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CONGRUENCE BETWEEN DUKE-IMAMOGLU-IKEDA LIFTS AND NON-DUKE-IMAMOGLU-IKEDA LIFTS

HIDENORI KATSURADA

1. Introduction

As is well known, there is a congruence between the Fourier coefficients of Eisenstein series and those of cuspidal Hecke eigenforms for $SL_2(\mathbb{Z})$. This type of congruence is not only interesting in its own right but also has an important application to number theory as shown by Ribet [Ri1]. Since then, there have been so many important results about the congruence of elliptic modular forms (cf. [Hi1],[Hi2],[Hi3].) In the case of Hilbert modular forms or Siegel modular forms, it is sometimes more natural and important to consider the congruence between the Hecke eigenvalues of Hecke eigenforms modulo a prime ideal $\mathfrak{p}$. We call such a $\mathfrak{p}$ a prime ideal giving the congruence or a congruence prime. For a cuspidal Hecke eigenform $f$ for $SL_2(\mathbb{Z})$, let $\hat{f}$ be a lift of $f$ to the space $\mathfrak{M}_l(\Gamma')$ of modular forms of weight $l$ for a modular group $\Gamma'$. Here we mean by the lift of $f$ a cuspidal Hecke eigenform whose certain $L$-function can be expressed in terms of certain $L$-functions of $f$. As typical examples of the lift we can consider the Doi-Naganuma lift, the Saito-Kurokawa lift, and the Duke-Imamoglu-Ikeda lift. We then consider the following problem:

**Problem.** Characterize the prime ideals giving the congruence between $\hat{f}$ and a cuspidal Hecke eigenform in $\mathfrak{M}_l(\Gamma')$ not coming from the lift. In particular characterize them in terms of special values of certain $L$-functions of $f$.

This type of problem was first investigated in the Doi-Naganuma lift case by Doi, Hida, and Ishii [D-H-I]. In our previous paper [Ka2], we considered the relationship between the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbb{Z})$ and the special values of their standard zeta functions. In particular, we proposed a conjecture concerning the congruence between Saito-Kurokawa lifts and non-Saito-Kurokawa lifts.
lifts, and proved it under certain condition. In this paper, we con-
sider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-
Imamoglu-Ikeda lifts, which is a generalization of our previous conje-
ture.

In Section 3, we review a result concerning the relationship between
the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbb{Z})$
and the special values of their standard zeta functions. In Section
4, we propose a conjecture concerning the congruence between Duke-
Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, and prove it
under a certain condition.

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fessor T. Yamauchi for their valuable comments.

Notation. For a commutative ring $R$, we denote by $M_{m,n}(R)$ the
set of $(m,n)$-matrices with entries in $R$. In particular put $M_{n}(R) =
M_{n,n}(R)$. Here we understand $M_{m,n}(R)$ the set of the empty matrix
if $m = 0$ or $n = 0$. For an $(m,n)$-matrix $X$ and an $(m,m)$-matrix $A$,
we write $A[X] = ^tXAX$, where $^tX$ denotes the transpose of $X$. Let
$a$ be an element of $R$. Then for an element $X$ of $M_{m,n}(R)$ we often
use the same symbol $X$ to denote the coset $X \mod aM_{m,n}(R)$. Put
$GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, where $\det A$ denotes the
determinant of a square matrix $A$, and $R^*$ denotes the unit group of
$R$. Let $S_n(R)$ denote the set of symmetric matrices of degree $n$ with
entries in $R$. Furthermore, for an integral domain $R$ of characteristic
different from 2, let $H_n(R)$ denote the set of half-integral matrices of
dergree $n$ over $R$, that is, $H_n(R)$ is the set of symmetric matrices of
dergree $n$ whose $(i,j)$-component belongs to $R$ or $\frac{1}{2}R$ according as $i = j$
or not. For a subset $S$ of $M_{m,n}(R)$ we denote by $S^\times$ the subset of $S$
consisting of non-degenerate matrices. In particular, if $S$ is a subset of
$S_n(R)$ with $R$ the field of real numbers, we denote by $S_{>0}$ (resp.
$S_{\geq 0}$) the subset of $S$ consisting of positive definite (resp. semi-positive
definite) matrices. Let $R'$ be a subring of $R$. Two symmetric matrices
$A$ and $A'$ with entries in $R$ are called equivalent over $R'$ with each
other and write $A \sim_{R'} A'$ if there is an element $X$ of $GL_{n}(R')$ such that
$A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For
square matrices $X$ and $Y$ we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

2. Standard zeta functions of Siegel modular forms

For a complex number $x$ put $e(x) = \exp(2\pi \sqrt{-1}x)$. Furthermore put
$J_n = \begin{pmatrix} O_n & -1^n \\ 1_n & O_n \end{pmatrix}$, where $1_n$ denotes the unit matrix of degree $n$. For
a subring $K$ of $\mathbb{R}$ put

$$GSp_n(K)^+ = \{ M \in GL_{2n}(K) \mid J_n[M] = \kappa(M)J_n \text{ with some } \kappa(M) > 0 \},$$

and

$$Sp_n(K) = \{ M \in GSp_n(K)^+ \mid J_n[M] = J_n \}.$$  

Furthermore, put

$$\Gamma^{(n)} = Sp_n(\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \bmod N \right\} \in \Gamma^{(n)} \mid C \equiv O_n \bmod N \}.$$  

Let $H_n$ be Siegel’s upper half-space. For each element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n(\mathbb{R})^+$ and $Z \in H_n$ put

$$M(Z) = (AZ + B)(CZ + D)^{-1}$$

and

$$j(M, Z) = \det(CZ + D).$$

Furthermore, for a function $f$ on $H_n$ and an integer $k$ we define $f|_k M$ as

$$(f|_k M)(Z) = \det(M)^{k/2} j(M, Z)^{-k} f(M(Z)).$$

For an integer or half integral $l$ and the subgroup $\Gamma_0^{(n)}(N)$ of $\Gamma^{(n)}$, we denote by $\mathbb{H}_k(\Gamma_0^{(n)}(N))$ (resp. $\mathbb{H}_k^\infty(\Gamma_0^{(n)}(N))$) the space of holomorphic (resp. $C^\infty$-) modular forms of weight $k$ with respect to $\Gamma_0^{(n)}(N)$. We denote by $\mathbb{S}_k(\Gamma_0^{(n)}(N))$ the sub-space of $\mathbb{H}_k(\Gamma_0^{(n)}(N))$ consisting of cusp forms. Let $f$ be a holomorphic modular form of weight $k$ with respect to $\Gamma_0^{(n)}(N)$. Then $f$ has the following Fourier expansion:

$$f(Z) = \sum_{A \in H(Z) \geq 0} a_f(A) e(\text{tr}(AZ)),$$

and in particular if $f$ is a cusp form, $f$ has the following Fourier expansion:

$$f(Z) = \sum_{A \in H(Z) \geq 0} a_f(A) e(\text{tr}(AZ)),$$

where $\text{tr}$ denotes the trace of a matrix. Let $dv$ denote the invariant volume element on $H_n$ defined by

$$dv = \det(\text{Im}(Z))^{-n-1} \wedge_{1 \leq i \leq n} (dx_{ji} \wedge dy_{ji}).$$

