On decay rate of quenching profile at space infinity for axisymmetric mean curvature flow*†

Yoshikazu Giga, Yukihiro Seki and Noriaki Umeda
Graduate School of Mathematical Sciences,
University of Tokyo,
3-8-1, Komaba, Meguro-ku, Tokyo, 153-8914, Japan

Abstract

We study the motion of noncompact hypersurfaces moved by their mean curvature obtained by a rotation around $x$-axis of the graph a function $y = u(x, t)$ (defined for all $x \in \mathbb{R}$). We are interested to estimate its profile when the hypersurface closes open ends at the quenching (pinching) time $T$. We estimate its profile at the quenching time from above and below. We in particular prove that $u(x, T) \sim |x|^{-a}$ as $|x| \to \infty$ if $u(x, 0)$ tends to its infimum with algebraic rate $|x|^{-2a}$ (as $|x| \to \infty$ with $a > 0$).

1 Introduction and main theorem

This is a continuation of our study [4] on motion of noncompact axisymmetric $n$-dimensional hypersurface $\Gamma_t$ moved by its mean curvature. Let $\Gamma_t$ be given by a rotation of the graph of a function $y = u(x, t)$ (defined on $x \in \mathbb{R}$) around the $x$-axis (cf [1, 2]). In our previous paper [4], among other results, we have proved that if $u(x, 0) \to m := \inf_{x \in \mathbb{R}} u(x, 0) > 0$ as $|x| \to \infty$, then $\Gamma_t$ closes open ends at the time $T(m)$, where $T(m)$ is the quenching (pinching) time of the regular cylinder with radius $m$. (Moreover, there is no neck-pinch in $\mathbb{R}$ at $t = T(m)$.) These results imply that

$$\lim_{x \to \infty} u(x, T(m)) = 0 \quad \text{or} \quad \lim_{x \to -\infty} u(x, T(m)) = 0,$$

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but it does not provide the convergence rate.

We are interested in studying the profile of $u(x, T(m))$, especially the behavior as $|x| \to \infty$ which is affected by initial data.

The equation for $u$ is of the form

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{n - 1}{u}, \quad x \in \mathbb{R}, \quad t > 0$$  \hspace{1cm} (1)

supplemented by initial data

$$u(x, 0) = u_0(x) > 0, \quad x \in \mathbb{R}.$$  \hspace{1cm} (2)

The function $u_0$ is assumed to satisfy

$$u_0 \text{ is bounded and uniformly continuous in } \mathbb{R},$$  \hspace{1cm} (3)

$$m := \inf_{x \in \mathbb{R}} u_0(x) > 0.$$  \hspace{1cm} (4)

The Cauchy problem (1)-(2) has a unique positive classical solution with the conditions (3)-(4) to the initial data (cf [4]). However, the solution quenches in finite time. For a given initial datum $u_0$, we see

$$T(u_0) = \sup\{t > 0; \inf_{x \in \mathbb{R}} u(x, t) > 0\} < \infty$$

and call it the quenching time of $u$. It is clear that

$$\lim_{t \to T(u_0)} \inf_{x \in \mathbb{R}} u(x, t) = 0.$$

Let $v$ be a solution of (1) with initial datum $m = \inf_{x \in \mathbb{R}} u_0(x)$. It is easily seen that

$$v' = -\frac{n - 1}{v}, \quad t > 0, \quad v(0) = m,$$  \hspace{1cm} (5)

and

$$v(t) = \sqrt{2(n - 1)(T(m) - t)} \quad \text{with} \quad T(m) = \frac{m^2}{2(n - 1)}.$$  \hspace{1cm} (6)

It is immediate that $T(u_0) \geq T(m)$ by a comparison argument. We treat the case $T(u_0) = T(m)$. The notion of “minimal quenching time” was defined in [4], which is recalled below.

**Definition 1.1.** A solution of the Cauchy problem (1)-(2) is said to have a minimal quenching time, if

$$T(u_0) = T(m).$$

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In [4] we characterized solutions of (1)-(2) quenching only at space infinity. The following conditions on initial data $u_0$ play essential roles in [4].

