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<th><strong>Title</strong></th>
<th>CONGRUENCE BETWEEN DUKE-IMAMOGLU-IKEDA LIFTS AND NON-DUKE-IMAMOGLU-IKEDA LIFTS</th>
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<tbody>
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CONGRUENCE BETWEEN DUKE-IMAMOGLU-IKEDA LIFTS AND NON-DUKE-IMAMOGLU-IKEDA LIFTS

HIDENORI KATSURADA

1. Introduction

As is well known, there is a congruence between the Fourier coefficients of Eisenstein series and those of cuspidal Hecke eigenforms for $SL_2(\mathbb{Z})$. This type of congruence is not only interesting in its own right but also has an important application to number theory as shown by Ribet [Ri1]. Since then, there have been so many important results about the congruence of elliptic modular forms (cf. [Hi1],[Hi2],[Hi3],). In the case of Hilbert modular forms or Siegel modular forms, it is sometimes more natural and important to consider the congruence between the Hecke eigenvalues of Hecke eigenforms modulo a prime ideal $\mathfrak{p}$. We call such a $\mathfrak{p}$ a prime ideal giving the congruence or a congruence prime. For a cuspidal Hecke eigenform $f$ for $SL_2(\mathbb{Z})$, let $\hat{f}$ be a lift of $f$ to the space $\mathfrak{M}(\Gamma')$ of modular forms of weight $l$ for a modular group $\Gamma'$. Here we mean by the lift of $f$ a cuspidal Hecke eigenform whose certain $L$-function can be expressed in terms of certain $L$-functions of $f$. As typical examples of the lift we can consider the Doi-Naganuma lift, the Saito-Kurokawa lift, and the Duke-Imamoglu-Ikeda lift. We then consider the following problem:

Problem. Characterize the prime ideals giving the congruence between $\hat{f}$ and a cuspidal Hecke eigenform in $\mathfrak{M}(\Gamma')$ not coming from the lift. In particular characterize them in terms of special values of certain $L$-functions of $f$.

This type of problem was first investigated in the Doi-Naganuma lift case by Doi, Hida, and Ishii [D-H-I]. In our previous paper [Ka2], we considered the relationship between the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbb{Z})$ and the special values of their standard zeta functions. In particular, we proposed a conjecture concerning the congruence between Saito-Kurokawa lifts and non-Saito-Kurokawa

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lifts, and proved it under certain condition. In this paper, we consider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, which is a generalization of our previous conjecture.

In Section 3, we review a result concerning the relationship between the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbb{Z})$ and the special values of their standard zeta functions. In Section 4, we propose a conjecture concerning the congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, and prove it under a certain condition.

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**Notation.** For a commutative ring $R$, we denote by $M_{mn}(R)$ the set of $(m,n)$-matrices with entries in $R$. In particular put $M_n(R) = M_{nn}(R)$. The empty matrix if $m = 0$ or $n = 0$. For an $(m,n)$-matrix $X$ and an $(m,m)$-matrix $A$, we write $A[X] = ^tXAX$, where $^tX$ denotes the transpose of $X$. Let $a$ be an element of $R$. Then for an element $X$ of $M_{mn}(R)$ we often use the same symbol $X$ to denote the coset $X \mod aM_{mn}(R)$. Put $GL_n(R) = \{ A \in M_n(R) \mid \det A \in R^* \}$, where $\det A$ denotes the determinant of a square matrix $A$, and $R^*$ denotes the unit group of $R$. Let $S_n(R)$ denote the set of symmetric matrices of degree $n$ with entries in $R$. Furthermore, for an integral domain $R$ of characteristic different from 2, let $H_n(R)$ denote the set of half-integral matrices of degree $n$ over $R$, that is, $H_n(R)$ is the set of symmetric matrices of degree $n$ whose $(i,j)$-component belongs to $R$ or $\frac{1}{2}R$ according as $i = j$ or not. For a subset $S$ of $M_n(R)$ we denote by $S^\times$ the subset of $S$ consisting of non-degenerate matrices. In particular, if $S$ is a subset of $S_n(\mathbb{R})$ with $\mathbb{R}$ the field of real numbers, we denote by $S_{\geq 0}$ (resp. $S_{> 0}$) the subset of $S$ consisting of positive definite (resp. semi-positive definite) matrices. Let $R'$ be a subring of $R$. Two symmetric matrices $A$ and $A'$ with entries in $R$ are called equivalent over $R'$ with each other and write $A \sim_{R'} A'$ if there is an element $X$ of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices $X$ and $Y$ we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

2. **Standard zeta functions of Siegel modular forms**

For a complex number $x$ put $e(x) = \exp(2\pi\sqrt{-1}x)$. Furthermore put $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$, where $1_n$ denotes the unit matrix of degree $n$. For
a subring $K$ of $\mathbb{R}$ put

$$GS_p_n(K)^+ = \{ M \in GL_{2n}(K) \mid J_n[M] = \kappa(M)J_n \text{ with some } \kappa(M) > 0 \},$$

and

$$Sp_n(K) = \{ M \in GS_p_n(K)^+ \mid J_n[M] = J_n \}.$$

Furthermore, put

$$\Gamma(n) = Sp_n(\mathbb{Z}) = \{ M \in GL_{2n}(\mathbb{Z}) \mid J_n[M] = J_n \}.$$

We sometimes write an element $M$ of $GS_p_n(K)$ as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in M_2(K)$. We define a subgroup $\Gamma_0^{(n)}(N)$ of $\Gamma^{(n)}$ as

$$\Gamma_0^{(n)}(N) = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \mid C \equiv O_n \mod N \}.$$

Let $H_n$ be Siegel’s upper half-space. For each element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GS_p_n(\mathbb{R})^+$ and $Z \in H_n$ put

$$M(Z) = (AZ + B)(CZ + D)^{-1}$$

and

$$j(M, Z) = \det(CZ + D).$$

Furthermore, for a function $f$ on $H_n$ and an integer $k$ we define $f|_k M$ as

$$(f|_k M)(Z) = \det(M)^{-k/2}j(M, Z)^{-k}f(M(Z)).$$

For an integer or half integral $l$ and the subgroup $\Gamma_0^{(n)}(N)$ of $\Gamma^{(n)}$, we denote by $\mathfrak{M}_k(\Gamma_0^{(n)}(N))$ (resp. $\mathfrak{M}_k^{\infty}(\Gamma_0^{(n)}(N))$) the space of holomorphic (resp. $C^\infty$-) modular forms of weight $k$ with respect to $\Gamma_0^{(n)}(N)$. We denote by $\mathfrak{S}_k(\Gamma_0^{(n)}(N))$ the sub-space of $\mathfrak{M}_k(\Gamma_0^{(n)}(N))$ consisting of cusp forms. Let $f$ be a holomorphic modular form of weight $k$ with respect to $\Gamma_0^{(n)}(N)$. Then $f$ has the following Fourier expansion:

