Extensions of the mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space

Shyuichi Izumiya and Handan Yıldırım

September 23, 2009

Abstract

In this paper we show one-parameter families of Legendrian dualities between pseudo-spheres in Lorentz-Minkowski space which are the extensions of four dualities in the previous research. Moreover, we construct new extrinsic differential geometries on spacelike hypersurfaces in these pseudo-spheres as applications of such extensions of the mandala.

1 Introduction

A theorem of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space has been shown in [9]. It is now a fundamental tool for the study of extrinsic differential geometry on submanifolds in these pseudo-spheres from the view point of Singularity theory (cf., [9, 11, 12, 15]). These dualities have been generalized into pseudo-spheres in general semi-Euclidean space [7]. In this paper, we extend such Legendrian dualities for continuous families of pseudo-spheres in Lorentz-Minkowski space. We do not consider semi-Euclidean space with general index here. However, we remark that by exactly the same way as in this paper we can easily generalize the results into the pseudo-spheres in semi-Euclidean space with general index, so that we omit them. The main results (cf., Theorems 3.1 and 3.2) are simple generalizations of the results in [9]. However, there are some new applications of such extended dualities. In §4, we only give some basic results on such applications. The detailed arguments on these applications will be appeared in the forthcoming papers [3, 16].

2 Basic notions

In this section we give basic notions and properties on Lorentz-Minkowski space. Let \( \mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) \mid x_i \in \mathbb{R}, i = 0, \ldots, n\} \) be an \((n+1)\)-dimensional vector space. For any vectors \( \mathbf{x} = (x_0, x_1, \ldots, x_n) \) and \( \mathbf{y} = (y_0, y_1, \ldots, y_n) \) in \( \mathbb{R}^{n+1} \), the pseudo scalar product of \( \mathbf{x} \) and \( \mathbf{y} \) is defined by \( \langle \mathbf{x}, \mathbf{y} \rangle = -x_0y_0 + \sum_{i=1}^{n} x_iy_i \). The space \( (\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle) \) is called Lorentz-Minkowski \((n+1)\)-space and denoted by \( \mathbb{R}^{n+1}_1 \). We say that a vector \( \mathbf{x} \) in \( \mathbb{R}^{n+1}_1 \setminus \{0\} \) is spacelike, null or...
timelike if $\langle x, x \rangle > 0$, = 0 or $< 0$, respectively. The norm of a vector $x \in \mathbb{R}_t^{n+1}$ is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. For a vector $v \in \mathbb{R}_t^{n+1} \setminus \{0\}$ and a real number $c$, we define a hyperplane with pseudo normal $v$ by

$$HP(v, c) = \{ x \in \mathbb{R}_t^{n+1} \mid \langle x, v \rangle = c \}.$$ 

We call $HP(v, c)$ a spacelike hyperplane, a timelike hyperplane or a lightlike hyperplane if $v$ is timelike, spacelike or lightlike, respectively. We have the following three kinds of pseudo-spheres in $\mathbb{R}_t^{n+1}$.

**Hyperbolic $n$-space** is defined by

$$H^n(-c^2) = \{ x \in \mathbb{R}_t^{n+1} \mid \langle x, x \rangle = -c^2 \},$$

**de Sitter $n$-space** by

$$S^n(1) = \{ x \in \mathbb{R}_t^{n+1} \mid \langle x, x \rangle = c^2 \}$$

and the (open) lightcone by

$$LC^+ = \{ x \in \mathbb{R}_t^{n+1} \setminus \{0\} \mid \langle x, x \rangle = 0 \},$$

for any real number $c$. Instead of $S^n(1)$, we usually write $S^n$.

## 3 Legendrian dualities

In this section we formulate theorems on Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space and give their proofs. For our purpose, we briefly review some properties of contact manifolds and Legendrian submanifolds. Let $N$ be a $(2n + 1)$-dimensional smooth manifold and $K$ be a tangent hyperplane field on $N$. Locally, such a field is defined as the field of zeros of a 1-form $\alpha$. The tangent hyperplane field $K$ is non-degenerate if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of $N$. We say that $(N, K)$ is a contact manifold if $K$ is a non-degenerate hyperplane field. In this case, $K$ is called a contact structure and $\alpha$ is a contact form. Let $\phi : N \to N'$ be a diffeomorphism between contact manifolds $(N, K)$ and $(N', K')$. We say that $\phi$ is a contact diffeomorphism if $d\phi(K) = K'$. Two contact manifolds $(N, K)$ and $(N', K')$ are contact diffeomorphic if there exists a contact diffeomorphism $\phi : N \to N'$. A submanifold $i : L \subset N$ of a contact manifold $(N, K)$ is said to be Legendrian if $dim L = n$ and $di_x(T_x L) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \to M$ is called a Legendrian fibration if its total space $E$ is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \to M$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \to M$ is called a Legendrian map. The image of the Legendrian map $\pi \circ i$ is called a wavefront set of $i$ which is denoted by $W(L)$. For any $z \in E$, it is known that there is a local coordinate system $(x, y, p) = (x_1, \ldots, x_m, y, p_1, \ldots, p_m)$ around $z$ such that $\pi(x, y, p) = (x, y)$ and the contact structure is given by the 1-form $\alpha = dy - \sum_{i=1}^{n} p_i dx_i$ (cf. [1], 20.3). In [9], the basic duality theorem for four Legendrian double fibrations which is the fundamental tool for the study of spacelike hypersurfaces in Lorentz-Minkowski pseudo-spheres has been shown.

Now, we consider a slight extension of these dualities by the following seven double fibrations:

1. (a) $H^n(-1) \times S^n \supset \Delta_1 = \{ (v, w) \mid \langle v, w \rangle = 0 \}$,
   (b) $\pi_{11} : \Delta_1 \to H^n(-1)$, $\pi_{12} : \Delta_1 \to S^n$,
   (c) $\theta_{11} = \langle dv, w \rangle |_{\Delta_1}$, $\theta_{12} = \langle v, dw \rangle |_{\Delta_1}$.
2. (a) $H^n(-1) \times LC^+ \supset \Delta_2^+ = \{ (v, w) \mid \langle v, w \rangle = \pm 1 \}$,
(b) \( \pi_{21}^\pm : \Delta_i^\pm \to H_n(-1) \), \( \pi_{22}^\pm : \Delta_i^\pm \to LC^* \),
(c) \( \theta_{21}^\pm = \langle dw, w \rangle|\Delta_i^\pm, \theta_{22}^\pm = \langle v, dw \rangle|\Delta_i^\pm \).
(3) (a) LC* \times S^n \supset \Delta_i^\pm = \{(v, w) \mid \langle v, w \rangle = \pm 1 \},
(b) \( \pi_{31}^\pm : \Delta_i^\pm \to LC^*, \pi_{32}^\pm : \Delta_i^\pm \to S^n \),
(c) \( \theta_{31}^\pm = \langle dw, w \rangle|\Delta_i^\pm, \theta_{32}^\pm = \langle v, dw \rangle|\Delta_i^\pm \).
(4) (a) LC* \times LC* \supset \Delta_i^\pm = \{(v, w) \mid \langle v, w \rangle = \pm 2 \},
(b) \( \pi_{41}^\pm : \Delta_i^\pm \to LC^*, \pi_{42}^\pm : \Delta_i^\pm \to LC^* \),
(c) \( \theta_{41}^\pm = \langle dv, w \rangle|\Delta_i^\pm, \theta_{42}^\pm = \langle v, dw \rangle|\Delta_i^\pm \).

