined by the formulas

$$\begin{aligned}
S_{-++}(+,\eta) &\mapsto S_{-++}(+,\eta) + S_{+-+}(+,+,\eta,-),\\
S_{-++}(-,\eta) &\mapsto S_{-++}(+,-,\eta,-) + S_{+-+}(-,+,\eta,-),\\
S_{**-} &\mapsto S_{**-},\\
S_{+-+}(+,+,-,+) &\mapsto S_{++-}(+,+),\\
S_{+-+}(+,-,+,+) &\mapsto S_{-++}(+,+) + S_{+-+}(+,+,+,-) + S_{++-}(+,+),\\
S_{+-+}(+,-,-,+) &\mapsto S_{-++}(+,-) + S_{+-+}(+,+,-,-) + S_{++-}(+,-),\\
S_{+-+}(-,+,+,+) &\mapsto S_{-++}(+,+) + S_{+-+}(+,+,+,-),\\
S_{+-+}(-,+,-,+) &\mapsto S_{-++}(+,-) + S_{+-+}(+,+,-,-) + S_{++-}(-,+),\\
S_{+-+}(-,-,+,+) &\mapsto S_{-++}(-,+) + S_{+-+}(+,-,+,-) + S_{+-+}(-,+,+,-)\\
&\qquad + S_{++-}(-,+),\\
S_{+-+}(-,-,-,+) &\mapsto S_{-++}(-,-) + S_{+-+}(+,-,-,-) + S_{+-+}(-,+,-,-)\\
&\qquad + S_{++-}(-,-),\\
S_{--+}(\epsilon,\zeta,\eta) &\mapsto S_{+--}(\epsilon,\zeta,\eta),\\
\text{otherwise} &\mapsto 0.
\end{aligned}$$

Second, consider the following composition (23) of the following isomorphism with ρ. The isomorphism $C' \to C$ is defined by the formulas

$$\begin{aligned}
S_{-++}(+,+) + S_{+-+}(+,+,+,-) &\mapsto T_{+-+}(+,+),\\
S_{-++}(+,-) + S_{+-+}(+,+,-,-) &\mapsto T_{+-+}(+,-) + T_{-++}(+,-),\\
S_{-++}(-,+) + S_{+-+}(+,-,+,-) + S_{+-+}(-,+,+,-) &\mapsto T_{+-+}(-,+) + T_{-++}(+,-),\\
S_{-++}(-,-) + S_{+-+}(+,-,-,-) + S_{+-+}(-,+,-,-) &\mapsto T_{+-+}(-,-) + T_{-++}(-,-),\\
S_{++-}(\epsilon,\eta) &\mapsto T_{++-}(\epsilon,\eta),\\
S_{-+-}(\epsilon) &\mapsto T_{-+-}(\epsilon),\\
S_{+--}(\epsilon,\zeta,\eta) &\mapsto T_{+--}(\epsilon,\zeta,\eta),\\
S_{---}(\epsilon,\eta) &\mapsto T_{---}(\epsilon,\eta).
\end{aligned}$$

Third, the map $h : C(P') \to C(P')$ such that $d \circ h + h \circ d = \mathrm{id} - \mathrm{in} \circ \rho$, is defined by the formulas

$$\begin{aligned}
S_{--+}(\epsilon,\zeta,\eta) &\mapsto S_{+-+}(\epsilon,\zeta,\eta,-),\\
S_{+-+}(\epsilon,\zeta,\eta,+) &\mapsto S_{+++}(\epsilon,\zeta,\eta),\\
\text{otherwise} &\mapsto 0.
\end{aligned}$$

The fact is, (III–1) – (III–5) prove that $KH^{i,j}((xAByACzBCt)) \simeq KH^{i,j}((xBAyCAzCBt))$ if $(|A|,|B|,|C|)$ is any of $\{(-1,-1,-1),\ (-1,1,1),\ (1,1,-1)\ \}$.