$$f(Z) = \sum_{A \in H(\mathbb{Z})_{\geq 0}} a_f(A)e(\text{tr}(AZ)),$$

and in particular if $f$ is a cusp form, $f$ has the following Fourier expansion:

$$f(Z) = \sum_{A \in H(\mathbb{Z})_{\geq 0}} a_f(A)e(\text{tr}(AZ)),$$

where $\text{tr}$ denotes the trace of a matrix. Let $dv$ denote the invariant volume element on $H_n$ defined by $dv = \det(\text{Im}(Z))^{-n-1} \wedge_{1 \leq j \leq l \leq n} (dx_{jl} \wedge dy_{jl})$. Here for $Z \in H_n$ we write $Z = (x_{jl}) + \sqrt{-1}(y_{jl})$ with real matrices.
For two $C^\infty$-modular forms $f$ and $g$ of weight $l$ with respect to $\Gamma_0^{(n)}(N)$ we define the Petersson scalar product $\langle f, g \rangle$ by

$$\langle f, g \rangle = [\Gamma^{(n)} : \Gamma_0^{(n)}(N)]^{-1} \int_{\Gamma_0^{(n)}(N) \backslash \mathbb{H}} f(Z)\overline{g(Z)} \det(\text{Im}(Z))^l dv,$$

provided the integral converges.

Let $\mathbb{L}_n = \mathbb{L}_Q(\text{GSp}_n(\mathbb{Q})^\dagger, \Gamma^{(n)})$ denote the Hecke algebra over $\mathbb{Q}$ associated with the Hecke pair $(\text{GSp}_n(\mathbb{Q})^\dagger, \Gamma^{(n)})$. Furthermore, let $\mathbb{L}_Q = \mathbb{L}_Q(\text{Sp}_n(\mathbb{Q}), \Gamma^{(n)})$ denote the Hecke algebra over $\mathbb{Q}$ associated with the Hecke pair $(\text{Sp}_n(\mathbb{Q}), \Gamma^{(n)})$. For each integer $m$ define an element $T(m)$ of $\mathbb{L}_n$ by

$$T(m) = \sum_{d_1, \ldots, d_n, e_1, \ldots, e_n} \Gamma^{(n)}(d_1 \perp \ldots \perp d_n \perp e_1 \perp \ldots \perp e_n) \Gamma^{(n)},$$

where $d_1, \ldots, d_n, e_1, \ldots, e_n$ run over all positive integer satisfying

$$d_i, d_{i+1}, e_{i+1} | e_i \ (i = 1, \ldots, n - 1), \ d_n | e_n, \ d_i e_i = m \ (i = 1, \ldots, n).$$

Furthermore, for $i = 1, \ldots, n$ and a prime number $p$ put

$$T_i(p^2) = \Gamma^{(n)}(1_{n-i} \perp p1_i \perp p^21_{n-i} \perp p1_i) \Gamma^{(n)},$$

and $(p^{\pm 1}) = \Gamma^{(n)}(p^{\pm 1}1_n) \Gamma^{(n)}$. As is well known, $\mathbb{L}_n$ is generated over $\mathbb{Q}$ by all $T(p)$, $T_i(p^2)$ $(i = 1, \ldots, n)$, and $(p^{\pm 1})$. We denote by $\mathbb{L}'_n$ the subalgebra of $\mathbb{L}_n$ generated by over $\mathbb{Z}$ by all $T(p)$ and $T_i(p^2)$ $(i = 1, \ldots, n)$. Let $T = \Gamma^{(n)}M\Gamma^{(n)}$ be an element of $\mathbb{L}_n \otimes \mathbb{C}$. Write $T$ as $T = \bigcup_p \Gamma^{(n)} \gamma$ and for $f \in \mathfrak{M}_k(\Gamma^{(n)})$ define the Hecke operator $|kT$ associated to $T$ as

$$f|kT = \det(M)^{k/2-(n+1)/2} \sum_{\gamma} f|k\gamma.$$

We call this action the Hecke operator as usual (cf. [A].) If $f$ is an eigenfunction of a Hecke operator $T \in \mathbb{L}_n \otimes \mathbb{C}$, we denote by $\lambda_f(T)$ its eigenvalue. Let $\mathbb{L}$ be a subalgebra of $\mathbb{L}_n$. We call $f \in \mathfrak{M}_k(\Gamma^{(n)})$ a Hecke eigenform for $\mathbb{L}$ if it is a common eigenfunction of all Hecke operators in $\mathbb{L}$. In particular if $\mathbb{L} = \mathbb{L}_n$ we simply call $f$ a Hecke eigenform. Furthermore, we denote by $\mathbb{Q}(f)$ the field generated over $\mathbb{Q}$ by eigenvalues of all $T \in \mathbb{L}_n$ as in Section 1. As is well known, $\mathbb{Q}(f)$ is a totally real algebraic number field of finite degree. Now, first we consider the integrality of the eigenvalues of Hecke operators. For an algebraic number field $K$, let $\mathfrak{O}_K$ denote the ring of integers in $K$. The following assertion has been proved in [Mi2] (see also [Ka2].)

**Theorem 2.1** Let $k \geq n + 1$. Let $f \in \mathfrak{M}_k(\Gamma^{(n)})$ be a common eigenform in $\mathbb{L}'_n$. Then $\lambda_f(T)$ belongs to $\mathfrak{O}_{\mathbb{Q}(f)}$ for any $T \in \mathbb{L}'_n$. 

Let \( L_{np} = L(GSp_n(\mathbb{Q})^+ \cap GL_{2n}(\mathbb{Z}[p^{-1}]), \Gamma(n)) \) be the Hecke algebra associated with the pair \((GSp_n(\mathbb{Q})^+ \cap GL_{2n}(\mathbb{Z}[p^{-1}]), \Gamma(n))\). \( L_{np} \) can be considered as a subalgebra of \( L_n \), and is generated over \( \mathbb{Q} \) by \( T(p) \) and \( T_i(p^2) \) \((i = 1, 2, \ldots, n)\). We now review the Satake \( p \)-parameters of \( L_{np} \); let \( \mathcal{P}_n = \mathbb{Q}[X_0^\pm, X_1^\pm, \ldots, X_n^\pm] \) be the ring of Laurent polynomials in \( X_0, X_1, \ldots, X_n \) over \( \mathbb{Q} \). Let \( \mathcal{W}_n \) be the group of \( \mathbb{Q} \)-automorphisms of \( \mathcal{P}_n \) generated by all permutations in variables \( X_1, \ldots, X_n \) and by the automorphisms \( \tau_1, \ldots, \tau_n \) defined by

\[
\tau_i(X_0) = X_0X_i, \quad \tau_i(X_i) = X_i^{-1}, \quad \tau_i(X_j) = X_j \ (j \neq i).
\]

Furthermore, a group \( \mathcal{W}_n \) isomorphic to \( \mathcal{W}_n \) acts on the set \( T_n = (\mathbb{C}^\times)^{n+1} \) in a way similarly to above. Then there exists a \( \mathbb{Q} \)-algebra isomorphism \( \Phi_{np} \), called the Satake isomorphism, from \( L_{np} \) to the \( \mathcal{W}_n \)-invariant subring \( \mathcal{P}_n^{W_n} \) of \( \mathcal{P}_n \). Then for a \( \mathbb{Q} \)-algebra homomorphism \( \lambda \) from \( L_{np} \) to \( \mathbb{C} \), there exists an element \((\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))\) of \( T_n \) satisfying

\[
\lambda(\Phi_{np}^{-1}(F(X_0, X_1, \ldots, X_n))) = F(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))
\]

for \( F \in \mathcal{P}_n^{W_n} \). The equivalence class of \((\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))\) under the action of \( \mathcal{W}_n \) is uniquely determined by \( \lambda \). We call this the Satake parameters of \( L_{np} \) determined by \( \lambda \).

