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<th>CONGRUENCE BETWEEN DUKE-IMAMOGLU-IKEDA LIFTS AND NON-DUKE-IMAMOGLU-IKEDA LIFTS</th>
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1. Introduction

As is well known, there is a congruence between the Fourier coefficients of Eisenstein series and those of cuspidal Hecke eigenforms for $SL_2(\mathbb{Z})$. This type of congruence is not only interesting in its own right but also has an important application to number theory as shown by Ribet [Ri1]. Since then, there have been so many important results about the congruence of elliptic modular forms (cf. [Hi1],[Hi2],[Hi3].) In the case of Hilbert modular forms or Siegel modular forms, it is sometimes more natural and important to consider the congruence between the Hecke eigenvalues of Hecke eigenforms modulo a prime ideal $\Psi$. We call such a $\Psi$ a prime ideal giving the congruence or a congruence prime. For a cuspidal Hecke eigenform $f$ for $SL_2(\mathbb{Z})$, let $\hat{f}$ be a lift of $f$ to the space $\mathfrak{M}_l(\Gamma')$ of modular forms of weight $l$ for a modular group $\Gamma'$. Here we mean by the lift of $f$ a cuspidal Hecke eigenform whose $\zeta$-function can be expressed in terms of certain $L$-functions of $f$. As typical examples of the lift we can consider the Doi-Naganuma lift, the Saito-Kurokawa lift, and the Duke-Imamoglu-Ikeda lift. We then consider the following problem:

**Problem.** Characterize the prime ideals giving the congruence between $\hat{f}$ and a cuspidal Hecke eigenform in $\mathfrak{M}_l(\Gamma')$ not coming from the lift. In particular characterize them in terms of special values of certain $L$-functions of $f$.

This type of problem was first investigated in the Doi-Naganuma lift case by Doi, Hida, and Ishii [D-H-I]. In our previous paper [Ka2], we considered the relationship between the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbb{Z})$ and the special values of their standard zeta functions. In particular, we proposed a conjecture concerning the congruence between Saito-Kurokawa lifts and non-Saito-Kurokawa
lifts, and proved it under certain condition. In this paper, we consider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, which is a generalization of our previous conjecture.

In Section 3, we review a result concerning the relationship between the congruence of cuspidal Hecke eigenforms with respect to $\text{Sp}_n(\mathbb{Z})$ and the special values of their standard zeta functions. In Section 4, we propose a conjecture concerning the congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, and prove it under a certain condition.

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Notation. For a commutative ring $R$, we denote by $M_{mn}(R)$ the set of $(m, n)$-matrices with entries in $R$. In particular put $M_n(R) = M_{nn}(R)$. Here we understand $M_{mn}(R)$ the set of the empty matrix if $m = 0$ or $n = 0$. For an $(m, n)$-matrix $X$ and an $(m, m)$-matrix $A$, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of $X$. Let $a$ be an element of $R$. Then for an element $X$ of $M_{mn}(R)$ we often use the same symbol $X$ to denote the coset $X \mod aM_{mn}(R)$. Put $\text{GL}_n(R) = \{A \in M_n(R) \mid \det A \in R^\times\}$, where $\det A$ denotes the determinant of a square matrix $A$, and $R^\times$ denotes the unit group of $R$. Let $S_n(R)$ denote the set of symmetric matrices of degree $n$ with entries in $R$. Furthermore, for an integral domain $R$ of characteristic different from 2, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree $n$ over $R$, that is, $\mathcal{H}_n(R)$ is the set of symmetric matrices of degree $n$ whose $(i, j)$-component belongs to $R$ or $\frac{1}{2}R$ according as $i = j$ or not. For a subset $S$ of $M_n(R)$ we denote by $S^\times$ the subset of $S$ consisting of non-degenerate matrices. In particular, if $S$ is a subset of $S_n(\mathbb{R})$ with $\mathbb{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of $S$ consisting of positive definite (resp. semi-positive definite) matrices. Let $R'$ be a subring of $R$. Two symmetric matrices $A$ and $A'$ with entries in $R$ are called equivalent over $R'$ with each other and write $A \sim_{R'} A'$ if there is an element $X$ of $\text{GL}_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices $X$ and $Y$ we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

2. Standard zeta functions of Siegel modular forms

For a complex number $x$ put $e(x) = \exp(2\pi \sqrt{-1}x)$. Furthermore put $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$, where $1_n$ denotes the unit matrix of degree $n$. For
a subring $K$ of $\mathbb{R}$ put

$$GSp_n(K)^+ = \{ M \in GL_{2n}(K) \mid J_n[M] = \kappa(M)J_n \text{ with some } \kappa(M) > 0 \},$$

and

$$Sp_n(K) = \{ M \in GSp_n(K)^+ \mid J_n[M] = J_n \}.$$

Furthermore, put

$$\Gamma^{(n)} = Sp_n(\mathbb{Z}) = fM_2GL_2n(\mathbb{Z})\mid J_n[M] = J_n.$$

We sometimes write an element $M$ of $GSp_n(K)$ as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \mid C \equiv O_n \text{ mod } N.$$

Let $\mathbf{H}_n$ be Siegel’s upper half-space. For each element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n(\mathbb{R})^+$ and $Z \in \mathbf{H}_n$ put

$$M(Z) = (AZ + B)(CZ + D)^{-1}$$

and

$$j(M, Z) = \det(CZ + D).$$

Furthermore, for a function $f$ on $\mathbf{H}_n$ and an integer $k$ we define $f|_k M$ as

$$(f|_k M)(Z) = \det(M)^{k/2}j(M, Z)^{-k}f(M(Z)).$$

For an integer or half integer $l$ and the subgroup $\Gamma^{(n)}_0(N)$ of $\Gamma^{(n)}$, we denote by $\mathfrak{M}_k(\Gamma^{(n)}_0(N))$ (resp. $\mathfrak{M}^{(n)}_k(\Gamma^{(n)}_0(N))$) the space of holomorphic (resp. $C^{\infty}$-) modular forms of weight $k$ with respect to $\Gamma^{(n)}_0(N)$. We denote by $E_k(\Gamma^{(n)}_0(N))$ the sub-space of $\mathfrak{M}_k(\Gamma^{(n)}_0(N))$ consisting of cusp forms. Let $f$ be a holomorphic modular form of weight $k$ with respect to $\Gamma^{(n)}_0(N)$. Then $f$ has the following Fourier expansion:

$$f(Z) = \sum_{A \in \mathfrak{H}(Z)_{\geq 0}} a_f(A)e(\text{tr}(AZ)),$$

and in particular if $f$ is a cusp form, $f$ has the following Fourier expansion:

$$f(Z) = \sum_{A \in \mathfrak{H}(Z)_{\geq 0}} a_f(A)e(\text{tr}(AZ)),$$

where $\text{tr}$ denotes the trace of a matrix. Let $dv$ denote the invariant volume element on $\mathbf{H}_n$ defined by

$$dv = \det(\text{Im}(Z))^{-n-1} \wedge_{1 \leq j \leq l \leq n} (dx_{jl} \wedge dy_{jl}).$$

