Abstract. In word and phrase theory of Turaev, we interpret links or virtual
links as equivalences of phrases over an alphabet consisting four letters. V. Tu-
raev constructed a version of the Jones polynomial for phrases. We study the
well-definedness of the Jones polynomial for phrases in word theory. On the
other hand, M. Khovanov introduced a collection of homology groups which is
a strictly stronger link invariant than the Jones polynomial and O. Viro recon-
structed these Khovanov homology groups. We construct phrase invariants as the
homology groups of certain chain complexes for phrases where the coefficients of
the Jones polynomial are the Euler characteristics of these complexes using the
Viro’s method of Khovanov theory. The invariance of these homology groups is
shown in only terminology of Turaev’s theory of phrases. Moreover, we apply
the homology groups to getting invariants for another type of phrases over an
alphabet consisting any letters.

Keywords Turaev’s homotopy theory of phrases, categorification, Jones poly-
nomial, homotopy invariants of phrases.

1. Introduction.

V. Turaev introduced a phrase over a set \( \alpha_* \) consisting four letters, which is one
to one corresponding to a stable equivalence class of a knot diagram on surfaces or
a virtual link [7, 8, 9]. In this study, we construct cohomology groups \( KH^{i,j}(P) \)
satisfying a Poincaré series (1) for a phrase \( P \), so-called a pseudolink, a projection
image of a nanophrase over \( \alpha_* \). \( \hat{J}(P) \) is a version of the Jones polynomial for phrases
defined by Turaev’s homotopy theory of phrases, we define in Section 3.

\[
\hat{J}(P) = \sum_{j=-\infty}^{\infty} q^j \sum_{i=-\infty}^{\infty} (-1)^i \text{rk}KH^{i,j}(P).
\]

\( \alpha_* \) is the set composed of 4 distinct elements \( a_+, a_-, b_+, b_- \) and \( \alpha_* \) has an invo-
lution \( \tau: a_+ \mapsto b_- \). Let \( S_* = \{(a_+, a_+, a_+), (a_+, a_-, a_+), (a_+, a_+, a_-), (b_+, b_+, b_+), (b_+, b_-, b_+), (b_+, b_+, b_-)\} \) where three upper signs or three lower signs should be
chosen in the double signs for each triple [8, Subsection 4.2]. \( \alpha_* \)-alphabet \( \mathcal{A} \) is a set
where every element \( A \) of \( \mathcal{A} \) has a projection \( |\ |: A \mapsto |A| \in \alpha_* \). A word of length
\( n \geq 1 \) in an alphabet \( \mathcal{A} \) is a mapping \( w: \hat{n} \rightarrow \mathcal{A} \) where \( \hat{n} = \{i \in \mathbb{N} \mid 1 \leq i \leq n\} \).
Such a word is encoded by the sequence \( w(1)w(2)\cdots w(n) \). By definition, there is
a unique word \( \emptyset \) of length 0. A word \( w: \hat{n} \rightarrow \mathcal{A} \) is a Gauss word if each element
of \( \mathcal{A} \) is the image of precisely two elements of \( \hat{n} \) or \( w = \emptyset \). A nanoword \((\mathcal{A}, w)\)

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over $\alpha_*$ is a pair (an $\alpha_*$-alphabet $A$, a Gauss word in the alphabet $A$). For a nanoword $(A, w = w_1w_2 \cdots w_k)$ over $\alpha_*$ consisting of subwords $w_i$ ($1 \leq i \leq k$) of $w$, a nanophrase of length $k \geq 0$ over $\alpha_*$ is defined as $(A, (w_1|w_2| \cdots |w_k))$.

For a nanophrase $(A, (w_1|w_2| \cdots |w_k))$ over $\alpha_*$, we associate $w$ to a pointed components of links and the order ($1 \leq i \leq k$) to a order of components of links. Then, by introducing an appropriate isomorphism and equivalence to nanophrases using a bijection between the set of stable equivalence classes of knot diagrams on surfaces and $\mathcal{P}(\alpha_*, \nu)$ that is $\mathcal{P}(\alpha_*, \nu)/S_\alpha$-homotopy. Moreover, for $\mathcal{P}(\alpha_*, \nu)$, let $\alpha_1 := \{1, -1\}$ and $S_1 := \{(+1, \pm 1), (+1, \mp 1), (\pm 1, \pm 1), (\pm 1, \mp 1), (\mp 1, \pm 1)\}$ where three upper signs or three lower signs should be chosen in the double signs for each triple.

We can consider the set $\mathcal{P}(\alpha_1, S_1, \text{id})$ that is an image of the projection: $\alpha_* \rightarrow \alpha_1$; $a_+, b_+ \mapsto 1$ and $a_-, b_- \mapsto -1$ induces $\mathcal{P}(\alpha_*, \nu) \rightarrow \mathcal{P}(\alpha_1, S_1, \text{id})$. Turaev construct the Jones polynomial as $S_\alpha$-homotopy invariants of elements, called pseudolinks, of $\mathcal{P}(\alpha_1, S_1, \text{id})$. However, Turaev’s definition of the Jones polynomial $J(P)$ for a pseudolink $P$ is depend on nanophrases over $\alpha_*$ [8, Section 8]. It is obvious the existence of $J(P)$ by using geometrical objects (i.e. links). However, it is not clear that the well-definedness of $J(P)$ in only word theory.

In this paper, we give $J(P)$, and $KH^{j\nu}(P)$ and show they are pseudolink invariants by using only $P$ of $\mathcal{P}(\alpha_1, S_1, \text{id})$. $KH^{j\nu}(P)$ preserve the property of the Khovanov homology group as follows: $KH^{j\nu}(P)$ is a strictly stronger invariant than $J(P)$. Moreover, we apply $KH^{j\nu}(P)$ to getting invariants for an other type of phrases over an alphabet consisting any letters.

2. Turaev’s theory of words

2.1. Nanowords and Nanophone. For our preliminary, we define nanophrases and their $S$-homotopy as the manner in Turaev’s original paper [7, Section 2], [8, Section 2], Gibson’s paper[3, Section 2], or Fukunaga’s paper [2, Section 2.1] that gives the detailed description of their terminology.

An alphabet is a finite set and letters are its elements. $\alpha$-alphabet $A$ is a set where every element $A$ of $A$ has a projection $| : A \mapsto |A| \in \alpha$. A word of length $n \geq 1$ in an alphabet $A$ is a mapping $w : \hat{n} \rightarrow A$ where $\hat{n} = \{i \in \mathbb{N} | 1 \leq i \leq n\}$. Such a word is encoded by the sequence $w(1)w(2) \cdots w(n)$. By definition, there is a unique word $\emptyset$ of length 0. We define opposite word by writing the letters of a word $w$ in the opposite order. For example, if $w = abc$, then $w^- = cba$. A word $w : \hat{n} \rightarrow A$ is a Gauss word in an alphabet $A$ if each element of $A$ is the image of precisely two elements of $\hat{n}$ or $w$ is $\emptyset$. A Gauss phrase in an alphabet $A$ is a sequence of words $x_1, x_2, \ldots, x_m$ in $A$ denoted by $(x_1|x_2| \cdots |x_m)$ such that $x_1x_2 \cdots x_m$ is a Gauss word in $A$. We call $x_i$ ith component of the Gauss phrase. In particular, if a Gauss phrase has only one component, that com ponent is a Gauss word. A nanoword $(A, w)$ over $\alpha$ is a pair (an $\alpha$-alphabet $A$, a Gauss word in the alphabet $A$). For a nanoword $(A, w = w_1w_2 \cdots w_k)$ over $\alpha$ consisting of subwords $w_i$ ($1 \leq i \leq k$) of $w$,
a nanophrase of length \( k \geq 0 \) over \( \alpha \) is defined as \((\mathcal{A}, (w_1|w_2|\cdots|w_k))\). Whenever possible, \((\mathcal{A}, (w_1|w_2|\cdots|w_k))\) is indicated by simple symbols: \((w_1|w_2|\cdots|w_k)\), \((\mathcal{A}, \mathcal{P})\) or \(\mathcal{P}\). We call \( w_i \) the \( i \)th component of the nanophrase.

An arbitrary nanoword \( w \) over \( \alpha \) yields a nanophrase \((w)\) of length 1. However, we distinguish between nanowords and nanophrases of length 1. By definition, there is a unique nanophrase of length 0. Pay attention to the fact that \((\emptyset)\) is not a nanophrase of length 0. (cf. [8, Subsection 6.1]. Turaev makes no difference between nanowords and nanophrases of length 1. We denote the nanophrase of length 0 by \( \emptyset \). Note that we distinguish the nanophrase \((\emptyset)|\emptyset|\cdots|\emptyset)\) of length \( k \) from the nanophrase \((\emptyset)|\emptyset|\cdots|\emptyset)\) of length \( l \) if \( k \neq l \).

An isomorphism of \( \alpha \)-alphabets \( \mathcal{A}_1, \mathcal{A}_2 \) is bijection \( f: \mathcal{A}_1 \rightarrow \mathcal{A}_2 \) such that \(|A| = |f(A)|\) for an arbitrary \( A \in \mathcal{A}_1 \). Two nanophrases \((\mathcal{A}_1, p_1 = (w_1|w_2|\cdots|w_k))\) and \((\mathcal{A}_2, p_2 = (w'_1|w'_2|\cdots|w'_{k'}))\) over \( \alpha \) are isomorphic if \( k = k' \) and there is an isomorphism of \( \alpha \)-alphabets \( f: \mathcal{A}_1 \rightarrow \mathcal{A}_2 \) such that \( w'_i = f w_i \) for every \( i \in \{1, 2, \ldots, k\} \).

### 2.2. Homotopy of nanophrases.

To define homotopy of nanophrases we fix a finite set \( \alpha \) with an involution \( \tau: \alpha \rightarrow \alpha \) and a subset \( S \subset \alpha \times \alpha \times \alpha \). We call the triple \((\alpha, \tau, S)\) a homotopy data. Turaev defines \( S \)-homotopy as follows (cf. [8, Section 2.2], [2, Section 2.1], [3, Section 2]).

**Definition 2.1.** Let \((\alpha, \tau, S)\) be a homotopy data. Two nanowords \((\mathcal{A}_1, w_1)\) and \((\mathcal{A}_2, w_2)\) are \( S \)-homotopic if one nanophrase is changed into the other by the finite sequence of the isomorphisms and the following three type deformations \((1)-(3)\), called homotopy moves, and their inverse. The relation \( S \)-homotopy is denoted by \( \equiv_s \).

- **(H1)** Replace \((\mathcal{A}, (xAAy))\) by \((\mathcal{A}\setminus\{A\}, (xy))\) for \( \mathcal{A} \) and \( x, y \) are words in \( \mathcal{A}\setminus\{A\} \) possibly including the \(| \cdot | \) character such that \((xy)\) is a Gauss phrase.
- **(H2)** Replace \((\mathcal{A}, (xAByBaz))\) by \((\mathcal{A}\setminus\{A,B\}, (xyz))\) if \( A, B \in \mathcal{A} \) with \( \tau(|A|) = |B| \) where \( x, y, z \) are words in \( \mathcal{A}\setminus\{A,B\} \) possibly including the \(| \cdot | \) character such that \((xyz)\) is a Gauss phrase.
- **(H3)** Replace \((\mathcal{A}, (xAByACzBC't))\) by \((\mathcal{A}, (xBAyCAzCB't))\) for \((|A|, |B|, |C|) \in S \) where \( x, y, z, t \) are words in \( \mathcal{A} \) possibly including the \(| \cdot | \) character such that \((xyzt)\) is a Gauss phrase.

Recall the following two lemmas from [8, Lemma 2.1, Lemma 2.2] (cf. [2, Lemma 2.4, Lemma 2.5]).

**Lemma 2.1.** Let \((\alpha, \tau, S)\) be a homotopy data and \( \mathcal{A} \) be an \( \alpha \)-alphabet. Let \( A, B, C \) be distinct letters in \( \mathcal{A} \) and let \( x, y, z, t \) be words possibly including the \(| \cdot | \) character in the alphabet \( \mathcal{A}\setminus\{A,B,C\} \) such that \((xyzt)\) is a Gauss phrase in this alphabet. Then,

- \((\mathcal{A}, (xAByCAzBC't)) \equiv_s (\mathcal{A}, (xBAyCAzCB't))\) for \((|A|, \tau(|B|), |C|) \in S\);
- \((\mathcal{A}, (xAByCAzBC't)) \equiv_s (\mathcal{A}, (xBAyACzBC't))\) for \((\tau(|A|), \tau(|B|), |C|) \in S\);
- \((\mathcal{A}, (xAByACzCB't)) \equiv_s (\mathcal{A}, (xBAyACzCB't))\) for \((\tau(|A|), |B|, |C|) \in S\).

**Lemma 2.2.** Suppose that \( S \cap (\alpha \times \{b\} \times \{b\}) \neq \emptyset \) for all \( b \in \alpha \). Let \((\mathcal{A}, (xAByABz))\) be a nanophrase over \( \alpha \) with \(|B| = \tau(|A|)\) where \( x, y, z \) words possibly including
Definition 2.2. Let \( \alpha \) be a finite set. Fix an involution \( \nu \) \( \alpha \to \alpha \) called the shift involution. The \( \nu \)-shift of a nanoword \( (A, w : \hat{n} \to A) \) over \( \alpha \) is the nanoword \( (A', w' : \hat{n} \to A') \) obtained by the following steps (1)–(3): (1) Let \( A := (A - \{\cdot\}) \cup \{A_\nu\} \) where \( A_\nu \) is a letter not belonging to \( A \).

(2) The projection \( A' \to \alpha \) extends the given projection \( A - \{\cdot\} \to \alpha \) by \( |A_\nu| = \nu(|A|) \).

(3) The word \( w' \) in the alphabet \( A' \) is defined by \( w' = xA_\nu yA_\nu \) for \( w = AxAy \).

We define \( \nu \)-shifts and \( \nu \)-permutations of words in a nanophrase \( P = (A, (w_1|w_2| \cdots |w_k)) \) over \( \alpha \) and define \( \mathcal{P}(\alpha, S, \nu) \) in the following manner as in [8, Subsection 6.2].

Fix a homomorphism data \((\alpha, \tau, S)\) and a shift involution in \( \alpha \).

Definition 2.3. For \( i = 1, \ldots, k \), the \( i \)th \( \nu \)-shift of a nanophrase \( P \) moves the first letter, say \( A_i \), of \( w_i \) to the end of \( w_i \) keeping \( |A| \in \alpha \) if \( A \) appears in \( w_i \) only once and applying \( \nu \) if \( A \) appears in \( w_i \) twice. All other words in \( P \) are preserved.

Definition 2.4. Given two words \( u, v \) on an \( \alpha \)-alphabet \( A \), consider the mapping \( A \to \alpha \) sending \( A \in A \) to \( \nu(|A|) \in \alpha \) if \( A \) appears both in \( u \) and \( v \) and sending \( A \) to \( |A| \) otherwise. The set \( A \) with this projection to \( \alpha \) is an \( \alpha \)-alphabet denoted by \( \mathcal{A}_{\alpha,u,v} \). For \( i = 1, \ldots, k-1 \), the \( \nu \)-permutation of the \( i \)th and \((i + 1)\)st words transforms a nanophrase \( P = (A, (w_1|w_2| \cdots |w_k)) \) into the nanophrase \( (A, (w_1|w_2| \cdots |w_{i-1}|w_{i+1}|w_i|w_{i+2}| \cdots |w_k)) \). The operation is involutive. The \( \nu \)-permutations define an action of the symmetric group \( S_k \) on the set of nanophrases of length \( k \).

Denote by \( \mathcal{P}(\alpha, S, \nu) \) the set of nanophrases over \( \alpha \) quotiented by the equivalence relation generated by \( S \)-homotopy, \( \nu \)-permutations and \( \nu \)-shifts on words.

Turaev defines pseudolinks in the following manner as in [8, Subsection 7.1].

Definition 2.5. Let \( \alpha_1 = \{-1, 1\} \) with involution \( \tau \) permuting \( 1 \) and \( -1 \) and let \( S_1 \subset \alpha_1 \times \alpha_1 \times \alpha_1 \) consists of the following six triples: \((1,1,1), (1,1,-1), (-1,1,1), (-1,-1,1), (1,-1,1), (1,-1,-1) \). Let \( \nu = \text{id} \). Nanophrases in \( \mathcal{P}(\alpha_1, S_1, \text{id}) \) is called pseudolinks.

Remark 2.1. Let \( \alpha_s \) be the set composed of 4 distinct elements \( a_+, a_-, b_+, b_- \) with involution \( \tau : a_+ \mapsto b_+ \). Let \( S_s = \{(a_+, a_+, a_+), (a_+, a_+, a_+), (a_+, a_+, a_+), (b_+, b_+, b_+), (b_+, b_+, b_+), (b_+, b_+, b_+), (b_+, b_+, b_+)\} \). A projection \( \alpha_s \to \alpha_1 := \{1,-1\} \); \( a_+, b_+ \mapsto 1 \) and \( a_-, b_- \mapsto -1 \) induces surjective mapping \( \mathcal{P}(\alpha_s, S_s, \nu) \to \mathcal{P}(\alpha_1, S_1, \text{id}) \).

In the last of this section, we prepare the notation \( \mathcal{A}_w \) as in [8, Subsection 6.2] and also prepare the notation \( \mathcal{P}_w \) as in [8, Subsection 8.2].

Notation 2.1. For a word \( w \), denote by \( \mathcal{A}_w \) the same alphabet \( A \) with new projection \( \alpha \) defined as follows: for \( A \in A \) set \( |A|_w = \tau(|A|) \) if \( A \) occurs once, \( |A|_w = \nu(|A|) \) if \( A \) occurs twice, and \( |A|_w = |A| \) otherwise. For a phrase \( P \) in an \( \alpha_1 \)-alphabet \( A \) and a word \( w \) on \( A \), denote by \( P_w \) the same phrase on the \( \alpha_1 \)-alphabet \( \mathcal{A}_w \).
3. The Jones Polynomial for Pseudolinks.

Turaev define the Jones polynomial for pseudolinks by using recursive relations for the bracket polynomial of nanophrases over $\alpha_*$ [8, Section 8]. In this section, we give a state sum representation of the Jones polynomial for pseudolinks.

**Definition 3.1.** For every pseudolink $P = (A, (w_1|w_2|\cdots|w_k))$, we assign $A$ with the sign $= -1$ or $1$ and call the sign the *marker* of $A$, denoted by $\text{mark}(A)$. Let a state $s$ of $P$ be $P$ with their markers for all the elements of $A$.

For an arbitrary pseudolink $P$ assigned state $s$, we consider the following deformation $(\ast)$:

$$
(\ast) \begin{cases}
(w_1|\cdots|AxAy|\cdots|w_k) \rightarrow & (w_1|\cdots|x|y|\cdots|w_k) \text{ if } \text{mark}(A) = |A| \\
(w_1|\cdots|x^{-y}|\cdots|w_k)x \text{ if } \text{mark}(A) = -|A| \\
(w_1|\cdots|xy|\cdots|w_k) \text{ if } \text{mark}(A) = |A| \\
(w_1|\cdots|x^{-y}|\cdots|w_k)x \text{ if } \text{mark}(A) = -|A|.
\end{cases}
$$

A pseudolink $(\emptyset|\cdots|\emptyset)$ is obtained by repeating these deformations from $P$. We denote the length of this pseudolink $(\emptyset|\cdots|\emptyset)$ by $|s|$.

**Notation 3.1.** We denote a letter $A$ with $|A| = 1$ and mark($A$) = $+1$ (respectively mark($A$) = $-1$) by $A_+$ (respectively $A_-$), and we denote a letter $A$ with $|A| = -1$ and mark($A$) = $+1$ (respectively mark($A$) = $-1$) by $\overline{A}_+$ (respectively $\overline{A}_-$).

**Example 3.1.** Consider $P = (ABAB)$ with $|A| = |B| = 1$. If mark($A$) = $1$ and mark($B$) = $-1$, $P$ is represented as $(A_+B_-A_+B_-)$ and $(A_+B_-A_+B_-) \xrightarrow{(\ast)} (B_-|B_-)$ \xrightarrow{(\ast)} $(\emptyset)$. If $P$ has mark($A$) = $1$ and mark($B$) = $-1$, $(A_-B_+A_-B_+) \xrightarrow{(\ast)} (\overline{B}_+\overline{B}_+) \xrightarrow{(\ast)} (\emptyset)$.

