



Title	A Comparison Principle for Hamilton-Jacobi equations with discontinuous Hamiltonians
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Citation	Hokkaido University Preprint Series in Mathematics, 955, 1-9
Issue Date	2010-2-25
DOI	10.14943/84102
Doc URL	<a href="http://hdl.handle.net/2115/69762">http://hdl.handle.net/2115/69762</a>
Type	bulletin (article)
File Information	pre955.pdf



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# A Comparison Principle for Hamilton-Jacobi equations with discontinuous Hamiltonians

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February 19, 2010

## Abstract

We show a comparison principle for viscosity super- and subsolutions to Hamilton-Jacobi equations with discontinuous Hamiltonians. The key point is that the Hamiltonian depends upon  $u$  and it has a special structure. The supersolution must enjoy some additional regularity.

**Key words:** Hamilton-Jacobi equation, viscosity solutions, discontinuous Hamiltonian

**2010 Mathematics Subject Classification.** Primary: 49L25

## 1 Introduction

The purpose of this note is to give a simple proof of a comparison principle for bounded, uniformly continuous sub-, supersolution solutions to the Hamilton-Jacobi equation

$$d_t + H(t, x, d, d_x) = 0 \quad \text{in } (0, T) \times \mathbb{R}, \quad (1)$$

when the Hamiltonian  $H$  is discontinuous and depends in non-trivial way on  $d$ . It is well-known that in general, if  $H$  is discontinuous in  $x$ , then the comparison principle may fail.

Here, we assume a special structure of  $H$  and its line of discontinuity. It comes from the singular curvature flow, considered in [GGR]. Namely, the equation studied there leads to the following form of  $H$ ,

$$H(t, x, u, p) = \begin{cases} -\sigma(t, r^*(t), u)m(p) & \text{if } |x| < r_0(t) \\ -\sigma(t, x, u)m(p) & \text{if } |x| \geq r_0(t), \end{cases} \quad (2)$$

Here, we explain the assumption starting from the line of discontinuity,

(R1)  $r_0$  and  $r^*$  belong to  $C^0(0, T)$  and  $r^*(t) > r_0(t)$  for all  $t \in [0, T]$ , in addition the set  $\{(t, r_0(t)) : t \in [0, T]\}$  is a Lipschitz curve.

Let us remark that  $r_0$  need not be Lipschitz continuous as a function of  $t$  as in the case of  $r_0(t) = \sqrt{t} - 1$ . The set in question is a subset of a parabola.

The conditions we present are not optimal, but they are simple enough and permit us to present the main argument. We have to specify restrictions on the other components. We assume that  $\sigma \in C^1$  is bounded, even as a function of  $x$  or  $u$  while other arguments are fixed. It is also increasing with respect to  $x$  as well as  $u$  provided that  $x, u > 0$ . In addition,

$$0 < \sigma_u(t, x, u) \leq M \quad (3)$$

and  $m$  is a positive convex function with linear growth at infinity. In the present paper, however, no conditions on  $m$  are necessary except for continuity.

In [GGR, Theorem 4.3] we showed a comparison principle for special bounded, even, Lipschitz continuous sub-, supersolution solutions to (1). We required in [GGR] that the supersolution be increasing on  $[0, +\infty)$ , while the subsolution be constant over  $[0, a(t)]$ , for  $a(t) \geq r_0(t)$  for all  $t \in [0, T]$ .

Here, we prove the Comparison Principle, see Theorem 1 without these structural restrictions on sub-, supersolution solutions, however, we impose moderate regularity assumptions. Before explaining our method, we will comment on the available literature.

Let us mention that while the notion of semicontinuous super- and subsolutions for discontinuous Hamiltonians is well-defined, (see [BP], [I] [St]), the authors frequently assume in the statements of their Comparison Principles that either supersolution or subsolution is at least Lipschitz continuous, [CS], [CH], [CR], [DE], [DZS], [T].

