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# On decay rate of quenching profile at space infinity for axisymmetric mean curvature flow<sup>\*†</sup>

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## Abstract

We study the motion of noncompact hypersurfaces moved by their mean curvature obtained by a rotation around  $x$ -axis of the graph a function  $y = u(x, t)$  (defined for all  $x \in \mathbf{R}$ ). We are interested to estimate its profile when the hypersurface closes open ends at the quenching (pinching) time  $T$ . We estimate its profile at the quenching time from above and below. We in particular prove that  $u(x, T) \sim |x|^{-a}$  as  $|x| \rightarrow \infty$  if  $u(x, 0)$  tends to its infimum with algebraic rate  $|x|^{-2a}$  (as  $|x| \rightarrow \infty$  with  $a > 0$ ).

## 1 Introduction and main theorem

This is a continuation of our study [4] on motion of noncompact axisymmetric  $n$ -dimensional hypersurface  $\Gamma_t$  moved by its mean curvature. Let  $\Gamma_t$  be given by a rotation of the graph of a function  $y = u(x, t)$  (defined on  $x \in \mathbf{R}$ ) around the  $x$ -axis (cf [1, 2]). In our previous paper [4], among other results, we have proved that if  $u(x, 0) \rightarrow m := \inf_{x \in \mathbf{R}} u(x, 0) > 0$  as  $|x| \rightarrow \infty$ , then  $\Gamma_t$  closes open ends at the time  $T(m)$ , where  $T(m)$  is the quenching (pinching) time of the regular cylinder with radius  $m$ . (Moreover, there is no neck-pinch in  $\mathbf{R}$  at  $t = T(m)$ .) These results imply that

$$\lim_{x \rightarrow \infty} u(x, T(m)) = 0 \quad \text{or} \quad \lim_{x \rightarrow -\infty} u(x, T(m)) = 0,$$

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but it does not provide the convergence rate.

We are interested in studying the profile of  $u(x, T(m))$ , especially the behavior as  $|x| \rightarrow \infty$  which is affected by initial data.

The equation for  $u$  is of the form

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{n-1}{u}, \quad x \in \mathbf{R}, \quad t > 0 \quad (1)$$

supplemented by initial data

$$u(x, 0) = u_0(x) > 0, \quad x \in \mathbf{R}. \quad (2)$$

The function  $u_0$  is assumed to satisfy

$$u_0 \text{ is bounded and uniformly continuous in } \mathbf{R}, \quad (3)$$

$$m := \inf_{x \in \mathbf{R}} u_0(x) > 0. \quad (4)$$

The Cauchy problem (1)-(2) has a unique positive classical solution with the conditions (3)-(4) to the initial data (cf [4]). However, the solution quenches in finite time. For a given initial datum  $u_0$ , we see

$$T(u_0) = \sup\{t > 0; \inf_{x \in \mathbf{R}} u(x, t) > 0\} < \infty$$

and call it the *quenching time* of  $u$ . It is clear that

$$\lim_{t \rightarrow T(u_0)} \inf_{x \in \mathbf{R}} u(x, t) = 0.$$

Let  $v$  be a solution of (1) with initial datum  $m = \inf_{x \in \mathbf{R}} u_0(x)$ . It is easily seen that

$$v' = -\frac{n-1}{v}, \quad t > 0, \quad v(0) = m, \quad (5)$$

and

$$v(t) = \sqrt{2(n-1)(T(m) - t)} \quad \text{with} \quad T(m) = \frac{m^2}{2(n-1)}. \quad (6)$$

It is immediate that  $T(u_0) \geq T(m)$  by a comparison argument. We treat the case  $T(u_0) = T(m)$ . The notion of “minimal quenching time” was defined in [4], which is recalled below.

**Definition 1.1.** *A solution of the Cauchy problem (1)-(2) is said to have a minimal quenching time, if*

$$T(u_0) = T(m).$$

In [4] we characterized solutions of (1)-(2) quenching only at space infinity. The following conditions on initial data  $u_0$  play essential roles in [4].