**Example 3.2.** Let us add two more examples. $(A_+\overline{B}_+A_+C_+\overline{B}_+C_+) \xrightarrow{(\ast)} (\overline{B}_+|C_+\overline{B}_+\overline{C}_+) \xrightarrow{(\ast)} (\emptyset|\emptyset)$. $(A_-\overline{B}_-A_-C_-\overline{B}_-C_-) \xrightarrow{(\ast)} (B_-|C_+B_-\overline{C}_+) \xrightarrow{(\ast)} (\overline{C}_+\overline{C}_+) \xrightarrow{(\ast)} (\emptyset)$.

**Lemma 3.1.** The $|s|$ is well-defined. In other words $|s|$ does not depend on the order of deleting letters.

**Proof.** On the case $A_+xA_+yB_+zB_+t$

If we delete $A$ first, then

$$
A_+xA_+yB_+zB_+t \rightarrow x|yB_+B_+t \rightarrow x|B_+B_+ty \rightarrow x|y|ty
$$

If we delete $B$ first, then


\[ A_+xA_+yB_+zB_+t \rightarrow B_+zB_+tA_+xA_+y \]
\[ \rightarrow z|tA_+xA_+y \]
\[ \rightarrow z|A_+xA_+yt \]
\[ \rightarrow z|x|yt \]

So in this case \(|s|\) does not depend on the order of deleting letters.

- On the case \(A_-xA_-yB_+zB_+t\)
  If we delete \(A\) first, then

\[ A_-xA_-yB_+zB_+t \rightarrow x^-yB_+zB_+t \]
\[ \rightarrow B_+zB_+tx^-y \]
\[ \rightarrow z|tx^-y \]

If we delete \(B\) first, then

\[ A_-xA_-yB_+zB_+t \rightarrow B_+zB_+tA_-xA_-y \]
\[ \rightarrow z|tA_-xA_-y \]
\[ \rightarrow z|A_+xA_+yt \]
\[ \rightarrow z|x^-yt \]

So in this case \(|s|\) does not depend on the order of deleting letters.

- On \(A_+xA_+yB_-zB_-t\).
  In this case we can prove similar as the case of \(A_-xA_-yB_+zB_+t\).

- On the case \(A_-xA_-yB_-zB_-t\)
  If we delete \(A\) first, then

\[ A_-xA_-yB_-zB_-t \rightarrow B_-zB_-tx^-y \]
\[ \rightarrow z^-tx^-y \]

If we delete \(B\) first, then

\[ A_-xA_-yB_-zB_-t \rightarrow z^-tA_-xA_-y \]
\[ \rightarrow A_-xA_yz^-t \]
\[ \rightarrow x^-yz^-t \]

So in this case \(|s|\) does not depend on the order of deleting letters.

- On the cases \(\overline{A}_{\epsilon_1}xA_{\epsilon_1}yB_{\epsilon_2}zB_{\epsilon_2}t\), where \(\epsilon_1, \epsilon_2 \in \{+, -\}\).
In this case we can prove similarly as the cases of $A_{-\epsilon_1}xA_{-\epsilon_1}yB_{\epsilon_2}zB_{\epsilon_2} t$.

- On the cases $A_{\epsilon_1}xA_{\epsilon_1}yB_{\epsilon_2}zB_{\epsilon_2} t$, where $\epsilon_1, \epsilon_2 \in \{+, -\}$.
  In this case we can prove similarly as the case of $A_{-\epsilon_1}xA_{-\epsilon_1}yB_{-\epsilon_2}zB_{-\epsilon_2} t$.

- On the case $A_+xB_+yA_+zB_+$.
  In this case, If we delete $A$ first, then

  \[
  A_+xB_+yA_+zB_+ t \rightarrow xB_+y|zB_+t \\
  \rightarrow B_+yx|B_+tz \\
  \rightarrow yxtz
  \]

  If we delete $B$ first, then

  \[
  A_+xB_+yA_+zB_+ t \rightarrow B_+yA_+zB_+tA_+x \\
  \rightarrow yA_+z|tA_+x \\
  \rightarrow A_+zy|A_+xt \\
  \rightarrow zyx
  \]

  So in this case $|s|$ does not depend on the order of deleting letters.

- On the case $A_+xB_+yA_+zB_-$.
  In this case we can prove similarly as the case of $A_+xB_+yA_+zB_-$.

- On the case $A_+xB_+yA_+zB_-$.
  In this case we can prove similarly as the case of $A_+xB_+yA_+zB_-$.

- On the case $A_+xB_+yA_+zB_-$.
  In this case we can prove similarly as the case of $A_+xB_+yA_+zB_-$.
In this case, If we delete $A$ first, then

$$A - x B - y A - z B - t \rightarrow y^{-} \overline{B} - x - z \overline{B} - t$$

$$\rightarrow \overline{B} - x - z \overline{B} - t y^{-}$$

$$\rightarrow x^{-} z | t y^{-}$$

If we delete $B$ first, then

$$A - x B + y A - z B + t \rightarrow B - y A - z B - t A - x$$

$$\rightarrow z^{-} \overline{A} - y^{-} t \overline{A} - x$$

$$\rightarrow \overline{A} - y^{-} t \overline{A} - x z^{-}$$

$$\rightarrow y t | x z^{-}$$

So in this case $|s|$ does not depend on the order of deleting letters.

- On the cases $\overline{A}_{e_1} x B_{e_2} y \overline{A}_{e_1} z B_{e_2}$, where $e_1, e_2 \in \{+, -\}$. In this cases we can prove similarly as the cases of $A_{-e_1} x B_{e_2} y A_{-e_1} z B_{e_2} t$.

- On the cases $\overline{A}_{e_1} x \overline{B}_{e_2} y \overline{A}_{e_1} z \overline{B}_{e_2}$, where $e_1, e_2 \in \{+, -\}$. In this cases we can prove similarly as the cases of $A_{-e_1} x B_{-e_2} y A_{-e_1} z B_{-e_2} t$.

- On the case $A_{+} x A_{+} y | B_{+} z B_{+} t$

If we delete $A$ first, then

$$A_{+} x A_{+} y | B_{+} z B_{+} t \rightarrow x y | B_{+} z B_{+} t$$

$$\rightarrow x y z | t.$$

If we delete $B$ first, then

$$A_{+} x A_{+} y | B_{+} z B_{+} t \rightarrow A_{+} x A_{+} y | z | t$$

$$\rightarrow x y z | t.$$

So in this case $|s|$ does not depend on the order of deleting letters.

- On the case $A_{-} x A_{-} y | B_{+} z B_{+} t$

If we delete $A$ first, then

$$A_{-} x A_{-} y | B_{+} z B_{+} t \rightarrow x^{-} y | B_{+} z B_{+} t$$

$$\rightarrow x^{-} y z | t.$$

If we delete $B$ first, then
\[ A_x A_y | B_z B_t \rightarrow A_x A_y | zt \]
\[ \rightarrow x^y z | t. \]

So in this case \(|s|\) does not depend on the order of deleting letters.

- On the case \(A_x A_y | B_z B_t\)
  In this case we can prove similarly as the case of \(A_x A_y | B_z B_t\).

- On the case \(A_x A_y | B_z B_t\)
  If we delete \(A\) first, then

\[ A_x A_y | B_z B_t \rightarrow x^y B_z B_t \]
\[ \rightarrow x^y z | t. \]

If we delete \(B\) first, then

\[ A_x A_y | B_z B_t \rightarrow A_x A_y | z^t \]
\[ \rightarrow x^y z | t. \]

So in this case \(|s|\) does not depend on the order of deleting letters.

- On the cases \(A_{\epsilon_1} x A_{\epsilon_1} y | B_{\epsilon_2} B_{\epsilon_2} t\) and \(A_{\epsilon_1} x A_{\epsilon_1} y | B_{\epsilon_2} B_{\epsilon_2} t\)
  where \(\epsilon_1, \epsilon_2 \in \{+, -\}\).
  We can prove similarly as the cases of \(A_{\epsilon_1} x A_{\epsilon_1} y | B_{\epsilon_2} B_{\epsilon_2} t\) and \(A_{-\epsilon_1} x A_{-\epsilon_1} y | B_{-\epsilon_2} B_{-\epsilon_2} t\) respectively.

- On the case \(A_x B_y | A_z B_t\)
  If we delete \(A\) first, then

\[ A_x B_y | A_z B_t \rightarrow x B_y z B_t \]
\[ \rightarrow B_y z B_t | x \]
\[ \rightarrow y z | tx. \]

If we delete \(B\) first, then

\[ A_x B_y | A_z B_t \rightarrow B_y A_x B_t A_z \]
\[ \rightarrow y A_x t A_z \]
\[ \rightarrow A_x t A_z y \]
\[ \rightarrow x t | z y. \]

So in this case \(|s|\) does not depend on the order of deleting letters.
• On the case $A_x B_+ y | A_z B_+ t$
If we delete $A$ first, then

$$A_x B_+ y | A_z B_+ t \rightarrow y \overline{B}_+ x^2 z \overline{B}_+ t$$
$$\rightarrow \overline{B}_+ x^2 z \overline{B}_+ t y^-$$
$$\rightarrow z^- x t y^- .$$

If we delete $B$ first, then

$$A_x B_+ y | A_z B_+ t \rightarrow B_+ y A_x | B_+ t A_z$$
$$\rightarrow y A_x t A_z$$
$$\rightarrow A_x t A_z y$$
$$\rightarrow t^- x^- z y .$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_x B_- | A_+ B_- .$
In this case we can prove similarly as the case of $A_x B_+ | A_+ B_-$.

• On the case $A_+ B_- | A_+ B_-$.

$$A_+ B_- y | A_+ z | A_+ B_+ t \rightarrow y \overline{B}_- x^2 z \overline{B}_- t$$
$$\rightarrow \overline{B}_- x^2 z \overline{B}_- t y^-$$
$$\rightarrow x^- z | t y^- .$$

If we delete $B$ first, then

$$A_+ B_- y | A_+ z | A_+ B_+ t \rightarrow B_- y A_+ x | B_- t A_+ z$$
$$\rightarrow x^- \overline{A}_- y^- t \overline{A}_- z$$
$$\rightarrow \overline{A}_- y^- t \overline{A}_- z x^-$$
$$\rightarrow y^- t | z x^- .$$

So in this case $|s|$ does not depend on the order of deleting letters.

• On the cases $\overline{A}_{e_1} x B_{e_2} y | \overline{A}_{e_1} z B_{e_2} t$ and $\overline{A}_{e_1} x B_{e_2} y | \overline{A}_{e_1} z \overline{B}_{e_2} t$ where $e_1, e_2 \in \{+,-\}$.
We can prove similarly as the cases of $A_{-e_1} x B_{e_2} y | A_{-e_1} z B_{e_2} t$ and $A_{-e_1} x B_{-e_2} y | A_{-e_1} z B_{-e_2} t$ respectively.

• On the case $A_+ x | A_+ y B_+ z B_+ t$
If we delete $A$ first, then
$A_x|A_yB_zB_t \rightarrow xyB_zB_t$
$\rightarrow B_zB_txy$
$\rightarrow z|txy.$

If we delete $B$ first, then

$A_x|A_yB_zB_t \rightarrow A_x|B_zB_tA_y$
$\rightarrow A_x|z|tA_y$
$\rightarrow A_x|A_y|z$
$\rightarrow xy|z.$

So in this case $|s|$ does not depend on the order of deleting letters.

- On the case $A_x|A_yB_zB_t$
If we delete $A$ first, then

$A_+x|A_+yB_zB_t \rightarrow x^-yB_zB_t$
$\rightarrow B_zB_tx^-y$
$\rightarrow z|tx^-y.$

If we delete $B$ first, then

$A_+x|A_+yB_zB_t \rightarrow A_+x|B_zB_tA_y$
$\rightarrow A_+x|z|tA_y$
$\rightarrow A_+x|A_y|z$
$\rightarrow x^-yt|z.$

So in this case $|s|$ does not depend on the order of deleting letters.

- On the case $A_+x|A_+yB_-zB_t$
If we delete $A$ first, then

$A_+x|A_+yB_-zB_t \rightarrow xyB_-zB_t$
$\rightarrow B_-zB_txy$
$\rightarrow z^-txy.$

If we delete $B$ first, then

$A_+x|A_+yB_-zB_t \rightarrow A_+x|B_-zB_tA_y$
$\rightarrow A_+x|z|tA_y$
$\rightarrow A_+x|A_y|z$
$\rightarrow x^-yt|z.$

So in this case $|s|$ does not depend on the order of deleting letters.
\begin{align*}
A_+x|A_+yB_-zB_-t & \rightarrow A_+x|B_-zB_-tA_+y \\
& \rightarrow A_+x|z^-tA_+y \\
& \rightarrow A_+x|A_+yz^-t \\
& \rightarrow xyz^-t.
\end{align*}

So in this case $|s|$ does not depend on the order of deleting letters.

* On the case $A_-x|A_-yB_-zB_-t$

If we delete $A$ first, then

\begin{align*}
A_-x|A_-yB_-zB_-t & \rightarrow x^-yB_-zB_-t \\
& \rightarrow B_-zB_-tx^-y \\
& \rightarrow z^-tx^-y.
\end{align*}

If we delete $B$ first, then

\begin{align*}
A_-x|A_-yB_-zB_-t & \rightarrow A_-x|B_-zB_-tA_-y \\
& \rightarrow A_-x|z^-tA_-y \\
& \rightarrow A_-x|A_-yz^-t \\
& \rightarrow x^-yz^-t.
\end{align*}

So in this case $|s|$ does not depend on the order of deleting letters.

* On the case $A_+x|A_+yB_+zB_+t$, $A_+x|A_+yB_+zB_+t$ and $A_+x|A_+yB_+zB_+t$ is proved similarly as the cases of above.

* On the case $A_+x|A_+yB_+zB_+t$

If we delete $A$ first, then

\begin{align*}
A_+x|A_+y|B_+zB_+t & \rightarrow xy|B_+zB_+t \\
& \rightarrow xy|z|t.
\end{align*}

If we delete $B$ first, then

\begin{align*}
A_+x|A_+y|B_+zB_+t & \rightarrow A_+x|A_+y|z|t \\
& \rightarrow xy|z|t.
\end{align*}

So in this case $|s|$ does not depend on the order of deleting letters.
• On the case $A_{-}x|A_{-}y|B_{+}zB_{+}t$

If we delete $A$ first, then

$$A_{-}x|A_{-}y|B_{+}zB_{+}t \rightarrow x^{-}y|B_{+}zB_{+}t$$
$$\quad \quad \quad \quad \quad \rightarrow x^{-}y|z|t.$$  

If we delete $B$ first, then

$$A_{-}x|A_{-}y|B_{+}zB_{+}t \rightarrow A_{-}x|A_{-}y|z|t$$
$$\quad \quad \quad \quad \quad \rightarrow x^{-}y|z|t.$$  

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_{+}x|A_{+}y|B_{-}zB_{-}t$

If we delete $A$ first, then

$$A_{+}x|A_{+}y|B_{-}zB_{-}t \rightarrow xy|B_{-}zB_{-}t$$
$$\quad \quad \quad \quad \quad \rightarrow xy|z^{-}t.$$  

If we delete $B$ first, then

$$A_{+}x|A_{+}y|B_{-}zB_{-}t \rightarrow A_{+}x|A_{+}y|z^{-}t$$
$$\quad \quad \quad \quad \quad \rightarrow xy|z^{-}t.$$  

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_{-}x|A_{-}y|B_{-}zB_{-}t$

If we delete $A$ first, then

$$A_{-}x|A_{-}y|B_{-}zB_{-}t \rightarrow x^{-}y|B_{-}zB_{-}t$$
$$\quad \quad \quad \quad \quad \rightarrow x^{-}y|z^{-}t.$$  

If we delete $B$ first, then

$$A_{-}x|A_{-}y|B_{-}zB_{-}t \rightarrow A_{-}x|A_{-}y|z^{-}t$$
$$\quad \quad \quad \quad \quad \rightarrow x^{-}y|z^{-}t.$$  

So in this case $|s|$ does not depend on the order of deleting letters.
• On the case $A_{e_1}x|A_{e_1}y|B_{e_2}zB_{e_2}t$, $A_{e_1}x|A_{e_1}y|\overline{B}_{e_2}z\overline{B}_{e_2}t$ and $A_{e_1}x|A_{e_1}y|\overline{B}_{e_2}z\overline{B}_{e_2}t$ is proved similarly as the cases of above.

• On the case $A_+x|B_+y|A_+zB_+t$

If we delete $A$ first, then

$$A_+x|B_+y|A_+zB_+t \rightarrow A_+x|A_+zB_+t|B_+y$$
$$\rightarrow xzB_+t|B_+y$$
$$\rightarrow B_+txz|B_+y$$
$$\rightarrow txzy.$$ 

If we delete $B$ first, then

$$A_+x|B_+y|A_+zB_+t \rightarrow A_+x|B_+y|B_+tA_+z$$
$$\rightarrow A_+x|ytA_+z$$
$$\rightarrow A_+x|A_+zyt$$
$$\rightarrow xzyt.$$ 

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_−x|B_+y|A_−zB_+t$

If we delete $A$ first, then

$$A_−x|B_+y|A_−zB_+t \rightarrow A_−x|A_−zB_+t|B_+y$$
$$\rightarrow x^{-}zB_+t|B_+y$$
$$\rightarrow B_+tx^{-}z|B_+y$$
$$\rightarrow tx^{-}zy.$$ 

If we delete $B$ first, then

$$A_−x|B_+y|A_−zB_+t \rightarrow A_−x|B_+y|B_+tA_−z$$
$$\rightarrow A_−x|ytA_−z$$
$$\rightarrow A_−x|A_−zyt$$
$$\rightarrow x^{-}zyt.$$ 

So in this case $|s|$ does not depend on the order of deleting letters.

• On the case $A_−x|B_+y|A_−zB_+t$
This case is proved similarly as the case of $A_-x|B_+y|A_-zB_+t$.

- On the case $A_-x|B_-y|A_-zB_-t$

  If we delete $A$ first, then

  $$A_-x|B_-y|A_-zB_-t \rightarrow A_-x|A_-zB_-t|B_+y$$

  $$\rightarrow x^{-}zB_-t|B_-y$$

  $$\rightarrow B_-tx^{-}z|B_-y$$

  $$\rightarrow z^{-}xt^{-}y.$$

  If we delete $B$ first, then

  $$A_-x|B_-y|A_-zB_-t \rightarrow A_-x|B_-y|B_-tA_-z$$

  $$\rightarrow A_-x|y^{-}tA_-z$$

  $$\rightarrow A_-x|A_-zy^{-}t$$

  $$\rightarrow t^{-}yz^{-}x.$$

  So in this case $|s|$ does not depend on the order of deleting letters.

- On the case $\bar{A}_{e_1}x|B_{e_2}y|\bar{A}_{e_1}zB_{e_2}t$ and $\bar{A}_{e_1}x|\bar{B}_{e_2}y|\bar{A}_{e_1}z\bar{B}_{e_2}t$ is proved similarly as the cases of above.

- On the case $A_+x|A_+y|B_+zB_+t$

  If we delete $A$ first, then

  $$A_+x|A_+y|B_+z|B_+t \rightarrow xy|B_+z|B_+t$$

  $$\rightarrow xy|zt.$$  

  If we delete $B$ first, then

  $$A_+x|A_+y|B_+z|B_+t \rightarrow A_+x|A_+y|zt$$

  $$\rightarrow xy|zt.$$

  So in this case $|s|$ does not depend on the order of deleting letters.

- On the case $A_-x|A_-y|B_+z|B_+t$

  If we delete $A$ first, then

  $$A_-x|A_-y|B_+z|B_+t \rightarrow x^{-}y|B_+z|B_+t$$

  $$\rightarrow x^{-}y|zt.$$
If we delete $B$ first, then

\[ A_-x|A_-y|B_+z|B_+t \quad \rightarrow \quad A_-x|A_-y|zt \]
\[ \quad \rightarrow \quad x^-y|zt. \]

So in this case $|s|$ does not depend on the order of deleting letters.

