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CONGRUENCE BETWEEN DUKE-IMAMOGLU-IKEDA LIFTS AND NON-DUKE-IMAMOGLU-IKEDA LIFTS

HIDENORI KATSURADA

1. Introduction

As is well known, there is a congruence between the Fourier coefficients of Eisenstein series and those of cuspidal Hecke eigenforms for $SL_2(\mathbb{Z})$. This type of congruence is not only interesting in its own right but also has an importan application to number theory as shown by Ribet [Ri1]. Since then, there have been so many important results about the congruence of elliptic modular forms (cf. [Hi1],[Hi2],[Hi3].) In the case of Hilbert modular forms or Siegel modular forms, it is sometimes more natual and important to consider the congruence between the Hecke eigenvalues of Hecke eigenforms modulo a prime ideal $\mathfrak{p}$. We call such a $\mathfrak{p}$ a prime ideal giving the congruence or a congruence prime. For a cuspidal Hecke eigenform $f$ for $SL_2(\mathbb{Z})$, let $\hat{f}$ be a lift of $f$ to the space $M_l(\Gamma')$ of modular forms of weight $l$ for a modular group $\Gamma'$. Here we mean by the lift of $f$ a cuspidal Hecke eigenform whose certain L-function can be expressed in terms of certain L-functions of $f$. As typical examples of the lift we can consider the Doi-Naganuma lift, the Saito-Kurokawa lift, and the Duke-Imamoglu-Ikeda lift. We then consider the following problem:

**Problem.** Characterize the prime ideals giving the congruence between $\hat{f}$ and a cuspidal Hecke eigenform in $M_l(\Gamma')$ not coming from the lift. In particular characterize them in terms of special values of certain L-functions of $f$.

This type of problem was first invetigated in the Doi-Naganuma lift case by Doi, Hida, and Ishii [D-H-I]. In our previous paper [Ka2], we considered the relationship between the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbb{Z})$ and the special values of their standard zeta functions. In particular, we proposed a conjecture concerning the congruence between Saito-Kurokawa lifts and non-Saito-Kurokawa...
lifts, and proved it under certain condition. In this paper, we consider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, which is a generalization of our previous conjecture.

In Section 3, we review a result concerning the relationship between the congruence of cuspidal Hecke eigenforms with respect to $\text{Sp}_n(\mathbb{Z})$ and the special values of their standard zeta functions. In Section 4, we propose a conjecture concerning the congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, and prove it under a certain condition.

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Notation. For a commutative ring $R$, we denote by $M_{mn}(R)$ the set of $(m, n)$-matrices with entries in $R$. In particular put $M_n(R) = M_{nn}(R)$. Here we understand $M_{mn}(R)$ the set of the empty matrix if $m = 0$ or $n = 0$. For an $(m, n)$-matrix $X$ and an $(m, m)$-matrix $A$, we write $A[X] = ^tXAX$, where $^tX$ denotes the transpose of $X$. Let $a$ be an element of $R$. Then for an element $X$ of $M_{mn}(R)$ we often use the same symbol $X$ to denote the coset $X \mod aM_{mn}(R)$. Put $GL_n(R) = \{ A \in M_n(R) \mid \det A \in R^* \}$, where $\det A$ denotes the determinant of a square matrix $A$, and $R^*$ denotes the unit group of $R$. Let $S_n(R)$ denote the set of symmetric matrices of degree $n$ with entries in $R$. Furthermore, for an integral domain $R$ of characteristic different from 2, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree $n$ over $R$, that is, $\mathcal{H}_n(R)$ is the set of symmetric matrices of degree $n$ whose $(i, j)$-component belongs to $R$ or $\frac{1}{2}R$ according as $i = j$ or not. For a subset $S$ of $M_n(R)$ we denote by $S^\times$ the subset of $S$ consisting of non-degenerate matrices. In particular, if $S$ is a subset of $S_n(R)$ with $R$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of $S$ consisting of positive definite (resp. semi-positive definite) matrices. Let $R'$ be a subring of $R$. Two symmetric matrices $A$ and $A'$ with entries in $R$ are called equivalent over $R'$ with each other and write $A \sim_{R'} A'$ if there is an element $X$ of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices $X$ and $Y$ we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

2. Standard zeta functions of Siegel modular forms

For a complex number $x$ put $e(x) = \exp(2\pi \sqrt{-1}x)$. Furthermore put $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$, where $1_n$ denotes the unit matrix of degree $n$. For
a subring $K$ of $\mathbb{R}$ put

$$GSp_n(K)^+ = \{ M \in GL_{2n}(K) \mid J_n[M] = \kappa(M)J_n \text{ with some } \kappa(M) > 0 \},$$

and

$$Sp_n(K) = \{ M \in GSp_n(K)^+ \mid J_n[M] = J_n \}.$$  

Furthermore, put

$$\Gamma^{(n)} = Sp_n(\mathbb{Z}) = \left\{ M \in GL_{2n}(\mathbb{Z}) \mid J_n[M] = J_n \right\}.$$ 

We sometimes write an element $M$ of $GSp_n(K)$ as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in M_2(K)$. We define a subgroup $\Gamma_0^{(n)}(N)$ of $\Gamma^{(n)}$ as

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \mid C \equiv O_n \mod N \right\}.$$ 

Let $\mathcal{H}_n$ be Siegel’s upper half-space. For each element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n(\mathbb{R})^+$ and $Z \in \mathcal{H}_n$ put

$$M(Z) = (AZ + B)(CZ + D)^{-1}$$

and

$$j(M,Z) = \det(CZ + D).$$

Furthermore, for a function $f$ on $\mathcal{H}_n$ and an integer $k$ we define $f|_k M$ as

$$(f|_k M)(Z) = \det(M)^{k/2} j(M,Z)^{-k} f(M(Z)).$$

For an integer or half integer $l$ and the subgroup $\Gamma_0^{(n)}(N)$ of $\Gamma^{(n)}$, we denote by $\mathcal{M}_k(\Gamma_0^{(n)}(N))$ (resp. $\mathcal{M}_k^\infty(\Gamma_0^{(n)}(N))$) the space of holomorphic (resp. $C^\infty$-) modular forms of weight $k$ with respect to $\Gamma_0^{(n)}(N)$. We denote by $\mathcal{E}_k(\Gamma_0^{(n)}(N))$ the sub-space of $\mathcal{M}_k(\Gamma_0^{(n)}(N))$ consisting of cusp forms. Let $f$ be a holomorphic modular form of weight $k$ with respect to $\Gamma_0^{(n)}(N)$. Then $f$ has the following Fourier expansion:

$$f(Z) = \sum_{A \in \mathcal{H}(Z) \geq 0} a_f(A) e(\text{tr}(AZ)),$$

and in particular if $f$ is a cusp form, $f$ has the following Fourier expansion:

$$f(Z) = \sum_{A \in \mathcal{H}(Z) \geq 0} a_f(A) e(\text{tr}(AZ)),$$

where tr denotes the trace of a matrix. Let $dv$ denote the invariant volume element on $\mathcal{H}_n$ defined by $dv = \det(\text{Im}(Z))^{n-1} \wedge_{1 \leq j \leq l \leq n} (dx_{jl} \wedge dy_{jl})$. Here for $Z \in \mathcal{H}_n$ we write $Z = (x_{jl}) + \sqrt{-1}(y_{jl})$ with real matrices
(x_j) and (y_j). For two $C^\infty$-modular forms $f$ and $g$ of weight $l$ with
respect to $\Gamma_0(n)(N)$ we define the Petersson scalar product $\langle f, g \rangle$ by
\[
\langle f, g \rangle = [\Gamma(n) : \Gamma_0(n)(N)]^{-1} \int_{\Gamma_0(n)(N)\mathbb{H}} f(Z)g(Z) \det(\text{Im}(Z))^l \, dv,
\]
provided the integral converges.

Let $\mathbb{L}_n = \mathbb{L}_Q(\text{GSp}_n(Q)^+, \Gamma^{(n)})$ denote the Hecke algebra over $Q$
associated with the Hecke pair $(\text{GSp}_n(Q)^+, \Gamma^{(n)})$. Furthermore, let $\mathbb{L}_n' =
\mathbb{L}_Q(\text{Sp}_n(Q), \Gamma^{(n)})$ denote the Hecke algebra over $Q$
associated with the Hecke pair $(\text{Sp}_n(Q), \Gamma^{(n)})$. For each integer $m$ define an element $T(m)$
of $\mathbb{L}_n$ by
\[
T(m) = \sum_{d_1, \ldots, d_n, e_1, \ldots, e_n} \Gamma^{(n)}(d_1 \perp \ldots \perp d_n \perp e_1 \perp \ldots \perp e_n) \Gamma^{(n)},
\]
where $d_1, \ldots, d_n, e_1, \ldots, e_n$ run over all positive integer satisfying
\[
d_i | d_{i+1}, \quad e_{i+1} | e_i \quad (i = 1, \ldots, n), \quad d_n | e_n, \quad d_i e_i = m \quad (i = 1, \ldots, n).
\]
Furthermore, for $i = 1, \ldots, n$ and a prime number $p$ put
\[
T_i(p^2) = \Gamma^{(n)}(1_n \perp p 1_i \perp p^2 1_{n-i} \perp p 1_i) \Gamma^{(n)},
\]
and $(p^{-1}) = \Gamma^{(n)}(p^{-1}) \Gamma^{(n)}$. As is well known, $\mathbb{L}_n$ is generated over $Q$
by all $T(p), T_i(p^2)$ ($i = 1, \ldots, n$), and $(p^{-1})$. We denote by $\mathbb{L}_n'$ the
subalgebra of $\mathbb{L}_n$ generated over $\mathbb{Z}$ by all $T(p)$ and $T_i(p^2)$ ($i = 1, \ldots, n$). Let $T = \Gamma^{(n)} M \Gamma^{(n)}$
be an element of $\mathbb{L}_n \otimes \mathbb{C}$. Write $T$ as $T = \bigcup \Gamma^{(n)} \gamma$
and for $f \in \mathbb{M}_k(\Gamma^{(n)})$ define the Hecke operator $T$ associated to $T$ as
\[
f| T = \text{det}(M)^{k/2 -(n+1)/2} \sum_{\gamma} f| \gamma.
\]
We call this action the Hecke operator as usual (cf. [A].) If $f$ is an
eigenfunction of a Hecke operator $T \in \mathbb{L}_n \otimes \mathbb{C}$, we denote by $\lambda_f(T)$ its
eigenvalue. Let $\mathbb{L}$ be a subalgebra of $\mathbb{L}_n$. We call $f \in \mathbb{M}_k(\Gamma^{(n)})$ a Hecke
eigenform for $\mathbb{L}$ if it is a common eigenfunction of all Hecke operators in $\mathbb{L}$. In particular if $\mathbb{L} = \mathbb{L}_n$
we simply call $f$ a Hecke eigenform. Furthermore, we denote by $\mathbb{Q}(f)$ the field generated over $Q$
by eigenvalues of all $T \in \mathbb{L}_n$ as in Section 1. As is well known, $\mathbb{Q}(f)$ is a totally real
algebraic number field of finite degree. Now, first we consider the integrality
of the eigenvalues of Hecke operators. For an algebraic number field $K$, let $\mathcal{O}_K$
denote the ring of integers in $K$. The following assertion has been proved in [Mi2] (see also [Ka2].)

**Theorem 2.1** Let $k \geq n + 1$. Let $f \in \mathbb{M}_k(\Gamma^{(n)})$ be a common eigenform
in $\mathbb{L}_n'$. Then $\lambda_f(T)$ belongs to $\mathbb{Q}(f)$ for any $T \in \mathbb{L}_n'$. 
Let $L_{np} = L(GSp_n(Q)^+ \cap GL_2n(Z[p^{-1}]), \Gamma(n))$ be the Hecke algebra associated with the pair $(GSp_n(Q)^+ \cap GL_2n(Z[p^{-1}]), \Gamma(n))$. $L_{np}$ can be considered as a subalgebra of $L_n$, and is generated over $Q$ by $T(p)$ and $T_i(p^2) (i = 1, 2, \ldots, n)$. We now review the Satake parameters of $L_{np}$; let $P_n = Q[X_0^+, X_1^+, \ldots, X_n^+]$ be the ring of Laurent polynomials in $X_0, X_1, \ldots, X_n$ over $Q$. Let $W_n$ be the group of $Q$-automorphisms of $P_n$ generated by all permutations in variables $X_1, \ldots, X_n$ and by the automorphisms $\tau_1, \ldots, \tau_n$ defined by

$$\tau_i(X_0) = X_0 X_i, \tau_i(X_i) = X_i^{-1}, \tau_i(X_j) = X_j (j \neq i).$$

Furthermore, a group $W_n$ isomorphic to $W_n$ acts on the set $T_n = (C^*)^{n+1}$ in a way similarly to above. Then there exists a $Q$-algebra isomorphism $\Phi_{np}$, called the Satake isomorphism, from $L_{np}$ to the $W_n$-invariant subring $P_n^{W_n}$ of $P_n$. Then for a $Q$-algebra homomorphism $\lambda$ from $L_{np}$ to $C$, there exists an element $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))$ of $T_n$ satisfying

$$\lambda(\Phi_{np}^{-1}(F(X_0, X_1, \ldots, X_n))) = F(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))$$

for $F \in P_n^{W_n}$. The equivalence class of $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))$ under the action of $W_n$ is uniquely determined by $\lambda$. We call this the Satake parameters of $L_{np}$ determined by $\lambda$.