Here for $Z \in H_n$ we write $Z = (x_{ji}) + \sqrt{-1}(y_{ji})$ with real matrices.
of $L$ Hecke pair $(L)$ to $[[1]]_{L}$ subalgebra of $Q$ associated with the Hecke pair $(\lambda)$. Let $Q$ by all and $(\Gamma)$ where $\gamma = \Gamma_{\gamma}$ in $L$ $\Gamma_{\gamma}$ be a subalgebra of $Q$ generated by over $n$. In particular if $f, g = \Gamma_{\gamma}(1) = \Gamma_{\gamma}(p)$ and for $\gamma = \Gamma_{\gamma}(n)$. Furthermore, let $\gamma = \Gamma_{\gamma}(1)$ be an element of $\gamma$ and for $f \in \mathfrak{M}_{k}(\Gamma_{\gamma}(n))$ define the Hecke operator $|kT|^ {\gamma}$ associated to $T$ as $f|kT = \det(M)^{k/2-(n+1)/2} \sum_{\gamma} f|k\gamma$. We call this action the Hecke operator as usual (cf. [A]). If $f$ is an eigenfunction of a Hecke operator $T \in \mathfrak{L}_{n} \otimes \mathbb{C}$, we denote by $\lambda_{f}(T)$ its eigenvalue. Let $L$ be a subalgebra of $\mathfrak{L}_{n}$. We call $f \in \mathfrak{M}_{k}(\Gamma_{\gamma}(1))$ a Hecke eigenform for $L$ if it is a common eigenfunction of all Hecke operators in $L$. In particular if $L = \mathfrak{L}_{n}$ we simply call $f$ a Hecke eigenform. Furthermore, we denote by $Q(f)$ the field generated over $Q$ by eigenvalues of all $T \in \mathfrak{L}_{n}$ as in Section 1. As is well known, $Q(f)$ is a totally real algebraic number field of finite degree. Now, first we consider the integrality of the eigenvalues of Hecke operators. For an algebraic number field $K$, let $\mathcal{O}_{K}$ denote the ring of integers in $K$. The following assertion has been proved in [Mi2] (see also [Ka2].)

**Theorem 2.1** Let $k \geq n + 1$. Let $f \in \mathfrak{M}_{k}(\Gamma_{\gamma}(n))$ be a common eigenform in $\mathfrak{L}_{n}$. Then $\lambda_{f}(T)$ belongs to $\mathcal{O}_{Q(f)}$ for any $T \in \mathfrak{L}_{n}$. 
Let $\mathbf{L}_{np} = \mathbf{L}(GSp_n(\mathbb{Q})^+ \cap GL_{2n}(\mathbb{Z}[p^{-1}]), \Gamma^{(n)})$ be the Hecke algebra associated with the pair $(GSp_n(\mathbb{Q})^+ \cap GL_{2n}(\mathbb{Z}[p^{-1}]), \Gamma^{(n)})$. $\mathbf{L}_{np}$ can be considered as a subalgebra of $\mathbf{L}_n$, and is generated over $\mathbb{Q}$ by $T_i(p^2)$ $(i = 1, 2, \ldots, n)$. We now review the Satake $p$-parameters of $\mathbf{L}_{np}$; let $\mathbf{P}_n = \mathbb{Q}[X_0^\pm, X_1^\pm, \ldots, X_n^\pm]$ be the ring of Laurent polynomials in $X_0, X_1, \ldots, X_n$ over $\mathbb{Q}$. Let $\mathbf{W}_n$ be the group of $\mathbb{Q}$-automorphisms of $\mathbf{P}_n$ generated by all permutations in variables $X_1, \ldots, X_n$ and by the automorphisms $\tau_1, \ldots, \tau_n$ defined by

$$
\tau_i(X_0) = X_0 X_i, \quad \tau_i(X_i) = X_i^{-1}, \quad \tau_i(X_j) = X_j \quad (j \neq i).
$$

Furthermore, a group $\mathbf{W}_n$ isomorphic to $\mathbf{W}_n$ acts on the set $T_n = (\mathbb{C}^*)^{n+1}$ in a way similarly to above. Then there exists a $\mathbb{Q}$-algebra isomorphism $\Phi_{np}$, called the Satake isomorphism, from $\mathbf{L}_{np}$ to the $\mathbf{W}_n$-invariant subring $\mathbf{P}_n^{W_n}$ of $\mathbf{P}_n$. Then for a $\mathbb{Q}$-algebra homomorphism $\lambda$ from $\mathbf{L}_{np}$ to $\mathbb{C}$, there exists an element $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))$ of $T_n$ satisfying

$$
\lambda(\Phi_{np}^{-1}(F(X_0, X_1, \ldots, X_n))) = F(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))
$$

for $F \in \mathbf{P}_n^{W_n}$. The equivalence class of $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))$ under the action of $\mathbf{W}_n$ is uniquely determined by $\lambda$. We call this the Satake parameters of $\mathbf{L}_{np}$ determined by $\lambda$.

Now assume that an element $f$ of $M_k(Sp_n(\mathbb{Z}))$ is a Hecke eigenform. Then for each prime number $p$, $f$ defines a $\mathbb{Q}$-algebra homomorphism $\lambda_{fp}$ from $\mathbf{L}_{np}$ to $\mathbb{C}$ in a usual way, and we denote by $\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda)$ the Satake parameters of $\mathbf{L}_{np}$ determined by $f$. We then define the standard zeta function $L(f, s, \mathbf{St})$ by

$$
L(s, f, \mathbf{St}) = \prod_p \prod_{i=1}^n \{(1 - p^{-s})(1 - \alpha_i(p)p^{-s})(1 - \alpha_i(p)^{-1}p^{-s})\}^{-1}.
$$

Let $f(z) = \sum_{A \in \Gamma_n(\mathbb{Z}) \setminus 0} a(A)e(\text{tr}(Az))$ be a Hecke eigenform in $\mathcal{E}_k(\Gamma^{(n)})$.

