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# Slant Geometry of Spacelike Hypersurfaces in Hyperbolic space and de Sitter space

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## Abstract

We consider a one-parameter family of new extrinsic differential geometries on hypersurfaces in Hyperbolic space. Recently, the second author and his collaborators have constructed a new geometry which is called *horospherical geometry* on Hyperbolic space. There is another geometry which is the famous Gauss-Bolyai-Robechevski geometry (i.e., the hyperbolic geometry) on Hyperbolic space. The slant geometry is a one-parameter family of geometries which connect these two geometries. Moreover, we construct a one-parameter family of geometries on spacelike hypersurfaces in de Sitter space.

## 1 Introduction

We construct one-parameter families of new extrinsic differential geometries on spacelike hypersurfaces in Hyperbolic space and de Sitter space. Recently, the second author and his collaborators have constructed a new geometry which is called a *horospherical geometry* on Hyperbolic space [5, 6, 7, 9]. Traditionally there is another geometry which is the non-Euclidean geometry of Gauss-Bolyai-Lobachevski (i.e., the hyperbolic geometry) on Hyperbolic space. We explain both the geometry in the case when the dimension is two (i.e., Hyperbolic plane). We consider the Poincaré disk model  $D^2$  of the hyperbolic plane which is an open unit disk in the  $(x, y)$  plane with Riemannian metric :  $ds^2 = 4(dx^2 + dy^2)/(1 - x^2 - y^2)^2$ . It is conformally equivalent to Euclidean plane, so that a circle in the Poincaré disk is also a circle in Euclidean plane. A geodesic in Poincaré disk is an Euclidean circle perpendicular to the ideal boundary (i.e., the unit circle). If we adopt geodesics as lines in the Poincaré disk, we have the model of Hyperbolic geometry. We have another class of curves in the Poincaré disk which have an analogous property with lines in Euclidean plane. A *horocycle* is an Euclidean circle which is tangent to the ideal boundary. We remark that a line in Euclidean plane can be considered as a limit of circles when the radii tend to infinity. A horocycle is also a curve as a limit of circles when the radii tend to infinity in the Poincaré disk. Therefore, horocycles are also an analogous notion of lines. If we adopt horocycles as lines, what kind of geometry do we obtain? We say that two horocycles are *parallel* if they have the common tangent point at the ideal boundary.

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Under this definition, the axiom of parallel is satisfied. However, for any two points in the disk, there are always two horocycles passing through the points, so that the axiom 1 of the Euclidean Geometry is not satisfied. In the case of general dimensions, we call this geometry a *horospherical geometry*. However, we have another kind of curves with the properties similar to those of Euclidean lines. A curve in the Poincaré disk is called an *equidistant curve* if it is a circle whose intersection with the ideal boundary consists of two points. Generally, the angle of an equidistant curve with the ideal boundary is  $\phi \in (0, \pi/2]$ . A geodesic is a special case of the equidistant curves with  $\phi = \pi/2$ . A horocycle is not an equidistant curve, but it is a circle with  $\phi = 0$ . In this paper, we consider a family of geometry depending on  $\phi$ . We call this geometry a *slant geometry* of hypersurfaces in Hyperbolic space. Moreover, we consider a slant geometry of spacelike hypersurfaces in de Sitter space.

On the other hand, the Legendrian dualities between pseudo-spheres in Minkowski space [8] have been generalized into pseudo-spheres in general semi-Euclidean space in [3] which are called the *mandala of Legendrian dualities for pseudo spheres*. These Legendrian dualities have been also extended for one-parameter families of pseudo-spheres in Lorentz-Minkowski space in [10]. There are some new applications of such Legendrian dualities. Some basic results of these new applications on the spacelike hypersurfaces in pseudo-spheres have been announced in [10]. In this paper, as one of the applications of the extended mandala of Legendrian dualities, we construct one-parameter families of new extrinsic differential geometries on spacelike hypersurfaces in Hyperbolic space and de Sitter space which include the results of [4, 12] as special cases. Moreover, we construct a  $\phi$ -de Sitter flat geometry of hypersurfaces in Hyperbolic space and a  $\phi$ -hyperbolic flat geometry of spacelike hypersurfaces in de Sitter space, for  $\phi \in [0, \pi/2]$ . For hypersurfaces in Hyperbolic space, 0-de Sitter flat geometry is the horospherical geometry (i.e., the *horizontal geometry*) and  $\pi/2$ -de Sitter flat geometry is the hyperbolic geometry (i.e., the *vertical geometry*). For spacelike hypersurfaces in de Sitter space, we also say the horizontal geometry for 0-hyperbolic flat geometry and the vertical geometry for  $\pi/2$ -hyperbolic flat geometry. Therefore, we call each of the  $\phi$ -de Sitter flat geometry and the  $\phi$ -hyperbolic flat geometry a *slant geometry* in Hyperbolic space and de Sitter space, respectively.

In this paper, we only construct the basic framework on the slant geometry in Hyperbolic space and de Sitter space from a contact view point for fixed  $\phi \in [0, \pi/2]$ . There are applications of the extended mandala of Legendrian dualities for the spacelike hypersurfaces not only in Hyperbolic space and de Sitter space, but also in the lightcone which has been appeared in the paper [11]. Other results of this new geometry in Hyperbolic space and de Sitter space will be appeared in the future research projects.

## 2 Basic notions

In this section, we give some basic notions related with Lorentz-Minkowski space and the contact geometry. Let  $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R}, i = 0, \dots, n\}$  be an  $(n+1)$ -dimensional vector space. For any vectors  $\mathbf{x} = (x_0, x_1, \dots, x_n)$  and  $\mathbf{y} = (y_0, y_1, \dots, y_n)$  in  $\mathbb{R}^{n+1}$ , the *pseudo scalar product* of  $\mathbf{x}$  and  $\mathbf{y}$  is defined by  $\langle \mathbf{x}, \mathbf{y} \rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i$ . The space  $(\mathbb{R}^{n+1}, \langle, \rangle)$  is called *Lorentz-Minkowski  $(n+1)$ -space* and denoted by  $\mathbb{R}_1^{n+1}$ . We say that a vector  $\mathbf{x}$  in  $\mathbb{R}_1^{n+1} \setminus \{\mathbf{0}\}$  is *spacelike*, *lightlike* or *timelike* if  $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ ,  $= 0$  or  $< 0$ , respectively. The norm of a vector  $\mathbf{x} \in \mathbb{R}_1^{n+1}$  is defined by  $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$ . For a vector  $\mathbf{v} \in \mathbb{R}_1^{n+1} \setminus \{\mathbf{0}\}$  and a real number  $c$ , we define a *hyperplane with pseudo normal  $\mathbf{v}$*  by  $HP(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{v} \rangle = c\}$ . We call  $HP(\mathbf{v}, c)$  a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if  $\mathbf{v}$  is timelike,

spacelike or lightlike, respectively. In  $\mathbb{R}_1^{n+1}$ , we have three kinds of pseudo-spheres which are called *Hyperbolic n-space*, *de Sitter n-space* and the *(open) lightcone* and defined respectively by

$$\begin{aligned} H^n(-c^2) &= \{ \mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -c^2 \}, \\ S_1^n(c^2) &= \{ \mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = c^2 \} \end{aligned}$$

and

$$LC^* = \{ \mathbf{x} \in \mathbb{R}_1^{n+1} \setminus \{ \mathbf{0} \} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0 \},$$

for any real number  $c$ . Instead of  $S_1^n(1)$ , we usually write  $S_1^n$ . For  $\phi \in [0, \pi/2]$ , we call  $H^n(-\sin^2 \phi)$  (respectively,  $S_1^n(\sin^2 \phi)$ ) a  $\phi$ -*hyperbolic space* (respectively,  $\phi$ -*de Sitter space*).

Now, we briefly review some properties of contact manifolds and Legendrian submanifolds. Let  $N$  be a  $(2n + 1)$ -dimensional smooth manifold and  $K$  be a tangent hyperplane field on  $N$ . Locally, such a field is defined as the field of zeros of a 1-form  $\alpha$ . The tangent hyperplane field  $K$  is *non-degenerate* if  $\alpha \wedge (d\alpha)^n \neq 0$  at any point of  $N$ . We say that  $(N, K)$  is a *contact manifold* if  $K$  is a non-degenerate hyperplane field. In this case,  $K$  is called a *contact structure* and  $\alpha$  is a *contact form*. Let  $\phi : N \rightarrow N'$  be a diffeomorphism between contact manifolds  $(N, K)$  and  $(N', K')$ . We say that  $\phi$  is a *contact diffeomorphism* if  $d\phi(K) = K'$ . Two contact manifolds  $(N, K)$  and  $(N', K')$  are *contact diffeomorphic* if there exists a contact diffeomorphism  $\phi : N \rightarrow N'$ . A submanifold  $i : L \subset N$  of a contact manifold  $(N, K)$  is said to be *Legendrian* if  $\dim L = n$  and  $di_x(T_x L) \subset K_{i(x)}$  at any  $x \in L$ . A smooth fiber bundle  $\pi : E \rightarrow M$  is called a *Legendrian fibration* if its total space  $E$  is furnished with a contact structure and its fibers are Legendrian submanifolds. Let  $\pi : E \rightarrow M$  be a Legendrian fibration. For a Legendrian submanifold  $i : L \subset E$ ,  $\pi \circ i : L \rightarrow M$  is called a *Legendrian map*. The image of the Legendrian map  $\pi \circ i$  is called a *wavefront set* of  $i$  which is denoted by  $W(L)$ . Here,  $L$  is called the *Legendrian lift* of  $W(L)$ . For any  $z \in E$ , it is known that there is a local coordinate system  $(x, y, p) = (x_1, \dots, x_m, y, p_1, \dots, p_m)$  around  $z$  such that  $\pi(x, y, p) = (x, y)$  and the contact structure is given by the 1-form  $\alpha = dy - \sum_{i=1}^m p_i dx_i$  (cf. [1], 20.3).

Throughout our study, we are interested in the following three double fibrations which have been given in [8, 10].

- (1) (a)  $H^n(-1) \times S_1^n \supset \Delta_1 = \{ (\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \}$ ,  
 (b)  $\pi_{11} : \Delta_1 \rightarrow H^n(-1)$ ,  $\pi_{12} : \Delta_1 \rightarrow S_1^n$ ,  
 (c)  $\theta_{11} = \langle d\mathbf{v}, \mathbf{w} \rangle |_{\Delta_1}$ ,  $\theta_{12} = \langle \mathbf{v}, d\mathbf{w} \rangle |_{\Delta_1}$ .
- (2) (a)  $H^n(-1) \times S_1^n(\sin^2 \phi) \supset \Delta_{21}^\pm(\phi) = \{ (\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = \pm \cos \phi \}$ ,  
 (b)  $\pi[\phi]_{(21)1}^\pm : \Delta_{21}^\pm(\phi) \rightarrow H^n(-1)$ ,  $\pi[\phi]_{(21)2}^\pm : \Delta_{21}^\pm(\phi) \rightarrow S_1^n(\sin^2 \phi)$ ,  
 (c)  $\theta[\phi]_{(21)1}^\pm = \langle d\mathbf{v}, \mathbf{w} \rangle |_{\Delta_{21}^\pm(\phi)}$ ,  $\theta[\phi]_{(21)2}^\pm = \langle \mathbf{v}, d\mathbf{w} \rangle |_{\Delta_{21}^\pm(\phi)}$ .
- (3) (a)  $H^n(-\sin^2 \phi) \times S_1^n \supset \Delta_{31}^\pm(\phi) = \{ (\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = \pm \cos \phi \}$ ,  
 (b)  $\pi[\phi]_{(31)1}^\pm : \Delta_{31}^\pm(\phi) \rightarrow H^n(-\sin^2 \phi)$ ,  $\pi[\phi]_{(31)2}^\pm : \Delta_{31}^\pm(\phi) \rightarrow S_1^n$ ,  
 (c)  $\theta[\phi]_{(31)1}^\pm = \langle d\mathbf{v}, \mathbf{w} \rangle |_{\Delta_{31}^\pm(\phi)}$ ,  $\theta[\phi]_{(31)2}^\pm = \langle \mathbf{v}, d\mathbf{w} \rangle |_{\Delta_{31}^\pm(\phi)}$ .

