On vorticity directions near singularities for the Navier-Stokes flows with infinite energy

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Abstract: We give a geometric nonblow up criterion on the direction of the vorticity for the three dimensional Navier-Stokes flow whose initial data is just bounded and may have infinite energy. We prove that under a restriction on behavior in time (type I condition) the solution does not blow up if the vorticity direction is uniformly continuous at place where vorticity is large. This improves the Lipschitz regularity condition for the vorticity direction first introduced by P. Constantin and C. Fefferman (1993) for finite energy (weak) solution. Our method is based on a simple blow up argument which says that the situation looks like two-dimensional under continuity of the vorticity direction. We also discuss about boundary value problems.

1 Introduction

There is a large number of work on regularity criterion or nonblow up criterion for the three dimensional Navier-Stokes flow started by Ohyama [O] and Serrin [Se]; see also [Le], [L]. Most of them are analytic in the sense that boundedness for some quantity is assumed. In 1993, P. Constantin and C. Fefferman [CF] gave a geometric condition for the vorticity direction $\zeta = \omega/|\omega|$, where ω is the vorticity. It says that if the vorticity is Lipschitz continuous in space uniformly in time in the region where vorticity is large, then the Leray-Hopf type weak solution is regular. Since then there are several improvement [BB], [B1], [B2], [CKL], [Z], [GrZ], [GrR], [Gr]. However, most of these work discuss a solution having a bounded kinetic energy.

In this paper we consider a smooth mild solution of the Navier-Stokes equations in $\mathbf{R}^3 \times (0, T)$, where a solution is just bounded in space direction and may not have a finite energy. We shall give a condition for the vorticity direction so that solution can be extended smoothly across $t = T$. Roughly speaking our result reads: the solution does not blow up at $t = T$ if the blow up is type I and the vorticity direction is uniformly continuous in space variables. For regularity assumptions on the vorticity direction our result improves existing one. However, we are forced to assume the growth of L^∞-norm of solution is bounded by a self-similar rate $(T-t)^{-1/2}$ which is called type I blow up.

All previous results are proved by integral estimates while ours is by a blow up argument. It was first used by De Giorgi [DG] (see also [Giu, Theorem 8.1]) for the study of the minimal surfaces. The first author [G] applied the same kind of argument for semilinear parabolic equations to derive a global uniform bounds (see also Giga-Kohn [GK] for derivation of blow up rate). The blow up argument also plays important roles for the analysis of the singularities for geometric flows like the harmonic map heat flow [St] and the Ricci flow [H]. Very recently, Koch, Nadirashvili, Seregin and Sverak [KNSS] applied it to show that type I axisymmetric Navier-Stokes flow must be regular; see [SS] for a local version and also [CSYT], [CSTY] for different proofs. We also note that the blow up argument is effectively studied to derive several estimates for semilinear equations [PQS].

Our blow up argument is not only simple but also clarifies the structure of the problem. The continuous alignment assumption on the vorticity directions eventually implies that the blow up limit is two dimensional flow. It is easy to guess that the solution is regular. Our argument justifies physical intuition.

We give an explicit form of our results. We consider the Navier-Stokes equations

$$\begin{aligned} u_t - \Delta u + (u, \nabla)u + \nabla p = 0 \quad &\text{in} \quad \Omega \times (-1, 0), \\ \operatorname{div} u = 0 \quad &\text{in} \quad \Omega \times (-1, 0), \end{aligned} \tag{NS}$$

where Ω is either $\mathbf{R}^3$ or a half space $\mathbf{R}^3_+$. We assume that the solution u is smooth and

$$||u||_\infty(t) = \sup_{x \in \Omega} |u(x,t)|$$

is bounded for all $t \in (-1, 0)$. Unfortunately, the solution is not unique even if we fix an initial data for $\Omega = \mathbf{R}^3$ if one allows solution with infinite energy. In fact, $u(x,t) = g(t),\ p = -g'(t) \cdot x$ always solve the equation. So as in [GIM] we only consider a mild solution, i.e., solution satisfying an integral equation, which is equivalent to require that $p = (-\Delta)^{-1}\nabla \cdot (u \otimes u)$ in some sense. (Such a relation is automatic for decaying solutions.) For the initial value problem there is a unique local in time mild solution for $\Omega = \mathbf{R}^3$ [GIM] and $\Omega = \mathbf{R}^3_+$ [So], [BJ] with the zero Dirichlet condition. The time $t = 0$ is considered as a possible blow up time. We say the blow up rate is type I if

$$||u||_\infty(t) \le C_0(-t)^{-1/2} \tag{I}$$

with some $C_0 > 0$ independent of $t \in (-1, 0)$. We consider a type I mild solution which is a mild solution satisfying (I). According to the estimate of existence time for the mild solution evolving from bounded initial data [GIM], we have $\|u\|_\infty(t) \ge \varepsilon_0(-t)^{-1/2}$ with some $\varepsilon_0 > 0$ if $t = 0$ is the blow up time. Thus if C_0 is small, then the solution cannot blow up at $t = 0$.