For a positive integer $m \leq k - n$ such that $m \equiv n \pmod{2}$ put

$$
\Lambda(f, m, \mathbf{St}) = (-1)^{(m+1)/2+1}2^{-4k+n+3m^2+n+(n-1)m+2}
\times \Gamma(m+1) \prod_{i=1}^n \Gamma(2k-n-i) \frac{L(f, m, \mathbf{St})}{\pi^{-n(n+1)/2+nk+(n+1)m}}.
$$

Then the following theorem is due to Böcherer [B2] and Mizumoto [Mi].

**Theorem 2.2.** Let $l, k$ and $n$ be positive integers such that $\rho(n) \leq l \leq k - n$, where $\rho(n) = 3$, or $1$ according as $n \equiv 1 \pmod{4}$ and $n \geq 5$, or not. Let $f \in \mathcal{E}_k(\Gamma^{(n)})$ be a Hecke eigenform. Then $\Lambda(f, m, \mathbf{St})$ belongs to $\mathbb{Q}(f)$. 
For later purpose, we consider a special element in $L_{np}$; the polynomial $X_0^2X_1X_2 \cdots X_n \sum_{i=1}^{n}(X_i + X_i^{-1})$ is an element of $P^n_{ \mathbb{W}}$, and thus we can define an element $\Phi_{np}^{-1}(X_0^2X_1X_2 \cdots X_n \sum_{i=1}^{n}(X_i + X_i^{-1}))$ of $L_{np}$, which is denoted by $r_1$.

**Proposition 2.3.** Under the above notation the element $r_1$ belongs to $L'_n$, and we have

$$\lambda_f(r_1) = p^{n(k-(n+1)/2)} \sum_{i=1}^{n}(\alpha_i(p) + \alpha_i(p)^{-1}).$$

**Proof.** By a careful analysis of the computation in page 159-160 of [A], we see that $r_1$ is a $\mathbb{Z}$-linear combination of $T_i(p^2)$ ($i = 1, \ldots, n$), and therefore we can prove the first assertion. Furthermore, by Lemma 3.3.34 of [A], we can prove the second assertion.

3. Congruence of modular forms and special values of the standard zeta functions

In this section we review a result concerning the congruence between the Hecke eigenvalues of modular forms of the same weight following [Ka2]. Let $K$ be an algebraic number field, and $\mathcal{O} = \mathcal{O}_K$ the ring of integers in $K$. For a prime ideal $\mathfrak{p}$ of $\mathcal{O}$, we denote by $\mathcal{O}(\mathfrak{p})$ the localization of $\mathcal{O}$ at $\mathfrak{p}$ in $K$. Let $\mathfrak{a}$ be a fractional ideal in $K$. If $\mathfrak{a} = \mathfrak{p}^{\mathfrak{A}}$ with $\mathfrak{A}\mathcal{O}(\mathfrak{p}) = \mathcal{O}(\mathfrak{p})$, we write $\text{ord}_{\mathfrak{p}} = e$. We simply write $\text{ord}_G(c) = \text{ord}_{\mathfrak{p}}(c)$ for $c \in K$. Now let $f$ be a Hecke eigenform in $\mathcal{S}_k(\Gamma(n))$ and $M$ be a subspace of $\mathcal{S}_k(\Gamma(n))$ stable under Hecke operators $T \in L_n$. Assume that $M$ is contained in $(Cf)^{\perp}$, where $(Cf)^{\perp}$ is the orthogonal complement of $Cf$ in $\mathcal{S}_k(\Gamma(n))$ with respect to the Petersson product. Let $K$ be an algebraic number field containing $\mathbb{Q}(f)$. A prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ is called a congruence prime of $f$ with respect to $M$ if there exists a Hecke eigenform $g \in M$ such that

$$\lambda_f(T) \equiv \lambda_g(T) \mod \tilde{\mathfrak{p}}$$

for any $T \in L_n'$, where $\tilde{\mathfrak{p}}$ is the prime ideal of $\mathcal{O}_K \mathbb{Q}(g)$ lying above $\mathfrak{p}$. If $M = (Cf)^{\perp}$, we simply call $\mathfrak{p}$ a congruence prime of $f$.

Now we consider the relation between the congruence primes and the standard zeta values. To consider this, we have to normalize the standard zeta value $\Lambda(f, l, \text{St})$ for a Hecke eigenform $f$ because it is not uniquely determined by the system of Hecke eigenvalues of $f$. We note that there is no reasonable normalization of cuspidal Hecke eigenform in the higher degree case unlike the elliptic modular case.
Thus we define the following quantities: for a Hecke eigenform \( f(z) = \sum_A a_f(A)e(\text{tr}(Az)) \) in \( \mathcal{S}_k(\Gamma(n)) \), let \( \mathfrak{A}_f \) be the \( \mathbb{Q}_f \)-module generated by all \( a_f(A) \)'s. Assume that there exists a complex number \( c \) such that all the Fourier coefficients \( cf \) belongs to \( \mathbb{Q}(f) \). Then \( \mathfrak{A}_f \) is a fractional ideal in \( \mathbb{Q}(f) \), and therefore, so is \( \Lambda(f, l, \mathfrak{A}_f) \mathfrak{A}_f \) if \( l \) satisfies the condition in Theorem 2.2. We note that this fractional ideal does not depend on the choice of \( c \). We also note that the value \( N_{\mathbb{Q}(f)}(\Lambda(f, l, \mathfrak{A}_f))N(\mathfrak{A}_f)^2 \) does not depend on the choice of \( c \), where \( N(\mathfrak{A}_f) \) is the norm of the ideal \( \mathfrak{A}_f \). Then, we have

**Theorem 3.1.** Let \( f \) be a Hecke eigenform in \( \mathcal{S}_k(\Gamma(n)) \). Assume that there exists a complex number \( c \) such that all the Fourier coefficients \( cf \) belongs to \( \mathbb{Q}(f) \). Let \( l \) be a positive integer satisfying the condition in Theorem 2.2. Let \( \mathfrak{P} \) be a prime ideal of \( \mathfrak{A} \). Assume that \( \text{ord}_{\mathfrak{P}}(\Lambda(f, l, \mathfrak{A}_f)) < 0 \) and that it does not divide \( (2l - 1)! \). Then \( \mathfrak{P} \) is a congruence prime of \( f \). In particular, if a rational prime number \( p \) divides the denominator of \( N_{\mathbb{Q}(f)}(\Lambda(f, l, \mathfrak{A}_f))N(\mathfrak{A}_f)^2 \), then \( p \) is divisible by some congruence prime of \( f \).