There are various kinds of motivation to study problems like (1). One stems from image analysis, like the ‘shape-from-shading’ problem [CR], [T], another is from flame propagation or etching [CH] or from game theory [DZS]. In those papers (1) is a general form of the eikonal equation and  $H$  does not depend upon  $u$ . For us (1) is a result of degeneration of a second order parabolic problem, see [GGR], where the dependence of  $H$  upon  $u$  is essential.

There is a spectrum of assumptions on admissible discontinuities with respect to  $x$  and  $t$ . Jump discontinuities across Lipschitz hypersurfaces are quite common, [CH], [DE], [DZS], [T], we may add that sometimes the authors admit triple junctions of the discontinuity set, see [DE]. In [CR] jump discontinuities are admitted along a set of vertical and horizontal intervals. The most general situation is considered in [CS], the authors must use tools from measure theory and they consider a slightly different notion of solution.

Our comparison principle does not require any special condition on the kind of dependence of  $H$  upon  $p$  except continuity. On the other hand [CS], [CR], [DE] need coercivity,

while the authors of [CS], [T] assume convexity in  $p$ . Sometimes other conditions are used like 1-homogeneity with respect to  $p$ , see [DZS], or linear growth at infinity, see [CH].

We deal, however, with the graph over the real line, so we have to control the behavior of supersolutions at infinity. For this purpose we introduce a convenient technical notion of supersolution at infinity. We also find it convenient to work with strict supersolution, but understood differently than in [T].

Our method of proof of Theorem 1 is based upon the idea of shifting the supersolution  $v$  away from the discontinuity of  $H$  so that the shifted  $v$  becomes a strict supersolution. We also regularize  $H$  in a proper manner, so that the strict supersolution will remain a supersolution and subsolutions will not lose this property. This permits us to use classical results and deduce our claim.

## 2 The Comparison Principle

We first recall from [GGR, Definition 1] (see also [BP], [I] and more recently by [CR]) the notion of a sub-/supersolution to (1) in case of a discontinuous Hamiltonian.

**Definition 1** (a) We shall say that a bounded, uniformly continuous function  $u : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$  is a *viscosity subsolution* of (1) provided that for all  $C^1$  functions  $\varphi : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $u - \varphi$  has a local maximum at  $(t_0, x_0)$ , then

$$\varphi_t(t_0, x_0) + H_*(t_0, x_0, u(t_0, x_0), \varphi_x(t_0, x_0)) \leq 0.$$

(b) We shall say that a bounded, uniformly continuous function  $v : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$  is a *viscosity supersolution* of (1) if it for all  $C^1$  functions  $\varphi : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $v - \varphi$  has a local minimum at  $(t_0, x_0)$ , then

$$\varphi_t(t_0, x_0) + H^*(t_0, x_0, v(t_0, x_0), \varphi_x(t_0, x_0)) \geq 0.$$

(c) We shall say that a bounded, uniformly continuous function  $d : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$  is a *viscosity solution* of (1) provided that it is a viscosity subsolution as well as a viscosity supersolution of (1).

For the sake of self-consistency we briefly recall the definitions of upper semicontinuous envelope,  $H^*$ , and lower semicontinuous envelope,  $H_*$ , for a locally bounded function  $H : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ . Namely, we set

$$H_*(z) = \liminf_{\zeta \rightarrow z} H(\zeta), \quad H^*(z) = \limsup_{\zeta \rightarrow z} H(\zeta).$$

We notice that Definition 1 is in the line of notion of sub-(super-)solution introduced by [BP], [I] and more recently by [CR] for discontinuous Hamiltonians.

We shall describe our assumptions on  $H$  which slightly generalize formula (2) above.

We begin with continuity requirements,

(H1) Hamiltonian  $H$  is lower semicontinuous in  $[0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ ;

(H2)  $H$  is continuous away from  $\Gamma = \{(t, \pm r_0(t)) : t \in [0, T]\}$  and it has a jump discontinuity at  $\Gamma$ .

(H3)  $H^*$  is continuous in  $G = \{(t, x) : |x| \geq r_0(t)\}$ , while  $H_*$  is continuous on the closure of  $[0, T] \times \mathbb{R} \setminus G$ .