A. There exists a sequence $\{x_k\} \in \mathbb{R}$ such that $x_k \to \infty$ and $u_0(x + x_k) \to m$ a.e. in $\mathbb{R}$ as $k \to \infty$.

B. There exists a sequence $\{x_k\} \in \mathbb{R}$ such that $x_k \to -\infty$ and $u_0(x + x_k) \to m$ a.e. in $\mathbb{R}$ as $k \to \infty$.

For an initial datum satisfying (3)-(4), we proved in [4] the following results for the Cauchy problem (1)-(2):

1. A solution of (1)-(2) has a minimal quenching time, if and only if the conditions A or B holds.

Moreover, if $u_0$ is not constant as well as the conditions A or B holds, then:

2. For an initial datum satisfying $u_0 \not\equiv m$, the solution (1)-(2) quenches only at space infinity.

3. There exists a function $u(\cdot, T(m)) \in C^\infty(\mathbb{R})$ such that $u(\cdot, t) \to u(\cdot, T(m))$ in the Frechét space $C^\infty(\mathbb{R})$ as $t \to T(m)$, $u(x, T(m)) > 0$ in the whole $\mathbb{R}$ and

$$\liminf_{x \to -\infty} u(x, T(m)) = 0 \quad \text{or} \quad \liminf_{x \to -\infty} u(x, T(m)) = 0.$$ 

For a solution $u$ of (1)-(2) with minimal quenching time $T(m)$, we call $u(\cdot, T(m))$ the profile of $u$ (at the quenching time $T(m)$). The hypersurface corresponding to $u(\cdot, T(m))$ is called limit surface.

These are related studies on blow-up at infinity for the reaction-diffusion equations [8, 5, 6, 3, 10, 9, 11] (see also [7]). We shall explain these papers at the end of this introduction. In particular, blow-up profile was discussed, for example, in [8] and [11] for a semilinear heat equation.

In this paper we consider the relation between the profile of a quenching solution at quenching time $T(m)$ and the form of initial data. Our goal, which is investigating the shape of limit surface, is similar to studying blow-up profile. Inspired by the method used in [8, §2b] and [11, Theorems 1.3 and 1.5], we construct a subsolution and a supersolution of the form $\phi(T(m) - t + g(x, t))$ with some function $g(x, t)$ decaying to zero at space infinity, where

$$\phi(s) = v(T(m) - s) = \sqrt{2(n - 1)s}, \quad (7)$$
in order to estimate the profile at the quenching time. Let \( \psi(x) \) be a positive function satisfying the following conditions:

\[
\sqrt{\psi(x)} \text{ is bounded and uniformly continuous in } \mathbb{R}; \quad (8)
\]

\[
\psi(x) > 0 \text{ for } x \in \mathbb{R}; \quad (9)
\]

\[
\lim_{x \to \infty} \psi(x) = 0 \text{ or } \lim_{x \to -\infty} \psi(x) = 0; \quad (10)
\]

there exist constants \( C_1 > 0 \) and \( C_2 > 0 \) such that

\[
\psi(x) \leq C_1 \max\{ \inf_{z \in [x-1,x]} \psi(z), \inf_{z \in [x,x+1]} \psi(z) \} \text{ for } x \in \mathbb{R}; \quad (11)
\]

\[
\psi(x - y) \leq C_2 \exp\left( a|y|^2 \right) \psi(x) \quad \text{for } x, y \in \mathbb{R}, \quad a \in \left( 0, \frac{1}{4T(m)} \right). \quad (12)
\]

**Example 1.2.** The functions \( \psi(x) = (|x|^2 + 1)^{-b/2}, e^{-b|x|} \) and \( \log(|x|+e))^{-b} \) with \( b > 0 \) satisfy (8)-(12).