Now for a Hecke eigenform \( f \) in \( \mathcal{S}_k(\Gamma(n)) \), let \( \mathfrak{E}_f \) denote the subspace of \( \mathcal{S}_k(\Gamma(n)) \) spanned by all Hecke eigenforms with the same system of the Hecke eigenvalues as \( f \).

**Corollary.** In addition to the above notation and the assumption, assume that \( \mathcal{S}_k(\Gamma(n)) \) has the multiplicity one property. Then \( \mathfrak{P} \) is a congruence prime of \( f \) with respect to \( \mathfrak{E}_f^{-} \). In particular, if a rational prime number \( p \) divides the denominator of \( N_{\mathbb{Q}(f)}(\Lambda(f, l, \mathfrak{A}_f))N(\mathfrak{A}_f)^2 \), then \( p \) is divisible by some congruence prime of \( f \) with respect to \( \mathfrak{E}_f^{-} \).

4. **Congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts**

In this section, we consider the congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts. Throughout this section and the next, we assume that \( n \) and \( k \) are even positive integers. Let

\[
 f(z) = \sum_{m=1}^{\infty} a(m)e(mz)
\]

be a normalized Hecke eigenform of weight \( 2k - n \) with respect to \( SL_2(\mathbb{Z}) \). For a Dirichlet character \( \chi \), we then define the L-function
$L(s, f)$ of $f$ twisted by $\chi$ by

$$L(s, f) = \prod_p \{(1 - \chi(p)\beta_p p^{k-n/2-1/2-s})(1 - \chi(p)\beta_p^{-1} p^{k-n/2-1/2-s})\}^{-1},$$

where $\beta_p$ is a non-zero complex number such that $\beta_p + \beta_p^{-1} = p^{-k+n/2+1/2}a(p)$. We simply write $L(s, f)$ as $L(s, f, \chi)$ if $\chi$ is the principal character. Furthermore, let $\tilde{f}$ be the cusp form of weight $k-n/2+1/2$ belonging to the Kohnen plus space corresponding to $f$ via the Shimura correspondence (cf. [Ko1]). Then $\tilde{f}$ has the following Fourier expansion:

$$\tilde{f}(z) = \sum c(e) e(ez),$$

where $e$ runs over all positive integers such that $(-1)^{k-n/2}e \equiv 0, 1 \mod 4$. We then put

$$a_{I_n(f)}(T) = c([T]) \prod_p (p^{k-n/2-1/2} \beta_p) \nu_\beta(F_p(T, p^{-n+1/2} \beta_p^{-1})).$$

We note that $a_{I_n(f)}(T)$ does not depend on the choice of $\beta_p$. Define a Fourier series $I_n(f)(Z)$ by

$$I_n(f)(Z) = \sum_{T \in \mathcal{H}_n(Z), \gamma > 0} a_{I_n(f)}(T)e(\text{tr}(TZ)).$$

In [Ik1] Ikeda showed that $I_n(f)(Z)$ is a cusp form of weight $k$ with respect to $\Gamma^{(n)}$ and a Hecke eigenform for $L_n$ such that

$$L(s, I_n(f), St) = \zeta(s) \prod_{i=1}^n L(s + k - i, f).$$

This was first conjectured by Duke and Imamoglu. Thus we call $I_n(f)$ the Duke-Imamoglu-Ikeda lift of $f$. We note that we have $Q(\tilde{f}) = Q(I_n(f)) = Q(f)$. Furthermore, we have $\mathfrak{Z}_j = \mathfrak{Z}_{I_n(f)}$, where $\mathfrak{Z}_j$ is the $\mathfrak{Q}(\tilde{f})$-module generated by all the Fourier coefficients of $\tilde{f}$.

Now to consider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, first we prove the following:

**Proposition 4.1** $I_n(f)$ is a Hecke eigenform.

We note that Ikeda proved in [Ik1] that $I_n(f)$ is a Hecke eigenform for $L_n$ but has not proved that it is a Hecke eigenform for $L_n$. This was pointed out to us by B. Heim (see [He]). We thank him for his comment. We also note that an explicit form of the spinor L-function of $I_n(f)$ was obtained by Murakawa [Mu] and Schmidt [Sch] assuming that $I_n(f)$ is a Hecke eigenform.
Proof of Proposition 4.1. We have only to prove that $I_n(f)$ is an eigenfunction of $T(p)$ for any prime $p$. The proof may be more or less well known, but for the convenience of the readers we here give the proof. For a modular form

$$F(Z) = \sum_B c_F(B) e(\text{tr}(BZ)),$$

let $c_F^{(p)}(B)$ be the $B$-th Fourier coefficient of $F|T(p)$. Then for any positive definite matrix $B$ we have

$$c_F^{(p)}(B) = p^{nk-n(n+1)/2} \sum_{d_1|d_2|\cdots|d_n|p} d_1^m d_2^{m-1} \cdots d_n$$

$$\times \sum_{D \in \Lambda_n(d_1 \cdots d_n) \Lambda_n} \det D^{-k} c_F(p^{-1}A[t^l D]),$$

where $\Lambda_n = GL_n(\mathbb{Z})$.

Now let $E_{n,k}(Z)$ be the Siegel Eisenstein series of degree $n$ and of weight $k$ defined by

$$E_{n,k}(Z) = \sum_{\gamma \in \Gamma_{n,\infty} \setminus \Gamma_n} j(\gamma, Z)^{-k}.$$

For $k \geq n+1$, the Siegel Eisenstein series $E_{n,k}(Z)$ is a holomorphic modular form of weight $k$ with respect to $\Gamma_n$. Furthermore, $E_{n,k}(Z)$ is a Hecke eigenform and in particular we have

$$E_{n,k}|T(p)(Z) = h_{n,p}(p^k) E_{n,k}(Z),$$

where

$$h_{n,p}(X) = 1 + \sum_{r=1}^{n} \sum_{1 \leq i_1 < \cdots < i_r \leq n} p^{-\sum_{j=1}^{r} i_j} X^r.$$