Now assume that an element \( f \) of \( \mathcal{M}_k(Sp_n(\mathbb{Z})) \) is a Hecke eigenform. Then for each prime number \( p \), \( f \) defines a \( \mathbb{Q} \)-algebra homomorphism \( \lambda_{f,p} \) from \( L_{np} \) to \( \mathbb{C} \) in a usual way, and we denote by \( \alpha_0(p), \alpha_1(p), \ldots, \alpha_n(p) \) the Satake parameters of \( L_{np} \) determined by \( f \). We then define the standard zeta function \( L(f, s, \text{St}) \) by

\[
L(s, f, \text{St}) = \prod_{p \in \mathbb{P}} \prod_{i=1}^{n} \left\{ (1 - p^{-s})(1 - \alpha_i(p)p^{-s})(1 - \alpha_i(p)^{-1}p^{-s}) \right\}^{-1}.
\]

Let \( f(z) = \sum_{A \in \mathcal{H}_n(\mathbb{Z})} a(A)e(\text{tr}(Az)) \) be a Hecke eigenform in \( \mathcal{E}_k(\Gamma(n)) \).

For a positive integer \( m \leq k - n \) such that \( m \equiv n \mod 2 \) put

\[
\Lambda(f, m, \text{St}) = (-1)^{m+1/2+1}2^{-4k+n+3m^2+n+(n-1)m+2} \times \Gamma(m+1) \prod_{i=1}^{n} \Gamma(2k-n-i) \frac{L(f, m, \text{St})}{< f, f >^{\pi-n(n+1)/2+mk+(n+1)m}}.
\]

Then the following theorem is due to Böcherer [B2] and Mizumoto [Mi].

**Theorem 2.2.** Let \( l, k \) and \( n \) be positive integers such that \( \rho(n) \leq l \leq k - n \), where \( \rho(n) = 3 \), or \( 1 \) according as \( n \equiv 1 \mod 4 \) and \( n \geq 5 \), or not. Let \( f \in \mathcal{E}_k(\Gamma(n)) \) be a Hecke eigenform. Then \( \Lambda(f, m, \text{St}) \) belongs to \( \mathbb{Q}(f) \).
For later purpose, we consider a special element in \( L_{np} \); the polynomial \( X_0^2 X_1 X_2 \cdots X_n \sum_{i=1}^{n}(X_i + X_i^{-1}) \) is an element of \( P_n^{W_n} \), and thus we can define an element \( \Phi_{np}^{-1}(X_0^2 X_1 X_2 \cdots X_n \sum_{i=1}^{n}(X_i + X_i^{-1})) \) of \( L_{np} \), which is denoted by \( r_1 \).

**Proposition 2.3.** Under the above notation the element \( r_1 \) belongs to \( L_0' \), and we have

\[
\lambda_f(r_1) = p^{n(n-1)/2} \sum_{i=1}^{n}(\alpha_i(p) + \alpha_i(p)^{-1}).
\]

**Proof.** By a careful analysis of the computation in page 159-160 of [A], we see that \( r_1 \) is a \( \mathbb{Z} \)-linear combination of \( T_i(p^2) \) \((i = 1, ..., n)\), and therefore we can prove the first assertion. Furthermore, by Lemma 3.3.34 of [A], we can prove the second assertion.

3. **Congruence of modular forms and special values of the standard zeta functions**

In this section we review a result concerning the congruence between the Hecke eigenvalues of modular forms of the same weight following [Ka2]. Let \( K \) be an algebraic number field, and \( \mathfrak{O} = \mathcal{O}_K \) the ring of integers in \( K \). For a prime ideal \( \mathfrak{p} \) of \( \mathcal{O} \), we denote by \( \mathcal{O}(\mathfrak{p}) \) the localization of \( \mathcal{O} \) at \( \mathfrak{p} \) in \( K \). Let \( \mathfrak{A} \) be a fractional ideal in \( K \). If \( \mathfrak{A} = \mathfrak{p}^e \mathcal{O}(\mathfrak{p}) \) with \( \mathfrak{p}(\mathfrak{p}) = \mathcal{O}(\mathfrak{p}) \) we write \( \text{ord}_{\mathfrak{p}} = e \). We simply write \( \text{ord}_{\mathfrak{p}}(c) = \text{ord}_{\mathfrak{p}}((c)) \) for \( c \in K \). Now let \( f \) be a Hecke eigenform in \( \mathfrak{S}_k(\Gamma(n)) \) and \( M \) be a subspace of \( \mathfrak{S}_k(\Gamma(n)) \) stable under Hecke operators \( T \in L_n \).

Assume that \( M \) is contained in \( (Cf)^\perp \), where \( (Cf)^\perp \) is the orthogonal complement of \( Cf \) in \( \mathfrak{S}_k(\Gamma(n)) \) with respect to the Petersson product. Let \( K \) be an algebraic number field containing \( \mathbb{Q}(f) \). A prime ideal \( \mathfrak{p} \) of \( \mathcal{O}_K \) is called a congruence prime of \( f \) with respect to \( M \) if there exists a Hecke eigenform \( g \in M \) such that

\[
\lambda_f(T) \equiv \lambda_g(T) \mod \mathfrak{p}
\]

for any \( T \in L_n' \), where \( \mathfrak{p} \) is the prime ideal of \( \mathcal{O}_{K\mathbb{Q}(f)} \) lying above \( \mathfrak{p} \). If \( M = (Cf)^\perp \), we simply call \( \mathfrak{p} \) a congruence prime of \( f \).