Here for $Z \in \mathbf{H}_n$ we write $Z = (x_{jl}) + \sqrt{-1}(y_{jl})$ with real matrices
(x_j) and (y_j). For two $C^\infty$-modular forms $f$ and $g$ of weight $l$ with respect to $\Gamma_0^{(n)}(N)$ we define the Petersson scalar product $\langle f, g \rangle$ by

$$\langle f, g \rangle = [\Gamma^{(n)}: \Gamma_0^{(n)}(N)]^{-1} \int_{\Gamma_0^{(n)}(N) \backslash \text{H}_n} f(Z)g(\overline{Z}) \det(\text{Im}(Z))^l \, dv,$$

provided the integral converges.

Let $L_n = L_\mathbf{Q}(GSp_n(\mathbf{Q})^+, \Gamma^{(n)})$ denote the Hecke algebra over $\mathbf{Q}$ associated with the Hecke pair $(GSp_n(\mathbf{Q})^+, \Gamma^{(n)})$. Furthermore, let $L_n' = L_\mathbf{Q}(Sp_n(\mathbf{Q}), \Gamma^{(n)})$ denote the Hecke algebra over $\mathbf{Q}$ associated with the Hecke pair $(Sp_n(\mathbf{Q}), \Gamma^{(n)})$. For each integer $m$ define an element $T(m)$ of $L_n$ by

$$T(m) = \sum_{d_1, \ldots, d_n, e_1, \ldots, e_n} \Gamma^{(n)}(d_1 \perp \cdots \perp d_n \perp e_1 \perp \cdots \perp e_n) \Gamma^{(n)},$$

where $d_1, \ldots, d_n, e_1, \ldots, e_n$ run over all positive integer satisfying

$$d_i | d_{i+1}, e_{i+1} | e_i \quad (i = 1, \ldots, n - 1), \quad d_n | e_n, d_i e_i = m \quad (i = 1, \ldots, n).$$

Furthermore, for $i = 1, \ldots, n$ and a prime number $p$ put

$$T_i(p^2) = \Gamma^{(n)}(1_{n-i} \perp p1_i \perp p^2 1_{n-i} \perp p1_i) \Gamma^{(n)},$$

and $(p^{\pm 1}) = \Gamma^{(n)}(p^{\pm 1}1_n) \Gamma^{(n)}$. As is well known, $L_n$ is generated over $\mathbf{Q}$ by all $T(p), T_i(p^2)$ $(i = 1, \ldots, n)$, and $(p^{\pm 1})$. We denote by $L'_n$ the subalgebra of $L_n$ generated by over $\mathbf{Z}$ by all $T(p)$ and $T_i(p^2)$ $(i = 1, \ldots, n)$. Let $T = \Gamma^{(n)} M \Gamma^{(n)}$ be an element of $L_n \otimes \mathbf{C}$. Write $T$ as $T = \bigcup_\gamma \Gamma^{(n)} \gamma$ and for $f \in \mathfrak{M}_k(\Gamma^{(n)})$ define the Hecke operator $|kT$ associated to $T$ as

$$f|_kT = \det(M)^{k/2-(n+1)/2} \sum_\gamma f|_{k\gamma}.$$

We call this action the Hecke operator as usual (cf. [A].) If $f$ is an eigenfunction of a Hecke operator $T \in L_n \otimes \mathbf{C}$, we denote by $\lambda_f(T)$ its eigenvalue. Let $L$ be a subalgebra of $L_n$. We call $f \in \mathfrak{M}_k(\Gamma^{(n)})$ a Hecke eigenform for $L$ if it is a common eigenfunction of all Hecke operators in $L$. In particular if $L = L_n$ we simply call $f$ a Hecke eigenform. Furthermore, we denote by $\mathbf{Q}(f)$ the field generated over $\mathbf{Q}$ by eigenvalues of all $T \in L_n$ as in Section 1. As is well known, $\mathbf{Q}(f)$ is a totally real algebraic number field of finite degree. Now, first we consider the integrality of the eigenvalues of Hecke operators. For an algebraic number field $K$, let $\mathfrak{O}_K$ denote the ring of integers in $K$. The following assertion has been proved in [Mi2] (see also [Ka2].)

**Theorem 2.1** Let $k \geq n + 1$. Let $f \in \mathfrak{M}_k(\Gamma^{(n)})$ be a common eigenform in $L'_n$. Then $\lambda_f(T)$ belongs to $\mathfrak{O}_K(\mathbf{Q}(f))$ for any $T \in L'_n$. 

Let $L_{np} = L(GSp_n(Q)^+ \cap GL_{2n}(Z[p^{-1}]), \Gamma(n))$ be the Hecke algebra associated with the pair $(GSp_n(Q)^+ \cap GL_{2n}(Z[p^{-1}]), \Gamma(n))$. $L_{np}$ can be considered as a subalgebra of $L_n$, and is generated over $Q$ by $T(p)$ and $T_i(p^2)$ ($i = 1, 2, \ldots, n$). We now review the Satake $p$-parameters of $L_{np}$; let $P_n = Q[\lambda, X_0^\pm, X_1^\pm, \ldots, X_n^\pm]$ be the ring of Laurent polynomials in $X_0, X_1, \ldots, X_n$ over $Q$. Let $W_n$ be the group of $Q$-automorphisms of $P_n$ generated by all permutations in variables $X_1, \ldots, X_n$ and by the automorphisms $\tau_1, \ldots, \tau_n$ defined by

$$\tau_i(X_0) = X_0 X_i, \tau_i(X_j) = X_j^{-1}, \tau_i(X_j) = X_j (j \neq i).$$

Furthermore, a group $W_n$ isomorphic to $W_n$ acts on the set $T_n = (C^x)^{n+1}$ in a way similarly to above. Then there exists a $Q$-algebra isomorphism $\Phi_{np}$, called the Satake isomorphism, from $L_{np}$ to the $W_n$-invariant subring $P_n^{W_n}$ of $P_n$. Then for a $Q$-algebra homomorphism $\lambda$ from $L_{np}$ to $C$, there exists an element $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))$ of $T_n$ satisfying

$$\lambda(\Phi_{np}^{-1}(F(X_0, X_1, \ldots, X_n))) = F(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))$$

for $F \in P_n^{W_n}$. The equivalence class of $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))$ under the action of $W_n$ is uniquely determined by $\lambda$. We call this the Satake parameters of $L_{np}$ determined by $\lambda$.