Now assume that an element $f$ of $M_k(Sp_n(Z))$ is a Hecke eigenform. Then for each prime number $p$, $f$ defines a $Q$-algebra homomorphism $\lambda_{f,p}$ from $L_{np}$ to $C$ in a usual way, and we denote by $\alpha_0(p), \alpha_1(p), \ldots, \alpha_n(p)$ the Satake parameters of $L_{np}$ determined by $f$. We then define the standard zeta function $L(f, s, St)$ by

$$L(s, f, St) = \prod_p \prod_{i=1}^n \{(1 - p^{-s})(1 - \alpha_i(p)p^{-s})(1 - \alpha_i(p)^{-1}p^{-s})\}^{-1}. $$

Let $f(z) = \sum_{A \in A_2(Z)} a(A)e(\text{tr}(Az))$ be a Hecke eigenform in $\mathfrak{S}_k(\Gamma(n))$. For a positive integer $m \leq k - n$ such that $m \equiv n \text{ mod } 2$ put

$$\Lambda(f, m, St) = (-1)^{n(m+1)/2+1} 2^{-4kn+3n^2+n+(n-1)m+2} \times \Gamma(m+1) \prod_{i=1}^n \Gamma(2k-n-i) \frac{L(f, m, St)}{<f, f> \pi^{-n(n+1)/2+nk+(n+1)m}}.$$ 

Then the following theorem is due to Böcherer [B2] and Mizimoto [Mi].

**Theorem 2.2.** Let $l, k$ and $n$ be positive integers such that $\rho(n) \leq l \leq k-n$, where $\rho(n) = 3$, or 1 according as $n \equiv 1 \text{ mod } 4$ and $n \geq 5$, or not. Let $f \in \mathfrak{S}_k(\Gamma(n))$ be a Hecke eigenform. Then $\Lambda(f, m, St)$ belongs to $Q(f)$. 
For later purpose, we consider a special element in $L_{np}$; the polynomial $X_0^2X_1X_2\cdots X_n\sum_{i=1}^n(X_i+X_i^{-1})$ is an element of $P_{n}^W$, and thus we can define an element $\Phi_{np}(X_0^2X_1X_2\cdots X_n\sum_{i=1}^n(X_i+X_i^{-1}))$ of $L_{np}$, which is denoted by $r_1$.

**Proposition 2.3.** Under the above notation the element $r_1$ belongs to $L'_n$, and we have

$$\lambda_f(r_1) = p^{n(k-(n+1)/2)}\sum_{i=1}^n(\alpha_i(p) + \alpha_i(p)^{-1}).$$

**Proof.** By a careful analysis of the computation in page 159-160 of [A], we see that $r_1$ is a $\mathbf{Z}$-linear combination of $T_i(p^2) \ (i = 1, ..., n)$, and therefore we can prove the first assertion. Furthermore, by Lemma 3.3.34 of [A], we can prove the second assertion.

3. **Congruence of modular forms and special values of the standard zeta functions**

In this section we review a result concerning the congruence between the Hecke eigenvalues of modular forms of the same weight following [Ka2]. Let $K$ be an algebraic number field, and $\mathcal{D} = \mathcal{O}_K$ the ring of integers in $K$. For a prime ideal $\mathfrak{p}$ of $\mathcal{D}$, we denote by $\mathcal{O}_\mathfrak{p}$ the localization of $\mathcal{D}$ at $\mathfrak{p}$ in $K$. Let $\mathfrak{a}$ be a fractional ideal in $K$. If $\mathfrak{a} = \mathfrak{p}\mathfrak{b}$ with $\mathfrak{b}\mathcal{O}_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$ we write $\text{ord}_\mathfrak{p} = e$. We simply write $\text{ord}_\mathfrak{p}(c) = \text{ord}_\mathfrak{p}(c)$. Now let $f$ be a Hecke eigenform in $\mathcal{S}_k(\Gamma(n))$ and $M$ be a subspace of $\mathcal{S}_k(\Gamma(n))$ stable under Hecke operators $T \in L_n$. Assume that $M$ is contained in $(Cf)^\perp$, where $(Cf)^\perp$ is the orthogonal complement of $Cf$ in $\mathcal{S}_k(\Gamma(n))$ with respect to the Petersson product. Let $K$ be an algebraic number field containing $\mathbf{Q}(f)$. A prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ is called a congruence prime of $f$ with respect to $M$ if there exists a Hecke eigenform $g \in M$ such that

$$\lambda_f(T) \equiv \lambda_g(T) \mod \tilde{\mathfrak{p}}$$

for any $T \in L'_n$, where $\tilde{\mathfrak{p}}$ is the prime ideal of $\mathcal{O}_{K\mathfrak{p}(g)}$ lying above $\mathfrak{p}$. If $M = (Cf)^\perp$, we simply call $\mathfrak{p}$ a congruence prime of $f$.

Now we consider the relation between the congruence primes and the standard zeta values. To consider this, we have to normalize the standard zeta value $\Lambda(f, l, \text{St})$ for a Hecke eigenform $f$ because it is not uniquely determined by the system of Hecke eigenvalues of $f$. We note that there is no reasonable normalization of cuspidal Hecke eigenform in the higher degree case unlike the elliptic modular case.
Thus we define the following quantities: for a Hecke eigenform $f(z) = \sum_A a_f(A)e(\text{tr}(Az))$ in $\mathfrak{S}_k(\Gamma(n))$, let $\mathfrak{I}_f$ be the $\mathcal{O}_{\mathbb{Q}(f)}$-module generated by all $a_f(A)$'s. Assume that there exists a complex number $c$ such that all the Fourier coefficients $cf$ belong to $\mathbb{Q}(f)$. Then $\mathfrak{I}_f$ is a fractional ideal in $\mathbb{Q}(f)$, and therefore, so is $\Lambda(f, l, S^2)\mathfrak{I}_f^2$ if $l$ satisfies the condition in Theorem 2.2. We note that this fractional ideal does not depend on the choice of $c$. We also note that the value $N_{\mathbb{Q}(f)}(\Lambda(f, l, S^2))N(\mathfrak{I}_f)^2$ does not depend on the choice of $c$, where $N(\mathfrak{I}_f)$ is the norm of the ideal $\mathfrak{I}_f$. Then, we have

**Theorem 3.1.** Let $f$ be a Hecke eigenform in $\mathfrak{S}_k(\Gamma(n))$. Assume that there exists a complex number $c$ such that all the Fourier coefficients $cf$ belong to $\mathbb{Q}(f)$. Let $l$ be a positive integer satisfying the condition in Theorem 2.2. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$. Assume that $\text{ord}_{\mathfrak{p}}(\Lambda(f, l, S^2)) < 0$ and that it does not divide $(2l - 1)!$. Then $\mathfrak{p}$ is a congruence prime of $f$. In particular, if a rational prime number $p$ divides the denominator of $N_{\mathbb{Q}(f)}(\Lambda(f, l, S^2))N(\mathfrak{I}_f)^2$, then $p$ is divisible by some congruence prime of $f$.

