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# Spacelike submanifolds in de Sitter space

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## Abstract

We investigate the differential geometry of spacelike submanifolds of codimension at least two in de Sitter space as an application of the theory of Legendrian singularities. We also discuss related geometric property of spacelike hypersurfaces in de Sitter space.

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*Key words:* de Sitter space, spacelike submanifold, spacelike hypersurface.

## 1 Introduction

It is known that de Sitter space is a Lorentzian space form with a positive curvature. Recently, Izumiya, Pei and Sano [3] investigated the extrinsic differential geometry of hypersurfaces in the hyperbolic space by applying the theory of Legendrian singularities. The main tool is a lightcone Gauss indicatrix, which is defined by a lightlike normal of hypersurface, and their singularity sets correspond to lightcone parabolic sets of hypersurfaces. For higher codimension case, the normal vector is not uniquely determined, however it is possible to construct hypersurfaces from the normal unit vector fields of the subspace. Izumiya, Pei, Romero Fuster and Takahashi [6] introduced the notion of canal hypersurfaces and horospherical hypersurfaces from the normal frames of submanifolds in the hyperbolic space, and investigated submanifolds of higher codimension in the hyperbolic space from the viewpoint of singularity theory. On the other hand, the differential geometry of de Sitter space is also studied. In [7] we introduced the notion of lightcone Gauss image which is an analogous tool introduced in [3], and investigate the case of spacelike hypersurface in de Sitter space. For codimension two case, Fusho and Izumiya [2] firstly introduced the notion of lightlike surface of a spacelike curve in the de Sitter three-space. In [8] we investigated singularities of lightlike hypersurface of spacelike submanifold of codimension two in de Sitter  $n$ -space for  $n \geq 3$ , which is an analogous study in the Minkowski space [4, 5].

In this paper, we argue an analogous study of the submanifolds of higher codimension in hyperbolic space [6] and introduce the notions of horospherical hypersurfaces and spacelike canal hypersurfaces by using timelike unit normal vector fields. The singular point of horospherical surface corresponds to the parabolic point of spacelike canal hypersurface, which we call a horospherical point, and the spacelike submanifold is tangent to a de Sitter hyperhorosphere

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at the horospherical point. If we assume a hypothesis of Theorem 6.5, then a contact type of a de Sitter hyperhorosphere and a spacelike submanifold corresponds to a singular type of horospherical hypersurface, and also corresponds to a singular type of lightcone Gauss image of spacelike canal surface. In §2 we review briefly the basic notions of differential geometry of spacelike hypersurfaces [7]. In §3,4 we define a timelike normal vector field of spacelike submanifolds in de Sitter space. and introduce a notion of horospherical height function and horospherical hypersurface. We also define a spacelike canal hypersurface, whose lightcone Gauss image is diffeomorphic to a horospherical hypersurface. In §5 we naturally interpret a horospherical hypersurfaces of a spacelike submanifold as a wave front set of horospherical height functions in the theory of Legendrian singularities. In §6 we use the theory of contacts between the submanifolds due to Montaldi [10], and we discuss geometric properties of singularities of horospherical hypersurfaces. We consider generic properties of spacelike submanifolds in §7.

## 2 Spacelike hypersurfaces in de Sitter space

In this section we review the extrinsic differential geometry of spacelike hypersurfaces in de Sitter space [7], which is an analogous study of [3]. Let  $\mathbb{R}^{n+1} = \{\mathbf{x} = (x_0, \dots, x_n) \mid x_i \in \mathbb{R} \ (i = 0, \dots, n)\}$  be an  $(n+1)$ -dimensional vector space. For any vectors  $\mathbf{x} = (x_0, \dots, x_n)$ ,  $\mathbf{y} = (y_0, \dots, y_n)$  in  $\mathbb{R}^{n+1}$ , the *pseudo scalar product* of  $\mathbf{x}$  and  $\mathbf{y}$  is defined by  $\langle \mathbf{x}, \mathbf{y} \rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i$ . We call  $(\mathbb{R}^{n+1}, \langle, \rangle)$  a *Minkowski  $(n+1)$ -space* and write  $\mathbb{R}_1^{n+1}$  instead of  $(\mathbb{R}^{n+1}, \langle, \rangle)$ .

We say that a vector  $\mathbf{x} \in \mathbb{R}_1^{n+1} \setminus \{\mathbf{0}\}$  is *spacelike*, *timelike* or *lightlike* if  $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ ,  $\langle \mathbf{x}, \mathbf{x} \rangle = 0$  or  $\langle \mathbf{x}, \mathbf{x} \rangle < 0$  respectively. The norm of the vector  $\mathbf{x} \in \mathbb{R}_1^{n+1}$  is defined by  $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$ . For a vector  $\mathbf{v} \in \mathbb{R}_1^{n+1} \setminus \{\mathbf{0}\}$  and a real number  $c$ , we define a *hyperplane with pseudo normal  $\mathbf{v}$*  in the Minkowski space by  $HP(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{v} \rangle = c\}$ . We say that a hyperplane  $HP(\mathbf{v}, c)$  is spacelike, timelike or lightlike if the vector  $\mathbf{v}$  is timelike, spacelike or lightlike.

We now respectively define *hyperbolic  $n$ -space* and *de Sitter  $n$ -space* by

$$\begin{aligned} H_{\pm}^n(-1) &= \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1, \operatorname{sgn}(x_0) = \pm 1\}, \\ S_1^n &= \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}, \end{aligned}$$

and we write  $H^n(-1) = H_+^n(-1) \cup H_-^n(-1)$ . For any  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in \mathbb{R}_1^{n+1}$ , we can define a vector  $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \dots \wedge \mathbf{x}_n$  with the property  $\langle \mathbf{x}, \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_n \rangle = \det(\mathbf{x}, \mathbf{x}_1, \dots, \mathbf{x}_n)$ , so that  $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_n$  is pseudo-orthogonal to any  $\mathbf{x}_i$  for  $i = 1, \dots, n$ . We also define *future* (resp. *past*) *lightcone* at the origin by

$$LC_{\pm}^* = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0, x_0 > 0 \ (x_0 < 0)\},$$

and we write  $LC^* = LC_+^* \cap LC_-^*$ .

Let  $\mathbf{X} : U \longrightarrow S_1^n$  be an embedding, where  $U \subset \mathbb{R}^{n-1}$  is an open subset. We say  $\mathbf{X}$  is a *spacelike hypersurface* in  $S_1^n$  if every non zero vector generated by  $\{\mathbf{X}_{u_i}(u)\}_{i=1}^{n-1}$  is always spacelike, where  $u = (u_1, \dots, u_{n-1})$  is an element of  $U$  and  $\mathbf{X}_{u_i}$  is a partial derivative of  $\mathbf{X}$  with respect to  $u_i$ . We denote  $M = \mathbf{X}(U)$  and identify  $M$  with  $U$  through the embedding  $\mathbf{X}$ . Since  $\langle \mathbf{X}, \mathbf{X} \rangle \equiv 1$ , we have  $\langle \mathbf{X}_{u_i}, \mathbf{X} \rangle \equiv 0$  for  $i = 1, \dots, n-1$ . It follows that a hyperplane spanned by  $\{\mathbf{X}, \mathbf{X}_{u_1}, \dots, \mathbf{X}_{u_{n-1}}\}$  is spacelike. We define a vector

$\mathbf{e}(u) = \mathbf{X}(u) \wedge \mathbf{X}_{u_1}(u) \wedge \dots \wedge \mathbf{X}_{u_{n-1}}(u) / \|\mathbf{X}(u) \wedge \mathbf{X}_{u_1}(u) \wedge \dots \wedge \mathbf{X}_{u_{n-1}}(u)\|$ . Then  $\mathbf{e}$  is pseudo orthogonal to  $\mathbf{X}$  and  $\mathbf{X}_{u_i}$  for  $i = 1, \dots, n-1$ . We define a map  $\mathbb{L}^\pm : U \longrightarrow LC_\pm^*$  by

$$\mathbb{L}^\pm(u) = \mathbf{X}(u) \pm \mathbf{e}(u),$$

which is called a *positive* (resp. *negative*) *lightcone Gauss image* of  $\mathbf{X}$ .

We now consider a hypersurface defined by  $HP(\mathbf{v}, c) \cap S_1^n$ . We say that  $HP(\mathbf{v}, c) \cap S_1^n$  is an *elliptic* hyperquadric or a *hyperbolic* hyperquadric if  $HP(\mathbf{v}, c)$  is spacelike or timelike respectively. We say that  $HP(\mathbf{v}, 1) \cap S_1^n$  is a *de Sitter hyperhorosphere* if  $HP(\mathbf{v}, 1)$  is lightlike. We have the following proposition analogous to ([3], Proposition 2.2).

**Proposition 2.1** ([7]). Let  $\mathbf{X} : U \longrightarrow S_1^n$  be a spacelike hypersurface in  $S_1^n$ . The lightcone Gauss image  $\mathbb{L}^\pm$  is constant if and only if the spacelike hypersurface  $M = \mathbf{X}(U)$  is a part of a de Sitter hyperhorosphere.