Symmetry of  $H$  is just for the sake of simplicity. That is, we impose,

(H4) For any  $\epsilon_1, \epsilon_2, \epsilon_3$  in  $\{-1, 1\}$  we have  $H(t, \epsilon_1 x, \epsilon_2 u, \epsilon_3 p) = H(t, x, u, p)$ .

Monotonicity of our Hamiltonian is crucial for our argument. We shall frequently use the following condition,

(H5) Hamiltonian  $H$  is strictly increasing with respect to  $u$ , i.e. there is a positive  $h_0$ , such that the following inequality holds for all  $u_2, u_1, x, t$  and  $p$ ,

$$H(t, x, u_2, p) - H(t, x, u_1, p) \geq h_0(u_2 - u_1). \quad (4)$$

(H6) For all  $t, u$  and  $p$  function  $x \mapsto H(t, x, u, p)$  is decreasing for  $x > r_0(t)$ , moreover  $H(t, x, u, p) = H(t, r^*(t), u, p)$  for  $x \in [-r_0(t), r_0(t)]$ .

**Remark 1** It is possible to convert our  $H$  into one satisfying (4), by means of the following change of variables  $v = e^{t\lambda}u$ , where  $\lambda = -2M$  and  $M$  is the constant appearing in (3). Nonetheless, even the transformed Hamiltonian,  $H_{new}$ , will have a jump in  $(t, x)$  at  $(t, \pm r_0(t))$ . It has the following form

$$H_{new}(t, x, v, p) = 2Mv + e^{-2Mt}H(t, x, e^{2Mt}v, e^{2Mt}p). \quad (5)$$

Interestingly, property (4) is inherited by  $H^*$  and  $H_*$ .

**Corollary 1** If  $H$  satisfies (4) so do  $H^*$  and  $H_*$  with same constant  $h_0$ .

*Proof.* By the definition of  $H^*(t, x, u, p)$  there is a sequence  $(t_n, x_n, u_n^1, p_n)$  converging to  $(t, x, u_1, p)$  such that

$$\lim_{n \rightarrow \infty} H(t_n, x_n, u_n^1, p_n) = H^*(t, x, u_1, p).$$

By (4) we have,

$$H(t_n, x_n, u_2, p_n) - H(t_n, x_n, u_n^1, p_n) \geq h_0(u_2 - u_n^1).$$

By definition of  $H^*$  the inequality  $H^*(t_n, x_n, u_2, p_n) \geq H(t_n, x_n, u_2, p_n)$  always holds. Since  $H^*$  is upper semicontinuous we have

$$\begin{aligned} H^*(t, x, u_2, p) - H^*(t, x, u_1, p) &\geq \limsup_{n \rightarrow \infty} H(t_n, x_n, u_2, p_n) - \lim_{n \rightarrow \infty} H(t_n, x_n, u_n^1, p_n) \\ &\geq \lim_{n \rightarrow \infty} h_0(u_2 - u_n^1) = h_0(u_2 - u_1). \end{aligned}$$

Hence,  $H^*$  indeed satisfies (4).

In order to show (4) for  $H_*$  we proceed in a similar way: we take a sequence  $(t_n, x_n, u_n^2, p_n)$  converging to  $(t, x, u_2, p)$  such that

$$\lim_{n \rightarrow \infty} H(t_n, x_n, u_n^2, p_n) = H^*(t, x, u, p).$$

Subsequently, we apply the lim inf to the inequality

$$H(t_n, x_n, u_n^2, p_n) - H_*(t_n, x_n, u_n^1, p_n) \geq h_0(u_n^2 - u_1).$$

Our claim follows.  $\square$

In order to state and establish our result we need a technical device, which is used to control the behavior of supersolution at infinity. This is so because our region has no boundary. This requires another condition on the Hamiltonian:

(H7)  $H$  converges to  $H^\infty \in C([0, T] \times \mathbb{R})$  locally uniformly with respect to  $(t, u) \in [0, T] \times \mathbb{R}$  and  $p$  near zero as  $|x| \rightarrow \infty$ , i.e.  $H^\infty$  does not depend upon  $p$ .