Let $c_{n,k}(B)$ be the $B$-th Fourier coefficient of $E_{n,k}(Z)$. Then we have

$$h_{n,p}(p^k)c_{n,k}(B) = p^{nk-n(n+1)/2} \sum_{d_1|d_2|\cdots|d_n|p} d_1^m d_2^{m-1} \cdots d_n$$

$$\times \sum_{D \in \Lambda_n(d_1 \cdots d_n) \Lambda_n} \det D^{-k} c_{n,k}(p^{-1}B[t^l D]).$$

Let $B$ be positive definite. Then we have

$$c_{n,k}(B) = a_{n,k}(\det 2B)^{k-(n+1)/2} L(k-n/2, \chi_B) \prod_q F_q(B, p^{-k}),$$

where $a_{n,k}$ is a non-zero constant depending only on $n$ and $k$. We note that we have

$$F_q(p^{-1}B[t^l D], X) = F_q(B, X)$$
for any $D \in \Lambda_n(d_1 \perp \cdots \perp d_n)\Lambda_n$ with $d_1 \cdots |d_n|p$ if $q \neq p$. Thus we have

$$h_{n,p}(p^k)F_p(B, p^{-k}) = \sum_{e_1 \leq e_2 \leq \cdots \leq e_n} p^{ne_1+(n-1)e_2+\cdots+e_n}p^{e_1+\cdots+e_n}(k-n-1)$$

$$\times \sum_{D \in \Lambda_n(p^{e_1+\cdots+e_n})\Lambda_n} F_p(p^{-1}B[^tD], p^{-k}).$$

The both-hand sides of the above are polynomials in $p^k$ and the equality holds for infinitely many $k$. Thus we have

$$h_{n,p}(X^{-1})F_p(B, X) = \sum_{e_1 \leq e_2 \leq \cdots \leq e_n} p^{ne_1+(n-1)e_2+\cdots+e_n}(X^{-1}p^{-n-1}(e_1+\cdots+e_n)$$

$$\times \sum_{D \in \Lambda_n(p^{e_1+\cdots+e_n})\Lambda_n} F_p(p^{-1}B[^tD], X)$$

as polynomials in $X$ and $X^{-1}$. Thus we have

$$(p^{k-(n+1)/2}X)^{n/2}h_{n,p}(p^{(n+1)/2}X^{-1})F_p(B_p) = p^{nk-n(n+1)/2}$$

$$\times \sum_{\beta \in \text{valuation}} \det D^{-k}(p^{k-(n+1)/2}X^{-1})\nu_p(B_p)F_p(p^{-1}B[^tD], p^{-n(n+1)/2}X).$$

We recall that we have

$$c_{n,p}(f) = c_f(\nu_p(\theta)) \frac{k^{1/2}p^{k-n(n+1)/2}}{\prod_q (\beta_q)^{\nu_q(\theta)}} F_q(B, q^{n(n+1)/2}p^{-1})$$

where $\beta_q$ is the Satake $q$-parameter of $f$. We also note that $c_f(\nu_p(\theta)) = c_f(\theta)$ for any $D$. Thus we have

$$F_p(B) = \sum_{D \in \Lambda_n(d_1 \perp \cdots \perp d_n)\Lambda_n} \det D^{-k}c_{n,p}(f)F_p(B[^tD])$$

This proves the assertion.

Let $f$ be a primitive form in $\mathfrak{S}_{2k-n}(\Gamma(1))$. Let $\{f_1, \ldots, f_d\}$ be a basis of $\mathfrak{S}_{2k-n}(\Gamma(1))$ consisting of primitive forms. Let $K$ be an algebraic number field containing $\mathbb{Q}(f_1) \cdots \mathbb{Q}(f_d)$, and $A = \mathfrak{O}_K$. To formulate our conjecture exactly, we introduce the Eichler-Shimura periods as follows (cf. Hida [Hi3].) Let $\mathfrak{P}$ be a prime ideal in $K$. Let $A_{\mathfrak{P}}$ be a valuation ring in $K$ corresponding to $\mathfrak{P}$. Assume that the residual characteristic of $A_{\mathfrak{P}}$ is greater than or equal to $5$. Let $L(2\kappa - n - 2, A_{\mathfrak{P}})$ be the module of homogeneous polynomials of degree $2\kappa - n - 2$ in the variables $X, Y$.
with coefficients in \( A_\mathfrak{p} \). We define the action of \( M_2(\mathbb{Z}) \cap GL_2(\mathbb{Q}) \) on \( L(2k - n - 2, A_\mathfrak{p}) \) via

\[
\gamma \cdot P(X, Y) = P(\gamma^t(X, Y)(\gamma)^t),
\]

where \( \gamma^t = (\det \gamma)^{-1} \). Let \( H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_\mathfrak{p})) \) be the parabolic cohomology group of \( \Gamma^{(1)} \) with values in \( L(2k - n - 2, A_\mathfrak{p}) \). Fix a point \( z_0 \in H_1 \). Let \( g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \) or \( g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \). We then define the differential \( \omega(g) \) as

\[
\omega(g)(z) = \begin{cases} 
2\pi i g(z)(X - zY)^n dz & \text{if } g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \\
2\pi \sqrt{-1} g(z)(X - \bar{z}Y)^n dz & \text{if } g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)}),
\end{cases}
\]

and define the cohomology class \( \delta(g) \) of the 1-cocycle of \( \Gamma^{(1)} \) as

\[
\gamma \in \Gamma^{(1)} \mapsto \int_{z_0}^{\gamma(z_0)} \omega(g).
\]

The mapping \( g \mapsto \delta(g) \) induces the isomorphism

\[
\delta : \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \oplus \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \rightarrow H^1_p(\Gamma^{(1)}, L(2k - n - 2, \mathbb{C})),
\]

which is called the Eichler-Shimura isomorphism. We can define the action of Hecke algebra \( \mathcal{L}_1 \) on \( H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_\mathfrak{p})) \) in a natural manner. Furthermore, we can define the action \( F_\infty \) on \( H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_\mathfrak{p})) \) as

\[
F_\infty(\delta(g)(z)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \delta(g)(-\bar{z}),
\]

and this action commutes with the Hecke action. For a primitive form \( f \) and \( j = \pm 1 \), we define the subspace \( H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_\mathfrak{p}))[f, j] \) of \( H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_\mathfrak{p})) \) as

\[
H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_\mathfrak{p}))[f, j] = \{ x \in H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_\mathfrak{p})); x|T = \lambda_f(T)x \text{ for } T \in \mathcal{L}_1 \text{, and } F_\infty(x) = jx \}.
\]