We now define the lightcone Gauss-Kronecker curvature and the lightcone mean curvature of the spacelike hypersurface  $M = \mathbf{X}(U)$ . For any  $p \in M$  and  $v \in T_pM$ , we can show  $D_{\mathbf{v}}\mathbf{e}, D_{\mathbf{v}}\mathbb{L}^\pm \in T_pM$ , where  $D_{\mathbf{v}}$  is the covariant derivative with respect to the tangent vector  $\mathbf{v}$ . Under the identification of  $U$  and  $M$ ,  $d\mathbb{L}^\pm(u_0)$  is a linear transformation on  $T_pM$ . We call  $S_p^\pm = -d\mathbb{L}^\pm(u_0)$  a *lightcone shape operator* of  $M = \mathbf{X}(U)$  of at  $p = \mathbf{X}(u_0)$ . The *lightcone Gauss-Kronecker curvature*  $K_\ell^\pm$  of  $M = \mathbf{X}(U)$  at  $p = \mathbf{X}(u_0)$  is defined to be the determinant of the lightcone shape operator.

Since  $\mathbf{X}_{u_i}$  (for  $i = 1, \dots, n-1$ ) are spacelike vectors, we have the Riemannian metric (*first fundamental form*)  $ds^2 = \sum_{i,j=1}^{n-1} g_{ij} du_i du_j$  on  $M = \mathbf{X}(U)$ , where  $g_{ij}(u) = \langle \mathbf{X}_{u_i}(u), \mathbf{X}_{u_j}(u) \rangle$  for any  $u \in U$ . We define a *lightcone second fundamental invariants* by  $\bar{h}_{ij}^\pm(u) = \langle -\mathbb{L}_{u_i}^\pm(u), \mathbf{X}_{u_j}(u) \rangle$  for any  $u \in U$ . In [7] we obtained explicit expression for the lightcone Gauss-Kronecker curvature:

$$K_\ell^\pm = \det(\bar{h}_{ij}^\pm) / \det(g_{\alpha\beta}).$$

We say that  $p = \mathbf{X}(u_0)$  is a *lightcone parabolic point* of  $\mathbf{X}$  if  $K_\ell^\pm(u_0) = 0$ .

We define a family of functions  $H : U \times LC^* \longrightarrow \mathbb{R}$  by

$$H(u, \mathbf{v}) = \langle \mathbf{X}(u), \mathbf{v} \rangle - 1,$$

which we call a *lightcone height function* of  $M$ . We have the following proposition analogous to ([3], Proposition 3.1).

**Proposition 2.2.** ([7]) Let  $H$  be a lightcone height function, then  $H(u, \mathbf{v}) = \partial H(u, \mathbf{v}) / \partial u_i = 0$  for  $i = 1, \dots, n-1$  if and only if  $\mathbf{v} = \mathbb{L}^\pm(u)$ .

We also naturally interpreted the lightcone Gauss image of a spacelike hypersurface as a wave front set in the frame work of contact geometry in [7]. This is the analogous way to the differential geometry of hypersurfaces in hyperbolic space [3].

Let  $\pi^\pm : PT^*(LC_\pm^*) \longrightarrow LC_\pm^*$  be the projective cotangent bundles with canonical contact structures. Consider the tangent bundle  $\tau^\pm : TPT^*(LC_\pm^*) \longrightarrow PT^*(LC_\pm^*)$  and the differential

map  $d\pi^\pm : TPT^*(LC_\pm^*) \longrightarrow T(LC_\pm^*)$  of  $\pi^\pm$ . For any  $X \in TPT^*(LC_\pm^*)$ , there exists an element  $\alpha \in T^*(LC_\pm^*)$  such that  $\tau^\pm(X) = [\alpha]$ . For  $v \in LC_\pm^*$  and  $V \in T_v(LC_\pm^*)$ , the property  $\alpha(V) = 0$  does not depend on the choice of representative of the class  $[\alpha]$ . Thus, we can define the canonical contact structure on  $PT^*(LC_\pm^*)$  by

$$K^\pm = \{X \in TPT^*(LC_\pm^*) \mid \tau^\pm(X)(d\pi^\pm(X)) = 0\}.$$

On the other hand, we consider a point  $\mathbf{v} = (v_0, v_1, \dots, v_n) \in LC_\pm^*$ , then we have the relation  $v_0 = \pm\sqrt{v_1^2 + \dots + v_n^2}$ . So we adopt the coordinate system  $(v_1, \dots, v_n)$  of the manifold  $LC_\pm^*$ . Then we have the trivialization  $PT^*(LC_\pm^*) \equiv LC_\pm^* \times P^*\mathbb{R}^{n-1}$ , and call  $((v_0, \dots, v_n), [\xi_1 : \dots : \xi_n])$  homogeneous coordinates of  $PT^*(LC_\pm^*)$ , where  $[\xi_1 : \dots : \xi_n]$  are the homogeneous coordinates of the dual projective space  $P^*\mathbb{R}^{n-1}$ .

It is easy to show that  $X_\bullet \in K_\bullet^\pm$  if and only if  $\sum_{i=1}^n \mu_i \xi_i = 0$ , where  $\bullet = (v, [\xi])$  and  $d\pi_\bullet^\pm(X_\bullet) = \sum_{i=1}^n \mu_i \partial/\partial v_i \in T_\bullet LC_\pm^*$ . An immersion  $i : L \longrightarrow PT^*(LC_\pm^*)$  is said to be a *Legendrian immersion* if  $\dim L = n - 1$  and  $di_q(T_q L) \subset K_{i(q)}$  for any  $q \in L$ . The map  $\pi \circ i$  is also called the *Legendrian map* and the image  $W(i) = \text{image}(\pi \circ i)$ , the *wave front* of  $i$ . Moreover,  $i$  (or the image of  $i$ ) is called the *Legendrian lift* of  $W(i)$ .

Let  $F : (\mathbb{R}^s \times \mathbb{R}^k, (u_0, v_0)) \longrightarrow (\mathbb{R}, 0)$  be a function germ. We say that  $F$  is a *Morse family* of hypersurfaces if the map germ  $\Delta^* F : (\mathbb{R}^s \times \mathbb{R}^k, (u_0, v_0)) \longrightarrow (\mathbb{R}^{s+1}, \mathbf{0})$  defined by

$$\Delta^* F = \left( F, \frac{\partial F}{\partial u_1}, \dots, \frac{\partial F}{\partial u_s} \right)$$

is non singular. In this case, we have a smooth  $(k - 1)$ -dimensional smooth submanifold,

$$\Sigma_*(F) = \left\{ (u, v) \in (\mathbb{R}^{n-r} \times \mathbb{R}^k, (u_0, v_0)) \mid F(u, v) = \frac{\partial F}{\partial u_1}(u, v) = \dots = \frac{\partial F}{\partial u_{n-r}}(u, v) = 0 \right\},$$

and the map germ  $\mathcal{L}_F : (\Sigma_*(F), (u_0, v_0)) \longrightarrow PT^*\mathbb{R}^k$  defined by

$$\mathcal{L}_F(u, v) = \left( v, \left[ \frac{\partial F}{\partial v_1}(u, v) : \dots : \frac{\partial F}{\partial v_k}(u, v) \right] \right)$$

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd and Zakalyukin [1, 12].

**Proposition 2.3.** All Legendrian submanifold germs in  $PT^*\mathbb{R}^k$  are constructed by the above method.

We call  $F$  a generating family of  $\mathcal{L}_F(\Sigma_*(F))$ . Therefore the wave front is

$$W(\mathcal{L}_F) = \left\{ v \in \mathbb{R}^k \mid \exists u \in \mathbb{R}^{n-r} \text{ such that } F(u, v) = \frac{\partial F}{\partial u_1}(u, v) = \dots = \frac{\partial F}{\partial u_{n-r}}(u, v) = 0 \right\}.$$

We call it the *discriminant set* of  $F$ . In [9] we showed that the lightcone height function  $H$  is a Morse family of hypersurface and its discriminant set is the image of Gauss images  $\mathbb{L}^\pm(U)$ . Therefore we have a immersion germ  $\mathcal{L}^\pm : (\Sigma_*^\pm(H), (u_0, \mathbf{v}_0^\pm)) \longrightarrow PT^*(LC_\pm^*)$  defined by

$$\mathcal{L}^\pm(u) = \left( \mathbf{v}^\pm, \left[ \frac{\partial H}{\partial v_1}(u, \mathbf{v}^\pm) : \dots : \frac{\partial H}{\partial v_n}(u, \mathbf{v}^\pm) \right] \right),$$

where  $\mathbf{v}^\pm = \mathbb{L}^\pm(u)$  and  $\Sigma_*^\pm(H)$  is a singular set of  $H$ .

### 3 Spacelike submanifolds in de Sitter space

In this section, we consider the differential geometry of spacelike submanifolds in de Sitter space, which is analogous to [6].

