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Great circular surfaces in the three-sphere

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Abstract

In this paper, we consider a special class of the surfaces in 3-sphere defined by one-parameter families of great circles. We give a generic classification of singularities of such surfaces and investigate the geometric meanings from the view point of spherical geometry.

1 Introduction

In this paper we investigate a special class of surfaces in 3-sphere which are called *great circular surfaces*. We say that a surface in 3-sphere is a *great circular surface* if it is given by a one-parameter family of great circles (cf., §4).

On the other hand, there appeared two kinds of curvatures in the previous theory of surfaces in 3-sphere, One is called the *extrinsic Gauss curvature* K_e and another is the *intrinsic Gauss curvature* K_I. The intrinsic Gauss curvature is nothing but the Gauss curvature defined by the induced Riemannian metric on the surface. The relation between these curvatures is known that $K_e = K_I - 1$. We can show that an extrinsic flat surface is (at least locally) parametrized as a great circular surface (cf., Theorem 3.3). Such a surface is an *extrinsic flat great circular surface* (briefly, we call an *E-flat great circular surface*). This is one of the motivation to investigate great circular surfaces. In Euclidean space, surfaces with the vanishing Gauss curvature are developable surfaces which belong to a special class of ruled surfaces [5, 6]. Therefore, the notion of great circular surfaces is one of the analogous notions with ruled surfaces in 3-sphere. In this paper, we study geometric properties and singularities of great circular surfaces. However, there is the canonical double covering $\pi : S^3 \longrightarrow \mathbb{RP}^3$ onto the projective space. A great circle corresponds to a projective line in $\mathbb{RP}^3$, so that the singularities of great circular surfaces are the same as those of ruled surfaces. There are a lot of researches on developable surfaces in $\mathbb{R}^3 \subset \mathbb{RP}^3$ from the view point of Projective differential geometry [2, 4, 12, 16]. We investigate the singularities of great circular surfaces from the view point of spherical geometry (i.e, $SO(4)$-invariant geometry).

For any smooth curve $A : I \longrightarrow SO(4)$ in the rotation group $SO(4)$, we can define a parametrization F_A of a great circular surface $M = \text{Image}\, F_A$ in 3-sphere. We can easily show that $C = A'A^{-1}$ is a smooth curve in the Lie algebra $\mathfrak{so}(4)$ of $SO(4)$. We can also obtain the

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curve A in $SO(4)$ with initial data $A(t_0) = A_0$ from C by the existence theorem of the linear ordinary differential equations. In this sense, $C(t)$ is a spherical invariant of great circular surfaces. We remark that $C(t)$ is an anti-symmetric matrix:

$$C(t) = \begin{pmatrix} 0 & c_1(t) & c_2(t) & c_3(t) \\ -c_1(t) & 0 & c_4(t) & c_5(t) \\ -c_2(t) & -c_4(t) & 0 & c_6(t) \\ -c_3(t) & -c_5(t) & -c_6(t) & 0 \end{pmatrix}.$$

Therefore we consider that the space of great circular surfaces is the space of smooth mappings $C^\infty(I, \mathfrak{so}(4))$ equiped with the Whitney C^∞-topology, where

$$\mathfrak{so}(4) = \left\{ C = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ -c_1 & 0 & c_4 & c_5 \\ -c_2 & -c_4 & 0 & c_6 \\ -c_3 & -c_5 & -c_6 & 0 \end{pmatrix} \middle| \; c_i \in \mathbb{R}, (i = 1, 2, 3, 4, 5, 6) \right\} = \mathbb{R}^6.$$

A generic classification of singularities of general great circular surfaces is given as follows (cf., Theorem 4.5 and Theorem 8.1):

Theorem 1.1. *There exists an open and dense subset $\mathcal{O} \subset C^\infty(I, \mathfrak{so}(4))$ such that $F_A(\theta, t)$ has only cross caps as singular points for any $C \in \mathcal{O}$.*

Here, we say that a singular point (θ, t) of F_A is the cross cap if the germ of the surface $F_A(\mathbb{R} \times I)$ at $F_A(\theta, t)$ is (locally) diffeomorphic to $CR = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = u, x_2 = uv, x_3 = v^2\}$

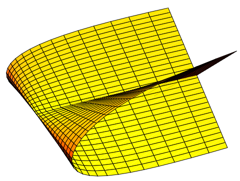


Fig.1: cross cap

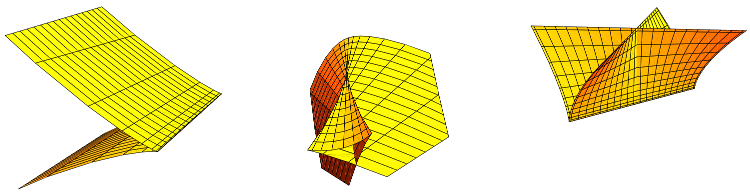
In §5 we show that a great circular surface ImageF_A is extrinsic flat if and only if $c_1(t) = c_3(t) = 0$. Therefore we may regard that the space of (parametrizations of) E-flat great circular surfaces is $C^\infty(I, \mathfrak{ef}(4))$ as a subspace of $C^\infty(I, \mathfrak{so}(4))$, where

$$\mathfrak{ef}(4) = \left\{ C = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ -c_1 & 0 & c_4 & c_5 \\ -c_2 & -c_4 & 0 & c_6 \\ -c_3 & -c_5 & -c_6 & 0 \end{pmatrix} \in \mathfrak{so}(4) \middle| \; c_1 = c_3 = 0 \right\} = \mathbb{R}^4.$$

One of the main results in this paper is a generic classification of singularities of extrinsic flat great circular surfaces by using the spherical invariant $C(t)$. Our classification theorem is summarized as follows (cf., Theorem 5.1 and Theorem 8.1):

Theorem 1.2. *There exists an open and dense subset $\mathcal{O} \subset C^\infty(I, \mathfrak{ef}(4))$ such that a singular point of $F_A(\theta, t)$ is the cuspidal edge, the swallowtail or the cuspidal cross cap for any $C \in \mathcal{O}$.*

Here, we say that a singular point (s,t) of F_A is the cuspidal edge (respectively swallowtail and cuspidal cross cap) if the germ of the surface $F_A(\mathbb{R}\times I)$ at $F_A(s,t)$ is (locally) diffeomorphic to $CE = \{(x_1,x_2,x_3)|{x_1}^2 = {x_2}^3\}$ (respectively, $SW = \{(x_1,x_2,x_3)|x_1 = 3u^4+u^2v, x_2 = 4u^3+2uv, x_3 = v\}$ and $CCR = \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1 = u, x_2 = uv^3, x_3 = v^2\}$.



The cuspidal edge The swallowtail The cuspidal cross cap
Fig. 2.

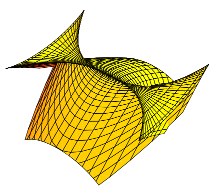
We have another interesting class of E-flat great circular surfaces. In §5, we show that each generating great circle is tangent to the regular part of the singular locus of the E-flat great circular surface $F_A(\theta,t)$ if and only if $c_1(t) = c_3(t) = c_4(t) = 0$. Such the surface is called a *tangential extrinsic flat great circular surface* (briefly, *T-E-flat great circular surface*). Therefore, we consider that the space of T-E-flat great circular surfaces is given by $C^\infty(I,\mathfrak{ef}_\tau(4))$ as a subspace of $C^\infty(I,\mathfrak{so}(4))$, where

$$\mathfrak{ef}_\tau(4) = \left\{ C = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ -c_1 & 0 & c_4 & c_5 \\ -c_2 & -c_4 & 0 & c_6 \\ -c_3 & -c_5 & -c_6 & 0 \end{pmatrix} \in \mathfrak{so}(4) \;\middle|\; c_1 = c_3 = c_4 = 0 \right\} = \mathbb{R}^3.$$

We have the following generic classification of singularities of T-E-flat great circular surfaces (cf., Theorem 5.2 and Theorem 8.1):

Theorem 1.3. *There exists an open and dense subset $\mathcal{O}\subset C^\infty(I,\mathfrak{ef}_\tau(4))$ such that a singular point of $F_A(\theta,t)$ is the cuspidal edge, the swallowtail, the cuspidal cross cap or the cuspidal beaks for any $C\in\mathcal{O}$.*

Here, we say that a singular point (s,t) of F_A is the cuspidal beaks if the germ of the surface $F_A(\mathbb{R}\times I)$ at $F_A(s,t)$ is (locally) diffeomorphic to $CBK = \{(x_{1,2}, x_3) \,|x_1 = v.x_2 = -2u^3+v^2u, x_3 = 3u^4-v^2u^2 \}$.



The cuspidal beaks
Fig. 3.

Therefore we can show that an extrinsic flat great circular surface is locally diffeomorphic to a developable surface in the Euclidean sense, so that generic singularities of extrinsic flat great circular surfaces are the same as those of developable surfaces. In §8, we present a dual relations among singularities of T-E-flat great circular surfaces. Comparing with the duality among surfaces in Euclidean 3-space, we can observe that the spherical duality gives beautiful dual relations [14]. In §9, we give three examples of great circular surfaces associated to a Frenet curve. Especially, the binormal great circular surface has the cross cap when $\tau_g(s_0) = 0$ and $\tau'_g(s_0) \neq 0$. In Euclidean 3-space, the binormal ruled surface is always non-singular, so that the situations are different.

All maps considered here are of class C^∞ unless otherwise stated.

2 Differential geometry of curves and surfaces in 3-sphere

We outline in this section the differential geometry of curves and surfaces in 3-sphere (cf., [13]).

Let S^3 be an 3-dimensional unit sphere in Euclidean space $\mathbb{R}^4$. Given a vector $\boldsymbol{n} \in \mathbb{R}^4 \setminus \{0\}$ and a real number c, the hyperplane with normal $\boldsymbol{n}$ is given by $HP(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^4 \mid \boldsymbol{n} \cdot \boldsymbol{x} = c\}$, where $\boldsymbol{v} \cdot \boldsymbol{v}$ is the canonical inner product. A sphere in S^3 is given by

$$S^2(\boldsymbol{n}, c) = S^3 \cap H(\boldsymbol{n}, c) = \{\boldsymbol{x} \in S^3 \mid \boldsymbol{n} \cdot \boldsymbol{x} = c\}.$$

We say that $S^2(\boldsymbol{n}, c)$ is a *great sphere* if $c = 0$, a *small hypersphere* if $c \neq 0$. For any $\boldsymbol{a}_{(i)} = \left(a^1_{(i)}, a^2_{(i)}, a^3_{(i)}, a^4_{(i)}\right) \in \mathbb{R}^4$ $(i = 1, 2, 3)$, the vector product $\boldsymbol{a}_{(1)} \times \boldsymbol{a}_{(2)} \times \boldsymbol{a}_{(3)}$ is defined by

$$\boldsymbol{a}_{(1)} \times \boldsymbol{a}_{(2)} \times \boldsymbol{a}_{(3)} = \det \begin{pmatrix} \boldsymbol{e}_{(1)} & \boldsymbol{e}_{(2)} & \boldsymbol{e}_{(3)} & \boldsymbol{e}_{(4)} \\ a^1_{(1)} & a^2_{(1)} & a^3_{(1)} & a^4_{(1)} \\ a^1_{(2)} & a^2_{(2)} & a^3_{(1)} & a^4_{(2)} \\ a^1_{(3)} & a^2_{(3)} & a^3_{(3)} & a^3_{(3)} \end{pmatrix},$$

where $\left\{\boldsymbol{e}_{(1)}, \boldsymbol{e}_{(2)}, \boldsymbol{e}_{(3)}, \boldsymbol{e}_{(4)}\right\}$ is the canonical basis of $\mathbb{R}^4$. We can easily show that $\boldsymbol{a}_{(1)} \times \boldsymbol{a}_{(2)} \times \boldsymbol{a}_{(3)}$ is orthogonal to any $\boldsymbol{a}_{(i)}$ $(i = 1, 2, 3)$.