Now assume that an element $f$ of $M_k(Spn(Z))$ is a Hecke eigenform. Then for each prime number $p$, $f$ defines a $Q$-algebra homomorphism $\lambda_{f,p}$ from $L_{np}$ to $C$ in a usual way, and we denote by $\alpha_0(p), \alpha_1(p), \ldots, \alpha_n(p)$ the Satake parameters of $L_{np}$ determined by $f$. We then define the standard zeta function $L(f, s, St)$ by

$$L(s, f, St) = \prod_p \prod_{i=1}^n (1 - (1 - \alpha_i(p)p^{-s})(1 - \alpha_i(p)^{-1}p^{-s}))^{-1}.$$ 

Let $f(z) = \sum_{A \in \Gamma(n)(Z)} a(A)e(\text{tr}(Az))$ be a Hecke eigenform in $\mathcal{S}_k(\Gamma(n))$.

For a positive integer $m \leq k - n$ such that $m \equiv n \mod 2$ put

$$\Lambda(f, m, St) = (-1)^{n(m+1)/2+1} 2^{-4kn+3n^2+n(n-1)m+2} \times \Gamma(m+1) \prod_{i=1}^n \Gamma(2k - n - i) \frac{L(f, m, St)}{\pi^{-n(n+1)/2+nk(n+1)}}.$$ 

Then the following theorem is due to Böcherer [B2] and Mizumoto [Mi].

**Theorem 2.2.** Let $l, k$ and $n$ be a positive integers such that $\rho(n) \leq l \leq k-n$, where $\rho(n) = 3$, or 1 according as $n \equiv 1 \mod 4$ and $n \geq 5$, or not. Let $f \in \mathcal{S}_k(\Gamma(n))$ be a Hecke eigenform. Then $\Lambda(f, m, St)$ belongs to $Q(f)$. 
For later purpose, we consider a special element in $L_{np}$: the polynomial $X^2_0X_1X_2\cdots X_n\sum_{i=1}^n(X_i+X_i^{-1})$ is an element of $P_n^{W_n}$, and thus we can define an element $\Phi_{np}^{-1}(X^2_0X_1X_2\cdots X_n\sum_{i=1}^n(X_i+X_i^{-1}))$ of $L_{np}$, which is denoted by $r_1$.

**Proposition 2.3.** Under the above notation the element $r_1$ belongs to $L_0'$, and we have

$$\lambda_f(r_1) = p^{n(k-(n+1)/2)}\sum_{i=1}^n(\alpha_i(p) + \alpha_i(p)^{-1}).$$

**Proof.** By a careful analysis of the computation in page 159-160 of [A], we see that $r_1$ is a $\mathbb{Z}$-linear combination of $T_i(p^2)$ $(i = 1, \ldots, n)$, and therefore we can prove the first assertion. Furthermore, by Lemma 3.3.34 of [A], we can prove the second assertion.

3. **Congruence of modular forms and special values of the standard zeta functions**

In this section we review a result concerning the congruence between the Hecke eigenvalues of modular forms of the same weight following [Ka2]. Let $K$ be an algebraic number field, and $\mathfrak{O} = \mathfrak{O}_K$ the ring of integers in $K$. For a prime ideal $\mathfrak{p}$ of $\mathfrak{O}$, we denote by $\mathfrak{O}(\mathfrak{p})$ the localization of $\mathfrak{O}$ at $\mathfrak{p}$ in $K$. Let $\mathfrak{A}$ be a fractional ideal in $K$. If $\mathfrak{A} = \mathfrak{p}^e\mathfrak{B}$ with $\mathfrak{B}\mathfrak{O}(\mathfrak{p}) = \mathfrak{O}(\mathfrak{p})$ we write $\text{ord}_{\mathfrak{p}} = e$. We simply write $\text{ord}_{\mathfrak{p}}((c)) = \text{ord}_{\mathfrak{p}}(c)$ for $c \in K$. Now let $f$ be a Hecke eigenform in $\mathfrak{H}_k(\Gamma(n))$ and $M$ be a subspace of $\mathfrak{H}_k(\Gamma(n))$ stable under Hecke operators $T \in L_n$. Assume that $M$ is contained in $(\mathcal{C}f)^\perp$, where $(\mathcal{C}f)^\perp$ is the orthogonal complement of $\mathcal{C}f$ in $\mathfrak{H}_k(\Gamma(n))$ with respect to the Petersson product. Let $K$ be an algebraic number field containing $\mathbb{Q}(f)$. A prime ideal $\mathfrak{p}$ of $\mathfrak{O}_K$ is called a congruence prime of $f$ with respect to $M$ if there exists a Hecke eigenform $g \in M$ such that

$$\lambda_f(T) \equiv \lambda_g(T) \mod \mathfrak{p}$$

for any $T \in L_n'$, where $\mathfrak{p}$ is the prime ideal of $\mathfrak{O}_{K\mathbb{Q}(f)}$ lying above $\mathfrak{p}$. If $M = (\mathcal{C}f)^\perp$, we simply call $\mathfrak{p}$ a congruence prime of $f$.

Now we consider the relation between the congruence primes and the standard zeta values. To consider this, we have to normalize the standard zeta value $\Lambda(f,l,\text{St})$ for a Hecke eigenform $f$ because it is not uniquely determined by the system of Hecke eigenvalues of $f$. We note that there is no reasonable normalization of cuspidal Hecke eigenform in the higher degree case unlike the elliptic modular case.
Thus we define the following quantities: for a Hecke eigenform \( f(z) = \sum_A a_f(A)e(\text{tr}(Az)) \) in \( \mathcal{S}_k(\Gamma(n)) \), let \( \mathfrak{A}_f \) be the \( \mathcal{O}_Q(f) \)-module generated by all \( a_f(A) \)'s. Assume that there exists a complex number \( c \) such that all the Fourier coefficients \( cf \) belongs to \( Q(f) \). Then \( \mathfrak{A}_f \) is a fractional ideal in \( Q(f) \), and therefore, so is \( \Lambda(f, l, St)^2 \mathfrak{A}_f \) if \( l \) satisfies the condition in Theorem 2.2. We note that this fractional ideal does not depend on the choice of \( c \). We also note that the value \( N_Q(f)(\Lambda(f, l, St)^2)N(\mathfrak{A}_f)^2 \) does not depend on the choice of \( c \), where \( N(\mathfrak{A}_f) \) is the norm of the ideal \( \mathfrak{A}_f \). Then, we have

**Theorem 3.1.** Let \( f \) be a Hecke eigenform in \( \mathcal{S}_k(\Gamma(n)) \). Assume that there exists a complex number \( c \) such that all the Fourier coefficients \( cf \) belongs to \( Q(f) \). Let \( l \) be a positive integer satisfying the condition in Theorem 2.2. Let \( \mathfrak{p} \) be a prime ideal of \( \mathcal{O} \). Assume that \( \text{ord}_{\mathfrak{p}}(\Lambda(f, l, St)^2) < 0 \) and that it does not divide \((2l - 1)!\). Then \( \mathfrak{p} \) is a congruence prime of \( f \). In particular, if a rational prime number \( p \) divides the denominator of \( N_Q(f)(\Lambda(f, l, St))N(\mathfrak{A}_f)^2 \), then \( p \) is divisible by some congruence prime of \( f \).