Now we consider the relation between the congruence primes and the standard zeta values. To consider this, we have to normalize the standard zeta value \( \Lambda(f, l, \text{St}) \) for a Hecke eigenform \( f \) because it is not uniquely determined by the system of Hecke eigenvalues of \( f \). We note that there is no reasonable normalization of cuspidal Hecke eigenform in the higher degree case unlike the elliptic modular case.
Thus we define the following quantities: for a Hecke eigenform \( f(z) = \sum_A a_f(A) e(\text{tr}(Az)) \) in \( \mathcal{S}_k(\Gamma(n)) \), let \( \mathfrak{F}_f \) be the \( \mathcal{O}(f) \)-module generated by all \( a_f(A) \)'s. Assume that there exists a complex number \( c \) such that all the Fourier coefficients \( cf \) belongs to \( \mathcal{O}(f) \). Then \( \mathfrak{F}_f \) is a fractional ideal in \( \mathcal{O}(f) \), and therefore, so is \( \Lambda(f, l, \mathfrak{F}) \mathfrak{F}_f^2 \) if \( l \) satisfies the condition in Theorem 2.2. We note that this fractional ideal does not depend on the choice of \( c \). We also note that the value \( N_{\mathcal{O}(f)}(\Lambda(f, l, \mathfrak{F}))N(\mathfrak{F}_f)^2 \) does not depend on the choice of \( c \), where \( N(\mathfrak{F}_f) \) is the norm of the ideal \( \mathfrak{F}_f \). Then, we have

**Theorem 3.1.** Let \( f \) be a Hecke eigenform in \( \mathcal{S}_k(\Gamma(n)) \). Assume that there exists a complex number \( c \) such that all the Fourier coefficients \( cf \) belongs to \( \mathcal{O}(f) \). Let \( l \) be a positive integer satisfying the condition in Theorem 2.2. Let \( \mathfrak{p} \) be a prime ideal of \( \mathcal{O} \). Assume that \( \text{ord}_{\mathfrak{p}}(\Lambda(f, l, \mathfrak{F})\mathfrak{F}_f^2) < 0 \) and that it does not divide \( (2l - 1)! \). Then \( \mathfrak{p} \) is a congruence prime of \( f \). In particular, if a rational prime number \( p \) divides the denominator of \( N_{\mathcal{O}(f)}(\Lambda(f, l, \mathfrak{F}))N(\mathfrak{F}_f)^2 \), then \( p \) is divisible by some congruence prime of \( f \).

Now for a Hecke eigenform \( f \) in \( \mathcal{S}_k(\Gamma(n)) \), let \( \mathcal{S}_f \) denote the subspace of \( \mathcal{S}_k(\Gamma(n)) \) spanned by all Hecke eigenforms with the same system of the Hecke eigenvalues as \( f \).

**Corollary.** In addition to the above notation and the assumption, assume that \( \mathcal{S}_k(\Gamma(n)) \) has the multiplicity one property. Then \( \mathfrak{p} \) is a congruence prime of \( f \) with respect to \( \mathcal{S}_f^+ \). In particular, if a rational prime number \( p \) divides the denominator of \( N_{\mathcal{O}(f)}(\Lambda(f, l, \mathfrak{F}))N(\mathfrak{F}_f)^2 \), then \( p \) is divisible by some congruence prime of \( f \) with respect to \( \mathcal{S}_f^+ \).

4. **Congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts**

In this section, we consider the congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts. Throughout this section and the next, we assume that \( n \) and \( k \) are even positive integers. Let

\[
  f(z) = \sum_{m=1}^{\infty} a(m) e(mz)
\]

be a normalized Hecke eigenform of weight \( 2k - n \) with respect to \( SL_2(\mathbb{Z}) \). For a Dirichlet character \( \chi \), we then define the L-function
$L(s, f)$ of $f$ twisted by $\chi$ by

$$L(s, f) = \prod_p \{(1 - \chi(p)\beta_p p^{k-n/2-1/2-s})(1 - \chi(p)\beta_p^{-1} p^{k-n/2-1/2-s})\}^{-1},$$

where $\beta_p$ is a non-zero complex number such that $\beta_p + \beta_p^{-1} = p^{-k+n/2+1/2}a(p)$. We simply write $L(s, f)$ as $L(s, f, \chi)$ if $\chi$ is the principal character. Furthermore, let $\tilde{f}$ be the cusp form of weight $k-n/2+1/2$ belonging to the Kohnen plus space corresponding to $f$ via the Shimura correspondence (cf. [Ko1]). Then $\tilde{f}$ has the following Fourier expansion:

$$\tilde{f}(z) = \sum_e c(e) e(ez),$$

where $e$ runs over all positive integers such that $(-1)^{k-n/2}e \equiv 0, 1 \mod 4$. We then put

$$a_{I_n(f)}(T) = c([T]) \prod_p (p^{k-n/2-1/2} \beta_p)^{v_p([T])} F_p(T, p^{-(n+1)/2} \beta_p^{-1}).$$

We note that $a_{I_n(f)}(T)$ does not depend on the choice of $\beta_p$. Define a Fourier series $I_n(f)(Z)$ by

$$I_n(f)(Z) = \sum_{T \in \mathcal{H}_n(Z), T > 0} a_{I_n(f)}(T) e(\text{tr}(TZ)).$$

In [Ik1] Ikeda showed that $I_n(f)(Z)$ is a cusp form of weight $k$ with respect to $\Gamma^{(n)}$ and a Hecke eigenform for $L_n^\infty$ such that

$$L(s, I_n(f), St) = \zeta(s) \prod_{i=1}^n L(s + k - i, f).$$

This was first conjecture by Duke and Imamoglu. Thus we call $I_n(f)$ the Duke-Imamoglu-Ikeda lift of $f$. We note that we have $Q(\tilde{f}) = Q(I_n(f)) = Q(f)$. Furthermore, we have $\mathfrak{F}_f = \mathfrak{F}_{I_n(f)}$, where $\mathfrak{F}_f$ is the $\mathcal{Q}(f)$-module generated by all the Fourier coefficients of $\tilde{f}$.

Now to consider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, first we prove the following:

**Proposition 4.1** $I_n(f)$ is a Hecke eigenform.

We note that Ikeda proved in [Ik1] that $I_n(f)$ is a Hecke eigenform for $L_n^\infty$ but has not proved that it is a Hecke eigenform for $L_n$. This was pointed to us by B. Heim (see [He]). We thank him for his comment. We also note that an explicit form of the spinor L-function of $I_n(f)$ was obtained by Murakawa [Mu] and Schmidt [Sch] assuming that $I_n(f)$ is a Hecke eigenform.
Proof of Proposition 4.1. We have only to prove that \( I_n(f) \) is an eigenfunction of \( T(p) \) for any prime \( p \). The proof may be more or less well known, but for the convenience of the readers we here give the proof. For a modular form \( F(Z) = \sum_B c_B(B) e(\text{tr}(BZ)) \),

let \( c_F^{(p)}(B) \) be the \( B \)-th Fourier coefficient of \( F(T(p)) \). Then for any positive definite matrix \( B \) we have

\[
c_F^{(p)}(B) = p^{nk-n(n+1)/2} \prod_{d_1 \mid d_2 \mid \cdots \mid d_n \mid p} d_1^n d_2^{n-1} \cdots d_n \times \sum_{D \in \Lambda_n(d_1 \cdots d_n)\Lambda_n} \det D^{-k} c_F(p^{-1}A[tD]),
\]

where \( \Lambda_n = GL_n(\mathbb{Z}) \).