We are now in position to state one of our main results.

Theorem 1.1. *Let u be a type I mild solution of (NS) for $\mathbf{R}^3 \times (-1, 0)$. For a given $d > 0$ let η be a modulus such that*

$$|\zeta(x,t) - \zeta(y,t)| \le \eta(|x - y|) \text{ for all } x, y \in \Omega_d(t),\ t \in (-1, 0) \tag{CA}$$

where $\Omega_d(t) = \{x \in \mathbf{R}^3 |\ |\omega(x,t)| > d\}$ and $\zeta = \omega/|\omega|,\ \omega = \operatorname{curl} u$. Then u does not blow up at $t = 0$.

Remark 1.2. Here by a modulus we simply mean that η is a nondecreasing continuous function defined in $[0, \infty)$ with $\eta(0) = 0$. For a uniformly continuous function f in $\mathbb{R}^n$ one defines a modulus of continuity by

$$m(\sigma) = \sup\{|f(x) - f(y)| \,|\, |x - y| \leq \sigma\}$$

this is of course, a modulus in our sense. Such m is moreover subadditive. Conversely if η is a modulus and subadditive, then η is a modulus of continuity of some f, see [Ku].

Remark 1.3. In [CF] η is taken $\eta(\sigma) = A\sigma$ in (CA) where A is a positive constant. In [BB] η is taken $\eta(\sigma) = A\sigma^{1/2}$. In [GrR] η is taken $\eta(\sigma) = A\sigma^{1/q}$ under the assumption $\int_{-1}^{0} \|\omega\|_q^{q/(q-1)}(t)dt$ is finite for some $q \geq 2$. These authors considered weak solutions and did not assume that the singularity is type I.

Remark 1.4. The continuous alignment assumption (CA) can be relaxed as follows:

$$|\zeta(x,t) - \zeta(y,t)| \leq \eta(o(1)\frac{|x-y|}{\sqrt{-t}}) \text{ for } x, y \in \Omega_d(t) \text{ and } t \in (-1, 0) \quad (CA')$$

as $t \uparrow 0$. We will prove the theorem under the condition (CA') in Section 2.1. We can replace these continuous alignment assumptions by the condition for $\nabla\zeta$. See Corollary 2.6.

Our blow up argument can be also applied for a local regularity criterion away from the boundary, which is regarded as a local version of Theorem 1.1. In order to prove the local regularity criterion, we will use a compactness theorem for a sequence of suitable weak solutions by Seregin and Sverak [SS]. For this reason, we have to assume local energy is finite uniformly in $(-1, 0)$. It should be noted that local energy is allowed to diverge as t tends to zero in Theorem 1.1. See Section 2.2.

A regularity criterion near the boundary turns to work for slip boundary condition by a simlar blow up argument. A key step for the blow up argument is to establish a Liouville type result. For the slip boundary value problem we may still use the Liouville type theorem for the flow without the boundary condition since our continuous alignment condition (CA) implies that the vorticity of the blow up limit is orthogonal to the boundary. We give an explicit statement in Section 3 when Ω is a half space. A corresponding result to [CF] for the slip boundary condition is discussed in [B1], where $\eta(\sigma) = A\sigma^{1/2}$ in (CA).

In the case of zero Dirichlet condition ($u = 0$) we need to study the Liouville type problem with the boundary. However, we do not know the answer since the vorticity on the boundary is not well controlled. In [B2] a regularity criteria is given for the Dirichlet problem by assuming further that $\int \partial|\omega|^2/\partial\overrightarrow{n}$ is small in some sense where $\overrightarrow{n}$ is the normal of the boundary. We do not know whether such an extra assumption is necessary. We shall discuss the Dirichlet problem in Section 3.

