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Hokkaido University Preprint Series in Mathematics, 958, 1-19
Issue Date: 2010-4-20
DOI: 10.14943/84105
Doc URL: http://hdl.handle.net/2115/69765
Type: bulletin (article)

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CONGRUENCE BETWEEN DUKE-IMAMOGLU-IKEDA LIFTS AND NON-DUKE-IMAMOGLU-IKEDA LIFTS

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1. Introduction

As is well known, there is a congruence between the Fourier coefficients of Eisenstein series and those of cuspidal Hecke eigenforms for $SL_2(\mathbb{Z})$. This type of congruence is not only interesting in its own right but also has an important application to number theory as shown by Ribet [Ri1]. Since then, there have been so many important results about the congruence of elliptic modular forms (cf. [Hi1],[Hi2],[Hi3].) In the case of Hilbert modular forms or Siegel modular forms, it is sometimes more natural and important to consider the congruence between the Hecke eigenvalues of Hecke eigenforms modulo a prime ideal $\mathfrak{p}$. We call such a $\mathfrak{p}$ a prime ideal giving the congruence or a congruence prime. For a cuspidal Hecke eigenform $f$ for $SL_2(\mathbb{Z})$, let $\hat{f}$ be a lift of $f$ to the space $\mathfrak{M}_l(\Gamma'')$ of modular forms of weight $l$ for a modular group $\Gamma''$. Here we mean by the lift of $f$ a cuspidal Hecke eigenform whose certain L-function can be expressed in terms of certain L-functions of $f$. As typical examples of the lift we can consider the Doi-Naganuma lift, the Saito-Kurokawa lift, and the Duke-Imamoglu-Ikeda lift. We then consider the following problem:

Problem. Characterize the prime ideals giving the congruence between $\hat{f}$ and a cuspidal Hecke eigenform in $\mathfrak{M}_l(\Gamma'')$ not coming from the lift. In particular characterize them in terms of special values of certain L-functions of $f$.

This type of problem was first investigated in the Doi-Naganuma lift case by Doi, Hida, and Ishii [D-H-I]. In our previous paper [Ka2], we considered the relationship between the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbb{Z})$ and the special values of their standard zeta functions. In particular, we proposed a conjecture concerning the congruence between Saito-Kurokawa lifts and non-Saito-Kurokawa
lifts, and proved it under certain condition. In this paper, we consider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, which is a generalization of our previous conjecture.

In Section 3, we review a result concerning the relationship between the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbb{Z})$ and the special values of their standard zeta functions. In Section 4, we propose a conjecture concerning the congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, and prove it under a certain condition.

The author thanks Professor H. Hida, Professor S. Yasuda, and Professor T. Yamauchi for their valuable comments.

**Notation.** For a commutative ring $R$, we denote by $M_{mn}(R)$ the set of $(m,n)$-matrices with entries in $R$. In particular put $M_n(R) = M_{nn}(R)$. Here we understand $M_{mn}(R)$ the set of the empty matrix if $m = 0$ or $n = 0$. For an $(m,n)$-matrix $X$ and an $(m,m)$-matrix $A$, we write $A[X] = ^tXAX$, where $^tX$ denotes the transpose of $X$. Let $a$ be an element of $R$. Then for an element $X$ of $M_{mn}(R)$ we often use the same symbol $X$ to denote the coset $X \mod aM_{mn}(R)$. Put $GL_n(R) = \{A \in M_n(R) \mid \det A \in R^*\}$, where $\det A$ denotes the determinant of a square matrix $A$, and $R^*$ denotes the unit group of $R$. Let $S_n(R)$ denote the set of symmetric matrices of degree $n$ with entries in $R$. Furthermore, for an integral domain $R$ of characteristic different from 2, let $H_n(R)$ denote the set of half-integral matrices of degree $n$ over $R$, that is, $H_n(R)$ is the set of symmetric matrices of degree $n$ whose $(i,j)$-component belongs to $R$ or $\frac{1}{2}R$ according as $i = j$ or not. For a subset $S$ of $M_n(R)$ we denote by $S^\times$ the subset of $S$ consisting of non-degenerate matrices. In particular, if $S$ is a subset of $S_n(R)$ with $R$ the field of real numbers, we denote by $S_{\geq 0}$ (resp. $S_{>0}$) the subset of $S$ consisting of positive definite (resp. semi-positive definite) matrices. Let $R'$ be a subring of $R$. Two symmetric matrices $A$ and $A'$ with entries in $R$ are called equivalent over $R'$ with each other and write $A \sim_{R'} A'$ if there is an element $X$ of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices $X$ and $Y$ we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

2. **Standard zeta functions of Siegel modular forms**

For a complex number $x$ put $e(x) = \exp(2\pi \sqrt{-1}x)$. Furthermore put

$$J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix},$$

where $1_n$ denotes the unit matrix of degree $n$. For
a subring $K$ of $\mathbb{R}$ put

$$GSp_n(K)^+ = \{ M \in GL_{2n}(K) \mid J_n[M] = \kappa(M)J_n \text{ with some } \kappa(M) > 0 \},$$

and

$$Sp_n(K) = \{ M \in GSp_n(K)^+ \mid J_n[M] = J_n \}.$$

Furthermore, put

$$\Gamma^{(n)} = Sp_n(Z) = fM_{2GL_{2n}}(Z)jJ_{n}[M] = \kappa(M)J_{n}g.$$  

We sometimes write an element $M$ of $GSp_n(K)$ as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in M_2(K)$. We define a subgroup $\Gamma^{(n)}_0(N)$ of $\Gamma^{(n)}$ as

$$\Gamma^{(n)}_0(N) = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \mid C \equiv O_n \text{ mod } N \}. $$

Let $H_n$ be Siegel’s upper half-space. For each element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n(\mathbb{R})^+$ and $Z \in H_n$ put

$$M(Z) = (AZ + B)(CZ + D)^{-1}$$

and

$$j(M, Z) = \det(CZ + D).$$

Furthermore, for a function $f$ on $H_n$ and an integer $k$ we define $f|_kM$ as

$$(f|_kM)(Z) = \det(M)^{k/2}j(M, Z)^{-k}f(M(Z)).$$

For an integer or half integral $l$ and the subgroup $\Gamma^{(n)}_0(N)$ of $\Gamma^{(n)}$, we denote by $\mathcal{M}_k(\Gamma^{(n)}_0(N))$ (resp. $\mathcal{M}_k^\infty(\Gamma^{(n)}_0(N))$) the space of holomorphic (resp. $C^\infty$-) modular forms of weight $k$ with respect to $\Gamma^{(n)}_0(N)$. We denote by $\mathcal{S}_k(\Gamma^{(n)}_0(N))$ the sub-space of $\mathcal{M}_k(\Gamma^{(n)}_0(N))$ consisting of cusp forms. Let $f$ be a holomorphic modular form of weight $k$ with respect to $\Gamma^{(n)}_0(N)$. Then $f$ has the following Fourier expansion:

$$f(Z) = \sum_{A \in \mathcal{H}(Z)_{\geq 0}} a_f(A)e(\text{tr}(AZ)),$$

and in particular if $f$ is a cusp form, $f$ has the following Fourier expansion:

$$f(Z) = \sum_{A \in \mathcal{H}(Z)_{\geq 0}} a_f(A)e(\text{tr}(AZ)),$$

where $\text{tr}$ denotes the trace of a matrix. Let $dv$ denote the invariant volume element on $H_n$ defined by $dv = \det(\text{Im}(Z))^{-n-1} \wedge_{1 \leq i \leq n} (dx_{ji} \wedge dy_{ji})$. Here for $Z \in H_n$ we write $Z = (x_{ji}) + \sqrt{-1}(y_{ji})$ with real matrices.
For two \( C^\infty \)-modular forms \( f \) and \( g \) of weight \( l \) with respect to \( \Gamma_0^{(n)}(N) \) we define the Petersson scalar product \( \langle f, g \rangle \) by

\[
\langle f, g \rangle = \langle \Gamma^{(n)} : \Gamma_0^{(n)}(N) \rangle^{-1} \int_{\Gamma_0^{(n)}(N) \backslash \mathbb{H}} f(Z) g(Z) \det(\text{Im}(Z))^l d\nu,
\]

provided the integral converges.

Let \( L_n = L_Q(GSp_n(Q)^+, \Gamma^{(n)}) \) denote the Hecke algebra over \( Q \) associated with the Hecke pair \( (GSp_n(Q)^+, \Gamma^{(n)}) \). Furthermore, let \( L_{n}^{0} = L_Q(Sp_{n}(Q), \Gamma^{(n)}) \) denote the Hecke algebra over \( Q \) associated with the Hecke pair \( (Sp_{n}(Q), \Gamma^{(n)}) \). For each integer \( m \) define an element \( T(m) \) of \( L_n \) by

\[
T(m) = \sum_{d_i \mid d_{i+1}, \ e_i | e_{i+1}} \Gamma^{(n)}(d_1 \perp \ldots \perp d_i \perp \ldots \perp e_1 \perp \ldots \perp e_n)\Gamma^{(n)},
\]

where \( d_1, \ldots, d_n, e_1, \ldots, e_n \) run over all positive integer satisfying

\[
d_i | d_{i+1}, \ e_i | e_{i+1} \quad (i = 1, \ldots, n-1), \ d_n | e_n, \ d_i e_i = m \quad (i = 1, \ldots, n).
\]

Furthermore, for \( i = 1, \ldots, n \) and a prime number \( p \) put

\[
T_i(p^2) = \Gamma^{(n)}(1_{n-i} \perp p 1_i \perp p^2 1_{n-i} \perp p 1_i)\Gamma^{(n)},
\]

and \( (p^{\pm 1}) = \Gamma^{(n)}(p^{\pm 1} 1_n)\Gamma^{(n)} \). As is well known, \( L_n \) is generated over \( Q \) by all \( T(p), T_i(p^2) \ (i = 1, \ldots, n) \), and \((p^{\pm 1})\). We denote by \( L_n' \) the subalgebra of \( L_n \) generated over \( \mathbb{Z} \) by all \( T(p) \) and \( T_i(p^2) \ (i = 1, \ldots, n) \). Let \( T = \Gamma^{(n)} M \Gamma^{(n)} \) be an element of \( L_n \otimes \mathbb{C} \). Write \( T \) as \( T = \bigcup_\gamma \Gamma^{(n)} \gamma \) and for \( f \in \mathfrak{M}_k(\Gamma^{(n)}) \) define the Hecke operator \( |_k T \) associated to \( T \) as

\[
f|_k T = \text{det}(M)^{k/2 - (n+1)/2} \sum_\gamma f|_k \gamma.
\]

We call this action the Hecke operator as usual (cf. [A]). If \( f \) is an eigenfunction of a Hecke operator \( T \in L_n \otimes \mathbb{C} \), we denote by \( \lambda_f(T) \) its eigenvalue. Let \( L \) be a subalgebra of \( L_n \). We call \( f \in \mathfrak{M}_k(\Gamma^{(n)}) \) a Hecke eigenform for \( L \) if it is a common eigenfunction of all Hecke operators in \( L \). In particular if \( L = L_n \) we simply call \( f \) a Hecke eigenform. Furthermore, we denote by \( Q(f) \) the field generated over \( Q \) by eigenvalues of all \( T \in L_n \) as in Section 1. As is well known, \( Q(f) \) is a totally real algebraic number field of finite degree. Now, first we consider the integrality of the eigenvalues of Hecke operators. For an algebraic number field \( K \), let \( \mathcal{O}_K \) denote the ring of integers in \( K \). The following assertion has been proved in [Mi2] (see also [Ka2].)

**Theorem 2.1** Let \( k \geq n + 1 \). Let \( f \in \mathfrak{M}_k(\Gamma^{(n)}) \) be a common eigenfunction in \( L_n' \). Then \( \lambda_f(T) \) belongs to \( \mathcal{O}_{Q(f)} \) for any \( T \in L_n' \).
Let $L_{np} = L(GSp_n(Q)^+ \cap GL_{2n}(Z[p^{-1}]), \Gamma(n))$ be the Hecke algebra associated with the pair $(GSp_n(Q)^+ \cap GL_{2n}(Z[p^{-1}]), \Gamma(n))$. $L_{np}$ can be considered as a subalgebra of $L_n$, and it is generated over $Q$ by $\langle T(p) \rangle$ and $\langle T_i(p^2) \rangle$ ($i = 1, 2, ..., n$). We now review the Satake $p$-parameters of $L_{np}$; let $P_n = Q[X_0^\pm, X_1^\pm, ..., X_n^\pm]$ be the ring of Laurent polynomials in $X_0, X_1, ..., X_n$ over $Q$. Let $W_n$ be the group of $Q$-automorphisms of $P_n$ generated by all permutations in variables $X_1, ..., X_n$ and by the automorphisms $\tau_1, ..., \tau_n$ defined by

$$\tau_i(X_0) = X_0 X_i, \tau_i(X_i) = X_i^{-1}, \tau_i(X_j) = X_j (j \neq i).$$

Furthermore, a group $W_n$ isomorphic to $W_n$ acts on the set $T_n = (C^\times)^{n+1}$ in a way similarly to above. Then there exists a $Q$-algebra isomorphism $\Phi_{np}$, called the Satake isomorphism, from $L_{np}$ to the $W_n$-invariant subring $P_n^{W_n}$ of $P_n$. Then for a $Q$-algebra homomorphism $\lambda$ from $L_{np}$ to $C$, there exists an element $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), ..., \alpha_n(p, \lambda))$ of $T_n$ satisfying

$$\lambda(\Phi_{np}^{-1}(F(X_0, X_1, ..., X_n))) = F(\alpha_0(p, \lambda), \alpha_1(p, \lambda), ..., \alpha_n(p, \lambda))$$

for $F \in P_n^{W_n}$. The equivalence class of $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), ..., \alpha_n(p, \lambda))$ under the action of $W_n$ is uniquely determined by $\lambda$. We call this the Satake parameters of $L_{np}$ determined by $\lambda$.