Now let \( E_{n,k}(Z) \) be the Siegel Eisenstein series of degree \( n \) and of weight \( k \) defined by

\[
E_{n,k}(Z) = \sum_{\gamma \in \Gamma_n \backslash \Gamma_n} j(\gamma, Z)^{-k}.
\]

For \( k \geq n + 1 \), the Siegel Eisenstein series \( E_{n,k}(Z) \) is a holomorphic modular form of weight \( k \) with respect to \( \Gamma_n \). Furthermore, \( E_{n,k}(Z) \) is a Hecke eigenform and in particular we have

\[
E_{n,k}[T(p)](Z) = h_{n,p}(p^k) E_{n,k}(Z),
\]

where

\[
h_{n,p}(X) = 1 + \sum_{r=1}^{n} \sum_{1 \leq i_1 < \cdots < i_r \leq n} p^{-\sum_{j=1}^{r} i_j} X^r.
\]

Let \( c_{n,k}(B) \) be the \( B \)-th Fourier coefficient of \( E_{n,k}(Z) \). Then we have

\[
h_{n,p}(p^k) c_{n,k}(B) = p^{nk-n(n+1)/2} \prod_{d_1 \mid d_2 \mid \cdots \mid d_n \mid p} d_1^n d_2^{n-1} \cdots d_n \times \sum_{D \in \Lambda_n(d_1 \cdots d_n)\Lambda_n} \det D^{-k} c_{n,k}(p^{-1}B[tD]).
\]

Let \( B \) be positive definite. Then we have

\[
c_{n,k}(B) = a_{n,k}(\det 2B)^{k-(n+1)/2} L(k - n/2, \chi_B) \prod_q F_q(B, p^{-k}),
\]

where \( a_{n,k} \) is a non-zero constant depending only on \( n \) and \( k \). We note that we have

\[
F_q(p^{-1}B[tD], X) = F_q(B, X)
\]
for any $D \in \Lambda_n(d_1 \perp \cdots \perp d_n)\Lambda_n$ with $d_1 | \cdots | d_n | p$ if $q \neq p$. Thus we have

$$h_{n,p}(p^k) F_p(B, p^{-k}) = \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{ne_1+(n-1) e_2 + \cdots + e_n} \sum_{D \in \Lambda_n \setminus \Lambda_n(p^{e_1 \perp \cdots \perp p^n})\Lambda_n} F_p(p^{-1} B[t_D], p^{-k}).$$

The both-hand sides of the above are polynomials in $p^k$ and the equality holds for infinitely many $k$. Thus we have

$$h_{n,p}(X^{-1}) F_p(B, X) = \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{ne_1+(n-1) e_2 + \cdots + e_n} \sum_{D \in \Lambda_n \setminus \Lambda_n(p^{e_1 \perp \cdots \perp p^n})\Lambda_n} F_p(p^{-1} B[t_D], X)$$

as polynomials in $X$ and $X^{-1}$. Thus we have

$$(p^{k-(n+1)/2} X)^n h_{n,p}(p^{(n+1)/2} X^{-1}) (p^{k-(n+1)/2} X^{-1}) \nu_{\beta} F_p(B, p^{-(n+1)/2} X) = p^{nk-n(n+1)/2} \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{ne_1+(n-1) e_2 + \cdots + e_n}$$

$$\times \sum_{D \in \Lambda_n \setminus \Lambda_n(p^{e_1 \perp \cdots \perp p^n})\Lambda_n} \det D^{-k} (p^{k-(n+1)/2} X^{-1}) \nu_{\beta} F_p(p^{-1} B[t_D], p^{-(n+1)/2} X).$$

We recall that we have

$$c_{I_n(f)}(B) = c_f([b_B])^{k-(n+1)/2} \prod_q (\beta_q)^{\nu_{\beta} F_q(B, q^{-(n+1)/2} p^{-1})},$$

where $\beta_q$ is the Satake $q$-parameter of $f$. We also note that $c_f([b_{p^{-1} B[t_D]}]) = c_f([b_B])$ for any $D$. Thus we have

$$h_{n,p}(p^{(n+1)/2} \alpha_p c_{I_n(f)}(B)) = p^{nk-n(n+1)/2} \sum_{d_1 | d_2 | \cdots | d_n | p} d_1^{n-1} d_2 \cdots d_n \sum_{D \in \Lambda_n(d_1 \perp \cdots \perp d_n)\Lambda_n} \det D^{-k} c_{I_n(f)}(p^{-1} B[t_D]).$$

This proves the assertion.

Let $f$ be a primitive form in $\mathfrak{S}_{2k-n}(\Gamma(1))$. Let $\{f_1, \ldots, f_d\}$ be a basis of $\mathfrak{S}_{2k-n}(\Gamma(1))$ consisting of primitive forms. Let $K$ be an algebraic number field containing $\mathbb{Q}(f_1) \cdots \mathbb{Q}(f_d)$, and $A = \mathcal{O}_K$. To formulate our conjecture exactly, we introduce the Eichler-Shimura periods as follows (cf. Hida [Hi3].) Let $\mathfrak{p}$ be a prime ideal in $K$. Let $A_{\mathfrak{p}}$ be a valuation ring in $K$ corresponding to $\mathfrak{p}$. Assume that the residual characteristic of $A_{\mathfrak{p}}$ is greater than or equal to 5. Let $L(2k-n-2, A_{\mathfrak{p}})$ be the module of homogeneous polynomials of degree $2k-n-2$ in the variables $X, Y$.
with coefficients in \(A_p\). We define the action of \(M_2(\mathbb{Z}) \cap GL_2(\mathbb{Q})\) on \(L(2k - n - 2, A_p)\) via
\[
g \cdot P(X, Y) = P^\prime(f(X, Y)(\gamma)^t),
\]
where \(\gamma^t = (\det \gamma)\gamma^{-1}\). Let \(H^1_p(\Gamma(1), L(2k - n - 2, A_p))\) be the parabolic cohomology group of \(\Gamma(1)\) with values in \(L(2k - n - 2, A_p)\). Fix a point \(z_0 \in \mathbb{H}_1\). Let \(g \in \mathfrak{E}_{2k-n}(\Gamma(1))\) or \(g \in \mathfrak{E}_{2k-n}(\Gamma(1))\). We then define the differential \(\omega(g)\) as
\[
\omega(g)(z) = \begin{cases} 
2\pi ig(z)(X - zY)^n \, dz & \text{if } g \in \mathfrak{E}_{2k-n}(\Gamma(1)) \\
2\pi\sqrt{-1}g(z)(X - \bar{z}Y)^n \, dz & \text{if } g \in \mathfrak{E}_{2k-n}(\Gamma(1)),
\end{cases}
\]
and define the cohomology class \(\delta(g)\) of the 1-cocycle of \(\Gamma(1)\). as
\[
\gamma \in \Gamma(1) \mapsto \int_{z_0}^{\gamma(z_0)} \omega(g).
\]
The mapping \(g \mapsto \delta(g)\) induces the isomorphism
\[
\delta : \mathfrak{E}_{2k-n}(\Gamma(1)) \oplus \mathfrak{E}_{2k-n}(\Gamma(1)) \to H^1_p(\Gamma(1), L(2k - n - 2, \mathbb{C})),
\]
which is called the Eichler-Shimura isomorphism. We can define the action of Hecke algebra \(L'_1\) on \(H^1_p(\Gamma(1), \mathfrak{E}_{2k-n}(\Gamma(1)))\) in a natural manner. Furthermore, we can define the action \(F_\infty\) on \(H^1_p(\Gamma(1), L(2k - n - 2, A_p))\) as
\[
F_\infty(\delta(g)(z)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \delta(g)(-\bar{z}),
\]
and this action commutes with the Hecke action. For a primitive form \(f\) and \(j = \pm 1\), we define the subspace \(H^1_p(\Gamma(1), L(2k - n - 2, A_p))[f, j]\) of \(H^1_p(\Gamma(1), L(2k - n - 2, A_p))\) as
\[
H^1_p(\Gamma(1), L(2k - n - 2, A_p))[f, j] = \{ x \in H^1_p(\Gamma(1), L(2k - n - 2, A_p)) : x|T = \lambda_f(T)x \text{ for } T \in L_1, \text{ and } F_\infty(x) = jx \}.
\]
Since \(A_p\) is a principal ideal domain, \(H^1_p(\Gamma(1), L(2k - n - 2, A_p))[f, j]\) is a free module of rank one over \(A_p\). For each \(j = \pm 1\) take a basis \(\eta(f, j, A_p)\) of \(H^1_p(\Gamma(1), (2k - n - 2, A_p))[f, j]\) and define a complex number \(\Omega(f, j; A_p)\) by
\[
(\delta(f) + jF_\infty(\delta(f)))/2 = \Omega(f, j; A_p)\eta(f, j, A_p).
\]
This \(\Omega(f, j; A_p)\) is uniquely determined up to constant multiple of units in \(A_p\). We call \(\Omega(f, +; A_p)\) and \(\Omega(f, -; A_p)\) the Eichler-Shimura periods. For \(j = \pm, 1 \leq l \leq 2k - n - 1\), and a Dirichlet character \(\chi\) such
that $\chi(-1) = j(-1)^{l-1}$, put