- **On the case** $A_+x|A_+y|B_-z|B_-t$
  
  If we delete $A$ first, then

\[ A_+x|A_+y|B_-z|B_-t \quad \rightarrow \quad x^-y|B_-z|B_-t \]
\[ \quad \rightarrow \quad xy|z^-t. \]

If we delete $B$ first, then

\[ A_+x|A_+y|B_-z|B_-t \quad \rightarrow \quad A_+x|A_+y|z^-t \]
\[ \quad \rightarrow \quad xy|z^-t. \]

So in this case $|s|$ does not depend on the order of deleting letters.

- **On the case** $A_-x|A_-y|B_-z|B_-t$
  
  If we delete $A$ first, then

\[ A_-x|A_-y|B_-z|B_-t \quad \rightarrow \quad x^-y|B_-z|B_-t \]
\[ \quad \rightarrow \quad x^-y|z^-t. \]

If we delete $B$ first, then

\[ A_-x|A_-y|B_-z|B_-t \quad \rightarrow \quad A_-x|A_-y|z^-t \]
\[ \quad \rightarrow \quad x^-y|z^-t. \]

So in this case $|s|$ does not depend on the order of deleting letters.

- **On the case** $\overline{A}_{e_1}x|\overline{A}_{e_1}y|B_{e_2}z|B_{e_2}t$ and $\overline{A}_{e_1}x|\overline{A}_{e_1}y|\overline{B}_{e_2}z|\overline{B}_{e_2}t$ is proved similarly as the cases of above.
  
  Now we have completed the proof. \qed

**Remark 3.1.** The deformation $(\ast)$ corresponds to smoothing crossings of a link diagrams in the following figures (cf. [11, Page 320, Figure 1]).
Definition 3.2. For an arbitrary pseudolink \( P \) and state \( s \) of \( P \), we define \([P], [P|s]\) ∈ \( \mathbb{Z}[t, u, d] \) by

\[
[P|s] := t^{|\text{positive marker}|} u^{|\text{negative marker}|} d^{|s|-1},
\]

(3)

\[
[P] := \sum_s [P|s].
\]

(4)

Proposition 3.1. The polynomial \([P]\) is invariant under \( S_1 \)-homotopy moves (H2) for an arbitrary pseudolink \( P \) if and only if \( u = t^{-1} \) and \( d = t^{-1} - t^{-2} \).

Proof. Consider a nanophrase \( P = (P_1|ABxBAY|P_2) \) with \( |A| = + \) and \( |B| = - \), where \( x \) and \( y \) are words not including “|” character. Then

\[
[(P_1|ABxBAY|P_2)] = t[(P_1|BxB|y|P_2)] + s[(P_1|Bx^{-}By|P_2)]
\]

\[
= (t^2 + td + s^2) [(P_1|x^{-}|y|P_2)] + st[(P_1|xy|P_2)]
\]

So if \([P]\) does not change by the second homotopy move, then \( t^2 + td + s^2 = 0 \) and \( st = 1 \). In other words \( s = t^{-1} \) and \( d = -t^2 - t^{-2} \).

Converse is checked easily by the above equation. \( \square \)

Remark 3.2. Substituting \( t^{-1} \) for \( u \) and \(-t^2 - t^{-2} \) for \( d \), we have

\[
[P] = \sum_s t^{\sigma(s)} (-t^2 - t^{-2})^{|s|-1}
\]

where \( \sigma(s) := |\text{ positive marker }| - |\text{ negative marker }| \).

Proposition 3.2. \([P]\) is invariant under \( S_1 \)-homotopy move (H3) for an arbitrary pseudolink \( P \).

Proof. First we consider the case of \((\epsilon(A), \epsilon(B), \epsilon(C)) = (\pm, \pm, \pm)\). Consider the 3rd homotopy move

\[
(P_1|ABxACyBCz|P_2) \rightarrow (P_1|BAxCAyCBz|P_2).
\]

Then

\[
[(P_1|ABxACyBCz|P_2)] = t^{3\epsilon(A)}[(P_1|xy|z|P_2)]
\]

\[
+ (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))[(P_1|zx^{-}y^{-}|P_2)]
\]

\[
+ t^{\epsilon(A)}[(P_1|x^{-}y|z|P_2)]
\]

\[
+ t^{-\epsilon(A)}[(P_1|xy^{-}z|P_2)]
\]

\[
+ t^{-\epsilon(A)}[(P_1|x^{-}yz|P_2)]
\]

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and
\[
([P_1|BAxCyCBz|P_2]) = t^{3\epsilon(A)}([P_1|xy|z|P_2]) \\
+ (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))(P_1|x^{-1}y|z|P_2) \\
+ t^{\epsilon(A)}([P_1|x^{-1}y|z|P_2]) \\
+ t^{-\epsilon(A)}([P_1|xy^{-1}z|P_2]) \\
+ t^{-\epsilon(A)}([P_1|z^{-1}y^{-1}x|P_2]).
\]

Note that
\[2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}) = t^{\epsilon(A)}\]

So \([P_1|ABxACyBCz|P_2]\) is equal to \([P_1|BAxCyCBz|P_2]\).

Consider the 3rd homotopy move
\[(P_1|ABx|ACyBCz|P_2) \longrightarrow (P_1|BAx|CAyCBz|P_2).
\]

Then
\[
([P_1|ABx|ACyBCz|P_2]) = t^{3\epsilon(A)}([P_1|xzy|P_2]) \\
+ (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))(P_1|z^{-1}x^{-1}y|P_2) \\
+ t^{-\epsilon(A)}([P_1|x^{-1}y^{-1}z|P_2]) \\
+ t^{\epsilon(A)}([P_1|x^{-1}y^{-1}z|P_2]) \\
+ t^{-\epsilon(A)}([P_1|z^{-1}y^{-1}z|P_2]).
\]

and
\[
([P_1|BAx|CAyCBz|P_2]) = t^{3\epsilon(A)}([P_1|xzy|P_2]) \\
+ (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))(P_1|z^{-1}x^{-1}y|P_2) \\
+ t^{-\epsilon(A)}([P_1|x^{-1}y^{-1}z|P_2]) \\
+ t^{-\epsilon(A)}([P_1|x^{-1}y^{-1}z|P_2]) \\
+ t^{\epsilon(A)}([P_1|x^{-1}y^{-1}z|P_2]).
\]

So \([P_1|ABx|ACyBCz|P_2]\) is equal to \([P_1|BAx|CAyCBz|P_2]\).

Consider the 3rd homotopy move
\[(P_1|ABxACy|BCz|P_2) \longrightarrow (P_1|BAxCy|CAz|P_2).
\]

Then
\[
([P_1|ABxACy|BCz|P_2]) = t^{3\epsilon(A)}([P_1|xzy|P_2]) \\
+ (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))(P_1|xy^{-1}z|P_2) \\
+ t^{-\epsilon(A)}([P_1|x^{-1}y^{-1}z|P_2]) \\
+ t^{\epsilon(A)}([P_1|x^{-1}y^{-1}z|P_2]) \\
+ t^{-\epsilon(A)}([P_1|yx^{-1}|z|P_2])
\]
and
\[
[(P_1|BAx|CAy|CBz|P_2)] = t^{3\epsilon(A)}[(P_1|xyz|P_2)] + (2t^{-\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))[(P_1|x^{-}yz|P_2)] + t^{-\epsilon(A)}[(P_1|xy^{-}z|P_2)] + t^{-\epsilon(A)}[(P_1|y^{-}xz|P_2)] + t^{-\epsilon(A)}[(P_1|yz-xz|P_2)].
\]

So \([(P_1|ABxACy|BCz|P_2)]\) is equal to \([(P_1|BAxCAy|CBz|P_2)]\).

Consider the 3rd homotopy move
\[
(P_1|ABx|ACy|BCz|P_2) \to (P_1|BAx|CAy|CBz|P_2).
\]

Then
\[
[(P_1|ABx|ACy|BCz|P_2)] = t^{3\epsilon(A)}[(P_1|yxz|P_2)] + (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))[(P_1|y^{-}xz|P_2)] + t^{-\epsilon(A)}[(P_1|x^{-}zy|P_2)] + t^{-\epsilon(A)}[(P_1|zx|P_2)] + t^{-\epsilon(A)}[(P_1|y^{-}xz|P_2)].
\]

and
\[
[(P_1|BAx|CAy|CBz|P_2)] = t^{3\epsilon(A)}[(P_1|yzx|P_2)] + (2t^{\epsilon(A)} - t^{-3\epsilon(A)} + t^{-\epsilon(A)}(-t^2 - t^{-2}))[(P_1|y^{-}xz|P_2)] + t^{-\epsilon(A)}[(P_1|x^{-}zy|P_2)] + t^{-\epsilon(A)}[(P_1|x^{-}y|P_2)] + t^{-\epsilon(A)}[(P_1|z^{-}yx|P_2)].
\]

So \([(P_1|ABx|ACy|BCz|P_2)]\) is equal to \([(P_1|BAxCAy|CBz|P_2)]\).

The cases of \((\epsilon(A), \epsilon(B), \epsilon(C)) = (\mp, \pm, \pm)\) and \((\epsilon(A), \epsilon(B), \epsilon(C)) = (\pm, \pm, \mp)\) are proved similarly as the above case.

**Proposition 3.3.** For an arbitrary pseudolink \(P\), the Jones polynomial \(J(P)\) for pseudolinks,
\[
J(P) = (-t)^{-3w(P)} \sum_{s: \text{states}} t^{\sigma(s)}(-t^2 - t^{-2})^{|s|-1}
\]
where \(w(P) = \sum_{\text{letters } A \text{ in } P} |A|\).

**Remark 3.3.** The Jones polynomial \(J(P)\) of a pseudolink \(P\) is given by using recursive relations for the bracket polynomial of nanophrases over \(\alpha\). [8, Section 8]. It is obvious the existence of \(J(P)\) by using geometrical objects (i.e. links). However, it is not clear that the well-definedness of \(J(P)\) is given in only word theory of Turaev. Then, we give the well-definedness by Lemma 3.1 and (5) using only \(P\) of \(\mathcal{P}(\alpha_1, S_1, \text{id})\).

Definition 3.3 of enhanced states is given in the manner as in [11, Page 326, Subsection 4.3].
Definition 3.3. By an enhanced state $S$ of pseudolink $P$ we mean a collection of markers constituting a state $s$ of $P$ enhanced by an assignment of a plus or minus sign to each of the components $(\emptyset | \cdots | \emptyset)$. (Recall that $(\emptyset | \cdots | \emptyset)$ is obtained by deformations $(\ast)$.) We denote $\emptyset$ with a positive marker $+$ by $\emptyset_+$ and $\emptyset$ with a negative marker $-$ by $\emptyset_-$.

Notation 3.2. We rewrite the deformation $(\ast)$ as follows:

\begin{align*}
(\ast) & \quad (w_1 \cdots | AxAy| \cdots | w_k) \rightarrow \begin{cases}
(w_1 \cdots | ax|ay| \cdots | w_k) & \text{if } \text{mark}(A) = |A| \\
(w_1 \cdots | ax^{-}ay| \cdots | w_k) & \text{if } \text{mark}(A) = -|A| \\
(w_1 \cdots | axay| \cdots | w_k) & \text{if } \text{mark}(A) = |A| \\
(w_1 \cdots | ax^{-}ay| \cdots | w_k) & \text{if } \text{mark}(A) = -|A|.
\end{cases}
\end{align*}

A pseudolink $(a_1^1 \cdots a_n^1 | a_2^2 \cdots a_{n_2}^2 | \cdots | a_{n_k}^{k'} \cdots a_{n_k'}^{k'})$ given by repeating these deformations $(\ast)$ from $P$ represents an enhanced state $S$ and the pseudolink is denoted by $B_1^1 \cdots B_{m_1}^1 \ B_2^2 \cdots B_{m_2}^2 \ B_3^{k'} \cdots B_{m_{k'}}^{k'}$. We denote $\emptyset_{e_1} | \emptyset_{e_2} | \cdots | \emptyset_{e_{k'}}$ where $B_1^i \cdots B_{m_i}^i$ is a word obtained by arranging all the distinct letters in $\{A_1^i, \ldots, A_{n_i}^i\}$ in any desired order. Note that $\{A_1^i, \ldots, A_{n_i}^i\}$ corresponds to $\{a_1^i, \ldots, a_{n_i}^i\}$.

Example 3.3. Example 3.2 is rewritten by using Notation 3.2. $(A_+B_+A+C_+B+C_+)$

\begin{align*}
(\ast) & \quad (aB_+ | aC_+ B_+ C_+) = (B_+ a | B_+ C_+ a C_+) \quad (\ast) & \quad (bab | aC_+ b) = (C_+ a C_+ b a) \quad (\ast) & \quad (ca | A C A B C) \\
(\ast) & \quad (c b a b) = (\emptyset | \emptyset). \quad (A_+ B_+ A_+ B_+ B_+ C_+) \quad (\ast) & \quad (aB_+ a C_+ B_+ C_+) = (B_+ a C_+ B_+ C_+) \quad (\ast) & \quad (AB_+ C_+ a b) \\
(\ast) & \quad (bC_+ a b C_+ a) = (C_+ a b C_+ a b) \quad (\ast) & \quad (c b a b) = (\emptyset | \emptyset).
\end{align*}

Notation 3.3. For an arbitrary enhanced state $S$ of pseudolink $P$, let

\begin{align*}
i(S) & := \frac{w(P) - \sigma(S)}{2} , \\
\tau(S) & := z\{\emptyset_+ \text{ in } P_S\} - z\{\emptyset_- \text{ in } P_S\} , \\
j(S) & := -\frac{\sigma(S) + 2\tau(S) - 3w(P)}{2} \in \mathbb{Z}.
\end{align*}
Let $s$ be a state of a pseudolink $P$, $S$ be an enhanced state of $P$ and $\hat{J}(P) = (-t^2 - t^{-2})J(P)$. By using these notations above we have

\begin{align}
\hat{J}(P) &= (-t)^{-3w(P)} \sum_{\text{states } s} t^{\sigma(s)}(-t^2 - t^{-2})|s| \\
&= (-t)^{-3w(P)} \sum_{\text{enhanced states } S} t^{\sigma(S)}(-t^2)^{\tau(S)} \\
&= \sum_{\text{enhanced states } S} (-1)^{w(P) + \tau(S)} t^{-2j(S)} \\
&= \sum_{\text{enhanced states } S} (-1)^{w(P) - \sigma(S)} \frac{q^{j(S)}}{q^{j(S)}} (q = -t^{-2}) \\
&= \sum_{\text{enhanced states } S} (-1)^{i(S)} q^{j(S)}.
\end{align}

**Remark 3.4.** Let $\alpha_0$ be the set $\{-1, 1\}$ with the involution $\tau_0 : \pm 1 \mapsto \mp 1$ and $S_0$ be $\{(-1, -1, -1), (1, 1, 1)\}$. Note that every $S_1$-homotopy invariant of pseudolinks is $S_0$-homotopy invariant of nanophrases over $\alpha_0$ because $S_0 \subset S_1$.

**Corollary 3.1.** $J(P)$ and $\hat{J}(P)$ are $S_0$-homotopy invariants for nanophrases $P$ over $\alpha_0$.

### 4. Khovanov Homology for Pseudolinks.

**Definition 4.1.** For an arbitrary pseudolink $P$, let $C(P)$ be a free abelian group generated by enhanced states of $P$. We define the subgroup $C^{i,j}(P)$ of $C(P)$ by

$$C^{i,j}(P) := \langle S : \text{enhanced states } | j(S) = j, i(S) = i \rangle \ (i, j \in \mathbb{Z}).$$

**Remark 4.1.** The Jones polynomial $\hat{J}(P) = \sum_{j=-\infty}^{\infty} q^j \sum_{i=-\infty}^{\infty} (-1)^{i} \text{rk} C^{i,j}(P)$.

Let us define the differential $d$ of bidegree $(1, 0)$ as follows:

$$d(S) = \sum_{\text{enhanced states } T} (S : T) T.$$

In other words, for two arbitrary enhanced states $S$ and $T$, we define incidence numbers $(S : T)$. We give the definition of differential in the manner as in [11, Section 5]. Assume that the order of letters in the alphabet of a pseudolink $P$ is given.

**Definition 4.2.** The incidence number $(S : T)$ is zero unless the markers of $S$ and $T$ differ at only one letter of $P$ and this letter is called the different part between $S$ and $T$. The marker of $S$ is positive, and that of $T$ is negative at this different part. If $(S : T) \neq 0$, the different part between $S$ and $T$ satisfy one of the six cases (15)–(20) in the following:
For (15)–(20), $(S : T)$ is defined as

$$(S : T) := 1.$$

**Theorem 4.1.** $d \cdot d = 0 \mod 2.$
Proof. Let $\epsilon_i$ be $i$th marker of $i$th letter and so $\epsilon_i$ is an element of $\{+, -\}$. Consider the tuple $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ consisting of all the markers of a phrase. If card \( \{ j \mid \epsilon_j = + \} \leq 1 \), $d^2(S) = 0$. So we can assume that card \( \{ j \mid \epsilon_j = + \} \geq 2 \) now.

To prove

\[
d \circ d(S) = \sum_{\text{enhanced states } T,U} (S : T)(T : U)U = 0,
\]

we show $\sum_{\text{enhanced states } T}(S : T)(T : U) = 0$.

Let $A$ and $B$ be different parts between $S$ and $U$. We can assume that the other letters in the phrase are already deleted by the deformation (**). We denote phrases which were consisted of letter replaced by the deformation (**) by $\alpha_j$ $(j \in \{1, \ldots, k\})$, $x,y,z$ and $t$. We denote a state $S$ by $S = (\text{a phrase } P \text{ with markers, a pseudolink given by repeating deformation (**) from } P \text{ to the end})$. We check following 26 cases:

(1)

\[
S = ((\alpha_1| \ldots |\alpha_{l-1}|A_+xA_B+yB_BzB_t|\alpha_{l+1}| \ldots |\alpha_k), \\
\ldots A \ldots AB \ldots A \ldots \\
(\varnothing_{\epsilon_1}| \ldots |\varnothing_{\epsilon_{l-1}} | \varnothing_{\epsilon_{l_1}} | \varnothing_{\epsilon_{l_2}} | \varnothing_{\epsilon_{l_3}} | \varnothing_{\epsilon_{l+1}} | \ldots |\varnothing_{\epsilon_k}))
\]

(2)

\[
S = ((\alpha_1| \ldots |\alpha_{l-1}|A_+xA_B+yB_BzB_t|\alpha_{l+1}| \ldots |\alpha_k), \\
\ldots AB \ldots B \ldots \\
(\varnothing_{\epsilon_1}| \ldots |\varnothing_{\epsilon_{l-1}} | \varnothing_{\epsilon_{l_1}} | \varnothing_{\epsilon_{l_2}} | \varnothing_{\epsilon_{l_3}} | \varnothing_{\epsilon_{l+1}} | \ldots |\varnothing_{\epsilon_k}))
\]

(3)

\[
S = ((\alpha_1| \ldots |\alpha_{l-1}|A_+xA_B+yB_BzB_t|\alpha_{l+1}| \ldots |\alpha_k), \\
\ldots AB \ldots \\
(\varnothing_{\epsilon_1}| \ldots |\varnothing_{\epsilon_{l-1}} | \varnothing_{\epsilon_{l_1}} | \varnothing_{\epsilon_{l_2}} | \varnothing_{\epsilon_{l_3}} | \varnothing_{\epsilon_{l+1}} | \ldots |\varnothing_{\epsilon_k}))
\]

(4)

\[
S = ((\alpha_1| \ldots |\alpha_{l-1}|A_+xA_B+yA_AzB_t|\alpha_{l+1}| \ldots |\alpha_k), \\
\ldots AB \ldots \\
(\varnothing_{\epsilon_1}| \ldots |\varnothing_{\epsilon_{l-1}} | \varnothing_{\epsilon_{l_1}} | \varnothing_{\epsilon_{l_2}} | \varnothing_{\epsilon_{l_3}} | \varnothing_{\epsilon_{l+1}} | \ldots |\varnothing_{\epsilon_k}))
\]