**Theorem 1.3.** Let \( \psi \) be a function satisfying (8)-(12). Assume that (3)-(4) hold and that there exist constants \( C_I > 0 \) and \( C_{II} > 0 \) such that

\[
u_0^2(x) - m^2 \geq C_I \psi(x) \quad \text{(or } \leq C_{II} \psi(x)\text{).} \quad (13)
\]

Then there exists \( C = C(C_1, C_2, a, T(m), C_I) > 0 \) (or \( C' = C'(C_1, C_2, a, T(m), C_{II}) > 0 \)) such that the solution of the Cauchy problem (1)-(2) satisfies

\[
u(x, T(m)) \geq C \sqrt{\psi(x)} \quad \text{(or } \leq C' \sqrt{\psi(x)}\text{).}
\]

By setting \( \psi(x) = \langle x \rangle^{-2a_1} \) (or \( \langle x \rangle^{-2a_2} \)) with \( \langle x \rangle = (1 + |x|^2)^{1/2} \), we obtain algebraic decay at the space infinity.

**Corollary 1.4.** Assume that there exist constants \( a_1 > 0, a_2 > 0, C_I > 0 \) and \( C_{II} > 0 \) such that

\[
u_0^2(x) - m^2 \geq C_I \langle x \rangle^{-2a_1} \quad \text{(or } \leq C_{II} \langle x \rangle^{-2a_2}\text{).} \quad (14)
\]

Then there exists \( C = C(a_1, T(m), C_I) > 0 \) (or \( C' = C'(a_2, T(m), C_{II}) > 0 \)) such that

\[
u(x, T(m)) \geq C \langle x \rangle^{-a_1} \quad \text{(or } \leq C' \langle x \rangle^{-a_2}\text{).}
\]

We conclude this introduction by giving a short review on blow-up (or quenching) at the space infinity. Lacey [8] considered problems in a half line of \( u_t = u_{xx} + f(u) \) in \( \mathbb{R}^+ = \{ x : x > 0 \} \) and constructed solutions blowing up only at space infinity. Gladkov [7] studied problems of the equation
\( u_t = u_{xx} + f(x, t, u) \) in \( \mathbb{R}^+ \) and showed that solutions of the problem uniformly converge as \( x \to \infty \) to the solution of the ODE obtained by dropping \( u_{xx} \) in the equation.

Giga-Umeda [5] proved that blow-up only at space infinity occurs under the condition \( \lim_{|x| \to \infty} u_0(x) = \sup_{x \in \mathbb{R}} u_0(x) =: M \) and \( u_0 \neq M \) for nonnegative solutions of \( u_t = \Delta u + u^p \) in \( \mathbb{R}^n \) (cf. also [12] for a related study). For generalization, see [6] and a review article by Giga-Seki-Umeda [3]. More recently, Shimojō [11] discussed blow-up profile \( u(x, T) := \lim_{t \to T} u(x, t) \) for \( x \in \mathbb{R}^n \). See also Seki-Suzuki-Umeda [10] and Seki [9] for quasilinear parabolic equations, which generalized the result of [6]. They also gave necessary and sufficient conditions for a solution to have “minimal blow-up time (or the least blow-up time)”. See [9, 10, 3] for the precise definition of the last notion.

2 Profile at quenching

In order to prove Theorem 1.3, we construct a subsolution and supersolution of the form \( \phi(T(m) - t + g(x, t)) \), as we have explained before. This is a modification of the method employed in [8] and [11] to study blow-up profile for a semilinear heat equation. The function

\[
g(x, t) = \int_{-\infty}^{\infty} G(x - y, t)\psi(y)dy
\]

with the Gauss kernel of heat equation

\[
G(x, t) = (4\pi t)^{-1/2} \exp \left( -\frac{x^2}{4t} \right)
\]

is used there. However, because the problem which we treat here is a quasilinear equation, the Gauss kernel is not appropriate in our problem. We use the following function instead of \( G(x, t) \):

\[
g_{\alpha, \beta}^{\gamma, \psi}(x, t) = g_{\alpha, \beta}(x, t) = \int_{-\infty}^{\infty} G_{\alpha, \beta}^{\gamma}(x - y, t)\psi(y)dy,
\]

where

\[
G_{\alpha, \beta}^{\gamma}(x, t) = \frac{|x|^{\beta}}{(t + \gamma)^{\alpha}} \exp \left( -\frac{x^2}{4(t + \gamma)} \right)
\]

with \( \alpha \geq 0, \beta \geq 0 \) and \( \gamma > 0 \) being constants. Note that this \( g_{\alpha, \beta}^{\gamma} \) may be expressed by