2 Blow up analysis

2.1 Proof of Theorem 1.1

To prove the main theorem it suffices to prove that

$$\lim_{t\uparrow 0} \sup_{-1\le\tau\le t} ||u||_\infty(\tau) < \infty$$

for example by [GIM]. Assume the contrary so that

$$\lim_{t\uparrow 0} \sup_{-1\le\tau\le t} ||u||_\infty(\tau) = \infty.$$

Then there is a sequence $\{t_k\}_{k=1}^\infty$ with $t_k \uparrow 0$ such that

$$\sup_{-1\le\tau\le t_k} ||u||_\infty(\tau) = ||u||_\infty(t_k) := M_k \uparrow \infty$$

as $k \to \infty$. We take x_k such that $|u(x_k, t_k)| \ge M_k - 1$. We rescale the solution by

$$u_k(x,t) = \lambda_k u(x_k + \lambda_k x,\ t_k + \lambda_k^2 t)$$

with $\lambda_k = 1/M_k$. Clearly, we know

$$|u_k(0,0)| \ge 1 - 1/M_k$$

and the uniform bound for the rescaled velocity is given as

$$|u_k(x,t)| \le 1 \text{ in } \mathbf{R}^3 \times (-M_k^2, 0].$$

By the scaling invariance properties of (NS), u_k is a mild solution of (NS) in $\mathbf{R}^3 \times (-M_k^2, 0)$. By the parabolic regularity theory (see [GIM], [GS], [MS]) also implies the uniform bound for the scale velocity.

$$||\partial_t^j u_k||_{L^\infty\left(\mathbf{R}^3\times(-M_k^2/2,0)\right)} + ||\nabla^j u_k||_{L^\infty\left(\mathbf{R}^3\times(-M_k^2/2,0)\right)} \leq C_j. \tag{2.1}$$

Note that [GS] discussed the case of decaying initial data but their argument easily extends to L^∞ case. Thus we can find a subsequence (still denoted u_k, ω_k) which converges to bounded continuous functions $\overline{u}$ and $\overline{\omega}$ locally uniformly in $\mathbf{R}^3 \times (-\infty, 0]$. Moreover, $\overline{u}$ is a mild solution for (NS) since $u^k \otimes u^k$ converges to $\overline{u} \otimes \overline{u}$ $*$-weakly in L^∞. The limit function $(\overline{u}, \overline{\omega})$ solves the vorticity equation:

$$\overline{\omega}_t - \Delta\overline{\omega} + (\overline{u}, \nabla)\overline{\omega} - (\overline{\omega}, \nabla)\overline{u} = 0.$$

Moreover, $\overline{u}$ and $\overline{\omega}$ have bounds

$$|\overline{u}| \leq 1, \ |\overline{\omega}| \leq C \text{ in } \mathbf{R}^3 \times (-\infty, 0]$$

and

$$|\overline{u}(0,0)| = 1.$$

Under the type I condition we will show that the backward global solution $\overline{\omega}$ is nontrivial.

Proposition 2.1. *Assume that u is a type I mild solution of (NS) in $\mathbf{R}^3 \times (-1, 0)$. Then $\overline{\omega} \not\equiv 0$ in $\mathbf{R}^3 \times (-\infty, 0]$.*

Proof. Suppose that $\overline{\omega} \equiv 0$ so that curl $\overline{\omega} = 0$. Since $-\Delta\overline{u} =$ curl $\overline{\omega}$ by div $\overline{u} = 0$, the Liouville theorem for harmonic functions yields $\overline{u}$ is spatially constant. Moreover since $\overline{u}$ is a mild solution, $\overline{u}$ is also constant in time. Then the condition $|\overline{u}(0,0)| = 1$ implies $||\overline{u}||_\infty(t) = 1$ for all $t < 0$.

From the type I assumption it follows that

$$\begin{aligned} ||u_k||_\infty(t) &\leq C_0 M_k^{-1}(|t_k| + M_k^{-2}|t|)^{-1/2} = C_0(M_k^2|t_k| + |t|)^{-1/2} \\ &\leq C_0(-t)^{-1/2}, \end{aligned}$$

which yields $||\overline{u}||_\infty(t) \leq C_0(-t)^{-1/2}$. This contradicts with the fact $||\overline{u}||_\infty(t) \equiv 1$. $\Box$.

We shall show that the continuous alignment condition (CA) or more generally (CA') implies $\overline{\omega} \equiv 0$.

Proposition 2.2. *Assume that u is a mild solution of (NS) in $\mathbf{R}^3 \times (-1, 0)$ which is bounded in $\mathbf{R}^3 \times (-1, -\delta)$ for all $\delta > 0$. If ζ satisfies (CA'), then $\overline{\omega} = 0$.*

These two propositions imply the main theorem (Theorem 1.1).