- A. There exists a sequence  $\{x_k\} \in \mathbf{R}$  such that  $x_k \rightarrow \infty$  and  $u_0(x+x_k) \rightarrow m$  a.e. in  $\mathbf{R}$  as  $k \rightarrow \infty$ .
- B. There exists a sequence  $\{x_k\} \in \mathbf{R}$  such that  $x_k \rightarrow -\infty$  and  $u_0(x+x_k) \rightarrow m$  a.e. in  $\mathbf{R}$  as  $k \rightarrow \infty$ .

For an initial datum satisfying (3)-(4), we proved in [4] the following results for the Cauchy problem (1)-(2):

- 1. A solution of (1)-(2) has a minimal quenching time, if and only if the conditions A or B holds.

Moreover, if  $u_0$  is not constant as well as the conditions A or B holds, then:

- 2. For an initial datum satisfying  $u_0 \not\equiv m$ , the solution (1)-(2) quenches only at space infinity.
- 3. There exists a function  $u(\cdot, T(m)) \in C^\infty(\mathbf{R})$  such that  $u(\cdot, t) \rightarrow u(\cdot, T(m))$  in the Frechét space  $C^\infty(\mathbf{R})$  as  $t \rightarrow T(m)$ ,  $u(x, T(m)) > 0$  in the whole  $\mathbf{R}$  and

$$\liminf_{x \rightarrow -\infty} u(x, T(m)) = 0 \quad \text{or} \quad \liminf_{x \rightarrow -\infty} u(x, T(m)) = 0.$$

For a solution  $u$  of (1)-(2) with minimal quenching time  $T(m)$ , we call  $u(\cdot, T(m))$  the profile of  $u$  (at the quenching time  $T(m)$ ). The hypersurface corresponding to  $u(\cdot, T(m))$  is called limit surface.

These are related studies on blow-up at infinity for the reaction-diffusion equations [8, 5, 6, 3, 10, 9, 11] (see also [7]). We shall explain these papers at the end of this introduction. In particular, blow-up profile was discussed, for example, in [8] and [11] for a semilinear heat equation.

In this paper we consider the relation between the profile of a quenching solution at quenching time  $T(m)$  and the form of initial data. Our goal, which is investigating the shape of limit surface, is similar to studying blow-up profile. Inspired by the method used in [8, §2b] and [11, Theorems 1.3 and 1.5], we construct a subsolution and a supersolution of the form  $\phi(T(m) - t + g(x, t))$  with some function  $g(x, t)$  decaying to zero at space infinity, where

$$\phi(s) = v(T(m) - s) = \sqrt{2(n-1)s}, \quad (7)$$

in order to estimate the profile at the quenching time. Let  $\psi(x)$  be a positive function satisfying the following conditions:

$$\sqrt{\psi(x)} \text{ is bounded and uniformly continuous in } \mathbf{R}; \quad (8)$$

$$\psi(x) > 0 \text{ for } x \in \mathbf{R}; \quad (9)$$

$$\lim_{x \rightarrow \infty} \psi(x) = 0 \text{ or } \lim_{x \rightarrow -\infty} \psi(x) = 0; \quad (10)$$

there exist constants  $C_1 > 0$  and  $C_2 > 0$  such that

$$\psi(x) \leq C_1 \max\left\{ \inf_{z \in [x-1, x]} \psi(z), \inf_{z \in [x, x+1]} \psi(z) \right\} \text{ for } x \in \mathbf{R}; \quad (11)$$

$$\psi(x-y) \leq C_2 \exp(a|y|^2) \psi(x) \text{ for } x, y \in \mathbf{R}, a \in \left(0, \frac{1}{4T(m)}\right). \quad (12)$$

**Example 1.2.** The functions  $\psi(x) = (|x|^2 + 1)^{-b/2}$ ,  $e^{-b|x|}$  and  $(\log(|x| + e))^{-b}$  with  $b > 0$  satisfy (8)-(12).