We now construct the extrinsic differential geometry on curves in S^3. Let $\boldsymbol{\gamma} : I \longrightarrow S^3$ be a regular curve. Since S^3 is a Riemannian manifold, we can reparametrize $\boldsymbol{\gamma}$ by the arc-length. Hence, we may assume that $\boldsymbol{\gamma}(s)$ is a unit speed curve. So we have the tangent vector $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ with $\|\boldsymbol{t}(s)\| = 1$. In the case when $\boldsymbol{t}'(s) \cdot \boldsymbol{t}'(s) \neq 1$, we have a unit vector $\boldsymbol{n}(s) = (\boldsymbol{t}'(s) + \boldsymbol{\gamma}(s))/(\|\boldsymbol{t}'(s) + \boldsymbol{\gamma}(s)\|)$. Moreover, define $\boldsymbol{e}(s) = \boldsymbol{\gamma}(s) \times \boldsymbol{t}(s) \times \boldsymbol{n}(s)$, then we have an orthonormal frame $\{\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ of $\mathbb{R}^4$ along $\boldsymbol{\gamma}$. By standard arguments, under the assumption that $\boldsymbol{t}'(s) \cdot \boldsymbol{t}'(s) \neq 1$, we have the following *Frenet-Serre type formulae*:

$$\begin{cases} \boldsymbol{\gamma}'(s) &= \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) &= \kappa_g(s)\boldsymbol{n}(s) - \boldsymbol{\gamma}(s) \\ \boldsymbol{n}'(s) &= -\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{e}(s) \\ \boldsymbol{e}'(s) &= -\tau_g(s)\boldsymbol{n}(s) \end{cases}, \tag{2.1}$$

where $\kappa_g(s) = \|\boldsymbol{t}'(s) + \boldsymbol{\gamma}(s)\|$ and $\tau_g(s) = -\dfrac{\det(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s))}{(\kappa_g(s))^2}$.

We can easily show that the condition $\boldsymbol{t}'(s) \cdot \boldsymbol{t}'(s) \neq 1$ is equivalent to the condition $\kappa_g(s) \neq 0$. We can show that the curve $\boldsymbol{\gamma}(s)$ satisfies the condition $\kappa_g(s) \equiv 0$ if and only if $\boldsymbol{\gamma}(s)$ is a great circle (i.e., the geodesic). We can study many properties of curves in the 3-sphere by using this fundamental equation.

On the other hand, we give a brief review on the extrinsic differential geometry on surfaces in S^3. Let $\boldsymbol{X} : U \longrightarrow S^3$ be a regular surface (i.e., an embedding), where $U \subset \mathbb{R}^2$ is an open subset. We denote that $M = \boldsymbol{X}(U)$ and identify M with U through the embedding $\boldsymbol{X}$. Define a vector

$$\boldsymbol{e}(u) = \frac{\boldsymbol{X}(u) \times \boldsymbol{X}_{u_1}(u) \times \boldsymbol{X}_{u_2}(u)}{\|\boldsymbol{X}(u) \times \boldsymbol{X}_{u_1}(u) \times \boldsymbol{X}_{u_2}(u)\|},$$

then we have $\boldsymbol{e} \cdot \boldsymbol{X}_{u_i} \equiv \boldsymbol{e} \cdot \boldsymbol{X} \equiv 0, \boldsymbol{e} \cdot \boldsymbol{e} \equiv 1$, where $\boldsymbol{X}_{u_i} = \partial \boldsymbol{X}/\partial u_i$. Therefore we have a mapping

$$\mathbb{G} : U \longrightarrow S^3$$

by $\mathbb{G}(u) = \boldsymbol{e}(u)$ which is called the *Gauss map* of $M = \boldsymbol{X}(U)$. It is easy to show that the surface $M = \boldsymbol{X}(U)$ is a part of a great sphere if and only if one of the Gauss map $\mathbb{G}$ is constant. It is well known that $D_v \mathbb{G} \in T_pM$ for any $p = \boldsymbol{X}(u_0) \in M$ and $\boldsymbol{v} \in T_pM$, where D_v denotes the *covariant derivative* with respect to the tangent vector $\boldsymbol{v}$. This means that $d\mathbb{G}(u_0)$ can be considered as a linear transformation of T_pM. We call the linear transformation $S_p = -d\mathbb{G}^{\pm}(u_0) : T_pM \longrightarrow T_pM$ the *shape operator* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$. We denote the eigenvalues of S_p by $\kappa_i(p)$ $(i = 1, 2)$. We call $\kappa_i(p)$ *principal curvatures* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$. We now describe the geometric meaning of the principal curvatures. Let $\boldsymbol{\gamma}(s) = \boldsymbol{X}(u_1(s), u_2(s))$ be a unit speed curve on $M = \boldsymbol{X}(U)$ with $p = \boldsymbol{\gamma}(s_0)$. We consider the *spherical curvature vector* $\boldsymbol{k}(s) = \boldsymbol{t}'(s) + \boldsymbol{\gamma}(s)$ and the *normal curvature*

$$\kappa_n(s_0) = \boldsymbol{k}(s_0) \cdot \mathbb{G}(u_1(s_0), u_2(s_0)) = \boldsymbol{t}'(s_0) \cdot \mathbb{G}(u_1(s_0), u_2(s_0))$$

of $\boldsymbol{\gamma}(s)$ at $p = \boldsymbol{\gamma}(s_0)$. We can show that the spherical normal curvature depends only on the point p and the unit tangent vector of M at p analogous to the Euclidean case. Therefore we have the maximum and the minimum of the spherical normal curvature at $p \in M$. We can also show that the principal curvatures $\kappa_i(p)$ are equal to the maximum or the minimum of the spherical normal curvature at p. Then we have the following spherical Rodoriges type formula: If $\boldsymbol{\gamma}(s) = \boldsymbol{X}(u_1(s), u_2(s))$ is a line of curvature, then $\kappa_n(s)$ is one of the principal curvatures at $\boldsymbol{\gamma}(s)$, so that we have

$$-\frac{d\mathbb{G}}{ds}(u_1(s), u_2(s)) = \kappa_n(s) \frac{d\boldsymbol{X}}{ds}(u_1(s), u_2(s)).$$

The *spherical Gauss-Kronecker curvature* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$ is defined to be

$$K_e(u_0) = \det S_p = \kappa_1(p)\kappa_2(p).$$

The *spherical mean curvature* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$ is defined to be

$$H_e(u_0) = \frac{1}{2}\mathrm{Trace} S_p = \frac{\kappa_1(p) + \kappa_2(p)}{2}.$$

We also call $K_e(u_0)$ the *extrinsic spherical Gauss curvature.*

We say that a point $u \in U$ or $p = \boldsymbol{X}(u)$ is an *umbilical point* if $\kappa_1(p) = \kappa_2(p)$. We say that $M = \boldsymbol{x}(U)$ is *totally umbilical* if all points on M are umbilical. The following proposition is a well-known result:

Proposition 2.1. *Suppose that $M = \boldsymbol{X}(U)$ is totally umbilic. Then $\kappa(p)$ is constant κ. Under this condition, we have the following classification:*

(1) *If $\kappa = 0$, then M is a part of a great hypersphere.*

(2) *If $\kappa \neq 0$, then M is a part of a small hypersphere.*

We establish next the spherical version of the Weingarten formula. We have the Riemannian metric (*spherical first fundamental form*) given by $ds^2 = \sum_{i=1}^{2} g_{ij} du_i du_j$ on $M = \boldsymbol{X}(U)$, where $g_{ij}(u) = \boldsymbol{X}_{u_i}(u) \cdot \boldsymbol{X}_{u_j}(u)$ and the *spherical second fundamental invariant* defined by $h_{ij}(u) = -\mathbb{G}_{u_i}(u) \cdot \boldsymbol{X}_{u_j}(u)$ for any $u \in U$. It is easy to show the following (cf., [13]):

Proposition 2.2. *Under the above notations, we have the following formula:*

$$\mathbb{G}_{u_i} = -\sum_{j=1}^{2} h_i^j \boldsymbol{X}_{u_j} \quad (The\ spherical\ Weingarten\ formula),$$

where $\left(h_i^j\right) = (h_{ik})\left(g^{kj}\right)$ *and* $\left(g^{kj}\right) = (g_{kj})^{-1}$.

As a corollary of the above proposition, we have an explicit expression of the spherical extrinsic Gauss curvature in terms of the Riemannian metric and the spherical second fundamental invariant.

Corollary 2.3. *Under the same notations as in the above proposition, we have the following formulae:*

$$K_e = \frac{\det(h_{ij})}{\det(g_{\alpha\beta})}.$$

We now consider the Riemannian curvature tensor

$$R_{ijk}^{\ell} = \frac{\partial}{\partial u_k}\begin{Bmatrix} \ell \\ i\ j \end{Bmatrix} - \frac{\partial}{\partial u_j}\begin{Bmatrix} \ell \\ i\ k \end{Bmatrix} + \sum_m \begin{Bmatrix} m \\ i\ j \end{Bmatrix}\begin{Bmatrix} \ell \\ m\ k \end{Bmatrix} - \sum_m \begin{Bmatrix} m \\ i\ k \end{Bmatrix}\begin{Bmatrix} \ell \\ m\ j \end{Bmatrix}.$$

We also consider the tensor $R_{ijk\ell} = \sum_m g_{im} R_{jk\ell}^m$. Standard calculations, analogous to those used in the study of the classical differential geometry on surfaces in Euclidean space, lead to the following:

Proposition 2.4. *Under the above notations, we have*

$$K_e = -\frac{R_{1212}}{g} - 1,$$

where $g = \det(g_{\alpha\beta})$.

We remark that $-R_{1212}/g$ is the *intrinsic Gaussian curvature* of the surface $M = \boldsymbol{X}(U)$. It is denoted by K_I, so that we have $K_e = K_I - 1$.

We now consider the spherical duality from the view point of contact geometry. We briefly review some properties of contact manifolds and Legendrian submanifolds [1, Part III]. Let W be a $2n+1$-dimensional smooth manifold and K be a tangent hyperplane field on W. Locally such a field is defined as the field of zeros of a 1-form α. If tangent hyperplane field K is non-degenerate, we say that (W, K) is a *contact manifold*. Here K is said to be *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of W. In this case K is called a *contact structure* and α is a *contact form.* A submanifold $i : L \subset W$ of a contact manifold (W, K) is a *Legendrian submanifold* if $\dim L = n$ and $di_p(T_pL) \subset K_{i(p)}$ at any point $p \in L$. We consider a smooth fiber bundle $\pi : N \to A$. The fiber bundle $\pi : N \to A$ is called a *Legendrian fibration* if its total space W is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : N \to A$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset N$, a map $\pi \circ i : L \to A$ is called a *Legendrian map* . The image of the Legendrian map $\pi \circ i$ is called a *wavefront set* of i which is denoted by $W(i)$.