Here,  $\pi_{11}(\mathbf{v}, \mathbf{w}) = \mathbf{v}$ ,  $\pi_{12}(\mathbf{v}, \mathbf{w}) = \mathbf{w}$ ,  $\pi[\phi]_{(ij)1}^\pm(\mathbf{v}, \mathbf{w}) = \mathbf{v}$  and  $\pi[\phi]_{(ij)2}^\pm(\mathbf{v}, \mathbf{w}) = \mathbf{w}$ , for  $(i, j) = (2, 1)$  and  $(3, 1)$ . Moreover,  $\langle d\mathbf{v}, \mathbf{w} \rangle = -w_0 dv_0 + \sum_{i=1}^n w_i dv_i$  and  $\langle \mathbf{v}, d\mathbf{w} \rangle = -v_0 dw_0 + \sum_{i=1}^n v_i dw_i$  are one-forms on  $\mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1}$ . We remark that  $\theta_{11}^{-1}(0)$  and  $\theta_{12}^{-1}(0)$  (respectively,  $\theta[\phi]_{(ij)1}^\pm^{-1}(0)$  and  $\theta[\phi]_{(ij)2}^\pm^{-1}(0)$ ) define the same tangent hyperplane field denoted by  $K_1$  (respectively,  $K[\phi]_{ij}^\pm$ ) over  $\Delta_1$  (respectively,  $\Delta_{ij}^\pm(\phi)$ ). In [10], the following theorem has been shown:

**Theorem 2.1** *Under the same notations as those of the previous paragraph,  $(\Delta_1, K_1)$  and  $(\Delta_{ij}^\pm(\phi), K[\phi]_{ij}^\pm)$   $((i, j) = (2, 1), (3, 1))$  are contact manifolds such that  $\pi_{1k}$  and  $\pi[\phi]_{(ij)k}^\pm$   $(k = 1, 2)$  are Legendrian fibrations. Moreover, these contact manifolds are contact diffeomorphic to each other.*

This theorem is a part of the assertions of Theorem 3.2 in [10]. Actually, we also have contact manifolds  $(\Delta_{ij}^\pm(\phi), K[\phi]_{ij}^\pm)$  for  $(i, j) = (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)$  in [10]. We remark that  $S_1^n(\sin^2 0) \setminus \{\mathbf{0}\} = H^n(-\sin^2 0) \setminus \{\mathbf{0}\} = LC^*$ . Suppose that we have a Legendrian immersion  $\mathcal{L}_{ij}[\phi] : U \rightarrow \Delta_{ij}^\pm(\phi)$  with the form  $\mathcal{L}_{ij}[\phi](u) = (L_1(u), L_2(u))$ . Then we say that  $L_1(u)$  and  $L_2(u)$  are the  $\Delta_{ij}^\pm(\phi)$ -dual. Especially, we say that  $L_2(u)$  is the  $\phi$ -de Sitter dual of  $L_1(u)$  if  $L_1(u)$  and  $L_2(u)$  are the  $\Delta_{21}^-(\phi)$ -dual and  $L_1(u)$  is the  $\phi$ -hyperbolic dual of  $L_2(u)$  if  $L_1(u)$  and  $L_2(u)$  are the  $\Delta_{31}^+(\phi)$ -dual.

### 3 Slant geometry of hypersurfaces in Hyperbolic space

In this section, we establish a new extrinsic differential geometry on hypersurfaces in Hyperbolic space with respect to the  $\phi$ -de Sitter duals as an application of the extended mandala of Legendrian dualities. We call this geometry a  *$\phi$ -de Sitter flat geometry*. Since all submanifolds in Hyperbolic space are spacelike, we consider general hypersurfaces here.

Let  $\mathbf{X}^h : U \rightarrow H^n(-1)$  be an embedding, where  $U \subset \mathbb{R}^{n-1}$  is an open subset and  $M^H = \mathbf{X}^h(U)$ . We define the following unit normal vector field along  $M^H$  by

$$\mathbf{X}^d(u) = \frac{\mathbf{X}^h(u) \wedge \mathbf{X}_{u_1}^h(u) \wedge \cdots \wedge \mathbf{X}_{u_{n-1}}^h(u)}{\|\mathbf{X}^h(u) \wedge \mathbf{X}_{u_1}^h(u) \wedge \cdots \wedge \mathbf{X}_{u_{n-1}}^h(u)\|}.$$

It is known that we have a Legendrian embedding  $\mathcal{L}_1 : U \rightarrow \Delta_1$  defined by  $\mathcal{L}_1(u) = (\mathbf{X}^h(u), \mathbf{X}^d(u))$  (cf., [8, 10]). Now, we define  $\mathbb{N}_\pm^d[\phi] : U \rightarrow S_1^n(\sin^2 \phi)$  by

$$\mathbb{N}_\pm^d[\phi](u) = \cos \phi \mathbf{X}^h(u) \pm \mathbf{X}^d(u).$$

We also define an embedding  $\mathcal{L}_{21}[\phi] : U \rightarrow \Delta_{21}^-(\phi)$  by

$$\mathcal{L}_{21}[\phi](u) = (\mathbf{X}^h(u), \mathbb{N}_\pm^d[\phi](u)).$$

Let us consider the contact manifold  $(\Delta_{21}^-(\phi), K[\phi]_{21}^-)$  and the contact diffeomorphism  $\Psi_{1(21)}^- : \Delta_1 \rightarrow \Delta_{21}^-(\phi)$  defined by

$$\Psi_{1(21)}^-(\mathbf{v}, \mathbf{w}) = (\mathbf{v}, \cos \phi \mathbf{v} \pm \mathbf{w}).$$

Since  $\Psi_{1(21)}^-$  is a contact diffeomorphism,  $\mathcal{L}_{21}[\phi] = \Psi_{1(21)}^- \circ \mathcal{L}_1$  is a Legendrian embedding. Consequently, we have  $\langle d\mathbf{X}^h(u), \mathbb{N}_\pm^d[\phi](u) \rangle = \mathcal{L}_{21}[\phi]^* \theta[\phi]_{(21)1}^- = 0$ . This means that  $\mathbb{N}_\pm^d[\phi](u)$  can be considered as a normal vector of  $M^H$  at  $p = \mathbf{X}^h(u)$ . Hence,  $\mathbb{N}_\pm^d[\phi]$  is the  $\phi^\pm$ -de Sitter dual of  $\mathbf{X}^h(U) = M^H$ . We remark that  $\mathbb{N}_\pm^d[0](u) = \mathbf{X}^h(u) \pm \mathbf{X}^d(u)$  is the hyperbolic Gauss indicatrix and  $\mathbb{N}_\pm^d[\pi/2](u) = \pm \mathbf{X}^d(u)$  is the de Sitter Gauss indicatrix introduced in [4]. Moreover, we have  $\langle \mathbf{X}^h(u), d\mathbb{N}_\pm^d[\phi](u) \rangle = \langle d\mathbf{X}^h(u), \mathbb{N}_\pm^d[\phi](u) \rangle = 0$ . By a straightforward calculation, we also have  $\langle \mathbf{X}^d(u), d\mathbb{N}_\pm^d[\phi](u) \rangle = 0$ . Since  $\{\mathbf{X}^h, \mathbf{X}^d, \mathbf{X}_{u_1}^h, \dots, \mathbf{X}_{u_{n-1}}^h\}$  is a basis of  $T_p \mathbb{R}^{n+1}$  and  $T_p M^H = \langle \mathbf{X}_{u_1}^h(u), \dots, \mathbf{X}_{u_{n-1}}^h(u) \rangle_{\mathbb{R}}$ ,  $d\mathbb{N}_\pm^d[\phi](u)$  can be considered as a linear transformation on  $T_p M^H$ .

We consider a hypersurface  $HQ^H(\mathbf{n}, c)$  in Hyperbolic space  $H^n(-1)$  defined by

$$HQ^H(\mathbf{n}, c) = HP(\mathbf{n}, c) \cap H^n(-1).$$

We say that  $HQ^H(\mathbf{n}, c)$  is a *hyperquadric* in Hyperbolic space. We respectively say that  $HQ^H(\mathbf{n}, c)$  is *hypersphere* (or, *elliptic hyperquadric*), *equidistant hypersurface* (or, *hyperbolic hyperquadric*) and *hyperhorosphere* (or, *parabolic hyperquadric*) if  $\mathbf{n}$  is timelike, spacelike and lightlike. A hyperbolic hyperquadric with  $c = 0$  is called a *hyperplane*. If  $c = -\cos \phi$  and  $\mathbf{n} \in S_1^n(\sin^2 \phi)$  for  $\phi \in [0, \pi/2]$ , we call  $HQ^H(\mathbf{n}, -\cos \phi)$  a  $\phi$ -*flat hyperbolic hyperquadric* which is a hyperhorosphere if  $\phi = 0$  and a hyperplane if  $\phi = \pi/2$ . It is easy to show the following proposition.

**Proposition 3.1** *If one of the  $\phi^\pm$ -de Sitter duals  $\mathbb{N}_\pm^d[\phi]$  of  $M^H = \mathbf{X}^h(U)$  is a constant map, then  $M^H$  is a subset of a  $\phi^\pm$ -flat hyperbolic hyperquadric.*

We only remark that  $M^H$  is a subset of  $HQ^H(\mathbf{n}, -\cos \phi)$  for  $\mathbf{n} = \mathbb{N}_\pm^d[\phi](u)$ . Therefore,  $HQ^H(\mathbf{n}, -\cos \phi)$  is a candidate of the flat model hypersurfaces in the geometry on hypersurfaces in Hyperbolic space such that  $\mathbb{N}_\pm^d[\phi]$  plays a similar role to the Gauss map for a hypersurface in Euclidean space. We call such a geometry a *slant geometry* of hypersurfaces in Hyperbolic space. Especially, it is the horospherical geometry [4, 7] if  $\phi = 0$  and it is the hyperbolic geometry if  $\phi = \pi/2$ . Consequently, we call the horospherical geometry and the hyperbolic geometry, the *horizontal geometry* and the *vertical geometry*, respectively.