**Theorem 1.3.** Let  $\psi$  be a function satisfying (8)-(12). Assume that (3)-(4) hold and that there exist constants  $C_I > 0$  and  $C_{II} > 0$  such that

$$u_0^2(x) - m^2 \geq C_I \psi(x) \quad (\text{or } \leq C_{II} \psi(x)). \quad (13)$$

Then there exists  $C = C(C_1, C_2, a, T(m), C_I) > 0$  (or  $C' = C'(C_1, C_2, a, T(m), C_{II}) > 0$ ) such that the solution of the Cauchy problem (1)-(2) satisfies

$$u(x, T(m)) \geq C \sqrt{\psi(x)} \quad \left(\text{or } \leq C' \sqrt{\psi(x)}\right).$$

By setting  $\psi(x) = \langle x \rangle^{-2a_1}$  (or  $\langle x \rangle^{-2a_2}$ ) with  $\langle x \rangle = (1 + |x|^2)^{1/2}$ , we obtain algebraic decay at the space infinity.

**Corollary 1.4.** Assume that there exist constants  $a_1 > 0$ ,  $a_2 > 0$ ,  $C_I > 0$  and  $C_{II} > 0$  such that

$$u_0^2(x) - m^2 \geq C_I \langle x \rangle^{-2a_1} \quad (\text{or } \leq C_{II} \langle x \rangle^{-2a_2}). \quad (14)$$

Then there exists  $C = C(a_1, T(m), C_I) > 0$  (or  $C' = C'(a_2, T(m), C_{II}) > 0$ ) such that

$$u(x, T(m)) \geq C \langle x \rangle^{-a_1} \quad (\text{or } \leq C' \langle x \rangle^{-a_2}).$$

We conclude this introduction by giving a short review on blow-up (or quenching) at the space infinity. Lacey [8] considered problems in a half line of  $u_t = u_{xx} + f(u)$  in  $\mathbf{R}^+ = \{x : x > 0\}$  and constructed solutions blowing up only at space infinity. Gladkov [7] studied problems of the equation

$u_t = u_{xx} + f(x, t, u)$  in  $\mathbf{R}^+$  and showed that solutions of the problem uniformly converge as  $x \rightarrow \infty$  to the solution of the ODE obtained by dropping  $u_{xx}$  in the equation.

Giga-Umeda [5] proved that blow-up only at space infinity occurs under the condition  $\lim_{|x| \rightarrow \infty} u_0(x) = \sup_{x \in \mathbf{R}} u_0(x) =: M$  and  $u_0 \not\equiv M$  for nonnegative solutions of  $u_t = \Delta u + u^p$  in  $\mathbf{R}^n$  (cf. also [12] for a related study). For generalization, see [6] and a review article by Giga-Seki-Umeda [3]. More recently, Shimojō [11] discussed blow-up profile  $u(x, T) := \lim_{t \rightarrow T} u(x, t)$  for  $x \in \mathbf{R}^n$ . See also Seki-Suzuki-Umeda [10] and Seki [9] for quasilinear parabolic equations, which generalized the result of [6]. They also gave necessary and sufficient conditions for a solution to have “minimal blow-up time (or the least blow-up time)”. See [9, 10, 3] for the precise definition of the last notion.

## 2 Profile at quenching

In order to prove Theorem 1.3, we construct a subsolution and supersolution of the form  $\phi(T(m) - t + g(x, t))$ , as we have explained before. This is a modification of the method employed in [8] and [11] to study blow-up profile for a semilinear heat equation. The function

$$g(x, t) = \int_{-\infty}^{\infty} G(x - y, t) \psi(y) dy$$

with the Gauss kernel of heat equation

$$G(x, t) = (4\pi t)^{-1/2} \exp\left(-\frac{x^2}{4t}\right)$$

is used there. However, because the problem which we treat here is a quasilinear equation, the Gauss kernel is not appropriate in our problem. We use the following function instead of  $G(x, t)$ :

$$g_{\alpha, \beta}^{\gamma}(x, t) = g_{\alpha, \beta}^{\gamma, \psi}(x, t) = \int_{-\infty}^{\infty} G_{\alpha, \beta}^{\gamma}(x - y, t) \psi(y) dy, \quad (15)$$

where

$$G_{\alpha, \beta}^{\gamma}(x, t) = \frac{|x|^{\beta}}{(t + \gamma)^{\alpha}} \exp\left(-\frac{x^2}{4(t + \gamma)}\right)$$

with  $\alpha \geq 0$ ,  $\beta \geq 0$  and  $\gamma > 0$  being constants. Note that this  $g_{\alpha, \beta}^{\gamma}$  may be expressed by

$$g_{\alpha, \beta}^{\gamma}(x, t) = \int_{-\infty}^{\infty} G_{\alpha, \beta}^{\gamma}(y, t) \psi(x - y) dy.$$