$$L(l, f, \chi) = L(l, f, \chi; A_{\mathfrak{q}}) = \frac{\Gamma(l)L(l, f, \chi)}{\tau(\chi)(2\pi\sqrt{-1})\Omega(f, j; A_{\mathfrak{q}})},$$

where $\tau(\chi)$ is the Gauss sum of $\chi$. In particular, put $L(l, f; A_{\mathfrak{q}}) = L(l, f, \chi; A_{\mathfrak{q}})$ if $\chi$ is the principal character. Furthermore, put

$$L(s, f, \text{St}) = 4(2\pi)^{-2s-2k+n+1}\Gamma(s)\Gamma(s + 2k - n - 1)L(s, f, \text{St}).$$

It is well-known that $L(l, f, \chi)$ belongs to the field $K(\chi)$ generated over $K$ by all the values of $\chi$, and $L(l, f, \text{St})$ belongs to $Q(f)$ (cf. [Bo].) Let $I_n(f)$ be the Duke-Imamoglu-Ikeda lift of $f$. Let $\mathfrak{E}_k(\Gamma^{(n)})^*$ be the subspace of $\mathfrak{E}_k(I_n) = \mathfrak{E}_k(\Gamma^{(2)})$ generated by all the Duke-Imamoglu-Ikeda lifts $I(g)^n$ of primitive forms $g \in \mathfrak{E}_{2k-n}(\Gamma^{(1)})$. We remark that $\mathfrak{E}_k(\Gamma^{(2)})^*$ is the Maass subspace of $\mathfrak{E}_k(\Gamma^{(2)})$.

**Conjecture A.** Let $K$ and $f$ be as above. Assume that $k > n$. Let $\mathfrak{p}$ be a prime ideal of $K$ not dividing $(2k - 1)!$. Then $\mathfrak{p}$ is a congruence prime of $I_n(f)$ with respect to $(\mathfrak{E}_k(\Gamma^{(n)})^*)^\perp$ if $\mathfrak{p}$ divides $L(k, f) \prod_{i=1}^{n/2-1} L(2i+1, f, \text{St}).$

**Remark.** This is an analogue of the Doi-Hida-Ishii conjecture concerning the congruence primes of the Doi-Naganuma lifting [D-H-I]. (See also [Ka1].) We also note that this type of conjecture has been proposed by Harder [Ha] in the case of vector valued Siegel modular forms.

Now to explain why our conjecture is reasonable, we refer to Ikeda’s conjecture on the Petersson inner product of the Duke-Imamoglu-Ikeda lifting. Let $f$ and $\tilde{f}$ be as above. Put

$$\tilde{\xi}(s) = 2(2\pi)^{-s}\Gamma(s)\zeta(s),$$

and

$$\Lambda(s, f) = 2(2\pi)^{-s}\Gamma(s)L(s, f).$$

**Theorem 4.2.** (Katsurada and Kawamura [K-K]) In addition to the above notation and the assumption, assume $k > n$. Then we have

$$\tilde{\xi}(n)\Lambda(k, f) \prod_{i=1}^{n/2-1} L(2i+1, f, \text{St})\tilde{\xi}(2i) = 2^n \frac{\langle I_n(f)f, I_n(f) \rangle}{\langle f, f \rangle^{n/2-1}}\langle \tilde{f}, \tilde{f} \rangle,$$

where $\alpha$ is an integer depending only on $n$ and $k$. 
We note that the above theorem was conjectured by Ikeda [Ik2] under more general setting. We note that the theorem has been proved by Kohnen and Skoruppa [K-S] in case $n = 2$.

**Proposition 4.3** Under the above notation and the assumption we have for any fundamental discriminant $D$ such that $(-1)^{n/2}D > 0$ and $L(k - n/2, f, \chi_D) \neq 0$ we have

$$
\frac{c(|D|)^2}{\langle I_n(f), I_n(f) \rangle} = a_{n,k} \frac{(f, f)^{n/2} |D|^{k-n/2} L(k - n/2, f, \chi_D)}{n/2 - 1} \left( \frac{L(k, f)}{\prod_{i=1}^{n/2-1} L(2i, f, \text{St})} \right)
$$

with some algebraic number $a_{n,k}$ depending only on $n, k$.

**Proof.** By the result in Kohnen-Zagier [K-Z], for any fundamental discriminant $D$ such that $(-1)^{n/2}D > 0$ we have

$$
\frac{c(|D|)^2}{\langle f, f \rangle} = 2^{k-n/2-1} |D|^{k-n/2-1/2} \Lambda(k - n/2, f, \chi_D).
$$

Thus the assertion holds.