Let  $r$  be an integer at least two and  $\mathbf{X} : U \longrightarrow S_1^n$  be an embedding from an open set  $U \subset \mathbb{R}^{n-r}$ . We say that  $\mathbf{X}$  is *spacelike* in  $S_1^n$  if every non zero vector generated by  $\{\mathbf{X}_{u_i}(u)\}_{i=1}^{n-r}$  is spacelike, where  $u \in U$  and  $\mathbf{X}_{u_i} = \partial\mathbf{X}/\partial u_i$ . We identify  $M = \mathbf{X}(U)$  with  $U$  through the embedding  $\mathbf{X}$  and call  $M$  a *spacelike submanifold of codimension  $r$*  in de Sitter space. Since  $\langle \mathbf{X}, \mathbf{X} \rangle \equiv 1$ , so that  $\langle \mathbf{X}_{u_i}, \mathbf{X} \rangle \equiv 0$  for  $i = 1, \dots, n-r$ . The tangent space of  $M$  at  $p = \mathbf{X}(u)$  is spanned by the vectors  $\mathbf{X}_{u_i}(u)$  for  $i = 1, \dots, n-r$ .

Let  $N_p M$  be the normal space of  $M$  at  $p$  in  $\mathbb{R}_1^{n+1}$  and we define  $N_p^*(M) = N_p M \cap T_p S_1^n$ . Let  $\mathbf{n}(u) : U \longrightarrow N_p^*(M)$  be a timelike unit normal of  $M$ . Since  $\langle \mathbf{n}, \mathbf{n} \rangle \equiv -1$  and  $\langle \mathbf{X}, \mathbf{n} \rangle \equiv 0$ ,  $\mathbf{n}_{u_i}$  is pseudo orthogonal to both of  $\mathbf{X}$  and  $\mathbf{n}$  for  $i = 1, \dots, n-r$ . Therefore we have  $\mathbf{n}_{u_i}(u) \in T_p M \oplus N_p^*(M)$ . Consider two pseudo orthonormal projections

$$\pi_p^t : T_p \mathbb{R}_1^{n+1} \longrightarrow T_p M, \quad \pi_p^n : T_p \mathbb{R}_1^{n+1} \longrightarrow N_p M.$$

Let  $d_u \mathbf{n}$  be the derivative of  $\mathbf{n}$  at  $u$ , under the identification of  $M$  and  $U$  through  $\mathbf{X}$ , we have the linear transformations on  $T_p M$

$$d_p \mathbf{n}^T = \pi_p^t \circ d_u \mathbf{n}, \quad d_p \mathbf{n}^N = \pi_p^n \circ d_u \mathbf{n}.$$

We respectively call the linear transformation  $A_p(\mathbf{n}) = -d_p \mathbf{n}^T$  and  $S_p(\mathbf{n}) = -(\text{id}_{T_p M} + d_p \mathbf{n}^T)$  an  *$\mathbf{n}$ -shape operator* and a *horospherical  $\mathbf{n}$ -shape operator* of  $M$  at  $p = \mathbf{X}(u)$ . We also call the linear map  $d_u \mathbf{n}^N$  a *normal connection* with respect to the timelike normal  $\mathbf{n}$  of  $M$ .

We denote eigenvalues of  $A_p(\mathbf{n})$  and  $S_p(\mathbf{n})$  by  $\kappa_p(\mathbf{n})$  and  $\bar{\kappa}_p(\mathbf{n})$ , which we respectively call an  *$\mathbf{n}$ -principal curvature* and a *horospherical  $\mathbf{n}$ -principal curvature*. The *horospherical Gauss-Kronecker curvature* with respect to  $\mathbf{n}$  at  $p = \mathbf{X}(u)$  is defined to be

$$K(\mathbf{n})(u) = \det S_p(\mathbf{n}).$$

We say that a point  $p_0 = \mathbf{X}(u_0)$  is an  *$\mathbf{n}$ -umbilic point* if  $\det S_{p_0}(\mathbf{n}) = \bar{\kappa}_{p_0}(\mathbf{n}) \text{id}_{T_{p_0} M}$ . Since the eigenvectors of  $\det S_{p_0}(\mathbf{n})$  and  $\det A_{p_0}(\mathbf{n})$  are the same, the above condition is equivalent to  $\det A_{p_0}(\mathbf{n}) = \kappa_{p_0}(\mathbf{n}) \text{id}_{T_{p_0} M}$ . We say that the spacelike submanifold  $M$  is *totally  $\mathbf{n}$ -umbilic* if every point on  $M$  is  $\mathbf{n}$ -umbilic. We also say that the timelike unit normal vector field  $\mathbf{n}$  is *parallel at  $p$*  if  $d_p \mathbf{n}^N = 0_{T_p M}$ . The timelike unit normal field  $\mathbf{n}$  is *parallel* if  $\mathbf{n}$  is parallel at any points on  $M$ . Then we have the following result which is analogous to ([6], Proposition 3.1).

**Proposition 3.1.** Let  $\mathbf{X} : U \longrightarrow S_1^n$  be a spacelike submanifold of codimension  $r \geq 2$ . Suppose that  $M = \mathbf{X}(U)$  is totally  $\mathbf{n}$ -umbilic and timelike unit normal vector field  $\mathbf{n}$  is parallel. Then  $\kappa_p(\mathbf{n}), \bar{\kappa}_p(\mathbf{n})$  are constant  $\kappa(\mathbf{n}), \bar{\kappa}(\mathbf{n})$  and there exists a vector  $v \in \mathbb{R}_1^{n+1}$  such that  $M$  is a part of hyperplanes  $HP(v, c) \cap S_1^n$  in de Sitter space for some vector  $v$  and real number  $c$ . Under this condition we have following cases:

- (1) If  $1 < |\bar{\kappa}(\mathbf{n}) + 1| = |\kappa(\mathbf{n})|$  then  $M$  is a part of a hyperbolic hyperquadric  $HP(v, +1)$ .

(2) If  $0 < |\bar{\kappa}(\mathbf{n}) + 1| = |\kappa(\mathbf{n})| < 1$  then  $M$  is a part of an elliptic hyperquadric  $HP(v, +1)$ .

(3) If  $\bar{\kappa}(\mathbf{n}) + 1 = \kappa(\mathbf{n}) = 0$  then  $M$  is a part of an elliptic hyperquadric  $HP(v, 0)$ .

(4) If  $\kappa(\mathbf{n}) = 1$  (namely  $\bar{\kappa}(\mathbf{n}) = 0$ ) then  $M$  is a part of a de Sitter hyperhorosphere  $HP(v, +1)$ .

*Proof.* By the assumption, we have  $d\mathbf{n}(u) \equiv \kappa_p \text{id}_{T_p M}$ . This means that  $\pi_p^T \circ \mathbf{n}_{u_i}(u) \equiv \kappa_p \mathbf{X}_{u_i}(u)$ . Since  $\mathbf{n}$  is parallel, we have  $\mathbf{n}_{u_i}(u) = \kappa_p \mathbf{X}_{u_i}(u)$ . So that  $\mathbf{n}_{u_i u_j}(u) = \kappa_{u_j, p} \mathbf{X}_{u_i}(u) + \kappa_p \mathbf{X}_{u_i u_j}(u)$  and  $\mathbf{n}_{u_j u_i}(u) = \kappa_{u_i, p} \mathbf{X}_{u_j}(u) + \kappa_p \mathbf{X}_{u_j u_i}(u)$ . It follows that  $\mathbf{X}_{u_i u_j} \equiv \mathbf{X}_{u_j u_i}$  and  $\mathbf{n}_{u_i u_j} \equiv \mathbf{n}_{u_j u_i}$ , then we have  $\kappa_{u_j, p} \mathbf{X}_{u_i}(u) = \kappa_{u_i, p} \mathbf{X}_{u_j}(u)$ . Since  $\mathbf{X}_i(u)$  and  $\mathbf{X}_j(u)$  are linearly independent, Therefore  $\kappa_{u_i, p} = \kappa_{u_j, p} = 0$ , this means that  $\kappa_p$  and  $\bar{\kappa}_p$  are constant  $\kappa, \bar{\kappa}$ .

We now assume that  $\bar{\kappa} + 1 = \kappa \neq 0$ . By the assumption, we have  $\mathbf{n}_{u_i}(u) = -\kappa \mathbf{X}_{u_i}(u)$ , so that there exists a constant vector  $v$  such that  $\mathbf{X}(u) = v - (1/\kappa)\mathbf{n}(u)$ . Then the vector  $v$  satisfies  $\langle v, v \rangle = 1 - 1/\kappa^2$  and  $\langle \mathbf{X}(u) - v, \mathbf{X}(u) - v \rangle = -1/\kappa^2$ , so that  $\langle \mathbf{X}(u), v \rangle = 1$  for any  $u \in U$ . This means that  $M$  is a part of hyperplanes in de Sitter space  $HP(v, +1)$ . Therefore we have (1), (2) and (4).