Now for a Hecke eigenform $f$ in $\mathfrak{S}_k(\Gamma(n))$, let $\mathfrak{S}_f$ denote the subspace of $\mathfrak{S}_k(\Gamma(n))$ spanned by all Hecke eigenforms with the same system of the Hecke eigenvalues as $f$.

**Corollary.** In addition to the above notation and the assumption, assume that $\mathfrak{S}_k(\Gamma(n))$ has the multiplicity one property. Then $\mathfrak{p}$ is a congruence prime of $f$ with respect to $\mathfrak{S}_f$. In particular, if a rational prime number $p$ divides the denominator of $N_{\mathbb{Q}(f)}(\Lambda(f, l, S^2))N(\mathfrak{I}_f)^2$, then $p$ is divisible by some congruence prime of $f$ with respect to $\mathfrak{S}_f$.

4. **Congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts**

In this section, we consider the congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts. Throughout this section and the next, we assume that $n$ and $k$ are even positive integers. Let

$$f(z) = \sum_{m=1}^{\infty} a(m)e(mz)$$

be a normalized Hecke eigenform of weight $2k - n$ with respect to $SL_2(\mathbb{Z})$. For a Dirichlet character $\chi$, we then define the L-function
$L(s, f)$ of $f$ twisted by $\chi$ by

$$L(s, f) = \prod_p \{(1 - \chi(p)\beta_p p^{k-n/2 - 1/2-s})(1 - \chi(p)\beta_p^{-1} p^{k-n/2 - 1/2-s})\}^{-1},$$

where $\beta_p$ is a non-zero complex number such that $\beta_p + \beta_p^{-1} = p^{k-n/2 + 1/2} a(p)$. We simply write $L(s, f)$ as $L(s, f, \chi)$ if $\chi$ is the principal character. Furthermore, let $\tilde{f}$ be the cusp form of weight $k - n/2 + 1/2$ belonging to the Kohnen plus space corresponding to $f$ via the Shimura correspondence (cf. [Ko1]). Then $\tilde{f}$ has the following Fourier expansion:

$$\tilde{f}(z) = \sum e c(e)e(ez),$$

where $e$ runs over all positive integers such that $(-1)^{k-n/2} e \equiv 0, 1 \mod 4$.

We then put

$$a_{I_n(f)}(T) = c(b_T) \prod_p (p^{k-n/2 - 1/2}\beta_p, l_T) F_p(T, p^{-(n+1)/2}\beta_p^{-1}).$$

We note that $a_{I_n(f)}(T)$ does not depend on the choice of $\beta_p$. Define a Fourier series $I_n(f)(Z)$ by

$$I_n(f)(Z) = \sum_{T \in \mathbb{H}_K} a_{I_n(f)}(T)e(\text{tr}(TZ)).$$

In [Ik1] Ikeda showed that $I_n(f)(Z)$ is a cusp form of weight $k$ with respect to $\Gamma^{(a)}$ and a Hecke eigenform for $L_n^e$ such that

$$L(s, I_n(f), \mathbb{S}) = \zeta(s) \prod_{i=1}^n L(s + k - i, f).$$

This was first conjecture by Duke and Imamoglu. Thus we call $I_n(f)$ the Duke-Imamoglu-Ikeda lift of $f$. We note that we have $Q(\tilde{f}) = Q(I_n(f)) = Q(f)$. Furthermore, we have $\mathfrak{F}\tilde{f} = \mathfrak{F}I_n(f)$, where $\mathfrak{F}\tilde{f}$ is the $Q(f)$-module generated by all the Fourier coefficients of $\tilde{f}$.

Now to consider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, first we prove the following:

**Proposition 4.1** $I_n(f)$ is a Hecke eigenform.

We note that Ikeda proved in [Ik1] that $I_n(f)$ is a Hecke eigenform for $L_n^e$ but has not proved that it is a Hecke eigenform for $L_n$. This was pointed to us by B. Heim (see [He].) We thank him for his comment. We also note that an explicit form of the spinor L-function of $I_n(f)$ was obtained by Murakawa [Mu] and Schmidt [Sch] assuming that $I_n(f)$ is a Hecke eigenform.
Proof of Proposition 4.1. We have only to prove that $I_n(f)$ is an eigenfunction of $T(p)$ for any prime $p$. The proof may be more or less well known, but for the convenience of the readers we here give the proof. For a modular form

$$F(Z) = \sum_B c_F(B)e(tr(BZ)),$$

let $c_F^{(p)}(B)$ be the $B$-th Fourier coefficient of $F|T(p)$. Then for any positive definite matrix $B$ we have

$$c_F^{(p)}(B) = p^{nk-n(n+1)/2} \sum_{d_1|d_2|\cdots|d_n|p} d_1^n d_2^{n-1} \cdots d_n$$

$$\times \sum_{D \in \Lambda_n(d_1 \perp \cdots \perp d_n)\Lambda_n} \det D^{-k} c_F(p^{-1}A[t^l D]),$$

where $\Lambda_n = GL_n(\mathbb{Z})$.