Here, \( \pi_{ij}(v, w) = v, \pi_{12}(v, w) = w, \pi_{i1}^\pm(v, w) = v \) and \( \pi_{i2}^\pm(v, w) = w \) \( (i=2,3,4) \). Moreover, \( \langle dv, w \rangle = -w_0dv_0 + \sum_{i=1}^n w_idv_i \) and \( \langle v, dw \rangle = -v_0dw_0 + \sum_{i=1}^n v_idw_i \) are one-forms on \( \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \). We remark that \( \theta_{11}^{-1}(0) \) and \( \theta_{12}^{-1}(0) \) define the same tangent hyperplane field denoted by \( K_1 \) over \( \Delta_1 \). And also \( \theta_{11}^{-1}(0) \) and \( \theta_{12}^{-1}(0) \) define the same tangent hyperplane field denoted by \( K_i^\pm \) over \( \Delta_i^\pm \) \( (i=2,3,4) \). We have the following basic duality theorem:

**Theorem 3.1** Under the same notations as the previous paragraph, \( (\Delta_1, K_1) \) and \( (\Delta_i^\pm, K_i^\pm) \) \( (i = 2, 3, 4) \) are contact manifolds such that \( \pi_{ij} \) and \( \pi_{ij}^\pm \) \( (j = 1, 2) \) are Legendrian fibrations. Moreover, these contact manifolds are contact diffeomorphic to each other.

**Proof.** By definition, we can easily show that \( \Delta_1 \) and \( \Delta_i^\pm \) \( (i = 2, 3, 4) \) are smooth submanifolds in \( \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \) and all of \( \pi_{ij} \) and \( \pi_{ij}^\pm \) \( (i = 2, 3, 4; j = 1, 2) \) are smooth fibrations.

In [9] it has been shown that \( (\Delta_1, K_1) \) is a contact manifold. We now give a brief review of the proof. Since \( H_n(-1) \) is a spacelike hypersurface in \( \mathbb{R}^{n+1}_1 \), \( \langle , \rangle|H_n(-1) \) is a Riemannian metric. Let \( \pi : S(TH^\ast(-1)) \to H_n(-1) \) be the unit tangent sphere bundle of \( H_n(-1) \). For any \( v \in H_n(-1) \), we have the local coordinates \( (v_1, \ldots, v_n) \) such that \( v = (\pm \sqrt{v_1^2 + \cdots + v_n^2} + 1, v_1, \ldots, v_n) \). We can represent the tangent vector \( w \in T_vH_n(-1) \) by

\[
\begin{align*}
\begin{array}{c}
w = (\pm 1) \sum_{i=1}^n w_i v_i, w_1, \ldots, w_n.
\end{array}
\end{align*}
\]

It follows that \( \langle w, v \rangle = (\pm \frac{1}{v_0} \sum_{i=1}^n w_i v_i)(\mp v_0) + \sum_{i=1}^n w_i v_i = 0 \). Therefore, \( w \in S(T_vH_n(-1)) \) if and only if \( \langle w, w \rangle = 1 \) and \( \langle v, w \rangle = 0 \). The last conditions are equivalent to the condition that \( (v, w) \in \Delta_1 \). This means that we can canonically identify \( S(TH^\ast(-1)) \) with \( \Delta_1 \). Moreover, the canonical contact structure on \( S(TH^\ast(-1)) \) is given by the one-form \( \theta(V) = \langle d\pi(V), \tau(V) \rangle \), where \( \tau : TS(TH^\ast(-1)) \to S(TH^\ast(-1)) \) is the tangent bundle of \( S(TH^\ast(-1)) \) (cf., [4, 6]). It can be represented by \( \langle dv, w \rangle|\Delta_1 \) through the above identification. Thus, \( (\Delta_1, \theta_{11}^{-1}(0)) \) is a contact manifold. For the other \( \Delta_i^\pm \) \( (i = 2, 3, 4) \), we define the smooth mappings \( \Psi_{12}^\pm : \Delta_1 \to \Delta_i^\pm \) by \( \Psi_{12}^\pm(v, w) = (v, \mp v + w) \), \( \Psi_{13}^\pm(v, w) = (v \pm w, w) \) and \( \Psi_{14}^\pm(v, w) = (v \pm w, \mp v + w) \). We can construct their reverse mappings, so that \( \Psi_{ij}^\pm \) are diffeomorphisms. Moreover, we have

\[
\begin{align*}
\Psi_{12}^\pm \theta_{21}^\pm &= \langle dv, \mp v + w \rangle|\Delta_1 \\
&= \langle dv, \mp v \rangle|\Delta_1 + \langle dv, w \rangle|\Delta_1 \\
&= \langle dv, w \rangle|\Delta_1 = \theta_{11}.
\end{align*}
\]

This means that \( (\Delta_2^\pm, K_2^\pm) \) is a contact manifold such that \( \Psi_{12}^\pm \) is a contact diffeomorphism. For \( \Delta_i^\pm \) \( (i = 3, 4) \), we have the similar calculations, so that \( (\Delta_i^\pm, K_i^\pm) \) \( (i = 3, 4) \) are contact manifolds such that \( \Psi_{1i}^\pm \) are contact diffeomorphisms. This completes the proof. \( \square \)
We can also give the contact diffeomorphisms $\Psi_{ij}^\pm : \Delta_{ij}^\pm \rightarrow \Delta_{ij}^\pm$ for the other pairs $(i, j)$ by $\Psi_{ij}^\pm = \Psi_{i1}^\pm \circ \Psi_{1j}^\pm$, where $\Psi_{i1}^\pm = (\Psi_{1i}^\pm)^{-1}$. It follows that we have a “mandala of Legendrian dualities” by the following commutative diagram:

The above mandala is a slight extension of the mandala given by the Legendrian dualities in [9]. However, we can extend it to infinite families of Legendrian dualities as follows:

(5) (a) $H^n(-1) \times S^1((\cos^2 \phi) \supset \Delta_{12}^\pm(\phi) = \{(v, w) \mid \langle v, w \rangle = \pm \sin \phi \}$,
(b) $\pi[\phi]_{1231} : \Delta_{12}^\pm(\phi) \rightarrow H^n(-1), \pi[\phi]_{1232} : \Delta_{12}^\pm(\phi) \rightarrow S^1((\cos^2 \phi)$,
(c) $\theta[\phi]_{1231} = (dv, w)|\Delta_{12}^\pm(\phi), \theta[\phi]_{1232} = (v, dw)|\Delta_{12}^\pm(\phi)$.