We now consider the following double fibrations of S^3:

$$\Delta = \{(\boldsymbol{v}, \boldsymbol{w}) \in S^3 \times S^3 | \boldsymbol{v} \cdot \boldsymbol{w} = 0\},$$
$$\pi_1 \ : \ \Delta \ni (\boldsymbol{v} \ , \ \boldsymbol{w}) \longmapsto \boldsymbol{v} \in S^3, \quad \pi_2 \ : \ \Delta \ni (\boldsymbol{v} \ , \ \boldsymbol{w}) \longmapsto \boldsymbol{w} \in S^3,$$
$$\theta_1 = d\boldsymbol{v} \cdot \boldsymbol{w}|_\Delta \quad \theta_2 = \boldsymbol{v} \cdot d\boldsymbol{w}|_\Delta.$$

Here, $d\boldsymbol{v} \cdot \boldsymbol{w} = \sum_{i=1}^4 w_i dv_i$ and $\boldsymbol{v} \cdot d\boldsymbol{w} = \sum_{i=1}^4 v_i dw_i$. Since $d(\boldsymbol{v} \cdot \boldsymbol{w}) = d\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot d\boldsymbol{w}$ and $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ on Δ, $\theta_1^{-1}(0)$ and $\theta_2^{-1}(0)$ define the same tangent hyperplane field over Δ which is denoted by K. The following proposition is well-known.

Proposition 2.5. *Under the above notation,* (Δ, K) *is a contact manifold and both of* π_i *are Legendrian fibrations.*

We now interpret the Gauss map of a hypersurface in S^3 as a wave front set in the above contact manifold. For any regular hypersurface $\boldsymbol{X} : U \to S^3$, we have $\boldsymbol{X}(u) \cdot \boldsymbol{e}(u) = 0$. Therefore we can define an embedding $\mathcal{L}_{\boldsymbol{X}} : U \to \Delta$ by $\mathcal{L}_{\boldsymbol{X}}(u) = (\boldsymbol{X}(u), \boldsymbol{e}(u)) = (\boldsymbol{X}(u), \mathbb{G}(u))$.

Proposition 2.6. *The mapping* $\mathcal{L}$ *is a Legendrian embedding to the contact manifold* (Δ, K).

Proof. Since $\boldsymbol{X} : U \to S^3$ is an embedding, $\mathcal{L}_{\boldsymbol{X}}$ is also an embedding and $\dim(\mathcal{L}_{\boldsymbol{X}}(U)) = 2$. Since $\mathcal{L}_{\boldsymbol{X}}^* \theta_1 = d\boldsymbol{X} \cdot \boldsymbol{e} = 0$, $\mathcal{L}_{\boldsymbol{X}}$ is a Legendrian embedding. This completes the proof. $\Box$

By definition, we have $\pi_2 \circ \mathcal{L}(U) = \mathbb{G}(U)$. Then we have the following corollary:

Corollary 2.7. *For any hypersurface* $\boldsymbol{X} : U \to S_3$, $\mathbb{G}(U)$ *is a wave front set of* $\mathcal{L}_{\boldsymbol{X}}(U)$ *with respect to the Legendrian fibration* π_2.

We say that a C^∞-mapping $\mathcal{L} : U \longrightarrow \Delta$ is an *isotropic mapping* if $\mathcal{L}^*\theta_i = 0$ $(i = 1$ or $2)$. We remark that the istropic mapping is Legendrian immersion if it is an immersion. If we have an isotropic mapping $\mathcal{L} : U \longrightarrow \Delta$, then we say that $\pi_1 \circ \mathcal{L}(U)$ and $\pi_2 \circ \mathcal{L}(U)$ are Δ-*dual* to each other. By Corollary 2.7, $\boldsymbol{X}$ and $\mathbb{G}$ are Δ-dual to each other.

Differential geometric properties of $\boldsymbol{X}$ corresponding to the A_k-singularities of its Gauss map $\mathbb{G}$ are investigated in [15].

3 Extrinsic flat surfaces

In this section we consider surfaces with vanishing extrinsic Gauss curvature. We say that a surface $M = \boldsymbol{x}(U)$ is an *extrinsic flat surface* (briefly, *E-flat surface*) if $K_e(p) = 0$ at any point $p \in M$. By Proposition 2.4, $K_e(p) = 0$ if and only if $K_I(p) = 1$.

One of the typical E-flat surfaces is the great sphere which is the totally umbilical surface with the vanishing curvature. If we suppose that a surface is umbilically free, then we have the following expression: Let $\boldsymbol{X} : U \longrightarrow S^3$ be an E-flat surface without umbilical points, where $U \subset \mathbb{R}^2$ is a neighborhood around the origin. In this case, we have two lines of curvature at each point and one of which corresponds to the vanishing principal curvature. We may assume that both the u-curve and the v-curve are the lines of curvature for the coordinate system $(u, v) \in U$.

Moreover, we assume that the u-curve corresponds to the vanishing principal curvature. By the spherical Weingarten formula (Proposition 2.2), we have

$$\mathbb{G}_u(u,v) = \mathbf{0} \quad \text{and} \quad \mathbb{G}_v(u,v) = -\kappa(u,v)\boldsymbol{X}_v(u,v),$$

where $\kappa(u,v) \neq 0$. It follows that $\mathbb{G}(0,v) = \mathbb{G}(u,v)$. We define a function $F : S^3 \times (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ by

$$F(\boldsymbol{x}, v) = \mathbb{G}(0,v) \cdot \boldsymbol{x},$$

for sufficiently small $\varepsilon > 0$. For any fixed $v \in (-\varepsilon, \varepsilon)$, we have a great sphere $S^2(\mathbb{G}(0,v), 0)$, so that $F = 0$ define a one-parameter family of great spheres. We have the following proposition.

Proposition 3.1. *Under the above notations, the surface $M = \boldsymbol{X}(U)$ is a part of the envelope of the family of greatspheres defined by $F = 0$.*

Proof. The envelope defined by $F = 0$ is the surface (might be singular) satisfying the condition $F = F_v = 0$. Here we have

$$F_v(\boldsymbol{x}, v) = \mathbb{G}_v(0,v) \cdot \boldsymbol{x} = -\kappa(0,v)(\boldsymbol{X}_v(0,v) \cdot \boldsymbol{x}).$$

We now consider the function $H(u,v) = F(\boldsymbol{X}(u,v), v)$, then

$$H(0,v) = F(\boldsymbol{X}(0,v), v) = \mathbb{G}(0,v) \cdot \boldsymbol{X}(0,v) = 0.$$

We also have $H_u(u,v) = \mathbb{G}(0,v) \cdot \boldsymbol{X}_u(u,v)$. Since $\mathbb{G}(0,v) = \mathbb{G}(u,v)$, we have $H_u(u,v) = \mathbb{G}(u,v) \cdot \boldsymbol{X}_u(u,v) = 0$ It follows that $H(u,v) = H(0,v) = 0$.

On the other hand, we consider a function $F_v(\boldsymbol{X}(u,v), v)$. By the same reason as the above arguments, we have $\mathbb{G}_v(u,v) = \mathbb{G}_v(0,v)$, so that

$$F_v(\boldsymbol{X}(u,v), v) = \mathbb{G}_v(0,v) \cdot \boldsymbol{X}(u,v) = \mathbb{G}_v(u,v) \cdot \boldsymbol{X}(u,v) = -\kappa(u,v)(\boldsymbol{X}_v(u,v) \cdot \boldsymbol{X}(u,v)).$$

Since $\boldsymbol{X}(u,v) \cdot \boldsymbol{X}(u,v) = 1$, we have $\boldsymbol{X}_v(u,v) \cdot \boldsymbol{X}(u,v) = 0$, so that $F_v(\boldsymbol{X}(u,v), v) = 0$. Therefore $\boldsymbol{X}(u,v)$ satisfies both the condition

$$F(\boldsymbol{X}(u,v), v) = F_v(\boldsymbol{X}(u,v), v) = 0.$$

This means that $M = \boldsymbol{X}(U)$ is a part of the envelope of the family of greatspheres defined by $F = 0$. $\square$

On the other hand, we consider a surface $\overline{\boldsymbol{X}} : J \times I \longrightarrow S^3$ defined by

$$\overline{\boldsymbol{X}}(\theta, v) = \cos\theta \boldsymbol{X}(0,v) + \sin\theta \frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|},$$

where $I \subset \mathbb{R}$ and $J \subset [0, 2\pi]$ are open intervals. We have the following proposition.

Proposition 3.2. *The surface $\overline{M} = \overline{\boldsymbol{X}}(J \times I)$ is a part of the envelope of the family of great spheres defined by $F = 0$.*

Proof. We remind that $\mathbb{G}(u,v)=\boldsymbol{e}(u,v)$ and $\boldsymbol{e}(u,v)$ is the unit normal of $M=\boldsymbol{X}(U)$ at $\boldsymbol{X}(u,v)$ with $\boldsymbol{e}(u,v)\cdot\boldsymbol{X}(u,v)=0$. It follows that

$$\mathbb{G}(0,v)\cdot\left(\cos\theta\boldsymbol{X}(0,v)+\sin\theta\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|}\right)=0,$$

so that $F(\overline{\boldsymbol{X}}(\theta,v),v)=0$. We remark that $\boldsymbol{X}(u,v)\cdot\boldsymbol{X}_v(u,v)=0$. Since $\mathbb{G}_v(0,v)=-\kappa(0,v)$ $\boldsymbol{X}_v(0,v)$, we have

$$\mathbb{G}_v(0,v)\cdot\left(\cos\theta\boldsymbol{X}(0,v)+\sin\theta\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|}\right)=-\frac{\sin\theta\kappa(0,v)}{\|\boldsymbol{X}_u(0,v)\|}(\boldsymbol{X}_v(0,v)\cdot\boldsymbol{X}_u(0,v)).$$

Since both the u-curve and the v-curve are the lines of curvature, $\boldsymbol{X}_v(0,v)\cdot\boldsymbol{X}_u(0,v)=0$. This means that $F_v(\overline{\boldsymbol{X}}(s,v),v)=0$. This completes the proof. □

By Propositions 3.1 and 3.2, an E-flat surface can be reparametrized (at least locally) by

$$\overline{\boldsymbol{X}}(\theta,v)=\cos\theta\boldsymbol{X}(0,v)+\sin\theta\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|}.$$

We now consider the meaning of the above parametrization. If we fix $v=v_0$, we denote that

$$\boldsymbol{a}_0=\boldsymbol{e}(0,v_0),\ \boldsymbol{a}_1=\boldsymbol{X}(0,v_0),\ \boldsymbol{a}_2=\frac{\boldsymbol{X}_v(0,v_0)}{\|\boldsymbol{X}_v(0,v_0)\|},\ \boldsymbol{a}_3=\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|}.$$

Then we have $\det(\boldsymbol{a}_0,\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_2)=1$. We define a curve by

$$\boldsymbol{\gamma}(\theta)=\cos\theta\boldsymbol{a}_1+\sin\theta\boldsymbol{a}_3.$$

Since $\boldsymbol{\gamma}'(\theta)=-\sin\theta\boldsymbol{a}_1+\cos\theta\boldsymbol{a}_3$, we have $\boldsymbol{\gamma}'(s)\cdot\boldsymbol{\gamma}'(s)=1$. Therefore $\boldsymbol{\gamma}(s)$ has the unit speed. Moreover, $\boldsymbol{\gamma}(\theta)$ is known to be the geodesic (the great circle) through $\boldsymbol{a}_1$ whose direction is given by $\boldsymbol{a}_3$. Therefore the E-flat surface is given by the one-parameter family of great circles. By the definition of $\boldsymbol{e}(u,v)$, we have $\det(\boldsymbol{a}_0,\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3)=1$, so that $A=(\boldsymbol{a}_0,\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3)\in SO(4)$. We say that a surface is a *great circular surface* if it is locally parametrized by one-parameter families of great circles around any point. Eventually we have the following theorem.