Now let $E_{n,k}(Z)$ be the Siegel Eisenstein series of degree $n$ and of weight $k$ defined by

$$E_{n,k}(Z) = \sum_{\gamma \in \Gamma_{n,\infty} \setminus \Gamma_n} j(\gamma, Z)^{-k}.$$

For $k \geq n + 1$, the Siegel Eisenstein series $E_{n,k}(Z)$ is a holomorphic modular form of weight $k$ with respect to $\Gamma_n$. Furthermore, $E_{n,k}(Z)$ is a Hecke eigenform and in particular we have

$$E_{n,k}|T(p)(Z) = h_{n,p}(p^k)E_{n,k}(Z),$$

where

$$h_{n,p}(X) = 1 + \sum_{r=1}^{n} \sum_{1 \leq i_1 < \cdots < i_r \leq n} p^{-\sum_{j=1}^{r} i_j} X^r.$$

Let $c_{n,k}(B)$ be the $B$-th Fourier coefficient of $E_{n,k}(Z)$. Then we have

$$h_{n,p}(p^k)c_{n,k}(B) = p^{nk-n(n+1)/2} \sum_{d_1|d_2|\cdots|d_n|p} d_1^n d_2^{n-1} \cdots d_n$$

$$\times \sum_{D \in \Lambda_n(d_1 \perp \cdots \perp d_n)\Lambda_n} \det D^{-k} c_{n,k}(p^{-1}B[t^l D]).$$

Let $B$ be positive definite. Then we have

$$c_{n,k}(B) = a_{n,k}(\det 2B)^{k-(n+1)/2} L(k-n/2, \chi_B) \prod_q F_q(B, p^{-k}),$$

where $a_{n,k}$ is a non-zero constant depending only on $n$ and $k$. We note that we have

$$F_q(p^{-1}B[t^l D], X) = F_q(B, X)$$
for any $D \in \Lambda_n(d_1 \cdot \ldots \cdot d_n)\Lambda_n$ with $d_1 | \cdots | d_n | p$ if $q \neq p$. Thus we have

$$h_{n,p}(k)F_p(B, p^{-k}) = \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{n e_1 + (n-1)e_2 + \cdots + e_n} p^{(k-n-1)}F_p(p^{-1}B[D], p^{-k}).$$

The both-hand sides of the above are polynomials in $p^k$ and the equality holds for infinitely many $k$. Thus we have

$$h_{n,p}(X^{-1})F_p(B, X) = \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{n e_1 + (n-1)e_2 + \cdots + e_n} (X^{-1}p^{-n-1})^{(e_1 + \cdots + e_n)}F_p(p^{-1}B[D], X)$$

as polynomials in $X$ and $X^{-1}$. Thus we have

$$(p^{k-(n+1)/2}X)^{n/2}h_{n,p}(p^{-(n+1)/2}X^{-1})(p^{k-(n+1)/2}X^{-1})^{\nu_p(B)}F_p(B, p^{-(n+1)/2}X)$$

$$= p^{nk-n(n+1)/2} \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{n e_1 + (n-1)e_2 + \cdots + e_n} \cdot \det D^{-k}((p^{k-(n+1)/2}X^{-1})^{\nu_p(B[D])}F_p(p^{-1}B[D], p^{-(n+1)/2}X).$$

We recall that we have

$$c_{I_n}(f)(B) = c_f([B])\frac{k-(n+1)/2}{B} \prod_q (\beta_q)^{\nu_q([B])}F_q(B, q^{-(n+1)/2}\beta_q^{-1}),$$

where $\beta_q$ is the Satake $q$-parameter of $f$. We also note that $c_f([B^{-1}B[D]]) = c_f([B])$ for any $D$. Thus we have

$$p^{nk-n(n+1)/2} \sum_{d_1 | d_2 | \cdots | d_n | p} d_1^{n}d_2^{n-1} \cdots d_n \sum_{D \in \Lambda_n(d_1 \cdot \ldots \cdot d_n)\Lambda_n} \det D^{-k}c_{I_n}(f)(p^{-1}B[D]).$$

This proves the assertion.

Let $f$ be a primitive form in $\mathfrak{S}_{2k-n}(\Gamma(1))$. Let $\{f_1, \ldots, f_d\}$ be a basis of $\mathfrak{S}_{2k-n}(\Gamma(1))$ consisting of primitive forms. Let $K$ be an algebraic number field containing $\mathbb{Q}(f_1) \cdot \cdots \cdot \mathbb{Q}(f_d)$, and $A = \mathcal{O}_K$. To formulate our conjecture exactly, we introduce the Eichler-Shimura periods as follows (cf. Hida [Hi3].) Let $\mathfrak{p}$ be a prime ideal in $K$. Let $A_{\mathfrak{p}}$ be a valuation ring in $K$ corresponding to $\mathfrak{p}$. Assume that the residual characteristic of $A_{\mathfrak{p}}$ is greater than or equal to 5. Let $L(2k-n-2, A_{\mathfrak{p}})$ be the module of homogeneous polynomials of degree $2k-n-2$ in the variables $X, Y$
with coefficients in $A_{\mathfrak{p}}$. We define the action of $M_2(\mathbb{Z}) \cap GL_2(\mathbb{Q})$ on $L(2k - n - 2, A_{\mathfrak{p}})$ via

$$\gamma \cdot P(X, Y) = P^{(f(X, Y)(\gamma)^*)},$$

where $\gamma^* = (\det \gamma)^{-1}$. Let $H^1_P(\Gamma(1), L(2k - n - 2, A_{\mathfrak{p}}))$ be the parabolic cohomology group of $\Gamma(1)$ with values in $L(2k - n - 2, A_{\mathfrak{p}})$. Fix a point $z_0 \in H_1$. Let $g \in \mathfrak{S}_{2k-n}(\Gamma(1))$ or $g \in \mathfrak{S}_{2k-n}(\Gamma(1))$. We then define the differential $\omega(g)$ as

$$\omega(g)(z) = \begin{cases} 2\pi i g(z)(X - zY)^n dz & \text{if } g \in \mathfrak{S}_{2k-n}(\Gamma(1)) \\ 2\pi \sqrt{-1} g(z)(X - zY)^n dz & \text{if } g \in \mathfrak{S}_{2k-n}(\Gamma(1)), \end{cases}$$

and define the cohomology class $\delta(g)$ of the 1-cocycle of $\Gamma(1)$ as

$$\gamma \in \Gamma(1) \mapsto \int_{z_0}^{\gamma(z_0)} \omega(g).$$

The mapping $g \mapsto \delta(g)$ induces the isomorphism

$$\delta : \mathfrak{S}_{2k-n}(\Gamma(1)) \oplus \mathfrak{S}_{2k-n}(\Gamma(1)) \to H^1_P(\Gamma(1), L(2k - n - 2, \mathbb{C})),
$$

which is called the Eichler-Shimura isomorphism. We can define the action of Hecke algebra $L'_1$ on $H^1_P(\Gamma(1), L(2k - n - 2, A_{\mathfrak{p}}))$ in a natural manner. Furthermore, we can define the action $F_\infty$ on $H^1_P(\Gamma(1), L(2k - n - 2, A_{\mathfrak{p}}))$ as

$$F_\infty(\delta(g)(z)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \delta(g)(-z),$$

and this action commutes with the Hecke action. For a primitive form $f$ and $j = \pm 1$, we define the subspace $H^1_P(\Gamma(1), L(2k - n - 2, A_{\mathfrak{p}}))[f, j]$ of $H^1_P(\Gamma(1), L(2k - n - 2, A_{\mathfrak{p}}))$ as

$$H^1_P(\Gamma(1), L(2k - n - 2, A_{\mathfrak{p}}))[f, j] = \{ x \in H^1_P(\Gamma(1), L(2k - n - 2, A_{\mathfrak{p}})) \mid x[T = \lambda_f(T)x \text{ for } T \in L_1, \text{ and } F_\infty(x) = jx \}.$$ 