(6) (a) $H^n(-\cos^2 \phi) \times S^1(\cos^2 \phi) \supset \Delta_{13}^\pm(\phi) = \{(v, w) \mid \langle v, w \rangle = \pm \sin \phi \}$,
(b) $\pi[\phi]_{1341} : \Delta_{13}^\pm(\phi) \rightarrow H^n(-\cos^2 \phi), \pi[\phi]_{1342} : \Delta_{13}^\pm(\phi) \rightarrow S^n$,
(c) $\theta[\phi]_{1341} = (dv, w)|\Delta_{13}^\pm(\phi), \theta[\phi]_{1342} = (v, dw)|\Delta_{13}^\pm(\phi)$.

(7) (a) $H^n(-\cos^2 \phi) \times S^1(\sin^2 \phi) \supset \Delta_{23}^\pm(\phi) = \{(v, w) \mid \langle v, w \rangle = \pm (\sin \phi + \cos \phi) \}$,
(b) $\pi[\phi]_{2341} : \Delta_{23}^\pm(\phi) \rightarrow H^n(-\cos^2 \phi), \pi[\phi]_{2342} : \Delta_{23}^\pm(\phi) \rightarrow S^n(\sin^2 \phi)$,
(c) $\theta[\phi]_{2341} = (dv, w)|\Delta_{23}^\pm(\phi), \theta[\phi]_{2342} = (v, dw)|\Delta_{23}^\pm(\phi)$.

(8) (a) $H^n(-\cos^2 \phi) \times LC^* \supset \Delta_{34}^\pm(\phi) = \{(v, w) \mid \langle v, w \rangle = \pm (\sin \phi + 1) \}$,
(b) $\pi[\phi]_{3451} : \Delta_{34}^\pm(\phi) \rightarrow H^n(-\cos^2 \phi), \pi[\phi]_{3452} : \Delta_{34}^\pm(\phi) \rightarrow LC^*$,
(c) $\theta[\phi]_{3451} = (dv, w)|\Delta_{34}^\pm(\phi), \theta[\phi]_{3452} = (v, dw)|\Delta_{34}^\pm(\phi)$.

(9) (a) $LC^* \times S^1(\cos^2 \phi) \supset \Delta_{34}^\pm(\phi) = \{(v, w) \mid \langle v, w \rangle = \pm (\sin \phi + 1) \}$,
(b) $\pi[\phi]_{3451} : \Delta_{34}^\pm(\phi) \rightarrow LC^*, \pi[\phi]_{3452} : \Delta_{34}^\pm(\phi) \rightarrow S^n(\cos^2 \phi)$,
(c) $\theta[\phi]_{3451} = (dv, w)|\Delta_{34}^\pm(\phi), \theta[\phi]_{3452} = (v, dw)|\Delta_{34}^\pm(\phi)$.

We also define the tangent hyperplane field $K[\phi]_{ij}^\pm$ over $\Delta_{ij}^\pm(\phi)$ by $K[\phi]_{ij}^\pm = \theta[\phi]_{ij}^\pm \mathbb{1} - (0) = \theta[\phi]_{ij}^\pm - (0)$. The main result in this paper is the following theorem:

**Theorem 3.2** Under the same notations as those of the previous paragraph, $(\Delta_1, K_1)$ and $(\Delta_{ij}^\pm(\phi), K[\phi]^\pm_{ij})$ $(i, j = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4))$ are contact manifolds such that $\pi_{1k}$ and $\pi[\phi]_{ij}^\pm(k = 1, 2)$ are Legendrian fibrations. Moreover, these contact manifolds are contact diffeomorphic to each other.

**Proof.** We can construct the diffeomorphisms $\Psi^\pm_{ij} : \Delta_{ij}^\pm(\phi) \rightarrow \Delta_1$ with $d\Psi^\pm_{ij}(K[\phi]^\pm_{ij}) = K_1$ as follows:
(5) We define a mapping 
\[ \Psi^\pm_{(12)} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} ; \quad \Psi^\pm_{(12)}(v, w) = (v, \pm \sin \phi v + w). \]

For any \((v, w) \in \Delta^\pm_{12}(\phi)\), we have 
\[ \langle \pm \sin \phi v + w, \pm \sin \phi v + w \rangle = -\sin^2 \phi + 2 \sin^2 \phi + \cos^2 \phi = 1 \]
and \((v, \pm \sin \phi v + w) = 0\). Therefore, we have \(\Psi^\pm_{(12)}(\Delta^\pm_{12}(\phi)) \subset \Delta_1\). We also define a mapping 
\[ \Psi^\pm_{1(12)} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} ; \quad \Psi^\pm_{1(12)}(v, w) = (v, \mp \sin \phi v + w). \]

We can easily calculate that \(\Psi^\pm_{1(12)}(\Delta_1) \subset \Delta^\pm_{12}(\phi)\), \(\Psi^\pm_{1(12)} \circ \Psi^\pm_{(12)}(\Delta^\pm_{12}(\phi)) = 1\), and \(\Psi^\pm_{(12)} \circ \Psi^\pm_{1(12)}(\Delta_1) = 1\). Moreover, we have 
\[ (\Psi^\pm_{(12)})^* \theta_{11} = (dv, \pm \sin \phi v + w)|_{\Delta^\pm_{12}(\phi)} = (dv, w)|_{\Delta^\pm_{12}(\phi)} = \theta|_{\Delta^\pm_{12}(\phi)}. \]

Therefore, \(K[\phi]^\pm_{12}\) is a contact structure on \(\Delta^\pm_{12}(\phi)\) such that \(\Psi^\pm_{1(12)}\) is a contact diffeomorphism.

For other cases, we can define the following mappings:

(6) \(\Psi^\pm_{(13)}(v, w) = (v \mp \sin \phi w, w)\).

(7) \(\Psi^\pm_{(14)}(v, w) = \frac{1}{\sin^2 \phi + 1}(v \mp \sin \phi w, \pm \sin \phi v + w)\).

(8) \(\Psi^\pm_{(23)}(v, w) = \frac{1}{\sin \phi \cos \phi + 1}(v \mp \sin \phi w, \pm \cos \phi v + w)\).

(9) \(\Psi^\pm_{(24)}(v, w) = \frac{1}{\sin \phi + 1}(v \mp \sin \phi w, \pm v + w)\).

(10) \(\Psi^\pm_{(34)}(v, w) = \frac{1}{\sin \phi + 1}(v \mp w, \pm \sin \phi v + w)\).

By straightforward calculations, we can show that \(\Psi^\pm_{(ij)}(\Delta^\pm_{ij}(\phi)) \rightarrow \Delta_1\), \(((i, j) = (1, 3), (1, 4), (2, 3), (2, 4), (3, 4))\) are diffeomorphisms such that \(d\Psi^\pm_{(ij)}(K[\phi]^\pm_{ij}) = K_1\). Therefore, \((\Delta_{ij}(\phi), K[\phi]^\pm_{ij})\) are contact manifolds which are contact diffeomorphic to \((\Delta_1, K_1)\).

We can write the above extension of the mandala as follows:

The extended Mandala of Legendrian Dualities
The above diagram is not a diagram for contact diffeomorphisms. If we add informations on the contact diffeomorphisms between $\Delta^\pm_{ij}$, the diagram might be very complicated, so that we omit the contact diffeomorphisms in the above diagram.