Theorem 3.3. *If $M\subset S^3$ is an umbilically free E-flat surface, then it is a great circular surface. Moreover, each great circle is the line of curvature with the vanishing principal curvature.*

Proof. The first part of the theorem is a direct consequence of the above arguments. For the second part, we assume that $M=\boldsymbol{X}(U)$ and both the u-curve and the v-curve are the lines of curvature which satisfy $\mathbb{G}_u(u,v)=0$ and $\mathbb{G}_v(u,v)=-\kappa(u,v)\boldsymbol{X}_v(u,v)$. We now consider the parametrization

$$\overline{\boldsymbol{X}}(\theta,v)=\cos\theta\boldsymbol{X}(0,v)+\sin\theta\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|}$$

of $M=\boldsymbol{X}(U)$. By a straightforward calculation, we have

$$\begin{aligned}\overline{\boldsymbol{X}}_\theta(\theta,v)&=-\sin\theta\boldsymbol{X}(0,v)+\cos\theta\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|},\\ \overline{\boldsymbol{X}}_v(s,v)&=\cos\theta\boldsymbol{X}_v(0,v)+\sin\theta\left(\frac{\boldsymbol{X}_{uv}(0,v)}{\|\boldsymbol{X}_u(0,v)\|}-\frac{2\boldsymbol{X}_u(0,v)\cdot\boldsymbol{X}_{uv}(0,v)}{\|\boldsymbol{X}_u(0,v)\|^2}\boldsymbol{X}_u(0,v)\right).\end{aligned}$$

Since $\mathbb{G}(0,v)\cdot \boldsymbol{X}_u(0,v)=0$, we have $\mathbb{G}_v(0,v)\cdot \boldsymbol{X}_u(0,v)+\mathbb{G}(0,v)\cdot \boldsymbol{X}_{uv}(0,v)=0$. By the assumption that v-curve is the line of curvature with $\mathbb{G}_v(0,v)=-\kappa(0,v)\boldsymbol{X}_v(0,v)$, we have $\mathbb{G}_v(0,v)\cdot\boldsymbol{X}_u(0,v)=-\kappa(0,v)(\boldsymbol{X}_v(0,v)\cdot\boldsymbol{X}_u(0,v))=0$. Therefore we have $\mathbb{G}(0,v)\cdot\boldsymbol{X}_{uv}(0,v)=0$. Since $\mathbb{G}(0,v)$ is the normal vector of $M=\boldsymbol{X}(U)$ at $\boldsymbol{X}(0,v)$, we have $\mathbb{G}(0,v)\cdot\overline{\boldsymbol{X}}_\theta(\theta,v)=\mathbb{G}(0,v)\cdot\overline{\boldsymbol{X}}_v(\theta,v)=0$. This means that $\mathbb{G}(0,v)$ is the normal of $M=\boldsymbol{X}(U)$ at $\overline{\boldsymbol{X}}(s,v)$. Therefore we have the unit normal $\mathbb{G}$ which is constant along the θ-curve. Since the θ-curve is a great circle, it is the line of curvature with vanishing principal curvature. □

Under the above notation, we remark that $\boldsymbol{e}(0,v)$ is a unit normal vector field of $\overline{\boldsymbol{X}}(\theta,v)$.

4 Great circular surfaces

In this section we study general properties of great circular surfaces. Let $\boldsymbol{a}_i : I \longrightarrow S^3$ $(i=0.1.2.3)$ be a smooth maps from an open interval I with $\boldsymbol{a}_i(t)\cdot\boldsymbol{a}_j(t)=\delta_{ij}$, so that we have an orthonormal frame $\{\boldsymbol{a}_0,\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3\}$ of $\mathbb{R}^4$. We now define a mapping

$$F_A : [0,2\pi]\times I \longrightarrow S^3$$

by

$$F_A(\theta,t)=\cos\theta\boldsymbol{a}_1(t)+\sin\theta\boldsymbol{a}_3(t),$$

where we assume that $A(t)=(\boldsymbol{a}_0(t),\boldsymbol{a}_1(t),\boldsymbol{a}_2(t),\boldsymbol{a}_3(t))\in SO(4)$. We have a great circle $F_A(\theta,t_0)$ for any fixed $t=t_0$. We call F_A (or the image of it) a *great circular surface.* We also call $\boldsymbol{a}_1(t)$ a *base curve* and $\boldsymbol{a}_3(t)$ a *directrix.* Each great circle $F_A(\theta,t_0)$ is called a *generating great circle.* By using the above orthonormal frame, we define the following fundamental invariants:

$$\begin{aligned}
c_1(t)&=\boldsymbol{a}_0'(t)\cdot\boldsymbol{a}_1(t)=-\boldsymbol{a}_0(t)\cdot\boldsymbol{a}_1'(t), & c_4(t)&=\boldsymbol{a}_1'(t)\cdot\boldsymbol{a}_2(t)=-\boldsymbol{a}_1(t)\cdot\boldsymbol{a}_2'(t),\\
c_2(t)&=\boldsymbol{a}_0'(t)\cdot\boldsymbol{a}_2(t)=-\boldsymbol{a}_0(t)\cdot\boldsymbol{a}_2'(t), & c_5(t)&=\boldsymbol{a}_1'(t)\cdot\boldsymbol{a}_3(t)=-\boldsymbol{a}_1(t)\cdot\boldsymbol{a}_3'(t),\\
c_3(t)&=\boldsymbol{a}_0'(t)\cdot\boldsymbol{a}_3(t)=-\boldsymbol{a}_0(t)\cdot\boldsymbol{a}_3'(t), & c_6(t)&=\boldsymbol{a}_2'(t)\cdot\boldsymbol{a}_3(t)=-\boldsymbol{a}_2(t)\cdot\boldsymbol{a}_3'(t).
\end{aligned}$$

We can show that the following fundamental differential equations for the horocyclic surface:

$$\begin{cases}
\boldsymbol{a}_0'(t) &= \ c_1(t)\boldsymbol{a}_1(t)+c_2(t)\boldsymbol{a}_2(t)+c_3(t)\boldsymbol{a}_3(t)\\
\boldsymbol{a}_1'(t) &= -c_1(t)\boldsymbol{a}_0(t)+c_4(t)\boldsymbol{a}_2(t)+c_5(t)\boldsymbol{a}_3(t)\\
\boldsymbol{a}_2'(t) &= -c_2(t)\boldsymbol{a}_0(t)-c_4(t)\boldsymbol{a}_1(t)+c_6(t)\boldsymbol{a}_3(t)\\
\boldsymbol{a}_3'(t) &= -c_3(t)\boldsymbol{a}_0(t)-c_5(t)\boldsymbol{a}_1(t)-c_6(t)\boldsymbol{a}_2(t).
\end{cases}$$

It can be written in the following form:

$$\begin{pmatrix}\boldsymbol{a}_0'(t)\\ \boldsymbol{a}_1'(t)\\ \boldsymbol{a}_2'(t)\\ \boldsymbol{a}_3'(t)\end{pmatrix}=\begin{pmatrix}0 & c_1(t) & c_2(t) & c_3(t)\\ -c_1(t) & 0 & c_4(t) & c_5(t)\\ -c_2(t) & -c_4(t) & 0 & c_6(t)\\ -c_3(t) & -c_5(t) & -c_6(t) & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{a}_0(t)\\ \boldsymbol{a}_1(t)\\ \boldsymbol{a}_2(t)\\ \boldsymbol{a}_3(t)\end{pmatrix}. \tag{4.1}$$

We remark that

$$C(t)=\begin{pmatrix}0 & c_1(t) & c_2(t) & c_3(t)\\ -c_1(t) & 0 & c_4(t) & c_5(t)\\ -c_2(t) & -c_4(t) & 0 & c_6(t)\\ -c_3(t) & -c_5(t) & -c_6(t) & 0\end{pmatrix}\in\mathfrak{so}(4),$$

where $\mathfrak{so}(4)$ is the Lie algebra of the rotation group $SO(4)$. If $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$ is a orthonormal frame field as the above, the 4×4-matrix determined by the frame defines a smooth curve $A : I \longrightarrow SO(4)$. Therefore we have the relation that $A'(t) = C(t)A(t)$. For the converse, let $A : I \longrightarrow SO(4)$ be a smooth curve, then we can show that $A'(t)A(t)^{-1} \in \mathfrak{so}(4)$. Moreover, for any smooth curve $C : I \longrightarrow \mathfrak{so}(4)$, we apply the existence theorem on the linear systems of ordinary differential equations, so that there exists a unique curve $A : I \longrightarrow SO(4)$ such that $C(t) = A'(t)A(t)^{-1}$ with an initial data $A(t_0) \in SO(4)$. Therefore, a smooth curve $C : I \longrightarrow \mathfrak{so}(4)$ might be identified with a great circular surface in S^3. Let $C : I \longrightarrow \mathfrak{so}(4)$ be a smooth curve with $C(t) = A'(t)A(t)^{-1}$ and $B \in SO(4)$, then we have $C(t) = (A(t)B)'(A(t)B)^{-1}$. This means that the curve $C : I \longrightarrow \mathfrak{so}(4)$ is a rotational invariant (spherical invariant) of the orthonormal frame $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$, so that it is a spherical invariant of the corresponding great circular surface.

Let $C^\infty(I, \mathfrak{so}(4))$ be the space of smooth curves into $\mathfrak{so}(4)$ equipped with Whitney C^∞-topology. By the above arguments, we may regard $C^\infty(I, \mathfrak{so}(4))$ as the space of great circular surfaces, where I is an open interval or the unit circle.

On the other hand, we consider the singularities of great circular surfaces. Let $F_A : \mathbb{R}\times I \longrightarrow S^3$ be a great circular surface defined by

$$F_A(\theta, t) = \cos\theta \boldsymbol{a}_1(t) + \sin\theta \boldsymbol{a}_3(t). \tag{4.2}$$

Then we have

$$\begin{aligned} \frac{\partial F_A}{\partial \theta}(\theta, t) &= -\sin\theta \boldsymbol{a}_1(t) + \cos\theta \boldsymbol{a}_3(t) \\ \frac{\partial F_A}{\partial t}(\theta, t) &= (-\cos\theta c_1(t) - \sin\theta c_3(t))\boldsymbol{a}_0(t) - \sin\theta c_5(t)\boldsymbol{a}_1(t) \\ &\quad +(\cos\theta c_4(t) - \sin\theta c_6(t))\boldsymbol{a}_2(t) + \cos\theta c_5(t)\boldsymbol{a}_3(t). \end{aligned}$$

Since (θ_0, t_0) is a singular point of F_A if and only if $(\partial F_A/\partial\theta)(\theta_0, t_0)$ and $(\partial F_A/\partial t)(\theta_0, t_0)$ are parallel, we have conditions

$$\begin{cases} \cos\theta_0 c_1(t_0) + \sin\theta_0 c_3(t_0) = 0 \\ -\lambda\sin\theta_0 + \sin\theta_0 c_5(t_0) = 0 \\ \cos\theta_0 c_4(t_0) - \sin\theta_0 c_6(t_0) = 0 \\ \lambda\cos\theta_0 - \cos\theta_0 c_5(t_0) = 0 \end{cases} \tag{4.3}$$

for some $\lambda \in \mathbb{R}$. It is equivalent to the conditions that

$$\begin{cases} \cos\theta_0 c_1(t_0) + \sin\theta_0 c_3(t_0) = 0 \\ \cos\theta_0 c_4(t_0) - \sin\theta_0 c_6(t_0) = 0. \end{cases} \tag{4.4}$$

The above relation means that $(\cos\theta_0, \sin\theta_0)$ is a non-trivial solution of the following simultaneous linear equation :

$$\begin{cases} c_1(t_0)x + c_3(t_0)y = 0 \\ c_4(t_0)x - c_6(t_0)y = 0. \end{cases} \tag{4.5}$$

It follows that we have $c_1(t_0)c_6(t_0) + c_3(t_0)c_4(t_0) = 0$. If t_0 satisfies this condition, there are non-trivial solutions (x, y), so that there exists θ_0 such that $(\cos\theta_0, \sin\theta_0)$ is a non-trivial solution of (4.5). Therefore we have the following proposition.