In our [GGR] we considered in fact piecewise  $C^1$  solutions. We need them here as well. We also make precise what we shall call here by a piecewise  $C^1$  function in order to make next notion meaningful.

**Definition 2** We shall say that a Lipschitz continuous function  $w$  is a *piecewise  $C^1$ -function*, (with discontinuity along  $\{|x| = r_0(t)\}$ ) provided that there are disjoint open sets,  $\mathcal{U}_i \subset (0, T) \times \mathbb{R}$ ,  $i = 1, \dots, N_w$ ,  $N_w \in \mathbb{N}$  such that: (a)  $[0, T] \times \mathbb{R} = \bigcup_{i=1}^{N_w} \bar{\mathcal{U}}_i$ , (b) each  $\mathcal{U}_i$  has Lipschitz boundary, (c) the set  $\{t \in [0, T] : (t, r_0(t)), (t, -r_0(t))\}$  is contained in  $\bigcup_i \partial \mathcal{U}_i$ , (d) there exist two indexes  $i_0$ , and  $j_0$  and a positive number  $\mu_0$  such that

$$(0, T) \times (-\infty, -\mu_0] \subset \bar{\mathcal{U}}_{i_0} \quad \text{and} \quad (0, T) \times [\mu_0, +\infty) \subset \bar{\mathcal{U}}_{j_0},$$

(e)  $w|_{\bar{\mathcal{U}}_i} \in C^1(\bar{\mathcal{U}}_i)$ , i.e. the derivatives can be extended to  $\bar{\mathcal{U}}_i$  as continuous functions.

Once we have imposed restrictions on the behavior of  $H$  by requiring (H7) we can introduce another notion.

**Definition 3** For  $H$  satisfying (H7) we shall say that a piecewise  $C^1$ -function  $w$  is a *supersolution at infinity* provided that  $w$  is a supersolution, the following limits exist and are uniform with respect to  $t \in [0, T]$ ,

$$w_t \rightarrow w_t^\infty, \quad w \rightarrow w^\infty, \quad w_x \rightarrow 0 \quad \text{as } |x| \rightarrow \infty$$

and

$$w_t^\infty(t) + H^\infty(t, w^\infty(t)) \geq 0. \tag{6}$$

We shall call  $w$  a *strict supersolution at infinity* if it is a supersolution at infinity and the inequality in (6) is strict.

Here is our main result. It is worth noticing that we do not impose on the Hamiltonian neither coercitivity nor convexity in  $p$ . In particular, Hamiltonian given in (5) satisfies all our conditions provided that  $\sigma$  in (2) converges uniformly, as  $|x| \rightarrow \infty$ , to  $\sigma^\infty \in C^1((0, T) \times \mathbb{R})$ .

**Theorem 1** *Let us assume that a measurable function  $H$  satisfies (R1) and (H1–H7) and for  $u, v \in BUC([0, T] \times \mathbb{R})$  following conditions are valid:*

- (a)  $v$  is a supersolution to (1),  $u$  is a subsolution to (1) and  $u(0, x) \leq v(0, x)$ .
- (b)  $v$  is a piecewise  $C^1$ -function.
- (c)  $v$  is a supersolution of (1) at infinity.

Then, for all  $t > 0$

$$u(t, x) \leq v(t, x).$$

The above statement is rather long, however, the content is rather simple: we have to impose condition permitting us to control the behavior of Hamiltonian  $H$  and super-, subsolutions at infinity. Moreover, we assume that the set of non-differentiability points of the supersolution is small and sets of discontinuities of  $H$  and  $v_x$  are ‘aligned’.

We shall proceed in several stages: we will move the problem away from the jump discontinuity of  $H$  by considering a “shifted supersolution”. We also modify  $H$  to make it a continuous function. Subsequently, we apply the classical results for continuous Hamiltonians.