(5)

\[
S = ((\alpha_1| \ldots |\alpha_{l-1}|A_+xA_B+yA_AzB_t|\alpha_{l+1}| \ldots |\alpha_k), \\
\ldots AB \ldots \\
(\varnothing_{\epsilon_1}| \ldots |\varnothing_{\epsilon_{l-1}} | \varnothing_{\epsilon_{l_1}} | \varnothing_{\epsilon_{l_2}} | \varnothing_{\epsilon_{l_3}} | \varnothing_{\epsilon_{l+1}} | \ldots |\varnothing_{\epsilon_k}))
\]

(6)

\[
S = ((\alpha_1| \ldots |\alpha_{l-1}|A_+xB_A+yA_AzB_t|\alpha_{l+1}| \ldots |\alpha_k), \\
\ldots AB \ldots AB \ldots \\
(\varnothing_{\epsilon_1}| \ldots |\varnothing_{\epsilon_{l-1}} | \varnothing_{\epsilon_{l_1}} | \varnothing_{\epsilon_{l_2}} | \varnothing_{\epsilon_{l_3}} | \varnothing_{\epsilon_{l+1}} | \ldots |\varnothing_{\epsilon_k}))
\]
\[ S = ((\alpha_1 \cdots | A_{i-1} A_i x A_i y B_i z B_i t | \alpha_{i+1} \cdots | \alpha_k), \]
\[ \cdots A \cdots B \cdots B \cdots \]
\[ (\theta_{e_1} \cdots | \theta_{e_{i-1}} | \theta_{e_{i+1}} | \cdots | \theta_{e_k})) \]

\[ S = ((\alpha_1 \cdots | A_{i-1} A_i x A_i y B_i z B_i t | \alpha_{i+1} \cdots | \alpha_k), \]
\[ \cdots A \cdots B \cdots \]
\[ (\theta_{e_1} \cdots | \theta_{e_{i-1}} | \theta_{e_{i+1}} | \theta_{e_{i+1}} | \cdots | \theta_{e_k})) \]

\[ S = ((\alpha_1 \cdots | A_{i-1} A_i x A_i y B_i z B_i t | \alpha_{i+1} \cdots | \alpha_k), \]
\[ \cdots A B \cdots A B \cdots \]
\[ (\theta_{e_1} \cdots | \theta_{e_{i-1}} | \theta_{e_{i+1}} | \theta_{e_{i+1}} | \cdots | \theta_{e_k})) \]

\[ S = ((\alpha_1 \cdots | A_{i-1} A_i x A_i y B_i z B_i t | \alpha_{i+1} \cdots | \alpha_k), \]
\[ \cdots A B \cdots \]
\[ (\theta_{e_1} \cdots | \theta_{e_{i-1}} | \theta_{e_{i+1}} | \cdots | \theta_{e_k})) \]

\[ S = ((\alpha_1 \cdots | A_{i-1} A_i x A_i y B_i z B_i t | \alpha_{i+1} \cdots | \alpha_k), \]
\[ \cdots A B \cdots A B \cdots \]
\[ (\theta_{e_1} \cdots | \theta_{e_{i-1}} | \theta_{e_{i+1}} | \theta_{e_{i+1}} | \cdots | \theta_{e_k})) \]

\[ S = ((\alpha_1 \cdots | A_{i-1} A_i x A_i y B_i z B_i t | \alpha_{i+1} \cdots | \alpha_k), \]
\[ \cdots A B \cdots A B \cdots \]
\[ (\theta_{e_1} \cdots | \theta_{e_{i-1}} | \theta_{e_{i+1}} | \cdots | \theta_{e_k})) \]

\[ S = ((\alpha_1 \cdots | A_{i-1} A_i x A_i y B_i z B_i t | \alpha_{i+1} \cdots | \alpha_k), \]
\[ \cdots A B \cdots A B \cdots \]
\[ (\theta_{e_1} \cdots | \theta_{e_{i-1}} | \theta_{e_{i+1}} | \cdots | \theta_{e_k})) \]

\[ S = ((\alpha_1 \cdots | A_{i-1} A_i x A_i y B_i z B_i t | \alpha_{i+1} \cdots | \alpha_k), \]
\[ \cdots A B \cdots \]
\[ (\theta_{e_1} \cdots | \theta_{e_{i-1}} | \theta_{e_{i+1}} | \cdots | \theta_{e_k})) \]
\begin{align}
S &= ((\alpha_1 \cdots |_{t-1} \bar{A}_x \mid \bar{A}_y \bar{B}_z \bar{t} \mid \alpha_{t+1} \cdots | \alpha_k), \\
    \cdots A \cdots B \cdots & (\theta_{\epsilon_1} \cdots | \theta_{\epsilon_{t-1}} \mid \theta_{\epsilon_{t+1}} \cdots | \theta_{\epsilon_k}))
\end{align}

\begin{align}
S &= ((\alpha_1 \cdots |_{t-1} A_x \mid A_y \mid B_z \bar{B}_t \mid \alpha_{t+1} \cdots | \alpha_k), \\
    \cdots A \cdots B \cdots & (\theta_{\epsilon_1} \cdots | \theta_{\epsilon_{t-1}} \mid \theta_{\epsilon_{t+1}} \cdots | \theta_{\epsilon_k}))
\end{align}

\begin{align}
S &= ((\alpha_1 \cdots |_{t-1} \bar{A}_x \mid \bar{A}_y \bar{B}_z \bar{t} \mid \alpha_{t+1} \cdots | \alpha_k), \\
    \cdots A \cdots B \cdots & (\theta_{\epsilon_1} \cdots | \theta_{\epsilon_{t-1}} \mid \theta_{\epsilon_{t+1}} \cdots | \theta_{\epsilon_k}))
\end{align}

\begin{align}
S &= ((\alpha_1 \cdots |_{t-1} A_x \mid B_y \mid A_z \bar{B}_t \mid \alpha_{t+1} \cdots | \alpha_k), \\
    \cdots A \cdots B \cdots & (\theta_{\epsilon_1} \cdots | \theta_{\epsilon_{t-1}} \mid \theta_{\epsilon_{t+1}} \cdots | \theta_{\epsilon_k}))
\end{align}

\begin{align}
S &= ((\alpha_1 \cdots |_{t-1} \bar{A}_x \mid \bar{B}_y \bar{A}_z \bar{t} \mid \alpha_{t+1} \cdots | \alpha_k), \\
    \cdots A \cdots B \cdots & (\theta_{\epsilon_1} \cdots | \theta_{\epsilon_{t-1}} \mid \theta_{\epsilon_{t+1}} \cdots | \theta_{\epsilon_k}))
\end{align}

\begin{align}
S &= ((\alpha_1 \cdots |_{t-1} \bar{A}_x \mid \bar{B}_y \bar{A}_z \bar{t} \mid \alpha_{t+1} \cdots | \alpha_k), \\
    \cdots A \cdots B \cdots & (\theta_{\epsilon_1} \cdots | \theta_{\epsilon_{t-1}} \mid \theta_{\epsilon_{t+1}} \cdots | \theta_{\epsilon_k}))
\end{align}

\begin{align}
S &= ((\alpha_1 \cdots |_{t-1} A_x \mid A_y \mid B_z \bar{B}_t \mid \alpha_{t+1} \cdots | \alpha_k), \\
    \cdots A \cdots B \cdots & (\theta_{\epsilon_1} \cdots | \theta_{\epsilon_{t-1}} \mid \theta_{\epsilon_{t+1}} \cdots | \theta_{\epsilon_k}))
\end{align}
\[ S = ((\alpha_1| \cdots |\alpha_{l-1}|A_+x|A_+y|B_+z|B_+t|\alpha_{l+1}| \cdots |\alpha_k), \]
\[ \cdots A \cdots B \cdots \]
\[ (\theta_{\epsilon_1}| \cdots |\theta_{\epsilon_{l-1}}| \theta_{\epsilon_{l+1}}| \cdots |\theta_{\epsilon_k}) \]
\]
\[ (\alpha_1| \cdots |\alpha_{l-1}|A_+x|A_+y|B_+z|B_+t|\alpha_{l+1}| \cdots |\alpha_k), \]
\[ \cdots A \cdots B \cdots \]
\[ (\theta_{\epsilon_1}| \cdots |\theta_{\epsilon_{l-1}}| \theta_{\epsilon_{l+1}}| \cdots |\theta_{\epsilon_k}) \]
\]

• Consider the case (1)

Let
\[ U = ((\alpha_1| \cdots |\alpha_{l-1}|A_+xA_+yB_+zB_+t|\alpha_{l+1}| \cdots |\alpha_k), \]
\[ \cdots AB \cdots \]
\[ (\theta_{\epsilon_1}| \cdots |\theta_{\epsilon_{l-1}}| \theta_{\epsilon_{l+1}}| \cdots |\theta_{\epsilon_k}) \]

It is sufficient to show that for each \((\epsilon_1, \epsilon_{12}, \epsilon_{13}) \in \{+, -\} \times \{+, -\} \times \{+, -\}\), the coefficient of \(U\) in \(d^2(S)\) is even for all \(\epsilon_{11} \in \{+, -\}\). Hence for \(S\) and \(U\), we have to check the total number of ways to get \(U\) from \(S\) is even (we denote the condition by \((\check{\epsilon})\)). Let us localize the problem of the difference parts of \(S\), \(A_+\) and \(B_+\). Two routes (i) and (ii) can be found to change \(A_+\) (respectively \(B_+\)) into \(A_-\) (respectively \(B_-\)) as follows:

(i)
\[ S = ((\alpha_1| \cdots |\alpha_{l-1}|A_+xA_+yB_+zB_+t|\alpha_{l+1}| \cdots |\alpha_k), \]
\[ \cdots A \cdots A \cdots B \cdots \]
\[ (\theta_{\epsilon_1}| \cdots |\theta_{\epsilon_{l-1}}| \theta_{\epsilon_{l+1}}| \cdots |\theta_{\epsilon_k}) \]

\[ \rightarrow T = ((\alpha_1| \cdots |\alpha_{l-1}|A_+xA_+yB_+zB_+t|\alpha_{l+1}| \cdots |\alpha_k), \]
\[ \cdots A \cdots B \cdots \]
\[ (\theta_{\epsilon_1}| \cdots |\theta_{\epsilon_{l-1}}| \theta_{\epsilon_{l+1}}| \cdots |\theta_{\epsilon_k}) \rightarrow U, \]

(ii)
\[ S = ((\alpha_1| \cdots |\alpha_{l-1}|A_+xA_+yB_+zB_+t|\alpha_{l+1}| \cdots |\alpha_k), \]
\[ \cdots A \cdots A \cdots B \cdots \]
\[ (\theta_{\epsilon_1}| \cdots |\theta_{\epsilon_{l-1}}| \theta_{\epsilon_{l+1}}| \cdots |\theta_{\epsilon_k}) \]

\[ \rightarrow T = ((\alpha_1| \cdots |\alpha_{l-1}|A_+xA_+yB_+zB_+t|\alpha_{l+1}| \cdots |\alpha_k), \]
\[ \cdots A \cdots A \cdots B \cdots \]
\[ (\theta_{\epsilon_1}| \cdots |\theta_{\epsilon_{l-1}}| \theta_{\epsilon_{l+1}}| \cdots |\theta_{\epsilon_k}) \rightarrow U. \]

Then the condition \((\check{\epsilon})\) can be also state that the sum of the contribution of (i) to the coefficient of \(U\) and the contribution of (ii) to the coefficient of \(U\) is even.

The case \((\epsilon_{11}, \epsilon_{12}, \epsilon_{13}) = (+, +, +)\).

In this case \((S, T) = 0\) for all \((\epsilon_{21}, \epsilon_{22})\) and \((\epsilon_{31}, \epsilon_{32})\). So the condition \((\check{\epsilon})\) holds.

The case \((\epsilon_{11}, \epsilon_{12}, \epsilon_{13}) = (-, +, +)\).
Consider the route (i), \((S, T)\) is not equal to 0 if and only if \((\varepsilon_{21}, \varepsilon_{22}) = (+, +)\). Then for this \(T, (T, U) = 0\) for all \(\varepsilon_{41} \in \{\pm\}\). On the other hand, in route (ii), \((S, T) = 0\) for all \(\varepsilon_{31}, \varepsilon_{32} \in \{\pm\}\). So the condition (\(\sharp\)) holds.

The case \((\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}) = (+, +, +)\).

Consider the route (i) (respectively the route (ii)), \((S, T)\) is not equal to 0 if and only if \((\varepsilon_{21}, \varepsilon_{22}) = (+, +)\) (respectively \((\varepsilon_{21}, \varepsilon_{22}) = (+, +)\)). Then for this \(T, (T, U) = 0\) for all \(\varepsilon_{41}\). So the condition (\(\sharp\)) holds.

The case \((\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}) = (+, +, -)\).

Consider the route (i), in this route \((S, T) = 0\) for all \(\varepsilon_{21}, \varepsilon_{22} \in \{\pm\}\). On the other hand, in route (ii), \((S, T)\) is not equal to 0 if and only if \((\varepsilon_{31}, \varepsilon_{32}) = (+, +)\). Then for this \(T, (T, U) = 0\) for all \(\varepsilon_{41}\). So the condition (\(\sharp\)) holds.

The case \((\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}) = (+, -, -)\).

Consider the route (i), \((S, T)\) is not equal to 0 if and only if \((\varepsilon_{21}, \varepsilon_{22}) = (+, -)\). Then for this \(T, (T, U)\) is not equal to 0 if and only if \(\varepsilon_{41} = +\). Similarly, in route (ii), \((S, T)\) is not equal to 0 for all \((\varepsilon_{31}, \varepsilon_{32}) = (+, -)\). Then for this \(T, (T, U)\) is not equal to 0 if and only if \(\varepsilon_{41} = +\). So the condition (\(\sharp\)) holds.

The case \((\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}) = (-, -, -)\).

Consider the route (i), \((S, T)\) is not equal to 0 if and only if \((\varepsilon_{21}, \varepsilon_{22}) = (-, +)\). Then for this \(T, (T, U)\) is not equal to 0 if and only if \(\varepsilon_{41} = +\). Similarly, in route (ii), \((S, T)\) is not equal to 0 for all \((\varepsilon_{31}, \varepsilon_{32}) = (-, +)\). Then for this \(T, (T, U)\) is not equal to 0 if and only if \(\varepsilon_{41} = +\). So the condition (\(\sharp\)) holds.

The case \((\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}) = (-, -, -)\).

Consider the route (i), \((S, T)\) is not equal to 0 if and only if \((\varepsilon_{21}, \varepsilon_{22}) = (-, -)\). Then for this \(T, (T, U)\) is not equal to 0 if and only if \(\varepsilon_{41} = +\). Similarly, in route (ii), \((S, T)\) is not equal to 0 for all \((\varepsilon_{31}, \varepsilon_{32}) = (-, -)\). Then for this \(T, (T, U)\) is not equal to 0 if and only if \(\varepsilon_{41} = +\). So the condition (\(\sharp\)) holds.

- Consider the case (2)

Let

\[
U = ((\alpha_1|\cdots|\alpha_{t-1}|A_{-}x\overline{A}_{-}yB_{-}zB_{-}t|\alpha_{t+1}|\cdots|\alpha_k), \\
\quad \cdots A \cdots AB \cdots \\
\quad (\theta_{\varepsilon_1}|\cdots|\theta_{\varepsilon_{t-1}}|\theta_{\varepsilon_{41}}|\theta_{\varepsilon_{42}}|\theta_{\varepsilon_{43}}|\cdots|\theta_{\varepsilon_k})).
\]

It is sufficient to show that for each \((\varepsilon_{11}, \varepsilon_{12}) \in \{\pm, \pm\}\) where double signs are arbitrary, the coefficient of \(U\) in \(\mathcal{d}^2(S)\) is even for all \(\varepsilon_{41}, \varepsilon_{42} \in \{\pm\}\). Hence for \(S\) and \(U\), we have to check the total number of ways to get \(U\) from \(S\) is even (we denote the condition by (\(\sharp\))). Let us localize the problem of the difference parts of \(S, A_+\) and \(B_+\). Two routes (i) and (ii) can be found to change \(A_+\) (respectively \(B_+\)) into \(A_-\) (respectively \(B_+\)) as follows:

(i)

\[
S = ((\alpha_1|\cdots|\alpha_{t-1}|A_{+}x\overline{A}_{+}yB_{+}zB_{+}t|\alpha_{t+1}|\cdots|\alpha_k), \\
\quad \cdots AB \cdots \cdots B \cdots \cdots A \cdots \\
\quad (\theta_{\varepsilon_1}|\cdots|\theta_{\varepsilon_{t-1}}|\theta_{\varepsilon_{41}}|\theta_{\varepsilon_{42}}|\theta_{\varepsilon_{43}}|\cdots|\theta_{\varepsilon_k})).
\]
Then the condition (\(\varepsilon\)) can be also state that the sum of the contribution of (i) to the coefficient of \(U\) and the contribution of (ii) to the coefficient of \(U\) is even.

The case (\(\varepsilon_{11}, \varepsilon_{12}\)) = (+, +).

Consider the route (i). Then \((S, T)\) is not equal to 0 if and only if \((\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (+, +, +)\). For this \(T\), \((T, U) = 0\) for all \(\varepsilon_{41}, \varepsilon_{42} \in \{\pm\}\). On the other hand, in route (ii), we obtain \((S, T) = 0\) for all \(\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23} \in \{\pm\}\). So the condition (\(\varepsilon\)) holds.

The case (\(\varepsilon_{11}, \varepsilon_{12}\)) = (+, -).

Consider the route (i). Then \((S, T)\) is not equal to 0 if and only if \((\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (+, +, +)\). For this \(T\), \((T, U) = 0\) if and only if \((\varepsilon_{41}, \varepsilon_{42}) = (+, +)\).

Consider the route (ii). Then \((S, T)\) is not equal to 0 if and only if \(\varepsilon_{31} = +\). For this \(T\), \((T, U) = 0\) if and only if \((\varepsilon_{41}, \varepsilon_{42}) = (+, +)\). So the condition (\(\varepsilon\)) holds.

The case (\(\varepsilon_{11}, \varepsilon_{12}\)) = (-, +).

Consider the route (i). Then \((S, T)\) is not equal to 0 if and only if \((\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (-, +, +)\). Put \((\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (-, +, +)\). For this \(T\), we obtain \((T, U) = 0\) for all \(\varepsilon_{41}, \varepsilon_{42} \in \{\pm\}\). Put \((\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (+, -, +)\). For this \(T\), \((T, U) = 0\) if and only if \((\varepsilon_{41}, \varepsilon_{42}) = (+, +)\) or \((\varepsilon_{41}, \varepsilon_{42}) = (+, -)\). Consider the route (ii). Then \((S, T)\) is not equal to 0 if and only if \(\varepsilon_{31} = +\). For this \(T\), \((T, U) = 0\) if and only if \((\varepsilon_{41}, \varepsilon_{42}) = (+, +)\). So the condition (\(\varepsilon\)) holds.

The case (\(\varepsilon_{11}, \varepsilon_{12}\)) = (-, -).

Consider the route (i). Then \((S, T)\) is not equal to 0 if and only if \((\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (-, +, +)\) or \((\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (+, -, +)\). Put \((\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}) = (-, +, +)\). For this \(T\), \((T, U) = 0\) if and only if \((\varepsilon_{41}, \varepsilon_{42}) = (+, +)\) or \((\varepsilon_{41}, \varepsilon_{42}) = (+, -)\). Consider the route (ii). Then \((S, T)\) is not equal to 0 if and only if \(\varepsilon_{31} = -\). For this \(T\), \((T, U) = 0\) if and only if \((\varepsilon_{41}, \varepsilon_{42}) = (+, -)\) or \((\varepsilon_{41}, \varepsilon_{42}) = (-, +)\). So the condition (\(\varepsilon\)) holds.