**Remark 3.3**

We can also define

$$
\Delta^\pm_{ij}(\phi) = \Delta^\pm_{ij}(\frac{\pi}{2} - \phi), \quad K[\phi]^\pm_{ij} = K[\frac{\pi}{2} - \phi]^\pm_{ij}, \quad \pi[\phi]^\pm_{(ij)k} = \pi[\frac{\pi}{2} - \phi]^\pm_{(ij)k}
$$

for $(i, j) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4)$ and $(3, 4)$. Then these are contact manifolds with $\Delta^\pm_{ii}(0) = \Delta^\pm_{ii}(\pi/2) = \Delta_1$ and $\Delta^\pm_{ij}(\pi/2) = \Delta^\pm_{ij}(i \neq 1)$. Moreover, all of them are canonically contact diffeomorphic to $(\Delta_1, K_1)$. Since these contact diffeomorphisms can be constructed by the canonical way, we omit to give the definitions here.

We can explicitly write these families of Legendrian dualities as follows:

$$
(5^*) \quad \begin{align*}
&\text{(a) } H^n(-1) \times S^n_1(\sin^2 \phi) \supset \Delta^\pm_{ij}(\phi) = \{(v, w) \mid \langle v, w \rangle = \pm \cos \phi \}, \\
&\text{(b) } \pi[\phi]_{(21)1}^\pm : \Delta^\pm_{21}(\phi) \rightarrow H^n(-1), \quad \pi[\phi]_{(21)2}^\pm : \Delta^\pm_{21}(\phi) \rightarrow S^n_1(\sin^2 \phi), \\
&\text{(c) } \theta[\phi]_{(21)1} = \langle dv, dw \rangle |_{\Delta^\pm_{21}(\phi)}, \quad \theta[\phi]_{(21)2} = \langle dv, dw \rangle |_{\Delta^\pm_{21}(\phi)}. \\
&\text{(6^*) } \begin{align*}
&\text{(a) } H^n(-\sin^2 \phi) \times S^n_1(\sin^2 \phi) \supset \Delta^\pm_{31}(\phi) = \{(v, w) \mid \langle v, w \rangle = \pm \cos \phi \}, \\
&\text{(b) } \pi[\phi]_{(31)1}^\pm : \Delta^\pm_{31}(\phi) \rightarrow H^n(-\sin^2 \phi), \quad \pi[\phi]_{(31)2}^\pm : \Delta^\pm_{31}(\phi) \rightarrow S^n_1(\sin^2 \phi), \\
&\text{(c) } \theta[\phi]_{(31)1} = \langle dv, dw \rangle |_{\Delta^\pm_{31}(\phi)}, \quad \theta[\phi]_{(31)2} = \langle dv, dw \rangle |_{\Delta^\pm_{31}(\phi)}. \\
&\text{(7^*) } \begin{align*}
&\text{(a) } H^n(-\sin^2 \phi) \times S^n_1(\sin^2 \phi) \supset \Delta^\pm_{ij}(\phi) = \{(v, w) \mid \langle v, w \rangle = \pm 2 \cos \phi \}, \\
&\text{(b) } \pi[\phi]_{(41)1}^\pm : \Delta^\pm_{41}(\phi) \rightarrow H^n(-\sin^2 \phi), \quad \pi[\phi]_{(41)2}^\pm : \Delta^\pm_{41}(\phi) \rightarrow S^n_1(\sin^2 \phi), \\
&\text{(c) } \theta[\phi]_{(41)1} = \langle dv, dw \rangle |_{\Delta^\pm_{41}(\phi)}, \quad \theta[\phi]_{(41)2} = \langle dv, dw \rangle |_{\Delta^\pm_{41}(\phi)}. \\
&\text{(8^*) } \begin{align*}
&\text{(a) } H^n(-\sin^2 \phi) \times S^n_1(\sin^2 \phi) \supset \Delta^\pm_{32}(\phi) = \{(v, w) \mid \langle v, w \rangle = \pm (\cos \phi + \sin \phi) \}, \\
&\text{(b) } \pi[\phi]_{(32)1}^\pm : \Delta^\pm_{32}(\phi) \rightarrow H^n(-\sin^2 \phi), \quad \pi[\phi]_{(32)2}^\pm : \Delta^\pm_{32}(\phi) \rightarrow S^n_1(\sin^2 \phi), \\
&\text{(c) } \theta[\phi]_{(32)1} = \langle dv, dw \rangle |_{\Delta^\pm_{32}(\phi)}, \quad \theta[\phi]_{(32)2} = \langle dv, dw \rangle |_{\Delta^\pm_{32}(\phi)}. \\
&\text{(9^*) } \begin{align*}
&\text{(a) } H^n(-\sin^2 \phi) \times LC^* \supset \Delta^\pm_{42}(\phi) = \{(v, w) \mid \langle v, w \rangle = \pm (\cos \phi + 1) \}, \\
&\text{(b) } \pi[\phi]_{(42)1}^\pm : \Delta^\pm_{42}(\phi) \rightarrow H^n(-\sin^2 \phi), \quad \pi[\phi]_{(42)2}^\pm : \Delta^\pm_{42}(\phi) \rightarrow LC^*, \\
&\text{(c) } \theta[\phi]_{(42)1} = \langle dv, dw \rangle |_{\Delta^\pm_{42}(\phi)}, \quad \theta[\phi]_{(42)2} = \langle dv, dw \rangle |_{\Delta^\pm_{42}(\phi)}. \\
&\text{(10^*) } \begin{align*}
&\text{(a) } LC^* \times S^n_1(\sin^2 \phi) \supset \Delta^\pm_{33}(\phi) = \{(v, w) \mid \langle v, w \rangle = \pm (\cos \phi + 1) \}, \\
&\text{(b) } \pi[\phi]_{(33)1}^\pm : \Delta^\pm_{33}(\phi) \rightarrow LC^*, \quad \pi[\phi]_{(33)2}^\pm : \Delta^\pm_{33}(\phi) \rightarrow S^n_1(\sin^2 \phi), \\
&\text{(c) } \theta[\phi]_{(33)1} = \langle dv, dw \rangle |_{\Delta^\pm_{33}(\phi)}, \quad \theta[\phi]_{(33)2} = \langle dv, dw \rangle |_{\Delta^\pm_{33}(\phi)}. 
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\end{
hypersurface $M^H = X^h(U)$ in $H^n(-1)$. We define $X^\ell_\pm(u) = X^h(u) \pm X^d(u)$. Then these are lightlike vectors. It follows that we have lightlike normal vector fields $X^\ell_\pm : U \to LC^*$ along $M^H$. We respectively call $X^d$ and $X^\ell$, the de Sitter Gauss image and the lightcone Gauss image of $M^H$. We define a map $L_2 : U \to \Delta_2^1$ by $L_2(u) = (X^h(u), X^\ell_\pm(u))$. It is easy to check that $L_2$ is a Legendrian embedding. In [8], we have constructed $X^d$ and $X^\ell$ by an explicit way and investigated the geometric meanings of the singularities of these Gauss images. Both of the de Sitter Gauss image $X^d$ and the lightcone Gauss image $X^\ell_\pm$ play similar roles with the Gauss map of a hypersurface in Euclidean space. We can interpret that $dX^d(u)$ is a linear transformation on $T_pM^H$ for $p = X^h(u)$. Since the derivative $dX^h(u)$ can be identified with the identity mapping $1_{T_pM^H}$ on the tangent space $T_pM^H$ under the identification of $U$ and $M^H$ through the embedding $X^h$, we have $dX^\ell(u) = 1_{T_pM^H} \pm dX^d(u)$, so that $dX^\ell(u)$ can be also interpreted as a linear transformation on $T_pM^H$. We call the linear transformations $A^p_d = -dX^d(u) : T_pM^H \to T_pM^H$ and $(S^\pm_h)_p = -dX^\ell_\pm(u) : T_pM^H \to T_pM^H$, the de Sitter shape operator and the lightcone shape operator of $M^H = X^h(U)$ at $p = X^h(u)$, respectively. The de Sitter Gauss-Kronecker curvature and the lightcone Gauss-Kronecker curvature of $M^H$ at $p = X^h(u)$ are defined to be $K_d(u) = \det A_p^d$ and $K^\ell_\pm(u) = \det (S^\pm_h)_p$, respectively. In [8], we have investigated the geometric meanings of the lightcone Gauss-Kronecker curvature from the contact viewpoint. The consequences of the results are that the de Sitter Gauss-Kronecker curvature (respectively, the lightcone Gauss-Kronecker curvature) estimates the contact of hypersurfaces with hyperplanes (respectively, hyperhorospheres). Here, a hyperplane is defined to be the intersection of $H^n(-1)$ with a timelike hyperplane through the origin and a hyperhorosphere is defined to be the intersection of $H^n(-1)$ with a lightlike hyperplane. We only remark here that $X^d$ is a constant vector if and only if $M^H$ is a part of a hyperplane. Moreover, one of $X^\ell_\pm$ is a constant vector if and only if $M^H$ is a part of a hyperhorosphere. These facts suggest us that there are two kinds of flat subjects in Hyperbolic space. One of them is a hyperplane and the other one is a hyperhorosphere. In the Poincaré ball model of Hyperbolic space, the hyperplane is a hypersphere as the Euclidean sense and it is orthogonal to the ideal boundary. The hyperhorosphere is also a hypersphere as the Euclidean sense, but it is tangent to the ideal boundary. We remark that the hyperplanes are totally flat hypersurfaces in the sense of Hyperbolic Geometry. What about hyperhorospheres? We emphasize that we discovered a new geometry which is called “Horospherical Geometry” in Hyperbolic space through the researches [5, 8, 10, 13, 15]. Hyperhorospheres are totally flat hypersurfaces in Hyperbolic space in the sense of Horospherical Geometry.