In order to state our next observation it is convenient to introduce the notion of a strict supersolution. It is known in the literature, see e.g. [T] for  $C^1$  sub-, supersolution, here however, we have to relax the regularity assumptions.

**Definition 4** We shall say that a supersolution  $v$  is a *strict supersolution* of (1), if for any test function  $\varphi \in C^1$  such that  $v - \varphi$  has a minimum at  $(t_0, x_0)$ , then

$$\varphi_t(t_0, x_0) + H^*(t_0, x_0, v(t_0, x_0), \varphi_x(t_0, x_0)) > 0.$$

We define a *strict subsolution* of (1) in a similar way.

We may now state our next observation as follows.

**Proposition 1** *Let us suppose that the assumptions (R1) and (H1)–(H7) hold. If  $v$  is a supersolution of (1) so is  $v + \epsilon$  for any positive  $\epsilon$ . Moreover,  $v + \epsilon$  is a strict supersolution.*

*Proof.* Since  $v$  is a supersolution then the inequality in Definition 4 is obvious due to the strict monotonicity of  $H^*$  shown in Corollary 1.  $\square$

Let us now define the regularized Hamiltonian. For  $\delta > 0$  we set

$$H^\delta(t, x, u, p) = \begin{cases} H(t, x, u, p) & |x| \geq r_0(t) + \delta, \\ (1 - \frac{\lambda}{\delta})H(t, r^*, u, p) + \frac{\lambda}{\delta}H(t, r_0 + \delta, u, p) & |x| = r_0(t) + \lambda, \lambda \in (0, \delta), \\ H(t, r^*(t), u, p) & |x| \leq r_0(t). \end{cases}$$

Note that  $\lambda$  depends on  $x$  and  $t$ . Here is our first observation on  $H^\delta$ .

**Lemma 1** *If  $u$  is a subsolution to (1), then it is also a subsolution to (7) below,*

$$d_t + H^\delta(t, x, d, d_x) = 0. \tag{7}$$

*Proof.* The claim follows immediately from the inequality

$$H^\delta(t, x, u, p) \leq H_*(t, x, u, p). \quad \square$$

We are ready for a definition of a shifted supersolution  $v^\delta$ . We set

$$v^\delta(x) = v(x - \delta).$$

We have to show that for a given  $\epsilon$ , then for a sufficiently small  $\delta$ , function  $v^\delta + \epsilon$  is indeed a supersolution to (7).

**Lemma 2** *Let us suppose that assumptions (R1) and (H1)–(H7) hold and:*

- (a) *w is piecewise a  $C^1$  function;*
- (b) *w is a supersolution of (1);*
- (c) *w is a supersolution at infinity of (1).*

*Then, for any  $\epsilon > 0$  there is such  $\delta_0(\epsilon) > 0$  that for any  $\delta \in (0, \delta_0(\epsilon))$ , function  $w^\delta + \epsilon$  is a supersolution of (7).*

*Proof.* By Proposition 1  $w + \epsilon$ ,  $\epsilon > 0$ , is a strict supersolution of (1). We claim that the restrictions imposed on the behavior of  $w$  at infinity permit us to show a stronger inequality than postulated by Definition 4. Namely, there exists  $\eta > 0$  such that for any test function  $\varphi$  such that the difference  $w - \varphi$  attains its minimum at  $(t, x) \in (0, T) \times \mathbb{R}$  we have

$$\varphi_t(t, x) + H^*(t, x, w(t, x) + \epsilon, \varphi_x(t, x)) \geq \eta > 0. \quad (8)$$

Indeed, we noticed that  $H^*$  satisfies (4) with the same  $h_0$  as  $H$  does. Thus, we deduce

$$\begin{aligned} & \varphi_t(t, x) + H^*(t, x, w(t, x) + \epsilon, \varphi_x(t, x)) \geq \\ & \varphi_t(t, x) + H^*(t, x, w(t, x), \varphi_x(t, x)) + h_0\epsilon \geq h_0\epsilon =: \eta > 0. \end{aligned}$$

In other words, (8) holds for all  $(t, x) \in (0, T) \times \mathbb{R}$  as desired.