Proof of Proposition 2.2. We argue by contradiction. Suppose that $\overline{\omega} \not\equiv 0$. Since $\overline{\omega}$ is continuous in $\mathbf{R}^3 \times (-\infty, 0)$, the set

$$U = \{(x,t) \in \mathbf{R}^3 \times (-\infty, 0) \mid \overline{\omega}(x,t) \neq 0\}$$

is a nonempty open set. Let K be a compact subset of U. Then there exists $\delta > 0$ and $k_0 \in \mathbf{N}$ such that

$$|\omega_k(x,t)| > \delta \text{ on } K$$

for $k \geq k_0$. Thus ζ_k is well-defined in K for $k \geq k_0$. For suffiently large k, say $k \geq k_1$ for some $k_1 (\geq k_0)$ we have $\delta > M_k^{-2} d$ where d is the constant in the definition of $\Omega_d(t)$. For $k \geq k_1$ we have

$$\begin{aligned} |\zeta_k(x,t) - \zeta_k(y,t)| &\leq \eta \left(o(1) \frac{|x-y|}{M_k \sqrt{|t_k| + M_k^{-2}|t|}} \right) \\ &\leq \eta \left(o(1) \frac{|x-y|}{\sqrt{M_k^2 |t_k| + |t|}} \right). \end{aligned}$$

By a local existence theorem for a bounded mild solution [GIM] we have a bound ε_0 such that

$$M_k^2 |t_k| > \varepsilon_0 > 0$$

for all $k > k_1$. Thus the difference is estimated as

$$|\overline{\zeta}(x,t) - \overline{\zeta}(y,t)| \leq \eta \left(o(1) \varepsilon_0^{-1/2} |x-y| \right).$$

The right hand side converges to zero as $t_k \uparrow 0$. We thus observe that $\zeta_k \to \overline{\zeta}$ with $\overline{\zeta} = \overline{\omega}/|\overline{\omega}|$ uniformly in K and

$$|\overline{\zeta}(x,t) - \overline{\zeta}(y,t)| = 0 \text{ for } x, y \in K.$$

Since K is an arbitrary compact set in U, we have

$$\overline{\omega}(x,t) = |\overline{\omega}(x,t)| \overline{\zeta}_0(t),$$

where $\overline{\zeta}_0(t)$ is a vector in $\mathbf{R}^3$ depending only on t. By rotation we may assume that $\overline{\omega}(x, t_0) = (0, 0, \overline{\omega}_3(x, t_0))$ for a given time t_0. If we assume (CA) instead of (CA'), we need not invoke a local existence theorem for a bounded mild solution in [GIM] to conclude that $\overline{\zeta}$ is a spatially constant vector.

We would like to prove that $\overline{\zeta}_0$ is also independent of time. At $t = t_0$ since $(\mathrm{curl}\, \overline{\omega})_3 = 0$ we have

$$-\Delta \overline{u}_3 = (\mathrm{curl}\, \overline{\omega})_3 = 0 \text{ in } \mathbf{R}^3.$$

By the Liouville theorem for harmonic functions this implies that $\overline{u}_3$ is spatially constant. Since $0 = \overline{\omega}_1 = \partial \overline{u}_3 / \partial x_2 - \partial \overline{u}_2 / \partial x_3$, $0 = \overline{\omega}_2 = \partial \overline{u}_1 / \partial x_3 - \partial \overline{u}_3 / \partial x_1$, we observe that u_1 and u_2 are independent of x_3. Thus the flow is two-dimensional at $t = t_0$. Once the flow is x_3-independent and u_3 is constant, by the unique local existence theorem of mild solution in [GIM], the solution stays two-dimensional (independent of x_3) for $t \geq t_0$. One may take t_0 arbitary so we conclude that $\overline{\zeta}_0$ is independent of t.

We thus conclude that $\overline{u} = (\overline{u}_1, \overline{u}_2, 0)$ and $\overline{\omega} = (0, 0, \overline{\omega}_3)$ solves the two-dimensional vorticity equation

$$\overline{\omega}_{3t} - \Delta \overline{\omega}_3 + (\overline{u}, \nabla) \overline{\omega}_3 = 0 \text{ in } \mathbf{R}^2 \times (-\infty, 0). \tag{2.2}$$

However, the following lemma which we may call a Liouville type theorem implies that $\overline{\omega} \equiv 0$ since we know that $\overline{\omega}$ and $\overline{u}$ are bounded and smooth in $\mathbf{R}^2 \times (-\infty, 0)$ and that $\mathrm{curl}\, \overline{u} = \overline{\omega}$ and $\mathrm{div}\, \overline{u} = 0$.