Since \( A_\mathfrak{p} \) is a principal ideal domain, \( H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_\mathfrak{p}))[f, j] \) is a free module of rank one over \( A_\mathfrak{p} \). For each \( j = \pm 1 \) take a basis \( \eta(f, j, A_\mathfrak{p}) \) of \( H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_\mathfrak{p}))[f, j] \) and define a complex number \( \Omega(f, j; A_\mathfrak{p}) \) by

\[
(\delta(f) + jF_\infty(\delta(f)))/2 = \Omega(f, j; A_\mathfrak{p})\eta(f, j; A_\mathfrak{p}).
\]

This \( \Omega(f, j; A_\mathfrak{p}) \) is uniquely determined up to constant multiple of units in \( A_\mathfrak{p} \). We call \( \Omega(f, +; A_\mathfrak{p}) \) and \( \Omega(f, -; A_\mathfrak{p}) \) the Eichler-Shimura periods. For \( j = \pm 1 \leq l \leq 2k - n - 1 \), and a Dirichlet character \( \chi \) such
that \( \chi(-1) = j(-1)^{l-1} \), put

\[
L(l, f, \chi) = L(l, f, \chi; A_{\mathfrak{P}}) = \frac{\Gamma(l) L(l, f, \chi)}{\tau(\chi)(2\pi \sqrt{-1})^l \Omega(f, j; A_{\mathfrak{P}})},
\]

where \( \tau(\chi) \) is the Gauss sum of \( \chi \). In particular, put \( L(l, f; A_{\mathfrak{P}}) = L(l, f, \chi; A_{\mathfrak{P}}) \) if \( \chi \) is the principal character. Furthermore, put

\[
L(s, f, \text{St}) = 4(2\pi)^{-s-2k+n+1} \Gamma(s) \Gamma(s + 2k - n - 1) L(s, f, \text{St}).
\]

It is well-known that \( L(l, f, \chi) \) belongs to the field \( K(\chi) \) generated over \( K \) by all the values of \( \chi \), and \( L(l, f, \text{St}) \) belongs to \( \mathbb{Q}(f) \) (cf. [Bo].) Let \( I_n(f) \) be the Duke-Imamoglu-Ikeda lift of \( f \). Let \( \mathcal{E}_k(\Gamma(n)^*) \) be the subspace of \( \mathcal{E}_k(I_n) \) generated by all the Duke-Imamoglu-Ikeda lifts \( I(g)^n \) of primitive forms \( g \in \mathcal{E}_{2k-n}(\Gamma(1)) \). We remark that \( \mathcal{E}_k(\Gamma(2)^*) \) is the Maass subspace of \( \mathcal{E}_k(\Gamma(2)) \).

**Conjecture A.** Let \( K \) and \( f \) be as above. Assume that \( k > n \). Let \( \mathfrak{P} \) be a prime ideal of \( K \) not dividing \( (2k-1)! \). Then \( \mathfrak{P} \) is a congruence prime of \( I_n(f) \) with respect to \( (\mathcal{E}_k(\Gamma(n)^*))^\perp \) if \( \mathfrak{P} \) divides \( L(k, f) \prod_{i=1}^{n/2-1} L(2i + 1, f, \text{St}) \).

**Remark.** This is an analogue of the Doi-Hida-Ishii conjecture concerning the congruence primes of the Doi-Naganuma lifting [D-H-I]. (See also [Ka1].) We also note that this type of conjecture has been proposed by Harder [Ha] in the case of vector valued Siegel modular forms.

Now to explain why our conjecture is reasonable, we refer to Ikeda’s conjecture on the Petersson inner product of the Duke-Imamoglu-Ikeda lifting. Let \( f \) and \( \tilde{f} \) be as above. Put

\[
\check{\xi}(s) = 2(2\pi)^{-s} \Gamma(s) \zeta(s),
\]

and

\[
\Lambda(s, f) = 2(2\pi)^{-s} \Gamma(s) L(s, f).
\]

**Theorem 4.2.** (Katsurada and Kawamura [K-K]) In addition to the above notation and the assumption, assume \( k > n \). Then we have

\[
\check{\xi}(n) \Lambda(k, f) \prod_{i=1}^{n/2-1} L(2i - 1, f, \text{St}) \check{\xi}(2i) = 2^n \frac{\langle I_n(f) f, I_n(f) \rangle}{\langle f, f \rangle^{n/2-1} \langle \tilde{f}, \tilde{f} \rangle},
\]

where \( \alpha \) is an integer depending only on \( n \) and \( k \).
We note that the above theorem was conjectured by Ikeda [Ik2] under more general setting. We note that the theorem has been proved by Kohnen and Skoruppa [K-S] in case $n = 2$.

**Proposition 4.3** Under the above notation and the assumption we have for any fundamental discriminant $D$ such that $(-1)^{n/2}D > 0$ and $L(k - n/2, f, \chi_D) \neq 0$ we have

$$
\frac{c(|D|)^2}{\langle I_n(f), I_n(f) \rangle} = a_{n,k} \frac{\langle f, f \rangle^{n/2} |D|^{k-n/2} L(k - n/2, f, \chi_D)}{\prod_{i=1}^{n/2-1} L(k, f) \xi(n) \prod_{i=1}^{n/2} L(2i + 1, f, St) \xi(2i)}
$$

with some algebraic number $a_{n,k}$ depending only on $n, k$.

**Proof.** By the result in Kohnen-Zagier [K-Z], for any fundamental discriminant $D$ such that $(-1)^{n/2}D > 0$ we have

$$
\frac{c(|D|)^2}{\langle f, f \rangle} = 2^{k-n/2-1} |D|^{k-n/2-1/2} \Lambda(k - n/2, f, \chi_D).
$$

Thus the assertion holds.

**Lemma 4.4.** Let $f$ be as above.

1. Let $r_1$ be an element of $L_n'$ in Proposition 2.3. Then we have

$$
\lambda_{I_n(f)}(r_1) = p^{(n-1)k-n(n+1)/2} a_f(p) \sum_{i=1}^{n} p^i.
$$

2. Let $n = 2$. Then we have

$$
\lambda_{I_2(f)}(T(p)) = a_f(p) + p^{2k-n-1} + p^{2k-n-2}.
$$

**Lemma 4.5.** Let $d$ be a fundamental discriminant such that $(-1)^{n/2}d > 0$.

1. Assume that $d \neq 1$. Then there exists a positive definite half integral matrix $A$ of degree $n$ such that $(-1)^{n/2} \det(2A) = d$.
2. Assume $n \equiv 0 \mod 8$. Then there exists a positive definite half integral matrix $A$ of degree $n$ such that $\det(2A) = 1$.
3. Assume that $n \equiv 4 \mod 8$. Then for any prime number $q$ there exists a positive definite half integral matrix $A$ of degree $n$ such that $\det(2A) = q^2$.