Now assume that an element $f$ of $M_k(Sp_n(Z))$ is a Hecke eigenform. Then for each prime number $p$, $f$ defines a $Q$-algebra homomorphism $\lambda_{f,p}$ from $L_{np}$ to $C$ in a usual way, and we denote by $\alpha_0(p), \alpha_1(p), ..., \alpha_n(p)$ the Satake parameters of $L_{np}$ determined by $f$. We then define the standard zeta function $L(f, s, \mathcal{St})$ by

$$L(s, f, \mathcal{St}) = \prod_p \prod_{i=1}^n (1 - p^{-s})(1 - \alpha_i(p)p^{-s})(1 - \alpha_i(p)^{-1}p^{-s})^{-1}.$$ 

Let $f(z) = \sum_{A \in \mathcal{K}_n(Z)_{>0}} a(A)e(\text{tr}(Az))$ be a Hecke eigenform in $\mathcal{E}_k(\Gamma(n))$. For a positive integer $m \leq k - n$ such that $m \equiv n \mod 2$ put

$$\Lambda(f, m, \mathcal{St}) = (-1)^{n(m+1)/2} \frac{L(f, m, \mathcal{St})}{\Gamma(2k - n - i)\Gamma(m + 1)} \frac{\Gamma(2k - n - i)}{\pi^{-n(n+1)/2+nk+(n+1)m}}.$$ 

Then the following theorem is due to Böcherer [B2] and Mizumoto [Mi].

**Theorem 2.2.** Let $l, k$ and $n$ be positive integers such that $\rho(n) \leq l \leq k - n$, where $\rho(n) = 3$, or $1$ according as $n \equiv 1 \mod 4$ and $n \geq 5$, or not. Let $f \in \mathcal{E}_k(\Gamma(n))$ be a Hecke eigenform. Then $\Lambda(f, m, \mathcal{St})$ belongs to $Q(f)$. 

**Remark.** Note that $\Lambda(f, m, \mathcal{St})$ is a special case of the constant term of the zeta function of the ordinary part of $f$. 

For later purpose, we consider a special element in $L_{np}$: the polynomial $X_0^2X_1X_2\cdots X_n\sum_{i=1}^n (X_i + X_i^{-1})$ is an element of $P_n^W$, and thus we can define an element $\Phi_{np}^{-1}(X_0^2X_1X_2\cdots X_n\sum_{i=1}^n (X_i + X_i^{-1}))$ of $L_{np}$, which is denoted by $r_1$.

**Proposition 2.3.** Under the above notation the element $r_1$ belongs to $L'_n$, and we have

$$\lambda_f(r_1) = p^{n(k-(n+1)/2)}\sum_{i=1}^n (\alpha_i(p) + \alpha_i(p)^{-1}).$$

**Proof.** By a careful analysis of the computation in page 159-160 of [A], we see that $r_1$ is a $\mathbb{Z}$-linear combination of $T_i(p^2) (i = 1, \ldots, n)$, and therefore we can prove the first assertion. Furthermore, by Lemma 3.3.34 of [A], we can prove the second assertion.

3. Congruence of modular forms and special values of the standard zeta functions

In this section we review a result concerning the congruence between the Hecke eigenvalues of modular forms of the same weight following [Ka2]. Let $K$ be an algebraic number field, and $\mathfrak{O} = \mathfrak{O}_K$ the ring of integers in $K$. For a prime ideal $\mathfrak{p}$ of $\mathfrak{O}$, we denote by $\mathfrak{O}(\mathfrak{p})$ the localization of $\mathfrak{O}$ at $\mathfrak{p}$ in $K$. Let $\mathfrak{a}$ be a fractional ideal in $K$. If $\mathfrak{a} = \mathfrak{p}\mathfrak{O}$ with $\mathfrak{a}\mathfrak{O}(\mathfrak{p}) = \mathfrak{O}(\mathfrak{p})$ we write $\text{ord}_{\mathfrak{p}} = e$. We simply write $\text{ord}_{\mathfrak{p}}(c) = \text{ord}_{\mathfrak{p}}((c))$ for $c \in K$. Now let $f$ be a Hecke eigenform in $\mathfrak{S}_k(\Gamma(n))$ and $M$ be a subspace of $\mathfrak{S}_k(\Gamma(n))$ stable under Hecke operators $T \in \mathfrak{L}_n$. Assume that $M$ is contained in $(\mathfrak{C}f)^\perp$, where $(\mathfrak{C}f)^\perp$ is the orthogonal complement of $\mathfrak{C}f$ in $\mathfrak{S}_k(\Gamma(n))$ with respect to the Petersson product. Let $K$ be an algebraic number field containing $\mathbb{Q}(f)$. A prime ideal $\mathfrak{p}$ of $\mathfrak{O}_K$ is called a congruence prime of $f$ with respect to $M$ if there exists a Hecke eigenform $g \in M$ such that

$$\lambda_f(T) \equiv \lambda_g(T) \mod \mathfrak{p}$$

for any $T \in \mathfrak{L}_n'$, where $\mathfrak{p}$ is the prime ideal of $\mathfrak{O}_{K\mathbb{Q}(f)}$ lying above $\mathfrak{p}$. If $M = (\mathfrak{C}f)^\perp$, we simply call $\mathfrak{p}$ a congruence prime of $f$.