Lemma 2.3. *Assume that $\overline{u} = (\overline{u}_1, \overline{u}_2)$ and $\overline{\omega}_3$ are bounded smooth solution of (2.2). Assume that* $\mathrm{curl}\, \overline{u} = \overline{\omega}$ *and* $\mathrm{div}\, \overline{u} = 0$. *Then $\overline{\omega} \equiv 0$.*

Corollary 2.4. *Assume that $\overline{u}$ is a bounded backward global mild smooth solutions of (NS) in $\mathbf{R}^2 \times (-\infty, 0)$. Then $\overline{u}$ must be a constant in space time.*

Remark 2.5. (i) Corollary 2.4 follows immediately from Lemma 2.3 since $\overline{\omega}$ is bounded in $\mathbf{R}^3 \times (-\infty, 0)$ (cf. [GIM]) and satisfies the vorticity equation. (If $\overline{\omega} \equiv 0$, as discussed above, $\overline{u}$ must be a spatially constant which implies that it is space-time constant for mild solutions.)
(ii) Corollary 2.4 is already proved in [KNSS] by using stability of the strong maximum principle. We shall give a shorter proof for Lemma 2.3 for completeness.

Proof of Lemma 2.3. We may assume that $\overline{u} = (\overline{u}_1, \overline{u}_2, 0)$ and $\overline{\omega} = (0, 0, \overline{\omega}_3)$ are defined in $(-\infty, 0]$ as a smooth continuous function by shifting in time as $\overline{u}_\varepsilon(x,t)=u(x,t-\varepsilon)$, $\overline{\omega}_\varepsilon(x,y)=\overline{\omega}(x,y-\varepsilon)$ for $\varepsilon>0$. (If we are able to prove that $\overline{\omega}_\varepsilon\equiv 0$ for $t\le 0$, for sufficiently small $\varepsilon>0$ this implies $\overline{\omega}_\varepsilon\equiv 0$ in $\mathbf{R}^2\times(-\infty,0)$).

Suppose that $L=||\overline{\omega}_\varepsilon||_{L^\infty(\mathbf{R}^2\times(-\infty,0])}>0$. Then there exists a sequence of points $(x_k,t_k)\in\mathbf{R}^2\times(-\infty,0]$ satisfying $\overline{\omega}_\varepsilon(x_k,t_k)\to L(\text{or}\ -L)$. Let $\overline{u}^{(k)}(x,t):=\overline{u}(x+x_k,\ t+t_k)$ and $\overline{\omega}^{(k)}(x,t):=\overline{\omega}(x+x_k,\ t+t_k)$. Then there exist $\tilde{u},\tilde{\omega}$ such that $\overline{u}^{(k)}$ and $\overline{\omega}^{(k)}$ subsequently converges to $\tilde{u}$ and $\tilde{\omega}$ respectively (in locally uniform sense) and $\tilde{\omega}_\varepsilon$ satisfies the two-dimensional vorticity equations in $\mathbf{R}^2\times(-\infty,0]$. (Here we have invoked L^∞ theory of the Navier-Stokes equations [GIM] or the vorticity equations (cf. [GGS])). By the choice of (x_k,t_k)

$$\tilde{\omega}_\varepsilon(0,0)=L\ (\text{or}\ -L)\ \text{and}\ |\tilde{\omega}_\varepsilon|\le L.$$

By the strong maximum principle we have $\tilde{\omega}_\varepsilon\equiv L(\text{or}\ -L)$. Since $-\Delta\tilde{u}=$ curl $\tilde{\omega}$, the Liouville theorem for harmonic functions yields $\tilde{u}$ is spatially constant which implies $\tilde{\omega}=$ curl $\tilde{\omega}=0$ which contradicts the $\tilde{\omega}_\varepsilon(0,0)=L$. □

Remark 2.5. The continuous alignment assumption (CA') can be replaced by the condition for $\nabla\zeta$ as follows:

Corollary 2.6 *Let u be a type I mild solution of (NS) for $\mathbf{R}^3\times(-1,0)$. For a given $d>0$ assume that*

$$\int_{-1}^{0}\|\nabla\zeta\|^2_{L^\infty(\Omega_d(t))}(t)dt<+\infty,$$

where $\Omega_d(t)$ is defined as Theorem 1.1. Then u does not blow up at $t=0$.

Remark 2.7. Our assumption implies that $\nabla\overline{\zeta}$ is identically zero. Hence $\overline{\zeta}$ turns out to be a constant vector as in the proof of Proposition 2.2. Thus the proof is reduced to one of Theorem 1.1. A similar criterion is established for solutions of the Euler equations by [CFM]. They assumed that above type integral bound for $\nabla\zeta$ in a bunch of trajectories. Moreover, they assumed that such neighborhood is large enough to capture the local intensification of the vorticity. Under these assumptions they proved nonblow up of solutions.