Now, we define a linear transformation  $S_\pm^d[\phi](p) = -d\mathbb{N}_\pm^d[\phi](u) : T_p M^H \rightarrow T_p M^H$  for  $p = \mathbf{X}^h(u)$ . We call  $S_\pm^d[\phi](p)$  a  $\phi^\pm$ -*de Sitter shape operator* of  $M^H$  at  $p = \mathbf{X}^h(u)$ . Under the identification of  $U$  and  $M^H$  through the embedding  $\mathbf{X}^h$ , the derivative  $d\mathbf{X}^h(u_0)$  is the identity mapping  $\text{id}_{T_p M^H}$  on  $T_p M^H$ , where  $p = \mathbf{X}^h(u_0)$ . We have the following relation:

$$d\mathbb{N}_\pm^d[\phi](u_0) = \cos \phi \text{id}_{T_p M^H} \pm d\mathbf{X}^d(u_0).$$

Therefore, we have the linear transformation  $S_\pm^d[\phi](p) = -\cos \phi \text{id}_{T_p M^H} \pm A_d(p)$ , where  $A_d(p) = -dX^d(u_0)$ . The linear transformation  $A_d(p)$  is called a *de Sitter shape operator* in [4]. We denote the eigenvalues of  $S_\pm^d[\phi](p)$  and  $A_d(p)$  by  $\bar{\kappa}_\pm^d[\phi](p)$  and  $\kappa_d(p)$ , respectively. It follows that  $S_\pm^d[\phi](p)$  and  $A_d(p)$  have the common eigen vectors. Moreover, we have a relation  $\bar{\kappa}_\pm^d[\phi](p) = -\cos \phi \pm \kappa_d(p)$ . We call  $\bar{\kappa}_\pm^d[\phi](p)$  and  $\kappa_d(p)$ , a  $\phi$ -*de Sitter principal curvature* and a *de Sitter principal curvature* of  $M^H = \mathbf{X}^h(U)$  at  $p = \mathbf{X}^h(u_0)$ , respectively. We give the following definitions of the curvatures of  $M^H = \mathbf{X}^h(U)$  at  $p = \mathbf{X}^h(u_0)$ :

$$K_\pm^d[\phi](u_0) = \det S_\pm^d[\phi](p); \phi^\pm\text{-de Sitter Gauss-Kronecker curvature,}$$

$$H_\pm^d[\phi](u_0) = \frac{1}{n-1} \text{Trace } S_\pm^d[\phi](p); \phi^\pm\text{-de Sitter mean curvature.}$$

We remark that the 0-de Sitter Gauss-Kronecker (respectively, mean) curvature is the *hyperbolic Gauss-Kronecker (respectively, mean) curvature* and the  $\pi/2$ -de Sitter Gauss-Kronecker (respectively, mean) curvature is the *de Sitter Gauss-Kronecker (respectively, mean) curvature* which are defined in [4].

Since  $\mathbf{X}_{u_i}^h$  ( $i = 1, \dots, n-1$ ) are spacelike vectors, the induced Riemannian metric (the *first fundamental form*) on  $M^H = \mathbf{X}^h(U)$  is given by

$$ds^2 = \sum_{i,j=1}^{n-1} g_{ij}^H du_i du_j,$$

where  $g_{ij}^H(u) = \langle \mathbf{X}_{u_i}^h(u), \mathbf{X}_{u_j}^h(u) \rangle$  for any  $u \in U$ . We also define the  $\phi^\pm$ -de Sitter second fundamental invariant by  $h_{\pm}^D[\phi]_{ij}(u) = \langle -(\mathbb{N}_{\pm}^d[\phi])_{u_i}(u), \mathbf{X}_{u_j}^h(u) \rangle$  for any  $u \in U$ . If we denote  $h_{ij}(u) = \langle -\mathbf{X}_{u_i}^h(u), \mathbf{X}_{u_j}^d(u) \rangle$ , then we have the following relation:

$$h_{\pm}^D[\phi]_{ij}(u) = -\cos \phi g_{ij}^H(u) \pm h_{ij}(u).$$

**Proposition 3.2** *Under the above notations, we have the following  $\phi^\pm$ -de Sitter Weingarten formula:*

$$(\mathbb{N}_{\pm}^d[\phi])_{u_i} = -\sum_{j=1}^{n-1} h_{\pm}^D[\phi]_{ij}^j \mathbf{X}_{u_j}^h,$$

where  $(h_{\pm}^D[\phi]_{ij}^j) = (h_{\pm}^D[\phi]_{ik})(g^H)^{kj}$  and  $((g^H)^{kj}) = (g_{kj}^H)^{-1}$ .

*Proof.* Since  $(\mathbb{N}_{\pm}^d[\phi])_{u_i}$  is a tangent vector of  $T_p M^H$ , there exist real numbers  $\Gamma_i^j$  such that

$$(\mathbb{N}_{\pm}^d[\phi])_{u_i} = \sum_{j=1}^{n-1} \Gamma_i^j \mathbf{X}_{u_j}^h.$$

By definition, we have

$$-h_{\pm}^D[\phi]_{i\beta} = \sum_{\alpha=1}^{n-1} \Gamma_i^\alpha \langle \mathbf{X}_{u_\alpha}^h, \mathbf{X}_{u_\beta}^h \rangle = \sum_{\alpha=1}^{n-1} \Gamma_i^\alpha g_{\alpha\beta}^H.$$

Hence, we have

$$-h_{\pm}^D[\phi]_{ij}^j = -\sum_{\beta=1}^{n-1} h_{\pm}^D[\phi]_{i\beta} (g^H)^{\beta j} = \sum_{\beta=1}^{n-1} \sum_{\alpha=1}^{n-1} \Gamma_i^\alpha g_{\alpha\beta}^H (g^H)^{\beta j} = \Gamma_i^j.$$

This completes the proof of the  $\phi^\pm$ -de Sitter Weingarten formula.  $\square$

As a corollary of the above proposition, we obtain an explicit expression of the  $\phi^\pm$ -de Sitter Gauss-Kronecker curvature by Riemannian metric and the  $\phi^\pm$ -de Sitter second fundamental invariant.

**Corollary 3.3** *Under the same notations as in the above proposition, the  $\phi^\pm$ -de Sitter Gauss-Kronecker curvature is given by*

$$K_{\pm}^d[\phi] = \frac{\det(h_{\pm}^D[\phi]_{ij})}{\det(g_{\alpha\beta}^H)}.$$

*Proof.* By the  $\phi^\pm$ -de Sitter Weingarten formula, the representation matrix of the  $\phi^\pm$ -de Sitter shape operator with respect to the basis  $\{\mathbf{X}_{u_1}^h(u), \dots, \mathbf{X}_{u_{n-1}}^h(u)\}$  is  $(h_{\pm}^D[\phi]_{ij}^j) = (h_{\pm}^D[\phi]_{i\beta})(g^H)^{\beta j}$ . It is obvious from this fact that

$$K_{\pm}^d[\phi] = \det S_{\pm}^d[\phi] = \det(h_{\pm}^D[\phi]_{ij}^j) = \det((h_{\pm}^D[\phi]_{i\beta})(g^H)^{\beta j}) = \frac{\det(h_{\pm}^D[\phi]_{ij})}{\det(g_{\alpha\beta}^H)}.$$

□

It has been given in [4] that a point  $u \in U$  or  $p = \mathbf{X}^h(u)$  is a *umbilic point* if  $A_d(p) = \kappa_d(p) \text{id}_{T_p M^H}$ . Since  $S_{\pm}^d[\phi](p)$  and  $A_d(p)$  have the common eigenvectors, a point  $u \in U$  or  $p = \mathbf{X}^h(u)$  is an umbilic point if and only if  $S_{\pm}^d[\phi](p) = \bar{\kappa}_{\pm}^d[\phi](p) \text{id}_{T_p M^H}$ . We also say that  $M^H = \mathbf{X}^h(U)$  is totally umbilic if all points of  $M^H$  are umbilic. The classification of totally umbilic hypersurfaces in Hypebolic space is well-known (see [4]). Here, we interpret the classification of totally umbilic hypersurfaces in Hyperbolic space by using the  $\phi^{\pm}$ -de Sitter principal curvature.

**Proposition 3.4** *Suppose that  $M^H = \mathbf{X}^h(U)$  is totally umbilic for fixed  $\phi \in [0, \frac{\pi}{2}]$ . Then  $\bar{\kappa}_{\pm}^d[\phi](p)$  is constant  $\bar{\kappa}_{\pm}^d[\phi]$ . Under this condition, we have the following classifications:*

(1) *Assume that  $\bar{\kappa}_{\pm}^d[\phi] \neq 0$ .*

(a) *If  $0 < |\bar{\kappa}_{\pm}^d[\phi] + \cos \phi| < 1$ , then  $M^H$  is a part of hyperbolic hyperquadric.*

(b) *If  $1 < |\bar{\kappa}_{\pm}^d[\phi] + \cos \phi|$ , then  $M^H$  is a part of elliptic hyperquadric.*

(c) *If  $|\bar{\kappa}_{\pm}^d[\phi] + \cos \phi| = 1$ , then  $M^H$  is a part of parabolic hyperquadric.*

(d) *If  $\bar{\kappa}_{\pm}^d[\phi] + \cos \phi = 0$ , then  $M^H$  is a part of flat hyperbolic hyperquadric.*

(2) *If  $\bar{\kappa}_{\pm}^d[\phi] = 0$ , then  $M^H$  is a part of  $\phi$ -flat hyperbolic hyperquadric.*

We remark that  $\pm \kappa_d(p) = \bar{\kappa}_{\pm}^d[\phi] + \cos \phi$  and the assertions directly follow from [4, Proposition 2.3].

We say that  $p = \mathbf{X}^h(u)$  is a  $\phi^{\pm}$ -de Sitter parabolic point if  $K_{\pm}^d[\phi](u) = 0$  and a  $\phi^{\pm}$ -de Sitter flat point if it is an umbilic point and  $K_{\pm}^d[\phi](u) = 0$  which are equivalent to the condition that the condition (2) in the above proposition is satisfied.

## 4 $\phi$ -de Sitter height functions

We define a family of functions

$$H_{\phi}^D : U \times S_1^n(\sin^2 \phi) \longrightarrow \mathbb{R}$$

by  $H_{\phi}^D(u, \mathbf{v}) = \langle \mathbf{X}^h(u), \mathbf{v} \rangle + \cos \phi$ . We call  $H_{\phi}^D$  a  $\phi$ -de Sitter height function on  $M^H = \mathbf{X}^h(U)$ .

**Proposition 4.1** *Let  $H_{\phi}^D : U \times S_1^n(\sin^2 \phi) \longrightarrow \mathbb{R}$  be a  $\phi$ -de Sitter height function on  $M^H = \mathbf{X}^h(U)$ . Then*

(1)  $H_{\phi}^D(u, \mathbf{v}) = 0$  if and only if  $(\mathbf{X}^h(u), \mathbf{v}) \in \Delta_{21}^-(\phi)$ .

(2)  $H_{\phi}^D(u, \mathbf{v}) = \frac{\partial H_{\phi}^D}{\partial u_i}(u, \mathbf{v}) = 0$  ( $i = 1, \dots, n-1$ ) if and only if  $\mathbf{v} = \mathbb{N}_{\pm}^d[\phi](u)$ .

*Proof.* The assertion (1) follows from the definition of  $H_{\phi}^D$  and  $\Delta_{21}^-(\phi)$ .

(2) There exist real numbers  $\lambda, \mu, \xi_j$  ( $j = 1, \dots, n-1$ ) such that  $\mathbf{v} = \lambda \mathbf{X}^h + \mu \mathbf{X}^d + \sum_{j=1}^{n-1} \xi_j \mathbf{X}_{u_j}^h$ .

Since  $\langle \mathbf{X}^h, \mathbf{X}^d \rangle = \langle \mathbf{X}_{u_i}^h, \mathbf{X}^h \rangle = 0$ , we obtain  $0 = H_{\phi}^D(u, \mathbf{v}) = -\lambda + \cos \phi$ , so that  $\mathbf{v} = \cos \phi \mathbf{X}^h(u) + \mu \mathbf{X}^d + \sum_{j=1}^{n-1} \xi_j \mathbf{X}_{u_j}^h$ . It follows from the fact  $\frac{\partial H_{\phi}^D}{\partial u_i}(u, \mathbf{v}) = \langle \mathbf{X}_{u_i}^h, \mathbf{v} \rangle$  that we have



$0 = \sum_{j=1}^{n-1} \xi_j g_{ij}^H(u)$ . Since  $g_{ij}^H$  is positive definite, we have  $\xi_j = 0$  ( $j = 1, \dots, n-1$ ). Moreover, we have  $\sin^2 \phi = \langle \mathbf{v}, \mathbf{v} \rangle = -\cos^2 \phi + \mu^2$ . Therefore,  $\mu = \pm 1$ . This means that  $\mathbf{v} = \mathbb{N}_{\pm}^d[\phi](u)$ . Thus, the proof is completed.  $\square$

Now, we study the extrinsic differential geometry of  $M^H = \mathbf{X}^h(U)$  by using  $\mathbb{N}_{\pm}^d[\phi]$  like as the Gauss map of a hypersurface in Euclidean space.