On the other hand, if  $\bar{\kappa} + 1 = \kappa = 0$  then there exists a constant timelike vector  $v$  such that  $\mathbf{n}(u) = v$  for any  $u \in U$ . So that  $\langle \mathbf{X}(u), v \rangle = \langle \mathbf{X}(u), \mathbf{n}(u) \rangle = 0$  for any  $u \in U$ . This means that  $M \subset HP(v, 0)$  Therefore (3) holds. This completes the proof.  $\square$

We now consider the following Weingarten type formula. Since  $\{\mathbf{X}_{u_i}\}_{i=1}^{n-r}$  spans a spacelike vector subspace, we induce a Riemannian metric (the *horospherical first fundamental form*) by  $ds^2 = \sum_{i,j=1}^{n-r} g_{ij} du_i du_j$  on  $M = \mathbf{X}(U)$ , where  $g_{ij} = \langle \mathbf{X}_{u_i}, \mathbf{X}_{u_j} \rangle$ . We respectively define the *second fundamental invariant* and *horospherical second fundamental invariant* with respect to the timelike unit normal vector field  $\mathbf{n}$  by  $h_{ij}(\mathbf{n}) = -\langle \mathbf{n}_{u_i}, \mathbf{X}_{u_j} \rangle$  and  $\bar{h}_{ij}(\mathbf{n}) = -\langle \mathbf{X}_{u_i} + \mathbf{n}_{u_i}, \mathbf{X}_{u_j} \rangle$ . We have the relation

$$\bar{h}_{ij}(\mathbf{n}) = -g_{ij} + h_{ij}(\mathbf{n}) \quad (\text{for } i, j = 1, \dots, n-r).$$

Under the above notations, we have the following *Weingarten type formula* with respect to the timelike unit normal vector field  $\mathbf{n}$ , which is analogous to ([6], Proposition 3.2).

$$\pi^T \circ (\mathbf{X} + \mathbf{n})_{u_i} = - \sum_{k=1}^{n-r} \bar{h}_i^j(\mathbf{n}) \mathbf{X}_{u_j},$$

where  $(\bar{h}_i^j(\mathbf{n}))_{ij} = (\bar{h}_{ik}(\mathbf{n}))_{ik} (g^{kj})_{kj}$  and  $(g^{kj}) = (g_{kj})^{-1}$ . Therefore, the Gauss-Kronecker curvature with respect to  $\mathbf{n}$  is given by

$$K(\mathbf{n}) = \det(\bar{h}_{ik}(\mathbf{n})) / \det(g_{kj}).$$

Since  $\langle \mathbf{X} + \mathbf{n}, \mathbf{X}_{u_j} \rangle \equiv 0$ , the coefficient of the second fundamental form with respect to the timelike parallel unit normal vector field  $\mathbf{n}$  is expressed by

$$\begin{aligned} \bar{h}_{ij}(\mathbf{n}) &= -\langle \mathbf{X}_{u_i} + \mathbf{n}_{u_i}, \mathbf{X}_{u_j} \rangle \\ &= -\partial \langle \mathbf{X} + \mathbf{n}, \mathbf{X}_{u_j} \rangle / \partial u_i + \langle \mathbf{X} + \mathbf{n}, \mathbf{X}_{u_i u_j} \rangle \\ &= \langle \mathbf{X} + \mathbf{n}, \mathbf{X}_{u_i u_j} \rangle. \end{aligned}$$

Therefore the horospherical second fundamental invariant at a point  $p_0 = \mathbf{X}(u_0)$  depends only on the timelike vector  $\mathbf{n}_0 = \mathbf{n}(u_0)$ . It is independent of the choice of timelike parallel unit normal vector field  $\mathbf{n}$  with  $\mathbf{n}_0 = \mathbf{n}(u_0)$ .

Let  $\mathbf{n}_0$  be a timelike unit normal vector, we say that a point  $p_0 = \mathbf{X}(u_0)$  is an  $\mathbf{n}_0$ -parabolic point (resp.  $\mathbf{n}_0$ -umbilic point) of  $M$  if  $K(\mathbf{n})(u_0) = 0$  ( $S_{p_0}(\mathbf{n}) = \bar{\kappa}_{p_0}(\mathbf{n})\text{id}_{T_{p_0}M}$ ) for some timelike parallel unit normal vector field  $\mathbf{n}$  with  $\mathbf{n}(u_0) = \mathbf{n}_0$ . We also say that  $p_0$  is an  $\mathbf{n}_0$ -horospherical point if it is an  $\mathbf{n}_0$ -parabolic point and an  $\mathbf{n}_0$ -umbilic point.

## 4 Horospherical hypersurfaces and horospherical height functions

In this section we introduce the notions of horospherical height function and horospherical hypersurface.

Let  $\mathbf{X} : U \rightarrow S_1^n$  be a spacelike submanifolds of codimension  $r \geq 2$  in de Sitter space and  $p = \mathbf{X}(u)$ , we choose unit orthonormal sections

$$N_p(M) = \langle \mathbf{X}(u), \mathbf{n}_0(u), \mathbf{n}_1(u), \dots, \mathbf{n}_{r-1}(u) \rangle,$$

where  $\mathbf{n}_0(u)$  is a timelike unit vector and  $\mathbf{n}_i(u)$  for  $i = 1, \dots, r-1$  are spacelike unit vectors. We define a map  $\mathbf{e} : U \times H^{r-1}(-1) \rightarrow H^{r-1}(-1)$  by

$$\mathbf{e}(u, \mu) = \mu_0 \mathbf{n}_0(u) + \sum_{i=1}^{r-1} \mu_i \mathbf{n}_i(u),$$

where  $\mu = (\mu_0, \dots, \mu_{r-1})$ . Let  $\theta$  be a fixed real number, we also define a map  $\bar{\mathbf{X}}_\theta : U \times H^{r-1}(-1) \rightarrow S_1^n$  by

$$\bar{\mathbf{X}}_\theta(u, \mu) = \cosh \theta \mathbf{X}(u) + \sinh \theta \mathbf{e}(u, \mu).$$

We remark that for any spacelike submanifold  $\mathbf{X}$  and point  $(u_0, \mu_0) \in U \times H^{r-1}(-1)$ , there are a real number  $\theta \neq 0$  and an open neighborhood  $V$  of  $(u_0, \mu_0)$  such that  $\bar{\mathbf{X}}_\theta$  is spacelike embedding on  $V$ . We assume that for any  $(u, \mu) \in V$  then  $(u, -\mu) \in V$ . We write  $CM$  as an image  $\bar{\mathbf{X}}_\theta(V)$  and call it a *spacelike canal hypersurface* of  $M = \mathbf{X}(U)$ . Izumiya, Pei, Romero Fuster, Takahashi [6] has introduced the notion of canal surfaces of submanifolds in the hyperbolic space.

We now consider the horospherical height function on a spacelike submanifold. For a spacelike submanifolds  $\mathbf{X}$  of codimension  $r$ , we define a function

$$H : U \times LC^* \rightarrow \mathbb{R}$$

by  $H(u, v) = \langle \mathbf{X}(u), v \rangle - 1$ , and we call  $H$  a *horospherical height function on  $M$* . For  $v_0 \in LC^*$  we denote  $h_{v_0}(u) = \langle \mathbf{X}(u), v_0 \rangle - 1$ . We have the following proposition which is analogous to ([6], Proposition 3.4).



**Proposition 4.1.** Let  $H : U \times LC^* \longrightarrow \mathbb{R}$  be a horospherical height function of a spacelike submanifold  $X : U \longrightarrow S_1^n$  of codimension  $r$ . Then  $H(u, v) = \partial H(u, v)/\partial u_i = 0$  for  $i = 1, \dots, n - r$  if and only if  $v = \mathbf{X}(u) + \mathbf{e}(u, \mu)$  for some  $\mu \in H^{r-1}(-1)$ .

The proof for above proposition are parallel to those of Proposition 3.4 in [6], so that we omit it. Therefore the discriminant set of the horospherical height function  $H$  is

$$D_H = \{ \mathbf{X}(u) + \mathbf{e}(u, \mu) \mid (u, \mu) \in U \times H^{r-1}(-1) \}.$$

We define a map  $HS_{\mathbf{X}} : U \times H^{r-1}(-1) \longrightarrow LC^*$  by

$$HS_{\mathbf{X}}(u, \mu) = \mathbf{X}(u) + \mathbf{e}(u, \mu),$$

which we call a *horospherical hypersurface of  $M$* . We remark that  $HS_{\mathbf{X}}$  depends on the choice of the orthonormal frames of  $N(M)$ . Let  $\{\mathbf{X}, \mathbf{n}_0, \dots, \mathbf{n}_{r-1}\}$  and  $\{\mathbf{X}, \mathbf{n}'_0, \dots, \mathbf{n}'_{r-1}\}$  be two orthonormal frames of  $N(M)$  with  $\mathbf{n}_0, \mathbf{n}'_0 \in H^{r-1}_+(-1)$ . Then we have  $\mathbf{n}_i = \sum_{j=0}^{r-1} \lambda_i^j \mathbf{n}'_j$ , where

$$\lambda_i^j(u) = \begin{cases} -\langle \mathbf{n}_i, \mathbf{n}'_j \rangle & \text{if } j = 0 \\ \langle \mathbf{n}_i, \mathbf{n}'_j \rangle & \text{if } j = 1, \dots, r-1. \end{cases}$$

So that we have a diffeomorphism  $\Phi : U \times H^{r-1}(-1) \longrightarrow U \times H^{r-1}(-1)$  defined by

$$\Phi(u, \mu) = \left( u, \left( \sum_{i=0}^{r-1} \lambda_i^0(u) \mu_i, \dots, \sum_{i=0}^{r-1} \lambda_i^{r-1}(u) \mu_i \right) \right).$$

We also define  $\mathbf{e}'(u, \mu) = \sum_{i=0}^{r-1} \mu_i \mathbf{n}'_i(u)$ . It follows from the above definition that  $\mathbf{e}(u, \mu) = \mathbf{e}' \circ \Phi(u, \mu)$  Therefore we have

$$HS_{\mathbf{X}}(u, \mu) = HS'_{\mathbf{X}} \circ \Phi(u, \mu),$$

where  $HS'_{\mathbf{X}} = \mathbf{X}(u) + \mathbf{e}'(u, \mu)$ . This means that  $HS_{\mathbf{X}}$  is independent to the choice of orthonormal frames of  $N(M)$  up to the diffeomorphic parametrization. We have a following proposition which is analogous to ([6], Proposition 3.5).