We notice that due to (4) the Hamiltonian at infinity  $H^\infty$  is also strictly increasing with respect to  $u$ . It is just sufficient to pass to the limit in (4) to deduce that

$$H^\infty(t, u_2) - H^\infty(t, u_1) \geq h_0(u_2 - u_1).$$

This inequality combined with (6) shows that  $w + \epsilon$  is a strict supersolution at infinity.

We will now show that  $w^\delta + \epsilon$  is a supersolution. We need to show for a test function such that  $w^\delta - \varphi$  attains its minimum at  $(t, x)$  that

$$\varphi_t + H^\delta(t, x, w^\delta + \epsilon, \varphi_x) \geq 0. \quad (9)$$

We consider first  $|x| > r_0(t) + \delta$ , then we have

$$\varphi_t(t, x) + H(t, x, w^\delta(t, x) + \epsilon, \varphi_x(t, x)) = \varphi_t(t, x) + H(t, x, w(t, x - \delta) + \epsilon, \varphi_x(t, x)).$$

We write  $y = x - \delta$ , hence  $|y| + \delta \geq r_0(t) + \delta$ . We notice that  $H$  is locally uniformly continuous in  $G \times \mathbb{R}^2$ . Indeed, because of the assumed uniform convergence of  $H$  to  $H^\infty$  for a given  $\eta$  we can find such  $R$  that for  $|y|, |z| \geq R \geq \mu_0$  we have  $|H(t, y, w, p) - H(t, z, w, p)| < \eta$ . Due to compactness of the set  $G \cap B_R(0)$  function  $H$  is uniformly continuous on  $\mathcal{F} = G \cap B_R(0) \times [-\|w\|_\infty - 1, \|w\|_\infty + 1] \times [-\|w_x\|_\infty, \|w_x\|_\infty]$ .

Let us now introduce a new test function by formula  $\psi(t, y) = \varphi(t, y + \delta)$ . We have to check that  $|\psi_x| \leq \|w_x\|_\infty$ . This is indeed so because the inequality  $(w(t, z) - \psi(t, z)) \geq 0$  for  $z \neq y$  in a neighborhood of  $y$ , implies that  $w_x^+(t, y) \geq \psi_x(t, y)$  and  $w_x^-(t, y) \leq \psi_x(t, y)$ . Since (8) holds, then we can find  $\delta$  so that

$$\psi_t(t, y) + H(t, y + \delta, w(t, y) + \epsilon, \psi_x) \geq \psi_t(t, y) + H(t, y, w(t, y) + \epsilon, \psi_x) - \eta \geq \eta - \eta = 0.$$

This proves the claim for  $|x| > r_0(t) + \delta$ .

Suppose now  $|x| \leq r_0$ , in this case  $H(t, x - \delta, u, p) = H(t, r^*(t), u, p)$  and it is by definition smaller than  $H^\delta(t, x, u, p)$ . Then, after setting  $y = x - \delta$  and introducing the same new test function  $\psi$  by (8), we have

$$\psi_t(t, y) + H^\delta(t, y + \delta, w(t, y) + \epsilon, \psi_x(t, y)) \geq \psi_t(t, y) + H(t, r^*, w(t, y), \psi_x(t, y)) \geq 0.$$



Our claim holds again.

Now, we take  $x \in [r_0(t), r_0(t) + \delta)$ , we proceed as before. We notice that

$$\begin{aligned} \varphi_t(t, x) + H^\delta(t, x, w^\delta(t, x) + \epsilon, \varphi_x(t, x)) &= \varphi_t(t, x) + H^\delta(t, x, w(t, x - \delta) + \epsilon, \varphi_x(t, x)) \\ &> \psi_t(t, y) + H^\delta(t, y + \delta, w(t, y), \psi_x(t, y)) \\ &\geq \psi_t(t, y) + H(t, r^*(t), w(t, y), \psi_x(t, y)) \geq 0. \end{aligned}$$