Consider $P' = (xAByACzBCt) \to (xBAyCAzCBt) = P$ where $(|A|, |B|, |C|) = (1, -1, -1)$.

$$\begin{aligned}
xAByACzBCt &\overset{\nu-\text{shift}}{\sim} xBCyABzACt \quad \text{with } (|A|, |B|, |C|) = (1, -1, -1) \\
&\overset{\text{isom}}{\simeq} xAByDAzDBt \quad \text{with } (|A|, |B|, |D|) = (-1, -1, 1) \\
&\overset{\text{Lemma 2.2}}{\sim} xAByDA\underline{CE}zDB\underline{CE}t \quad \text{with } |C| = -1, |E| = 1 \\
&\overset{H3}{\sim} x\underline{BA}yD\underline{CA}EzD\underline{CB}Et \quad \text{with } (|A|, |B|, |C|) = (-1, -1, -1) \\
&\overset{H2}{\sim} xBAyAEzBEt \quad \text{with } |C| = -1, |D| = 1 \\
&\overset{\text{isom}}{\simeq} xCByBAzCAt \quad \text{with } (|A|, |B|, |C|) = (1, -1, -1) \\
&\overset{\nu-\text{shift}}{\sim} xBAyCAzCBt \quad \text{with } (|A|, |B|, |C|) = (1, -1, -1)
\end{aligned}$$

We has already showed that the invariance of $KH^{i,j}$ under these moves above and then $KH^{i,j}$ is preserved under the third homotopy move H3 and its inverse move with $(|A|, |B|, |C|) = (1, -1, -1)$. In particular, in this case, we use the invariance of $KH^{i,j}$ under H3 and its inverse with $(|A|, |B|, |C|) = (-1, -1, -1)$. By using the invariance under H3 and its inverse with $(|A|, |B|, |C|) = (1, 1, -1)$ (resp. $(-1, 1, 1)$), we can verify that the invariance of $KH^{i,j}$ under H3 and its inverse with $(|A|, |B|, |C|) = (1, 1, 1)$ (resp. $(-1, -1, 1)$).

We conclude the proof that $KH^{i,j}(P') \simeq KH^{i,j}(P)$ for $P' \simeq_{S_1} P$. □

The following corollary is a similar result to Corollary 3.1.

Corollary 5.1. *$KH^{i,j}(P)$ is a S_0-homotopy invariant for nanophrases P over α_0.*

6. An application of $KH^{i,j}$ via words to nanophrases over any α.

In the previous sections, we discuss S_1-homotopy invariants $\hat{J}(P)$ and $KH^{i,j}(P)$ of pseudolinks. Here, we construct homotopy invariants of nanophrases over any α from $\hat{J}(P)$ and $KH^{i,j}(P)$.

Let α be an arbitrary alphabet, τ be $\alpha \to \alpha$; involution, Δ_α be $\{(a, a, a)\}_{a \in \alpha}$ and $\alpha/\tau := \{\tilde{a}_1, \dots, \tilde{a}_m\}$. We fix a complete residue system $\{a_1, \dots, a_m\}$ of α/τ and denote $\{a_1, \dots, a_m\}$ by $\mathrm{crs}(\alpha/\tau)$.

We use the notion of an 6.1 as in [7, Section 4.1].

Definition 6.1. An *orbit* of the involution $\tau : \alpha \to \alpha$ is a subset of α consisting either of one element preserved by τ or of two elements permuted by τ; in the latter case the orbit is *free.*

Definition 6.2. For $A \in \mathcal{A}$, we define sign of A by

$$\mathrm{sign}_L(A) := \begin{cases} 1 \text{ if } |A| \in L; |\tilde{A}| : \text{a free orbit} \\ -1 \text{ if } |A| \in \tau(L); |\tilde{A}| : \text{a free orbit} \\ 0 \quad \text{otherwise} \end{cases} \tag{24}$$

where L is a nonempty subset of $\mathrm{crs}(\alpha/\tau)$.

Let $\mathcal{P}_k(\alpha,\tau)$ be a set of nanophrases of length k over α with τ.

Definition 6.3. For an arbitrary (α,τ) and an arbitrary subset $L \subset \mathrm{crs}(\alpha/\tau)$, $\mathcal{U}_L : \mathcal{P}_k(\alpha,\tau) \to \mathcal{P}_k(\alpha_0,\tau_0)$; $P \mapsto P_0$ is defined by the following two steps:

(Step 1) Remove $A \in \mathcal{A}$ such that $\mathrm{sign}_L(A) = 0$ from $(\mathcal{A}, P) \in \mathcal{P}_k(\alpha,\tau)$.

(Step 2) Let the nanophrase be $(\mathcal{A}', P')$ after removing letters from $(\mathcal{A}, P)$ by using (Step 1). We consider an α_0-alphabet $\mathcal{B}$ such that $\mathrm{card}\mathcal{B} = \mathrm{card}\mathcal{A}'$ and $\mathcal{A}' \cap \mathcal{B}$ is the empty set. Transpose each letter of $(\mathcal{A}', P')$ and a letter in $\mathcal{B}$ as follows:

$$\begin{cases} \text{transform A with } \mathrm{sign}(A) = 1 \text{ into } B \in \mathcal{B} \text{ with } |B| = 1 \\ \text{transform A with } \mathrm{sign}(A) = -1 \text{ into } B \in \mathcal{B} \text{ with } |B| = -1. \end{cases} \tag{25}$$

By accomplishing (1) and (2), the nanophrase over α_0 derived from $(\mathcal{A}, P)$ is denoted by $\mathcal{U}_L((\mathcal{A}, P))$ or simply $\mathcal{U}_L(P)$.