It is easily seen that the derivatives are calculated and estimated as follows:

$$|\partial_x g_{\alpha,0}^\gamma| \leq \frac{g_{\alpha+1,1}^\gamma}{2}, \quad (16)$$

$$\partial_{xx} g_{\alpha,0}^\gamma = \frac{g_{\alpha+2,2}^\gamma}{4} - \frac{g_{\alpha+1,0}^\gamma}{2}, \quad (17)$$

$$\partial_t g_{\alpha,0}^\gamma = \frac{g_{\alpha+2,2}^\gamma}{4} - \alpha g_{\alpha+1,0}^\gamma, \quad (18)$$

and

$$g_{\alpha,\beta}^\gamma(x,t) = \frac{g_{0,\beta}^\gamma}{(t+\gamma)^\alpha}. \quad (19)$$

Before proving the Theorem 1.3 we prepare two propositions.

**Proposition 2.1.** *Let  $\psi$  be a positive bounded uniformly continuous function. For any  $C > 0$  and  $\gamma > 0$  the function*

$$W(x,t) = \phi(T(m) - t + Cg_{0,0}^\gamma(x,t)) \quad (20)$$

is a supersolution of (1) in  $\mathbf{R} \times (0, T(m))$ , where  $\phi$  is defined in (7).

*Proof.* By a direct calculation we have

$$\begin{aligned} W_t - \frac{W_{xx}}{1+W_x^2} + \frac{n-1}{W} \\ = -\phi' + C\phi'\partial_t g_{0,0}^\gamma - \frac{C\phi'\partial_{xx} g_{0,0}^\gamma + (C\partial_x g_{0,0}^\gamma)^2 \phi''}{1+(C\phi'\partial_x g_{0,0}^\gamma)^2} + \frac{n-1}{\phi}. \end{aligned}$$

Noting that  $\phi'\partial_t g_{0,0}^\gamma \geq 0$  from (18) and  $\phi' = (n-1)/\phi$ , we obtain

$$W_t - \frac{W_{xx}}{1+W_x^2} + \frac{n-1}{W} \geq \frac{C\phi'\partial_t g_{0,0}^\gamma - C\phi'\partial_{xx} g_{0,0}^\gamma - (C\partial_x g_{0,0}^\gamma)^2 \phi''}{1+(C\phi'\partial_x g_{0,0}^\gamma)^2}.$$

Since  $(\partial_t - \partial_{xx})g_{0,0}^\gamma = g_{1,0}^\gamma/2$  by (17)-(18), we have

$$W_t - \frac{W_{xx}}{1+W_x^2} + \frac{n-1}{W} \geq \frac{1}{1+(C\phi'\partial_x g_{0,0}^\gamma)^2} \left( \frac{C\phi'g_{1,0}^\gamma}{2} - (C\partial_x g_{0,0}^\gamma)^2 \phi'' \right).$$

Due to the fact that  $\phi'' \leq 0$ , we see that  $W$  is a supersolution of (1).  $\square$

**Proposition 2.2.** Assume that  $\psi$  is a function satisfying (8)-(12) and

$$\gamma \in \left(0, \frac{1}{a} - 4T(m)\right) \quad (21)$$

with the constant  $a$  in (12). Then, for each constant  $C > 0$ , the function

$$w(x, t) = \phi(T(m) - t + Cg_{\alpha_0}^\gamma(x, t)) \quad (22)$$

is a subsolution of (1) in  $\mathbf{R} \times (0, T(m))$  provided that  $\alpha$  satisfies  $\alpha \geq \alpha_0$  with some constant  $\alpha_0 = \alpha_0(C_1, C_2, a, T(m), \gamma) > 0$ , where  $\phi$  is the function defined in (7).

Before proving Proposition 2.2, we prepare a lemma on estimates for  $g_{0,\beta}^\gamma$ .