Now we consider the relation between the congruence primes and the standard zeta values. To consider this, we have to normalize the standard zeta value $\Lambda(f, l, \text{St})$ for a Hecke eigenform $f$ because it is not uniquely determined by the system of Hecke eigenvalues of $f$. We note that there is no reasonable normalization of cuspidal Hecke eigenform in the higher degree case unlike the elliptic modular case.
Thus we define the following quantities: for a Hecke eigenform \( f(z) = \sum A a_f(A)e(\text{tr}(Az)) \) in \( \mathfrak{S}_k(\Gamma([n])) \), let \( \mathfrak{Z}_f \) be the \( \mathfrak{S}_{Q(f)} \)-module generated by all \( a_f(A) \)'s. Assume that there exists a complex number \( c \) such that all the Fourier coefficients \( cf \) belongs to \( \mathbb{Q}(f) \). Then \( \mathfrak{Z}_f \) is a fractional ideal in \( \mathbb{Q}(f) \), and therefore, so is \( \Lambda(f, l, \mathfrak{S})\mathfrak{Z}_f^2 \) if \( l \) satisfies the condition in Theorem 2.2. We note that this fractional ideal does not depend on the choice of \( c \). We also note that the value \( N_{\mathfrak{Q}(f)}(\Lambda(f, l, \mathfrak{S})) N(\mathfrak{Z}_f)^2 \) does not depend on the choice of \( c \), where \( N(\mathfrak{Z}_f) \) is the norm of the ideal \( \mathfrak{Z}_f \). Then, we have

**Theorem 3.1.** Let \( f \) be a Hecke eigenform in \( \mathfrak{S}_k(\Gamma([n])) \). Assume that there exists a complex number \( c \) such that all the Fourier coefficients \( cf \) belongs to \( \mathbb{Q}(f) \). Let \( l \) be a positive integer satisfying the condition in Theorem 2.2. Let \( \mathfrak{P} \) be a prime ideal of \( \mathfrak{S} \). Assume that \( \text{ord}_p(\Lambda(f, l, \mathfrak{S})\mathfrak{Z}_f^2) < 0 \) and that it does not divide \( (2l - 1)! \). Then \( \mathfrak{P} \) is a congruence prime of \( f \). In particular, if a rational prime number \( p \) divides the denominator of \( N_{\mathfrak{Q}(f)}(\Lambda(f, l, \mathfrak{S})) N(\mathfrak{Z}_f)^2 \), then \( p \) is divisible by some congruence prime of \( f \).

Now for a Hecke eigenform \( f \) in \( \mathfrak{S}_k(\Gamma([n])) \), let \( \mathfrak{X}_f \) denote the subspace of \( \mathfrak{S}_k(\Gamma([n])) \) spanned by all Hecke eigenforms with the same system of the Hecke eigenvalues as \( f \).

**Corollary.** In addition to the above notation and the assumption, assume that \( \mathfrak{S}_k(\Gamma([n])) \) has the multiplicity one property. Then \( \mathfrak{P} \) is a congruence prime of \( f \) with respect to \( \mathfrak{X}_f \). In particular, if a rational prime number \( p \) divides the denominator of \( N_{\mathfrak{Q}(f)}(\Lambda(f, l, \mathfrak{S})) N(\mathfrak{Z}_f)^2 \), then \( p \) is divisible by some congruence prime of \( f \) with respect to \( \mathfrak{X}_f \).

4. **Congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts**

In this section, we consider the congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts. Throughout this section and the next, we assume that \( n \) and \( k \) are even positive integers. Let

\[
f(z) = \sum_{m=1}^{\infty} a(m)e(mz)
\]

be a normalized Hecke eigenform of weight \( 2k - n \) with respect to \( SL_2(\mathbb{Z}) \). For a Dirichlet character \( \chi \), we then define the L-function

\[
L(f, s) = \sum_{m=1}^{\infty} \frac{a(m)^2}{m^s}
\]
\[ L(s, f) \text{ of } f \text{ twisted by } \chi \text{ by} \]
\[ L(s, f) = \prod_p \{(1 - \chi(p)\beta_pp^{k-n/2-1/2-s})(1 - \chi(p)\beta_p^{-1}p^{k-n/2-1/2-s})\}^{-1}, \]
where \( \beta_p \) is a non-zero complex number such that \( \beta_p + \beta_p^{-1} = p^{-k+n/2+1/2}a(p) \).

We simply write \( L(s, f) \) as \( L(s, f, \chi) \) if \( \chi \) is the principal character. Furthermore, let \( \tilde{f} \) be the cusp form of weight \( k-n/2+1/2 \) belonging to the Kohnen plus space corresponding to \( f \) via the Shimura correspondence (cf. [Ko1]). Then \( \tilde{f} \) has the following Fourier expansion:
\[ \tilde{f}(z) = \sum e^{c(e)\xi(ez)}, \]
where \( e \) runs over all positive integers such that \((-1)^{k-n/2}e \equiv 0, 1 \mod 4 \).

We then put
\[ a_{I_n(f)}(T) = c([T])\prod_p (p^{k-n/2-1/2}\beta_p)^{\nu_s([T])}F_p(T, p^{-(n+1)/2}\beta_p^{-1}). \]

We note that \( a_{I_n(f)}(T) \) does not depend on the choice of \( \beta_p \). Define a Fourier series \( I_n(f)(Z) \) by
\[ I_n(f)(Z) = \sum_{T \in \mathcal{H}_n(Z), \nu_s} a_{I_n(f)}(T) e(\text{tr}(TZ)). \]

In [Ik1] Ikeda showed that \( I_n(f)(Z) \) is a cusp form of weight \( k \) with respect to \( \Gamma^{(n)} \) and a Hecke eigenform for \( \mathbf{L}_n^\chi \) such that
\[ L(s, I_n(f), St) = \zeta(s) \prod_{i=1}^n L(s+k-i, f). \]

This was first conjecture by Duke and Imamoglu. Thus we call \( I_n(f) \) the Duke-Imamoglu-Ikeda lift of \( f \). We note that we have \( Q(\tilde{f}) = Q(I_n(f)) = Q(f) \). Furthermore, we have \( \mathfrak{F}_\tilde{f} = \mathfrak{F}_{I_n(f)} \), where \( \mathfrak{F}_\tilde{f} \) is the \( \mathfrak{S}_{Q(f)}\)-module generated by all the Fourier coefficients of \( \tilde{f} \).

Now to consider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, first we prove the following:

**Proposition 4.1** \( I_n(f) \) is a Hecke eigenform.

We note that Ikeda proved in [Ik1] that \( I_n(f) \) is a Hecke eigenform for \( \mathbf{L}_n^\chi \) but has not proved that it is a Hecke eigenform for \( \mathbf{L}_n^\omega \). This was pointed to us by B. Heim (see [He].) We thank him for his comment. We also note that an explicit form of the spinor \( L \)-function of \( I_n(f) \) was obtained by Murakawa [Mu] and Schmidt [Sch] assuming that \( I_n(f) \) is a Hecke eigenform.
Proof of Proposition 4.1. We have only to prove that $I_n(f)$ is
an eigenfunction of $T(p)$ for any prime $p$. The proof may be more or
less well known, but for the convenience of the readers we here give the
proof. For a modular form

$$F(Z) = \sum_B c_F(B) e(\text{tr}(BZ)),$$

let $c_F(B)$ be the $B$-th Fourier coefficient of $F|T(p)$. Then for any
positive definite matrix $B$ we have

$$c_F(B) = p^{nk-n(n+1)/2} \sum_{d_1|d_2|\cdots|d_n|p} \frac{d^m}{d_2^m} \cdots d_n$$

$$\times \sum_{D \in \Lambda_n(d_1 \cdots d_n) \Lambda_n} \det D^{-k} c_F(p^{-1}A[tD]),$$

where $\Lambda_n = GL_n(\mathbb{Z})$.