Theorem 6.1. *For an arbitrary $L \subset \mathrm{crs}(\alpha/\tau)$ and for two arbitrary nanophrases $(\mathcal{A}_1, P_1)$ and $(\mathcal{A}_2, P_2)$,*

$$(\mathcal{A}_1, P_1) \simeq_{\Delta_\alpha} (\mathcal{A}_2, P_2) \Longrightarrow \mathcal{U}_L((\mathcal{A}_1, P_1)) \simeq_{S_0} \mathcal{U}_L((\mathcal{A}_2, P_2)).$$

Proof. First, isomorphism does not change $\mathcal{U}_L(P)$ up to isomorphic is clear.

Consider the first homotopy move

$$P_1 := (\mathcal{A}, (xAAy)) \longrightarrow P_2 := (\mathcal{A} \setminus \{A\}, (xy))$$

where x and y are words on $\mathcal{A}$, possibly including "|" character. Suppose $\mathrm{sign}(A) \neq 0$. Then

$$\mathcal{U}_L(P_1) = x_L AA y_L \simeq x_L y_L = \mathcal{U}_L(P_2)$$

where x_L and y_L are words which obtained by deleting all letters $X \in \mathcal{A}$ such that $\mathrm{sign}(X) = 0$ from x and y respectively.

Suppose $\mathrm{sign}(A) = 0$. Then

$$\mathcal{U}_L(P_1) = x_L y_L = \mathcal{U}_L(P_2).$$

So the first homotopy move does not change the homotopy class of $\mathcal{U}_L(P)$.

Consider the second homotopy move

$$P_1 := (\mathcal{A}, (xAByBAz)) \longrightarrow (\mathcal{A} \setminus \{A, B\}, (xyz))$$

where $|A| = \tau(|B|)$, and x, y and z are words on $\mathcal{A}$ possibly including "|" character. Suppose $|A| \in L \cup \tau(L)$ and $|\tilde{A}|$ is free orbit. Then $|B| \in L \cup \tau(L)$ and $|\tilde{A}|$ is free orbit since $|A| = \tau(|B|)$. So

$$\mathcal{U}_L(P_1) = x_L AB y_L BA z_L \simeq x_L y_L z_L = \mathcal{U}_L(P_2).$$

where x_L, y_L and z_L are words which obtained by deleting all letters $X \in \mathcal{A}$ such that $\mathrm{sign}(X) = 0$ from x, y and z respectively. Suppose $|A| \notin L \cup \tau(L)$ or $|A|$ is a fixed point of τ. Then $|B| \notin L \cup \tau(L)$ or $|B|$ is a fixed point of τ since $|A| = \tau(|B|)$. So

$$\mathcal{U}_L(P_1) = x_L y_L z_L = \mathcal{U}_L(P_2).$$

By the above, the second homotopy move does not change the homotopy class of $\mathcal{U}_L(P)$.

Consider the third homotopy move

$$P_1 := (\mathcal{A}, (xAByACzBCt)) \to P_2 := (\mathcal{A}, (xBAyCAzCBt))$$

where $|A| = |B| = |C|$, and x, y, z and t are words on $\mathcal{A}$ possibly including "|" character. Suppose $\mathrm{sign}(A) \neq 0$. Then $\mathrm{sign}(B), \mathrm{sign}(C) \neq 0$ since $|A| = |B| = |C|$. So we obtain

$$\mathcal{U}_L(P_1) = x_L AB y_L AC z_L AC t_L \simeq x_L BA y_L CA z_L CB t_L = \mathcal{U}_L(P_2).$$

where x_L, y_L, z_L and t_L are words which obtained by deleting all letters $X \in \mathcal{A}$ such that $\mathrm{sign}(X) = 0$ from x, y, z and t respectively. Suppose $\mathrm{sign}(A) = 0$. Then $\mathrm{sign}(B), \mathrm{sign}(C) = 0$ since $|A| = |B| = |C|$. So we obtain

$$\mathcal{U}_L(P_1) = x_L y_L z_L t_L = \mathcal{U}_L(P_2).$$

So the third homotopy move does not change the homotopy class of $\mathcal{U}_L(P)$.