Now for a Hecke eigenform \( f \) in \( \mathcal{S}_k(\Gamma(n)) \), let \( \mathfrak{E}_f \) denote the subspace of \( \mathcal{S}_k(\Gamma(n)) \) spanned by all Hecke eigenforms with the same system of the Hecke eigenvalues as \( f \).

**Corollary.** In addition to the above notation and the assumption, assume that \( \mathcal{S}_k(\Gamma(n)) \) has the multiplicity one property. Then \( \mathfrak{p} \) is a congruence prime of \( f \) with respect to \( \mathfrak{E}_f \). In particular, if a rational prime number \( p \) divides the denominator of \( N_Q(f)(\Lambda(f, l, St))N(\mathfrak{A}_f)^2 \), then \( p \) is divisible by some congruence prime of \( f \) with respect to \( \mathfrak{E}_f \).


In this section, we consider the congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts. Throughout this section and the next, we assume that \( n \) and \( k \) are even positive integers. Let

\[
f(z) = \sum_{m=1}^{\infty} a(m)e(mz)
\]

be a normalized Hecke eigenform of weight \( 2k - n \) with respect to \( SL_2(\mathbb{Z}) \). For a Dirichlet character \( \chi \), we then define the L-function
$L(s, f)$ of $f$ twisted by $\chi$ by

$$L(s, f) = \prod_p \{(1 - \chi(p)\beta_p p^{k-n/2-1/2-s})(1 - \chi(p)\beta_p^{-1} p^{k-n/2-1/2-s})\}^{-1},$$

where $\beta_p$ is a non-zero complex number such that $\beta_p + \beta_p^{-1} = p^{-k+n/2+1/2}a(p)$. We simply write $L(s, f)$ as $L(s, f, \chi)$ if $\chi$ is the principal character. Furthermore, let $\tilde{f}$ be the cusp form of weight $k-n/2+1/2$ belonging to the Kohnen plus space corresponding to $f$ via the Shimura correspondence (cf. [Ko1]). Then $\tilde{f}$ has the following Fourier expansion:

$$\tilde{f}(z) = \sum c(e)e(ez),$$

where $e$ runs over all positive integers such that $(-1)^{k-n/2}e \equiv 0, 1 \mod 4$. We then put

$$a_{I_n(f)}(T) = c([_TJ]\prod_p (p^{k-n/2-1/2}\beta_p)^{\nu_p(T)} F_p(T, p^{-n+1/2}\beta_p^{-1}).$$

We note that $a_{I_n(f)}(T)$ does not depend on the choice of $\beta_p$. Define a Fourier series $I_n(f)(Z)$ by

$$I_n(f)(Z) = \sum_{T \in K_n(Z), \nu_0 > 0} a_{I_n(f)}(T)e(\text{tr}(TZ)).$$

In [Ik1] Ikeda showed that $I_n(f)(Z)$ is a cusp form of weight $k$ with respect to $\Gamma^{(n)}$ and a Hecke eigenform for $L^\infty$ such that

$$L(s, I_n(f), St) = \zeta(s) \prod_{i=1}^n L(s + k - i, f).$$

This was first conjecture by Duke and Imamoglu. Thus we call $I_n(f)$ the Duke-Imamoglu-Ikeda lift of $f$. We note that we have $Q(\hat{f}) = Q(I_n(f)) = Q(f)$. Furthermore, we have $\mathfrak{S}_f = \mathfrak{S}_{I_n(f)}$, where $\mathfrak{S}_f$ is the $\mathfrak{S}_{Q(f)}$-module generated by all the Fourier coefficients of $\hat{f}$.

Now to consider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, first we prove the following:

**Proposition 4.1** $I_n(f)$ is a Hecke eigenform.

We note that Ikeda proved in [Ik1] that $I_n(f)$ is a Hecke eigenform for $L^\infty$ but has not proved that it is a Hecke eigenform for $L_n$. This was pointed to us by B. Heim (see [He].) We thank him for his comment. We also note that an explicit form of the spinor L-function of $I_n(f)$ was obtained by Murakawa [Mu] and Schmidt [Sch] assuming that $I_n(f)$ is a Hecke eigenform.
Proof of Proposition 4.1. We have only to prove that \( I_n(f) \) is an eigenfunction of \( T(p) \) for any prime \( p \). The proof may be more or less well known, but for the convenience of the readers we here give the proof. For a modular form

\[
F(Z) = \sum_B c_F(B) e(\text{tr}(BZ)),
\]

let \( c_F^{(p)}(B) \) be the \( B \)-th Fourier coefficient of \( F|T(p) \). Then for any positive definite matrix \( B \) we have

\[
c_F^{(p)}(B) = p^{nk-n(n+1)/2} \sum_{d_1|d_2|\ldots|d_n|p} d_1^n \ldots d_n^n \det D^{-k} c_F(p^{-1}A[t \cdot D]),
\]

where \( \Lambda_n = GL_n(\mathbb{Z}) \).

Now let \( E_{n,k}(Z) \) be the Siegel Eisenstein series of degree \( n \) and of weight \( k \) defined by

\[
E_{n,k}(Z) = \sum_{\gamma \in \Gamma_n} j(\gamma, Z)^{-k}.
\]

For \( k \geq n + 1 \), the Siegel Eisenstein series \( E_{n,k}(Z) \) is a holomorphic modular form of weight \( k \) with respect to \( \Gamma_n \). Furthermore, \( E_{n,k}(Z) \) is a Hecke eigenform and in particular we have

\[
E_{n,k}|T(p)(Z) = h_{n,p}(p^k) E_{n,k}(Z),
\]

where

\[
h_{n,p}(X) = 1 + \sum_{r=1}^{n} \sum_{1 \leq i_1 < \ldots < i_r \leq n} p^{-\sum_{j=1}^r i_j} X^r.
\]

Let \( c_{n,k}(B) \) be the \( B \)-th Fourier coefficient of \( E_{n,k}(Z) \). Then we have

\[
h_{n,p}(p^k) c_{n,k}(B) = p^{nk-n(n+1)/2} \sum_{d_1|d_2|\ldots|d_n|p} d_1^n \ldots d_n^n \det D^{-k} c_{n,k}(p^{-1}B[t \cdot D]).
\]