Now let $E_{n,k}(Z)$ be the Siegel Eisenstein series of degree $n$ and of
weight $k$ defined by

$$E_{n,k}(Z) = \sum_{\gamma \in \Gamma_{n,\infty} \backslash \Gamma_n} j(\gamma, Z)^{-k}.$$

For $k \geq n + 1$, the Siegel Eisenstein series $E_{n,k}(Z)$ is a holomorphic
modular form of weight $k$ with respect to $\Gamma_n$. Furthermore, $E_{n,k}(Z)$ is
a Hecke eigenform and in particular we have

$$E_{n,k}|T(p)(Z) = h_{n,p}(p^k)E_{n,k}(Z),$$

where

$$h_{n,p}(X) = 1 + \sum_{r=1}^{n} \sum_{1 \leq i_1 < \cdots < i_r \leq n} p^{-\sum_{j=1}^{r} i_j} X^r.$$

Let $c_{n,k}(B)$ be the $B$-th Fourier coefficient of $E_{n,k}(Z)$. Then we have

$$h_{n,p}(p^k)c_{n,k}(B) = p^{nk-n(n+1)/2} \sum_{d_1|d_2|\cdots|d_n|p} \frac{d^m}{d_2^m} \cdots d_n$$

$$\times \sum_{D \in \Lambda_n(d_1 \cdots d_n) \Lambda_n} \det D^{-k} c_{n,k}(p^{-1}B[tD]).$$

Let $B$ be positive definite. Then we have

$$c_{n,k}(B) = a_{n,k} \left( \det 2B \right)^{k-(n+1)/2} L(k - n/2, \chi_B) \prod_q F_q(B, p^{-k}),$$

where $a_{n,k}$ is a non-zero constant depending only on $n$ and $k$. We note
that we have

$$F_q(p^{-1}B[tD], X) = F_q(B, X)$$
for any $D \in \Lambda_n(d_1 \perp \cdots \perp d_n)\Lambda_n$ with $d_1 \cdots |d_n| p$ if $q \neq p$. Thus we have

$$h_{n,p}(p^k)F_p(B, p^{-k}) = \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{n e_1 + (n-1)e_2 + \cdots + e_n} p^{(k-n-1)}$$

$$\times \sum_{D \in \Lambda_n \setminus \Lambda_n(p^{e_1} \perp \cdots \perp p^{e_n})\Lambda_n} F_p(p^{-1}B^{[D]}, p^{-k}).$$

The both-hand sides of the above are polynomials in $p^k$ and the equality holds for infinitely many $k$. Thus we have

$$h_{n,p}(X^{-1})F_p(B, X) = \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{n e_1 + (n-1)e_2 + \cdots + e_n} (X^{-1} p^{-n-1})^{(e_1 + \cdots + e_n)}$$

$$\times \sum_{D \in \Lambda_n \setminus \Lambda_n(p^{e_1} \perp \cdots \perp p^{e_n})\Lambda_n} F_p(p^{-1}B^{[D]}, X)$$

as polynomials in $X$ and $X^{-1}$. Thus we have

$$(p^{k-(n+1)/2} X)^{n/2}h_{n,p}(p^{(n+1)/2} X^{-1})(p^{k-(n+1)/2} X^{-1})^{\nu_p(||B||)} F_p(B, p^{-(n+1)/2} X)$$

$$= p^{nk-n(n+1)/2} \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{n e_1 + (n-1)e_2 + \cdots + e_n}$$

$$\times \sum_{D \in \Lambda_n \setminus \Lambda_n(p^{e_1} \perp \cdots \perp p^{e_n})\Lambda_n} \det D^{-k}(p^{k-(n+1)/2} X^{-1})^{\nu_p(||B^{[D]}||)} F_p(p^{-1}B^{[D]}, p^{-(n+1)/2} X).$$

We recall that we have

$$c_{I_n(f)}(B) = c_f(||B||) \prod_q (\beta_q)^{\nu_q(||B||)} F_q(B, q^{-(n+1)/2} \beta^{-1}),$$

where $\beta_q$ is the Satake $q$-parameter of $f$. We also note that $c_f(||B^{[D]}||) = c_f(||B||)$ for any $D$. Thus we have

$$p^{nk-n(n+1)/2} \sum_{d_1 |d_2| \cdots |d_n| p} d_1^{n} d_2^{n-1} \cdots d_n^{n} \sum_{D \in \Lambda_n(d_1 \perp \cdots \perp d_n)\Lambda_n} \det D^{-k} c_{I_n(f)}(p^{-1}B^{[D]}).$$

This proves the assertion.

Let $f$ be a primitive form in $E_{2k-n}(\Gamma(1))$ Let $\{f_1, \ldots, f_d\}$ be a basis of $E_{2k-n}(\Gamma(1))$ consisting of primitive forms. Let $K$ be an algebraic number field containing $\mathbb{Q}(f_1) \cdots \mathbb{Q}(f_d)$, and $A = \mathcal{O}_K$. To formulate our conjecture exactly, we introduce the Eichler-Shimura periods as follows (cf. Hida [Hi3].) Let $\mathfrak{p}$ be a prime ideal in $K$. Let $A_{\mathfrak{p}}$ be a valuation ring in $K$ corresponding to $\mathfrak{p}$. Assume that the residual characteristic of $A_{\mathfrak{p}}$ is greater than or equal to $5$. Let $L(2k-n-2, A_{\mathfrak{p}})$ be the module of homogeneous polynomials of degree $2k-n-2$ in the variables $X, Y$
with coefficients in $A_\mathfrak{p}$. We define the action of $M_2(\mathbb{Z}) \cap GL_2(\mathbb{Q})$ on $L(2k - n - 2, A_\mathfrak{p})$ via

$$\gamma \cdot P(X,Y) = P(\gamma(X,Y)(\gamma)^t),$$

where $\gamma' = (\det \gamma)\gamma^{-1}$. Let $H^1_p(\Gamma(1), L(2k - n - 2, A_\mathfrak{p}))$ be the parabolic cohomology group of $\Gamma(1)$ with values in $L(2k - n - 2, A_\mathfrak{p})$. Fix a point $z_0 \in H_1$. Let $g \in \mathfrak{S}_{2k-n}(\Gamma(1))$ or $g \in \mathfrak{S}_{2k-n}(\Gamma(1))$. We then define the differential $\omega(g)$ as

$$\omega(g)(z) = \begin{cases} 2\pi ig(z)(X - zY)^ndz & \text{if } g \in \mathfrak{S}_{2k-n}(\Gamma(1)) \\
2\pi \sqrt{-1}g(z)(X - zY)^ndz & \text{if } g \in \mathfrak{S}_{2k-n}(\Gamma(1)), \end{cases}$$