By the above, $\mathcal{U}_L$ is a homotopy invariant of nanophrases. $\square$

Corollary 6.1. *Let I be a S_0-homotopy invariant of nanophrase over α_0. For $P \in \mathcal{P}_k(\alpha, \tau)$, we define I' by*

$$I'(P) := \{I(\mathcal{U}_L(P))\}_{L \subset \mathrm{crs}(\alpha/\tau)}.$$

I' is a Δ_α-homotopy invariant of $P \in \mathcal{P}_k(\alpha, \tau)$. In particular, for $(\mathcal{A}, P) \in \mathcal{P}_k(\alpha_0, \tau_0)$, $I'(P) = \{I(P)\}$ if we take $\mathrm{crs}(\alpha_0/\tau_0) = \{1\}$.

Theorem 6.1 implies the following Corollaries.

Corollary 6.2. *Let α be an arbitrary alphabet. $\hat{J}(\mathcal{U}_L(P))$ is Δ_α-homotopy invariants for nanophrases P over α.*

Corollary 6.3. *Let α be an arbitrary alphabet. $KH^{i,j}(\mathcal{U}_L(P))$ is Δ_α-homotopy invariants for nanophrases P over α.*

Remark 6.1. $\hat{J}(\mathcal{U}_L(P)) = \sum_{j=-\infty}^{\infty} q^j \sum_{i=-\infty}^{\infty} (-1)^i \mathrm{rk} KH^{i,j}(\mathcal{U}_L(P))$.

We give some examples of calculating of $KH^{i,j}(P)$ or $KH^{i,j}(\mathcal{U}_L(P))$.

Example 6.1. For two pseudolinks $P_1 = ABCDEABCDE$ with $|A| = |B| = |C| = |D| = |E| = -1$ and $P_2 = ABCDEFBGDHFIJEHCGAIJ$ with $|A| = |C| = |E| = |G| = |H| = |I| = |J| = -1$ and $|B| = |D| = |D| = |F| = 1$, $\hat{J}(P_1) = \hat{J}(P_2)$. However, $KH^{-7,15}(P_1) \simeq 0$ and $KH^{-7,15}(P_2) \simeq \mathbb{Z}_2$. (cf. [1, 10].)

Theorem 6.2. *$KH^{i,j}(P)$ is a strictly stronger invariant than $\hat{J}(P)$.*

In [7], Turaev constructed a Δ_α-homotopy invariant λ for nanophrases over α.

Example 6.2. Let a, b, c be elements (possibly coinciding) of any alphabet α and A, B, C are letters with $|A| = a$, $|B| = b$ and $|C| = c$. If $a = c = \tau(b) \neq b$, $\lambda(ABACBC) = \lambda(ACAC) = a + a_\bullet - a{a_\bullet}^2 - a^2 a_\bullet$. However, $KH^{0,2}(\mathcal{U}_{\{a\}}(ACAC)) \simeq 0$ and $KH^{0,2}(\mathcal{U}_{\{a\}}(ABACBC)) \simeq \mathbb{Z}_2$.

Turaev constructed a strictly stronger Δ_α-homotopy invariant $f \circ v_+$ than λ for nanophrases over α [7].

Example 6.3. Let a, b, c, d be elements (possibly coinciding) of any alphabet α and A, B, C, D are letters with $|A| = a$, $|B| = b$, $|C| = c$ and $|D| = d$. If $a = b$ and $c = \tau(b) = d$, $f(v_+(ABCDCDAB)) = f(v_+(\emptyset)) = \underline{1}$. However, $KH^{0,3}(\mathcal{U}_{\{c\}}(\emptyset)) \simeq 0$ and $KH^{0,3}(\mathcal{U}_{\{c\}}(ABCDCDAB)) \simeq \mathbb{Z}_2$.

Theorem 6.3. *Let α be an arbitrary alphabet and S be Δ_α. $KH^{i,j}(\mathcal{U}_L(P))$ is independent of $f \circ v_+$ for nanophrases P over α.*

We has left the following problems unsolved: Is $KH^{i,j}(\mathcal{U}_L(P))$ strictly stronger than $\hat{J}(\mathcal{U}_L(P))$? What relation is $KH^{i,j}(P)$ of a pseudolink P to Manturov's categorification [5, 6]?

Acknowledgment– The authors would like to express the deep gratitude to Professors Toshiyuki Akita, Goo Ishikawa and Jun Murakami for their support.

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