**Lemma 2.3.** Assume the same hypotheses as in Proposition 2.2. Then for  $\beta = 0, 1, 2$ , there exist constants  $C_3 = C_3(C_1, \gamma) > 0$  and  $C_4 = C_4(C_2, a, T(m), \gamma) > 0$  such that

$$C_3\psi(x) \leq g_{0,\beta}^\gamma(x, t) \leq C_4\psi(x) \quad \text{in } \mathbf{R} \times [0, T(m)],$$

where  $C_1$  and  $C_2$  are the constants in (11) and (12), respectively.

*Proof.* First we show  $g_{0,\beta}^\gamma \geq C_3\psi(x)$  with some  $C_3 > 0$ . From (11)

$$\psi(x) \leq C_1 \inf_{z \in [x-1, x]} \psi(z) \quad (23)$$

or

$$\psi(x) \leq C_1 \inf_{z \in [x, x+1]} \psi(z) \quad (24)$$

for each  $x \in \mathbf{R}$ . If (23) holds, then there exists a constant  $C_3 > 0$  such that

$$\begin{aligned} g_{0,\beta}^\gamma(x, t) &\geq \inf_{z \in (x-1, x)} \psi(z) \times \int_0^1 |y|^\beta \exp\left(-\frac{|y|^2}{4\gamma}\right) dy \\ &\geq \psi(x) \frac{1}{C_1} \int_0^1 |y|^\beta \exp\left(-\frac{|y|^2}{4\gamma}\right) dy. \end{aligned}$$

Set

$$C_3 = \min_{\beta=0,1,2} \frac{1}{C_1} \int_0^1 |y|^\beta \exp\left(-\frac{|y|^2}{4\gamma}\right) dy = \frac{1}{C_1} \int_0^1 |y|^2 \exp\left(-\frac{|y|^2}{4\gamma}\right) dy.$$

We then see that

$$g_{0,\beta}^\gamma(x, t) \geq C_3\psi(x).$$



A similar argument shows that if (24) holds, then

$$g_{0,\beta}^\gamma(x, t) \geq C_3\psi(x).$$

Thus we see that

$$g_{0,\beta}^\gamma(x, t) \geq C_3\psi(x)$$

for any  $x \in \mathbf{R}$ .

We next prove  $g_{0,\beta}^\gamma(x, t) \leq C_4\psi(x)$  with some  $C_4 > 0$ . For (21) it is possible to take a constant  $\gamma > 0$  depending only  $a$  and  $m$  that satisfies

$$\frac{1}{4(T(m) + \gamma)} - a > 0.$$

Thus we see that from (12)

$$g_{0,\beta}^\gamma(x, t) \leq C_2\psi(x) \int_{-\infty}^{\infty} |y|^\beta \exp \left\{ - \left( \frac{1}{4(T(m) + \gamma)} - a \right) |y|^2 \right\} dy$$

for  $t \in [0, T(m)]$ . Let

$$C_4 = \max_{\beta=0,1,2} C_2 \int_{-\infty}^{\infty} |y|^\beta \exp \left\{ - \left( \frac{1}{4(T(m) + \gamma)} - a \right) |y|^2 \right\} dy.$$

Then we see

$$g_{0,\beta}^\gamma(x, t) \leq C_4\psi(x)$$

for  $t \in [0, T(m)]$ . □

*Proof of Proposition 2.2.* As before, for  $\phi = \phi(T(m) - t + Cg_{\alpha,0}^\gamma(x, t))$  we have

$$\begin{aligned} w_t - \frac{w_{xx}}{1 + w_x^2} + \frac{n-1}{w} &= -\phi' + C\phi' \partial_t g_{\alpha,0}^\gamma - \frac{C\phi' \partial_{xx} g_{\alpha,0}^\gamma + (C\partial_x g_{\alpha,0}^\gamma)^2 \phi''}{1 + (C\phi' \partial_x g_{\alpha,0}^\gamma)^2} + \frac{n-1}{\phi} \\ &\leq \frac{C(n-1) \partial_t g_{\alpha,0}^\gamma}{\phi} + \frac{C(n-1) |\partial_{xx} g_{\alpha,0}^\gamma|}{\phi} + \frac{\{C(n-1) \partial_x g_{\alpha,0}^\gamma\}^2}{\phi^3} \end{aligned} \quad (25)$$

by using the fact that  $\phi' = (n-1)/\phi$  and  $\phi'' = -(n-1)^2/\phi^3$ . It is easily seen that

$$\phi^2 = 2(n-1)(T(m) - t + Cg_{\alpha,0}^\gamma) \geq 2(n-1)(Cg_{\alpha,0}^\gamma). \quad (26)$$