**Proof.** (1) For a non-degenerate symmetric matrix $A$ with entries in $\mathbb{Q}_p$ let $h_p(A)$ be the Hasse invariant of $A$. First let $n \equiv 2 \mod 4$ and $d = -4$. Take a family $\{A_p\}_p$ of half integral matrices over $\mathbb{Z}_p$ of
any matrix $A$ as in (1) we can show that there exists a positive definite half integral matrices $A_p$ divisible by $1_n$ if $p \neq 2$, and $A_2 = (-1)^{(n-2)/4} 1_2 \perp H_{n/2-1}$, where $H_r = H \perp \cdots \perp H$ with $H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$. Then we have $\det A = 2^{2-n} \in \mathbb{Q}_p^×/(\mathbb{Q}_p^×)^2$ for any $p$, and $h_p(A) = 1$ for any $p$. Thus by [I-S, Proposition 2.1], there exists an element $A$ of $\mathcal{L}_{n,2}\mathbb{Z}$ such that $A \sim A_p$ for any $p$. In particular we have $(-1)^{n/2} \det(2A) = -4$. Next let $d = (-1)^{n/2}$. We take $A_p = (1)^{n/2} 1 , -1$ if $p \neq 2$. We can take $\xi \in \mathbb{Z}_2^×$ such that $(2, \xi) = (-1)^{(n+4)/8}$, and put $A_2 = 2\xi \perp (-\xi) \perp H_{n/2-1}$. Then we have $\det A = (-1)^{n/2} 2^{3-n} \in \mathbb{Q}_p^×/(\mathbb{Q}_p^×)^2$ for any $p$, and $h_p(A) = 1$ for any $p$. Thus again by [I-S, Proposition 2.1], we prove the assertion for this case. Finally assume that $d$ contains an odd prime factor $q$. For $p \neq p_0$ we take a matrix $A_p$ so that $\det A_p = 2^{-n} d \in \mathbb{Q}_q^×/(\mathbb{Q}_q^×)^2$. Then for almost all $p$ we have $h_p(A_p) = 1$. We take $\xi \in \mathbb{Z}_q^×$ such that $(q, -\xi) = \prod_{p \neq q} h_p(A_p)$, and put $A_q = \xi d \perp \xi \perp 1_{n-2}$. Then we have $\det A_q 2^{-n} d \in \mathbb{Q}_q^×/(\mathbb{Q}_q^×)^2$, and $h_p(A_q) \prod_{p \neq q} = 1$. Thus again by [I-S, Proposition 2.1], we prove the assertion for this case.

(2) It is well known that there exists a positive definite half-integral matrix $E_8$ of degree 8 such that $\det(2E_8) = 1$. Thus $A = E_8 \perp \cdots \perp E_8$ satisfies the required condition.

(3) Let $q \neq 2$. Then, take a family $\{A_p\}$ of half-integral matrices over $\mathbb{Z}_p$ of degree $n$ such that $A_q \sim_{\mathbb{Z}_q} q \perp (-q\xi) \perp (-\xi) \perp 1_{n-3}$ with $(\xi_q) = -1, A_2 = H_{n/2}$, and $A_p = 1_n$ for $p \neq 2$. Then by the same argument as in (1) we can show that there exists a positive definite half integral matrix $A$ of degree $n$ such that $\det(2A) = q^2$ such that $A \sim_{\mathbb{Z}_q} A_p$ for any $p$. Let $q = 2$. Then the matrix $A' = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \end{pmatrix}$ is a positive definite and $\det(2A') = 4$. Thus the matrix $A' \perp E_8 \perp \cdots \perp E_8$ satisfies the required condition.

**Proposition 4.6.** Let $k$ and $n$ be positive even integer. Let $d$ be a fundamental discriminant. Let $f$ be a primitive form in $S_2(k-\text{n}(\Gamma_1))$. Let $\mathfrak{p}$ be a prime ideal in $K$. Then there exists a positive definite half integral matrices $A$ of degree $n$ such that $c_{I_n(f)}(A) = c_f(|d|)q$ with $q$ not divisible by $\mathfrak{p}$.
Proof. First assume that $d \neq 1$, or $n \not\equiv 4 \mod 8$. (1) By (1) and (2) of Lemma 4.5, there exists a matrix $A$ such that $b_A = d$. Thus we have $c_{I_n(f)}(A) = c_f(|d|)$. This proves the assertion.

Next assume that $n \equiv 4 \mod 8$ and that $d = 1$. Assume that $c_f(q) + q^{k-n/2-1}(-q - 1)$ is divisible by $\mathfrak{p}$ for any prime number $q$. Let $p$ be a prime number divisible by $\mathfrak{p}$. Fix an imbedding $\iota_p : \mathbb{Q} \to \overline{\mathbb{Q}}_p$, and let $\rho_{f,p} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Q}_p)$ be the Galois representation attached to $f$. Then by Chebotarev density theorem, the semi-simplification $\overline{\rho}^{ss}_{f,p}$ of $\overline{\rho}_{f,p}$ can be expressed as

$$\overline{\rho}^{ss}_{f,p} = \chi_p^{k-n/2} \oplus \chi_p^{k-n/2-1}$$

with $\chi_p$ the $p$-adic mod $p$ cyclotomic character. On the other hand, by the Fontaine-Messing [Fo-Me] and Fontaine-Laffaille [Fo-La], $\overline{\rho}_{f,p}$ should be $\chi_p^{2k-n-1} \oplus 1$ or $\omega_2^{2k-n-1} \oplus \omega_2^{p(2k-n-1)}$ with $\omega_2$ the fundamental character of level 2, where $I_p$ denotes the inertia group of $p$ in $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This is impossible because $k > 2$. Thus there exists a prime number $q$ such that $c_f(q) + q^{k-n/2-1}(-q - 1)$ is not divisible by $\mathfrak{p}$. For such a $q$, take a positive definite matrix $A$ in (3) of Lemma 4.5. Then

$$c_{I_n(f)}(A) = c(1)q^{k-(n+1)/2} \beta_q F_q(A, q^{-(n+1)/2} \beta_q^{-1}).$$

By [Ka1], we have

$$F_q(B, X) = 1 - Xq^{(n-2)/2}(q^2 + q) + q^3(Xq^{(n-2)/2})^2.$$ 

Thus we have

$$c_{I_n(f)}(A) = c(1)(c_f(q) + q^{k-n/2-1}(-q - 1)).$$

Thus the assertion holds.