**Proposition 4.2.** Let  $\mathbf{X} : U \longrightarrow S_1^n$  be a spacelike hypersurface of codimension  $r \geq 2$  in de Sitter space, then  $HS_{\mathbf{X}}(u, \mu) = \mathbf{X}(u) + \mathbf{e}(u, \mu)$  is constant map for some smooth map  $\mu : U \longrightarrow H^{r-1}(-1)$  if and only if  $M$  is a part of de Sitter hyperhorosphere  $HP(v, 1) \cap S_1^n$ . By Proposition 3.1, If  $M$  is totally  $\mathbf{e}(u, \mu(u))$ -umbilic for some parallel normal vector field  $\mathbf{e}(u, \mu(u))$  and  $K(\mathbf{e}(u, \mu(u)))(u) = 0$ , then the above assertion holds.

*Proof.* Suppose that  $v_0 = \mathbf{X}(u) + \mathbf{e}(u, \mu)$  is a constant vector. Since  $\mathbf{e}(u, \mu)$  is pseudo orthogonal to  $\mathbf{X}(u)$ , then we have  $\langle \mathbf{X}(u), v_0 \rangle = +1$  for any  $u \in U$ . This means that  $M$  is a part of a de Sitter hyperhorosphere  $HP(v_0, 1) \cap S_1^n$ . On the other hand, if  $M \subset HP(v_0, 1) \cap S_1^n$  for some lightlike vector, then  $\langle v_0 - \mathbf{X}(u), \mathbf{X}(u) \rangle = 0$  for any  $u \in U$ . Since  $\mathbf{X}(u)$  is pseudo orthogonal to

$\mathbf{X}_{u_i}(u)$ , it follows that  $\langle v_0 - \mathbf{X}(u), \mathbf{X}_{u_i}(u) \rangle = 0$ . This means that  $\mathbf{X}(u) - v_0$  is a normal vector of  $M$  at  $p = \mathbf{X}(u)$ . We define a function  $\mu(u)$  by

$$\mu(u) = -\langle \mathbf{X}(u) - v_0, \mathbf{n}_0(u) \rangle \mathbf{n}_0(u) + \sum_{i=1}^{r-1} \langle \mathbf{X}(u) - v_0, \mathbf{n}_i(u) \rangle \mathbf{n}_i(u).$$

Then we have  $v_0 - \mathbf{X}(u) = \mathbf{e}(u, \mu)$ . This completes the proof.  $\square$

Since the image of  $HS_{\mathbf{X}}$  is the discriminant set of the horospherical height function  $H$  on  $M$ , the singular set of  $HS_{\mathbf{X}}$  corresponds to the null set of the Hessian matrix of the horospherical height function with the fixed parameter  $v$  at each point. Therefore we have the following proposition which is analogous to ([6], Proposition 3.6).

**Proposition 4.3.** The singular set of  $HS_{\mathbf{X}}$  is given by

$$\Sigma(HS_{\mathbf{X}}) = \{(u, \mu) \in U \times H^{r-1}(-1) \mid K(\mathbf{e}(u, \mu))(u) = 0\}.$$

*Proof.* Let  $h_v(u)$  be a horospherical height function with  $v \in LC^*$ , then we have  $\text{Hess } h_v(u) = \langle \mathbf{X}_{u_i u_j}(u), v \rangle$ . Suppose that  $(u, v) \in \Sigma_*(H)$ , then  $v = \mathbf{X}(u) + \mathbf{e}(u, \mu)$  for some  $\mu \in H^{r-1}(-1)$ . We recall that  $\bar{h}_{ij}(v)(u) = \langle \mathbf{X}_{u_i u_j}(u), \mathbf{X}(u) + \mathbf{e}(u, \mu) \rangle$ , where  $(\bar{h}_{ij}(v)(u_0))$  is the horospherical second fundamental invariant with respect to the timelike direction  $\mathbf{e}(u_0, \mu_0)$ . The horospherical Gauss-Kronecker curvature is  $K(\mathbf{e}(u, \mu))(u) = \det(\langle \mathbf{X}_{u_i u_j}(u), \mathbf{X}(u) + \mathbf{e}(u) \rangle) / \det(g_{ij}(u)) = \det \text{Hess } h_v(u) / \det(g_{ij}(u))$ , where  $(g_{ij}(u))$  is the first fundamental invariant of  $M$ . Therefore  $\text{Hess } h_v(u) = 0$  if and only if  $K(\mathbf{e}(u, \mu))(u) = 0$ . This completes the proof.  $\square$

Therefore the singular set of  $HS_{\mathbf{X}}$  corresponds to the parabolic set of  $M$  with respect to some timelike parallel normal vector field  $\mathbf{e}(u, \mu)$ . By the proof of above proposition, we have  $\text{rank Hess } h_{v_0}(u_0) = \text{rank}(\bar{h}_{ij}(u_0))_{ij}$ . Therefore we also have the following proposition which is analogous to ([6], Proposition 3.7).

**Proposition 4.4.** For any spacelike submanifold  $\mathbf{X}$  of codimension  $r \geq 2$  and lightlike vector  $v_0 = \mathbf{X}(u_0) + \mathbf{e}(u_0, \mu_0)$ , we have the following assertions.

- (1) A point  $p_0 = \mathbf{X}(u_0)$  is an  $\mathbf{e}(u_0, \mu_0)$ -parabolic point if and only if  $\det \text{Hess } h_{v_0}(u_0) = 0$ .
- (2) A point  $p_0$  is an  $\mathbf{e}(u_0, \mu_0)$ -horospherical point if and only if  $\text{rank Hess } h_{v_0}(u_0) = 0$ .

Here  $\text{Hess } h_{v_0}(u_0)$  is a Hessian matrix of  $h_{v_0}(u)$  at  $u = u_0$ .

We now consider the lightcone height function and the lightcone Gauss image of spacelike canal hypersurface  $\bar{\mathbf{X}}_{\theta} : V \longrightarrow S_1^n$  with  $V \subset U \times H^{r-1}(-1)$ . The lightcone height function  $\bar{H} : V \times LC^* \longrightarrow \mathbb{R}$  of the spacelike hypersurface  $\bar{\mathbf{X}}_{\theta}$  is

$$\bar{H}((u, \mu), v) = \langle \bar{\mathbf{X}}_{\theta}(u, \mu), v \rangle - 1.$$

We denote  $\bar{h}_v(u) = \bar{H}((u, \mu), v)$  for any  $v \in LC^*$ . Now we define a map  $\bar{\mathbf{e}} : V \longrightarrow H^{n-1}(-1)$  by  $\bar{\mathbf{e}}(u, \mu) = \sinh \theta \mathbf{X}(u) + \cosh \theta \mathbf{e}(u, \mu)$ . Then we have  $\langle \bar{\mathbf{e}}(u, \mu), \bar{\mathbf{X}}_{\theta}(u) \rangle = \langle \bar{\mathbf{e}}(u, \mu), \bar{\mathbf{X}}_{\theta, u_i}(u) \rangle = 0$

for any  $(u, \mu) \in V$  and  $i = 1, \dots, n - r$ . Therefore  $\bar{\mathbf{e}}$  is a lightlike normal of  $CM$ . The positive lightcone Gauss image  $\mathbb{L}_{CM} : V \longrightarrow LC^*$  is defined by

$$\mathbb{L}_{CM}(u, \mu) = \bar{\mathbf{X}}_\theta(u) + \bar{\mathbf{e}}(u, \mu) = (\cosh \theta + \sinh \theta)(\mathbf{X}(u) + \mathbf{e}(u, \mu)).$$

By Proposition 2.2,  $\bar{H}((u, \mu), v) = \bar{H}_{u_i}((u, \mu), v) = \bar{H}_{\mu_j}((u, \mu), v) = 0$  for  $i = 1, \dots, n - r$  and  $j = 0, \dots, r - 1$  if and only if  $v = \bar{\mathbf{X}}_\theta(u) \pm \bar{\mathbf{e}}(u, \mu) = e^{\pm\theta}(\mathbf{X}(u) + \mathbf{e}(u, \pm\mu))$ . By assumption,  $(u, -\mu)$  is also an element of  $V$ . Therefore the discriminant set of the lightcone height function  $\bar{H}$  is

$$D(\bar{H}) = \{e^{\pm\theta}(\mathbf{X}(u) + \mathbf{e}(u, \mu)) \mid (u, \mu) \in V\}.$$

We now define a diffeomorphism

$$\mathcal{M}_c : LC^* \longrightarrow LC^*$$

given by  $\mathcal{M}_c(v) = cv$  for a fixed positive real number  $c$ . Then we have the following lemma, which is analogous to ([6], Proposition 3.9).