Proposition 4.1. *A point* (θ_0, t_0) *is a singular point of* $F_A(\theta, t)$ *if and only if* $c_1(t_0)c_6(t_0) + c_3(t_0)c_4(t_0) = 0$ *and* $(\cos\theta_0, \sin\theta_0)$ *is a nontrivial solution of* (4.5).

We now investigate geometric properties of the function $c_1(t)c_6(t) + c_3(t)c_4(t)$. We consider the vector defined by

$$\boldsymbol{n}(\theta, t) = (\cos\theta c_4(t) - \sin\theta c_6(t))\boldsymbol{a}_0(t) + (\cos\theta c_1(t) + \sin\theta c_3(t))\boldsymbol{a}_3(t).$$

We can easily to show that

$$\frac{\partial F_A}{\partial \theta}(\theta, t) \cdot \boldsymbol{n}(\theta, t) = \frac{\partial F_A}{\partial t}(\theta, t) \cdot \boldsymbol{n}(\theta, t) = 0.$$

Thus, $\boldsymbol{n}(\theta_0, t_0)$ is a normal vector of $F_A(\theta, t)$ at $\theta_0, t_0)$. Therefore, we have the unit normal vector field

$$\boldsymbol{e}(\theta, t) = \lambda(\theta, t)\boldsymbol{n}(\theta, t), \text{ where } \lambda(\theta, t) = \frac{1}{\sqrt{(\cos\theta c_4(t) - \sin\theta c_6(t))^2 + (\cos\theta c_1(t) + \sin\theta c_3(t))^2}}$$

under the assumption that (θ, t) is not a singular point of F_A. Moreover, we have

$$\boldsymbol{e}_\theta(\theta, t) = \lambda_\theta(\theta, t)\boldsymbol{n}(\theta, t) + \lambda(\theta, t)\{(-\sin\theta c_4(t) - \cos\theta c_6(t))\boldsymbol{a}_0(t) + (-\sin\theta c_1(t) + \cos\theta c_3(t))\boldsymbol{a}_2(t)\}.$$

It follows that

$$\frac{\partial F_A}{\partial \theta}(\theta, t) \cdot \boldsymbol{e}_\theta(\theta, t) = 0, \text{ and } \frac{\partial F_A}{\partial t}(\theta, t) \cdot \boldsymbol{e}_\theta(\theta, t) = \lambda(\theta, t)(c_1(t)c_6(t) + c_3(t)c_4(t)),$$

so that the second fundamental matrix is given by

$$(h_{ij}(\theta, t)) = \begin{pmatrix} 0 & \lambda(\theta, t)(c_1(t)c_6(t) + c_3(t)c_4(t)) \\ \lambda(\theta, t)(c_1(t)c_6(t) + c_3(t)c_4(t)) & * \end{pmatrix}.$$

We also have

$$g(\theta, t) = \det(g_{ij}(\theta, t)) = \frac{1}{\lambda^2(\theta, t)}.$$

Thus, we have the following proposition:

Proposition 4.2. *Let* (θ, t) *be a regular point of* F_A. *Then the extrinsic Gauss curvature is*

$$K_e(\theta, t) = \frac{-(c_1(t)c_6(t) + c_3(t)c_4(t))^2}{((\cos\theta c_4(t) - \sin\theta c_6(t))^2 + (\cos\theta c_1(t) + \sin\theta c_3(t))^2)^2}.$$

Corollary 4.3. *If* (θ_0, t_0) *is a singular point, then* $K_e(\theta, t_0) = 0$ *for any regular point* (θ, t_0). *Moreover, if* $K_e(\theta_0, t_0) = 0$, *then there exists* θ_1 *such that* (θ_1, t_0) *is a singular point of* F_A.

If $(c_1(t_0), c_3(t_0), c_4(t_0), c_6(t_0)) = (0, 0, 0, 0)$, then all points on the great circle $F_A(\theta, t_0)$ are the singularities. We say that F_A is *non-cyclic* if $(c_1(t), c_3(t), c_4(t), c_6(t)) \neq (0, 0, 0, 0)$.

By the above results, the function $c_1(t)c_6(t) + c_3(t)c_4(t)$ has a special meaning. We denote that $c_\kappa(t) = c_1(t)c_6(t) + c_3(t)c_4(t)$.

Corollary 4.4. *Let* (θ_0, t_0) *be a regular point of* $F_A(\theta, t)$. *Then* $F_A(\theta, t)$ *is extrinsic flat at* (θ_0, t_0) *if and only if* $c_\kappa(t_0) = 0$.

If $c_\kappa(t_0) = 0$, then there exists θ_1 such that (θ_1, t_0) is a singular point of $F_A(\theta, t)$. For classifications of singularities of general great circular surfaces, we have the following:

Theorem 4.5. *Let F_A be a non-cyclic great circular surface. A point (θ_0, t_0) is $\mathcal{A}$-equivalent to the cross cap if and only if $c_\kappa(t_0) = 0$, θ_0 satisfies* (4.4)*, and $c'_\kappa(t_0) \neq 0$.*

Two map germs $f_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ $(i = 1, 2)$ are *$\mathcal{A}$-equivalent* (or *locally diffeomorphic*) if there exist germs of C^∞ diffeomorphisms $d_s : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $d_t : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $d_t \circ f_1 = f_2 \circ d_s$ holds.

Proof. A point (t_0, θ_0) is a singular point of F_A if and only if $c_{\mathrm{cr}}(t_0) = 0$ and θ_0 satisfies the conditions (4.4). It has been known in [17, p.161 (b)] that F_A at (t_0, θ_0) is $\mathcal{A}$-equivalent to the cross cap if and only if (t_0, θ_0) is a singular point and satisfies the condition that

$$\det\left(F_A,\ \frac{\partial F_A}{\partial\theta},\ \frac{\partial^2 F_A}{\partial\theta\partial t},\ \frac{\partial^2 F_A}{\partial t^2}\right)(t_0, \theta_0) \neq 0.$$

Under the assumption that $(c_1c_6 + c_3c_4)(t_0) = 0$ and the relation (4.4), we can calculate that

$$\begin{aligned}
&\det\left(F_A,\ \frac{\partial F_A}{\partial\theta},\ \frac{\partial^2 F_A}{\partial\theta\partial t},\ \frac{\partial^2 F_A}{\partial t^2}\right)(t_0, \theta_0)\\
&= \sin\theta_0 c_1(t_0)c_4'(t_0) - \sin\theta c_1(t_0)c_6'(t_0) - \cos^2\theta_0 c_3(t_0)c_4'(t_0)\\
&+ \cos\theta_0\sin\theta_0 c_3(t_0)c_6'(t_0) - \sin\theta_0\cos\theta_0 c_4(t_0)c_1'(t_0) - \sin^2\theta_0 c_4(t_0)c_3'(t_0)\\
&- \cos^2\theta_0 c_6(t_0)c_1'(t_0) - \cos\theta_0\sin\theta_0 c_6(t_0)c_3'(t_0)\\
&= -(c_3(t_0)c_4'(t_0) + c_1(t_0)c_6'(t_0) + c_6(t_0)c_1'(t_0) + c_4(t_0)c_3'(t_0)) = -(c_1c_6 + c_3c_4)'(t_0).
\end{aligned}$$

This completes the proof. □

This theorem shows that generic singularities of great circular surfaces are cross cap (cf., §8, Proposition 8.1). Remark that this theorem implies if (θ_0, t_0) is the cross cap, then $(\theta_0 + \pi, t_0)$ is also the cross cap. Since a great circular surface is a double covering of a ruled surface in $\mathbb{RP}^n$, generic classifications of singularities are the same as those of ruled surfaces (see, [5, 6]). However, we emphasize that we give an exact condition for the cross cap by using the invariant $c_\kappa(t)$. Moreover, the above theorem shows that great circular surfaces have a different property with the circular surfaces in $\mathbb{R}^3$ (see [7]).

On the other hand, we consider a parameter transformation $\Theta = \theta - \theta(t)$, $T = t$ for any smooth function $\theta(t)$. We define $\overline{A} = (\overline{\boldsymbol{a}}_0(t), \overline{\boldsymbol{a}}_1(t), \overline{\boldsymbol{a}}_2(t), \overline{\boldsymbol{a}}_3(t))$ by

$$\begin{aligned}
&\overline{\boldsymbol{a}}_0(T) = \boldsymbol{a}_0(t),\ \overline{\boldsymbol{a}}_1(T) = \cos\theta(t)\boldsymbol{a}_1(t) + \sin\theta(t)\boldsymbol{a}_3(t)\\
&\overline{\boldsymbol{a}}_2(T) = \boldsymbol{a}_2(t) \text{ and } \overline{\boldsymbol{a}}_3(T) = -\sin\theta(t)\boldsymbol{a}_1(t) + \cos\theta(t)\boldsymbol{a}_2(t).
\end{aligned}$$

Then $\overline{A}(T) \in SO(4)$ and $F_A(\theta, t) = F_{\overline{A}}(\Theta, T)$. By straight forward calculations, we have

$$\begin{cases}
\bar{c}_1(T) = c_1(t)\cos\theta(t) + c_3(t)\sin\theta(t)\\
\bar{c}_2(T) = c_2(t)\\
\bar{c}_3(T) = -c_1(t)\sin\theta(t) + c_3(t)\cos\theta(t)\\
\bar{c}_4(T) = c_4(t)\cos\theta(t) - c_6(t)\sin\theta(t)\\
\bar{c}_5(T) = -\theta'(t) - c_5(t)\\
\bar{c}_6(T) = c_4(t)\sin\theta(t) + c_6(t)\cos\theta(t).
\end{cases} \tag{4.6}$$

We call the above parameter transformation an *adapted parameter transformation* of F_A.