Remark 2.8. In [BB], regularity of a weak solution of (NS) is established under the assumption that

$$\int_{-1}^{0}\|\nabla\zeta\|^a_{L^b(\Omega_d(t))}(t)dt<+\infty$$

for $2/a + 3/b = 1/2$. This assumption is not scaling invariant while ours is scaling invariant. It is easy to generalize our assumption in this form with $2/a + 3/b = 1$ and $2 \le a < \infty$.

2.2 Local regularity

We will give a remark on a local version of Theorem 1.1. For the purpose, we introduce the notion of suitable weak solutions following [Lin], [SS], which is originally introduced by [CKN].

Definition 2.9. *Let $B(r)$ be a ball centered at the origin with radius $r > 0$ and $Q = B(1) \times (-1, 0)$ be a unit parabolic cylinder in $\mathbf{R}^3 \times (-1, 0)$. We say the pair u and p is a suitable weak solution of (NS) in Q if the following the assumptions are satisfied*

$$u \in L^\infty(-1, 0; L^2(B(1))) \cap L^2(-1, 0; W^{1,2}(B(1))),$$
$$p \in L^{3/2}(-1, 0; L^{3/2}(B(1))),$$

and (u, p) satisfies (NS) in the sense of distributions and the local energy inequality

$$\int_{B(1)} \varphi(x,t)|u(x,t)|^2 dx + 2\int_{-1}^{t}\int_{B(1)} \varphi|\nabla u|^2 dxds$$
$$\le \int_{-1}^{t}\int_{B(1)} \{|u|^2(\Delta\varphi + \partial_t\varphi) + u\cdot\nabla\varphi(|u|^2 + 2p)\}dxds$$

for all nonnegative functions $\varphi \in C_0^\infty(Q)$ and almost all $t \in (-1, 0)$.

We now state our main result in this subsection.

Theorem 2.10. *Let (u, p) be a suitable weak solution for (NS) on $B(1) \times (-1, 0)$ safistying*

$$\|u\|_{L^\infty(B(r_0))}(t) \le C_0(-t)^{-1/2}$$

with some positive constants $r_0 \in (0, 1)$ and $C_0 > 0$ independent of $t \in (-1, 0)$.

For a given $d > 0$ let η be a modulus such that

$$|\zeta(x,t) - \zeta(y,t)| \le \eta(|x-y|) \qquad \text{for all} \quad x, y \in \Omega_d(t) \quad t \in (-1, 0), \quad \text{(D)}$$

where $\Omega_d(t) = \{x \in B(r_0); |\omega(x,t)| > d\}$. Then u is regular at $(x,t) = (0,0)$.

Remark 2.11. We do not prove Theorem 2.10 because it is almost parallel as the proof of Theorem 1.1. The only difference is the compactness argument which justifies convergence of sequences of the rescaled local solutions. For the purpose we invoke a result in Seregin and Sverak [SS, Theorem 2.8], which says that the rescaled solutions constructed by same way as in Section 2.1 are Hölder continuous locally uniformly in $\mathbf{R}^3 \times (-\infty, 0]$, and then the subsequence converges to some backward global mild solution $\overline{u}$ locally uniformly in $\mathbf{R}^3 \times (-\infty, 0]$.

We notice that since local existence theory in general domains is not known, it is not easy to weaken the assumption of the vorticity directions as (CA') in the local case.

3 Effect of boundary

We now consider the boundary value problem for (NS) when ω is the half space $\mathbf{R}^3_+$. If one imposes the slip boundary condition for example

$$\overrightarrow{n} T(u) \overrightarrow{\tau} = 0 \text{ and } u \cdot \overrightarrow{n} = 0,$$

where $\overrightarrow{n}$ is the normal $(0,0,-1)$ and $\overrightarrow{\tau}$ is a tangential vector of $\partial\mathbf{R}^3_+$, then we have

$$\frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = u_3 = 0 \text{ on } \{x_3 = 0\}. \tag{3.1}$$

This in particular implies that

$$\omega_1 = \omega_2 = 0 \text{ on } \{x_3 = 0\}. \tag{3.2}$$

where $\omega = (\omega_1, \omega_2, \omega_3)$. Here $T(u)$ is the stress tensor defined by

$$T_{ij}(u) = \left((\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) - p\delta_{ij} \right), 1 \le i, j \le 3.$$

Arguing in the same way to prove Thorem 1.1 we have

Theorem 3.1. *Let u be a type I mild solution of (NS) with $\Omega = \mathbf{R}^3_+$ where a slip boundary condition (3.1) is imposed. For a given $d > 0$ let η be a modulus satisfying the continuous alignment condition (CA) for ζ. Then u does not blow up at $t = 0$.*

The L^∞-theory of the half space with the slip boundary condition is not explicit in the literature. However, since the Stokes operator with this boundary condition is the same as the heat operator with the same boundary condition. We are able to establish L^∞ theory on the same as the whole space case.