\[
g_{\alpha, \beta}^{\gamma}(x, t) = \int_{-\infty}^{\infty} G_{\alpha, \beta}^{\gamma}(y, t)\psi(x - y)dy.
\]
It is easily seen that the derivatives are calculated and estimated as follows:

\[ |\partial_x g^\gamma_{\alpha,0}| \leq \frac{g^\gamma_{\alpha+1,1}}{2}, \quad (16) \]
\[ \partial_{xx} g^\gamma_{\alpha,0} = \frac{g^\gamma_{\alpha+2,2}}{4} - \frac{g^\gamma_{\alpha+1,0}}{2}, \quad (17) \]
\[ \partial_t g^\gamma_{\alpha,0} = \frac{g^\gamma_{\alpha+2,2}}{4} - \alpha g^\gamma_{\alpha+1,0}, \quad (18) \]

and

\[ g^\gamma_{\alpha,\beta}(x,t) = \frac{g^\gamma_{0,\beta}}{(t+\gamma)^\alpha}. \quad (19) \]

Before proving the Theorem 1.3 we prepare two propositions.

**Proposition 2.1.** Let \( \psi \) be a positive bounded uniformly continuous function. For any \( C > 0 \) and \( \gamma > 0 \) the function

\[ W(x,t) = \phi(T(m) - t + Cg^\gamma_{0,0}(x,t)) \quad (20) \]

is a supersolution of \((1)\) in \( \mathbb{R} \times (0, T(m)) \), where \( \phi \) is defined in \((7)\).

**Proof.** By a direct calculation we have

\[
W_t - \frac{W_{xx}}{1 + W^2_x} + \frac{n - 1}{W} = -\phi' + C\phi'\partial_t g^\gamma_{0,0} - \frac{C\phi'\partial_{xx} g^\gamma_{0,0} + (C\partial_x g^\gamma_{0,0})^2\phi''}{1 + (C\phi'\partial_x g^\gamma_{0,0})^2} + \frac{n - 1}{\phi}. 
\]

Noting that \( \phi'\partial_t g^\gamma_{0,0} \geq 0 \) from \((18)\) and \( \phi' = (n - 1)/\phi \), we obtain

\[
W_t - \frac{W_{xx}}{1 + W^2_x} + \frac{n - 1}{W} \geq \frac{C\phi'\partial_t g^\gamma_{0,0} - C\phi'\partial_{xx} g^\gamma_{0,0} - (C\partial_x g^\gamma_{0,0})^2\phi''}{1 + (C\phi'\partial_x g^\gamma_{0,0})^2}. 
\]

Since \( (\partial_t - \partial_{xx})g^\gamma_{0,0} = g^\gamma_{1,0}/2 \) by \((17)\)-(18), we have

\[
W_t - \frac{W_{xx}}{1 + W^2_x} + \frac{n - 1}{W} \geq \frac{1}{1 + (C\phi'\partial_x g^\gamma_{0,0})^2} \left( \frac{C\phi' g^\gamma_{1,0}}{2} - (C\partial_x g^\gamma_{0,0})^2\phi'' \right). 
\]

Due to the fact that \( \phi'' \leq 0 \), we see that \( W \) is a supersolution of \((1)\). \( \Box \)
Proposition 2.2. Assume that \( \psi \) is a function satisfying (8)-(12) and
\[
\gamma \in \left(0, \frac{1}{a} - 4T(m)\right)
\] with the constant \( a \) in (12). Then, for each constant \( C > 0 \), the function
\[
w(x, t) = \phi(T(m) - t + Cg_{\alpha,0}^\gamma(x, t))
\] is a subsolution of (1) in \( \mathbb{R} \times (0, T(m)) \) provided that \( \alpha \) satisfies \( \alpha \geq \alpha_0 \) with some constant \( \alpha_0 = \alpha_0(C_1, C_2, a, T(m), \gamma) > 0 \), where \( \phi \) is the function defined in (7).

Before proving Proposition 2.2, we prepare a lemma on estimates for \( g_{0,\beta}^\gamma \).