Let

\[
U = ((\alpha_1| \cdots |\alpha_{l-1} | A \cdot x \cdot y \cdot B \cdot z \cdot t | \alpha_{l+1}| \cdots | \alpha_k),
\]

\[
\cdots A \cdots AB \cdots B \cdots \]

\[
(\theta_1| \cdots |\theta_{l-1} | \theta_{e_{21}} | \theta_{e_{22}} | \theta_{e_{23}} | \theta_{e_{l+1}}| \cdots |\theta_{e_k})) \rightarrow U.
\]
It is sufficient to show that for each \( \varepsilon_{11} \in \{ \pm \} \) the coefficient of \( U \) in \( d^2(S) \) is even for all \( \varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43} \in \{ \pm \} \). Hence for \( S \) and \( U \), we have to check the total number of ways to get \( U \) from \( S \) is even (we denote the condition by \( \ddagger \)). Let us localize the problem of the difference parts of \( S, A_+ \) and \( B_+ \). Two routes (i) and (ii) can be found to change \( A_+ \) (respectively \( B_+ \)) into \( A_- \) (respectively \( B_+ \)) as follows:

(i)
\[
S = ((\alpha_1|\cdots|\alpha_{l-1}|\bar{A}_+ x A_+ y B_+ z B_+ t|\alpha_{l+1}|\cdots|\alpha_k),
\cdots AB \cdots
(\emptyset_{\varepsilon_1}|\cdots|\emptyset_{\varepsilon_{l-1}}|\emptyset_{\varepsilon_{l+1}}|\cdots|\emptyset_{\varepsilon_k}))
\]
\[
\rightarrow T = ((\alpha_1|\cdots|\alpha_{l-1}|\bar{A}_- x A_- y B_+ z B_+ t|\alpha_{l+1}|\cdots|\alpha_k),
\cdots A \cdots AB \cdots
(\emptyset_{\varepsilon_1}|\cdots|\emptyset_{\varepsilon_{l-1}}|\emptyset_{\varepsilon_{l+1}}|\cdots|\emptyset_{\varepsilon_k})) \rightarrow U,
\]

(ii)
\[
S = ((\alpha_1|\cdots|\alpha_{l-1}|\bar{A}_+ x A_+ y B_+ z B_+ t|\alpha_{l+1}|\cdots|\alpha_k),
\cdots AB \cdots
(\emptyset_{\varepsilon_1}|\cdots|\emptyset_{\varepsilon_{l-1}}|\emptyset_{\varepsilon_{l+1}}|\cdots|\emptyset_{\varepsilon_k}))
\]
\[
\rightarrow T = ((\alpha_1|\cdots|\alpha_{l-1}|\bar{A}_+ x A_+ y B_+ z B_+ t|\alpha_{l+1}|\cdots|\alpha_k),
\cdots A \cdots AB \cdots
(\emptyset_{\varepsilon_1}|\cdots|\emptyset_{\varepsilon_{l-1}}|\emptyset_{\varepsilon_{l+1}}|\cdots|\emptyset_{\varepsilon_k})) \rightarrow U.
\]

Then the condition \( \ddagger \) can be also state that the sum of the contribution of (i) to the coefficient of \( U \) and the contribution of (ii) to the coefficient of \( U \) is even.

The case \( \varepsilon_{11} = + \).

Consider the route (i). Then \( (S, T) \) is not equal to 0 if and only if \( (\varepsilon_{21}, \varepsilon_{22}) = (+, +) \). For this \( T, (T, U) \) is not equal to 0 if and only if \( (\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (+, +, +) \).

Consider the route (ii). Then \( (S, T) \) is not equal to 0 if and only if \( (\varepsilon_{31}, \varepsilon_{32}) = (+, +) \). For this \( T, (T, U) \) is not equal to 0 if and only if \( (\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (+, +, +) \). So in this case the condition \( \ddagger \) holds.

The case \( \varepsilon_{11} = - \).

Consider the route (i). In this case \( (S, T) \) is not equal to 0 if and only if \( (\varepsilon_{21}, \varepsilon_{22}) = (+, -) \) or \( (\varepsilon_{21}, \varepsilon_{22}) = (-, +) \). Put \( (\varepsilon_{21}, \varepsilon_{22}) = (+, -) \), then for this \( T, (T, U) \) is not equal to 0 if and only if \( (\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (+, +, -) \). Put \( (\varepsilon_{21}, \varepsilon_{22}) = (-, +) \), then for this \( T, (T, U) \) is not equal to 0 if and only if \( (\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (-, +, +) \) or \( (\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (-, +, -) \) or \( (\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (+, +, -) \). On the other hand, in route (ii) \( (S, T) \) is not equal to 0 if and only if \( (\varepsilon_{31}, \varepsilon_{32}) = (+, -) \) or \( (\varepsilon_{31}, \varepsilon_{32}) = (-, +) \). Put \( (\varepsilon_{31}, \varepsilon_{32}) = (+, -) \), then for this \( T, (T, U) \) is not equal to 0 if and only if \( (\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (+, +, -) \). Put \( (\varepsilon_{31}, \varepsilon_{32}) = (-, +) \), then for this \( T, (T, U) \) is not equal to 0 if and only if \( (\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (-, +, +) \) or \( (\varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}) = (+, +, +) \). So in this case the condition \( \ddagger \) holds.

- Consider the case (4)
Let
\[
U = ((\alpha_1 \cdots | \alpha_{l-1} | A x B y A \cdots zB_t | \alpha_{l+1} \cdots | \alpha_k), \\
\cdots AB \cdots \\
(\theta_{e_1} \cdots | \theta_{e_{l-1}} | \theta_{e_{l_1}} | \theta_{e_{l+1}} \cdots | \theta_{e_k})).
\]

It is sufficient to show that for each \( \varepsilon_{11} \in \{ \pm \} \) the coefficient of \( U \) in \( d^2(S) \) is even for all \( \varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43} \in \{ \pm \} \). Hence for \( S \) and \( U \), we have to check the total number of ways to get \( U \) from \( S \) is even (we denote the condition by (\#)). Let us localize the problem of the difference parts of \( S, A_+ \) and \( B_+ \). Two routes (i) and (ii) can be found to change \( A_+ \) (respectively \( B_+ \)) into \( A_- \) (respectively \( B_- \)) as follows:

(i) \[
S = ((\alpha_1 \cdots | \alpha_{l-1} | A_+ x B_+ y A_+ zB_t | \alpha_{l+1} \cdots | \alpha_k), \\
\cdots AB \cdots \\
(\theta_{e_1} \cdots | \theta_{e_{l-1}} | \theta_{e_{l_1}} | \theta_{e_{l+1}} \cdots | \theta_{e_k})),
\]

\[ \to T = ((\alpha_1 \cdots | \alpha_{l-1} | A_+ x B_+ y A_+ zB_t | \alpha_{l+1} \cdots | \alpha_k), \\
\cdots AB \cdots \\
(\theta_{e_1} \cdots | \theta_{e_{l-1}} | \theta_{e_{l_1}} | \theta_{e_{l+1}} \cdots | \theta_{e_k})) \to U, \]

(ii) \[
S = ((\alpha_1 \cdots | \alpha_{l-1} | A_+ x B_+ y A_+ zB_t | \alpha_{l+1} \cdots | \alpha_k), \\
\cdots AB \cdots \\
(\theta_{e_1} \cdots | \theta_{e_{l-1}} | \theta_{e_{l_1}} | \theta_{e_{l+1}} \cdots | \theta_{e_k})),
\]

\[ \to T = ((\alpha_1 \cdots | \alpha_{l-1} | A_+ x B_+ y A_+ zB_t | \alpha_{l+1} \cdots | \alpha_k), \\
\cdots AB \cdots \\
(\theta_{e_1} \cdots | \theta_{e_{l-1}} | \theta_{e_{l_1}} | \theta_{e_{l+1}} \cdots | \theta_{e_k})) \to U. \]

Then the condition (\#) can be also state that the sum of the contribution of (i) to the coefficient of \( U \) and the contribution of (ii) to the coefficient of \( U \) is even.

In this case, it is clear that \( (S, T)(T, U) = 0 \) for all \( T \) by the definition of \( d \).

- Consider the case (5)

Let
\[
U = ((\alpha_1 \cdots | \alpha_{l-1} | \overline{A} x B y \overline{A} \cdots zB_t | \alpha_{l+1} \cdots | \alpha_k), \\
\cdots AB \cdots \\
(\theta_{e_1} \cdots | \theta_{e_{l-1}} | \theta_{e_{l_1}} | \theta_{e_{l+1}} \cdots | \theta_{e_k})).
\]

It is sufficient to show that for each \( \varepsilon_{11} \in \{ \pm \} \) the coefficient of \( U \) in \( d^2(S) \) is even for all \( \varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43} \in \{ \pm \} \). Hence for \( S \) and \( U \), we have to check the total number of ways to get \( U \) from \( S \) is even (we denote the condition by (\$)). Let us localize the problem of the difference parts of \( S, A_+ \) and \( B_+ \). Two routes (i) and (ii) can be found to change \( A_+ \) (respectively \( B_+ \)) into \( A_- \) (respectively \( B_- \)) as follows:
The route (ii). In this case (respectively (i)) is not equal to 0 if and only if \( (\varepsilon_{31}, \varepsilon_{32}) = (+, +) \). For this \( T \), \( (T, U) = 0 \) for all \( \varepsilon_{41} \in \{\pm\} \). So in this case the condition (\( \sharp \)) holds.

The case \( \varepsilon_{11} = + \)

On the route (i), we obtain \( (S, T) = 0 \) for all \( T \) by the definition of \( d \). Consider the route (ii). In this case \( (S, T) \) is not equal to 0 if and only if \( (\varepsilon_{31}, \varepsilon_{32}) = (+, +) \). For this \( T \), \( (T, U) = 0 \) for all \( \varepsilon_{41} \in \{\pm\} \). So in this case the condition (\( \sharp \)) holds.

The case \( \varepsilon_{11} = - \)

Consider the route (i). Then \( (S, T) = 0 \) for all \( T \). Consider the route (ii). Then \( (S, T) \) is not equal to 0 if and only if \( (\varepsilon_{31}, \varepsilon_{32}) = (+, -) \) or \( (\varepsilon_{31}, \varepsilon_{32}) = (-, +) \). Put \( (\varepsilon_{31}, \varepsilon_{32}) = (+, +) \). Then for this \( T \), we obtain \( (T, U) \) is not equal to 0 if and only if \( \varepsilon_{41} = + \). Then for this \( T \), we obtain \( (T, U) \) is not equal to 0 if and only if \( \varepsilon_{41} = + \). So the condition (\( \sharp \)) holds.

Consider the case \( (6) \)

Let

\[
S = \left( (\alpha_1 \cdots |\alpha_{l-1}|A_+xB_yA_+zB_t|\alpha_{l+1} \cdots |\alpha_k), \right.
\]

\[
\cdots AB \cdots
\]

\[
(\theta_{\varepsilon_1} \cdots |\theta_{\varepsilon_{l-1}}| \theta_{\varepsilon_{l+1}} \cdots |\theta_{\varepsilon_k})
\]

\[
\rightarrow T = \left( (\alpha_1 \cdots |\alpha_{l-1}|A_+xB_yA_+zB_t|\alpha_{l+1} \cdots |\alpha_k), \right.
\]

\[
\cdots AB \cdots
\]

\[
(\theta_{\varepsilon_1} \cdots |\theta_{\varepsilon_{l-1}}| \theta_{\varepsilon_{l+1}} \cdots |\theta_{\varepsilon_k}) \rightarrow U,
\]

\( (ii) \)

\[
S = \left( (\alpha_1 \cdots |\alpha_{l-1}|A_+xB_yA_+zB_t|\alpha_{l+1} \cdots |\alpha_k), \right.
\]

\[
\cdots AB \cdots
\]

\[
(\theta_{\varepsilon_1} \cdots |\theta_{\varepsilon_{l-1}}| \theta_{\varepsilon_{l+1}} \cdots |\theta_{\varepsilon_k})
\]

\[
\rightarrow T = \left( (\alpha_1 \cdots |\alpha_{l-1}|A_+xB_yA_+zB_t|\alpha_{l+1} \cdots |\alpha_k), \right.
\]

\[
\cdots AB \cdots
\]

\[
(\theta_{\varepsilon_1} \cdots |\theta_{\varepsilon_{l-1}}| \theta_{\varepsilon_{l+1}} \cdots |\theta_{\varepsilon_k}) \rightarrow U.
\]

Then the condition (\( \sharp \)) can be also state that the sum of the contribution of (i) to the coefficient of \( U \) and the contribution of (ii) to the coefficient of \( U \) is even.

It is sufficient to show that for each \( (\varepsilon_{11}, \varepsilon_{12}) \in \{(\pm, \pm)\} \) where double signs are arbitrary, the coefficient of \( U \) in \( d^2(S) \) is even for all \( \varepsilon_{41} \in \{\pm\} \). Hence for \( S \) and \( U \), we have to check the total number of ways to get \( U \) from \( S \) is even (we denote the condition by (\( \sharp \))). Let us localize the problem of the difference parts of \( S, A_+ \) and \( B_+ \). Two routes (i) and (ii) can be found to change \( A_+ \) (respectively \( B_+ \)) into \( A_- \) (respectively \( B_- \)) as follows:
In this case, we can easily check that (\(\text{the coefficient of}\) (ii) can be also stated that the sum of the contribution of (i) to the coefficient of \(U\) and the contribution of (ii) to the coefficient of \(U\) is even.

In this case, we can easily check that \((S, T)(T, U) = 0\) for all \(T\) by the definition of \(T\).

Consider the case (7).

Let

\[
S = ((\alpha_1) \cdots | \alpha_k), \quad \cdots AB \cdots AB \ldots (\theta_{\varepsilon_1} \cdots | \theta_{\varepsilon_{k-1}} | \theta_{\varepsilon_{k+1}} \cdots | \theta_{\varepsilon_k})
\]

\[
T = ((\alpha_1) \cdots | \alpha_k), \quad \cdots AB \cdots AB \ldots (\theta_{\varepsilon_1} \cdots | \theta_{\varepsilon_{k-1}} | \theta_{\varepsilon_{k+1}} \cdots | \theta_{\varepsilon_k}) \rightarrow U,
\]

\[
S = ((\alpha_1) \cdots | \alpha_k), \quad \cdots AB \cdots AB \ldots (\theta_{\varepsilon_1} \cdots | \theta_{\varepsilon_{k-1}} | \theta_{\varepsilon_{k+1}} \cdots | \theta_{\varepsilon_k}) \rightarrow U,
\]

\[
T = ((\alpha_1) \cdots | \alpha_k), \quad \cdots AB \cdots AB \ldots (\theta_{\varepsilon_1} \cdots | \theta_{\varepsilon_{k-1}} | \theta_{\varepsilon_{k+1}} \cdots | \theta_{\varepsilon_k}) \rightarrow U.
\]

Then the condition (\(\#\)) can be also stated that the sum of the contribution of (i) to the coefficient of \(U\) and the contribution of (ii) to the coefficient of \(U\) is even.

In this case, we can easily check that \((S, T)(T, U) = 0\) for all \(T\) by the definition of \(T\).

Consider the case (7).

Let

\[
U = ((\alpha_1) \cdots | \alpha_k), \quad \cdots A \cdots A \ldots B \cdots B \ldots (\theta_{\varepsilon_1} \cdots | \theta_{\varepsilon_{k-1}} | \theta_{\varepsilon_{k+1}} \cdots | \theta_{\varepsilon_k}).
\]

It is sufficient to show that for each \((\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{14}) \in \{ (\pm, \pm, \pm, \pm) \} \) where double signs are arbitrary, the coefficient of \(U\) in \(d^2(S)\) is even for all \(\varepsilon_{11}, \varepsilon_{12} \in \{ \pm \}.\) Hence for \(S\) and \(U\), we have to check the total number of ways to get \(U\) from \(S\) is even (we denote the condition by \(\#\)). Let us localize the problem of the difference parts of \(S, A_+\) and \(B_+\). Two routes (i) and (ii) can be found to change \(A_+\) (respectively \(B_+\)) into \(A_-\) (respectively \(B_+\)) as follows:

(i)

\[
S = ((\alpha_1) \cdots | \alpha_k), \quad \cdots A \cdots A \cdots B \cdots B \ldots (\theta_{\varepsilon_1} \cdots | \theta_{\varepsilon_{k-1}} | \theta_{\varepsilon_{k+1}} \cdots | \theta_{\varepsilon_k})
\]

\[
T = ((\alpha_1) \cdots | \alpha_k), \quad \cdots A \cdots B \cdots B \ldots (\theta_{\varepsilon_1} \cdots | \theta_{\varepsilon_{k-1}} | \theta_{\varepsilon_{k+1}} \cdots | \theta_{\varepsilon_k}) \rightarrow U,
\]
In this case we can easily check that the condition (ii)

\[ S = (\alpha_1 \cdots | \alpha_{l-1} | A_+ x A_+ y | B_+ z B_+ t | \alpha_{l+1} | \cdots | \alpha_k), \]

\[ \cdots A \cdots A \cdots B \cdots \]

\[ (\theta_{e_1} | \cdots | \theta_{e_{l-1}} | \theta_{e_{l+1}} | \theta_{e_{l+2}} | \theta_{e_{l+3}} | \theta_{e_{l+4}} | \theta_{e_{l+1}} | \cdots | \theta_{e_k})) \]

\[ \rightarrow T = (\alpha_1 \cdots | \alpha_{l-1} | A_+ x A_+ y | B_+ z B_+ t | \alpha_{l+1} | \cdots | \alpha_k), \]

\[ \cdots A \cdots A \cdots B \cdots \]

\[ (\theta_{e_1} | \cdots | \theta_{e_{l-1}} | \theta_{e_{l+1}} | \theta_{e_{l+2}} | \theta_{e_{l+3}} | \theta_{e_{l+4}} | \theta_{e_{l+1}} | \cdots | \theta_{e_k})) \rightarrow U. \]

Then the condition (\#) can be also state that the sum of the contribution of (i) to the coefficient of \( U \) and the contribution of (ii) to the coefficient of \( U \) is even.

In this case we can easily check that the condition (\#) holds since empty words which relates \( A \) and empty words which relates \( B \) are independent.

- **Consider the cases (8) and (9).**
  In this cases the condition (\#) holds similarly as the case (7).
- **Consider the case (10)**

Let

\[ U = (\alpha_1 \cdots | \alpha_{l-1} | A_+ x B_+ y | A_+ z B_+ t | \alpha_{l+1} | \cdots | \alpha_k), \]

\[ \cdots A B \cdots A B \cdots \]

\[ (\theta_{e_1} | \cdots | \theta_{e_{l-1}} | \theta_{e_{l+1}} | \theta_{e_{l+2}} | \theta_{e_{l+3}} | \theta_{e_{l+4}} | \theta_{e_{l+1}} | \cdots | \theta_{e_k})). \]

It is sufficient to show that for each \( (\varepsilon_{11}, \varepsilon_{12}) \in \{(\pm, \pm)\} \) where double signs are arbitrary, the coefficient of \( U \) in \( d^2(S) \) is even for all \( \varepsilon_{41}, \varepsilon_{42} \in \{\pm\} \). Hence for \( S \) and \( U \), we have to check the total number of ways to get \( U \) from \( S \) is even (we denote the condition by (\#)). Let us localize the problem of the difference parts of \( S, A_+ \) and \( B_+ \). Two routes (i) and (ii) can be found to change \( A_+ \) (respectively \( B_+ \)) into \( A_+ \) (respectively \( B_+ \)) as follows:

(i)

\[ S = (\alpha_1 \cdots | \alpha_{l-1} | A_+ x B_+ y | A_+ z B_+ t | \alpha_{l+1} | \cdots | \alpha_k), \]

\[ \cdots A B \cdots A B \cdots \]

\[ (\theta_{e_1} | \cdots | \theta_{e_{l-1}} | \theta_{e_{l+1}} | \theta_{e_{l+2}} | \theta_{e_{l+3}} | \theta_{e_{l+4}} | \theta_{e_{l+1}} | \cdots | \theta_{e_k})). \]

\[ \rightarrow T = (\alpha_1 \cdots | \alpha_{l-1} | A_+ x B_+ y | A_+ z B_+ t | \alpha_{l+1} | \cdots | \alpha_k), \]

\[ \cdots A B \cdots \]

\[ (\theta_{e_1} | \cdots | \theta_{e_{l-1}} | \theta_{e_{l+1}} | \theta_{e_{l+2}} | \theta_{e_{l+3}} | \theta_{e_{l+4}} | \theta_{e_{l+1}} | \cdots | \theta_{e_k})) \rightarrow U, \]

(ii)

\[ S = (\alpha_1 \cdots | \alpha_{l-1} | A_+ x B_+ y | A_+ z B_+ t | \alpha_{l+1} | \cdots | \alpha_k), \]

\[ \cdots A B \cdots A B \cdots \]

\[ (\theta_{e_1} | \cdots | \theta_{e_{l-1}} | \theta_{e_{l+1}} | \theta_{e_{l+2}} | \theta_{e_{l+3}} | \theta_{e_{l+4}} | \theta_{e_{l+1}} | \cdots | \theta_{e_k})). \]

\[ \rightarrow T = (\alpha_1 \cdots | \alpha_{l-1} | A_+ x B_+ y | A_+ z B_+ t | \alpha_{l+1} | \cdots | \alpha_k), \]

\[ \cdots A B \cdots \]

\[ (\theta_{e_1} | \cdots | \theta_{e_{l-1}} | \theta_{e_{l+1}} | \theta_{e_{l+2}} | \theta_{e_{l+3}} | \theta_{e_{l+4}} | \theta_{e_{l+1}} | \cdots | \theta_{e_k})) \rightarrow U. \]
Then the condition (‡) can be also state that the sum of the contribution of (i) to the coefficient of $U$ and the contribution of (ii) to the coefficient of $U$ is even.