**Lemma 4.4.** Let $f$ be as above.

(1) Let $r_1$ be an element of $L'_n$ in Proposition 2.3. Then we have

$$
\lambda_{I_n(f)}(r_1) = p^{(n-1)k-n(n+1)/2} a_f(p) \sum_{i=1}^{n} p^i.
$$

(2) Let $n = 2$. Then we have

$$
\lambda_{I_2(f)}(T(p)) = a_f(p) + p^{2k-n-1} + p^{2k-n-2}.
$$

**Lemma 4.5.** Let $d$ be a fundamental discriminant such that $(-1)^{n/2}d > 0$.

(1) Assume that $d \neq 1$. Then there exists a positive definite half integral matrix $A$ of degree $n$ such that $(-1)^{n/2} \det(2A) = d$.

(2) Assume $n \equiv 0 \mod 8$. Then there exists a positive definite half integral matrix $A$ of degree $n$ such that $\det(2A) = 1$.

(3) Assume that $n \equiv 4 \mod 8$. Then for any prime number $q$ there exists a positive definite half integral matrix $A$ of degree $n$ such that $\det(2A) = q^2$.

**Proof.** (1) For a non-degenerate symmetric matrix $A$ with entries in $\mathbb{Q}_p$ let $h_p(A)$ be the Hasse invariant of $A$. First let $n \equiv 2 \mod 4$ and $d = -4$. Take a family $\{A_p\}_p$ of half integral matrices over $\mathbb{Z}_p$ of
degree \( n \) such that \( A_p = 1_n \) if \( p \neq 2 \), and \( A_2 = (-1)^{(n-2)/4} 1_2 \perp H_{n/2-1} \), where \( H_n = \overline{H \perp \cdots \perp H} \) with \( H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \). Then we have \( \det A = 2^{2-n} \in Q_p^\times/(Q_p^\times)^2 \) for any \( p \), and \( h_p(A) = 1 \) for any \( p \). Thus by [I-S, Proposition 2.1], there exists an element \( A \) of \( L_{n,2}^{n/8} \) such that \( A \sim A_p \) for any \( p \). In particular we have \((-1)^{n/2} \det(2A) = -4 \). Next let \( d = (-1)^{n/2}8 \). We take \( A_p = (-1)^{n/2}2 \perp 1_{n-1} \) if \( p \neq 2 \). We can take \( \xi \in Z_2^* \) such that \( (2, \xi) = (-1)^{(n-2)(n+4)/8} \), and put \( A_2 = 2\xi \perp (-\xi) \perp H_{n/2-1} \). Then we have \( \det A = (-1)^{n/2}23^{-n} \in Q_p^\times/(Q_p^\times)^2 \) for any \( p \), and \( h_p(A) = 1 \) for any \( p \). Thus again by [I-S, Proposition 2.1], we prove the assertion for this case. Finally assume that \( d \) contains an odd prime factor \( q \). For \( p \neq p_0 \) we take a matrix \( A_p \) so that \( \det A_p = 2^{-n}d \in Q_q^\times/(Q_q^\times)^2 \). Then for almost all \( p \) we have \( h_p(A_p) = 1 \). We take \( \xi \in Z_q^* \) such that \( (q, -\xi) = \prod_{p 
eq q} h_p(A_p) \), and put \( A_q = \xi d \perp \xi \perp 1_{n-2} \). Then we have \( \det A_q = 2^{-n}d \in Q_q^\times/(Q_q^\times)^2 \), and \( h_p(A_q) \prod_{p 
eq q} = 1 \). Thus again by [I-S, Proposition 2.1], we prove the assertion for this case.

(2) It is well known that there exists a positive definite half-integral matrix \( E_8 \) of degree 8 such that \( \det(2E_8) = 1 \). Thus \( A = \overline{E_8 \perp \cdots \perp E_8}^{n/8} \) satisfies the required condition.

(3) Let \( q \neq 2 \). Then, take a family \( \{A_p\} \) of half-intrgral matrices over \( Z_p \) of degree \( n \) such that \( A_q \sim Z_q 2 \perp (-q\xi) \perp (-\xi) \perp 1_{n-3} \) with \( (\xi_q) = -1, A_2 = H_{n/2} \), and \( A_p = 1_n \) for \( p \neq 2 \). Then by the same argument as in (1) we can show that there exists a positive definite half integral matrix \( A \) of degree \( n \) such that \( \det(2A) = q^2 \) such that \( A \sim Z_n A_p \) for any \( p \). Let \( q = 2 \). Then the matrix \( A' = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \end{pmatrix}^{(n-4)/8} \) is a positive definite and \( \det(2A') = 4 \). Thus the matrix \( A' \perp \overline{E_8 \perp \cdots \perp E_8}^{n/8} \) satisfies the required condition.

**Proposition 4.6.** Let \( k \) and \( n \) be positive even integer. Let \( d \) be a fundamental discriminant. Let \( f \) be a primitive form in \( S_{2k-n}(\Gamma_1) \). Let \( \Psi \) be a prime ideal in \( K \). Then there exists a positive definite half integral matrices \( A \) of degree \( n \) such that \( c_{I_n(f)}(A) = c_f(|d|)q \) with \( q \) not divisible by \( \Psi \).
Proof. First assume that $d \neq 1$, or $n \not\equiv 4 \pmod{8}$. (1) By (1) and (2) of Lemma 4.5, there exists a matrix $A$ such that $\tau_A = d$. Thus we have $c_{\tau_A}(f)(A) = c_f(|d|)$. This proves the assertion.

Next assume that $n \equiv 4 \pmod{8}$ and that $d = 1$. Assume that $c_f(q) + q^{k-n/2-1}(-q - 1)$ is divisible by $\psi$ for any prime number $q$. Let $p$ be a prime number divisible by $\psi$. Fix an embedding $\iota_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$, and let $\rho_{f,p} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}_p})$ be the Galois representation attached to $f$. Then by Chebotarev density theorem, the semi-simplification $\overline{\rho}_{f,p}$ of $\overline{\rho}_{f,p}$ can be expressed as

$$\overline{\rho}_{f,p} = \chi_p^{k-n/2} \oplus \chi_p^{k-n/2-1}$$

with $\chi_p$ the $p$-adic mod $p$ cyclotomic character. On the other hand, by the Fontaine-Messing [Fo-Me] and Fontaine-Laffaille [Fo-La], $\overline{\rho}_{f,p}/I_p$ should be $\overline{\chi}_p^{2k-n-1} \oplus 1$ or $\omega_2^{2k-n-1} \oplus \omega_2^{p(2k-n-1)}$ with $\omega_2$ the fundamental character of level 2, where $I_p$ denotes the inertia group of $p$ in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. This is impossible because $k > 2$. Thus there exists a prime number $q$ such that $c_f(q) + q^{k-n/2-1}(-q - 1)$ is not divisible by $\psi$. For such a $q$, take a positive definite matrix $A$ in (3) of Lemma 4.5. Then

$$c_{\tau_A}(f)(A) = c(1)q^{k-(n+1)/2} \beta_q F_q(A, q^{-(n+1)/2} \beta_q^{-1}).$$

By [Ka1], we have

$$F_q(B, X) = 1 - Xq^{(n-2)/2}(q^2 + q) + q^3(Xq^{(n-2)/2}).$$

Thus we have

$$c_{\tau_A}(f)(A) = c(1)(c_f(q) + q^{k-n/2-1}(-q - 1)).$$

Thus the assertion holds.