We denote the Hessian matrix of the  $\phi$ -de Sitter height function  $h_{\phi, v_0}^D(u) = H_{\phi}^D(u, \mathbf{v}_0)$  at  $u_0$  by  $\text{Hess}(h_{\phi, v_0}^D)(u_0)$ .

**Proposition 4.2** *Let  $\mathbf{X}^h : U \rightarrow H^n(-1)$  be a hypersurface in Hyperbolic space and  $\mathbf{v}_0 = \mathbb{N}_{\pm}^d[\phi](u_0)$ . Then we have the following:*

- (1)  $p = \mathbf{X}^h(u_0)$  is a parabolic point if and only if  $\det \text{Hess}(h_{\phi, v_0}^D)(u_0) = 0$ .
- (2)  $p = \mathbf{X}^h(u_0)$  is a flat point if and only if  $\text{rank} \text{Hess}(h_{\phi, v_0}^D)(u_0) = 0$ .

*Proof.* By definition, we have  $h_{\phi, v_0}^D(u_0) = \langle \mathbf{X}^h(u_0), \mathbf{v}_0 \rangle + \cos \phi$ . Using this equation, we get

$$\begin{aligned} \frac{\partial^2 h_{\phi, v_0}^D}{\partial u_i \partial u_j}(u_0) &= \left\langle \mathbf{X}_{u_i u_j}^h(u_0), \mathbf{v}_0 \right\rangle \\ &= \cos \phi \left\langle \mathbf{X}_{u_i u_j}^h(u_0), \mathbf{X}^h(u_0) \right\rangle \pm \left\langle \mathbf{X}_{u_i u_j}^h(u_0), \mathbf{X}^d(u_0) \right\rangle \\ &= -\cos \phi \left\langle \mathbf{X}_{u_i}^h(u_0), \mathbf{X}_{u_j}^h(u_0) \right\rangle \mp \left\langle \mathbf{X}_{u_i}^h(u_0), \mathbf{X}_{u_j}^d(u_0) \right\rangle \\ &= -\cos \phi g_{ij}^H(u_0) \pm h_{ij}(u_0) \\ &= h_{\pm}^D[\phi]_{ij}(u_0). \end{aligned}$$

This means that  $\text{Hess}(h_{\phi, v_0}^D)(u_0) = (h_{\pm}^D[\phi]_{ij}(u_0))$ . Hence, we obtain

$$K_{\pm}^d[\phi](u_0) = \frac{\det(h_{\pm}^D[\phi]_{ij}(u_0))}{\det(g_{\alpha\beta}^H(u_0))} = \frac{\det \text{Hess}(h_{\phi, v_0}^D)(u_0)}{\det(g_{\alpha\beta}^H(u_0))}.$$

As a result, the first assertion follows from this formula.

For the second assertion, by the  $\phi$ -de Sitter Weingarten formula,  $p = \mathbf{X}^h(u_0)$  is an umbilic point if and only if there exists an orthogonal matrix  $A$  such that  $A^t (h_{\pm}^D[\phi]_i^{\alpha}) A = \bar{\kappa}_{\pm}^d[\phi] I$ . Therefore, we have

$$(h_{\pm}^D[\phi]_i^{\alpha}) = A \bar{\kappa}_{\pm}^d[\phi] A^t = \bar{\kappa}_{\pm}^d[\phi] I,$$

so that

$$\text{Hess}(h_{\phi, v_0}^D) = (h_{\pm}^D[\phi]_{ij}) = (h_{\pm}^D[\phi]_i^{\alpha}) (g_{\alpha j}^H) = \bar{\kappa}_{\pm}^d[\phi] (g_{ij}^H).$$

Thus,  $p$  is a flat point (i.e.,  $\bar{\kappa}_{\pm}^d[\phi](u_0) = 0$ ) if and only if  $\text{rank} \text{Hess}(h_{\phi, v_0}^D)(u_0) = 0$ .  $\square$

## 5 The $\phi$ -de Sitter dual as a wave front

In order to investigate the  $\phi$ -de Sitter dual of a hypersurface in Hyperbolic space as a wave front set, we give a quick review on the Legendrian singularity theory due to Arnol'd-Zakalyukin

[1, 16]. Let  $\pi : PT^*(M) \rightarrow M$  be the projective cotangent bundle over an  $n$ -dimensional manifold  $M$ . This fibration can be considered as a Legendrian fibration with the canonical contact structure  $K$  on  $PT^*(M)$ . Now, we review geometric properties of this space. Let us consider the tangent bundle  $\tau : TPT^*(M) \rightarrow PT^*(M)$  and the differential map  $d\pi : TPT^*(M) \rightarrow TM$  of  $\pi$ . For any  $X \in TPT^*(M)$ , there exists an element  $\alpha \in T^*(M)$  such that  $\tau(X) = [\alpha]$ . For an element  $V \in T_x(M)$ , the property  $\alpha(V) = 0$  does not depend on the choice of representative of the class  $[\alpha]$ . Thus, we can define the canonical contact structure on  $PT^*(M)$  by  $K = \{X \in TPT^*(M) | \tau(X)(d\pi(X)) = 0\}$ . For a local coordinate neighbourhood  $(U, (x_1, \dots, x_n))$  on  $M$ , we have a trivialization  $PT^*(U) \cong U \times P(\mathbb{R}^{n-1})^*$  and we call  $((x_1, \dots, x_n), [\xi_1 : \dots : \xi_n])$  *homogeneous coordinates*, where  $[\xi_1 : \dots : \xi_n]$  are homogeneous coordinates of the dual projective space  $P(\mathbb{R}^{n-1})^*$ . It is easy to show that  $X \in K_{(x, [\xi])}$  if and only if  $\sum_{i=1}^n \mu_i \xi_i = 0$ , where  $d\pi(X) = \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i}$ . It is known that any Legendrian fibration is locally equivalent to  $\pi : PT^*(M) \rightarrow M$ , (cf. [1], Part III).

The main tool of the theory of Legendrian singularities is the notion of generating families. Since we only consider local properties, we may assume that  $M = \mathbb{R}^n$ . Let  $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}, \mathbf{0})$  be a function germ. We say that  $F$  is a *Morse family of hypersurfaces* if the map germ

$$\Delta^*F = \left( F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where  $(q, x) = (q_1, \dots, q_k, x_1, \dots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ . In this case, we have a smooth  $(n-1)$ -dimensional submanifold germ

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \mid F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \dots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\} = (\Delta^*F)^{-1}(0)$$

and a map germ  $\mathcal{L}_F : (\Sigma_*(F), \mathbf{0}) \rightarrow PT^*\mathbb{R}^n$  defined by

$$\mathcal{L}_F(q, x) = \left( x, \left[ \frac{\partial F}{\partial x_1}(q, x) : \dots : \frac{\partial F}{\partial x_n}(q, x) \right] \right)$$

which is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd-Zakalyukin [1, 16].

**Proposition 5.1** *All Legendrian submanifold germs in  $PT^*\mathbb{R}^n$  are constructed by the above method.*

We call  $F$  a *generating family* of  $\mathcal{L}_F(\Sigma_*(F))$ . Consequently, the wave front is

$$W(\mathcal{L}_F) = \left\{ x \in \mathbb{R}^n \mid \exists q \in \mathbb{R}^k \text{ such that } F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \dots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}.$$

We also denote that  $\mathcal{D}_F = W(\mathcal{L}_F)$  and call it the *discriminant set* of  $F$ .

Let us consider a point  $\mathbf{v} = (v_0, v_1, \dots, v_n) \in S_1^n(\sin^2 \phi)$ . Then we have  $(v_1, \dots, v_n) \neq (0, \dots, 0)$ . Without loss of generality, we suppose that  $v_1 > 0$ . We choose the local coordinate neighbourhood system  $(V_+^1, U^1, \psi)$ , where

$$V_+^1 = \{\mathbf{v} \in S_1^n(\sin^2 \phi) \mid v_1 > 0\}, \quad U^1 = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 - \sum_{i=2}^n x_i^2 + \sin^2 \phi > 0\}$$

and  $\psi : V_+^1 \longrightarrow U^1$  is induced by the canonical projection. We consider the projective cotangent bundle  $\pi : PT^*(S_1^n(\sin^2 \phi)) \longrightarrow S_1^n(\sin^2 \phi)$  with the canonical contact structure. By using the above coordinate system, we have a trivialization as follows:

$$\Phi : PT^*(V_+^1) \equiv V_+^1 \times P(\mathbb{R}^{n-1})^* ; \Phi\left(\left[\sum_{i=1}^n \xi_i dv_i\right]\right) = ((v_0, v_1, \dots, v_n), [\xi_1 : \dots : \xi_n]).$$

On the other hand, we define the mapping

$$\Psi : \Delta_{21}^-(\phi)|(H^n(-1) \times V_+^1) \longrightarrow V_+^1 \times P(\mathbb{R}^{n-1})^*$$

by

$$\Psi(\mathbf{v}, \mathbf{w}) = (\mathbf{w}, [-v_0 w_1 + v_1 w_0 : v_2 w_1 - v_1 w_2 : \dots : v_n w_1 - v_1 w_n]).$$

For the canonical contact form  $\theta = \sum_{i=1}^n \xi_i dx_i$  on  $PT^*(V_+^1)$ , we get

$$\begin{aligned} \Psi^* \theta &= ((-v_0 w_1 + v_1 w_0)dw_0 + (v_2 w_1 - v_1 w_2)dw_2 + \dots + (v_n w_1 - v_1 w_n)dw_n)|\Delta_{21}^-(\phi) \\ &= w_1(-v_0 dw_0 + v_1 dw_1 + \dots + v_n dw_n)|\Delta_{21}^-(\phi) = w_1 \langle \mathbf{v}, d\mathbf{w} \rangle |\Delta_{21}^-(\phi) = w_1 \theta|_{(\phi)_{(21)2}^-}, \end{aligned}$$

where  $w_1 = \sqrt{w_0^2 - \sum_{i=2}^n w_i^2 + \sin^2 \phi}$ . Thus,  $\Psi$  is a contact morphism.

**Proposition 5.2** *The  $\phi$ -de Sitter height function  $H_\phi^D : U \times S_1^n(\sin^2 \phi) \longrightarrow \mathbb{R}$  is a Morse family of hypersurfaces.*

*Proof.* We consider the local coordinate neighborhood  $V_+^1$ . For any  $\mathbf{v} = (v_0, v_1, \dots, v_n) \in V_+^1$ , we have  $v_1 = \sqrt{v_0^2 - \sum_{i=2}^n v_i^2 + \sin^2 \phi}$ , so that

$$H_\phi^D(u, \mathbf{v}) = -x_0(u)v_0 + x_1(u)\sqrt{v_0^2 - \sum_{i=2}^n v_i^2 + \sin^2 \phi} + x_2(u)v_2 + \dots + x_n(u)v_n + \cos \phi,$$

where  $\mathbf{X}^h(u) = (x_0(u), \dots, x_n(u))$ . We define a mapping  $\Delta^* H_\phi^D : U \times S_1^n(\sin^2 \phi) \longrightarrow \mathbb{R} \times \mathbb{R}^{n-1}$  by  $\Delta^* H_\phi^D = \left(H_\phi^D, \frac{\partial H_\phi^D}{\partial u_1}, \dots, \frac{\partial H_\phi^D}{\partial u_{n-1}}\right)$ . We have to prove that  $\Delta_* H_\phi^D$  is non-singular at any point on  $\Sigma_*(H_\phi^D) = (\Delta^* H_\phi^D)^{-1}(0)$ . By exactly the same reason as the proof of [4, Proposition 4.2], it is enough to show that  $\det A \neq 0$ , where

$$\mathbf{A} = \begin{pmatrix} -x_0 + \frac{v_0}{v_1} x_1 & x_2 - \frac{v_2}{v_1} x_1 & \cdots & x_n - \frac{v_n}{v_1} x_1 \\ -x_{0u_1} + \frac{v_0}{v_1} x_{1u_1} & x_{2u_1} - \frac{v_2}{v_1} x_{1u_1} & \cdots & x_{nu_1} - \frac{v_n}{v_1} x_{1u_1} \\ \vdots & \vdots & \vdots & \vdots \\ -x_{0u_{n-1}} + \frac{v_0}{v_1} x_{1u_{n-1}} & x_{2u_{n-1}} - \frac{v_2}{v_1} x_{1u_{n-1}} & \cdots & x_{nu_{n-1}} - \frac{v_n}{v_1} x_{1u_{n-1}} \end{pmatrix}.$$

We can also show that  $\det A \neq 0$  for any  $(u, \mathbf{v}) \in \Sigma_*(H_\phi^D)$  by the same calculations as those in the proof of [4, Proposition 4.2].