Let \( B \) be positive definite. Then we have

\[
c_{n,k}(B) = a_{n,k}(\det 2B)^{-k-n(n+1)/2} L(k-n/2, \chi_B) \prod_q F_q(B, p^{-k}),
\]

where \( a_{n,k} \) is a non-zero constant depending only on \( n \) and \( k \). We note that we have

\[
F_q(p^{-1}B[t \cdot D], X) = F_q(B, X)
\]
for any $D \in \Lambda_n(d_1 \perp \cdots \perp d_n)\Lambda_n$ with $d_1 | \cdots | d_n | p$ if $q \neq p$. Thus we have

$$h_{n,p}(p^k) F_p(B, p^{-k}) = \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{n_e + (n-1)e_2 + \cdots + e_n} p^{(k-n)(k-n-1)}$$

$$\times \sum_{D \in \Lambda_n \setminus \Lambda_n(p^{e_1 \perp \cdots \perp p^n})\Lambda_n} F_p(p^{-1} B[D^t], p^{-k}).$$

The both-hand sides of the above are polynomials in $p^k$ and the equality holds for infinitely many $k$. Thus we have

$$h_{n,p}(X^{-1}) F_p(B, X) = \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{n_e + (n-1)e_2 + \cdots + e_n} (X^{-1} p^{-n} - (e_1 + \cdots + e_n)$$

$$\times \sum_{D \in \Lambda_n \setminus \Lambda_n(p^{e_1 \perp \cdots \perp p^n})\Lambda_n} F_p(p^{-1} B[D^t], X)$$

as polynomials in $X$ and $X^{-1}$. Thus we have

$$(p^{k-(n+1)/2} X)^{n/2} h_{n,p}(p^{(n+1)/2} X^{-1}) (p^{k-(n+1)/2} X^{-1})^{\nu_B(B)} F_p(B, p^{-(n+1)/2} X)$$

$$= p^{nk-n(n+1)/2} \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{n_e + (n-1)e_2 + \cdots + e_n}$$

$$\times \sum_{D \in \Lambda_n \setminus \Lambda_n(p^{e_1 \perp \cdots \perp p^n})\Lambda_n} \det D^{-k} (p^{k-(n+1)/2} X^{-1})^{\nu_D(B[D^t])} F_p(p^{-1} B[D^t], p^{-(n+1)/2} X).$$

We recall that we have

$$c_{I_n}(f)(B) = c_f([B]) F_B^{k-(n+1)/2} \prod_q \nu_B(q) F_q(B, q^{-(n+1)/2} \alpha_q^{-1}),$$

where $\alpha_q$ is the Satake $q$-parameter of $f$. We also note that $c_f([B]) = c_f([B])$ for any $D$. Thus we have

$$(p^{k-(n+1)/2} \alpha_p^{-1})^{n/2} \sum_{d_1 | d_2 | \cdots | d_n | p} d_1^{n_1} d_2^{n_2} \cdots d_n^{n_n}$$

$$\times \sum_{D \in \Lambda_n \setminus \Lambda_n(d_1 \perp \cdots \perp d_n)\Lambda_n} \det D^{-k} c_{I_n}(f)(p^{-1} B[D^t]).$$

This proves the assertion.

Let $f$ be a primitive form in $\mathcal{S}_{2k-n}(\Gamma(1))$. Let $\{f_1, \ldots, f_d\}$ be a basis of $\mathcal{S}_{2k-n}(\Gamma(1))$ consisting of primitive forms. Let $K$ be an algebraic number field containing $\mathbb{Q}(f_1) \cdots \mathbb{Q}(f_d)$, and $A = \mathcal{O}_K$. To formulate our conjecture exactly, we introduce the Eichler-Shimura periods as follows (cf. Hida [Hi3].) Let $\mathfrak{P}$ be a prime ideal in $K$. Let $A_{\mathfrak{P}}$ be a valuation ring in $K$ corresponding to $\mathfrak{P}$. Assume that the residual characteristic of $A_{\mathfrak{P}}$ is greater than or equal to 5. Let $L(2k-n-2, A_{\mathfrak{P}})$ be the module of homogeneous polynomials of degree $2k-n-2$ in the variables $X, Y.$
with coefficients in $A_{\mathfrak{p}}$. We define the action of $M_2(\mathbb{Z}) \cap GL_2(\mathbb{Q})$ on $L(2k - n - 2, A_{\mathfrak{p}})$ via

$$
\gamma \cdot P(X, Y) = P(\gamma'(X, Y)(\gamma)^{-1}),
$$

where $\gamma' = (\det \gamma)\gamma^{-1}$. Let $H^1_P(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))$ be the parabolic cohomology group of $\Gamma^{(1)}$ with values in $L(2k - n - 2, A_{\mathfrak{p}})$. Fix a point $z_0 \in H_1$. Let $g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)})$ or $g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)})$. We then define the differential $\omega(g)$ as

$$
\omega(g)(z) = \begin{cases} 
2\pi i g(z)(X - zY)^n \, dz & \text{if } g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \\
2\pi \sqrt{-1} g(z)(X - \bar{z}Y)^n \, dz & \text{if } g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)}),
\end{cases}
$$

and define the cohomology class $\delta(g)$ of the 1-cocycle of $\Gamma^{(1)}$ as

$$
\gamma \in \Gamma^{(1)} \rightarrow \int_{z_0}^{\gamma(z_0)} \omega(g).
$$

The mapping $g \rightarrow \delta(g)$ induces the isomorphism

$$
\delta : \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \oplus \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \rightarrow H^1_P(\Gamma^{(1)}, L(2k - n - 2, \mathbb{C})),
$$

which is called the Eichler-Shimura isomorphism. We can define the action of Hecke algebra $L'_1$ on $H^1_P(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))$ in a natural manner. Furthermore, we can define the action $F_\infty$ on $H^1_P(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))$ as

$$
F_\infty(\delta(g)(z)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \delta(g)(-\bar{z}),
$$

and this action commutes with the Hecke action. For a primitive form $f$ and $j = \pm 1$, we define the subspace $H^1_P(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))[f, j]$ of $H^1_P(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))$ as

$$
H^1_P(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))[f, j] = \{ x \in H^1_P(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}})) ; x|T = \lambda f(T)x \text{ for } T \in L'_1, \text{ and } F_\infty(x) = jx \}. \quad (1)
$$

Since $A_{\mathfrak{p}}$ is a principal ideal domain, $H^1_P(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))[f, j]$ is a free module of rank one over $A_{\mathfrak{p}}$. For each $j = \pm 1$ take a basis $\eta(f, j, A_{\mathfrak{p}})$ of $H^1_P(\Gamma^{(1)}, (2k - n - 2, A_{\mathfrak{p}}))[f, j]$ and define a complex number $\Omega(f, j; A_{\mathfrak{p}})$ by

$$
(\delta(f) + jF_\infty(\delta(f))) / 2 = \Omega(f, j; A_{\mathfrak{p}}) \eta(f, j; A_{\mathfrak{p}}).
$$