5 Extrinsic flat great circular surfaces

In this section we consider extrinsic flat great circular surfaces. By Proposition 4.2, $F_A(\theta,t)$ is extrinsic flat if and only if $c_\kappa(t)=0$ for any t. Suppose that $F_A(\theta,t)$ is non-cyclic and extrinsic flat. Since $\boldsymbol{e}(\theta,t)$ is independent of θ, we have the following new orthonormal frame:

$$\begin{cases} \widetilde{\boldsymbol{a}}_0(t)=\boldsymbol{e}(\theta,t)=\lambda(\theta,t)((\cos\theta c_4(t)-\sin\theta c_6(t))\boldsymbol{a}_0(t)+(\cos\theta c_1(t)+\sin\theta c_3(t))\boldsymbol{a}_2(t)), \\ \widetilde{\boldsymbol{a}}_1(t)=\boldsymbol{a}_1(t) \\ \widetilde{\boldsymbol{a}}_2(t)=\lambda(\theta,t)((\cos\theta c_4(t)-\sin\theta c_6(t))\boldsymbol{a}_2(t)-(\cos\theta c_1(t)+\sin\theta c_3(t))\boldsymbol{a}_0(t)) \\ \widetilde{\boldsymbol{a}}_3(t)=\boldsymbol{a}_3(t). \end{cases} \tag{5.1}$$

It follows that we have

$$\begin{cases} \widetilde{c}_1(t)=-\widetilde{\boldsymbol{a}}_0(t)\cdot\boldsymbol{a}_1'(t)=-\lambda(\theta,t)\sin\theta(c_1(t)c_6(t)+c_3(t)c_4(t))=0, \\ \widetilde{c}_3(t)=-\widetilde{\boldsymbol{a}}_0(t)\cdot\boldsymbol{a}_3'(t)=-\lambda(\theta,t)\cos\theta(c_1(t)c_6(t)+c_3(t)c_4(t)=0, \\ \widetilde{c}_4(t)=\boldsymbol{a}_1'(t)\cdot\widetilde{\boldsymbol{a}}_2(t)=\lambda(\theta,t)\{\cos\theta(c_4^2(t)+c_1^2(t))+\sin\theta(c_1(t)c_3(t)-c_4(t)c_6(t))\}. \end{cases}$$

Moreover, we have $F_A(\theta,t)=F_{\widetilde{A}}(\theta,t)$ and $\widetilde{\boldsymbol{a}}_0(t)$ is the unit normal vector of $F_{\widetilde{A}}(\theta,t)$ at regular point (θ,t), where $\widetilde{A}(t)=(\widetilde{\boldsymbol{a}}_0(t),\widetilde{\boldsymbol{a}}_1(t),\widetilde{\boldsymbol{a}}_2(t),\widetilde{\boldsymbol{a}}_3(t))\in SO(4)$.

On the other hand, we can easily calculate that

$$\frac{\partial F_A}{\partial\theta}(\theta,t)\cdot\boldsymbol{a}_0(t)=0 \text{ and } \frac{\partial F_A}{\partial t}(\theta,t)\cdot\boldsymbol{a}_0(t)=-\cos\theta c_1(t)-\sin\theta c_3(t).$$

Therefore $\boldsymbol{a}_0(t)$ is a unit normal of F_A at any (θ,t) if and only if $c_1(t)\equiv c_3(t)\equiv 0$. By the same arguments, $\boldsymbol{a}_2(t)$ is a unit normal at any point (θ,t) if and only if $c_4(t)\equiv c_6(t)\equiv 0$. Suppose that $\boldsymbol{a}_2(t)$ is a unit normal of F_A at any point (θ,t). If we have another orthonormal frame $\widetilde{A}(t)=(\widetilde{\boldsymbol{a}}_0(t),\widetilde{\boldsymbol{a}}_1(t),\widetilde{\boldsymbol{a}}_2(t),\widetilde{\boldsymbol{a}}_3(t))$ defined by $\widetilde{\boldsymbol{a}}_0(t)=-\boldsymbol{a}_2(t),\widetilde{\boldsymbol{a}}_1(t)=-\boldsymbol{a}_1(t),\widetilde{\boldsymbol{a}}_2(t)=\boldsymbol{a}_0(t),\widetilde{\boldsymbol{a}}_3(t)=-\boldsymbol{a}_3(t)$, then we have $F_A(\theta,t)=F_{\widetilde{A}}(\theta,t)$ and

$$\widetilde{c}_1(t)=c_4(t),\ \widetilde{c}_2(t)=c_2(t),\ \widetilde{c}_3(t)=-c_6(t),\ \widetilde{c}_4(t)=-c_1(t),\ \widetilde{c}_5(t)=c_5(t),\ \widetilde{c}_6(t)=c_3(t).$$

It follows that $c_4(t)\equiv c_6(t)\equiv 0$ if and only if $\widetilde{c}_1(t)\equiv\widetilde{c}_3(t)\equiv 0$.

Throughout the remainder in this paper, we say that a great circular surface $F_A(\theta,t)$ is *extrinsic flat* (briefly, *E-flat*) if $c_1(t)=c_3(t)=0$.

Suppose that F_A is an E-flat great circular surface with $(c_4,c_6)(t)\neq(0,0)$. Let $\theta(t)$ be a smooth function with

$$c_4(t)\cos\theta(t)-c_6(t)\sin\theta(t)=0. \tag{5.2}$$

By the adopted parameter transformation $\Theta=\theta-\theta(t)$, $T=t$, we have $\bar{c}_1(T)\equiv\bar{c}_3(T)\equiv\bar{c}_4(T)\equiv 0$. Therefore, we assume that $c_1(t)\equiv c_3(t)\equiv c_4(t)\equiv 0$ for an E-flat great circular surface.

Let $\boldsymbol{\sigma}(t)=\cos\theta(t)\boldsymbol{a}_1(t)+\sin\theta(t)\boldsymbol{a}_3(t)$ be a curve on the great circular surface F_A defined by the condition (5.2). If $(c_4(t),c_6(t))\neq(0,0)$, the function $\theta(t)$ is well-defined. The condition $c_1(t)\equiv c_4(t)\equiv 0$ is equivalent to the condition that all generating great circles are always tangent to the curve $\boldsymbol{\sigma}(t)$ at regular points of the curve.

Theorem 5.1. *Suppose that $c_1\equiv c_3\equiv 0$ and $(c_4,c_6)(t_0)\neq(0,0)$. Then $p_0=(\theta_0,t_0)\in S(F_A)$ if and only if $\theta_0=\theta(t_0)$, where $\theta(t)$ is the function defined by (5.2). The following assertions hold.*

- *F_A at p_0 is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $c_2(t_0)(c_5(t_0)+\theta'(t_0)) \neq 0$.*
- *F_A at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if*

$$c_5(t_0)+\theta'(t_0)=0 \text{ and } c_2(t_0)(c_5'(t_0)+\theta''(t_0)) \neq 0.$$

- *F_A at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if*

$$c_2(t_0)=0 \text{ and} c_2'(t_0)(c_5(t_0)+\theta'(t_0)) \neq 0.$$

By the previous arguments, if $c_1(t)=c_3(t)=0$ and $(c_4(t),c_6(t)) \neq (0,0)$, then we have $\widetilde{c}_1(t)=\widetilde{c}_3(t)=\widetilde{c}_4(t)=0$ and $\widetilde{c}_6(t) \neq 0$ by choosing the different orthonormal frame $\widetilde{A}(t)$ with $F_A(\theta,t)=F_{\widetilde{A}}(\theta,t)$. Therefore, we consider the case when $c_1 \equiv c_3 \equiv c_4 \equiv 0$ without the assumption $c_6(t) \neq 0$.

Theorem 5.2. *Suppose that $c_1 \equiv c_3 \equiv c_4 \equiv 0$ and $p_0=(\theta_0,t_0) \in S(F_A)$.*
(1) *If $c_6(t_0) \neq 0$. Then $\theta_0=0$ or π and the following assertions hold.*

- *F_A at p_0 is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $c_2(t_0)c_5(t_0) \neq 0$.*
- *F_A at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_5(t_0)=0$ and $c_2(t_0)c_5'(t_0) \neq 0$.*
- *F_A at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if*

$$c_2(t_0)=0 \text{ and } c_5(t_0)c_2'(t_0) \neq 0.$$

(2) *If $c_6(t_0)=0$, then $\theta_0=\theta(t_0)$ and the following assertions hold.*

- *F_A at p_0 is $\mathcal{A}$-equivalent to the cuspidal edge if and only if*

$$\theta_0 \neq 0, \pi \text{ and } c_2(t_0)c_5(t_0)c_6'(t_0) \neq 0.$$

- *F_A at p_0 is $\mathcal{A}$-equivalent to the cuspidal beaks if and only if*

$$\theta_0=0 \text{ or } \theta_0=\pi \text{ and } c_2(t_0)c_5(t_0)c_6'(t_0) \neq 0.$$

- *F_A at p_0 is never $\mathcal{A}$-equivalent to the swallowtail, cuspidal lips, and cuspidal cross cap.*

We say that F_A is a *tangent extrinsic flat great circular surface* (briefly, a *T-E-flat great circular surface*) if $c_1(t)=c_3(t)=c_4(t)=0$.

6 Proofs of Theorems 5.1 and 5.2

In this section, we prove Theorems 5.1 and 5.2 by using criteria for singularities of fronts. For the detailed descriptions of fronts, see [1]. Let (M,g) be a 3-dimensional Riemannian manifold. The unit cotangent bundle T_1^*M is canonically identified with the unit tangent bundle T_1M and has the canonical contact structure. Let U be an open domain of $\mathbb{R}^2$. A map $f:U \to M$ is a *frontal* if there exists a unit vector field $\nu : U \to T_1M$ along f such that $g(df(X),\nu)(p)=0$ holds for any $X \in T_pU$. This condition is equivalent to that $L_f=(f,\nu):U \to T_1M$ is isotropic with respect to the canonical contact structure of T_1M. A frontal is a *front* if L_f is an immersion, namely the image of f is a wavefront set of L_f. Let (u,v) be a local coordinate system of U and f a frontal. The *signed area density* λ of f is defined by

$$\lambda(u,v) = \Omega(f_u, f_v, \nu),$$

where Ω is a never vanishing 3-form of M. Needless to say, $\lambda^{-1}(0) = S(f)$ holds. A singular point p is *non-degenerate* if $d\lambda(p) \neq 0$ holds. Let p be a non-degenerate singular point then there exists a regular curve $\gamma : (-\varepsilon, \varepsilon) \to U$ such that $\gamma(0) = p$ and $\mathrm{image}(\gamma) = S(f)$ hold near p. If $\mathrm{rank}(df_p) = 1$, we have a never vanishing vector field η near p satisfying $\langle \eta_q \rangle_{\mathbb{R}} = \ker df_q$ at $q \in S(f)$. We call η the *null vector field.* If p is non-degenerate, η can be regarded as a vector field along γ. In this case, we write $\eta|_{\gamma(t)} = \eta(t)$. Under these settings, the following Theorem holds:

Theorem 6.1. *Let $f : U \to M$ be a frontal and $p \in U$ a singular point of f.*

(A) *Let p be non-degenerate singular point, and $p = \gamma(0)$. If f is a front at p, then f at p is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $\mu_{\mathrm{sw}}(0) \neq 0$ holds. Here,*

$$\mu_{\mathrm{sw}}(t) = \det(\gamma', \eta)(t).$$

(B) *Let p be non-degenerate singular point, and $p = \gamma(0)$. If f is a front at p, then f at p is $\mathcal{A}$-equivalent to the swallowtail if and only if $\mu_{\mathrm{sw}}(0) = 0$ and $\mu'_{\mathrm{sw}}(0) \neq 0$ hold.*

(C) *If f is a front at p. Then f at p is $\mathcal{A}$-equivalent to the cuspidal beaks if and only if $d\lambda(p) = 0$, $\det \mathrm{Hess}\, \lambda(p) < 0$ and $\eta\eta\lambda(p) \neq 0$.*

(D) *If p is non-degenerate and $p = \gamma(0)$, then f at p is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\mu_{\mathrm{sw}}(0) \neq 0$, $\mu_{\mathrm{ccr}}(0) = 0$ and $\mu'_{\mathrm{ccr}}(0) \neq 0$ hold. Here,*

$$\mu_{\mathrm{ccr}}(t) = \Omega\big((f \circ \gamma)'(t),\ \nu \circ \gamma(t),\ d\nu(\eta(t))\big).$$

For the detailed descriptions and proofs of (A) and (B), see [11]. For the proof of (C), see [10]. For the proof of (D), see [3].