On the other hand, an equidistant hypersurface is defined to be the intersection of $H^n(-1)$ with a timelike hyperplane which does not contain the origin. It is well known that a non-compact complete totally umbilic hypersurface in Hyperbolic space is a hyperplane, an equidistant hypersurface or a hyperhorosphere (cf., [8]). Here, we consider a natural question.

**Question.** Can we construct a geometry such that an equidistant hypersurface is a totally flat hypersurface?

In order to give an answer to this question, we consider the contact manifold $(\Delta_{21}^\omega(\phi), K^\omega_{[\phi]_{21}})$ and the contact diffeomorphism $\Psi^\omega_{[21]} : \Delta_{21} \to \Delta_{21}(\phi)$ defined by $\Psi^\omega_{[21]}(v, \cos \phi v \pm w)$. We define $N^d_\pm[\phi] : U \to S^\omega_1(\sin^2 \phi)$ by

$$N^d_\pm[\phi](u) = \cos \phi X^h(u) \pm X^d(u),$$

for $\phi \in [0, \pi/2]$. It follows that $N^d_\pm[0] = X^d_\pm, N^d_\pm[\pi/2] = \pm X^d$ and $\langle X^h(u), N^d_\pm[\phi](u) \rangle = -\cos \phi.$
We also define an embedding $\mathcal{L}_{21}[\phi] : U \rightarrow \Delta_{21}(\phi)$ by $\mathcal{L}_{21}[\phi](u) = (X^h(u), N^d_+[\phi](u))$. Then we have $\mathcal{L}_{21}[\phi] = \Psi_{1(21)}^{-1} \circ \mathcal{L}_1^{-1}$, so that $\mathcal{L}_{21}[\phi]$ is a Legendrian embedding. Therefore, we have $\langle dX^h, N^d_+[\phi] \rangle = \mathcal{L}_{21}[\phi]^* \theta|_{\Delta_{21}} = 0$. This means that $N^d_+[\phi](u)$ is a normal vector of $M^H$ at $p = X^h(u)$. We call $N^d_+[\phi](u) : U \rightarrow S^n_+(\sin^2 \phi)$ the $\phi$-de Sitter Gauss image of $M^H$. By definition, we have $dN^d_+[\phi](u) = \cos \phi_1 T_p M^H \pm dX^d(u)$ which can be considered as a linear transformation on $T_p M^H$. We call $S^d_+[\phi]_p = -dN^d_+[\phi](u) : T_p M^H \rightarrow T_p M^H$ a $\phi$-de Sitter shape operator (or $\phi$-de Sitter Weingarten map) of $M^H$ at $p = X^h(u)$. The $\phi$-de Sitter Gauss-Kronecker curvature of $M^H$ at $p = X^h(u)$ is defined to be $K^d_+[\phi](u) = \det S^d_+[\phi](u)$. The geometry related to $\phi$-de Sitter Gauss image is called the $\phi$-geometry of hypersurfaces in Hyperbolic space. Since the 0-geometry is the horospherical geometry and $\pi/2$-geometry is the hyperbolic geometry, we call the $\phi$-geometry a slant geometry in Hyperbolic space if $\phi \in (0, \pi/2)$. The detailed investigation of the slant geometry in Hyperbolic space will be appeared in the forthcoming paper [3]. Here, we only consider the most degenerate case.

**Proposition 4.1** For a hypersurface $M^H$, one of $N^d_+[\phi](u)$ is a constant vector if and only if $M^H$ is a part of an equidistant hypersurface $H^n(-1) \cap HP(v, -\cos \phi)$ with $v \in S^n_+(\sin^2 \phi)$.

**Proof.** Suppose that $N^d_+[\phi](u) = \text{constant} = v$. Then we have

\[
\langle X^h(u), v \rangle = \langle X^h(u), N^d_+[\phi](u) \rangle = 0.
\]

This means that $M^H \subset H^n(-1) \cap HP(v, -\cos \phi)$. If $N^d_+[\phi](u) = \text{constant} = v$, we have the similar result. For the converse, suppose that $M^H \subset H^n(-1) \cap HP(v, -\cos \phi)$ with $v \in S^n_+(\sin^2 \phi)$. Since $v$ is a normal vector of $M^H$, there exist $\lambda, \mu$ such that $v = \lambda X^h(u) + \mu X^d(u)$. By definition, we have $-\cos \phi = \langle X^h(u), v \rangle = -\lambda$ and $\sin^2 \phi = -\lambda^2 + \mu^2$. It follows that $v = N^d_+[\phi](u)$ or $v = N^d_+[\phi](u)$. \hfill \Box

We remark that the above proposition asserts that a totally flat hypersurface in the $\phi$-geometry is a part of equidistant hypersurface $H^n(-1) \cap HP(v, -\cos \phi)$ with $v \in S^n_+(\sin^2 \phi)$. We call it a $\phi$-equidistant hypersurface in Hyperbolic space $H^n(-1)$. A $\phi$-equidistant hypersurface is a hyperhorosphere (respectively, a hyperplane), when $\phi = 0$ (respectively, $\phi = \pi/2$).