**Theorem 4.7.** Let $k \geq 2n+4$. Let $K$ and $f$ be as above. Assume that the Conjecture $B$ holds for $f$. Let $\psi$ be a prime ideal of $K$. Furthermore assume that

1. $\psi$ divides $L(k, f) \prod_{i=1}^{n/2-1} L(2i + 1, f, St)$.
2. $\psi$ does not divide

$$\tilde{\xi}(2m) \prod_{i=1}^{n} L(2m + k - i, f)L(k - n/2, f, \chi_D)D(k - 1)!$$

for some integer $n/2 + 1 \leq m \leq k/2 - n/2 - 1$, and for some fundamental discriminant $D$ such that $(-1)^{n/2}D > 0$.

Then $\psi$ is a congruence prime of $I_n(f)$ with respect to $\mathbb{C}I_n(f)^\perp$. Furthermore assume that the following condition hold:
(3) $\mathfrak{P}$ does not divide 
\[ C_{k,n} \frac{\langle f, f \rangle}{\Omega(f, +; A_{\mathfrak{P}}) \Omega(f, -; A_{\mathfrak{P}})}, \]
where $C_{k,n} = 1 \text{ or } \prod_{q \leq (2k-n)/12} (1 + q + \cdots + q^{n-1})$ according as $n = 2$ or not.

Then $\mathfrak{P}$ is a congruence prime of $I_n(f)$ with respect to $(\mathfrak{E}_k(I_n))^\perp$.

Proof. Let $\mathfrak{P}$ be a prime ideal satisfying the condition (1) and (2). For the $D$ above, take a matrix $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$ so that $c_{I_n(f)}(A) = c_f(|D|)q$ with $q$ not divisible by $\mathfrak{P}$. Then by Proposition 4.3, we have
\[ \Lambda(2m, I_n(f), \mathfrak{S}t)|c_{I_n(f)}(A)|^2 = \Lambda(2m, I_n(f), \mathfrak{S}t)|c_f(|D|)|^2 q^2 \]
\[ = \epsilon_{k,m} \prod_{i=1}^{n} \mathcal{L}(2m + k - i, f)|D|^{k-n/2} \mathcal{L}(k - n/2, f, \chi_D) \]
\[ \times \frac{\Omega(f, +; \mathfrak{P}) \Omega(f, -; A_{\mathfrak{P}})}{(f, f)} n/2, \]
where $\epsilon_{k,m}$ is a rational number whose numerator is not divided by $\mathfrak{P}$. We note that $\frac{(f, f)}{\Omega(f, +; A_{\mathfrak{P}}) \Omega(f, -; A_{\mathfrak{P}})}$ is $\mathfrak{P}$-integral. Thus by assumptions (1) and (2), $\mathfrak{P}$ divides $(\Lambda(2m, I_n(f), \mathfrak{S}t)c_{I_n(f)}(A))^2$, and thus it divides $(\Lambda(2m, I_n(f), \mathfrak{S}t)c_{I_n(f)}(A))^2$. We note that $I_n(f)$ satisfies the assumption in Theorem 3.1. Thus by Theorem 3.1, there exits a Hecke eigenform $G \in \mathbb{C}(I_n(f))^\perp$ such that
\[ \lambda_G(T) \equiv \lambda_{I_n(f)}(T) \mod \mathfrak{P} \]
for any $T \in \mathcal{L}'_n$. Assume that we have $G = I_n(g)$ with some primitive form $g(z) = \sum_{m=1}^{\infty} a_g(m)e(mz) \in \mathfrak{E}_{2k-n}(\Gamma(1))$. Let $n = 2$. Then by (1) of Proposition 4.2, $\mathfrak{P}$ is also a congruence prime of $f$. Let $n \geq 4$. Then by (1) of Proposition 4.4, we have
\[ (p^{n-1} + \cdots + p + 1)a_f(p) \equiv (p^{n-1} + \cdots + p + 1)a_g(p) \mod \mathfrak{P} \]
for any prime number $p$ not divisible by $\mathfrak{P}$. By assumption (3), in particular, for any $p \leq (2k-n)/12$, we have
\[ a_f(p) = a_g(p) \mod \mathfrak{P}. \]
Thus by Sturm [Stur], $\mathfrak{P}$ is also a congruence prime of $f$. Thus by [Hi2] and [Ri2], $\mathfrak{P}$ divides $\frac{(f, f)}{\Omega(f, +; A_{\mathfrak{P}}) \Omega(f, -; A_{\mathfrak{P}})}$, which contradicts the assumption (3). Thus $\mathfrak{P}$ is a congruence prime of $I_n(f)$ with respect to $(\mathfrak{E}_k(\Gamma(n))^\perp$. 
**Example** Let \( n = 4 \) and \( k = 18 \). Then we have \( \dim S_{18}(\Gamma_4) \approx 16 \) (cf. Poor and Yuen [P-Y]) and \( \dim S_{18}(\Gamma_4)^* = \dim S_{32}(\Gamma_1) = 2 \). Take a primitive form \( f \in S_{32}(\Gamma_1) \). Then we have \([\mathbb{Q}(f) : \mathbb{Q}] = 2\), and 211 = \( \mathfrak{p} \mathfrak{p}' \) in \( \mathbb{Q}(f) \). Then we have

\[
N_{\mathbb{Q}(f)/\mathbb{Q}}(L(18, f)) = 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 211,
\]

\[
N_{\mathbb{Q}(f)/\mathbb{Q}}(\prod_{i=1}^{4} L(24-i, f)) = 2^{19} \cdot 3^{13} \cdot 5^5 \cdot 7^8 \cdot 11^2 \cdot 13^5 \cdot 17^5 \cdot 19^3 \cdot 23 \cdot 503 \cdot 1307 \cdot 14243,
\]

\[
\tilde{\xi}(6) = 2^{-2} \cdot 3^{-2} \cdot 7^{-1}
\]

and

\[
N_{\mathbb{Q}(f)/\mathbb{Q}}(L(16, f, \chi_1)) = 2^5 \cdot 3^2 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13^2.
\]

(cf. Stein [Ste].) Furthermore, by a direct computation we see neither \( \mathfrak{p} \) nor \( \mathfrak{p}' \) is a congruence prime of \( \tilde{f} \) with respect to \( \mathbb{C}\tilde{g} \) for another primitive form \( g \in S_{32}(\Gamma_1) \). Thus by Theorem 4.7, \( \mathfrak{p} \) or \( \mathfrak{p}' \) is a congruence prime of \( \tilde{f} \) with respect to \( S_{18}(\Gamma_4)^* \).

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18 Hidenori Katsurada

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