Proof of Theorem 5.1. We apply Theorem 6.1 considering $(M, g) = (S^3, \ \cdot \ )$ and $\Omega = \det(p, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$ for $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) \in (T_pS^3)^3$. By the assumption $c_1 \equiv c_3 \equiv 0$, we can take $\nu = \boldsymbol{a}_0$ as the normal vector field of F_A. Since we have

$$(F_A)_\theta = -\sin\theta \boldsymbol{a}_1 + \cos\theta \boldsymbol{a}_2, \quad (F_A)_t = -\sin\theta c_5 \boldsymbol{a}_1 + (\cos\theta c_4 - \sin\theta c_6)\boldsymbol{a}_2 + \cos\theta c_5 \boldsymbol{a}_3, \quad (6.1)$$

and $(c_4, c_6) \neq (0,0)$, we see that $S(F_A) = \{(\theta(t), t)\}$, where $\theta(t)$ is a function satisfying $\cos\theta(t)c_4(t) - \sin\theta(t)c_6(t) = 0$. Let we set $s(t) = (\theta(t), t)$. Then $s(t)$ is a parameterization of $S(F_A)$. We have $\lambda(\theta, t) = \det(F_A, (F_A)_\theta, (F_A)_t, \boldsymbol{a}_0) = \cos\theta c_4 - \sin\theta c_6$. Then we see that $\lambda_\theta = \sin\theta c_4 - \cos\theta c_6 \neq 0$ on $s(t)$. So $p_0 \in S(F_A)$ is non-degenerate.

By (6.1) again, we may take $\eta(t) = -c_5(t)(\partial/\partial\theta) + (\partial/\partial t)$ as the null vector field on $s(t)$. Since $\eta\nu = c_2\boldsymbol{a}_2$, F_A to be a front near $s(t_0)$ if and only if $c_2(t_0) \neq 0$. Since $\eta(t) = -c_5(t)(\partial/\partial\theta) + (\partial/\partial t)$ and $s'(t) = \theta'(\partial/\partial\theta) + (\partial/\partial t)$, we have $\mu_{\mathrm{sw}}(t) = \theta' + c_5$. Thus we have assertions (a) and (b) using (A) and (B) of Theorem 6.1.

On the other hand, assume that a point p_0 satisfies $\mu_{\mathrm{sw}} = \theta' + c_5 \neq 0$. Then since $\mu_{\mathrm{ccr}}(t) = \det(F_A(s(t)), (d/dt)F_A(s(t)), \boldsymbol{a}_0, c_2\boldsymbol{a}_2) = c_2(\theta' + c_5)$, we see that $\mu_{\mathrm{ccr}}(t_0) = 0$ and $\mu'_{\mathrm{ccr}}(t_0) \neq 0$ if and only if $c_2(t_0) = 0$ and $c'_2(t_0) \neq 0$. By (D) of Theorem 6.1, we have the assertion (c). $\square$

Proof of Theorem 5.2. Putting $c_4 = 0$ in Theorem 5.1, one can easily show (d), (e) and (f) of Theorem 5.2. We shall prove (g), (h) and (i). Assume that $c_1 \equiv c_3 \equiv c_4 \equiv 0$, $c_6(t_0) = 0$ hold and take $p_0 = (\theta_0, t_0)$. Put $\tilde{s}(\theta) = (\theta, t_0)$. Then $\tilde{s}(\theta)$ is a parameterization of $S(F_A)$ near p_0. Like as the proof of Theorem 5.1, the null vector field η is given by $\eta(\theta) = -c_5(t)(\partial/\partial\theta) + (\partial/\partial t)$ on $\tilde{s}(\theta)$. Since we see $\mu_{\mathrm{ccr}}(\theta) = c_2(t_0)$, if $c_2(t_0) \neq 0$, then F_A at p_0 to be a front, but if $c_2(t_0) = 0$, then $(d/d\theta)\mu_{\mathrm{ccr}}(\theta) = 0$ hold. Thus p_0 never $\mathcal{A}$-equivalent to the cuspidal cross cap. Now we assume $c_1 \equiv c_3 \equiv c_4 \equiv 0$, $c_6(t_0) = 0$ and $c_2(t_0) \neq 0$. Then we have $\lambda = \sin\theta c_6$, the singular point p_0 is non-degenerate if and only if $(d/dt)\lambda(p_0) = \sin\theta_0 c_6'(t_0) \neq 0$. On the other hand, $\mu_{\mathrm{sw}}(\theta) = c_5(t_0)$ holds. Summerizing the above arguments, we have (g), and that F_A is never $\mathcal{A}$-equivalent to the swallowtail. Assume $\sin\theta_0 c_6'(t_0) = 0$ in addition. Then

$$\det \operatorname{Hess} \lambda(p_0) = \det \begin{pmatrix} -\sin\theta c_6 & \cos\theta c_6' \\ \cos\theta c_6' & \sin\theta c_6'' \end{pmatrix} (p_0) = -\cos^2\theta_0 (c_6'(t_0))^2 \leq 0$$

holds. Thus F_A at p_0 is never $\mathcal{A}$-equivalent to the cuspidal lips and we have (i). We assume that $c_6'(t_0) \neq 0$. Then $\sin\theta_0 = 0$ by the assumption $\sin\theta_0 c_6'(t_0) = 0$. In this case, $\eta\eta\lambda(p_0) = -2c_5(t_0)c_6'(t_0)\cos\theta_0$ holds. By (C) of Theorem 6.1, we have the assertion (h). □

7 Duality of singularities

In this section, we consider the Δ-dual surface of the locus of singular values of F_A under the assumption that $c_1 \equiv c_4 \equiv 0$ and $c_6 \neq 0$. By the equation (4.4), the singular point of F_A is $(t, 0)$ and (t, π) so that the singular value is $\boldsymbol{a}_1(t)$. We consider a great circular surface defined by

$$F_A^\sharp(\theta, t) = \cos\theta \boldsymbol{a}_0(t) + \sin\theta \boldsymbol{a}_2(t).$$

Then $F_A^\sharp(\theta, t) \cdot \boldsymbol{a}_1(t) = 0$, so that we have a mapping $\mathcal{L} : J \times I \longrightarrow \Delta$ defined by $\mathcal{L}(\theta, t) = (F_A^\sharp(\theta, t), \boldsymbol{a}_1(t))$. It follows that $\mathcal{L}^*\theta_2 = F_A^\sharp(\theta, t) \cdot \boldsymbol{a}_1'(t) = -\cos\theta c_1(t) + \sin\theta c_4(t) = 0$. Therefore $\mathcal{L}$ is an isotropic mapping. Thus, $F_A^\sharp(\theta, t)$ and $\boldsymbol{a}_1(t)$ are Δ-dual to each other. Let $\psi(t)$ be a function satisfying the condition that $\cos\psi(t)c_3(t) + \sin\psi(t)c_6(t) = 0$. Then we have that $S(F_A^\sharp) = \{(\psi(t), t)\}$. If we consider the orthonormal frame $\bar{A}(t) = (\boldsymbol{a}_1(t), \boldsymbol{a}_0(t), \boldsymbol{a}_3(t), \boldsymbol{a}_2(t)) \in SO(4)$, $F_A^\sharp$ is equal to $F_{\bar{A}}$ and we have

$$\bar{c}_1 = -c_1, \quad \bar{c}_2 = c_5, \quad \bar{c}_3 = c_4, \quad \bar{c}_4 = c_3, \quad \bar{c}_5 = c_2, \quad \bar{c}_6 = -c_6, \tag{7.1}$$

where $\bar{C}$ is the fundamental invariants of $\bar{A}$. Thus $F_A^\sharp$ is an E-flat great circular surface if and only if $c_1 \equiv c_4 \equiv 0$ and $F_A^\sharp$ is a T-E-flat great circular surface if and only if $c_1 \equiv c_3 \equiv c_4 \equiv 0$. In this case F_A is also a T-E-flat great circular surface. If we assume $c_1 \equiv c_3 \equiv c_4 \equiv 0$, then $F_A^\sharp(S(F_A^\sharp)) = \{\boldsymbol{a}_0(t) \mid t \in J\}$ and we have the following diagram:

$$\begin{array}{ccc} F_A & \xrightarrow{\text{taking singular value}} & \boldsymbol{a}_1 \\ \text{dual} \uparrow & & \downarrow \text{dual} \\ \boldsymbol{a}_0 & \xleftarrow{\text{taking singular value}} & F_A^\sharp \end{array} .$$

Under the assumption $c_1 \equiv c_3 \equiv c_4 \equiv 0$, by Theorem 5.2 and (7.1), we have the following corollary.

Corollary 7.1. *Suppose that* $c_1 \equiv c_3 \equiv c_4 \equiv 0$ *and* $p_0 = (\theta_0, t_0) \in S(F^\sharp_A)$.
(1) *If* $c_6(t_0) \neq 0$, *then* $\theta_0 = 0$ *or* π *and the following assertions hold.*

- $F^\sharp_A$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal edge if and only if* $c_2(t_0)c_5(t_0) \neq 0$.
- $F^\sharp_A$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the swallowtail if and only if* $c_2(t_0) = 0$ *and* $c_5(t_0)c'_2(t_0) \neq 0$.
- $F^\sharp_A$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal cross cap if and only if*

$$c_5(t_0) = 0 \text{ and } c_2(t_0)c'_5(t_0) \neq 0.$$

(2) *If* $c_6(t_0) = 0$, *then the following assertions hold.*

- $F^\sharp_A$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal edge if and only if*

$$\theta_0 \neq 0, \pi \text{ and } c_2(t_0)c_5(t_0)c'_6(t_0) \neq 0.$$

- $F^\sharp_A$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal beaks if and only if*

$$\theta_0 = 0 \text{ or } \theta_0 = \pi \text{ and } c_2(t_0)c_5(t_0)c'_6(t_0) \neq 0.$$

- $F^\sharp_A$ *at* p_0 *is never* $\mathcal{A}$*-equivalent to the swallowtail, cuspidal lips and the cuspidal cross cap.*

Comparing with Theorem 5.2, we can observe a duality of singularities between the swallowtail and the cuspidal cross cap as we pointed out in [9] (cf., [16, 3]). We can also observe a self duality of cuspidal beaks. We summerize this situation on the Table 1. In the table, we explain the conditions for the singularities at the point $(\theta_0, t_0) \in J \times I$. We observe the complete correspondence between the singularities for F_A and $F^\sharp_A$ by exchanging the invariants c_2 and c_5.

	CE $c_6 \neq 0$	SW $c_6 \neq 0$	CCR $c_6 \neq 0$	CE $c_6 = 0$	CBK $c_6 = 0$
$F_A(\theta_0, t_0)$	$c_2c_5 \neq 0$, $\theta_0 = 0, \pi$	$c_5 = 0$, $c_2c'_5 \neq 0$, $\theta_0 = 0, \pi$	$c_2 = 0$, $c'_2c_5 \neq 0$, $\theta_0 = 0, \pi$	$c_2c_5c'_6 \neq 0$, $\theta_0 \neq 0, \pi$	$c_2c_5c'_6 \neq 0$, $\theta_0 = 0, \pi$
$F^\sharp_A(\theta_0, t_0)$	$c_2c_5 \neq 0$ $\theta_0 = 0, \pi$	$c_2 = 0$, $c'_2c_5 \neq 0$, $\theta_0 = 0, \pi$	$c_5 = 0$, $c_2c'_5 \neq 0$, $\theta_0 = 0, \pi$	$c_2c_5c'_6 \neq 0$, $\theta_0 \neq 0, \pi$	$c_2c_5c'_6 \neq 0$, $\theta_0 = 0, \pi$

Table 1: Dualities of condition for singularity.