and define the cohomology class $\delta(g)$ of the 1-cocycle of $\Gamma(1)$ as

$$\gamma \in \Gamma(1) \longrightarrow \int_{z_0}^{\gamma(z_0)} \omega(g).$$

The mapping $g \longrightarrow \delta(g)$ induces the isomorphism

$$\delta : \mathfrak{S}_{2k-n}(\Gamma(1)) \oplus \mathfrak{S}_{2k-n}(\Gamma(1)) \longrightarrow H^1_p(\Gamma(1), L(2k - n - 2, \mathbb{C})), $$

which is called the Eichler-Shimura isomorphism. We can define the action of Hecke algebra $L_1$ on $H^1_p(\Gamma(1), L(2k - n - 2, A_\mathfrak{p}))$ in a natural manner. Furthermore, we can define the action $F_\infty$ on $H^1_p(\Gamma(1), L(2k - n - 2, A_\mathfrak{p}))$ as

$$F_\infty(\delta(g)(z)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \delta(g)(-z),$$

and this action commutes with the Hecke action. For a primitive form $f$ and $j = \pm 1$, we define the subspace $H^1_p(\Gamma(1), L(2k - n - 2, A_\mathfrak{p}))[f,j]$ of $H^1_p(\Gamma(1), L(2k - n - 2, A_\mathfrak{p}))$ as

$$H^1_p(\Gamma(1), L(2k - n - 2, A_\mathfrak{p}))[f,j] = \{ x \in H^1_p(\Gamma(1), L(2k - n - 2, A_\mathfrak{p})) \mid x[T] = \lambda_f(T)x \text{ for } T \in L_1, \text{ and } F_\infty(x) = jx \}. $$

Since $A_\mathfrak{p}$ is a principal ideal domain, $H^1_p(\Gamma(1), L(2k - n - 2, A_\mathfrak{p}))[f,j]$ is a free module of rank one over $A_\mathfrak{p}$. For each $j = \pm 1$ take a basis $\eta(f,j,A_\mathfrak{p})$ of $H^1_p(\Gamma(1), (2k - n - 2, A_\mathfrak{p}))[f,j]$ and define a complex number $\Omega(f,j; A_\mathfrak{p})$ by

$$\delta(f) + jF_\infty(\delta(f))/2 = \Omega(f,j; A_\mathfrak{p})\eta(f,j; A_\mathfrak{p}).$$

This $\Omega(f,j; A_\mathfrak{p})$ is uniquely determined up to constant multiple of units in $A_\mathfrak{p}$. We call $\Omega(f,+; A_\mathfrak{p})$ and $\Omega(f,-; A_\mathfrak{p})$ the Eichler-Shimura periods. For $j = \pm 1 \leq l \leq 2k - n - 1$, and a Dirichlet character $\chi$ such
that $\chi(-1) = j(-1)^{l-1}$, put

$$L(l, f, \chi) = L(l, f, \chi; A_{\mathfrak{p}}) = \frac{\Gamma(l)L(l, f, \chi)}{\tau(\chi)(2\pi\sqrt{-1})^l\Omega(f, j; A_{\mathfrak{p}})},$$

where $\tau(\chi)$ is the Gauss sum of $\chi$. In particular, put $L(l, f; A_{\mathfrak{p}}) = L(l, f, \chi; A_{\mathfrak{p}})$ if $\chi$ is the principal character. Furthermore, put

$$L(s, f, \text{St}) = (2\pi)^{-2s-2k+n+1} \Gamma(s) \Gamma(s + 2k - n - 1)L(s, f, \text{St}).$$

It is well-known that $L(l, f, \chi)$ belongs to the field $K(\chi)$ generated over $K$ by all the values of $\chi$, and $L(l, f, \text{St})$ belongs to $\mathbb{Q}(f)$ (cf. [Bo].) Let $I_n(f)$ be the Duke-Imamoglu-Ikeda lift of $f$. Let $\mathfrak{E}_k(\Gamma(n))^*$ be the subspace of $\mathfrak{E}_k(\Gamma)$ generated by all the Duke-Imamoglu-Ikeda lifts $I(g)^*$ of primitive forms $g \in \mathfrak{E}_{2k-n}(\Gamma(1))$. We remark that $\mathfrak{E}_k(\Gamma(2))^*$ is the Maass subspace of $\mathfrak{E}_k(\Gamma(2))$.

**Conjecture A.** Let $K$ and $f$ be as above. Assume that $k > n$. Let $\mathfrak{p}$ be a prime ideal of $K$ not dividing $(2k-1)!$. Then $\mathfrak{p}$ is a congruence prime of $I_n(f)$ with respect to $(\mathfrak{E}_k(\Gamma(n))^*)^\perp$ if $\mathfrak{p}$ divides $L(k, f) \prod_{i=1}^{n/2-1} L(2i+1, f, \text{St})$.

**Remark.** This is an analogue of the Doi-Hida-Ishii conjecture concerning the congruence primes of the Doi-Naganuma lifting [D-H-I]. (See also [Ka1].) We also note that this type of conjecture has been proposed by Harder [Ha] in the case of vector valued Siegel modular forms.

Now to explain why our conjecture is reasonable, we refer to Ikeda’s conjecture on the Petersson inner product of the Duke-Imamoglu-Ikeda lifting. Let $f$ and $\tilde{f}$ be as above. Put

$$\tilde{\xi}(s) = 2(2\pi)^{-s}\Gamma(s)\zeta(s),$$

and

$$\Lambda(s, f) = 2(2\pi)^{-s}\Gamma(s)L(s, f).$$

**Theorem 4.2.** (Katsurada and Kawamura [K-K]) In addition to the above notation and the assumption, assume $k > n$. Then we have

$$\tilde{\xi}(n)\Lambda(k, f) \prod_{i=1}^{n/2-1} L(2i-1, f, \text{St})\tilde{\xi}(2i) = 2^{n} \frac{\langle I_n(f) f, I_n(f) f \rangle}{\langle f, f \rangle^{n/2-1}\langle \tilde{f}, \tilde{f} \rangle},$$

where $\alpha$ is an integer depending only on $n$ and $k$. 
We note that the above theorem was conjectured by Ikeda [Ik2] under more general setting. We note that the theorem has been proved by Kohnen and Skoruppa [K-S] in case \( n = 2 \).

**Proposition 4.3** Under the above notation and the assumption we have for any fundamental discriminant \( D \) such that \((-1)^{n/2}D > 0 \) and \( L(k - n/2, f, \chi_D) \neq 0 \) we have

\[
\frac{c(|D|)^2}{\langle I_n(f), I_n(f) \rangle} = a_{n,k} \frac{\langle f, f \rangle^{n/2} |D|^{k - n/2} L(k - n/2, f, \chi_D)}{\prod_{i=1}^{n/2-1} L(k; f)\tilde{\xi}(n) \sum_{i=1}^{n} L(2i + 1, f, St)\tilde{\xi}(2i)}
\]

with some algebraic number \( a_{n,k} \) depending only on \( n, k \).