Since $A_{\mathfrak{p}}$ is a principal ideal domain, $H^1_P(\Gamma(1), L(2k - n - 2, A_{\mathfrak{p}}))[f, j]$ is a free module of rank one over $A_{\mathfrak{p}}$. For each $j = \pm 1$ take a basis $\eta(f, j, A_{\mathfrak{p}})$ of $H^1_P(\Gamma(1), (2k - n - 2, A_{\mathfrak{p}}))[f, j]$ and define a complex number $\Omega(f, j; A_{\mathfrak{p}})$ by

$$(\delta(f) + jF_\infty(\delta(f)))/2 = \Omega(f, j; A_{\mathfrak{p}})\eta(f, j; A_{\mathfrak{p}}).$$

This $\Omega(f, j; A_{\mathfrak{p}})$ is uniquely determined up to constant multiple of units in $A_{\mathfrak{p}}$. We call $\Omega(f, +; A_{\mathfrak{p}})$ and $\Omega(f, -; A_{\mathfrak{p}})$ the Eichler-Shimura periods. For $j = \pm 1$, $1 \leq l \leq 2k - n - 1$, and a Dirichlet character $\chi$ such
that $\chi(-1) = j(-1)^{l-1}$, put

$$L(l, f, \chi) = L(l, f, \chi ; A_{\mathfrak{p}}) = \frac{\Gamma(l)L(l, f, \chi)}{\tau(\chi)(2\pi \sqrt{-1})^{l}\Omega(f, j; A_{\mathfrak{p}})},$$

where $\tau(\chi)$ is the Gauss sum of $\chi$. In particular, put $L(l, f ; A_{\mathfrak{p}}) = L(l, f, \chi ; A_{\mathfrak{p}})$ if $\chi$ is the principal character. Furthermore, put

$$L(s, f, \text{St}) = \frac{4(2\pi)^{-2s-2k+n+1}\Gamma(s)\Gamma(s + 2k - n - 1)L(s, f, \text{St})}{\Gamma(s)\Gamma(s + 2k - n + 1)L(s, f, \text{St})}.$$ 

It is well-known that $L(l, f, \chi)$ belongs to the field $K(\chi)$ generated over $K$ by all the values of $\chi$, and $L(l, f, \text{St})$ belongs to $Q(f)$ (cf. [Bo].) Let $I_n(f)$ be the Duke-Imamoglu-Ikeda lift of $f$. Let $\mathfrak{E}_k(\Gamma(n)^*)$ be the subspace of $\mathfrak{E}_k(I_n)$ generated by all the Duke-Imamoglu-Ikeda lifts $I(g)^n$ of primitive forms $g \in \mathfrak{E}_{2k-n}(\Gamma(1))$. We remark that $\mathfrak{E}_k(\Gamma(2)^*)$ is the Maass subspace of $\mathfrak{E}_k(\Gamma(2))$.

**Conjecture A.** Let $K$ and $f$ be as above. Assume that $k > n$. Let $\mathfrak{p}$ be a prime ideal of $K$ not dividing $(2k - 1)!$. Then $\mathfrak{p}$ is a congruence prime of $I_n(f)$ with respect to $(\mathfrak{E}_k(\Gamma(n)^*))^\perp$ if $\mathfrak{p}$ divides

$$L(k, f) \prod_{i=1}^{n/2-1} L(2i + 1, f, \text{St}).$$

**Remark.** This is an analogue of the Doi-Hida-Ishii conjecture concerning the congruence primes of the Doi-Naganuma lifting [D-H-I]. (See also [Ka1].) We also note that this type of conjecture has been proposed by Harder [Ha] in the case of vector valued Siegel modular forms.

Now to explain why our conjecture is reasonable, we refer to Ikeda’s conjecture on the Petersson inner product of the Duke-Imamoglu-Ikeda lifting. Let $f$ and $\tilde{f}$ be as above. Put

$$\tilde{\xi}(s) = 2(2\pi)^{-s}\Gamma(s)\zeta(s),$$

and

$$\Lambda(s, f) = 2(2\pi)^{-s}\Gamma(s)L(s, f).$$

**Theorem 4.2.** (Katsurada and Kawamura [K-K]) In addition to the above notation and the assumption, assume $k > n$. Then we have

$$\tilde{\xi}(n)\Lambda(k, f) \prod_{i=1}^{n/2-1} L(2i - 1, f, \text{St})\tilde{\xi}(2i) = 2^n \frac{\langle I_n(f), f \rangle}{\langle f, f \rangle^{n/2-1} \langle \tilde{f}, \tilde{f} \rangle},$$

where $\alpha$ is an integer depending only on $n$ and $k$. 
We note that the above theorem was conjectured by Ikeda [Ik2] under more general setting. We note that the theorem has been proved by Kohnen and Skoruppa [K-S] in case \( n = 2 \).

**Proposition 4.3** Under the above notation and the assumption we have for any fundamental discriminant \( D \) such that \((-1)^{n/2} D > 0\) and \( L(k - n/2, f, \chi_D) \neq 0 \) we have

\[
\frac{c(|D|)^2}{\langle I_n(f), I_n(f) \rangle} = a_{n,k} \frac{\langle f, f \rangle^{n/2} |D|^{k-n/2} L(k - n/2, f, \chi_D)}{\prod_{i=1}^{n/2-1} L(k, f)}
\]

with some algebraic number \( a_{n,k} \) depending only on \( n, k \).

**Proof.** By the result in Kohnen-Zagier [K-Z], for any fundamental discriminant \( D \) such that \((-1)^{n/2} D > 0\) we have

\[
\frac{c(|D|)^2}{\langle f, f \rangle} = 2^{k-n/2-1} |D|^{k-n/2-1/2} \Lambda(k - n/2, f, \chi_D)
\]

Thus the assertion holds.