This $\Omega(f, j; A_{\mathfrak{p}})$ is uniquely determined up to constant multiple of units in $A_{\mathfrak{p}}$. We call $\Omega(f, +; A_{\mathfrak{p}})$ and $\Omega(f, -; A_{\mathfrak{p}})$ the Eichler-Shimura periods. For $j = \pm 1$, $1 \leq l \leq 2k - n - 1$, and a Dirichlet character $\chi$ such
that \( \chi(-1) = j(-1)^{l-1} \), put

\[
L(l, f, \chi) = L(l, f; A_\Psi) = \frac{\Gamma(l) L(l, f, \chi)}{\tau(\chi)(2\pi \sqrt{-1}) \Omega(f, j; A_\Psi)},
\]

where \( \tau(\chi) \) is the Gauss sum of \( \chi \). In particular, put \( L(l, f; A_\Psi) = L(l, f, \chi; \Psi) \) if \( \chi \) is the principal character. Furthermore, put

\[
L(s, f, St) = 4(2\pi)^{-2s-2k+n+1} \Gamma(s) \Gamma(s + 2k - n - 1)L(s, f, St).
\]

It is well-known that \( L(l, f, \chi) \) belongs to the field \( K(\chi) \) generated over \( K \) by all the values of \( \chi \), and \( L(l, f, St) \) belongs to \( Q(f) \) (cf. [Bo].)

Let \( I_n(f) \) be the Duke-Imamoglu-Ikeda lift of \( f \). Let \( \mathcal{E}_k(\Gamma(n))^* \) be the subspace of \( \mathcal{E}_k(I_n) \) generated by all the Duke-Imamoglu-Ikeda lifts \( I(g) \) of primitive forms \( g \in \mathcal{E}_{2k-n}(\Gamma(1)) \). We remark that \( \mathcal{E}_k(\Gamma(2))^* \) is the Maass subspace of \( \mathcal{E}_k(\Gamma(2)) \).

**Conjecture A.** Let \( K \) and \( f \) be as above. Assume that \( k > n \). Let \( \mathfrak{p} \) be a prime ideal of \( K \) not dividing \((2k-1)!)\). Then \( \mathfrak{p} \) is a congruence prime of \( I_n(f) \) with respect to \((\mathcal{E}_k(\Gamma(n))^*)^\perp \) if \( \mathfrak{p} \) divides

\[
L(k, f) \prod_{i=1}^{n/2-1} L(2i+1, f, St) = 2^{n} \langle I_n(f) f, I_n(f) \rangle \langle f, f \rangle^{n/2-1} \langle \tilde{f}, \tilde{f} \rangle,
\]

where \( \alpha \) is an integer depending only on \( n \) and \( k \).

**Remark.** This is an analogue of the Doi-Hida-Ishii conjecture concerning the congruence primes of the Doi-Naganuma lifting [D-H-I]. (See also [Ka1].) We also note that this type of conjecture has been proposed by Harder [Ha] in the case of vector valued Siegel modular forms.

Now to explain why our conjecture is reasonable, we refer to Ikeda’s conjecture on the Petersson inner product of the Duke-Imamoglu-Ikeda lifting. Let \( f \) and \( \tilde{f} \) be as above. Put

\[
\tilde{\xi}(s) = 2(2\pi)^{-s} \Gamma(s) \zeta(s),
\]

and

\[
\Lambda(s, f) = 2(2\pi)^{-s} \Gamma(s) L(s, f).
\]

**Theorem 4.2.** (Katsurada and Kawamura [K-K]) In addition to the above notation and the assumption, assume \( k > n \). Then we have

\[
\tilde{\xi}(n) \Lambda(k, f) \prod_{i=1}^{n/2-1} L(2i-1, f, St) \tilde{\xi}(2i) = 2^n \frac{\langle I_n(f) f, I_n(f) \rangle}{\langle f, f \rangle^{n/2-1} \langle \tilde{f}, \tilde{f} \rangle},
\]

where \( \alpha \) is an integer depending only on \( n \) and \( k \).
We note that the above theorem was conjectured by Ikeda [Ik2] under more general setting. We note that the theorem has been proved by Kohnen and Skoruppa [K-S] in case \( n = 2 \).

**Proposition 4.3** Under the above notation and the assumption we have for any fundamental discriminant \( D \) such that \((-1)^{n/2}D > 0\) and \( L(k - n/2, f, \chi_D) \neq 0 \) we have

\[
\frac{c(|D|)^2}{\langle I_n(f), I_n(f) \rangle} = a_{n,k} \frac{\langle f, f \rangle^{n/2} |D|^{k-n/2} L(k - n/2, f, \chi_D)}{\prod_{i=1}^{n/2-1} L(k, f) \xi(n) L(2i + 1, f, \text{St}) \xi(2i)}
\]

with some algebraic number \( a_{n,k} \) depending only on \( n, k \).

**Proof.** By the result in Kohnen-Zagier [K-Z], for any fundamental discriminant \( D \) such that \((-1)^{n/2}D > 0\) we have

\[
\frac{c(|D|)^2}{\langle f, f \rangle} = \frac{2^{k-n/2-1} |D|^{k-n/2-1/2} \Lambda(k - n/2, f, \chi_D)}{\langle f, f \rangle}.
\]

Thus the assertion holds.

**Lemma 4.4.** Let \( f \) be as above.

1. Let \( r_1 \) be an element of \( \mathbf{L}_n' \) in Proposition 2.3. Then we have

\[
\lambda_{\mathbf{L}_n(f)}(r_1) = p^{(n-1)k-n(n+1)/2}a_f(p) \sum_{i=1}^{n} p^i.
\]

2. Let \( n = 2 \). Then we have

\[
\lambda_{\mathbf{L}_2(f)}(T(p)) = a_f(p) + p^{2k-n-1} + p^{2k-n-2}.
\]

**Lemma 4.5.** Let \( d \) be a fundamental discriminant such that \((-1)^{n/2}d > 0\).

1. Assume that \( d \neq 1 \). Then there exists a positive definite half integral matrix \( A \) of degree \( n \) such that \((-1)^{n/2} \det(2A) = d \).
2. Assume \( n \equiv 0 \mod 8 \). Then there exists a positive definite half integral matrix \( A \) of degree \( n \) such that \( \det(2A) = 1 \).
3. Assume that \( n \equiv 4 \mod 8 \). Then for any prime number \( q \) there exists a positive definite half integral matrix \( A \) of degree \( n \) such that \( \det(2A) = q^2 \).