**Lemma 4.5.** Under the above notations, we have

$$\mathcal{M}_c \circ HS_{\mathbf{X}}(u, \mu) = \mathbb{L}_{CM}(u, \mu)$$

on  $V \subset U \times H^{r-1}(-1)$ , where  $c = e^{\pm\theta}$ .

By the above lemma, the horospherical hypersurface  $HS_{\mathbf{X}}$  is locally diffeomorphic to the lightcone Gauss image of the spacelike canal hypersurface  $\bar{\mathbf{X}}_\theta$ .

## 5 Horospherical hypersurfaces as wave fronts

In this section we naturally interpret the horospherical hypersurfaces of  $M$  as a wave front set of the horospherical height functions in the theory of Legendrian singularities.

By proceeding arguments in §2, the horospherical hypersurface  $HS_{\mathbf{X}}$  is the discriminant set of the horospherical height function  $H$ , and the singular point set of the horospherical hypersurface is the horospherical point set. We have the following proposition which is analogous to ([6], Proposition 4.1).

**Proposition 5.1.** Let  $\mathbf{X} : U \longrightarrow S_1^n$  be a spacelike submanifold of codimension  $r \geq 2$  and  $H : U \times LC^* \longrightarrow \mathbb{R}$  be a horospherical height function of  $M$ . Then  $H$  is a Morse family.

*Proof.* We denote  $\mathbf{X}(u) = (X_0(u), \dots, X_n(u))$  and  $\mathbf{X}_{u_i}(u) = (X_{0,u_i}(u), \dots, X_{n,u_i}(u))$ . For any  $v = (v_0, \dots, v_n) \in LC^*$ , we have  $v_0 \neq 0$ . Without loss of generality, we assume that  $v_0 = \sqrt{v_1^2 + \dots + v_n^2} > 0$ , so that we have

$$H(u, v) = \langle \mathbf{X}(u), v \rangle - 1 = -X_0 v_0 + \sum_{i=1}^n X_n v_n - 1.$$

We now prove a map

$$\Delta^* H = \left( H, \frac{\partial H}{\partial u_1}, \dots, \frac{\partial H}{\partial u_{n-r}} \right)$$

is non singular at any  $(u, v) \in \Sigma_*(H)$ . The Jacobian matrix of  $\Delta^* H$  is

$$J\Delta^* H(u, v) = \left( \begin{array}{c|c} * & \frac{\partial H}{\partial v_j}(u, v)_{j=1, \dots, n} \\ \hline * & \left( \frac{\partial^2 H}{\partial u_i \partial v_j}(u, v) \right)_{\substack{j=1, \dots, n \\ i=1, \dots, n-r}} \end{array} \right).$$

We denote an  $(n-r+1) \times n$  matrix  $B$  by  $J\Delta^* H = (* | B)$ . It is sufficient to show that  $\text{rank } B = n-r+1$  at  $(u, v) \in \Sigma_*(H)$ . We also denote an  $(n-r+3) \times (n+1)$  matrix  $C$  by

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ v_0 & v_1 & \cdots & v_n \\ X_0 & X_1 & \cdots & X_n \\ X_{0,u_1} & X_{1,u_1} & \cdots & X_{n,u_1} \\ \vdots & \vdots & \ddots & \vdots \\ X_{0,u_{n-r}} & X_{1,u_{n-r}} & \cdots & X_{n,u_{n-r}} \end{pmatrix}.$$

We now show that the rank of the matrix  $C$  is equal to  $n-r+3$ . Since  $v, \mathbf{X}(u)$  and  $\mathbf{X}_{u_i}(u)$  are linearly independent for all  $(u, v) \in \Sigma_*(H)$ , it is sufficient to show that timelike unit vector  $e = (1, 0, \dots, 0)$  can not be written by a linear combination of  $v, \mathbf{X}(u)$  and  $\mathbf{X}_{u_i}(u)$ . If that is not so, there exists some real numbers  $\eta, \mu, \xi_i$  such that  $e = \eta v + \mu \mathbf{X}(u) + w$  and  $w = \sum_{i=1}^{n-r} \xi_i \mathbf{X}_{u_i}(u)$ . Then we have  $\langle e, e \rangle = \mu^2 + \langle w, w \rangle$ . However,  $w$  is a spacelike vector, so that  $\langle e, e \rangle$  would not be negative, which contradicts our assumption. This means that  $e, v, \mathbf{X}(u)$  and  $\mathbf{X}_{u_i}(u)$  are linearly independent, therefore we have  $\text{rank } C = n-r+3$ .

We now show  $\text{rank } B = \text{rank } C' - 2$ . We subtract the second row multiplied by  $X_0/v_0$  from the third row of the matrix  $C$ , and add the second row multiplied by  $X_{0,u_k}(u)/v_0$  from the  $(3+k)$ -th row for  $k = 1, \dots, n-r$ . Then we have a matrix

$$C' = \left( \begin{array}{c|c} 1 & 0 \cdots 0 \\ v_0 & v_1 \cdots v_n \\ \hline 0 & B \\ \vdots & \\ 0 & \end{array} \right).$$

Therefore we have  $\text{rank } B = \text{rank } C' - 2 = n-r+1$ . This completes the proof.  $\square$

Since  $H$  is a Morse family of hypersurfaces, we have the Legendrian immersion germ  $\mathcal{L}_H : (\Sigma_*(H), (u_0, v_0)) \rightarrow PT^*(LC^*)$  defined by

$$\mathcal{L}_H(u, v) = \left( v, \left[ \frac{\partial H}{\partial u_1}(u, v) : \dots : \frac{\partial H}{\partial u_{n-r}}(u, v) \right] \right).$$

We remark that the wave front set of the Legendrian immersion germ  $\mathcal{L}_H$  is the horospherical hypersurfaces  $HS_{\mathbf{X}}$  of  $M$ . On the other hand, we define a contact diffeomorphism  $\tilde{\mathcal{M}}_c : PT^*(LC^*) \longrightarrow PT^*(LC^*)$  by

$$\tilde{\mathcal{M}}_c(\mathbf{v}, [\xi]) = (c\mathbf{v}, [\xi]),$$

where  $c$  is a fixed real parameter with  $c \neq 0$ . By definition, we have the following theorem.

**Theorem 5.2.** For a spacelike submanifold  $\mathbf{X} : U \longrightarrow S_1^n$ , we have

$$\tilde{\mathcal{M}}_c \circ \mathcal{L}_H = \mathcal{L}_{\bar{H}},$$

where  $c = e^{\pm\theta}$  and  $\mathcal{L}_{\bar{H}}$  is a Legendrian lift of the lightcone Gauss image  $\mathbb{L}_{CM}$  of the spacelike canal hypersurface of  $M$ .

By the above theorem, the Legendrian lift of the lightcone Gauss image  $\mathbb{L}_{CM}$  is  $\mathcal{A}$ -equivalent to the Legendrian lift of the horospherical hypersurface  $HS_{\mathbf{X}}$  of  $M$ .

## 6 Contact with de Sitter hyperhorospheres

In this section we use the theory of contacts between the spacelike submanifolds and the de Sitter hyperhorospheres due to Montaldi [10].

Let  $X_i$  and  $Y_i$  ( $i = 1, 2$ ) be submanifolds of  $\mathbb{R}^n$  with  $\dim X_1 = \dim X_2$ ,  $\dim Y_1 = \dim Y_2$  and  $y_i \in X_i \cap Y_i$  for  $i = 1, 2$ . We say that *the contact* of  $X_1$  and  $Y_1$  at  $y_1$  is *the same type* as *the contact* of  $X_2$  and  $Y_2$  at  $y_2$  if there is a diffeomorphism germ  $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$  such that  $\Phi((X_1, y_1)) = (X_2, y_2)$  and  $\Phi((Y_1, y_1)) = (Y_2, y_2)$ . In this case we write  $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ . Two function germs  $g_1, g_2 : (\mathbb{R}^n, a_i) \longrightarrow (\mathbb{R}, 0)$  ( $i = 1, 2$ ) are  $\mathcal{K}$ -equivalent if there are a diffeomorphism germ  $\Phi : (\mathbb{R}^n, a_1) \longrightarrow (\mathbb{R}^n, a_2)$  and a function germ  $\lambda : (\mathbb{R}^n, a_1) \longrightarrow \mathbb{R}$  with  $\lambda(a_1) \neq 0$  such that  $f_1 = \lambda \cdot (g_2 \circ \Phi)$ . In [10] Montaldi has shown the following theorem.

**Theorem 6.1.** ([10]) Let  $X_i$  and  $Y_i$  ( $i = 1, 2$ ) be submanifolds of  $\mathbb{R}^n$  with  $\dim X_1 = \dim X_2$ ,  $\dim Y_1 = \dim Y_2$  and  $y_i \in X_i \cap Y_i$  for  $i = 1, 2$ . Let  $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$  be immersion germs and  $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}, 0)$  be submersion germs with  $(Y_i, y_i) = (f_i^{-1}(0), y_i)$ . Then  $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$  if and only if  $f_1 \circ g_1$  and  $f_2 \circ g_2$  are  $\mathcal{K}$ -equivalent.