**Proof.** By the result in Kohnen-Zagier [K-Z], for any fundamental discriminant \( D \) such that \((-1)^{n/2}D > 0 \) we have

\[
\frac{c(|D|)^2}{\langle f, f \rangle} = 2^{k-n/2-1} |D|^{k-n/2-1/2} \Lambda(k - n/2, f, \chi_D). 
\]

Thus the assertion holds.

**Lemma 4.4.** Let \( f \) be as above.

(1) Let \( r_1 \) be an element of \( L_n' \) in Proposition 2.3. Then we have

\[
\lambda_{I_n(f)}(r_1) = p^{(n-1)(k-n(n+1)/2)} a_f(p) \sum_{i=1}^{n} p^i. 
\]

(2) Let \( n = 2 \). Then we have

\[
\lambda_{I_2(f)}(T(p)) = a_f(p) + p^{2k-n-1} + p^{2k-n-2}. 
\]

**Lemma 4.5.** Let \( d \) be a fundamental discriminant such that \((-1)^{n/2}d > 0 \).

(1) Assume that \( d \neq 1 \). Then there exists a positive definite half integral matrix \( A \) of degree \( n \) such that \((-1)^{n/2} \det(2A) = d \).

(2) Assume \( n \equiv 0 \mod 8 \). Then there exists a positive definite half integral matrix \( A \) of degree \( n \) such that \( \det(2A) = 1 \).

(3) Assume that \( n \equiv 4 \mod 8 \). Then for any prime number \( q \) there exists a positive definite half integral matrix \( A \) of degree \( n \) such that \( \det(2A) = q^2 \).

**Proof.** (1) For a non-degenerate symmetric matrix \( A \) with entries in \( Q_p \) let \( h_p(A) \) be the Hasse invariant of \( A \). First let \( n \equiv 2 \mod 4 \) and \( d = -4 \). Take a family \( \{A_p\}_p \) of half integral matrices over \( Z_p \) of
degree \( n \) such that \( A_p = 1_n \) if \( p \neq 2 \), and \( A_2 = (-1)^{(n-2)/4} 1_{2\perp H_{n/2-1}} \), where \( H_n = \underbrace{H \perp \cdots \perp H}_r \) with \( H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \). Then we have \( \det A = 2^{2-n} \in \mathbb{Q}_p^\times/\langle \mathbb{Q}_p^\times \rangle^2 \) for any \( p \), and \( h_p(A) = 1 \) for any \( p \). By [I-S, Proposition 2.1], there exists an element \( A = 1_{E_{n/2}} \). We take the assertion for this case. Finally assume that \( d \) contains an odd prime factor \( q \). For \( n \neq p \), we take a matrix \( A_p \) so that \( \det A_p = 2^{-n/2} d \in \mathbb{Q}_q^\times/\langle \mathbb{Q}_q^\times \rangle^2 \). Then for almost all \( p \) we have \( h_p(A_p) = 1 \). We take \( \xi \in \mathbb{Z}_q^\times \) such that \( (q, -\xi) = \prod_{p \neq q} h_p(A_p) \), and put \( A_q = \xi d \perp \xi \perp 1_{n-2} \). Then we have \( \det A_q 2^{-n/2} d \in \mathbb{Q}_q^\times/\langle \mathbb{Q}_q^\times \rangle^2 \), and \( h_p(A_q) \prod_{p \neq q} = 1 \). Thus again by [I-S, Proposition 2.1], we prove the assertion for this case.

(2) It is well known that there exists a positive definite half-integral matrix \( E_8 \) of degree 8 such that \( \det(2E_8) = 1 \). Thus \( A = (E_8 \perp \cdots \perp E_8) \) satisfies the required condition.

(3) Let \( q \neq 2 \). Then, take a family \( \{A_p\} \) of half-integral matrices over \( \mathbb{Z}_q \) of degree \( n \) such that \( A_q \sim_{\mathbb{Z}_q} q \perp (-q\xi) \perp (-\xi) \perp 1_{n-3} \) with \( (\xi) = -1, A_2 = H_{n/2}, \) and \( A_p = 1_n \) for \( p \neq 2 \). Then by the same argument as in (1) we can show that there exits a positive definite half integral matrix \( A \) of degree \( n \) such that \( \det(2A) = q^2 \) such that \( A \sim_{\mathbb{Z}_q} A_p \) for any \( p \). Let \( q = 2 \). Then the matrix \( A' = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \end{pmatrix}^{(n-4)/4} \) is a positive definite and \( \det(2A') = 4 \). Thus the matrix \( A' \perp (E_8 \perp \cdots \perp E_8) \) satisfies the required condition.

**Proposition 4.6.** Let \( k \) and \( n \) be positive even integer. Let \( d \) be a fundamental discriminant. Let \( f \) be a primitive form in \( S_{2k-n}(\Gamma_1) \). Let \( \mathfrak{p} \) be a prime ideal in \( K \). Then there exists a positive definite half integral matrices \( A \) of degree \( n \) such that \( c_{I_n(f)}(A) = c_f(|d|)q \) with \( q \) not divisible by \( \mathfrak{p} \).
Proof. First assume that $d \neq 1$, or $n \not \equiv 4 \mod 8$. (1) By (1) and (2) of Lemma 4.5, there exists a matrix $A$ such that $\delta_A = d$. Thus we have $c_{\tau_n(f)}(\mathcal{A}) = c_f(|d|)$. This proves the assertion.

Next assume that $n \equiv 4 \mod 8$ and that $d = 1$. Assume that $c_f(q) + q^{k-n/2-1}(-q-1)$ is divisible by $\wp$ for any prime number $q$. Let $p$ be a prime number divisible by $\wp$. Fix an imbedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let $\rho_{f,p} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$ be the Galois representation attached to $f$. Then by Chebotarev density theorem, the semi-simplification $\overline{\rho}_{f,p}^ss$ of $\overline{\rho}_{f,p}$ can be expressed as

$$\overline{\rho}_{f,p}^ss = \overline{\chi}_p^{k-n/2} \oplus \overline{\chi}_p^{k-n/2-1}$$

with $\overline{\chi}_p$ the $p$-adic mod $p$ cyclotomic character. On the other hand, by the Fontaine-Messing [Fo-Me] and Fontaine-Laffaille [Fo-La], $\overline{\rho}_{f,p}/I_p$ should be $\overline{\chi}_p^{2k-n-1} \oplus 1$ or $\omega_2^{2k-n-1} \oplus \omega_2^{p(2k-n-1)}$ with $\omega_2$ the fundamental character of level 2, where $I_p$ denotes the inertia group of $p$ in $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This is impossible because $k > 2$. Thus there exists a prime number $q$ such that $c_f(q) + q^{k-n/2-1}(-q-1)$ is not divisible by $\wp$. For such a $q$, take a positive definite matrix $A$ in (3) of Lemma 4.5. Then

$$c_{\tau_n(f)}(\mathcal{A}) = c(1)q^{k-(n+1)/2} \beta qF_q(A,q^{-(n+1)/2}\beta^{-1}).$$

By [Ka1], we have

$$F_q(B,X) = 1 - Xq^{(n-2)/2}(q^2 + q) + q^3(Xq^{(n-2)/2})^2.$$

Thus we have

$$c_{\tau_n(f)}(\mathcal{A}) = c(1)(c_f(q) + q^{k-n/2-1}(-q-1)).$$

Thus the assertion holds.