If we adopt the other local coordinates, we obtain the similar calculations to the above. This completes the proof.  $\square$

**Theorem 5.3** For any hypersurface  $\mathbf{X}^h : U \rightarrow H^n(-1)$ , the  $\phi$ -de Sitter height function  $H_\phi^D : U \times S_1^n(\sin^2 \phi) \rightarrow \mathbb{R}$  of  $M^H = \mathbf{X}^h(U)$  is a generating family of the Legendrian immersion  $\mathcal{L}_{21}[\phi](U) \subset \Delta_{21}^-(\phi)$  with respect to the Legendrian fibration  $\pi[\phi]_{(21)2}^- : \Delta_{21}^-(\phi) \rightarrow S_1^n(\sin^2 \phi)$ .

*Proof.* We remember the contact morphism

$$\Psi : \Delta_{21}^-(\phi) | (H^n(-1) \times V_+^1) \rightarrow V_+^1 \times P(\mathbb{R}^{n-1})^*.$$

Since the  $\phi$ -de Sitter height function  $H_\phi^D : U \times V_+^1 \rightarrow \mathbb{R}$  is a Morse family of hypersurfaces, we have a Legendrian immersion

$$\mathcal{L}_{H_\phi^D} : \Sigma_*(H_\phi^D) \rightarrow V_+^1 \times P(\mathbb{R}^{n-1})^*$$

defined by

$$\mathcal{L}_{H_\phi^D}(u, \mathbf{v}) = \left( \mathbf{v}, \left[ \frac{\partial H_\phi^D}{\partial v_0} : \frac{\partial H_\phi^D}{\partial v_2} : \cdots : \frac{\partial H_\phi^D}{\partial v_n} \right] \right),$$

where  $\mathbf{v} = (v_0, \dots, v_n)$  and  $v_1 = \sqrt{v_0^2 - \sum_{i=2}^n v_i^2 + \sin^2 \phi}$ . By Proposition 4.1, we get

$$\Sigma_*(H_\phi^D) = \{(u, \mathbb{N}_\pm^d[\phi](u)) \in U \times V_+^1 \mid u \in U\}.$$

Since  $\mathbf{v} = \mathbb{N}_\pm^d[\phi](u)$  and  $v_1 = \sqrt{v_0^2 - \sum_{i=2}^n v_i^2 + \sin^2 \phi}$ , we obtain

$$\frac{\partial H_\phi^D}{\partial v_0}(u, \mathbb{N}_\pm^d[\phi](u)) = -x_0(u) + \frac{n_0^\pm(u)}{n_1^\pm(u)} x_1(u), \quad \frac{\partial H_\phi^D}{\partial v_i}(u, \mathbb{N}_\pm^d[\phi](u)) = x_i(u) - \frac{n_i^\pm(u)}{n_1^\pm(u)} x_1(u),$$

where  $i = 2, \dots, n$ ,  $\mathbf{X}^h(u) = (x_0(u), \dots, x_n(u))$  and  $\mathbb{N}_\pm^d[\phi](u) = (n_0^\pm(u), \dots, n_n^\pm(u))$ . It follows that

$$\mathcal{L}_{H_\phi^D}(u, \mathbb{N}_\pm^d[\phi](u)) = (\mathbb{N}_\pm^d[\phi](u), [\boldsymbol{\xi}] ),$$

where

$$[\boldsymbol{\xi}] = [-x_0(u)n_1^\pm(u) + n_0^\pm(u)x_1(u) : x_2(u)n_1^\pm(u) - n_2^\pm(u)x_1(u) : \cdots : x_n(u)n_1^\pm(u) - n_n^\pm(u)x_1(u)].$$

Therefore, we have  $\Psi \circ \mathcal{L}_{21}[\phi](u) = \mathcal{L}_{H_\phi^D}(u)$ . This means that  $H_\phi^D$  is a generating family of  $\mathcal{L}_{21}[\phi](U) \subset \Delta_{21}^-(\phi)$  with respect to the Legendrian fibration  $\pi[\phi]_{(21)2}^- : \Delta_{21}^-(\phi) \rightarrow S_1^n(\sin^2 \phi)$ . This completes the proof.  $\square$

## 6 Contact with $\phi$ -de Sitter flat hyperquadrics

In this section, we consider the contact of hypersurfaces in Hyperbolic space with  $\phi$ -de Sitter flat hyperquadrics. For our purpose, we briefly review the theory of contact due to Montaldi [14]. Let  $X_i$  and  $Y_i$  ( $i = 1, 2$ ) be submanifolds of  $\mathbb{R}^n$  with  $\dim X_1 = \dim X_2$  and  $\dim Y_1 = \dim Y_2$ . We say that the *contact of*  $X_1$  and  $Y_1$  at  $y_1$  is the same type as the *contact of*  $X_2$  and  $Y_2$  at  $y_2$  if there is a diffeomorphism germ  $\Phi : (\mathbb{R}^n, y_1) \rightarrow (\mathbb{R}^n, y_2)$  such that  $\Phi(X_1) = X_2$  and  $\Phi(Y_1) = Y_2$ . In this case, we write  $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ . It is clear that in the definition,  $\mathbb{R}^n$  can be replaced by any manifold. In [14], Montaldi gives a characterization of the

notion of contact by using the terminology of singularity theory. Let  $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$  be map germs. We say that  $f$  and  $g$  are  $\mathcal{K}$ -equivalent if there exists a diffeomorphism germ  $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$  such that  $I(f \circ \phi) = I(g)$ , where  $I(f) = \langle f_1, \dots, f_p \rangle \varepsilon_n$  is the ideal generated by the component function germs  $f_1, \dots, f_p$  of  $f$  (i.e.,  $f = (f_1, \dots, f_p)$ ) in the local ring  $\mathcal{E}_n = \{h \mid h : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}\}$  of function germs at 0.

**Theorem 6.1** *Let  $X_i$  and  $Y_i$  ( $i = 1, 2$ ) be submanifolds of  $\mathbb{R}^n$  with  $\dim X_1 = \dim X_2$  and  $\dim Y_1 = \dim Y_2$ . Let  $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$  be immersion germs and  $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$  be submersion germs with  $(Y_i, y_i) = (f_i^{-1}(0), y_i)$ . Then  $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$  if and only if  $f_1 \circ g_1$  and  $f_2 \circ g_2$  are  $\mathcal{K}$ -equivalent.*

Now, we consider a function  $\mathcal{H}_\phi^D : H^n(-1) \times S_1^n(\sin^2 \phi) \longrightarrow \mathbb{R}$  defined by  $\mathcal{H}_\phi^D(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle + \cos \phi$ . For any  $\mathbf{v}_0 \in S_1^n(\sin^2 \phi)$ , we denote that  $(\mathfrak{h}_\phi^D)_{\mathbf{v}_0}(\mathbf{u}) = \mathcal{H}_\phi^D(\mathbf{u}, \mathbf{v}_0)$  and we have a  $\phi$ -de Sitter flat hyperquadric  $(\mathfrak{h}_\phi^D)_{\mathbf{v}_0}^{-1}(0) = HP(\mathbf{v}_0, -\cos \phi) \cap H^n(-1) = DHQ^H(\mathbf{v}_0, -\cos \phi)$ . For any  $u_0 \in U$ , we consider the spacelike vector  $\mathbf{v}_0 = \mathbb{N}_\pm^d[\phi](u_0)$ . Then we have

$$(\mathfrak{h}_\phi^D)_{\mathbf{v}_0} \circ \mathbf{X}^h(u_0) = \mathcal{H}_\phi^D \circ (\mathbf{X}^h \times \text{id}_{S_1^n(\sin^2 \phi)})(u_0, \mathbf{v}_0) = \mathcal{H}_\phi^D(u_0, \mathbb{N}_\pm^d[\phi](u_0)) = 0.$$

By Proposition 4.1, we also have the following relations for  $i = 1, \dots, n-1$  :

$$\frac{\partial ((\mathfrak{h}_\phi^D)_{\mathbf{v}_0} \circ \mathbf{X}^h)}{\partial u_i}(u_0) = \frac{\partial \mathcal{H}_\phi^D}{\partial u_i}(u_0, \mathbb{N}_\pm^d[\phi](u_0)) = 0.$$

This means that the  $\phi$ -de Sitter flat hyperquadric

$$(\mathfrak{h}_\phi^D)_{\mathbf{v}_0}^{-1}(0) = HQ^H(\mathbf{v}_0, -\cos \phi)$$

is tangent to  $M^H = \mathbf{X}^h(U)$  at  $p = \mathbf{X}^h(u_0)$ . In this case, we call  $HQ^H(\mathbf{v}_0, -\cos \phi)$  the *tangent  $\phi$ -de Sitter flat hyperquadric* of  $M^H = \mathbf{X}^h(U)$  at  $p = \mathbf{X}^h(u_0)$  (or,  $u_0$ ), which we write  $T_{DHQ}[\phi](M^H, p)$  (or,  $T_{DHQ}[\phi](\mathbf{X}^h, u_0)$ ).