We now consider the function  $\mathcal{H} : S_1^n \times LC^* \longrightarrow \mathbb{R}$  defined by  $\mathcal{H}(x, v) = \langle x, v \rangle - 1$ . Given  $v_0 \in LC^*$ , we denote  $\mathfrak{h}_{v_0}(x) = \mathcal{H}(x, v_0)$ , so that we have  $\mathfrak{h}_{v_0}^{-1}(0) = HP(v_0, +1) \cap S_1^n$ . Let  $\mathbf{X} : U \longrightarrow \mathbb{R}_1^n$  be a spacelike submanifold of codimension  $r \geq 2$ . For any  $u_0 \in U$  and  $\mu_0 \in H^{r-1}(-1)$ , we take a point  $v_0 = \mathbf{X}(u_0) + \mathbf{e}(u_0, \mu_0)$ . By Proposition 4.1, we have

$$\begin{aligned} (\mathfrak{h}_{v_0} \circ \mathbf{X})(u_0) &= \mathcal{H} \circ (\mathbf{X} \times \text{id}_{LC^*})(u_0, v_0) = H(u_0, v_0) = 0. \\ \frac{\partial(\mathfrak{h}_{v_0} \circ \mathbf{X})}{\partial u_i}(u_0) &= \frac{\partial H}{\partial u_i}(u_0, \mathbf{X}(u_0) + \mathbf{e}(u_0, \mu_0)) = 0. \end{aligned}$$

It follows that the de Sitter hyperhorosphere  $\mathfrak{h}_{v_0}^{-1}(0) = HP(v_0, +1) \cap S_1^n$  is tangent to  $M$  at  $p_0 = \mathbf{X}(u_0)$ . In this case we call  $HP(v_0, +1) \cap S_1^n$  a *tangent de Sitter hyperhorosphere* (briefly,

tangent hyperhorosphere) with respect to  $\mathbf{X}(u_0) + \mathbf{e}(u_0, \mu_0)$ . We may also consider the contacts of the spacelike canal surface  $CM = \bar{\mathbf{X}}(V)$  and the de Sitter horospheres. (see [7])

We now review some notions of Legendrian singularity theory to study the contact between hypersurfaces and de Sitter hyperhorospheres. We say that Legendrian immersion germs  $\iota_i : (U_i, u_i) \longrightarrow (PT^*\mathbb{R}^n, p_i)$  ( $i = 1, 2$ ) are *Legendrian equivalent* if there are a contact diffeomorphism germ  $H : (PT^*\mathbb{R}^n, p_1) \longrightarrow (PT^*\mathbb{R}^n, p_2)$  and a diffeomorphic germ  $\tau : (U_1, u_1) \longrightarrow (U_2, u_2)$  such that  $H$  preserves fibers of  $\pi$  and  $H \circ \iota_1 = \iota_2 \circ \tau$ . A Legendrian immersion germ at a point is said to be *Legendrian stable* if for every map with the given germ there are a neighborhood in the space of Legendrian immersions (in the Whitney  $C^\infty$ -topology) and a neighborhood of the original point such that each Legendrian map belonging to the first neighborhood has in the second neighborhood a point at which its germ is Legendrian equivalent to the original germ. (see [1])

**Proposition 6.2.** ([13]) Let  $i_1, i_2$  be Legendrian immersion germs such that regular sets of  $\pi \circ i_1$  and  $\pi \circ i_2$  are respectively dense. Then  $i_1, i_2$  are Legendrian equivalent if and only if corresponding wave front sets  $W(i_1)$  and  $W(i_2)$  are diffeomorphic as set germs.

Let  $F_i : (\mathbb{R}^n \times \mathbb{R}^k, (a_i, b_i)) \longrightarrow (\mathbb{R}, c)$  ( $k = 1, 2$ ) be  $k$ -parameter unfoldings of function germs  $f_i$ , we say  $F_1$  and  $F_2$  are  $\mathcal{P}$ - $\mathcal{K}$ -equivalent if there exists a diffeomorphism germ  $\Phi : (\mathbb{R}^n \times \mathbb{R}^k, (a_1, b_1)) \longrightarrow (\mathbb{R}^n \times \mathbb{R}^k, (a_2, b_2))$  of the form  $\Phi(u, x) = (\phi_1(u, x), \phi_2(x))$  for  $(u, x) \in \mathbb{R}^n \times \mathbb{R}^k$  and a function germ  $\lambda : (\mathbb{R}^n \times \mathbb{R}^k, (a_1, b_1)) \longrightarrow \mathbb{R}$  such that  $\lambda(a_1, b_1) \neq 0$  and  $F_1(u, x) = \lambda(u, x) \cdot (F_2 \circ \Phi)(u, x)$ .

**Theorem 6.3.** ([1, 12]) Let  $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$  be Morse families and denote the corresponding Legendrian immersion germs by  $\mathcal{L}_F, \mathcal{L}_G$ . Then

- (1)  $\mathcal{L}_F$  and  $\mathcal{L}_G$  are Legendrian equivalent if and only if  $F$  and  $G$  are  $\mathcal{P}$ - $\mathcal{K}$ -equivalent.
- (2)  $\mathcal{L}_F$  is Legendrian stable if and only if  $F$  is  $\mathcal{K}$ -versal deformation of  $f$ .

Let  $G_i : (\mathbb{R}^m, a_i) \longrightarrow (\mathbb{R}^n, b_i)$  (for  $i = 1, 2$ ) be map germs. We say that  $G_1$  and  $G_2$  are  $\mathcal{A}$ -equivalent if and only if there exist diffeomorphism germs  $\phi : (\mathbb{R}^m, a_1) \longrightarrow (\mathbb{R}^m, a_2)$  and  $\Phi : (\mathbb{R}^n, b_1) \longrightarrow (\mathbb{R}^n, b_2)$  such that  $\Phi \circ G_1 = G_2 \circ \phi$ .

We denote  $h_{i, v_i} : (U, u_i) \longrightarrow (\mathbb{R}, \mathbf{0})$  ( $i = 1, 2$ ) by  $h_{i, v_i}(u) = H_i(u, v_i)$ . Then we have  $h_{i, v_i}(u) = (\mathfrak{h}_{i, v_i} \circ \mathbf{X}_i)(u)$ . By Theorem 6.1,

$$K(\mathbf{X}_1(U), HP(v_1, 1) \cap S_1^n; p_1) = K(\mathbf{X}_2(U), HP(v_2, 1) \cap S_1^n; p_2)$$

if and only if  $h_{1, v_1}$  and  $h_{2, v_2}$  are  $\mathcal{K}$ -equivalent.

Let  $Q(\mathbf{X}, u_0)$  be the local ring of the horospherical height function germ  $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$  defined by

$$Q(\mathbf{X}, u_0; \mu_0) = C_{u_0}^\infty(U) / \langle h_{v_0} \rangle_{C_{u_0}^\infty(U)},$$

where  $v_0 = \mathbf{X}(u_0) + \mathbf{e}(u_0, \mu_0)$ ,  $\mu_0 \in H^{r-1}(-1)$  and  $C_{u_0}^\infty(U)$  is the local ring of function germs at  $u_0$  with the unique maximal ideal  $\mathfrak{M}$ . We also denote  $Q(\bar{\mathbf{X}}_\theta, (u_0, \mu_0))$  as the local ring of the lightcone height function germ  $\bar{h}_{v'_0} : (U \times H^{r-1}(-1), (u_0, \mu_0)) \longrightarrow (\mathbb{R}, 0)$  of the canal hypersurface  $\bar{\mathbf{X}}_\theta$ , where  $v'_0 = \mathbb{L}_{CM}(u_0, \mu_0)$ .

**Proposition 6.4.** ([3], Proposition A.4) Let  $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$  be Morse families. Suppose that Legendrian immersion germs  $\mathcal{L}_F$  and  $\mathcal{L}_G$  are Legendrian stable, then the following conditions are equivalent:

- (1)  $(W(\mathcal{L}_F), \lambda)$  and  $(W(\mathcal{L}_G), \lambda')$  are diffeomorphic as set germs.
- (2)  $\mathcal{L}_F$  and  $\mathcal{L}_G$  are Legendrian equivalent.
- (3)  $Q(f)$  and  $Q(g)$  are isomorphic as  $\mathbb{R}$ -algebras, where  $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$  and  $g = G|_{\mathbb{R}^k \times \{\mathbf{0}\}}$ .

By the above arguments, we have following theorem.