The blow up limit $(\overline{u}, \overline{\omega})$ should fulfill either the boundary value problem or whole space problem. It depends on the behavior of dist $(x_k, \partial \mathbf{R}^3_+) M_k$. If

$$\limsup_{k\to\infty} \text{ dist } (x_k, \partial \mathbf{R}^3_+) M_k = +\infty,$$

then the problem is reduced to the whole space case cf. Section 2. We may assume that

$$\lim_{k\to\infty} \text{ dist } (x_k, \partial \mathbf{R}^3_+) M_k < +\infty$$

by taking a subsequence. In this case the limit $\overline{u}$ becomes a bounded backward global mild solution of (NS) in the half space with the slip boundary condition (cf. [G]). Since $\overline{\omega}_1 = \overline{\omega}_2 = 0$ on the boundary by (3.2), the continuous alignment condition (CA) implies that $\overline{\omega} = (0, 0, \overline{\omega}_3)$. The proof of Theorem 3.1 is completed by the following Liouville type theorem.

Lemma 3.2. *Assume that the pair* $\overline{u} = (\overline{u}_1, \overline{u}_2, \overline{u}_3)$ *and* $\overline{\omega} = (0, 0, \overline{\omega}_3)$ *is a bounded backward global mild solution of*

$$\overline{\omega}_{3t} - \Delta \overline{\omega}_3 + (\overline{u}, \nabla) \overline{\omega}_3 - \overline{\omega}_3 \partial_{x_3} \overline{u}_3 = 0 \text{ in } \mathbf{R}^3_+ \times (-\infty, 0) \qquad (3.3)$$

with $\overline{u}_3 = 0$ on $\{x_3 = 0\}$, *where* $\operatorname{curl} \overline{u} = \overline{\omega}$, $\operatorname{div} \overline{u} = 0$. *Then* $\overline{\omega} = 0$.

Proof. We first observe that $(\text{curl } \overline{\omega})_3 = 0$. Since $\overline{\omega}_3$ solves

$$-\Delta \overline{u}_3 = (\text{curl } \overline{\omega})_3 \text{ in } \mathbf{R}^3_+$$

and $\overline{u}_3$ is bounded with $\overline{u}_3 = 0$ on $\{x_3 = 0\}$, a classical Liouville theorem implies that $\overline{u}_3$ is a constant therefore $\overline{u}_3 \equiv 0$.

Since $0 = \overline{\omega}_1 = \partial \overline{u}_3 / \partial x_2 - \partial \overline{u}_2 / \partial x_3$, $0 = \overline{\omega}_2 = \partial \overline{u}_1 / \partial x_3 - \partial \overline{u}_3 / \partial x_1$, we conclude that $\overline{u}_1, \overline{u}_2$ are independent of x_3 so that $\overline{\omega}_3$ is independent of x_3.

$$\overline{\omega}_{3t} - \Delta \overline{\omega}_3 + (\overline{u}, \nabla) \overline{\omega}_3 = 0 \text{ in } \mathbf{R}^2 \times (-\infty, 0)$$

with parameter x_3. Thus the problem is reduced to the Liouville problem for the whole space. $\square$

Remark 3.3. Even if the domain has a curved boundary by blow up argument the limit $\overline{u}$ is expected to solve the problem with flat boundary as in

[G]. However, one should establish necessary L^∞ theory for curved boundary case.

We now consider the Dirichlet boundary condition

$$u_1 = u_2 = u_3 = 0 \text{ on } \{x_3 = 0\}.$$

We argue in the same way as for the slip boundary condition. The case we should discuss is the case:

$$\limsup_{k\to\infty} \operatorname{dist}(x_k, \partial \mathbf{R}^3_+) M_k < +\infty.$$

In this case the limit $\overline{u}$ becomes a bounded backward global mild solution of (NS) in the half space with the Dirichlet boundary condition. Since $\overline{\omega}_3 = 0$ by $\overline{u}_1 = \overline{u}_2 = 0$ on the boundary, (CA) implies that $\overline{\omega} = (\overline{\omega}_1, 0, 0)$ by rotating coordinates in x_1, x_2 space. Arguing in the same way as Lemma 3.2 we conclude that the problem is two-dimensional, i.e., $\overline{u}_1 = 0, \overline{u}_2, \overline{u}_3$ is independent of x_1. If the following Liouville type problem is solved, we are able to establish a regularity criterion on the vorticity direction for the Dirichlet problem similar to Theorem 3.1.