Eventually, we have tools for the study of the contact between hypersurfaces and  $\phi$ -de Sitter flat hyperquadrics. Let  $(\mathbb{N}_\pm^d[\phi])_i : (U, u_i) \longrightarrow (S_1^n(\sin^2 \phi), \mathbf{v}_i)$  ( $i = 1, 2$ ) be  $\phi$ -de Sitter dual germs of hypersurface germs  $(\mathbf{X}^h)_i : (U, u_i) \longrightarrow (H^n(-1), \mathbf{u}_i)$ . We say that  $(\mathbb{N}_\pm^d[\phi])_1$  and  $(\mathbb{N}_\pm^d[\phi])_2$  are  $\mathcal{A}$ -equivalent if there exist diffeomorphism germs  $\phi : (U, u_1) \longrightarrow (U, u_2)$  and  $\Phi : (S_1^n(\sin^2 \phi), \mathbf{v}_1) \longrightarrow (S_1^n(\sin^2 \phi), \mathbf{v}_2)$  such that  $\Phi \circ (\mathbb{N}_\pm^d[\phi])_1 = (\mathbb{N}_\pm^d[\phi])_2 \circ \phi$ . We remark that the  $\mathcal{A}$ -equivalence preserve the singularities of the both map-germs. In order to understand the geometric meanings of the  $\mathcal{A}$ -equivalence among the  $\phi$ -de Sitter dual germs, we need the theory of Legendrian equivalence [1, 16, 17]. Let  $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$  and  $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$  be Legendrian immersion germs. Then we say that  $i$  and  $i'$  are *Legendrian equivalent* if there exists a contact diffeomorphism germ  $H : (PT^*\mathbb{R}^n, p) \longrightarrow (PT^*\mathbb{R}^n, p')$  such that  $H$  preserves fibers of  $\pi$  and  $H(L) = L'$ . A Legendrian immersion germ  $i : (L, p) \subset PT^*\mathbb{R}^n$  (or, a *Legendrian map*  $\pi \circ i$ ) at a point is said to be *Legendrian stable* if for every map with the given germ there is a neighbourhood in the space of Legendrian immersions (in the Whitney  $C^\infty$  topology) and a neighbourhood of the original point such that each Legendrian immersion belonging to the first neighbourhood has a point in the second neighbourhood at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift  $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$  is uniquely determined on the regular part of the wave front  $W(i)$ , we have the following simple but significant property of Legendrian immersion germs [17]:

**Proposition 6.2** *Let  $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$  and  $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$  be Legendrian immersion germs such that the representatives of both of the germs are proper mappings and the regular sets of the projections  $\pi \circ i$  and  $\pi \circ i'$  are dense. Then  $i$  and  $i'$  are Legendrian equivalent if and only if wave front sets  $W(i)$  and  $W(i')$  are diffeomorphic as set germs.*

The assumption in the above proposition is a generic condition for  $i$  and  $i'$ . Specially, if  $i$  and  $i'$  are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We consider the unique maximal ideal  $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$  of the local ring  $\mathcal{E}_n$ . Let  $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}, \mathbf{0})$  be function germs. We say that  $F$  and  $G$  are  $P$ - $\mathcal{K}$ -equivalent if there exists a diffeomorphism germ  $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$  of the form  $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$  for  $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$  such that  $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$ . Here  $\Psi^* : \mathcal{E}_{k+n} \rightarrow \mathcal{E}_{k+n}$  is the pull back  $\mathbb{R}$ -algebra isomorphism defined by  $\Psi^*(h) = h \circ \Psi$ .

Let  $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}, \mathbf{0})$  be a function germ. We say that  $F$  is a  $\mathcal{K}$ -versal deformation of  $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$  if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1} \Big|_{\mathbb{R}^k \times \{\mathbf{0}\}}, \dots, \frac{\partial F}{\partial x_n} \Big|_{\mathbb{R}^k \times \{\mathbf{0}\}} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k},$$

(See [13]). The main result in Arnol'd-Zakalyukin's theory [1, 16] is the following:

**Theorem 6.3** *Let  $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}, 0)$  be Morse families of hypersurfaces. Then*

(1)  $\mathcal{L}_F$  and  $\mathcal{L}_G$  are Legendrian equivalent if and only if  $F$  and  $G$  are  $P$ - $\mathcal{K}$ -equivalent.

(2)  $\mathcal{L}_F$  is Legendrian stable if and only if  $F$  is a  $\mathcal{K}$ -versal deformation of  $F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$ .

Since  $F$  and  $G$  are function germs on the common space germ  $(\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ , we do not need the notion of stably  $P$ - $\mathcal{K}$ -equivalences under this situation (cf., [1]).

If both of the regular sets of  $(\mathbb{N}_{\pm}^d[\phi])_i$  ( $i = 1, 2$ ) are dense in  $(U, u_i)$ , it follows from Proposition 6.2 that  $(\mathbb{N}_{\pm}^d[\phi])_1$  and  $(\mathbb{N}_{\pm}^d[\phi])_2$  are  $\mathcal{A}$ -equivalent if and only if the corresponding Legendrian immersion germs  $\mathcal{L}_{21}^1[\phi] : (U, u_1) \rightarrow (\Delta_{21}^-(\phi), z_1)$  and  $\mathcal{L}_{21}^2[\phi] : (U, u_2) \rightarrow (\Delta_{21}^-(\phi), z_2)$  are Legendrian equivalent. This condition is also equivalent to the condition that two generating families  $(\mathcal{H}_{\phi}^D)_1$  and  $(\mathcal{H}_{\phi}^D)_2$  are  $P$ - $\mathcal{K}$ -equivalent by Theorem 6.3. Here,  $(\mathcal{H}_{\phi}^D)_i : (U \times S_1^n(\sin^2 \phi), (u_i, \mathbf{v}_i)) \rightarrow \mathbb{R}$  are the  $\phi$ -de Sitter height function germs of  $(\mathbf{X}^h)_i$ .

On the other hand, if we denote that  $(h_{\phi}^D)_{i, \mathbf{v}_i}(u) = (H_{\phi}^D)_i(u, \mathbf{v}_i)$ , then we have  $(h_{\phi}^D)_{i, \mathbf{v}_i}(u) = (h_{\phi}^D)_{\mathbf{v}_i} \circ (\mathbf{X}^h)_i(u)$ . By Theorem 6.1, for  $p_i = (\mathbf{X}^h)_i(u_i)$

$$K((M^H)_1, T_{DHQ}((M^H)_1, p_1), p_1) = K((M^H)_2, T_{DHQ}((M^H)_2, p_2), p_2)$$

if and only if  $(h_{\phi}^D)_{1, \mathbf{v}_1}$  and  $(h_{\phi}^D)_{2, \mathbf{v}_2}$  are  $\mathcal{K}$ -equivalent.

**Theorem 6.4** *Let  $(\mathbf{X}^h)_i : (U, u_i) \rightarrow (H^n(-1), p_i)$  ( $i = 1, 2$ ) be hypersurface germs such that the corresponding Legendrian map germs*

$$(\mathbb{N}_{\pm}^d[\phi])_i = \pi[\phi]_{(21)2}^- \circ \mathcal{L}_{21}^i[\phi] : (U, u_i) \rightarrow (S_1^n(\sin^2 \phi), \mathbf{v}_i)$$

are Legendrian stable. Then the following conditions are equivalent:

- (1)  $(\mathbb{N}_{\pm}^d[\phi])_1$  and  $(\mathbb{N}_{\pm}^d[\phi])_2$  are  $\mathcal{A}$ -equivalent.
- (2)  $((\mathbb{N}_{\pm}^d[\phi])_1(U), z_1)$  and  $((\mathbb{N}_{\pm}^d[\phi])_2(U), z_2)$  are diffeomorphic as set germs.
- (3)  $\mathcal{L}_{21}^1[\phi] : (U, u_1) \rightarrow (\Delta_{21}^-(\phi), z_1)$  and  $\mathcal{L}_{21}^2[\phi] : (U, u_2) \rightarrow (\Delta_{21}^-(\phi), z_2)$  are Legendrian equivalent.
- (4)  $(H_{\phi}^D)_1$  and  $(H_{\phi}^D)_2$  are  $P$ - $\mathcal{K}$ -equivalent.
- (5)  $(h_{\phi}^D)_{1,v_1}$  and  $(h_{\phi}^D)_{2,v_2}$  are  $\mathcal{K}$ -equivalent.
- (6)  $K((M^H)_1, T_{DHQ}((M^H)_1, p_1), p_1) = K((M^H)_2, T_{DHQ}((M^H)_2, p_2), p_2)$ .

*Proof.* By the previous arguments (mainly from Theorem 6.1), it has been already shown that conditions (5) and (6) are equivalent. By Theorem 6.3, the conditions (3) and (4) are equivalent. By definition, the condition (4) implies the condition (5). Suppose that  $(\mathbb{N}_{\pm}^d[\phi])_i$  are Legendrian stable. By the uniqueness result of the  $P$ - $\mathcal{K}$ -versal deformation, the condition (5) implies the condition (4). Moreover, by Proposition 6.2 and Theorem 6.3, the conditions (1),(2) and (3) are equivalent.  $\square$

## 7 Surfaces in Hyperbolic 3-space

In this section, we consider a classification of singularities of  $\phi$ -de Sitter dual  $\mathbb{N}_{\pm}^d[\phi]$  for fixed  $\phi \in [0, \pi/2]$ , when  $n = 3$ . By the standard arguments of the jet-transversality theorem of Wasserman (cf., [4, 15]), we have the following theorem:

**Theorem 7.1** *There exists an open dense subset  $\mathcal{O} \subset \text{Emb}(U, H^3(-1))$  such that for any  $\mathbf{X}^h \in \mathcal{O}$ , the following conditions hold:*

(1) *The  $\phi$ -de Sitter parabolic set  $(K_{\pm}^d[\phi])^{-1}(0)$  is a regular curve. We call such a curve the  $\phi$ -de Sitter parabolic curve.*

(2) *The  $\phi$ -de Sitter dual  $\mathbb{N}_{\pm}^d[\phi]$  along the  $\phi$ -de Sitter parabolic curve is a cuspidal edge except at isolated points. At these points,  $\mathbb{N}_{\pm}^d[\phi]$  is a swallowtail.*

Here, a map germ  $f : (\mathbb{R}^2, \mathbf{a}) \rightarrow (\mathbb{R}^3, \mathbf{b})$  is called a cuspidal edge if it is  $\mathcal{A}$ -equivalent to the germ  $(u_1, u_2^2, u_2^3)$  (cf., Fig.1) and a swallowtail if it is  $\mathcal{A}$ -equivalent to the germ  $(3u_1^4 + u_1^2u_2, 4u_1^3 + 2u_1u_2, u_2)$  (cf., Fig.1).

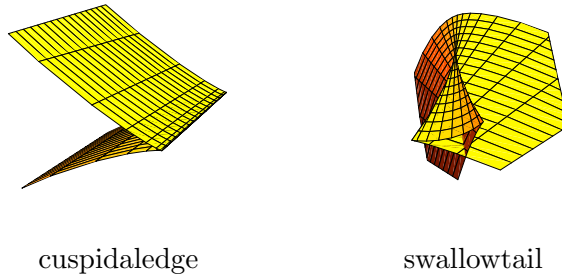


Fig.1

The assertion of Theorem 7.1 can be interpreted that the Legendrian lift  $\mathcal{L}_{21}[\phi]$  of the  $\phi$ -de Sitter dual  $\mathbb{N}_{\pm}^d[\phi]$  of  $\mathbf{X}^h \in \mathcal{O}$  is Legendrian stable at each point. We can give the geometric

meanings of cuspidal edges and swallowtails of the  $\phi$ -de Sitter dual  $\mathbb{N}_{\pm}^d[\phi]$  analogous to the results in (Banchoff et al., [2, 4]). We only give the following corollary without the proof.

**Corollary 7.2** *Let  $\mathcal{O}$  be the open and dense set in  $\text{Emb}(U, H^3(-1))$  given by Theorem 7.1. For  $\mathbf{X}^h \in \mathcal{O}$ , we have the  $\phi$ -de Sitter height function  $h_{\phi, v_0}^d$  with  $\mathbf{v}_0^{\pm} = \mathbb{N}_{\pm}^d[\phi](u_0)$ . Then we have the following:*

(1) *Suppose that  $u_0$  is a  $\phi^{\pm}$ -de Sitter parabolic point of  $\mathbf{X}^h$ . Then the following conditions are equivalent:*

(a)  $\mathbb{N}_{\pm}^d[\phi]$  has the cuspidal edge at  $u_0$ .

(b) *Tangent  $\phi^{\pm}$ -de Sitter flat indicatrix is the ordinary cusp, where a curve  $C \subset \mathbb{R}^2$  is called the ordinary cusp if it is diffeomorphic to the curve given by  $\{(u_1, u_2) \mid u_1^2 - u_2^3 = 0\}$ .*

(2) *Suppose that  $u_0$  is a  $\phi^{\pm}$ -de Sitter parabolic point of  $\mathbf{X}^h$ . Then the following conditions are equivalent:*

(a)  $\mathbb{N}_{\pm}^d[\phi]$  has the swallowtail at  $u_0$ .

(b) *Tangent  $\phi^{\pm}$ -de Sitter flat indicatrix is a point or the tachnodal, where a curve  $C \subset \mathbb{R}^2$  is called the tachnodal if it is diffeomorphic to the curve given by  $\{(u_1, u_2) \mid u_1^2 - u_2^4 = 0\}$ .*

The pictures of the ordinary cusp and the tachnodal are given in Fig.2.