**Lemma 4.4.** Let \( f \) be as above.

1. Let \( r_1 \) be an element of \( L'_n \) in Proposition 2.3. Then we have

\[
\lambda_{I_n(f)}(r_1) = p^{(n-1)k-n(n+1)/2} a_f(p) \sum_{i=1}^{n} p^i.
\]

2. Let \( n = 2 \). Then we have

\[
\lambda_{I_2(f)}(T(p)) = a_f(p) + p^{2k-n-1} + p^{2k-n-2}.
\]

**Lemma 4.5.** Let \( d \) be a fundamental discriminant such that \((-1)^{n/2} d > 0\).

1. Assume that \( d \neq 1 \). Then there exists a positive definite half integral matrix \( A \) of degree \( n \) such that \((-1)^{n/2} \det(2A) = d \).
2. Assume \( n \equiv 0 \mod 8 \). Then there exists a positive definite half integral matrix \( A \) of degree \( n \) such that \( \det(2A) = 1 \).
3. Assume that \( n \equiv 4 \mod 8 \). Then for any prime number \( q \) there exists a positive definite half integral matrix \( A \) of degree \( n \) such that \( \det(2A) = q^2 \).

**Proof.** (1) For a non-degenerate symmetric matrix \( A \) with entries in \( \mathbb{Q}_p \) let \( h_p(A) \) be the Hasse invariant of \( A \). First let \( n \equiv 2 \mod 4 \) and \( d = -4 \). Take a family \( \{A_p\}_p \) of half integral matrices over \( \mathbb{Z}_p \) of...
degree $n$ such that $A_p = 1_n$ if $p \neq 2$, and $A_2 = (-1)^{(n-2)/4}1_2 \perp H_{n/2-1}$, where $H_n = \underbrace{H \perp \cdots \perp H}_r$ with $H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$. Then we have det $A = 2^{2-n} \in Q_p^\times/(Q_p^\times)^2$ for any $p$, and $h_p(A) = 1$ for any $p$. By [I-S, Proposition 2.1], there exists an element $A$ of $\mathcal{L}_{n,2\geq 0}$ such that $A \sim A_p$ for any $p$. In particular we have $(-1)^{n/2} \det(2A) = -4$. Next let $d = (-1)^{n/2}$. We take $A_p = (-1)^{n/2}2 \perp 1_{n-1}$ if $p \neq 2$. We can take $\xi \in \mathbb{Z}_2^*$ such that $(2, \xi) = (-1)^{(n-2)(n+4)/8}$, and put $A_2 = 2\xi \perp (-\xi) \perp H_{n/2-1}$. Then we have det $A = (-1)^{n/2}2^{3-n} \in Q_p^\times/(Q_p^\times)^2$ for any $p$, and $h_p(A) = 1$ for any $p$. Thus again by [I-S, Proposition 2.1], we prove the assertion for this case. Finally assume that $d$ contains an odd prime factor $q$. For $p \neq p_0$ we take a matrix $A_p$ so that det $A_p = 2^{-n}d \in Q_q^\times/(Q_q^\times)^2$. Then for almost all $p$ we have $h_p(A_p) = 1$. We take $\xi \in \mathbb{Z}_q^*$ such that $(q, -\xi) = \prod_{p \neq q} h_p(A_p)$, and put $A_q = \xi d \perp \xi \perp 1_{n-2}$. Then we have det $A_q 2^{-n}d \in Q_q^\times/(Q_q^\times)^2$, and $h_p(A_q) \prod_{p \neq q} = 1$. Thus again by [I-S, Proposition 2.1], we prove the assertion for this case.

(2) It is well known that there exists a positive definite half-integral matrix $E_8$ of degree 8 such that det$(2E_8) = 1$. Thus $A = \overbrace{E_8 \perp \cdots \perp E_8}^{n/8}$ satisfies the required condition.

(3) Let $q \neq 2$. Then, take a family $\{A_p\}$ of half-integral matrices over $\mathbb{Z}_p$ of degree $n$ such that $A_q \sim_{\mathbb{Z}_q} q \perp (-q\xi) \perp (-\xi) \perp 1_{n-3}$ with $(\xi_q) = -1, A_2 = H_{n/2}$, and $A_p = 1_n$ for $p /\nmid 2$. Then by the same argument as in (1) we can show that there exists a positive definite half integral matrix $A$ of degree $n$ such that det$(2A) = q^2$ such that $A \sim_{\mathbb{Z}_q} A_p$ for any $p$. Let $q = 2$. Then the matrix $A' = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \end{pmatrix}^{(n-4)/8}$ is a positive definite and det$(2A') = 4$. Thus the matrix $A' \perp E_8 \perp \cdots \perp E_8$ satisfies the required condition.

**Proposition 4.6.** Let $k$ and $n$ be positive even integer. Let $d$ be a fundamental discriminant. Let $f$ be a primitive form in $S_{2k-n}(\Gamma_1)$. Let $\mathfrak{P}$ be a prime ideal in $K$. Then there exists a positive definite half integral matrices $A$ of degree $n$ such that $c_{I_n(f)}(A) = c_f(|d|)q$ with $q$ not divisible by $\mathfrak{P}$. 
Proof. First assume that $d \neq 1$, or $n \neq 4 \mod 8$. (1) By (1) and (2) of Lemma 4.5, there exists a matrix $A$ such that $b_A = d$. Thus we have $c_{I_n(f)}(A) = c_f(|d|)$. This proves the assertion.

Next assume that $n \equiv 4 \mod 8$ and that $d = 1$. Assume that $c_f(q) + q^{k-n/2-1}(-q - 1)$ is divisible by $\mathfrak{P}$ for any prime number $q$. Let $p$ be a prime number divisible by $\mathfrak{P}$. Fix an imbedding $\iota_p : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$, and let $\rho_{f,p} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\overline{\mathbb{Q}}_p)$ be the Galois representation attached to $f$. Then by Chebotarev density theorem, the semi-simplification $\overline{\rho}_{f,p}$ of $\overline{\rho}_{f,p}$ can be expressed as

\[
\overline{\rho}_{f,p} = \overline{\chi}_p^{k-n/2} \oplus \overline{\chi}_p^{k-n/2-1}
\]

with $\overline{\chi}_p$ the $p$-adic mod $p$ cyclotomic character. On the other hand, by the Fontaine-Messing [Fo-Me] and Fontaine-Laffaille [Fo-La], $\overline{\rho}_{f,p}|I_p$ should be $\overline{\chi}_p^{2k-n-1} \oplus 1$ or $\omega_2^{2k-n-1} \oplus \omega_2^{p(2k-n-1)}$ with $\omega_2$ the fundamental character of level 2, where $I_p$ denotes the inertia group of $p$ in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. This is impossible because $k > 2$. Thus there exists a prime number $q$ such that $c_f(q) + q^{k-n/2-1}(-q - 1)$ is not divisible by $\mathfrak{P}$. For such a $q$, take a positive definite matrix $A$ in (3) of Lemma 4.5. Then

\[
c_{I_n(f)}(A) = c(1)q^{-n/2} \beta_q F_q(A, q^{n/2} \beta_q^{-1}).
\]

By [Ka1], we have

\[
F_q(B, X) = 1 - Xq^{(n-2)/2}(q^2 + q) + q^3(Xq^{(n-2)/2}).
\]

Thus we have

\[
c_{I_n(f)}(A) = c(1)(c_f(q) + q^{k-n/2-1}(-q - 1)).
\]

Thus the assertion holds.