**Theorem 6.5.** Let  $\mathbf{X}_i : (U_i, u_i) \longrightarrow (S_1^n, p_i)$  ( $i = 1, 2$ ) be spacelike submanifold germs of codimension at least two in de Sitter space. For  $\mu_i \in H^{r-1}(-1)$  ( $i = 1, 2$ ), we denote  $v_i = HS_i(u_i, \mu_i)$ ,  $v'_i = \mathbb{L}_{CM_i}(u_i, \mu_i)$ ,  $h_{i,v_i} = H_i|_{U \times \{v_i\}}$ ,  $\bar{h}_{i,v'_i} = \bar{H}_i|_{U \times \{v'_i\}}$  and  $p'_i = \bar{\mathbf{X}}_{i,\theta_i}(u_i)$ . If the corresponding Legendrian immersion germs  $\mathcal{L}_{H_i}$  are Legendrian stable, then the following conditions are equivalent:

- (1) Horospherical hypersurface germs  $HS_{\mathbf{X}_1}$  and  $HS_{\mathbf{X}_2}$  are  $\mathcal{A}$ -equivalent.
- (2) Legendrian immersion germs  $\mathcal{L}_{H_1}$  and  $\mathcal{L}_{H_2}$  are Legendrian equivalent.
- (3) Horospherical height function germs  $H_1$  and  $H_2$  are  $\mathcal{P}$ - $\mathcal{K}$ -equivalent.
- (4)  $h_{1,v_1}$  and  $h_{2,v_2}$  are  $\mathcal{K}$ -equivalent.
- (5)  $K(\mathbf{X}_1(U), HP(v_1, 1) \cap S_1^n; p_1) = K(\mathbf{X}_2(U), HP(v_2, 1) \cap S_1^n; p_2)$ .
- (6)  $Q(\mathbf{X}_1, u_1)$  and  $Q(\mathbf{X}_2, u_2)$  are isomorphic as  $\mathbb{R}$ -algebras.
- (7) Lightcone Gauss image germs  $\mathbb{L}_{CM_1}$  and  $\mathbb{L}_{CM_2}$  are  $\mathcal{A}$ -equivalent.
- (8) Legendrian immersion germs  $\mathcal{L}_{\bar{H}_1}$  and  $\mathcal{L}_{\bar{H}_2}$  are Legendrian equivalent.
- (9) Lightcone height function germs  $\bar{H}_1$  and  $\bar{H}_2$  are  $\mathcal{P}$ - $\mathcal{K}$ -equivalent.
- (10)  $\bar{h}_{1,v'_1}$  and  $\bar{h}_{2,v'_2}$  are  $\mathcal{K}$ -equivalent.
- (11)  $K(CM_1, HP(v'_1, +1) \cap S_1^n; p'_1) = K(CM_2, HP(v'_2, +1) \cap S_1^n; p'_2)$ ,
- (12)  $Q(\bar{\mathbf{X}}_{\theta_1}, (u_1, \mu_1))$  and  $Q(\bar{\mathbf{X}}_{\theta_2}, (u_2, \mu_2))$  are isomorphic as  $\mathbb{R}$ -algebras.

In this case  $(\mathbf{X}_1^{-1}(HP(v_1, 1) \cap S_1^n), u_1)$  and  $(\mathbf{X}_2^{-1}(HP(v_2, 1) \cap S_1^n), u_2)$  are diffeomorphic as set germs.

*Proof.* Since  $\mathcal{L}_{H_1}$  and  $\mathcal{L}_{H_2}$  are Legendrian stable, regular sets of  $HS_{\mathbf{X}_1}$  and  $HS_{\mathbf{X}_2}$  are respectively dense, by applying Proposition 6.2, the conditions (1) and (2) are equivalent. And we apply Theorem 6.3, the conditions (2) and (3) are equivalent. By the previous arguments from Theorem 6.1, the conditions (4) and (5) are equivalent. If we assume the condition (3), then the  $\mathcal{P}$ - $\mathcal{K}$ -equivalence of  $H_i$  ( $i = 1, 2$ ) preserves the  $\mathcal{K}$ -equivalence of  $h_{i,v_i}$ , so that the condition (4) holds. Since the local rings  $Q(\mathbf{X}_i, u_i)$  are  $\mathcal{K}$ -invariant, this means that the condition (6) holds. By Proposition 6.4, the condition (6) implies the condition (2). Therefore the conditions from (1) to (6) are equivalent.

By theorem 5.2, (2) and (8) are equivalent. Since  $\mathcal{L}_{H_i}$  are Legendrian stable,  $\mathcal{L}_{\bar{H}_i}$  are also Legendrian stable. So that we may similarly show the equivalence of the conditions from (7) to (12). On the other hand,  $h_{i,v_i}^{-1}(0) = (\mathbf{X}_i^{-1}(HP(v_i, 1) \cap S_1^n), u_i)$  and  $\mathcal{K}$ -equivalence preserves the zero level sets, so that  $(\mathbf{X}_i^{-1}(HP(v_i, 1) \cap S_1^n), u_i)$  ( $i = 1, 2$ ) are diffeomorphic as set germs. This completes the proof.  $\square$

## 7 Generic properties

In this section we consider generic properties of spacelike submanifolds of codimension  $r \geq 2$  in  $S_1^n$ . Let  $U$  be an open subset of  $\mathbb{R}^{n-r}$ . We consider the space of spacelike embeddings  $\text{Sp-Emb}(U, S_1^n)$  with Whitney  $C^\infty$ -topology. We define a function  $\mathcal{H} : S_1^n \times LC^* \rightarrow \mathbb{R}$  by  $\mathcal{H}(x, v) = \langle x, v \rangle$ , and denote  $\mathfrak{h}_v(x) = \mathcal{H}(x, v)$ . Then  $\mathfrak{h}_v$  is a submersion for any  $v \in LC^*$ . For spacelike submanifolds  $\mathbf{X} \in \text{Sp-Emb}(U, S_1^n)$ , we have  $H = \mathcal{H} \circ (\mathbf{X} \times \text{id}_{LC^*})$ . We also have the  $\ell$ -jet extension  $j_1^\ell H : U \times S_1^n \rightarrow J^\ell(U, \mathbb{R})$  defined by  $j_1^\ell H(x, v) = j^\ell \mathfrak{h}_v(u)$ . We consider the trivialization  $J^\ell(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^\ell(n-r, 1)$ . For any submanifold  $Q \subset J^\ell(n-r, 1)$ , we denote  $\tilde{Q} = U \times \{0\} \times Q$ . Then we have the following proposition as a corollary of Lemma 6 of Wassermann [11].

**Proposition 7.1.** ([11]) Let  $Q$  be a submanifold of  $J^\ell(n-r, 1)$ . Then the set

$$T_Q = \{x \in \text{Sp-Emb}(U, S_1^n) \mid j_1^\ell H \text{ is transversal to } \tilde{Q}\}$$

is a residual subset of  $\text{Sp-Emb}(U, S_1^n)$ . If  $Q$  is a closed subset, then  $T_Q$  is open.

We remark that if the corresponding horospherical height function  $h_{v_0}$  is  $\ell$ -determined relative to  $\mathcal{K}$ , then  $H$  is a  $\mathcal{K}$ -versal deformation if and only if  $j_1^\ell H$  is transversal to  $\tilde{\mathcal{K}}_{h,v_0}^\ell$ , where  $\mathcal{K}_{h,v_0}^\ell$  is the  $\mathcal{K}$ -orbit through  $j^\ell h_{v_0}(\mathbf{0}) \in J^\ell(n-r, 1)$ . Applying Theorem 6.3, this condition is equivalent to the condition that the corresponding Legendrian immersion germ is Legendrian stable. From the previous arguments and §5 in [6], we have the following proposition. (See also [1].)

**Theorem 7.2.** If  $n \leq 6$ , there exists an open subset  $\mathcal{O} \subset \text{Sp-Emb}(U, S_1^n)$  such that for any  $\mathbf{X} \in \mathcal{O}$ , the corresponding Legendrian immersion germ  $\mathcal{L}$  is Legendrian stable.

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## References

- [1] V.I. Arnold, S.M. Gusein-Zade and A.N. Varchenko, Singularities of Differential Maps, Volume I, Birkhäuser, Basel, 1986.
- [2] T. Fusho and S. Izumiya, Lightlike surfaces of spacelike curves in de Sitter 3-space, *J. Geom.* 88 (2008), 19–29.
- [3] S. Izumiya, D. Pei and T. Sano, Singularities of hyperbolic Gauss maps, *Proc. London Math Soc.* 86 (2003) 485–512.
- [4] S. Izumiya, M. Kossowski, D. Pei and M.C. Romero Fuster, Singularities of Lightlike Hypersurfaces in Minkowski four-space, *Tohoku Math J.* 58 (2006), 71–88.
- [5] S. Izumiya and M.C. Romero Fuster, The lightlike flat geometry on spacelike submanifolds of codimension two in Minkowski space, *Sel. math. NS.* 13 (2007) 23–55.
- [6] S. Izumiya, D. Pei, M.C. Romero Fuster and M. Takahashi, The horospherical geometry of submanifolds in hyperbolic space, *J. London Math. Soc.* (2) 71 (2005) 779–800.
- [7] M. Kasedou, Singularities of lightcone Gauss images of spacelike hypersurfaces in de Sitter space, *J. of Geom* 94 Issue 1 (2009) 107–121.
- [8] M. Kasedou, Spacelike submanifolds of codimension two in de Sitter space, *J. of Geom. and Physics* (2009), doi:10.1016/j.geomphys.2009.08.005 (in press).
- [9] M. Kasedou, Spacelike submanifolds of codimension at most two in de Sitter space, preprint.
- [10] J.A. Montaldi, On contact between submanifolds, *Michigan Math. J.* 33 (1986) 195–199.
- [11] G. Wassermann, Stability of Caustics, *Math. Ann.* 216 (1975) 43–50.
- [12] V.M. Zakalyukin, Lagrangian and Legendrian singularities, *Funct. Anal. Appl.* 10 (1976) 26–36.
- [13] V.M. Zakalyukin, Reconstructions of fronts and caustics depending one parameter and versality of mappings, *J. Soviet. Math.* 27 (1984) 2713–2735.

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