**Theorem 4.7.** Let $k \geq 2n+4$. Let $K$ and $f$ be as above. Assume that the Conjecture $B$ holds for $f$. Let $\wp$ be a prime ideal of $K$. Furthermore assume that

1. $\wp$ divides $L(k,f)\prod_{i=1}^{n/2-1} L(2i+1,f,St)$.
2. $\wp$ does not divide

$$\hat{\xi}(2m)\prod_{i=1}^{n} L(2m+k-i,f)L(k-n/2,f,\chi_D)D(2k-1)!$$

for some integer $n/2 + 1 \leq m \leq k/2 - n/2 - 1$, and for some fundamental discriminant $D$ such that $(-1)^{n/2}D > 0$.

Then $\wp$ is a congruence prime of $I_{\tau_n(f)}$ with respect to $CI_{\tau_n(f)}$. Furthermore assume that the following condition hold:
(3) $\Psi$ does not divide
\[
\langle f, f \rangle \frac{C_{k,n}}{\Omega(f, +, A_\Psi)} \Omega(f, -, A_\Psi),
\]
where $C_{k,n} = 1$ or $\prod_{q \leq (2k-n)/12}(1 + q + \cdots + q^{n-1})$ according as $n = 2$ or not.

Then $\Psi$ is a congruence prime of $I_n(f)$ with respect to $(\mathfrak{S}_k(\Gamma_n)^*)^\perp$.

Proof. Let $\Psi$ be a prime ideal satisfying the condition (1) and (2). For the $D$ above, take a matrix $A \in H_n(\mathbb{Z})_{> 0}$ so that $c_{I_n(f)}(A) = c_f(|D|)q$ with $q$ not divisible by $\Psi$. Then by Proposition 4.3, we have
\[
\Lambda(2m, I_n(f), \mathfrak{S} f)|c_{I_n(f)}(A)|^2 = \Lambda(2m, I_n(f), \mathfrak{S} f)|c_f(|D|)|^2 q^2
\]
\[
\epsilon_{k,m} \prod_{i=1}^n L(2m + k - i, f)|D|^{k-n/2}L(k - n/2, f, \chi_D)\frac{L(k; f)\xi(n)\prod_{i=1}^{n/2-1}L(2i + 1, f, \mathfrak{S} f)\xi(2i)}{\Omega(f, +; A_\Psi)\Omega(f, -; A_\Psi)^n/2},
\]
where $\epsilon_{k,m}$ is a rational number whose numerator is not divided by $\Psi$.
We note that $\langle f, f \rangle \frac{\Omega(f, +; A_\Psi)}{\Omega(f, -, A_\Psi)}$ is $\Psi$-integral. Thus by assumptions (1) and (2), $\Psi$ divides $(\Lambda(2m, I_n(f), \mathfrak{S} f)c_{I_n(f)}(A))^2$, and thus it divides $(\Lambda(2m, I_n(f), \mathfrak{S} f))^{2}I_{I_n(f)}$. We note that $I_n(f)$ satisfies the assumption in Theorem 3.1. Thus by Theorem 3.1, there exits a Hecke eigenform $G \in \mathbb{C}(I_n(f))^{\perp}$ such that
\[
\lambda_G(T) \equiv \lambda_{I_n(f)}(T) \mod \Psi
\]
for any $T \in L_n'$. Assume that we have $G = I_n(g)$ with some primitive form $g(z) = \sum_{m=1}^{\infty} a_g(m)e(mz) \in \mathfrak{S}_{2k-n}(\Gamma(1))$. Let $n = 2$. Then by (1) of Proposition 4.2, $\Psi$ is also a congruence prime of $f$. Let $n \geq 4$. Then by (1) of Proposition 4.4, we have
\[
(p^{n-1} + \cdots + p + 1)a_f(p) \equiv (p^{n-1} + \cdots + p + 1)a_g(p) \mod \Psi
\]
for any prime number $p$ not divisible by $\Psi$. By assumption (3), in particular, for any $p \leq (2k - n)/12$, we have
\[
a_f(p) \equiv a_g(p) \mod \Psi.
\]
Thus by Sturm [Stur], $\Psi$ is also a congruence prime of $f$. Thus by [Hi2] and [Ri2], $\Psi$ divides $\frac{\langle f, f \rangle}{\Omega(f, +; A_\Psi)\Omega(f, -; A_\Psi)}$, which contradicts the assumption (3). Thus $\Psi$ is a congruence prime of $I_n(f)$ with respect to $(\mathfrak{S}_k(\Gamma(n))^*)^{\perp}$. 
Example Let $n = 4$ and $k = 18$. Then we have $\dim S_{18}(\Gamma_4) \approx 16$ (cf. Poor and Yuen [P-Y]) and $\dim S_{18}(\Gamma_4)^* = \dim S_{32}(\Gamma_1) = 2$. Take a primitive form $f \in S_{32}(\Gamma_1)$. Then we have $[Q(f) : Q] = 2$, and $211 = \mathfrak{p}'$ in $Q(f)$. Then we have

$$N_{Q(f)/Q}(L(18, f)) = 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 211,$$

$$N_{Q(f)/Q}(\prod_{i=1}^4 L(24-i, f)) = 2^{19} \cdot 3^{13} \cdot 5^5 \cdot 7^8 \cdot 11^2 \cdot 13^5 \cdot 17^5 \cdot 19^3 \cdot 23 \cdot 503 \cdot 1307 \cdot 14243,$$

$$\tilde{\chi}(6) = 2^{-2} \cdot 3^{-2} \cdot 7^{-1}$$

and

$$N_{Q(f)/Q}(L(16, f, \chi_1)) = 2^5 \cdot 3^2 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13^2.$$

(cf. Stein [Ste].) Furthermore, by a direct computation we see neither $\mathfrak{p}$ nor $\mathfrak{p}'$ is a congruence prime of $\tilde{f}$ with respect to $S_{18}(\Gamma_4)^*$. Thus by Theorem 4.7, $\mathfrak{p}$ or $\mathfrak{p}'$ is a congruence prime of $\tilde{f}$ with respect to $S_{18}(\Gamma_4)^*$.}

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