**Theorem 4.7.** Let $k \geq 2n+4$. Let $K$ and $f$ be as above. Assume that the Conjecture $B$ holds for $f$. Let $\mathfrak{P}$ be a prime ideal of $K$. Furthermore assume that

1. $\mathfrak{P}$ divides $L(k, f) \prod_{i=1}^{n/2-1} L(2i+1, f, St)$.
2. $\mathfrak{P}$ does not divide

\[
\tilde{\xi}(2m) \prod_{i=1}^{n} L(2m + k - i, f)L(k - n/2, f, \chi_D)D(2k - 1)!
\]

for some integer $n/2 + 1 \leq m \leq k/2 - n/2 - 1$, and for some fundamental discriminant $D$ such that $(-1)^{n/2}D > 0$.

Then $\mathfrak{P}$ is a congruence prime of $I_n(f)$ with respect to $CI_n(f)^{\perp}$. Furthermore assume that the following condition hold:
(3) $Ψ$ does not divide

$$
\langle f, f \rangle
$$

where $C_{k,n} = 1$ or $\prod_{q \leq (2k-n)/12} (1 + q + \cdots + q^{n-1})$ according as $n = 2$ or not.

Then $Ψ$ is a congruence prime of $I_n(f)$ with respect to $(Σ_k(I_n))^{-1}$.

Proof. Let $Ψ$ be a prime ideal satisfying the condition (1) and (2). For the $D$ above, take a matrix $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$ so that $c_{I_n(f)}(A) = c_f(|D|)q$ with $q$ not divisible by $Ψ$. Then by Proposition 4.3, we have

$$
\Lambda(2m, I_n(f), St) | c_{I_n(f)}(A)|^2 = \Lambda(2m, I_n(f), St) | c_f(|D|)|^2 q^2
$$

$$
= \epsilon_{k,m} \prod_{i=1}^{n} L(2m + k - i, f) |D|^{k-n/2} L(k - n/2, f, \chi D)
$$

$$
\times \left( \frac{Ω(f, +; Ψ) Ω(f, -; A_Ψ)}{Ω(f, f)} \right)^{n/2},
$$

where $\epsilon_{k,m}$ is a rational number whose numerator is not divided by $Ψ$.

We note that $\frac{\langle f, f \rangle}{Ω(f, +; A_Ψ) Ω(f, -; A_Ψ)}$ is $Ψ$-integral. Thus by assumptions (1) and (2), $Ψ$ divides $(\Lambda(2m, I_n(f), St) c_{I_n(f)}(A)^2)^{-1}$, and thus it divides $(\Lambda(2m, I_n(f), St) S_{I_n(f)}^2)^{-1}$. We note that $I_n(f)$ satisfies the assumption in Theorem 3.1. Thus by Theorem 3.1, there exits a Hecke eigenform $G \in C(I_n(f))$ such that

$$
λ_G(T) \equiv λ_{I_n(f)}(T) \mod Ψ
$$

for any $T \in L_n$. Assume that we have $G = I_n(g)$ with some primitive form $g(z) = \sum_{m=1}^{∞} a_g(m) e(mz) \in Σ_{2k-n}(Γ^{(1)})$. Let $n = 2$. Then by (1) of Proposition 4.2, $Ψ$ is also a congruence prime of $f$. Let $n \geq 4$. Then by (1) of Proposition 4.4, we have

$$
(p^{n-1} + \cdots + p + 1) a_f(p) \equiv (p^{n-1} + \cdots + p + 1) a_g(p) \mod Ψ
$$

for any prime number $p$ not divisible by $Ψ$. By assumption (3), in particular, for any $p \leq (2k - n)/12$, we have

$$
a_f(p) \equiv a_g(p) \mod Ψ.
$$

Thus by Sturm [Stur], $Ψ$ is also a congruence prime of $f$. Thus by [Hi2] and [Ri2], $Ψ$ divides $\frac{\langle f, f \rangle}{Ω(f, +; A_Ψ) Ω(f, -; A_Ψ)}$, which contradicts the assumption (3). Thus $Ψ$ is a congruence prime of $I_n(f)$ with respect to $(Σ_k(Γ^{(n)}))^{-1}$. 

Example Let \( n = 4 \) and \( k = 18 \). Then we have \( \dim S_{18}(\Gamma_4) \approx 16 \) (cf. Poor and Yuen[P-Y]) and \( \dim S_{18}(\Gamma_4)^* = \dim S_{32}(\Gamma_1) = 2 \). Take a primitive form \( f \in S_{32}(\Gamma_1) \). Then we have \([Q(f) : \mathbb{Q}] = 2\), and \( 211 = \mathfrak{p} \mathfrak{p}' \) in \( Q(f) \). Then we have

\[
N_{Q(f)/Q}(L(18, f)) = 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 211,
\]

\[
N_{Q(f)/Q}(\prod_{i=1}^{4} L(24-i, f)) = 2^{19} \cdot 3^{13} \cdot 5^5 \cdot 7^8 \cdot 11^2 \cdot 13^5 \cdot 17^5 \cdot 19^3 \cdot 23 \cdot 503 \cdot 1307 \cdot 14243,
\]

and

\[
\tilde{\xi}(6) = 2^{-2} \cdot 3^{-2} \cdot 7^{-1}
\]

(cf. Stein [Ste].) Furthermore, by a direct computation we see neither \( \mathfrak{p} \) nor \( \mathfrak{p}' \) is a congruence prime of \( \hat{f} \) with respect to \( C \hat{g} \) for another primitive form \( g \in S_{32}(\Gamma_1) \). Thus by Theorem 4.7, \( \mathfrak{p} \) or \( \mathfrak{p}' \) is a congruence prime of \( \hat{f} \) with respect to \( S_{18}(\Gamma_4)^* \).

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