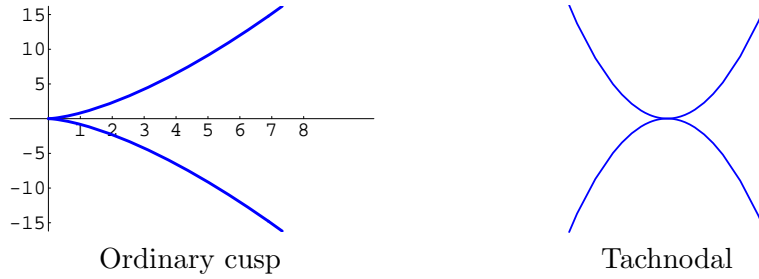


Fig.2

We can study more detailed properties of the surfaces in Hyperbolic 3-space. The bifurcations of the singularities of  $\phi^{\pm}$ -de Sitter dual  $\mathbb{N}_{\pm}^d[\phi]$  depending on  $\phi$  are specially interesting. Nevertheless, we stop here. These will be discussed in the forthcoming papers.

## 8 Hyperbolic Monge forms and examples

In order to give examples of hypersurfaces in Hyperbolic space, we review the notion of hyperbolic Monge form which was introduced in [4].

We consider a function  $f(u_1, \dots, u_{n-1})$  with  $f(0) = 0$  and  $f_{u_i}(0) = 0$ . Then we have a hypersurface in  $H_+^n(-1)$  defined by

$$\mathbf{X}_f^h(u_1, \dots, u_{n-1}) = (\sqrt{f^2(u_1, \dots, u_{n-1}) + u_1^2 + \dots + u_{n-1}^2 + 1}, f(u_1, \dots, u_{n-1}), u_1, \dots, u_{n-1}).$$

We can easily calculate that  $\mathbf{X}_f^h(0) = (1, 0, \dots, 0)$  and  $\mathbf{X}_f^d(0) = (0, -1, 0, \dots, 0)$ , therefore  $\mathbb{N}_{\pm}^d[\phi](0) = (\cos \phi, \mp 1, 0, \dots, 0)$ . We call  $\mathbf{X}_f^h$  a *hyperbolic Monge form* (briefly, *H-Monge form*). In [4], it has been shown that any hypersurface in  $H^n(-1)$  is locally given by the H-Monge form.



For the lightlike vector  $\mathbf{v}_0^\pm = (1, \mp 1, 0, \dots, 0)$ , we consider the hyperhorosphere  $HS(\mathbf{v}_0^\pm, -1)$ . Then we have the following H-Monge form of  $HS(\mathbf{v}_0^\pm, -1)$ :

$$\mathbf{h}^\pm(u_1, \dots, u_{n-1}) = \left( \frac{u_1^2 + \dots + u_{n-1}^2 + 2}{2}, \mp \frac{u_1^2 + \dots + u_{n-1}^2}{2}, u_1, \dots, u_{n-1} \right).$$

Here, we can easily check the relation that  $\langle \mathbf{v}_0^\pm, \mathbf{h}^\pm(u) \rangle = -1$ . Moreover, we consider the spacelike vector  $\mathbf{v}_0^\pm[\phi] = (\cos \phi, \mp 1, 0, \dots, 0)$ . Then we have the following H-Monge form of the  $\phi^\pm$ -flat hyperbolic hyperquadric  $HQ^H(\mathbf{v}_0^\pm[\phi], -\cos \phi)$ :

$$\mathbf{X}_\pm^{HQ^H}[\phi](u_1, \dots, u_{n-1}) = \left( \frac{\sin^2 \phi - 1 + \sqrt{1 + \sin^2 \phi(u_1^2 + \dots + u_{n-1}^2)}}{\sin^2 \phi}, \right. \\ \left. \frac{\pm \cos \phi \left( 1 - \sqrt{1 + \sin^2 \phi(u_1^2 + \dots + u_{n-1}^2)} \right)}{\sin^2 \phi}, u_1, \dots, u_{n-1} \right),$$

for  $\phi \in (0, \pi/2]$ .

On the other hand,  $\mathbf{X}_\pm^{HQ^H}[\phi](0) = (1, 0, \dots, 0) = p$  and  $(\mathbf{X}_\pm^{HQ^H}[\phi])_{u_i}(0) = (\mathbf{X}_f^h)_{u_i}(0)$  for  $i = 1, 2, \dots, n-1$ . This means that  $T_p M^H = T_p(\mathbf{X}_\pm^{HQ^H}[\phi](U))$ . Therefore,  $\mathbf{X}_\pm^{HQ^H}[\phi](U) = HQ^H(\mathbf{v}_0^\pm[\phi], -\cos \phi)$  is the tangent  $\phi^\pm$ -de Sitter flat hyperquadric of  $M^H = \mathbf{X}_f^h(U)$  at  $p = \mathbf{X}_f^h(0)$ . It follows from this fact that the tangent  $\phi^\pm$ -de Sitter flat hyperquadric indicatrix of the Monge form germ  $(\mathbf{X}_f^h, 0)$  is given as follows:

$$(\mathbf{X}_f^h)^{-1}(HQ^H(\mathbf{v}_0^\pm[\phi], -\cos \phi)) = \{(u_1, \dots, u_{n-1}) \in U \mid \pm \sin^2 \phi f(u_1, \dots, u_{n-1}) \\ = \cos \phi \left( 1 - \sqrt{1 + \sin^2 \phi(u_1^2 + \dots + u_{n-1}^2)} \right)\}.$$

Since the  $\phi$ -de Sitter height function of  $\mathbf{X}_f^h$  at  $\mathbf{v}_0^\pm[\phi]$  is

$$h_{\mathbf{v}_0^\pm[\phi]}^D(u) = -\cos \phi \sqrt{f^2(u_1, \dots, u_{n-1}) + u_1^2 + \dots + u_{n-1}^2 + 1} \mp f(u_1, \dots, u_{n-1}) + \cos \phi,$$

we can calculate the Hessian matrix. Then we have  $\text{Hess}(h_{\mathbf{v}_0^\pm[\phi]}^D)(0) = \mp \text{Hess}(f)(0) - \cos \phi I_{n-1}$ .

On the other hand, since  $f(0) = f_{u_i}(0) = 0$ , we may write

$$f(u_1, \dots, u_{n-1}) = \frac{\kappa_1}{2} u_1^2 + \dots + \frac{\kappa_{n-1}}{2} u_{n-1}^2 + g(u_1, \dots, u_{n-1}),$$

where  $g \in \mathfrak{M}_{n-1}^3$  and  $\kappa_1, \dots, \kappa_{n-1}$  are eigenvalues of  $\text{Hess}(f)(0)$ . Under this representation, we can easily calculate that  $\mathbf{X}_{f, u_i u_j}^h(0) = (\delta_{ij}, f_{u_i u_j}(0), 0, \dots, 0)$ . It follows from this fact that

$$h_{\pm}^D[\phi]_{ij}(0) = \delta_{ij}(-\cos \phi \mp \kappa_i) \text{ and } g_{ij}^H(0) = \delta_{ij}.$$

Therefore, we have  $\bar{\kappa}^\pm[\phi]_i(0) = -\cos \phi \mp \kappa_i$  and

$$K_\pm^d[\phi](0) = \prod_{i=1}^{n-1} \bar{\kappa}^\pm[\phi]_i(0) = \prod_{i=1}^{n-1} (-\cos \phi \mp \kappa_i).$$

The tangent  $\phi^\pm$ -flat hyperbolic hyperquadric indicatrix is given by

$$\begin{aligned} (\mathbf{X}_f^h)^{-1}(HQ^H(\mathbf{v}_0^\pm[\phi], -\cos\phi)) &= \{(u_1, \dots, u_{n-1}) \mid \sin^2\phi f^2(u_1, \dots, u_{n-1}) \\ &= -\cos\phi(\sum_{i=1}^{n-1} \bar{\kappa}^\pm[\phi]_i(0)u_i^2 \mp 2g(u_1, \dots, u_{n-1}))\}. \end{aligned}$$

In the last part of this section, we give some examples and draw their pictures for the case  $n = 3$ . In order to draw pictures, we consider the Poincaré ball model. Let  $D = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 < 1\}$  be the Poincaré ball model of Hyperbolic space. Here, we consider the hyperbolic metric

$$ds^2 = \frac{4(dx_1^2 + dx_2^2 + dx_3^2)}{1 - x_1^2 - x_2^2 - x_3^2}.$$

It has been known that there is a canonical isometric diffeomorphism  $\Phi : H_+^3(-1) \rightarrow D$  given by

$$\Phi(x_0, x_1, x_2, x_3) = \left( \frac{x_1}{x_0 + 1}, \frac{x_2}{x_0 + 1}, \frac{x_3}{x_0 + 1} \right).$$

For a H-Monge form

$$\mathbf{X}_f^h(u_1, u_2) = (\sqrt{f^2(u_1, u_2) + u_1^2 + u_2^2 + 1}, f(u_1, u_2), u_1, u_2),$$

the composition  $\Phi \circ \mathbf{X}_f^h(u_1, u_2)$  is given by

$$\left( \frac{f(u_1, u_2)}{\sqrt{f^2(u_1, u_2) + u_1^2 + u_2^2 + 1}}, \frac{u_1}{\sqrt{f^2(u_1, u_2) + u_1^2 + u_2^2 + 1}}, \frac{u_2}{\sqrt{f^2(u_1, u_2) + u_1^2 + u_2^2 + 1}} \right).$$

Therefore, for any function  $f(u_1, u_2)$  with  $f(0) = f_{u_i}(0) = 0$ , we can draw the picture of the surface in  $D$ . For the  $\phi^\pm$ -flat hyperbolic quadric  $HQ^H(\mathbf{v}_0^\pm[\phi], -\cos\phi)$ , we have

$$\begin{aligned} \Phi \circ \mathbf{X}_\pm^{HQ^H}[\phi](u_1, u_2) &= \left( \frac{\pm \cos\phi \left(1 - \sqrt{1 + \sin^2\phi(u_1^2 + u_2^2)}\right)}{2 \sin^2\phi - 1 + \sqrt{1 + \sin^2\phi(u_1^2 + u_2^2)}}, \right. \\ &\quad \frac{\sin^2\phi u_1}{2 \sin^2\phi - 1 + \sqrt{1 + \sin^2\phi(u_1^2 + u_2^2)}}, \\ &\quad \left. \frac{\sin^2\phi u_2}{2 \sin^2\phi - 1 + \sqrt{1 + \sin^2\phi(u_1^2 + u_2^2)}} \right). \end{aligned}$$

The pictures of  $\Phi \circ \mathbf{X}_\pm^{HQ^H}[\phi](U)$  for  $\phi = \pi/4$  and  $\pi/2$  are given by the pictures in Fig.3.