Lemma 2.3. Assume the same hypotheses as in Proposition 2.2. Then for \( \beta = 0, 1, 2 \), there exist constants \( C_3 = C_3(C_1, \gamma) > 0 \) and \( C_4 = C_4(C_2, a, T(m), \gamma) > 0 \) such that
\[
C_3\psi(x) \leq g_{0,\beta}^\gamma(x, t) \leq C_4\psi(x) \quad \text{in} \quad \mathbb{R} \times [0, T(m)],
\]
where \( C_1 \) and \( C_2 \) are the constants in (11) and (12), respectively.

Proof. First we show \( g_{0,\beta}^\gamma \geq C_3\psi(x) \) with some \( C_3 > 0 \). From (11)
\[
\psi(x) \leq C_1 \inf_{z \in [x-1,x]} \psi(z)
\]
or
\[
\psi(x) \leq C_1 \inf_{z \in [x,x+1]} \psi(z)
\]
for each \( x \in \mathbb{R} \). If (23) holds, then there exists a constant \( C_3 > 0 \) such that
\[
g_{0,\beta}^\gamma(x, t) \geq \inf_{z \in [x-1,x]} \psi(z) \times \int_0^1 |y|^\beta \exp \left(-\frac{|y|^2}{4\gamma}\right) dy \\
\geq \psi(x) \frac{1}{C_1} \int_0^1 |y|^\beta \exp \left(-\frac{|y|^2}{4\gamma}\right) dy.
\]
Set
\[
C_3 = \min_{\beta=0,1,2} \frac{1}{C_1} \int_0^1 |y|^\beta \exp \left(-\frac{|y|^2}{4\gamma}\right) dy = \frac{1}{C_1} \int_0^1 |y|^2 \exp \left(-\frac{|y|^2}{4\gamma}\right) dy.
\]
We then see that
\[
g_{0,\beta}^\gamma(x, t) \geq C_3\psi(x).
\]
A similar argument shows that if (24) holds, then
\[ g_{0,\beta}^\gamma(x, t) \geq C_3 \psi(x). \]
Thus we see that
\[ g_{0,\beta}^\gamma(x, t) \geq C_3 \psi(x) \]
for any \( x \in \mathbb{R} \).

We next prove \( g_{0,\beta}^\gamma(x, t) \leq C_4 \psi(x) \) with some \( C_4 > 0 \). For (21) it is possible to take a constant \( \gamma > 0 \) depending only on \( a \) and \( m \) that satisfies
\[ \frac{1}{4(T(m) + \gamma)} - a > 0. \]
Thus we see that from (12)
\[ g_{0,\beta}^\gamma(x, t) \leq C_2 \psi(x) \int_{-\infty}^{\infty} |y|^\beta \exp \left\{ - \left( \frac{1}{4(T(m) + \gamma)} - a \right) |y|^2 \right\} dy \]
for \( t \in [0, T(m)] \). Let
\[ C_4 = \max_{\beta=0,1,2} C_2 \int_{-\infty}^{\infty} |y|^\beta \exp \left\{ - \left( \frac{1}{4(T(m) + \gamma)} - a \right) |y|^2 \right\} dy. \]
Then we see
\[ g_{0,\beta}^\gamma(x, t) \leq C_4 \psi(x) \]
for \( t \in [0, T(m)] \).