For  $-x \in [r_0(t), r_0(t) + \delta)$  the calculations are essentially the same,

$$\begin{aligned} \varphi_t(t, x) + H^\delta(t, x, w^\delta(t, x) + \epsilon, \varphi_x(t, x)) &= \varphi_t(t, x) + H^\delta(t, x, w(x - \delta) + \epsilon, \varphi_x(t, x)) \\ &= \psi_t(t, y) + H^\delta(t, y + \delta, w(t, y) + \epsilon, \psi_x(t, y)) \\ &> \psi_t(t, y) + H(t, r^*(t), w(t, y), \psi_x(t, y)) \geq 0. \end{aligned}$$

Finally, we consider points  $(t, x) = (t, \pm(r_0(t) + \delta))$ . These are translations of special points of  $H$ . Let us consider first  $(t, x) = (t, r_0(t) + \delta)$ . For any test function  $\varphi$  we have to show that

$$L := \varphi_t + H^\delta(t, r_0(t) + \delta, w^\delta(r_0(t) + \delta, t) + \epsilon, \varphi_x) = \varphi_t + H^*(t, r_0(t) + \delta, w(r_0(t), t), \varphi_x) \geq 0.$$

Since  $H^*$  is uniformly continuous in  $G \cap [0, T] \times [0, R]$ , then we can find  $\delta(\epsilon)$ , so that (8) implies

$$\begin{aligned} L &\geq \psi_t(t, r_0(t)) + H^*(t, r_0(t), w(t, r_0(t)) + \epsilon, \psi_x(t, r_0(t))) - \eta \\ &\geq \eta - \eta = 0. \end{aligned}$$

We recall that  $H^*(t, r_0(t), v, p) = H(t, r_0(t)^+, v, p)$ . Our claim follows.  $\square$

**Remark.** The assumption that  $v + \epsilon$  is a strict supersolution at infinity is technical only for the sake of dealing with unbounded domain.

We are now ready for the *proof of our main result*. Let us suppose a subsolution  $u$  and supersolution  $v$  satisfy the assumptions. By Proposition 1, for any positive  $\epsilon$ ,  $v + \epsilon$  is a strict supersolution, moreover  $v + \epsilon$  is a strict supersolution at infinity. Since  $v + \epsilon$  is uniformly continuous over  $\mathbb{R}$ , then there is  $\delta_0(\epsilon)$ , such that for all  $\delta \in (0, \delta_0(\epsilon))$  we have  $v^\delta(0, x) + \epsilon > u(0, x)$ . By Lemma 1,  $u$  is a subsolution to (7), while  $v^\delta + \epsilon$  is a supersolution to the same equation, possibly for a smaller  $\delta$ . Since  $H^\delta$  is strictly increasing with respect to  $u$ , we may apply the classical comparison principle to conclude that

$$u(t, x) \leq v^\delta(t, x) + \epsilon,$$

for all  $t \in [0, T]$ . Since  $\delta_0(\epsilon)$  goes to zero when  $\epsilon \rightarrow 0$  and  $v^\delta + \epsilon$  converges to  $v$ , we conclude that

$$u(t, x) \leq v(t, x), \quad \text{for all } t \in [0, T],$$

as desired.  $\square$

We notice that our comparison principle improves the uniqueness result of [GGR, Theorem 1.1], because there we assumed unnecessary structure of our super- and sub-solutions. Here, they are removed, but we restrict the behavior of super- and sub-solutions at infinity. However, the complete analysis of this issue will be presented elsewhere.

**Acknowledgment** The first and third authors acknowledge stimulating conversations with Professor H.Ishii on the subject of this paper.

The work of the first author was partly supported by a Grant-in-Aid for Exploratory Research (20654017) and a Grant-in-Aid for Scientific Research (S) (21224001) from the Japan Society for the Promotion of Science. PR thanks Hokkaido University for its hospitality, for a part of the research was performed while PR was visiting the university. The last two authors were partly supported by the Polish Ministry of Science grant N N2101 268935, PG enjoyed also a partial support from Fondecyt 3100019.

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