### 4.2 De Sitter space

We also consider the Legendrian embedding $\mathcal{L}_1 : U \rightarrow \Delta_1$ and suppose that $X^d : U \rightarrow S^n_1$ is an embedding. In this case, all the tangent vectors of $M^D = X^d(U)$ are spacelike, so that $X^d$ is a spacelike embedding. In [17] Kasedou constructed the extrinsic differential geometry on the spacelike hypersurfaces in $S^n_1$ analogous to the theory in [8]. We can interpret his framework by using the mandala of Legendrian dualities. We consider the lightlike vectors $\pm X_{\pm}^d(u) = X^d(u) \pm X^h(u)$. We respectively call $X^h$ and $\pm X_{\pm}^d$, the hyperbolic Gauss image and the lightcone Gauss image of $M^D = X^d(U)$. We also define a map $\mathcal{L}_3 : U \rightarrow \Delta_3^+$ by $\mathcal{L}_3(u) = (\pm X_{\pm}^d(u), X^d(u))$. It is a Legendrian embedding and $d(\pm X_{\pm}^d)(u) = 1_{T_p M^D} \pm dX^d(u)$ for $p = X^d(u)$. Since $dX^h(u)$ is considered to be a linear transformation on $T_p M^D$, $d(\pm X_{\pm}^d)(u)$ is also a linear transformation on $T_p M^D$. We call $(S^d_+)_p = -d(\pm X_{\pm}^d)(u) : T_p M^D \rightarrow T_p M^D$ and $A^h_p = -dX^h(u) : T_p M^D \rightarrow T_p M^D$, the lightcone shape operator and the hyperbolic shape operator of $M^D$ at $p = X^d(u)$, respectively. Geometric characterizations of the singularities of the lightcone Gauss image $\pm X_{\pm}^d$ of $M^D$ from the view point of the contact with model hypersurfaces (cf., Montaldi, [19]) are one of the main results in [17]. Especially, Theorem 5.6 in
[17] is obtained by applying the theory of Legendrian singularities for $\pm X^f_\pm(u)$. For definitions and basic properties of the theory of Legendrian singularities, see (Part III, [1]). Here, we can interprete the results in [17] by using the mandala of Legendrian dualities. Let $\Phi_{23}^\pm : \Delta_2^\pm \rightarrow \Delta_3^+$ be the mappings defined by $\Phi_{23}^\pm (v, w) = (\pm v, \pm (u - v))$. Then we have $\pi_{31} \circ \Phi_{23}^\pm = \pm \pi_{22}$. It is easy to show that $\Phi_{23}^\pm$ are contact diffeomorphisms. By definition, we have

$$\Phi_{23}^\pm \circ \mathcal{L}_2(u) = (\pm X^f_\pm(u), \pm (X^f_\pm(u) - X^h(u))) = \mathcal{L}_3^\pm(u).$$

This means that Legendrian maps $\pi_{22} \circ \mathcal{L}_2$ and $\pi_{31} \circ \mathcal{L}_3^\pm$ are Legendrian equivalent. We only remark here that all conditions in Theorem 6.3 in [8] and Theorem 5.6 in [17] are invariant under the Legendrian equivalence. Therefore, the assertions of these theorems are equivalent.

On the other hand, we consider the contact manifold $(\Delta_3^+(\phi), K[\phi]^+_3)$ and the contact diffeomorphism $\Psi_{1(31)}^+ : \Delta_1 \rightarrow \Delta_3^+(\phi)$ defined by $\Psi_{1(31)}^+(\phi)(v, w) = (\pm v + \cos \phi w, w)$. We define a map $N^h_{\pm} : U \rightarrow H^n(-\sin^2 \phi)$ by

$$N^h_{\pm}[\phi](u) = \cos \phi X^d(u) \pm X^h(u),$$

for $\phi \in [0, \pi/2]$. It follows that $N^h_{\pm}[0] = \pm X^f_\pm$, $N^h_{\pm}[\pi/2] = \pm X^h$ and $\langle X^d(u), N^h_{\pm}[\phi](u) \rangle = \cos \phi$. We also define an embedding $\mathcal{L}_{31}[\phi] : U \rightarrow \Delta_3^+(\phi)$ by $\mathcal{L}_{31}[\phi](u) = (N^h_{\pm}[\phi](u), X^d(u))$. Then we have $\Delta_{31}[\phi] = \Psi_{1(31)}^+ \circ \mathcal{L}_1$, so that $\mathcal{L}_{31}[\phi]$ is a Legendrian embedding. Therefore, we have $\langle dX^d, N^h_{\pm}[\phi] \rangle = \mathcal{L}_{31}[\phi]^* \theta[\phi]^+_3 = 0$. By exactly the same way as the hyperbolic case, we can construct the $\phi$-hyperbolic shape operator $S^h_{\pm}[\phi] = -dN^h_{\pm}[\phi](u) : T_p M^D \rightarrow T_p M^D$ and the $\phi$-hyperbolic Gauss-Kronecker curvature $K^h_{\pm}[\phi](u)$ of $M^D$ at $p = X^d(u)$. The geometry related to the Gauss image $N^h_{\pm}[\phi]$ is called a $\phi$-geometry of the spacelike hypersurfaces in de Sitter space. We also consider the most degenerate case here.

**Proposition 4.2** For a spacelike hypersurface $M^D \subset S^m_1$, one of $N^h_{\pm}[\phi](u)$ is a constant vector if and only if $M^D$ is a part of an elliptic hyperquadric $S^m_1 \cap HP(v, \cos \phi)$ with $v \in H^n(-\sin^2 \phi)$.

**Proof.** Suppose that $N^h_{\pm}[\phi](u) = \text{constant} = v$. Then we have

$$\langle X^d(u), v \rangle = \langle X^d(u), N^h_{\pm}[\phi](u) \rangle = 0.$$

This means that $M^D \subset S^m_1 \cap HP(v, \cos \phi)$. If $N^h_{\pm}[\phi](u) = \text{constant} = v$, we have the similar result. For the converse, suppose that $M^D \subset S^m_1 \cap HP(v, \cos \phi)$ with $v \in H^n(-\sin^2 \phi)$. Since $v$ is a normal vector of $M^D$, there exist $\lambda, \mu$ such that $v = \lambda X^h(u) + \mu X^d(u)$. By definition, we have $\cos \phi = \langle X^d(u), v \rangle = \mu$ and $-\sin^2 \phi = -\lambda^2 + \mu^2$. It follows that $v = N^h_{\pm}[\phi](u)$ or $v = N^h_{\pm}[\phi](u)$. \hfill $\square$

We also remark that the above proposition asserts that a totally flat spacelike hypersurface in the $\phi$-geometry is a part of a hyperquadric $S^m_1 \cap HP(v, \cos \phi)$ with $v \in H^n(-\sin^2 \phi)$. We call it a $\phi$-hyperquadric in de Sitter space $S^m_1$. By definition, the 0-hyperquadric is $S^m_1 \cap HP(v, 1)$ for $v \in LC^*$ and $\pi/2$-hyperquadric is $S^m_1 \cap HP(v, 0)$ for $v \in H^n(-1)$. The 0-hyperquadric is called a de Sitter hyperhorosphere which is nothing but a parabolic hyperquadric. We call the $\pi/2$-hyperquadric a small elliptic hyperquadric. We remark that a small elliptic hyperquadric is a spacelike geodesic, when $n = 2$. We also call the geometry related to the Gauss image $N^h_{\pm}[\phi]$ a slant geometry of spacelike hypersurfaces in de Sitter space if $\phi \in (0, \pi/2)$. 