**Proof.** (1) For a non-degenerate symmetric matrix \( A \) with entries in \( \mathbf{Q}_p \) let \( h_p(A) \) be the Hasse invariant of \( A \). First let \( n \equiv 2 \mod 4 \) and \( d = -4 \). Take a family \( \{A_p\}_p \) of half integral matrices over \( \mathbf{Z}_p \) of
degree \( n \) such that \( A_\ell = 1_n \) if \( \ell \neq 2 \), and \( A_2 = (-1)^{(n-2)/4} L_{n/2-1} \),
where \( H_n = \underbrace{H \cdots H}_{\ell} \) with \( H = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \). Then we have \( \det A = 2^{2-n} \in \mathcal{O}_p^\times/(\mathcal{O}_p^\times)^2 \) for any \( p \), and \( h_p(A) = 1 \) for any \( p \). Thus by [I-S, Proposition 2.1], there exists an element \( A \) of \( \mathcal{L}_{n,2} \) such that \( A \sim A_\ell \) for any \( \ell \). In particular, we have \((-1)^{n/2} \det(2A) = -4 \). Next let \( d = (-1)^{n/2} 8 \). We take \( A_p = (-1)^{n/2} 8 1_{n-1} \) if \( \ell \neq 2 \). We can take \( \xi \in \mathbb{Z}_2^* \) such that \( (2,\xi) = (-1)^{(n-2)(n+4)/8} \), and put \( A_2 = 2 \xi L (-\xi) \ell H_{n/2-1} \). Then we have \( \det A = (-1)^{n/2} 2^{3-n} \in \mathcal{O}_p^\times/(\mathcal{O}_p^\times)^2 \) for any \( p \), and \( h_p(A) = 1 \) for any \( p \). Thus again by [I-S, Proposition 2.1], we prove the assertion for this case. Finally assume that \( d \) contains an odd prime factor \( q \). For \( \ell \neq p_0 \) we take a matrix \( A_\ell \) so that \( \det A_\ell = -n q \in \mathcal{O}_q^\times/(\mathcal{O}_q^\times)^2 \). Then for almost all \( p \) we have \( h_p(A_\ell) = 1 \). We take \( \xi \in \mathbb{Z}_2^* \) such that \( (q, -\xi) = \prod_{\ell \neq q} h_p(A_\ell) \), and put \( A_q = \xi q q \ell 1_{n-2} \). Then we have \( \det A_q 2^{n} \in \mathcal{O}_q^\times/(\mathcal{O}_q^\times)^2 \), and \( h_p(A_q) \prod_{\ell \neq q} = 1 \). Thus again by [I-S, Proposition 2.1], we prove the assertion for this case.

(2) It is well known that there exists a positive definite half-integral matrix \( E_8 \) of degree 8 such that \( \det(2E_8) = 1 \). Thus \( A = E_8 \ell \cdots \ell E_8 \) satisfies the required condition.

(3) Let \( q \neq 2 \). Then, take a family \( \{A_\ell\} \) of half-integral matrices over \( \mathbb{Z}_p \) of degree \( n \) such that \( A_q = \mathbf{Z}_q \ell q \ell (-q \xi) \ell (-\xi) \ell 1_{n-3} \) with \( (\xi) = -1 \), \( A_2 = H_{n/2} \), and \( A_\ell = 1_n \) for \( \ell \neq 2 \). Then by the same argument as in (1) we can show that there exists a positive definite half integral matrix \( A \) of degree \( n \) such that \( \det(2A) = q^2 \) such that \( A \sim \mathbf{Z}_p \) for any \( p \). Let \( q = 2 \). Then the matrix \( A' = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \end{pmatrix}^{(n-4)/8} \) is a positive definite and \( \det(2A') = 4 \). Thus the matrix \( A' \ell E_8 \ell \cdots \ell E_8 \) satisfies the required condition.

**Proposition 4.6.** Let \( k \) and \( n \) be positive even integer. Let \( d \) be a fundamental discriminant. Let \( f \) be a primitive form in \( S_{2k-n}(\Gamma_1) \). Let \( \mathfrak{p} \) be a prime ideal in \( K \). Then there exists a positive definite half integral matrices \( A \) of degree \( n \) such that \( c_{L_n(f)}(A) = c_f(|d|) q \) with \( q \) not divisible by \( \mathfrak{p} \).
Proof. First assume that \( d \neq 1 \), or \( n \neq 4 \) mod 8. (1) By (1) and (2) of Lemma 4.5, there exists a matrix \( A \) such that \( v_A = d \). Thus we have \( c_{\ell_n(f)}(A) = c_f(|d|) \). This proves the assertion.

Next assume that \( n \equiv 4 \) mod 8 and that \( d = 1 \). Assume that \( c_f(q) + q^{k-n/2-1}(-q - 1) \) is divisible by \( \wp \) for any prime number \( q \). Let \( p \) be a prime number divisible by \( \wp \). Fix an imbedding \( \iota_p : \mathbb{Q} \rightarrow \overline{\mathbb{Q}}_p \), and let \( \rho_{f,p} : Gal(\mathbb{Q}/\mathbb{Q}) \rightarrow GL_2(\overline{\mathbb{Q}}_p) \) be the Galois representation attached to \( f \). Then by Chebotarev density theorem, the semi-simplification \( \overline{\rho}_{f,p}^{ss} \) of \( \overline{\rho}_{f,p} \) can be expressed as

\[
\overline{\rho}_{f,p}^{ss} = \chi_p^{k-n/2} \oplus \chi_p^{k-n/2-1}
\]

with \( \chi_p \) the \( p \)-adic mod \( p \) cyclotomic character. On the other hand, by the Fontaine-Messing [Fo-Me] and Fontaine-Laffaille [Fo-La], \( \overline{\rho}_{f,p}/I_p \) should be \( \chi_p^{2k-n-1} \oplus 1 \) or \( \omega_2^{2k-n-1} \oplus \omega_2^{p(2k-n-1)} \) with \( \omega_2 \) the fundamental character of level 2, where \( I_p \) denotes the inertia group of \( p \) in \( Gal(\mathbb{Q}/\mathbb{Q}) \). This is impossible because \( k > 2 \). Thus there exists a prime number \( q \) such that \( c_f(q) + q^{k-n/2-1}(-q - 1) \) is not divisible by \( \wp \). For such a \( q \), take a positive definite matrix \( A \) in (3) of Lemma 4.5. Then

\[
c_{\ell_n(f)}(A) = c(1)q^{k-(n+1)/2}B_q(A, q^{-(n+1)/2}B_q^{-1}).
\]

By [Ka1], we have

\[
F_q(B, X) = 1 - Xq^{(n-2)/2}(q^2 + q) + q^3(Xq^{(n-2)/2}).
\]

Thus we have

\[
c_{\ell_n(f)}(A) = c(1)(c_f(q) + q^{k-n/2-1}(-q - 1)).
\]

Thus the assertion holds.