The case $(\varepsilon_{11}, \varepsilon_{12}) = (+, +)$.

In this case, both in the route (i) and in the route (ii), $(S, T) = 0$ for all $T$. So the condition (‡) holds.

The case $(\varepsilon_{11}, \varepsilon_{12}) = (+, -)$.

Consider the route (i). In this route $(S, T)$ is not equal to 0 if and only if $\varepsilon_{21} = +$. Then for this $T$, $(T, U)$ is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+, +)$. Consider the route (ii). In this route $(S, T)$ is not equal to 0 if and only if $\varepsilon_{31} = +$. Then for this $T$, $(T, U)$ is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+, +)$. So the condition (‡) holds.

The case $(\varepsilon_{11}, \varepsilon_{12}) = (-, +)$.

Consider the route (i). Then $(S, T)$ is not equal to 0 if and only if $\varepsilon_{21} = +$. Moreover for this $T$, $(T, U)$ is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+, +)$. Consider the route (ii). Then $(S, T)$ is not equal to 0 if and only if $\varepsilon_{31} = +$. Moreover for this $T$, $(T, U)$ is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+, +)$. So the condition (‡) holds.

The case $(\varepsilon_{11}, \varepsilon_{12}) = (-, -)$.

Consider the route (i). Then $(S, T)$ is not equal to 0 if and only if $\varepsilon_{21} = -$. Moreover for this $T$, $(T, U)$ is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+, -)$ or $(\varepsilon_{41}, \varepsilon_{42}) = (-, +)$. Consider the route (ii). Then $(S, T)$ is not equal to 0 if and only if $\varepsilon_{31} = -$. Moreover for this $T$, $(T, U)$ is not equal to 0 if and only if $(\varepsilon_{41}, \varepsilon_{42}) = (+, -)$. $(\varepsilon_{41}, \varepsilon_{42}) = (-, +)$. So the condition (‡) holds.

- Consider the case (11)

Let

$$U = ((\alpha_1 | \cdots | \alpha_{l-1} | \overline{A}_- x B_- y | \overline{A}_- z B_- t | \alpha_{l+1} | \cdots | \alpha_k), \quad \cdots AB \cdots \quad (\theta_{\varepsilon_{11}} | \cdots | \theta_{\varepsilon_{l-1}} | \theta_{\varepsilon_{41}} \quad | \theta_{\varepsilon_{l+1}} | \cdots | \theta_{\varepsilon_k})).$$

It is sufficient to show that for each $\varepsilon_{11} \in \{\pm\}$, the coefficient of $U$ in $d^2(S)$ is even for all $\varepsilon_{41} \in \{\pm\}$. Hence for $S$ and $U$, we have to check the total number of ways to get $U$ from $S$ is even (we denote the condition by (‡)). Let us localize the problem of the difference parts of $S$, $A_+$ and $B_+$. Two routes (i) and (ii) can be found to change $A_+$ (respectively $B_+$) into $A_-$ (respectively $B_-$) as follows:

(i)

$$S = ((\alpha_1 | \cdots | \alpha_{l-1} | \overline{A}_+ x B_+ y | \overline{A}_+ z B_+ t | \alpha_{l+1} | \cdots | \alpha_k), \quad \cdots AB \cdots \quad (\theta_{\varepsilon_{11}} | \cdots | \theta_{\varepsilon_{l-1}} | \theta_{\varepsilon_{11}} \quad | \theta_{\varepsilon_{l+1}} | \cdots | \theta_{\varepsilon_k})) \quad \rightarrow T = ((\alpha_1 | \cdots | \alpha_{l-1} | \overline{A}_- x B_- y | \overline{A}_- z B_- t | \alpha_{l+1} | \cdots | \alpha_k), \quad \cdots AB \cdots \cdots AB \cdots \quad (\theta_{\varepsilon_{21}} | \cdots | \theta_{\varepsilon_{l-1}} | \theta_{\varepsilon_{21}} \quad | \theta_{\varepsilon_{l+1}} | \cdots | \theta_{\varepsilon_k})) \rightarrow U,$$
In this case we can choose $\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{31}, \varepsilon_{32} \in \{\pm\}$ so that $(\varepsilon_{21}, \varepsilon_{22}) = (\varepsilon_{31}, \varepsilon_{32})$ and $(S, T)$ is not equal to 0. Moreover $T$s in the route (i) and in the route (ii) have same form. So the condition (i) holds.

• Consider the case (12)

Let

$$U = ((\alpha_1) \cdots |\alpha_{l-1}|A_+ x B_{y} | A_+ z B_{t} | \alpha_{l+1}) \cdots |\alpha_{k}),$$

$$\cdots AB \cdots (\theta_{i_1} | \cdots | \theta_{l_{i-1}} | \theta_{i_1} \cdots | \theta_{l_{i-1}} | \theta_{l_{i+1}} | \cdots | \theta_{k})).$$

Then the condition (i) can be also state that the sum of the contribution of (i) to the coefficient of $U$ and the contribution of (ii) to the coefficient of $U$ is even.

It is sufficient to show that for each $(\varepsilon_{11}, \varepsilon_{12}) \in \{\pm, \pm\}$ where double signs are arbitrary, the coefficient of $U$ in $d^2(S)$ is even for all $\varepsilon_{41}, \varepsilon_{42} \in \{\pm\}$. Hence for $S$ and $U$, we have to check the total number of ways to get $U$ from $S$ is even (we denote the condition by (12)). Let us localize the problem of the difference parts of $S$, $A_+$ and $B_+$. Two routes (i) and (ii) can be found to change $A_+$ (respectively $B_+$) into $A_-$ (respectively $B_+$) as follows:

(i) 

$$S = ((\alpha_1) \cdots |\alpha_{l-1}|A_+ x B_{y} | A_+ z B_{t} | \alpha_{l+1}) \cdots |\alpha_{k}),$$

$$\cdots AB \cdots (\theta_{i_1} | \cdots | \theta_{l_{i-1}} | \theta_{i_1} \cdots | \theta_{l_{i-1}} | \theta_{l_{i+1}} | \cdots | \theta_{k})).$$

$$\rightarrow T = ((\alpha_1) \cdots |\alpha_{l-1}|A_+ x B_{y} | A_+ z B_{t} | \alpha_{l+1}) \cdots |\alpha_{k}),$$

$$\cdots AB \cdots (\theta_{i_1} | \cdots | \theta_{l_{i-1}} | \theta_{i_1} \cdots | \theta_{l_{i+1}} | \cdots | \theta_{k})).$$

(ii) 

$$S = ((\alpha_1) \cdots |\alpha_{l-1}|A_+ x B_{y} | A_+ z B_{t} | \alpha_{l+1}) \cdots |\alpha_{k}),$$

$$\cdots AB \cdots (\theta_{i_1} | \cdots | \theta_{l_{i-1}} | \theta_{i_1} \cdots | \theta_{l_{i-1}} | \theta_{l_{i+1}} | \cdots | \theta_{k})).$$

$$\rightarrow T = ((\alpha_1) \cdots |\alpha_{l-1}|A_+ x B_{y} | A_+ z B_{t} | \alpha_{l+1}) \cdots |\alpha_{k}),$$

$$\cdots AB \cdots (\theta_{i_1} | \cdots | \theta_{l_{i-1}} | \theta_{i_1} \cdots | \theta_{l_{i+1}} | \cdots | \theta_{k})).$$

Then the condition (i) can be also state that the sum of the contribution of (i) to the coefficient of $U$ and the contribution of (ii) to the coefficient of $U$ is even.
This case is completely same as the case (ii).

- The cases (13) - (23)
  We can easily check the condition (z) by the definition of \( d \).

- The cases (24) - (26)
  In this case we can prove the condition (z) holds same as the case (7).

Now we have completed the proof. \( \square \)

**Definition 4.3.** We denote the mapping \( d \) modulo 2 : \( C^{i,j}(P; \mathbb{Z}_2) \rightarrow C^{i+1,j}(P; \mathbb{Z}_2) \) by \( d_2^i \) for \( i \) and \( j \). The Khovanov homology group \( KH^{i,j}(P) \) for a pseudolink \( P \) is defined as

\[
KH^{i,j}(P) := \text{Ker} \ d_2^i/\text{Im} \ d_2^{i-1}.
\]

**Remark 4.2.** \( KH^{i,j}(P) \) is independent of the order of the letters of \( P \) because the incidence number \((S: T)\) is always either 0 or 1 modulo 2 for enhanced states \( S \) and \( T \).

5. **Invariance under \( S_1 \)-homotopy moves.**

**Theorem 5.1.** \( KH^{i,j}(P) \) is \( S_1 \)-homotopy invariants for pseudolinks.

**Proof.** By construction of \( KH^{i,j}(P) \), \( KH^{i,j}(P) \) is not depend on an arbitrary isomorphism of \( P \). Then \( KH^{i,j} \) is invariant under isomorphisms. It remains to prove that if a nanophrase \( P \) is obtained from a nanophrase \( P' \) by a homotopy move then \( KH^{i,j}(P') \cong KH^{i,j}(P) \).

(I) Consider the first homotopy move \((xAy) \rightarrow (xy)\) and its inverse move where \(|A| = 1\). For \( P' \) and \( P \), denote by \( S_+(\epsilon, \eta) \) the state \((u | \emptyset_\epsilon | \emptyset_\eta | v)\) of \( P' \) with \( \text{mark}(A) = 1 \) and denote by \( S_-(\epsilon) \) the state \((u | \emptyset_\epsilon | v)\) of \( P' \) with \( \text{mark}(A) = -1 \) where \( \epsilon, \eta \in \{+, -\} \). The subcomplex \( C' \) of \( C(P') \) is defined by \( C' := C(S_+(+, +), S_+(+, -) - S_+(-, +)) \).

First, the retraction

\[
\rho : C(P') \rightarrow C(S_+(+, +), S_+(+, -) - S_+(-, +))
\]

is defined by the formulas

\[
S_+(+, +) \mapsto S_+(+, +),
S_+(-, +) \mapsto S_+(-, +) - S_+(+, -),
\text{otherwise} \mapsto 0.
\]

Second, the isomorphism

\[
C(S_+(+, +), S_+(+, -) - S_+(-, +)) \rightarrow C(P) = C((u | \emptyset_+ | v), (u | \emptyset_- | v))
\]

is defined by the formulas

\[
S_+(+, +) \mapsto (u | \emptyset_+ | v),
S_+(+, -) \mapsto (u | \emptyset_- | v).
\]

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Third, consider the following composition of this isomorphism with $\rho$

$$C(P') \xrightarrow{\rho} C' \xrightarrow{\text{isom}} C(P).$$

The map $h : C(P') \to C(P')$ such that $d \circ h + h \circ d = \text{id} - \in C\rho$, is defined by the formulas

$$
\begin{align*}
S_-(+) & \mapsto S_+(+, -), \\
S_-(--) & \mapsto S_+(-, -), \\
\text{otherwise} & \mapsto 0.
\end{align*}
$$

(II) Consider the second homotopy move $P' = (xAByBAz) \to (xyz) = P$ and its inverse move where $(|A|, |B|) = (1, -1)$. It is necessary to consider two distinct cases (II-1), (II-2) as follows.

(II-1) Consider the case where the state of $P'$ with $(\text{mark}(A), \text{mark}(B)) = (1, 1)$ is represented as $(u| \emptyset_e | v)$.

Denote by $S_{+-}(\epsilon, \eta)$ the state $(u| \emptyset_\epsilon | \emptyset_\eta | v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B)) = (1, -1)$, denote by $S_{-+}(\epsilon, \eta)$ the state $(u| \emptyset_\epsilon | \emptyset_\eta | v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B)) = (-1, 1)$, denote by $S_{++}(\epsilon)$ the state $(u| \emptyset_\epsilon | \emptyset_\eta | v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B)) = (1, 1)$ and denote by $S_{--}(\epsilon)$ the state $(u| \emptyset_\epsilon | \emptyset_\eta | v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B)) = (-1, -1)$ where $\epsilon, \eta \in \{+, -\}$. The subcomplex $C'$ of $C(P')$ is defined by $C' := C(S_{+-}(+, +), S_{++}(+, -) + S_{+-}(+, -), S_{-+}(-, +) + S_{+-}(+, -), S_{--}(-, -) + S_{+-}(-, -))$.

First, the retraction $\rho : C(P') \to C'$ is defined by the formulas

$$
\begin{align*}
S_{++}(+, +) & \mapsto S_{++}(+, +), \\
S_{+-}(+, -) & \mapsto S_{++}(+, -) + S_{+-}(+, -), \\
S_{-+}(-, +) & \mapsto S_{++}(-, +) + S_{+-}(-, +), \\
S_{--}(-, -) & \mapsto S_{++}(-, -) + S_{+-}(-, -), \\
S_{+-}(+, -) & \mapsto S_{++}(+, +), \\
S_{++}(+, -) & \mapsto S_{+-}(+, -) + S_{+-}(+, -), \\
\text{otherwise} & \mapsto 0.
\end{align*}
$$

Second, the isomorphism

$$C' \to C(P) = C((u|\emptyset_\epsilon |\emptyset_\eta | v))$$
is defined by the formulas

\[\begin{align*}
S_{-+}(+,-) &\mapsto (u|\emptyset_+|\emptyset_-|v), \\
S_{-+}(-,-) &\mapsto (u|\emptyset_-|\emptyset_+|v),
\end{align*}\]

\[\begin{align*}
S_{-+}(+-,+) &\mapsto (u|\emptyset_+|\emptyset_-|v), \\
S_{-+}(-,-,+) &\mapsto (u|\emptyset_-|\emptyset_+|v),
\end{align*}\]

Third, consider the following composition of this isomorphism with \(\rho\)

\[C(P') \xrightarrow{\rho} C' \xrightarrow{\text{isom}} C(P).\]

The map \(h : C(P') \to C(P')\) such that \(d \circ h + h \circ d = \text{id} - \text{in} \circ \rho\), is defined by the formulas

\[\begin{align*}
S_{-+}(\epsilon) &\mapsto S_{++}(\epsilon), \\
S_{++}(\epsilon) &\mapsto S_{++}(\epsilon),
\end{align*}\]

otherwise \(\mapsto 0.\)

(II-2) Consider the case where the state of \(P'\) with \((\text{mark}(A), \text{mark}(B)) = (1, 1)\)

\[Aw ABt\]

is represented as \((u|\emptyset_e|\emptyset_\eta|v).\)

Denote by \(S_{+-}(\epsilon, \zeta, \eta)\) the state \((u|\emptyset_e|\emptyset_\zeta|\emptyset_\eta|v)\) of \(P'\) with \((\text{mark}(A), \text{mark}(B)) = (1, -1)\), denote by \(S_{-+}(\epsilon)\) the state \((u|\emptyset_e|\emptyset_\eta|v)\) of \(P'\) with \((\text{mark}(A), \text{mark}(B)) = (1, 1)\) and denote by \(S_{++}(\epsilon, \eta)\) the state \((u|\emptyset_e|\emptyset_\eta|v)\) of \(P'\) with \((\text{mark}(A), \text{mark}(B)) = (-1, 1)\) where \(\epsilon, \eta \in \{+, -\}\) and the word \(t'\) is obtained by deleting from \(t\) all letters which appear in \(w\). The subcomplex \(C''\) of \(C(P')\) is defined by \(C'' := C(\ S_{++}(+) + S_{+-}(+-,+), S_{++}(-) + S_{+-}(+-,-) + S_{+-}(-,-,+))\).

First, the retraction \(\rho : C(P') \to C''\) is defined by the formulas

\[\begin{align*}
S_{++}(+) &\mapsto S_{++}(+) + S_{+-}(+-,+), \\
S_{++}(-) &\mapsto S_{++}(-) + S_{+-}(+-,-) + S_{+-}(-,-,+), \\
S_{+-}(+,+) &\mapsto S_{+-}(+) + S_{+-}(+-,+), \\
S_{+-}(-,+) &\mapsto S_{+-}(+) + S_{+-}(+-,+), \\
S_{+-}(-,-) &\mapsto S_{+-}(-) + S_{+-}(+-,-) + S_{+-}(-,-,+), \\
\text{otherwise} &\mapsto 0.
\end{align*}\]
Second, the isomorphism

$$C' \to C(P) = C((u|\emptyset|v))$$

is defined by the formulas

$$S_{-+}(+) + S_{+-}(+, -, +) \mapsto (u|\emptyset|v),$$

$$S_{-+}(-) + S_{+-}(+, -, -) + S_{-+}(-, -, -) \mapsto (u|\emptyset|v).$$

Third, consider the following composition of this isomorphism with \( \rho \)

$$C(P') \overset{\rho}{\to} C' \overset{\text{isom}}{\to} C(P).$$

The map \( h : C(P') \to C(P') \) such that \( d \circ h + h \circ d = \id - \in \circ \rho \), is defined by the formulas

$$S_{-}(-)(\epsilon, \eta) \mapsto S_{+}(-)(\epsilon, \eta),$$

$$S_{+}(-)(\epsilon, +, \eta) \mapsto S_{+}(\epsilon, \eta),$$

otherwise \( \mapsto 0. \)

By using (II-1) and (II-2), we proved \( KH^{i,j}((xAByBAz)) \simeq KH^{i,j}((xyz)) \) if \( (|A|, |B|) = (1, -1) \). In addition, the fact is, (II-1) and (II-2) prove that \( KH^{i,j}((xAByABz)) \simeq KH^{i,j}((xyz)) \) if \( (|A|, |B|) = (-1, 1) \). Moreover, by exchanging \( A, \ B \) in the proofs above, (II-1) and (II-2) prove that \( KH^{i,j}((xAByABz)) \simeq KH^{i,j}((xyz)) \) if \( (|A|, |B|) = (1, -1) \).

Here, consider

$$xABy \overset{H_1}{\sim} xAB \sim xAy \quad \text{with} \quad |A| = -1, |B| = 1$$

$$\overset{H_2}{\sim} xy.$$ We have already showed that the invariance of \( KH^{i,j} \) under these moves above and then \( KH^{i,j} \) is preserved under the first homotopy move \( xAy \to xy \) with \( |A| = -1 \) and its inverse move.

(III) Consider the third homotopy move \( P' = (xAByACzBCt) \to (xBAyCAzCBt) = P \) and its inverse move where \( (|A|, |B|, |C|) = (-1, -1, -1) \). For the letters \( A, \ B \) and \( C \), we define \( w_{ABC}, w_{AB}, w_{AC}, w_{BC}, w_A, w_B \) and \( w_C \) in the following. Let \( w_{ABC} \) be a word containing \( A, \ B \) and \( C \). Let \( (X, Y, Z) = \{(A, B, C), (A, C, B), (B, C, A)\} \). Denote by \( w_{XY} \) a word containing \( X \) and \( Y \) and not containing \( Z \) and denote by \( w_Z \) a word containing \( Z \) and not containing \( X \) and \( Y \).

(III–1) Consider the case where the state of \( P' \) with (mark(\( A \)), mark(\( B \)), mark(\( C \))) \( = (1, 1, 1) \) is represented as \( (u|\emptyset|v) \).
Denote by $S_{+++}(\epsilon)$ the state $(u|\varnothing_\epsilon |v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, 1)$; denote by $S_{--+}(\epsilon, \eta)$ the state $(u|\varnothing_\epsilon |\varnothing_\eta |v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, 1, 1)$; denote by $S_{+-+}(\epsilon, \eta)$ the state $(u|\varnothing_\epsilon |\varnothing_\eta |v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, -1, 1)$; denote by $S_{++-}(\epsilon, \eta)$ the state $(u|\varnothing_\epsilon |\varnothing_\eta |v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, -1)$; denote by $S_{--+}(\epsilon, \zeta, \eta)$ the state $(u|\varnothing_\epsilon |\varnothing_\zeta |\varnothing_\eta |v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, -1, -1)$.