Problem 3.4. *Assume that $\overline{u} = (0, \overline{u}_2, \overline{u}_3)$ and $\overline{\omega} = (\overline{\omega}_1, 0, 0)$ are bounded smooth (up to the boundary) solutions of*

$$\overline{\omega}_{1t} - \Delta \overline{\omega}_1 + (\overline{u}, \nabla)\overline{\omega}_1 = 0 \text{ in } \mathbf{R}^2_+ \times (-\infty, 0),$$

where $\overline{\omega}_1, \overline{u}_2, \overline{u}_3$ is independent of x_1 and $\mathbf{R}^2_+ = \{(x_2, x_3) | x_3 > 0\}$. Assume that curl $\overline{u} = \overline{\omega}$ *and* div $\overline{u} = 0$. *Assume furthermore that $\overline{u} = 0$ on the boundary $\partial\mathbf{R}^2_+ \times (-\infty, 0)$. Are there any (nontrivial) solutions other than $\overline{u} \equiv 0$?*

Remark 3.5. If $\overline{u}$ decays rapidly enough as $t \to -\infty$ and $|x| \to \infty$, by the standard energy inequality we know $\overline{u}$ must be zero. We do not know the decay of $\overline{u}$ and $\overline{\omega}$. If we apply the similar argument to prove the Liouville type theorem for the whole space, we are able to prove the following decay estimate:

Proposition 3.6. *Under the assumption of Problem 3.4, we have*

$$\lim_{|m|\to\infty} \sup_{x_2 \in \mathbf{R}, |x_3| \geq m, t<0} |\overline{\omega}_1(x_2, x_3, t)| = 0.$$

Proof. Suppose that there exist positive constants $m_0 > 0$ and $\alpha > 0$ and a sequence $(x_k, t_k) \in \mathbb{R}^2_+ \times (-\infty, 0)$ such that $x_{k,3} \uparrow +\infty$ and

$$|\overline{\omega}_1(x_k, t_k)| \geq \alpha.$$

Let $\overline{u}^{(k)}(x,t) := \overline{u}(x + x_k, t + t_k)$ and $\overline{\omega}^{(k)}(x,t) := \overline{\omega}(x + x_k, t + t_k)$. By a standard compactness argument there exists the limit $(\tilde{u}, \tilde{\omega})$ (by taking an appropriate subsequence of $(\overline{u}^{(k)}, \overline{\omega}^{(k)})$) which is a classical solution of the vorticity equation in the whole plane for $t < 0$. Indeed, since $\overline{u}^{(k)}$ and $\overline{\omega}^{(k)}$ are bounded, a standard parabolic theory [LSU] implies that $\overline{\omega}^{(k)}$ is locally bounded in $W^{2,1}_p$ for any $p > 1$. This already implies that $(\overline{u}^{(k)}, \overline{\omega}^{(k)})$ converges to $(\tilde{u}, \tilde{\omega})$ in a distribution sense. Since $u \cdot \nabla\omega = \nabla \cdot (u\,\omega)$, this observation yields that $\tilde{u}$ and $\tilde{\omega}$ solve the vorticity equations in the whole plane for $t < 0$. It is a classical solution by a regularity theory of parabolic and Poisson equation for u.

Since $W^{2,1}_p$ is compactly embedded in a Hölder space locally ([LSU], Chap II, Lemma 3.3), we may conclude that $\overline{\omega}^{(k)}$ converges its limit at least uniformly. This implies that $|\tilde{\omega}(0,0)| \geq \alpha$, and so $\tilde{\omega}$ is not identically zero. Therefore we get a nontrivial bounded solution for the vorticity equation in the whole plane for $t < 0$. This contradicts with the Liouville type theorem Lemma 2.3. $\square$

Remark 3.7. If we assume in addition that

$$\left(||\partial\omega_{\tan}/\partial\overrightarrow{n}||_\infty/||u||^2_\infty\right)(t) \to 0$$

as $t \uparrow 0$ other than (CA), then we get $\partial\overline{\omega}_1/\partial\overrightarrow{n} = 0$ at the limit problem. If we have such a condition for our Liouville problem, the similar argument to prove the Liouville problem for the whole space yields nonexistence of nontrivial solution since $\overline{\omega}_1$ cannot attain its maximum on the boundary. Here $\omega_{\tan}$ denotes the tangential component of ω.

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