8 Generic properties

In this section we stick to the study of the generic singularities of great circular surfaces. In §1 and §4, we have shown that the space of great circular surfaces are regarded as $C^\infty(I, \mathfrak{so}(4))$ and E-flat surfaces are regarded as $C^\infty(I, \mathfrak{ef}(4))$. In §5, we have defined the notion of T-E-flat great circular surfaces and regarded $C^\infty(I, \mathfrak{ef}_\tau(4))$ as the space of T-E-flat great circular surfaces. The topology of these spaces are given by the Whitney C^∞-topology. In this section, we prove the following theorem.

Theorem 8.1. (1) *There exists a residual subset $\mathcal{O}_1 \subset C^\infty(I, \mathfrak{so}(4))$ such that for any $C \in \mathcal{O}_1$, F_A is non-cyclic at any point and singularities of F_A are only cross cap.*
(2) *There exists a residual subset $\mathcal{O}_2 \subset C^\infty(I, \mathfrak{ef}(4))$ such that for any $C \in \mathcal{O}_2$, singularities of F_A are only cuspidal edge, swallowtail and cuspidal cross cap.*
(3) *There exists a residual subset $\mathcal{O}_3 \subset C^\infty(I, \mathfrak{tef}_\tau(4))$ such that for any $C \in \mathcal{O}_3$, singularities of F_A are only cuspidal edge, swallowtail, cuspidal beaks and cuspidal cross cap.*

Here, A is an orthonormal frame obtained from the data C by the equation (4.1).

Proof. (1) We consider the 1-jet space

$$J^1(I, \mathfrak{so}(4)) \cong I \times \mathbb{R}^6 \times \mathbb{R}^6 = \{(t, c, d) \mid t \in I, c, d \in \mathbb{R}^6 \},$$

where $c = (c_1, \dots, c_6), d = (d_1, \dots, d_6)$. Define

$$S_1 = \{c_1 = c_3 = 0\},\ S_2 = \{c_4 = c_6 = 0\}$$
$$S_3 = \{(c_3c_4 + c_1c_6)(t) = 0\},\ S_4 = \{c_3d_4 + d_3c_4 + c_1d_6 + d_1c_6 = 0\}.$$

Then we see that S_1, S_2 are codimension two submanifolds and S_3 and S_4 are algebraic subsets of codimension one. Moreover, $S_3 \cap S_4$ is an algebraic subset of codimension two. Therefore, we have stratifications of S_3 and $S_3 \cap S_4$. We say that j^1C is transverse to S_3 (or, $S_3 \cap S_4$) if j^1C is transverse to all strata of these stratifications. By the Thom jet transversality theorem, $\mathcal{O}_1 = \{C \in C^\infty(I, \mathfrak{so}(4)) \,|\, j^1C$ is transverse to $S_1,\ S_2,\ S_3$ and $S_3 \cap S_4\}$ is an open and dense in $C^\infty(I, \mathfrak{so}(4))$. On the other hand, one can easily see that Theorem 4.5 implies that $\mathcal{O}_1$ satisfies the required condition.
(2) Suppose that $(c_4(t), c_6(t)) \neq (0, 0)$. Then we have a function $\theta(t)$ defined by $c_4(t)\cos\theta(t) - c_6(t)\sin\theta(t) = 0$. It follows that

$$\theta'(t) = \frac{c_4'(t)\cos\theta(t) - c_6'(t)\sin\theta(t)}{c_4(t)\sin\theta(t) + c_6(t)\cos\theta(t)}.$$

Therefore, $c_5(t) + \theta'(t) = 0$ if and only if

$$c_4'(t)c_6(t) + c_5(t)c_6^2(t) - c_6'(t)c_4(t) - c_5(t)c_4^2(t) = 0.$$

Moreover, we can show that $c_5'(t) + \theta''(t) = 0$ if and only if

$$c_5'(t)(c_4^2(t) + c_6^2(t))^2 + (c_4''(t)c_6(t) - c_6''(t)c_4(t))(c_4^2(t) + c_6^2(t))$$
$$-2(c_4'(t)c_4(t) + c_6'(t)c_6(t))(c_4'(t)c_6(t) - c_6'(t)c_4(t)) = 0$$

We now consider the 2-jet space

$$J^2(I, \mathfrak{ef}(4)) \cong \{I \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 = \{(t, c, d) \mid t \in I, c, d \in \mathbb{R}^4 \},$$

where $c = (c_2, c_4, c_5, c_6), d = (d_2, d_4, d_5, d_6), e = (e_2, e_4, e_5, e_6)$. Define

$$S_1 = \{c_4 = c_6 = 0\},\ S_2 = \{c_2 = 0\},\ S_3 = \{d_4c_6 + c_5c_6^2 - d_6c_4 - c_5c_4^2 = 0\},$$
$$S_4 = \{d_2 = 0\},\ S_5 = \{d_5(c_4^2 + c_6^2)^2 + (e_4c_6 - e_6c_4)(c_4^2 + c_6^2) - 2(d_4c_4 + d_6c_6)(d_4c_6 - d_6c_4) = 0\}.$$

By the similar reason to the case (1), we have an open dense subset

$$\mathcal{O}_2 = \{C \in C^\infty(I, \mathfrak{ef}(4)) \,|\, j^1C \text{ is transverse to } S_1,\ S_2,\ S_3,\ S_2 \cap (S_3 \cup S_4) \text{ and } S_3 \cap (S_4 \cup S_5)\}.$$

By Theorem 5.1, $\mathcal{O}_2$ satisfies the required condition.
(3) In this case, we consider the 1-jet space

$$J^1(I, \mathfrak{ef}_\tau(4)) \cong I \times \mathbb{R}^3 \times \mathbb{R}^3 = \{(t, c, d) \mid t \in I, c, d \in \mathbb{R}^3\},$$

where $c = (c_2, c_5, c_6), d = (d_2, d_5, d_6)$.

We also define

$$S_0 = \{c_6 = 0\},\ S_1 = \{c_2 = 0\},\ S_2 = \{c_5 = 0\},\ S_3 = \{d_2 = 0\},\ S_4 = \{d_5 = 0\} \text{ and } S_5 = \{d_6 = 0\}.$$

We have the following open dense subset of $C^\infty(I, \mathfrak{ef}_\tau(4))$:

$$\begin{aligned}\mathcal{O}_3 \;=\; & \{C \in C^\infty(I, \mathfrak{ef}_\tau(4)) \,|\, j^1C \text{ is transverse to } S_0,\ S_0 \cap S_2,\ S_0 \cap S_3, \\ & S_2 \cap (S_1 \cup S_4), S_1 \cap (S_2 \cup S_3) \text{ and } S_0 \cap (S_1 \cup S_2 \cup S_5)\}.\end{aligned}$$

By Theorem 5.2, $\mathcal{O}_3$ satisfies the required condition. □

9 Great circular surfaces associated to the Frenet frame

In §2 we defined the Frenet frame for a unit speed curve in S^3. We have three kinds of great circular surfaces associated to the Frenet frame.

Let $\boldsymbol{\gamma} : I \longrightarrow S^3$ be a unit speed curve with $\kappa_g(s) \neq 0$. We consider the Frenet frame $\{\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ which is defined in §2. We define canonical great circular surfaces associated with to the Frenet frame as follows:

(1) $F_T(\theta, s) = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\boldsymbol{t}(s)$: the tangent great circular surface,

(2) $F_N(\theta, s) = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\boldsymbol{n}(s)$: the principal normal great circular surface,

(3) $F_E(\theta, s) = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\boldsymbol{e}(s)$: the binormal great circular surface.

(1) Tangent great circular surfaces. In this case, we consider the orthonormal frame $T =^t (\boldsymbol{e}(s), \boldsymbol{\gamma}(s), \boldsymbol{n}(s), \boldsymbol{t}(s)) \in SO(4)$. By the Frenet-Serret type formulae, we have

$$T'(s) = \begin{pmatrix} 0 & 0 & -\tau_g(s) & 0 \\ 0 & 0 & 0 & 1 \\ \tau_g(s) & 0 & 0 & -\kappa_g(s) \\ 0 & -1 & \kappa_g(s) & 0 \end{pmatrix} T(s)$$

Therefore, we have $c_1 = 0, c_2 = -\tau_g, c_3 = 0, c_4 = 0, c_5 = 1, c_6 = -\kappa_g$. By Theorem 5.2, we have the following proposition:

Proposition 9.1. *The singular point of the tangent great circular surface $F_T(\theta, s)$ is $\theta = 0, \pi$. Both of the germs of tangent great circular surface F_T at $(0, s_0), (\pi, s_0)$ are $\mathcal{A}$-equivalent to the following germs:*

- *The cuspidal edge if $\tau_g(s_0) \neq 0$*
- *The cuspidal cross cap if $\tau_g(s_0) = 0, \tau_g'(s_0) \neq 0$.*
- *The swallowtail does not appear.*

We remark that this proposition corresponds to the result of Cleave[2].

(2) Principal normal great circular surfaces. In this case, we consider the orthonormal frame $N = {}^t(\boldsymbol{t}(s), \boldsymbol{\gamma}(s), \boldsymbol{e}(s), \boldsymbol{n}(s)) \in SO(4)$. By the Frenet-Serret type formulae, we have

$$N'(s) = \begin{pmatrix} 0 & -1 & 0 & \kappa_g(s) \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tau_g(s) \\ -\kappa_g(s) & 0 & \tau_g(s) & 0 \end{pmatrix} N(s)$$

Therefore, we have $c_1 = -1, c_2 = 0, c_3 = \kappa_g, c_4 = 0, c_5 = 0, c_6 = -\tau_g$. By Theorem 4.5, we have the following proposition:

Proposition 9.2. *The singular point $(\theta_0.s_0)$ of the principal normal great circular surface $F_N(\theta, s)$ is given by $\tan\theta_0 = -1/\kappa_g(s_0)$ and $\tau_g(s_0) = 0$. The germ of principal normal great circular surface F_N at (θ_0, s_0) is $\mathcal{A}$-equivalent to the cross cap if $\tau_g(s_0) = 0, \tau_g'(s_0) \neq 0$.*

We remark that this proposition corresponds to the result [6, Theorem 5.3].

(3) Binormal great circular surfaces. In this case, we consider the orthonormal frame $E = {}^t(\boldsymbol{n}(s), \boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{e}(s)) \in SO(4)$. By the Frenet-Serret type formulae, we have

$$E'(s) = \begin{pmatrix} 0 & 0 & -\kappa_g(s) & \tau_g(s) \\ 0 & 0 & 1 & 0 \\ \kappa_g(s) & -1 & 0 & 0 \\ -\tau_g(s) & 0 & 0 & 0 \end{pmatrix} E(s)$$

Therefore, we have $c_1 = 0, c_2 = -\kappa_g, c_3 = \tau_g, c_4 = 1, c_5 = 0, c_6 = 0$. By Theorem 4.5, we have the following proposition:

Proposition 9.3. *The singular point $(\theta_0.s_0)$ of the binormal great circular surface $F_E(\theta, s)$ is given by $\theta_0 = \pi/2, 3\pi/2$ and $\tau_g(s_0) = 0$. The germ of binormal great circular surface F_E at (θ_0, s_0) is $\mathcal{A}$-equivalent to the cross cap if $\theta_0 = \pi/2, 3\pi/2$ and $\tau_g(s_0) = 0, \tau_g'(s_0) \neq 0$.*

We remark that binormal ruled surfaces in $\mathbb{R}^3$ are always non-singular, so that we have a completely different situation for binormal great circular surfaces in S^3.

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