The subcomplex $C'$ of $C(P')$ is defined by $C' := C(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, -1, 1)$; denote by $T_{+-+}(\epsilon, \eta)$ the state $(u|\varnothing_\epsilon |\varnothing_\eta |v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, -1)$; denote by $T_{++-}(\epsilon, \zeta, \eta)$ the state $(u|\varnothing_\epsilon |\varnothing_\zeta |\varnothing_\eta |v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, 1, -1)$; denote by $T_{+-+}(\epsilon, \zeta, \eta)$ the state $(u|\varnothing_\epsilon |\varnothing_\zeta |\varnothing_\eta |v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, -1, -1)$; denote by $T_{++-}(\epsilon, \zeta, \eta)$ the state $(u|\varnothing_\epsilon |\varnothing_\zeta |\varnothing_\eta |v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, -1, -1)$; denote by $T_{++-}$ every states of $P$ with mark$(C) = -1$.

The subcomplex $C$ of $C(P)$ is defined by $C := C(T_{+-+}(+, \eta) + T_{++-}(+, +, \eta, -), T_{+-+}(-, \eta) + T_{++-}(+, -, \eta, -) + T_{++-}(-, +, \eta, -), T_{++-})$.  

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First, the retraction $\rho : C(P') \to C'$ is defined by the formulas
\[
\begin{align*}
S_{++}(+, +) &\mapsto S_{++}(+, +), \\
S_{++}(+, -) &\mapsto S_{++}(+, -) + S_{++}(+, -), \\
S_{++}(-, +) &\mapsto S_{++}(-, +) + S_{++}(-, +), \\
S_{++}(-, -) &\mapsto S_{++}(-, -) + S_{++}(-, -), \\
S_{+-} &\mapsto S_{+-}, \\
S_{+-}(+, +) &\mapsto S_{++}(+, +) + S_{++}(+, +), \\
S_{+-}(-, +) &\mapsto S_{++}(-, +) + S_{++}(-, +) + S_{++}(-, +) + S_{++}(-, +), \\
S_{+-}(\epsilon) &\mapsto S_{+-}(\epsilon), \\
\text{otherwise} &\mapsto 0.
\end{align*}
\]

Second, consider the following composition of the following isomorphism with $\rho$
\[
(23) \quad C(P') \xrightarrow{\rho} C' \xrightarrow{\text{isom}} C \xrightarrow{i} C(P).
\]
The isomorphism $C' \to C$ is defined by the formulas
\[
\begin{align*}
S_{++}(+, +) &\mapsto T_{++}(+, +) + T_{++}(+, +), \\
S_{++}(+, -) + S_{++}(+, -) &\mapsto T_{++}(+, +) + T_{++}(+, +), \\
S_{++}(-, +) + S_{++}(-, +) &\mapsto T_{++}(-, +) + T_{++}(-, +) + T_{++}(-, +), \\
S_{++}(-, -) + S_{++}(-, -) &\mapsto T_{++}(-, -) + T_{++}(-, -) + T_{++}(-, -), \\
S_{+-}(\epsilon, \eta) &\mapsto T_{+-}(\epsilon, \eta), \\
S_{+-}(\epsilon, \zeta; \eta) &\mapsto T_{+-}(\epsilon, \zeta, \eta), \\
S_{+-}(\epsilon) &\mapsto T_{+-}(\epsilon), \\
S_{--}(\epsilon, \eta) &\mapsto T_{--}(\epsilon, \eta).
\end{align*}
\]

Third, the map $h : C(P') \to C(P')$ such that $d \circ h + h \circ d = \text{id} - \text{in} \circ \rho$, is defined by the formulas
\[
\begin{align*}
S_{--}(\epsilon) &\mapsto S_{++}(\epsilon, -), \\
S_{++}(\epsilon, +) &\mapsto S_{++}(\epsilon), \\
\text{otherwise} &\mapsto 0.
\end{align*}
\]

(III–2) Consider the case where the state of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C))$
\[
= (1, 1, 1)
\]
is represented as $(u| \emptyset \epsilon | \emptyset \eta | v)$.

Denote by $S_{+++}(\epsilon, \eta)$ the state $(u| \emptyset \epsilon | \emptyset \eta | v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, 1)$, denote by $S_{++}(\epsilon)$ the state $(u| \emptyset \epsilon | v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, 1, 1)$, denote by $S_{+-}(\epsilon, \zeta; \eta)$ the state $(u| \emptyset \epsilon | \emptyset \zeta | \emptyset \eta | v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, -1, 1)$, denote by $S_{--}(\epsilon, \eta)$
the state \((u \mid \emptyset_{\epsilon} \mid \emptyset_{\eta} \mid v)\) of \(P'\) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, -1, 1)\), denote by \(S_{+++}(\epsilon)\) the state \((u \mid \emptyset_{\epsilon} \mid v)\) of \(P'\) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, -1)\), denote by \(S_{--}(\epsilon, \eta)\) the state \((u \mid \emptyset_{\epsilon} \mid \emptyset_{\eta} \mid v)\) of \(P'\) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, 1, -1)\), denote by \(S_{+-}(\epsilon, \eta)\) the state \((u \mid \emptyset_{\epsilon} \mid \emptyset_{\eta} \mid v)\) of \(P'\) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, -1, -1)\), denote by \(S_{-}\) denotes every states with mark\((C) = -1\).

Denote by \(T_{++}(\epsilon)\) the state \((u \mid \emptyset_{\epsilon} \mid v)\) of \(P\) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, 1)\), denote by \(T_{+-}(\epsilon, \zeta, -)\) the state \((u \mid \emptyset_{\epsilon} \mid \emptyset_{\eta} \mid v)\) of \(P\) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, 1, 1)\), denote by \(T_{++}(\epsilon)\) the state \((u \mid \emptyset_{\epsilon} \mid v)\) of \(P\) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, -1)\), denote by \(T_{--}(\epsilon, \eta)\) the state \((u \mid \emptyset_{\epsilon} \mid \emptyset_{\eta} \mid v)\) of \(P\) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, 1, -1)\), denote by \(T_{++}(\epsilon, \zeta, \eta)\) the state \((u \mid \emptyset_{\epsilon} \mid \emptyset_{\eta} \mid v)\) of \(P\) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, -1, -1)\), denote by \(T_{++}\) every states of \(P\) with mark\((C) = -1\).

The subcomplex \(C'\) of \(C(P')\) is defined by \(C' := C'(S_{++}(+) + S_{++}(+,-,+), S_{++}(+) + S_{++}(+,,-,-) + S_{++}(+,-,+), S_{**} )\) where \(S_{**}\) denotes every states with mark\((C) = -1\).

First, the retraction \(\rho : C(P') \rightarrow C'\) is defined by the formulas

\[
\begin{align*}
S_{++}(+) & \mapsto S_{++}(+) + S_{++}(+,-,+), \\
S_{--}(+) & \mapsto S_{++}(+) + S_{++}(+,-,-) + S_{--}(-,-,+), \\
S_{**} & \mapsto S_{**}, \\
S_{++}(+,-,-) & \mapsto S_{++}(+) + S_{++}(+,-,+), \\
S_{++}(+,-,+) & \mapsto S_{++}(+) + S_{++}(+,-,-) + S_{++}(+), \\
S_{++}(+,-,+) & \mapsto S_{++}(+) + S_{++}(+,-,+) + S_{++}(+), \\
S_{++}(+-,-) & \mapsto S_{++}(+) + S_{++}(+,-,-) + S_{++}(+), \\
S_{++}(+-,+) & \mapsto S_{++}(+) + S_{++}(+,-,-) + S_{++}(+), \\
S_{++}(\epsilon, \eta) & \mapsto S_{++}(\epsilon, \eta), \\
\text{otherwise} & \mapsto 0.
\end{align*}
\]
Second, consider the following composition (23) of the following isomorphism with \( \rho \). The isomorphism \( C' \to C \) is defined by the formulas

\[
S_{++} (+) + S_{++} (+, -, +) \mapsto T_{++} (+) + T_{++} (+, -, -),
\]

\[
S_{++} (-) + S_{++} (+, -, -) + S_{++} (-, -, +) \mapsto T_{++} (-) + T_{++} (+, -, -) + T_{++} (-, +, -),
\]

\[
S_{++} (\epsilon) \mapsto T_{++} (\epsilon),
\]

\[
S_{++} (\epsilon, \eta) \mapsto T_{++} (\epsilon, \eta),
\]

\[
S_{++} (\epsilon, \eta) \mapsto T_{++} (\epsilon, \eta),
\]

\[
S_{++} (\epsilon, \eta) \mapsto T_{++} (\epsilon, \eta).
\]

Third, the map \( h : C(P') \to C(P) \) such that \( d \circ h + h \circ d = \text{id} - \text{id} \circ \rho \), is defined by the formulas

\[
S_{--} (\epsilon, \eta) \mapsto S_{--} (\epsilon, -\eta),
\]

\[
S_{++} (\epsilon, +\eta) \mapsto S_{++} (\epsilon, \eta),
\]

\[
\text{otherwise} \mapsto 0.
\]

(III–3) Consider the case where the state of \( P' \) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C))\)

\[
w_A w_{ABC}
\]

\[
= (1, 1, 1)
\]

is represented as \((u| \theta_{\epsilon} | \theta_{\eta} | v)\).

Denote by \( S_{++} (\epsilon, \eta) \) the state \((u| \theta_{\epsilon} | \theta_{\eta} | v)\) of \( P' \) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C))\) \(= (1, 1, 1)\), denote by \( S_{--} (\epsilon) \) the state \((u| \theta_{\epsilon} | \theta_{\eta} | v)\) of \( P' \) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C))\) \(= (-1, 1, 1)\), denote by \( S_{--} (\epsilon, \zeta, \eta) \) the state \((u| \theta_{\epsilon} | \theta_{\zeta} | \theta_{\eta} | v)\) of \( P' \) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C))\) \(= (1, -1, 1)\), denote by \( S_{++} (\epsilon, \zeta, \eta) \) the state \((u| \theta_{\epsilon} | \theta_{\zeta} | \theta_{\eta} | v)\) of \( P' \) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C))\) \(= (-1, -1, 1)\), denote by \( S_{--} (\epsilon) \) the state \((u| \theta_{\epsilon} | \theta_{\eta} | v)\) of \( P' \) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C))\) \(= (1, 1, -1)\), denote by \( S_{--} (\epsilon, \eta) \) the state \((u| \theta_{\epsilon} | \theta_{\eta} | v)\) of \( P' \) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C))\) \(= (-1, 1, -1)\), denote by \( S_{++} (\epsilon) \) the state \((u| \theta_{\epsilon} | v)\) of \( P' \) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C))\) \(= (1, 1, 1)\), denote by \( S_{--} (\epsilon) \) the state \((u| \theta_{\epsilon} | \theta_{\eta} | v)\) of \( P' \) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C))\) \(= (1, -1, -1)\).

The subcomplex \( C' \) of \( C(P') \) is defined by \( C' := C \left( S_{++} (+) + S_{++} (+, -, +), S_{++} (-) + S_{++} (+, -, -) + S_{++} (-, -, +), S_{++} (-, +, -) \right) \) where \( S_{++} \) denotes every states with \( \text{mark}(C) = -1 \).

Denote by \( T_{++} (\epsilon) \) the state \((u| \theta_{\epsilon} | v)\) of \( P \) with \((\text{mark}(A), \text{mark}(B), \text{mark}(C))\) \(= (1, 1, 1)\), denote by \( T_{++} (\epsilon, -, \eta) \) the state \((u| \theta_{\epsilon} | \theta_{-} | \theta_{\eta} | v)\) of \( P \) with
(mark(A), mark(B), mark(C)) = (-1, 1, 1), denote by \( T_{+++}(\epsilon, \zeta, \eta) \) the state \( w_{AC} w_B w_{ABC} \) of \( P \) with (mark(A), mark(B), mark(C)) = (1, 1, -1), denote by \( T_{---}(\epsilon, \eta) \) the state (\( u|\emptyset_\epsilon|\emptyset_\zeta|\emptyset_\eta|v \)) of \( P \) with (mark(A), mark(B), mark(C)) = (-1, 1, -1), denote by \( T_{+-+}(\epsilon, \eta) \) the state (\( u|\emptyset_\epsilon|\emptyset_\eta|v \)) of \( P \) with (mark(A), mark(B), mark(C)) = (1, -1, -1), denote by \( T_{++-}(\epsilon) \) the state (\( u|\emptyset_\epsilon|\emptyset_\eta|v \)) of \( P \) with (mark(A), mark(B), mark(C)) = (-1, 1, -1), denote by \( T_{++-} \) every states of \( P \) with mark(C) = -1.

The subcomplex \( C \) of \( C(P) \) is defined by \( C := C( T_{++-}(+)+T_{++-}(+, -, +), T_{++-}(-)+T_{++-}(+, -, -)+T_{++-}(-, -, +), T_{++-} ). \)

First, the retraction \( \rho : C(P') \to C' \) is defined by the formulas

\[
S_{++-}(+) \mapsto S_{++-}(+) + S_{++-}(+, -, +),
S_{++-}(-) \mapsto S_{++-}(-) + S_{++-}(+, -, -) + S_{++-}(-, -, +),
S_{++-}(+, +, +) \mapsto S_{++-}(+, +, +),
S_{++-}(+, +, -) \mapsto S_{++-}(+) + S_{++-}(+, -, +) + S_{++-}(+, +, -) + S_{++-}(+, -, +),
S_{++-}(-, +, +) \mapsto S_{++-}(+) + S_{++-}(-, +, +) + S_{++-}(-, +, +),
S_{++-}(-, +, -) \mapsto S_{++-}(-) + S_{++-}(+, -, -) + S_{++-}(-, -, +) + S_{++-}(-, +, -) + S_{++-}(-, -, +),
S_{++-}(\epsilon, \eta) \mapsto S_{++-}(\epsilon, \eta),
\text{otherwise} \mapsto 0.
\]

Second, consider the following composition (23) of the following isomorphism with \( \rho \). The isomorphism \( C' \to C \) is defined by the formulas

\[
S_{++-}(+) + S_{++-}(+, -, +) \mapsto T_{++-}(+) + T_{++-}(+, -, +),
S_{++-}(-) + S_{++-}(+, -, -) + S_{++-}(-, -, +) \mapsto T_{++-}(-) + T_{++-}(+, -, -) + T_{++-}(-, -, +),
S_{++-}(\epsilon) \mapsto T_{++-}(\epsilon),
S_{++-}(\epsilon, \eta) \mapsto T_{++-}(\epsilon, \eta),
S_{++-}(\epsilon, \eta) \mapsto T_{++-}(\epsilon, \eta),
S_{++-}(\epsilon, \zeta, \eta) \mapsto T_{++-}(\epsilon, \zeta, \eta).
\]

Third, the map \( h : C(P') \to C(P') \) such that \( d \circ h + h \circ d = \text{id} - \text{id} \circ \rho \), is defined by the formulas

\[
S_{---}(\epsilon, \eta) \mapsto S_{---}(\epsilon, -, \eta),
S_{+++}(\epsilon, +, \eta) \mapsto S_{+++}(\epsilon, \eta),
\text{otherwise} \mapsto 0.
\]
(III–4) Consider the case where the state of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, 1)$ is represented as $(u|\emptyset_e|\emptyset_\eta|v)$. 

Denote by $S_{+++}(\epsilon, \eta)$ the state $(u|\emptyset_e|\emptyset_\eta|v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, 1)$, denote by $S_{+++}(\epsilon, \zeta, \eta)$ the state $(u|\emptyset_e|\emptyset_\zeta|\emptyset_\eta|v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, 1)$, denote by $S_{++-}(\epsilon, \zeta, \eta)$ the state $(u|\emptyset_e|\emptyset_\zeta|\emptyset_\eta|v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, 1, 1)$, denote by $S_{---}(\epsilon, \eta)$ the state $(u|\emptyset_e|\emptyset_\eta|v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, -1, 1)$, denote by $S_{--}(\epsilon, \eta)$ the state $(u|\emptyset_e|\emptyset_\eta|v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, -1)$, denote by $S_{--}(\epsilon, \eta)$ the state $(u|\emptyset_e|\emptyset_\eta|v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, -1)$, denote by $S_{-++}(\epsilon, \eta)$ the state $(u|\emptyset_e|\emptyset_\eta|v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, -1, 1)$, denote by $S_{-++}(\epsilon, \eta)$ the state $(u|\emptyset_e|\emptyset_\eta|v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, -1)$, denote by $S_{-++}(\epsilon, \eta)$ the state $(u|\emptyset_e|\emptyset_\eta|v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, -1, 1)$, denote by $S_{-++}(\epsilon, \eta)$ the state $(u|\emptyset_e|\emptyset_\eta|v)$ of $P'$ with $(\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, -1, 1)$. 

The subcomplex $C' \text{ of } C(P')$ is defined by $C' := C( S_{+++}(+, +, \eta), S_{+++}(+,-,\eta) \nn \+ S_{++-}(+,-,\eta), S_{++-}(-,+,\eta) \nn \+ S_{---}(+,+,\eta), S_{---}(-,-,\eta))$. 

The subcomplex $C$ of $C(P)$ is defined by $C := C( T_{++-}(+, +, \eta), T_{++-}(+,-,\eta) \nn \+ T_{++-}(-,+ \eta), T_{++-}(-,- \eta), T_{---}(+,-,\eta), T_{---}(-,+ \eta), T_{---}(-,-,\eta), T_{---})$. 

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First, the retraction \( \rho : C(P') \to C' \) is defined by the formulas

\[
\begin{align*}
S_{++}(+,+,\eta) & \mapsto S_{++}(+,+,\eta), \\
S_{++}(+,-,\eta) & \mapsto S_{++}(+,-,\eta) + S_{+-}(+,-,\eta), \\
S_{++}(-,+,\eta) & \mapsto S_{++}(-,+,\eta) + S_{++}(+,-,\eta), \\
S_{++}(-,-,\eta) & \mapsto S_{++}(-,-,\eta) + S_{++}(-,-,\eta), \\
S_{++}(\epsilon,\eta) & \mapsto S_{++}(\epsilon,\eta), \\
S_{+-} & \mapsto S_{+-}, \\
S_{++}(+,+,+) & \mapsto S_{++}(+,+,+), \\
S_{++}(+,+,+) & \mapsto S_{++}(+,+,+), \\
S_{++}(+,+,-) & \mapsto S_{++}(+,+,-) + S_{++}(+,+,-) + S_{++}(+,+,-). \\
S_{++}(\epsilon,\eta) & \mapsto S_{++}(\epsilon,\eta), \\
\text{otherwise} & \mapsto 0.
\end{align*}
\]

Second, consider the following composition (23) of the following isomorphism with \( \rho \). The isomorphism \( C' \to C \) is defined by the formulas

\[
\begin{align*}
S_{++}(+,+,\eta) & \mapsto T_{++}(+,+,\eta), \\
S_{++}(+,+,-) & \mapsto T_{++}(+,+,-) + T_{++}(+,+,-), \\
S_{++}(-,+,\eta) & \mapsto T_{++}(-,+,\eta) + T_{++}(-,+,\eta), \\
S_{++}(-,-,\eta) & \mapsto T_{++}(-,-,\eta) + T_{++}(-,-,\eta), \\
S_{++}(\epsilon,\eta) & \mapsto T_{++}(\epsilon,\eta), \\
S_{+-} & \mapsto T_{+-}(\epsilon,\eta), \\
S_{++} & \mapsto T_{++}(\epsilon,\eta), \\
\text{otherwise} & \mapsto 0.
\end{align*}
\]

Third, the map \( h : C(P') \to C(P') \) such that \( d \circ h + h \circ d = \text{id} - \text{in} \circ \rho \), is defined by the formulas

\[
\begin{align*}
S_{--}(\epsilon,\eta) & \mapsto S_{+-}(\epsilon, -,\eta), \\
S_{++}(\epsilon, +,\eta) & \mapsto S_{++}(\epsilon, +,\eta), \\
\text{otherwise} & \mapsto 0.
\end{align*}
\]

(III–5) Consider the case where the state of \( P' \) with \( (\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, 1) \) is represented as \( (u|\emptyset_e|\emptyset_e|\emptyset_e|v) \).