**Theorem 4.7.** Let \( k \geq 2n+4 \). Let \( K \) and \( f \) be as above. Assume that the Conjecture B holds for \( f \). Let \( \wp \) be a prime ideal of \( K \). Furthermore assume that

1. \( \wp \) divides \( L(k, f) \prod_{i=1}^{n/2-1} L(2i + 1, f, St) \).
2. \( \wp \) does not divide

\[
\tilde{\xi}(2m) \prod_{i=1}^{n} L(2m + k - i, f)L(k - n/2, f, \chi_D)D(2k - 1)!
\]

for some integer \( n/2 + 1 \leq m \leq k/2 - n/2 - 1 \), and for some fundamental discriminant \( D \) such that \((-1)^{n/2}D > 0 \).

Then \( \wp \) is a congruence prime of \( I_{\wp}(f) \) with respect to \( CI_{\wp}(f) \). Furthermore assume that the following condition hold:
(3) \( \mathfrak{P} \) does not divide

\[
\langle f, f \rangle \frac{\Omega(f, +, A_{\mathfrak{P}})\Omega(f, -; A_{\mathfrak{P}})}{C_{k,n}},
\]

where \( C_{k,n} = 1 \) or \( \prod_{q \leq (2k-n)/12} (1 + q + \cdots + q^{n-1}) \) according as \( n = 2 \) or not.

Then \( \mathfrak{P} \) is a congruence prime of \( I_n(f) \) with respect to \( (\mathfrak{S}_k(\Gamma_n)^*)^\perp \).

Proof. Let \( \mathfrak{P} \) be a prime ideal satisfying the condition (1) and (2). For the \( D \) above, take a matrix \( A \in \mathcal{H}_n(\mathbb{Z}_{>0}) \) so that \( c_{I_n(f)}(A) = c_f(|D|)q \) with \( q \) not divisible by \( \mathfrak{P} \). Then by Proposition 4.3, we have

\[
\Lambda(2m, I_n(f), St)|c_{I_n(f)}(A)|^2 = \Lambda(2m, I_n(f), St)|c_f(|D|)|^2 q^2
\]

\[
\epsilon_{k,m} \frac{\prod_{i=1}^n L(2m + k - i, f)|D|^{k-n/2}L(k-n/2, f, \chi_D)}{L(k, f)\xi(n)\prod_{i=1}^{n/2-1} L(2i + 1, f, St)\xi(2i)} \times \frac{\Omega(f, +; \mathfrak{P})\Omega(f, -; A_{\mathfrak{P}})}{\langle f, f \rangle^2},
\]

where \( \epsilon_{k,m} \) is a rational number whose numerator is not divided by \( \mathfrak{P} \).

We note that \( \frac{\langle f, f \rangle}{\Omega(f, +; A_{\mathfrak{P}})\Omega(f, -; A_{\mathfrak{P}})} \) is \( \mathfrak{P} \)-integral. Thus by assumptions (1) and (2), \( \mathfrak{P} \) divides \( \Lambda(2m, I_n(f), St)c_{I_n(f)}(A)^2 \), and thus it divides \( \Lambda(2m, I_n(f), St)c_{I_n(f)}^2 \). We note that \( I_n(f) \) satisfies the assumption in Theorem 3.1. Thus by Theorem 3.1, there exits a Hecke eigenform \( G \in \mathbb{C}(I_n(f)) \) such that

\[
\lambda_G(T) \equiv \lambda_{I_n(f)}(T) \mod \mathfrak{P}
\]

for any \( T \in \mathbb{L}_n' \). Assume that we have \( G = I_n(g) \) with some primitive form \( g(z) = \sum_{m=1}^{\infty} a_g(m)e(mz) \in \mathfrak{S}_{2k-n}(\Gamma(1)) \). Let \( n = 2 \). Then by (1) of Proposition 4.2, \( \mathfrak{P} \) is also a congruence prime of \( f \). Let \( n \geq 4 \). Then by (1) of Proposition 4.4, we have

\[
(p^{n-1} + \cdots + p + 1)a_f(p) \equiv (p^{n-1} + \cdots + p + 1)a_g(p) \mod \mathfrak{P}
\]

for any prime number \( p \) not divisible by \( \mathfrak{P} \). By assumption (3), in particular, for any \( p \leq (2k-n)/12 \), we have

\[
a_f(p) \equiv a_g(p) \mod \mathfrak{P}.
\]

Thus by Sturm [Stur], \( \mathfrak{P} \) is also a congruence prime of \( f \). Thus by [Hi2] and [Ri2], \( \mathfrak{P} \) divides \( \frac{\langle f, f \rangle}{\Omega(f, +; A_{\mathfrak{P}})\Omega(f, -; A_{\mathfrak{P}})} \), which contradicts the assumption (3). Thus \( \mathfrak{P} \) is a congruence prime of \( I_n(f) \) with respect to \( (\mathfrak{S}_k(\Gamma(n))^*)^\perp \).
Example Let \( n = 4 \) and \( k = 18 \). Then we have \( \text{dim } S_{18}(\Gamma_4) \approx 16 \) (cf. Poor and Yuen[PY]) and \( \text{dim } S_{18}(\Gamma_4)^* = \text{dim } S_{32}(\Gamma_1) = 2 \). Take a primitive form \( f \in S_{32}(\Gamma_1) \). Then we have \([Q(f) : Q] = 2\), and \( 211 = \mathfrak{p}' \) in \( Q(f) \). Then we have

\[
N_{Q(f)/Q}(L(18, f)) = 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 211,
\]

\[
N_{Q(f)/Q}(\prod_{i=1}^{4} L(24-i, f)) = 2^{19} \cdot 3^{13} \cdot 5^5 \cdot 7^8 \cdot 11^2 \cdot 13^5 \cdot 17^5 \cdot 19^3 \cdot 23 \cdot 503 \cdot 1307 \cdot 14243,
\]

\[
\tilde{\xi}(6) = 2^{-2} \cdot 3^{-2} \cdot 7^{-1}
\]

and

\[
N_{Q(f)/Q}(L(16, f, \chi_1)) = 2^5 \cdot 3^2 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13^2.
\]

(cf. Stein [Ste].) Furthermore, by a direct computation we see neither \( \mathfrak{p} \) nor \( \mathfrak{p}' \) is a congruence prime of \( \hat{f} \) with respect to \( Cg \) for another primitive form \( g \in S_{32}(\Gamma_1) \). Thus by Theorem 4.7, \( \mathfrak{p} \) or \( \mathfrak{p}' \) is a congruence prime of \( \hat{f} \) with respect to \( S_{18}(\Gamma_4)^{\perp} \).

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