**Theorem 4.7.** Let $k \geq 2n+4$. Let $K$ and $f$ be as above. Assume that the Conjecture B holds for $f$. Let $\mathfrak{p}$ be a prime ideal of $K$. Furthermore assume that

1. $\mathfrak{p}$ divides $L(k, f) \prod_{i=1}^{n/2-1} L(2i + 1, f, \text{St}).$
2. $\mathfrak{p}$ does not divide $\xi(2m) \prod_{i=1}^{n} L(2m + k - i, f)L(k - n/2, f, \chi_D)D(2k - 1)!$

for some integer $n/2 + 1 \leq m \leq k/2 - n/2 - 1$, and for some fundamental discriminant $D$ such that $(-1)^{n/2}D > 0$.

Then $\mathfrak{p}$ is a congruence prime of $I_n(f)$ with respect to $CI_n(f)$. Furthermore assume that the following condition hold:
(3) $\mathfrak{P}$ does not divide

$$
C_{k,n} \frac{\langle f, f \rangle}{\Omega(f, +, A_{\mathfrak{P}}) \Omega(f, -, A_{\mathfrak{P}})},
$$

where $C_{k,n} = 1$ or $\prod_{q \leq (2k - n)/12} (1 + q + \cdots + q^{a-1})$ according as $n = 2$ or not.

Then $\mathfrak{P}$ is a congruence prime of $I_n(f)$ with respect to $(\mathfrak{E}_k(\Gamma_n)\dagger)^\perp$.

Proof. Let $\mathfrak{P}$ be a prime ideal satisfying the condition (1) and (2). For the $D$ above, take a matrix $A \in \mathcal{H}_n(\mathfrak{Z})_>0$ so that $c_{I_n(f)}(A) = c_f(|D|)q$ with $q$ not divisible by $\mathfrak{P}$. Then by Proposition 4.3, we have

$$
\Lambda(2m, I_n(f), \text{St})|c_{I_n(f)}(A)|^2 = \Lambda(2m, I_n(f), \text{St})|c_f(|D|)|^2 q^2
$$

$$
= \epsilon_{k,m} \prod_{i=1}^{n} L(2m + k - i, f) |D|^{k - n/2} L(k - n/2, f, \chi_D) \frac{L(k; f) \xi(n) \prod_{i=1}^{n/2-1} L(2i + 1, f, \text{St}) \xi(2i)}{(f, f)}
$$

where $\epsilon_{k,m}$ is a rational number whose numerator is not divided by $\mathfrak{P}$.

We note that $\frac{\langle f, f \rangle}{\Omega(f, +, A_{\mathfrak{P}}) \Omega(f, -, A_{\mathfrak{P}})}$ is $\mathfrak{P}$-integral. Thus by assumptions (1) and (2), $\mathfrak{P}$ divides $(\Lambda(2m, I_n(f), \text{St})c_{I_n(f)}(A))^2$, and thus it divides $(\Lambda(2m, I_n(f), \text{St})c_{I_n(f)}(A))$. We note that $I_n(f)$ satisfies the assumption in Theorem 3.1. Thus by Theorem 3.1, there exits a Hecke eigenform $G \in \mathbb{C}(I_n(f))\dagger$ such that

$$
\lambda_G(T) \equiv \lambda_{I_n(f)}(T) \pmod{\mathfrak{P}}
$$

for any $T \in \mathbb{L}_n'$. Assume that we have $G = I_n(f)$ with some primitive form $g(z) = \sum_{m=1}^{\infty} a_g(m) e(mz) \in \mathfrak{E}_{2k-n}(\Gamma^1(\mathfrak{N}))$. Let $n = 2$. Then by (1) of Proposition 4.2, $\mathfrak{P}$ is also a congruence prime of $f$. Let $n \geq 4$. Then by (1) of Proposition 4.4, we have

$$(p^{n-1} + \cdots + p + 1)a_f(p) \equiv (p^{n-1} + \cdots + p + 1)a_g(p) \pmod{\mathfrak{P}}$$

for any prime number $p$ not divisible by $\mathfrak{P}$. By assumption (3), in particular, for any $p \leq (2k - n)/12$, we have

$$a_f(p) \equiv a_g(p) \pmod{\mathfrak{P}}.$$

Thus by Sturm [Stur], $\mathfrak{P}$ is also a congruence prime of $f$. Thus by [Hi2] and [Ri2], $\mathfrak{P}$ divides $\frac{\langle f, f \rangle}{\Omega(f, +, A_{\mathfrak{P}}) \Omega(f, -, A_{\mathfrak{P}})}$, which contradicts the assumption (3). Thus $\mathfrak{P}$ is a congruence prime of $I_n(f)$ with respect to $(\mathfrak{E}_k(\Gamma_n)\dagger)^\perp$. 
ExampleLet $n = 4$ and $k = 18$. Then we have $\dim S_{18}(\Gamma_4) \approx 16$ (cf. Poor and Yuen [P-Y]) and $\dim S_{18}(\Gamma_4)^* = \dim S_{32}(\Gamma_1) = 2$. Take a primitive form $f \in S_{32}(\Gamma_1)$. Then we have $[Q(f) : Q] = 2$, and $211 = \mathfrak{P}'$ in $Q(f)$. Then we have

$$N_{Q(f)/Q}(\mathcal{L}(18, f)) = 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 211,$$

$$N_{Q(f)/Q}(\prod_{i=1}^{4} \mathcal{L}(24-i, f)) = 2^{19} \cdot 3^{13} \cdot 5^3 \cdot 7^8 \cdot 11^2 \cdot 13^5 \cdot 17^5 \cdot 19^3 \cdot 23 \cdot 503 \cdot 1307 \cdot 14243,$$

and

$$\tilde{\xi}(6) = 2^{-2} \cdot 3^{-2} \cdot 7^{-1}$$

(cf. Stein [Ste].) Furthermore, by a direct computation we see neither $\mathfrak{P}$ nor $\mathfrak{P}'$ is a congruence prime of $\hat{f}$ with respect to $C\hat{g}$ for another primitive form $g \in S_{32}(\Gamma_1)$. Thus by Theorem 4.7, $\mathfrak{P}$ or $\mathfrak{P}'$ is a congruence prime of $\hat{f}$ with respect to $S_{18}(\Gamma_4)^{\perp}$.

References


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