Denote by \( S_{+++}(\epsilon,\eta) \) the state \( (u|\emptyset_e|\emptyset_e|\emptyset_e|\emptyset_e|v) \) of \( P' \) with \( (\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, 1) \), denote by \( S_{--}(\epsilon,\zeta,\eta) \) the state \( (u|\emptyset_e|\emptyset_e|\emptyset_e|\emptyset_e|v) \) of \( P' \) with \( (\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, 1, 1) \), denote by \( S_{+-}(\epsilon,\zeta,\eta,\theta) \) the state...
\( w_A B \, w_C \, ABC \)

\begin{align*}
(u| \emptyset_\varepsilon | \emptyset_\zeta | \emptyset_\eta | v) \text{ of } P' \text{ with } (\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, -1, 1), \text{ denote by } S_{-\cdots}(\varepsilon, \zeta, \eta) \text{ the state } (u| \emptyset_\varepsilon | \emptyset_\zeta | \emptyset_\eta | v) \text{ of } P' \text{ with } (\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, -1, 1), \text{ denote by } S_{+\cdots}(\varepsilon, \eta) \text{ the state } (u| \emptyset_\varepsilon | \emptyset_\eta | v) \text{ of } P' \text{ with } (\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, 1, -1), \text{ denote by } S_{-\cdots}(\varepsilon, \eta) \text{ the state } (u| \emptyset_\varepsilon | \emptyset_\eta | v) \text{ of } P' \text{ with } (\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, -1, -1), \text{ denote by } S_{-\cdots}(\varepsilon, \eta) \text{ the state } (u| \emptyset_\varepsilon | \emptyset_\eta | v) \text{ of } P' \text{ with } (\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, 1, -1).

The subcomplex \( C'(P') \) is defined by \( C' := C \left(S_{-\cdots}(\varepsilon, \eta) + S_{+\cdots}(+, +, \eta, -), S_{-\cdots}(\varepsilon, \eta) + S_{-\cdots}(\varepsilon, -, \eta, -) + S_{-\cdots}(\varepsilon, +, \eta, -), S_{-\cdots} \right) \) where \( S_{-\cdots} \) denotes every states with \( \text{mark}(C) = -1 \).

Denote by \( T_{+\cdots}(\varepsilon, \eta) \) the state \( (u| \emptyset_\varepsilon | \emptyset_\eta | v) \) of \( P \) with \( (\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, -1, 1), \text{ denote by } T_{-\cdots}(\varepsilon, \eta) \text{ the state } (u| \emptyset_\varepsilon | \emptyset_\eta | v) \text{ of } P \text{ with } (\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, 1, -1), \text{ denote by } T_{-\cdots}(\varepsilon, \eta) \text{ the state } (u| \emptyset_\varepsilon | \emptyset_\eta | v) \text{ of } P \text{ with } (\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (1, -1, -1), \text{ denote by } T_{-\cdots}(\varepsilon, \eta) \text{ the state } (u| \emptyset_\varepsilon | \emptyset_\eta | v) \text{ of } P \text{ with } (\text{mark}(A), \text{mark}(B), \text{mark}(C)) = (-1, 1, -1), \text{ denote by } T_{-\cdots} \text{ every states of } P \text{ with } \text{mark}(C) = -1.

The subcomplex \( C \) of \( C(P) \) is defined by \( C := C \left(T_{+\cdots}(\varepsilon, \eta) + T_{-\cdots}(\varepsilon, \eta) \right) \).
First, the retraction \( \rho : C(P') \to C' \) is defined by the formulas

\[
S_{--}(+, \eta) \mapsto S_{--}(+, \eta) + S_{+-}(+, +, -), \\
S_{--}(-, \eta) \mapsto S_{++}(+, -, -) + S_{++}(-, +, -),
\]

\[
S_{+--}(+, +, -, +) \mapsto S_{++}(-, +, +) + S_{++}(+,-, +) + S_{+-}(+,-, +), \\
S_{+--}(+, -, -, +) \mapsto S_{++}(+, -, +) + S_{++}(+,-, -) + S_{+-}(+,-, -),
\]

\[
S_{+--}(-, +, +) \mapsto S_{++}(+, +, -) + S_{++}(+,-, -) + S_{+-}(+,-, -), \\
S_{+--}(-, -, +) \mapsto S_{++}(-, +, -) + S_{++}(+,-, -) + S_{+-}(+,-, -), \\
S_{+--}(-, -, -, +) \mapsto S_{++}(-, +, -) + S_{++}(+,-, -) + S_{+-}(+,-, -) + S_{+-}(+,-, -),
\]

\[
S_{+--}(\epsilon, \zeta, \eta) \mapsto S_{+--}(\epsilon, \zeta, \eta), \\
\text{otherwise} \mapsto 0.
\]

Second, consider the following composition (23) of the following isomorphism with \( \rho \). The isomorphism \( C' \to C \) is defined by the formulas

\[
S_{++}(+, +) + S_{++}(+, +, -) \mapsto T_{++}(+, +), \\
S_{++}(-, -) + S_{--}(+, +, -) \mapsto T_{--}(+,-, +) + T_{--}(+,-, -),
\]

\[
S_{++}(+, +, -) + S_{--}(+, +, -) + S_{--}(-, +, -) \mapsto T_{++}(-, +) + T_{--}(+,-, +), \\
S_{++}(+, +, -) + S_{--}(+, +, -) + S_{--}(-, +, -) \mapsto T_{--}(-, -) + T_{--}(+,-, +),
\]

\[
S_{++}(\epsilon, \eta) \mapsto T_{++}(\epsilon, \eta), \\
S_{--}(\epsilon) \mapsto T_{--}(\epsilon), \\
S_{--}(\epsilon, \zeta, \eta) \mapsto T_{--}(\epsilon, \zeta, \eta), \\
S_{++}(\epsilon, \eta) \mapsto T_{--}(\epsilon, \eta).
\]

Third, the map \( h : C(P') \to C(P') \) such that \( d \circ h + h \circ d = id - \circ \rho \) is defined by the formulas

\[
S_{+-}(\epsilon, \zeta, \eta) \mapsto S_{+-}(\epsilon, \zeta, \eta), \\
S_{++}(\epsilon, \zeta, \eta) \mapsto S_{++}(\epsilon, \zeta, \eta), \\
\text{otherwise} \mapsto 0.
\]

The fact is, (III–1) - (III-5) prove that \( KH^{i,j}((xAByACzBCt)) \simeq KH^{i,j}((xBAyCAzCBt)) \) if \( (|A|, |B|, |C|) \) is any of \( \{(-1, -1, -1), (-1, 1, 1), (1, 1, -1) \} \).
Consider $P' = (xAByACzBCt) \rightarrow (xBAYCAzCBt) = P$ where $(|A|, |B|, |C|) = (1, -1, -1)$.

$xAByACzBCt \xrightarrow{\nu\text{-shift}} xBCyABzACt$ with $(|A|, |B|, |C|) = (1, -1, -1)$

$xAByDAzDBt \cong xAByDAzDBt$ with $(|A|, |B|, |C|) = (1, -1, 1)$

$\text{Lemma } 2.2 \text{ implies } xAByDAzDBt \cong xAByDAzDBt$ with $(|A|, |B|, |C|) = (1, -1, -1)$

$H_3 \cong xAByDAzDBt \cong xAByDAzDBt$ with $(|A|, |B|, |C|) = (1, -1, 1)$

$H_3 \cong xAByDAzDBt$ with $(|A|, |B|, |C|) = (1, -1, -1)$

$\nu\text{-shift } xAByACzCBt$ with $(|A|, |B|, |C|) = (1, -1, -1)$

We have already showed that the invariance of $KH_3$ under these moves above and then $KH_3$ is preserved under the third homotopy move $H_3$ and its inverse move with $(|A|, |B|, |C|) = (1, -1, -1)$. In particular, in this case, we use the invariance of $KH_3$ under $H_3$ and its inverse move with $(|A|, |B|, |C|) = (1, -1, -1)$. By using the invariance under $H_3$ and its inverse with $(|A|, |B|, |C|) = (1, 1, -1)$ (resp. $(1, -1, 1)$), we can verify that the invariance of $KH_3$ under $H_3$ and its inverse move with $(|A|, |B|, |C|) = (1, 1, 1, 1)$ (resp. $(1, -1, -1, 1)$).

We conclude the proof that $KH_3(P') \cong KH_3(P)$ for $P' \cong S_t P$.

The following corollary is a similar result to Corollary 3.1.

**Corollary 5.1.** $KH_3(P)$ is a $S_0$-homotopy invariant for nanophrases $P$ over $\alpha_0$.

6. AN APPLICATION OF $KH_3$ VIA WORDS TO NANOPHRASES OVER ANY $\alpha$.

In the previous sections, we discuss $S_1$-homotopy invariants $\hat{J}(P)$ and $KH_3(P)$ of pseudolinks. Here, we construct homotopy invariants of nanophrases over any $\alpha$ from $\hat{J}(P)$ and $KH_3(P)$.

Let $\alpha$ be an arbitrary alphabet, $\tau$ be $\alpha \rightarrow \alpha$; involution, $\Delta_\alpha$ be $\{(a, a, a)\}_{a \in \alpha}$ and $\alpha/\tau := \{\tilde{a}_1, \ldots, \tilde{a}_m\}$. We fix a complete residue system $\{a_1, \ldots, a_m\}$ of $\alpha/\tau$ and denote $\{a_1, \ldots, a_m\}$ by $\text{crs}(\alpha/\tau)$.

We use the notion of an 6.1 as in [7, Section 4.1].

**Definition 6.1.** An orbit of the involution $\tau : \alpha \rightarrow \alpha$ is a subset of $\alpha$ consisting either of one element preserved by $\tau$ or of two elements permuted by $\tau$; in the latter case the orbit is free.

**Definition 6.2.** For $A \in \mathcal{A}$, we define sign of $A$ by

$$\text{sign}_L(A) := \begin{cases} 
1 & \text{if } |A| \in L; \hat{A} : \text{a free orbit} \\
-1 & \text{if } |A| \in \tau(L); \hat{A} : \text{a free orbit} \\
0 & \text{otherwise}
\end{cases}$$

where $L$ is a nonempty subset of $\text{crs}(\alpha/\tau)$. 49
Let $P_k(\alpha, \tau)$ be a set of nanophrases of length $k$ over $\alpha$ with $\tau$.

**Definition 6.3.** For an arbitrary $(\alpha, \tau)$ and an arbitrary subset $L \subset \text{crs}(\alpha/\tau)$, $U_L : P_k(\alpha, \tau) \rightarrow P_k(\alpha_0, \tau_0)$; $P \mapsto P_0$ is defined by the following two steps:

(Step 1) Remove $A \in \mathcal{A}$ such that $\text{sign}_L(A) = 0$ from $(A, P) \in P_k(\alpha, \tau)$.

(Step 2) Let the nanophrase be $(A', P')$ after removing letters from $(A, P)$ by using (Step 1). We consider an $\alpha_0$-alphabet $\mathcal{B}$ such that $\text{card}\mathcal{B} = \text{card}\mathcal{A}'$ and $\mathcal{A}' \cap \mathcal{B}$ is the empty set. Transpose each letter of $(A', P')$ and a letter in $\mathcal{B}$ as follows:

$$U_L : P_k(\alpha, \tau) \rightarrow P_k(\alpha_0, \tau_0); \quad P \mapsto P_0$$

$$U_L : \mathcal{A} \rightarrow \mathcal{A}_0; \quad A \mapsto A_0$$

$$U_L : \mathcal{B} \rightarrow \mathcal{B}_0; \quad B \mapsto B_0$$

By accomplishing (1) and (2), the nanophrase over $\alpha_0$ derived from $(A, P)$ is denoted by $U_L((A, P))$ or simply $U_L(P)$.

**Theorem 6.1.** For an arbitrary $L \subset \text{crs}(\alpha/\tau)$ and for two arbitrary nanophrases $(A_1, P_1)$ and $(A_2, P_2)$,

$$(A_1, P_1) \simeq_{\Delta_L} (A_2, P_2) \implies U_L((A_1, P_1)) \simeq_{\Delta_L} U_L((A_2, P_2)).$$

**Proof.** First, isomorphism does not change $U_L(P)$ up to isomorphic is clear.

Consider the first homotopy move

$$P_1 := (A, (xAAAy)) \rightarrow P_2 := (A \setminus \{A\}, (xy))$$

where $x$ and $y$ are words on $\mathcal{A}$, possibly including “|” character. Suppose $\text{sign}(A) \neq 0$. Then

$$U_L(P_1) = x_LAAYL \simeq x_Ly_L = U_L(P_2)$$

where $x_L$ and $y_L$ are words which obtained by deleting all letters $X \in \mathcal{A}$ such that $\text{sign}(X) = 0$ from $x$ and $y$ respectively.

Suppose $\text{sign}(A) = 0$. Then

$$U_L(P_1) = x_Ly_L = U_L(P_2).$$

So the first homotopy move does not change the homotopy class of $U_L(P)$.

Consider the second homotopy move

$$P_1 := (A, (xAByBAz)) \rightarrow (A \setminus \{A, B\}, (xyz))$$

where $|A| = \tau(|B|)$, and $x$, $y$ and $z$ are words on $\mathcal{A}$ possibly including “|” character. Suppose $|A| \in L \cup \tau(L)$ and $\tilde{A}$ is free orbit. Then $|B| \in L \cup \tau(L)$ and $|\tilde{A}|$ is a fixed point of $\tau$. Then $|B| \notin L \cup \tau(L)$ or $|B|$ is a fixed point of $\tau$ since $|A| = \tau(|B|)$. So

$$U_L(P_1) = x_LABy_LBAz_L \simeq x_Ly_Lz_L = U_L(P_2).$$

where $x_L$, $y_L$ and $z_L$ are words which obtained by deleting all letters $X \in \mathcal{A}$ such that $\text{sign}(X) = 0$ from $x$, $y$ and $z$ respectively. Suppose $|A| \notin L \cup \tau(L)$ or $|\tilde{A}|$ is a fixed point of $\tau$. Then $|B| \notin L \cup \tau(L)$ or $|B|$ is a fixed point of $\tau$ since $|A| = \tau(|B|)$. So

$$U_L(P_1) = x_Ly_Lz_L = U_L(P_2).$$

By the above, the second homotopy move does not change the homotopy class of $U_L(P)$.
Consider the third homotopy move
\[ P_1 := (\mathcal{A}, xAByACzBCt) \rightarrow P_2 := (\mathcal{A}, xBAyCAzCBt) \]
where \(|A| = |B| = |C|\), and \(x, y, z\) and \(t\) are words on \(\mathcal{A}\) possibly including “\(|\)” character. Suppose sign(\(A\)) \(\neq 0\). Then sign(\(B\)), sign(\(C\)) \(\neq 0\) since |\(A\)| = |\(B\)| = |\(C\)|. So we obtain
\[ \mathcal{U}_L(P_1) = x_LABy_LACz_LACt_L \simeq x_LBAy_LCAz_LCBt_L = \mathcal{U}_L(P_2) \]
where \(x_L, y_L, z_L\) and \(t_L\) are words which obtained by deleting all letters \(X \in \mathcal{A}\) such that sign(\(X\)) \(\neq 0\) from \(x, y, z\) and \(t\) respectively. Suppose sign(\(A\)) = 0. Then sign(\(B\)), sign(\(C\)) = 0 since |\(A\)| = |\(B\)| = |\(C\)|. So we obtain
\[ \mathcal{U}_L(P_1) = x_Ly_Lz_L t_L = \mathcal{U}_L(P_2) \]
So the third homotopy move does not change the homotopy class of \(\mathcal{U}_L(P)\).

By the above, \(\mathcal{U}_L\) is a homotopy invariant of nanophrases.

\[ \square \]

**Corollary 6.1.** Let \(\mathcal{I}\) be a \(S_0\)-homotopy invariant of nanophrase over \(\alpha_0\). For \(P \in \mathcal{P}_k(\alpha, \tau)\), we define \(I'\) by
\[ I'(P) := \{ I(\mathcal{U}_L(P)) \}_{L \in \text{crs}(\alpha/\tau)} \]
\(I'\) is a \(\Delta_\alpha\)-homotopy invariant of \(P \in \mathcal{P}_k(\alpha, \tau)\). In particular, for \((\mathcal{A}, P) \in \mathcal{P}_k(\alpha_0, \tau_0)\), \(I'(P) = \{ I(P) \}\) if we take \(\text{crs}(\alpha_0/\tau_0) = \{1\}\).

Theorem 6.1 implies the following Corollaries.

**Corollary 6.2.** Let \(\alpha\) be an arbitrary alphabet. \(\hat{J}(\mathcal{U}_L(P))\) is \(\Delta_\alpha\)-homotopy invariants for nanophrases \(P\) over \(\alpha\).

**Corollary 6.3.** Let \(\alpha\) be an arbitrary alphabet. \(KH^{i,j}(\mathcal{U}_L(P))\) is \(\Delta_\alpha\)-homotopy invariants for nanophrases \(P\) over \(\alpha\).

**Remark 6.1.** \(\hat{J}(\mathcal{U}_L(P)) = \sum_{j=-\infty}^{\infty} q^j \sum_{i=-\infty}^{\infty} (-1)^i \text{rk} KH^{i,j}(\mathcal{U}_L(P))\).

We give some examples of calculating of \(KH^{i,j}(P)\) or \(KH^{i,j}(\mathcal{U}_L(P))\).

**Example 6.1.** For two pseudolinks \(P_1 = ABCDEABCD\) with |\(A\)| = |\(B\)| = |\(C\)| = |\(D\)| = |\(E\)| = |\(I\)| = |\(J\)| = |\(L\)| and |\(B\)| = |\(D\)| = |\(F\)| = 1, \(\hat{J}(P_1) = \hat{J}(P_2)\). However, \(KH^{-7,15}(P_1) \simeq 0\) and \(KH^{-7,15}(P_2) \simeq \mathbb{Z}_2\). (cf. [1, 10].)

**Theorem 6.2.** \(KH^{i,j}(P)\) is a strictly stronger invariant than \(\hat{J}(P)\).

In [7], Turaev constructed a \(\Delta_\alpha\)-homotopy invariant \(\lambda\) for nanophrases over \(\alpha\).

**Example 6.2.** Let \(a, b, c\) be elements (possibly coinciding) of any alphabet \(\alpha\) and \(A, B, C\) are letters with |\(A\)| = \(a\), |\(B\)| = \(b\) and |\(C\)| = \(c\). If \(a = c = \tau(b) \neq b\), \(\lambda(ABACBC) = \lambda(ACAC) = a + a - aa - a^2 a\). However, \(KH^{0,2}(\mathcal{U}_\{a\})(ACAC)\) \(\simeq 0\) and \(KH^{0,2}(\mathcal{U}_\{a\})(ABACBC)\) \(\simeq \mathbb{Z}_2\).

Turaev constructed a strictly stronger \(\Delta_\alpha\)-homotopy invariant \(f \circ v_+\) than \(\lambda\) for nanophrases over \(\alpha\) [7].
Example 6.3. Let $a, b, c, d$ be elements (possibly coinciding) of any alphabet $\alpha$ and $A, B, C, D$ are letters with $|A| = a, |B| = b, |C| = c$ and $|D| = d$. If $a = b$ and $c = \tau(b) = d$, $f(v_+(ABCDCDAB)) = f(v_+\emptyset) = 1$. However, $KH^{0,3}(U_{\{c\}}(\emptyset)) \simeq 0$ and $KH^{0,3}(U_{\{c\}}(ABCDCDAB)) \simeq \mathbb{Z}_2$.

Theorem 6.3. Let $\alpha$ be an arbitrary alphabet and $S$ be $\Delta_\alpha$. $KH^{i,j}(U_L(P))$ is independent of $f \circ v_+$ for nanophrases $P$ over $\alpha$.

We has left the following problems unsolved: Is $KH^{i,j}(U_L(P))$ strictly stronger than $\hat{J}(U_L(P))$? What relation is $KH^{i,j}(P)$ of a pseudolink $P$ to Manturov’s categorification [5, 6]?

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