From Lemma 2.3, (16), (19) and (26), it follows that

$$\left| \frac{\partial_x g_{\alpha,0}^\gamma}{\phi^2} \right| \leq \frac{g_{0,1}^\gamma}{4(n-1)(t+\gamma)g_{0,0}^\gamma} \leq \frac{C_4}{4\gamma(n-1)CC_3}. \quad (27)$$

Substituting (27) for (25), and using (17)-(19), we have

$$\begin{aligned} w_t - \frac{w_{xx}}{1+w_x^2} + \frac{n-1}{w} &\leq \frac{C(n-1)}{2(t+\gamma)^{\alpha+2}\phi} \left[ g_{0,2}^\gamma + (t+\gamma) \left\{ -2\alpha g_{0,0}^\gamma + g_{0,0}^\gamma + \frac{C_4 g_{0,1}^\gamma}{4C_3} \right\} \right] \\ &\leq \frac{C(n-1)\psi}{2(t+\gamma)^{\alpha+2}\phi} \left[ -2\alpha\gamma C_3 + C_4 \left\{ 1 + (T(m) + \gamma) \left( 1 + \frac{C_4}{4C_3} \right) \right\} \right] \end{aligned}$$

in  $\mathbf{R} \times [0, T(m)]$ . If  $\alpha$  satisfies

$$\alpha \geq \alpha_0 \equiv \frac{C_4}{2\gamma C_3} \left\{ 1 + (T(m) + \gamma) \left( 1 + \frac{C_4}{4C_3} \right) \right\}, \quad (28)$$

then  $w$  is a subsolution of (1) in  $\mathbf{R} \times (0, T(m))$ .  $\square$

*Proof of Theorem 1.3.* There exist positive constants  $c_1 = c_1(C_2, a, \gamma, \alpha)$  and  $c_2 = c_2(C_1, \gamma)$  such that

$$g_{\alpha,0}^\gamma(x, 0) \leq c_1\psi(x), \quad g_{0,0}^\gamma(x, 0) \geq c_2\psi(x)$$

by Lemma 2.3 and (19), and thus

$$u_0^2(x) \geq m^2 + C_l g_{\alpha,0}^\gamma(x, 0) \quad (\text{or } \leq m^2 + C_h g_{0,0}^\gamma(x, 0))$$

with  $C_l = C_I/c_1$  (or  $C_h = C_{II}/c_2$ ).

Since  $m^2 = 2(n-1)T(m)$  by (6), we have

$$\begin{aligned} u_0(x) &\geq \sqrt{2(n-1)T(m) + C_l g_{\alpha,0}^\gamma(x, 0)} \geq w(x, 0) \\ &\quad (\text{or } \leq \sqrt{2(n-1)T(m) + C_h g_{0,0}^\gamma(x, 0)} \leq W(x, 0)). \end{aligned}$$

Propositions 2.1, 2.2 and the comparison principle yield

$$u(x, t) \geq w(x, t) \quad (\text{or } \leq W(x, t)) \quad \text{in } \mathbf{R} \times [0, T(m)].$$

We thereby get

$$u(x, T(m)) \geq \sqrt{C_l g_{\alpha,0}^\gamma(x, T(m))} \quad (\text{or } \leq \sqrt{C_h g_{0,0}^\gamma(x, T(m))}).$$

By using Lemma 2.3 and letting  $C = \sqrt{C_l C_3}$  and  $C' = \sqrt{C_h C_4}$ , we obtain

$$u(x, T(m)) \geq C\psi^{1/2}(x) \quad (\text{or } \leq C'\psi^{1/2}(x)).$$

We may choose

$$\gamma = \frac{1}{2a} - 2T(m), \quad \alpha = \alpha_0$$

with  $\alpha_0$  in (28), and then the constant  $C$  (or  $C'$ ) depends only on  $C_1, C_2, a, T(m), C_I$  (or  $C_1, C_2, a, T(m), C_{II}$ ).  $\square$

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