9
4.3 The lightcone

In [9] we have considered an extrinsic differential geometry on spacelike hypersurfaces in the lightcone motivated by the result of [2]. However, the induced metric on the nullcone is degenerate, so that we cannot apply ordinary submanifold theory of semi-Riemannian geometry. The $\Delta^+_1$-duality is really useful in this case. Let $\mathcal{L}_4: U \rightarrow \Delta^+_1$ be a Legendrian embedding with $\mathcal{L}_4(u) = (X^t_+(u), X^t_-(u))$ for an open subset $U \subset \mathbb{R}^{n-1}$. Suppose that $X^t_+ : U \rightarrow \mathbb{L}^* \mathbb{C}$ is a spacelike embedding. In [9] the Legendrian embedding $\mathcal{L}_4$ has been used for the construction of the extrinsic differential geometry on spacelike hypersurfaces $\mathbb{M}^L = X^t_+(U)$ in the lightcone. It has been shown that for any spacelike embedding $X^t_+ : U \rightarrow \mathbb{L}^* \mathbb{C}$, there exists a unique Legendrian embedding $\mathcal{L}_4: U \rightarrow \Delta^+_1$ such that $\pi_{41} \circ \mathcal{L}_4 = X^t_+$. Since $\mathcal{L}_4$ is Legendrian, $X^t_+(u)$ is a lightlike normal vector of $\mathbb{M}^L$ at $p = X^t_+(u)$. We call it a lightcone normal vector of $\mathbb{M}^L$. If $X^t_-$ is an embedding, then $X^t_-(u)$ is called a lightcone normal vector of $\mathbb{M}^L = X^t_-(U)$ at $p = X^t_-(u)$. We define two part fields

$$X^h(u) = \frac{X^t_+(u) + X^t_-(u)}{2} \quad \text{and} \quad X^d(u) = \frac{X^t_-(u) - X^t_+(u)}{2}.$$ 

Then $X^h(u) \in H^n(-1)$ and $X^d(u) \in S^n_1$. Moreover, we have mappings $\mathcal{L}_1: U \rightarrow \Delta_1$, $\mathcal{L}^\pm_2: U \rightarrow \Delta^\pm_2$ and $\mathcal{L}^\pm_3: U \rightarrow \Delta^\pm_3$ which are defined by $\mathcal{L}_1(u) = (X^h(u), X^d(u))$, $\mathcal{L}^+_2(u) = (X^h(u), X^+_2(u))$ and $\mathcal{L}^-_3(u) = (X^-_3(u), X^d(u))$, respectively. It is easy to show that $\mathcal{L}_4$ and $\mathcal{L}^\pm_i$ ($i = 2, 3$) are Legendrian embeddings. We now define mappings $\Phi^\pm_i: \Delta^+_4 \rightarrow \Delta^\pm_i$ by $\Phi^+_4(v, w) = \left(\frac{v + w}{2}, v\right)$ and $\Phi^-_4(v, w) = \left(\frac{v + w}{2}, w\right)$. Then we have $\pi_{22} \circ \Phi^+_4 = \pi_{42}$ and $\pi_{22} \circ \Phi^-_4 = \pi_{41}$. We can show that $\Phi^\pm_i$ are contact diffeomorphisms and $\Phi^\pm_4 \circ \mathcal{L}_4 = \mathcal{L}^\pm_i$. Therefore, $\pi_{41} \circ \mathcal{L}_4$ and $\pi_{22} \circ \mathcal{L}^\pm_3$ (respectively, $\pi_{41} \circ \mathcal{L}_4$ and $\pi_{22} \circ \mathcal{L}^\pm_3$) are Legendrian equivalent. It follows that the assertions of Theorem 6.3 in [8] and Theorem 6.6 in [9] are equivalent. By the arguments in Subsection 4.2, the assertions of Theorem 5.6 in [17] and Theorem 6.6 in [9] are also equivalent. However, we can directly define the Legendrian equivalence between $\pi_{41} \circ \mathcal{L}_4$ ($i = 1, 2$) and $\pi_{31} \circ \mathcal{L}^\pm_3$ as follows:

Let $\Phi^\pm_4: \Delta^+_4 \rightarrow \Delta^\pm_3$ be a mapping defined by $\Phi^+_4(v, w) = \left(v, \frac{w - v}{2}\right)$ and $\Phi^-_4(v, w) = \left(w, \frac{w - v}{2}\right)$. By exactly the same reasons as above, we can show that $\Phi^\pm_4$ give Legendrian equivalences between $\pi_{4i} \circ \mathcal{L}_4$ ($i = 1, 2$) and $\pi_{31} \circ \mathcal{L}^\pm_3$.

On the other hand, we have a proposition as a special case of Proposition 3.7 in [9] as follows:

**Proposition 4.3** Let $\mathcal{L}_4: U \rightarrow \Delta^+_4$ be a Legendrian embedding with $\mathcal{L}_4(u) = (X^t_+(u), X^t_-(u))$.

1. Suppose that $X^t_+$ is an embedding. Then $X^d(u)$ is a constant vector if and only if $M^L_+$ is a part of $\mathbb{L}^* \mathbb{C} \cap \mathbb{H} \mathbb{P}(\mathbb{v}, -1)$ with $\mathbb{v} \in S^n_1$.
2. Suppose that $X^t_+$ is an embedding. Then $X^t_-(u)$ is a constant vector if and only if $M^L_-$ is a part of $\mathbb{L}^* \mathbb{C} \cap \mathbb{H} \mathbb{P}(\mathbb{v}, -2)$ with $\mathbb{v} \in \mathbb{L}^* \mathbb{C}$.
3. Suppose that $X^t_-$ is an embedding. Then $X^h(u)$ is a constant vector if and only if $M^L_-$ is a part of $\mathbb{L}^* \mathbb{C} \cap \mathbb{H} \mathbb{P}(\mathbb{v}, -1)$ with $\mathbb{v} \in H^n(-1)$.