**Proof of Proposition 2.2.** As before, for \( \phi = \phi(T(m) - t + C g_{\alpha,0}^\gamma(x, t)) \) we have
\begin{align*}
w_t - \frac{w_{xx}}{1 + w_x^2} + \frac{n - 1}{w} & = -\phi' + C \phi' \partial_{t} g_{\alpha,0}^\gamma - \frac{C \phi' \partial_{xx} g_{\alpha,0}^\gamma + (C \partial_{x} g_{\alpha,0}^\gamma)^2 \phi'' + n - 1}{\phi} \\
& \leq \frac{C(n - 1) \partial_{t} g_{\alpha,0}^\gamma}{\phi} + \frac{C(n - 1) |\partial_{xx} g_{\alpha,0}^\gamma|}{\phi} + \frac{\{C(n - 1) \partial_{x} g_{\alpha,0}^\gamma\}^2}{\phi^3} \tag{25}
\end{align*}
by using the fact that \( \phi' = (n - 1)/\phi \) and \( \phi'' = -(n - 1)/\phi^3 \). It is easily seen that
\[ \phi^2 = 2(n - 1)(T(m) - t + C g_{\alpha,0}^\gamma) \geq 2(n - 1)C g_{\alpha,0}^\gamma. \tag{26} \]
From Lemma 2.3, (16), (19) and (26), it follows that
\[
\left| \frac{\partial_x g^\gamma_{\alpha,0}}{\phi^2} \right| \leq \frac{g^\gamma_{0,1}}{4(n-1)(t+\gamma)g^\gamma_{0,0}} \leq \frac{C_4}{4\gamma(n-1)\rho C_3}. \tag{27}
\]
Substituting (27) for (25), and using (17)-(19), we have
\[
w_t - \frac{w_{xx}}{1 + w_x^2} + \frac{n-1}{w} \leq \frac{C(n-1)}{2(t+\gamma)^{\alpha+2}+\phi} \left[ g_{0,2}^\gamma + (t + \gamma) \left\{ -2\alpha g_{0,0}^\gamma + g_{0,0}^\gamma + \frac{C_4 g_{0,1}^\gamma}{4C_3} \right\} \right] \\
\leq \frac{C(n-1)}{2(t+\gamma)^{\alpha+2}+\phi} \left[ -2\alpha \gamma C_3 + C_4 \left\{ 1 + (T(m) + \gamma) \left( 1 + \frac{C_4}{4C_3} \right) \right\} \right]
\]
in \( \mathbb{R} \times [0, T(m)] \). If \( \alpha \) satisfies
\[
\alpha \geq \alpha_0 \equiv \frac{C_4}{2\gamma C_3} \left\{ 1 + (T(m) + \gamma) \left( 1 + \frac{C_4}{4C_3} \right) \right\}, \tag{28}
\]
then \( w \) is a subsolution of (1) in \( \mathbb{R} \times (0, T(m)) \).

**Proof of Theorem 1.3.** There exist positive constants \( c_1 = c_1(C_2, a, \gamma, \alpha) \) and \( c_2 = c_2(C_1, \gamma) \) such that
\[
g^\gamma_{\alpha,0}(x, 0) \leq c_1 \psi(x), \quad g^\gamma_{0,0}(x, 0) \geq c_2 \psi(x)
\]
by Lemma 2.3 and (19), and thus
\[
u_0^\gamma(x) \geq m^2 + C_1 g^\gamma_{\alpha,0}(x, 0) \quad (\text{or } \leq m^2 + C_h g^\gamma_{0,0}(x, 0))
\]
with \( C_1 = C_1/c_1 \) (or \( C_h = C_{II}/c_2 \)).

Since \( m^2 = 2(n-1)T(m) \) by (6), we have
\[
u_0(x) \geq \sqrt{2(n-1)T(m) + C_1 g^\gamma_{\alpha,0}(x, 0)} \geq w(x, 0) \\
(\text{or } \leq \sqrt{2(n-1)T(m) + C_h g^\gamma_{0,0}(x, 0)} \leq W(x, 0)).
\]
Propositions 2.1, 2.2 and the comparison principle yield
\[
u(x, t) \geq w(x, t) \quad (\text{or } \leq W(x, t)) \quad \text{in } \mathbb{R} \times [0, T(m)].
\]
We thereby get
\[
u(x, T(m)) \geq \sqrt{C_1 g^\gamma_{\alpha,0}(x, T(m))} \quad (\text{or } \leq \sqrt{C_h g^\gamma_{0,0}(x, T(m))}).
\]
By using Lemma 2.3 and letting $C = \sqrt{C_1 C_3}$ and $C' = \sqrt{C_1 C_4}$, we obtain
\[ u(x, T(m)) \geq C \psi^{1/2}(x) \quad \text{or} \quad C' \psi^{1/2}(x) \]

We may choose
\[ \gamma = \frac{1}{2\alpha} - 2T(m), \quad \alpha = \alpha_0 \]
with $\alpha_0$ in (28), and then the constant $C$ (or $C'$) depends only on $C_1, C_2, a, T(m), C_I$ (or $C_1, C_2, a, T(m), C_{II}$).

**References**