We respectively call $\mathbb{L}^* \mathbb{C} \cap \mathbb{H} \mathbb{P}(\mathbb{v}, -1)$ with $\mathbb{v} \in S^n_1$, $\mathbb{L}^* \mathbb{C} \cap \mathbb{H} \mathbb{P}(\mathbb{v}, -2)$ with $\mathbb{v} \in \mathbb{L}^* \mathbb{C}$ and $\mathbb{L}^* \mathbb{C} \cap \mathbb{H} \mathbb{P}(\mathbb{v}, -1)$ with $\mathbb{v} \in H^n(-1)$, a de Sitter flat hyperbolic hyperquadric, a lightcone flat parabolic hyperquadric and a hyperbolic flat elliptic hyperquadric. In [9], the lightcone Gauss-Kronecker curvature for a spacelike hypersurface $M^L_+$ was introduced by using $X^t_-$ as
a Gauss map. Actually, it is defined by \( K^\ell(u) = \det(-dX^\ell(u)) \). The lightcone flat parabolic hyperquadric is a totally flat in this sense. By the above proposition, we have three kinds of totally flat spacelike hypersurfaces in the lightcone. Therefore, we are interested in the relations among these flatness.

We consider the contact manifold \( (\Delta_{43}^-\!(\phi), K[\phi]_{43}^-) \) and the contact diffeomorphism \( \Psi_{43}^- \colon \Delta_4^- \to \Delta_{43}^-\!(\phi) \) defined by

\[
\Psi_{43}^- (v, w) = \left( v, \frac{1}{2} ((\cos \phi - 1)v + (\cos \phi + 1)w) \right).
\]

We define a map \( N^d_\ell[\phi] : U \to S^0_1(\sin^2 \phi) \) by

\[
N^d_\ell[\phi](u) = \frac{1}{2}((\cos \phi - 1)X^\ell_+(u) + (\cos \phi + 1)X^\ell(u)),
\]

for \( \phi \in [0, \pi/2] \). We also define an embedding \( \mathcal{L}_{43}[\phi] : U \to \Delta_{43}^-\!(\phi) \) by

\[
\mathcal{L}_{43}[\phi](u) = (X^\ell_+(u), N^d_\ell[\phi](u)).
\]

Then we have \( \mathcal{L}_{43}[\phi] = \Psi_{43}^- \circ \mathcal{L}_4 \), so that \( \mathcal{L}_{43}[\phi] \) is a Legendrian embedding. Therefore, we have

\[
\langle dX^\ell_+, N^d_\ell[\phi] \rangle = \mathcal{L}_{43}[\phi]^* \theta[\phi]_{43}^- = 0.
\]

This means that \( N^d_\ell[\phi](u) \) can be considered as a normal vector of \( M^L_+ \) at \( p = X^\ell_+(u) \). We remark that \( N^d_\ell[0](u) = X^\ell(u) \) and \( N^d_\ell[\pi/2](u) = X^d(u) \). Then we have the following proposition.

**Proposition 4.4** Suppose that \( X^\ell_+ \) is an embedding. Then \( N^d_\ell[\phi](u) \) is a constant vector if and only if \( M^L_+ \) is a part of \( LC^* \cap HP(v, -(\cos \phi + 1)) \) with \( v \in S^0_1(\sin^2 \phi) \).

**Proof.** Suppose that \( \langle dX^\ell_+, N^d_\ell[\phi] \rangle = v \). Then we have \( \langle X^\ell_+(u), v \rangle = \langle X^\ell_+(u), N^d_\ell[\phi](u) \rangle = 0 \). This means that \( M^L_+ \subset LC^* \cap HP(v, -(\cos \phi + 1)) \). For the converse, suppose that \( M^L_+ \subset LC^* \cap HP(v, -(\cos \phi + 1)) \) with \( v \in S^0_1(\sin^2 \phi) \). Since \( v \) is a normal vector of \( M^L_+ \) in \( \mathbb{R}^{n+1}_1 \), there exist \( \lambda, \mu \) such that \( v = \lambda X^\ell_+(u) + \mu X^\ell(u) \). By definition, we have \(-2 = \langle X^\ell_+(u), v \rangle = \lambda \mu \) and \( \sin^2 \phi = -4\lambda \mu \), so that \( 2\lambda = \cos \phi - 1 \). It follows that \( v = N^d_\ell[\phi](u) \).

We call \( LC^* \cap HP(v, -(\cos \phi + 1)) \) with \( v \in S^0_1(\sin^2 \phi) \) a \( \phi \)-de Sitter flat hyperbolic hyperquadric.

On the other hand, we consider the contact manifold \( (\Delta_{42}^-\!(\phi), K[\phi]_{42}^-) \) and the contact diffeomorphism \( \Psi_{42}^- \colon \Delta_4^- \to \Delta_{42}^-\!(\phi) \) defined by

\[
\Psi_{42}^- (v, w) = \left( \frac{1}{2} ((1 + \cos \phi)v + (1 - \cos \phi)w) , \ w \right).
\]

We define a map \( N^h_\ell[\phi] : U \to H^n(-\sin^2 \phi) \) by

\[
N^h_\ell[\phi](u) = \frac{1}{2}((1 + \cos \phi)X^\ell_+(u) + (1 - \cos \phi)X^\ell(u)),
\]

for \( \phi \in [0, \pi/2] \) and have a map \( \mathcal{L}_{42}[\phi] : U \to \Delta_{42}^-\!(\phi) \) defined by \( \mathcal{L}_{42}[\phi](u) = (N^h_\ell[\phi](u), X^\ell_+(u)) \).

By exactly the same reason as the above case, \( \mathcal{L}_{42}[\phi] \) is a Legendrian embedding, so that \( N^h_\ell[\phi](u) \) can be considered as a normal vector of \( M^L_+ \) at \( p = X^\ell_+(u) \). We remark that \( N^h_\ell[0](u) = X^\ell(u) \) and \( N^h_\ell[\pi/2](u) = X^h(u) \). Then we have the following proposition.
Proposition 4.5 Suppose that $X^\ell_-$ is an embedding. Then $N^d_\ell[\phi](u)$ is a constant vector if and only if $M_\ell^h$ is a part of $LC^* \cap HP(v, -(1 + \cos \phi))$ with $v \in H^n(-\sin^2 \phi)$.

Since the proof of Proposition 4.5 is given by exactly the same arguments as those of Proposition 4.4, we omit it. We call $LC^* \cap HP(v, -(1 + \cos \phi))$ with $v \in H^n(-\sin^2 \phi)$ a $\phi$-hyperbolic flat elliptic hyperquadric.

We call both the geometry related to the Gauss maps $N^d_\ell[\phi]$ and $N^h_\ell[\phi]$ a slant geometry of spacelike hypersurfaces in the lightcone. The detailed arguments on the slant geometry will be appeared in the forthcoming paper [16].

Acknowledgements. The first author was partly supported by Grant-in-Aid for Scientific Research (B) (18340013) and Grant-in-Aid for Challenging Exploratory Research (21654007). Moreover, the second author was supported by BÖDEB-2214 Programme of TUBITAK-BAYG during her researches in Hokkaido University.

References


Shyuichi Izumiya, Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan.
e-mail:izumiya@math.sci.hokudai.ac.jp

Handan Yıldırım, Department of Mathematics, Faculty of Science, Istanbul University, 34134, Vezneciler, Istanbul, Turkey.
e-mail:handanyildirim@istanbul.edu.tr