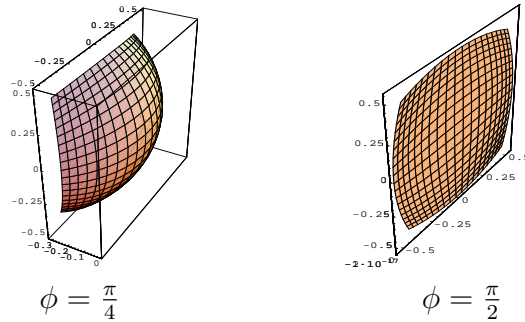
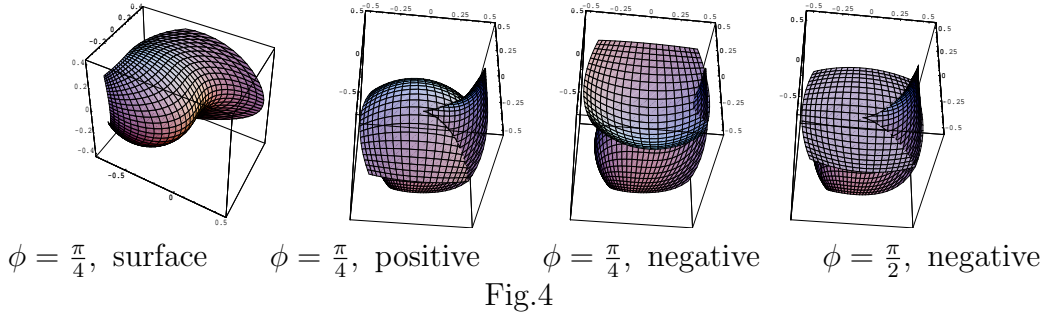


Fig.3

**Example 8.1** Suppose that

$$f(u_1, u_2) = \frac{\cos \phi}{\sin^2 \phi} \left( 1 - \sqrt{1 + \sin^2 \phi (u_1^2 + u_2^2)} \right) + \frac{2}{3} u_1^3 - u_2^2.$$

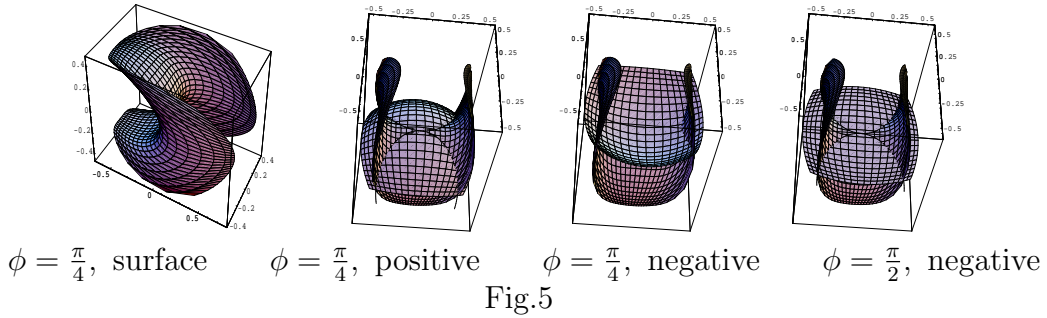
Then  $\kappa_1 = -\cos \phi$  and  $\kappa_2 = -\cos \phi - 2$ , so that we have  $K_+^d[\phi](0) = 0$  and  $K^d(0) = 4 \cos \phi (\cos \phi + 1)$ . Therefore, the origin is a  $\phi^+$ -de Sitter parabolic point for any  $\phi \in (0, \pi/2]$ . However,  $\bar{\kappa}_1^+(0) = -1$  and  $\bar{\kappa}_2^+(0) = 0$ . The tangent  $\phi^+$ -flat (or, positive  $\phi$ -flat) hyperbolic hyperquadric indicatrix is the ordinary cusp  $u_2^2 = \frac{2}{3} u_1^3$ . Consequently, the  $\phi^+$  (or positive  $\phi$ ) -de Sitter dual  $\mathbb{N}_+^d[\phi]$  is a cuspidal edge at the origin. However, the  $\phi^-$  (or negative  $\phi$ ) -de Sitter dual  $\mathbb{N}_-^d[\phi]$  is non-singular at the origin for  $\phi \neq \pi/2$ . Since the tangent hyperplane ( $\pi/2$ -flat hyperbolic quadric) is unique, the negative  $\pi/2$ -de Sitter dual  $\mathbb{N}_-^d[\pi/2]$  is also a cuspidal edge at the origin. The surface and the intersection of it with the positive  $\phi$ -de Sitter hyperquadric are depicted in Fig.4.



**Example 8.2** Consider a function

$$f(u_1, u_2) = \frac{\cos \phi}{\sin^2 \phi} \left( 1 - \sqrt{1 + \sin^2 \phi (u_1^2 + u_2^2)} \right) + u_1^4 - u_2^2.$$

Then  $\kappa_1 = -\cos \phi$  and  $\kappa_2 = -\cos \phi - 2$ . By the same reason as the previous example, the origin is a  $\phi^+$ -de Sitter parabolic point for any  $\phi \in (0, \pi/2]$  but it is an  $H^+$ -parabolic point. The positive  $\phi$ -flat hyperbolic hyperquadric indicatrix is the tachnode  $u_2^2 = u_1^4$ . So, the positive  $\phi$ -de Sitter dual  $\mathbb{N}_+^d[\phi]$  is a swallowtail at the origin. However, the negative  $\phi$ -de Sitter dual  $\mathbb{N}_-^d[\phi]$  is non-singular at the origin for  $\phi \neq \pi/2$ . By the same reason as the previous example, the negative  $\pi/2$ -de Sitter dual  $\mathbb{N}_-^d[\pi/2]$  is a swallowtail at the origin. The surface and the intersection of it with the positive horosphere are also depicted in Fig.5.



## 9 Slant geometry of spacelike hypersurfaces in de Sitter space

In this section, we establish a new extrinsic differential geometry on spacelike hypersurfaces in de Sitter space with respect to the  $\phi$ -hyperbolic duals as an application of the extended mandala of Legendrian dualities. We call this geometry a  $\phi$ -hyperbolic flat geometry. The results are analogous to those of the previous sections. So, from now on, we omit almost all of the proofs except some special cases for the assertions.

We consider the contact manifold  $(\Delta_{31}^+(\phi), K[\phi]_{31}^+)$  and the contact diffeomorphism  $\Psi_{1(31)}^+ : \Delta_1 \rightarrow \Delta_{31}^+(\phi)$  defined by

$$\Psi_{1(31)}^+(\mathbf{v}, \mathbf{w}) = (\pm\mathbf{v} + \cos\phi\mathbf{w}, \mathbf{w}).$$

Suppose that  $\mathbf{X}^d : U \rightarrow S_1^n$  is a spacelike embedding. Then we define a map  $\mathbb{N}_{\pm}^h[\phi] : U \rightarrow H^n(-\sin^2\phi)$  by

$$\mathbb{N}_{\pm}^h[\phi](u) = \pm\mathbf{X}^h(u) + \cos\phi\mathbf{X}^d(u)$$

for  $\phi \in [0, \pi/2]$  and have a map  $\mathcal{L}_{31}[\phi] : U \rightarrow \Delta_{31}^+(\phi)$  defined by  $\mathcal{L}_{31}[\phi](u) = (\mathbb{N}_{\pm}^h[\phi](u), \mathbf{X}^d(u))$ . Since  $\mathcal{L}_{31}[\phi] = \Psi_{1(31)}^+ \circ \mathcal{L}_1$ ,  $\mathcal{L}_{31}[\phi]$  is a Legendrian embedding, so that  $\mathbb{N}_{\pm}^h[\phi](u)$  can be considered as a normal vector of  $M^D$  at  $p = \mathbf{X}^d(u)$ . Moreover,  $\mathbb{N}_{\pm}^h[\phi](u)$  is the  $\phi^{\pm}$ -hyperbolic dual of  $\mathbf{X}^d(U) = M^D$ . We remark that  $\mathbb{N}_{\pm}^h[0](u) = \mathbf{X}^d(u) \pm \mathbf{X}^h(u)$  and  $\mathbb{N}_{\pm}^h[\pi/2](u) = \pm\mathbf{X}^h(u)$ .

We consider a hypersurface  $HQ^D(\mathbf{n}, c)$  in de Sitter space  $S_1^n$  defined by

$$HQ^D(\mathbf{n}, c) = HP(\mathbf{n}, c) \cap S_1^n.$$

We say that  $HQ^D(\mathbf{n}, c)$  is a *hyperquadric* in de Sitter space. We respectively say that  $HQ^D(\mathbf{n}, c)$  is an *elliptic hyperquadric*, a *hyperbolic hyperquadric* and a *parabolic hyperquadric* (or, *de Sitter hyperhorosphere*) if  $\mathbf{n}$  is timelike, spacelike and lightlike. An elliptic hyperquadric with  $c = 0$  is called a *small elliptic hyperquadric*. If  $c = \cos\phi$  and  $\mathbf{n} \in H^n(-\sin^2\phi)$  for  $\phi \in [0, \pi/2]$ , we call  $HQ^D(\mathbf{n}, \cos\phi)$  a  $\phi$ -flat *elliptic hyperquadric* which is a hyperhorosphere if  $\phi = 0$  and a hyperplane if  $\phi = \pi/2$ . It is easy to show the following proposition.

**Proposition 9.1** *If one of the  $\phi^{\pm}$ -hyperbolic duals  $\mathbb{N}_{\pm}^h[\phi]$  of  $M^D = \mathbf{X}^d(U)$  is a constant map, then  $M^D$  is a subset of a  $\phi^{\pm}$ -flat elliptic hyperquadric.*

We only remark that  $M^D$  is a subset of  $HQ^D(\mathbf{n}, \cos\phi)$  for  $\mathbf{n} = \mathbb{N}_{\pm}^h[\phi](u)$ . Therefore,  $HQ^D(\mathbf{n}, \cos\phi)$  is a candidate of the flat model hypersurfaces in the geometry on spacelike hypersurfaces in de Sitter space such that  $\mathbb{N}_{\pm}^h[\phi]$  plays a similar role to the Gauss map for a hypersurface in Euclidean space. We call this geometry a *slant geometry of spacelike hypersurfaces* in de Sitter space. When  $\phi = 0$ , the Legendrian dual is  $\mathbb{N}_{\pm}^h[0] = \mathbf{X}^d(u) \pm \mathbf{X}^h(u)$  which has been investigated by Kasedou [12]. We call this geometry a *horizontal geometry* of spacelike hypersurfaces in de Sitter space. We also call the slant geometry with  $\phi = \pi/2$  a *vertical geometry* of spacelike hypersurfaces in de Sitter space.

Now, we study the extrinsic differential geometry of  $M^D = \mathbf{X}^d(U)$  by using  $\mathbb{N}_{\pm}^h[\phi]$  like as the Gauss map of a hypersurface in Euclidean space. By exactly the same reasons as in §3,  $d\mathbb{N}_{\pm}^h[\phi](u_0)$  can be considered as a linear transformation on  $T_p M^D$ , where  $p = \mathbf{X}^d(u_0)$ . Under

the identification of  $U$  and  $M^D$  through the embedding  $\mathbf{X}^d$ , the derivative  $d\mathbf{X}^d(u_0)$  is the identity mapping  $\text{id}_{T_p M^D}$  on  $T_p M^D$ . We have the following relation:

$$d\mathbb{N}_{\pm}^h[\phi](u_0) = \pm d\mathbf{X}^h(u_0) + \cos\phi \text{id}_{T_p M^D}.$$

We call the linear transformations  $S_{\pm}^h[\phi](p) = -d\mathbb{N}_{\pm}^h[\phi](p) : T_p M^D \rightarrow T_p M^D$  and  $A_h(p) = -d\mathbf{X}^h(p) : T_p M^D \rightarrow T_p M^D$ , a  $\phi$ -hyperbolic shape operator and a hyperbolic shape operator, respectively. We denote the eigenvalues of  $S_{\pm}^h[\phi](p)$  and  $A_h(p)$  by  $\bar{\kappa}_{\pm}^h[\phi](p)$  and  $\kappa_h(p)$ , respectively. Because of the relation  $S_{\pm}^h[\phi](p) = \pm A_h(p) - \cos\phi \text{id}_{T_p M^D}$ ,  $S_{\pm}^h[\phi](p)$  and  $A_h(p)$  have the common eigen vectors. As a result, we get a relation  $\bar{\kappa}_{\pm}^h[\phi](p) = \pm\kappa_h(p) - \cos\phi$ . We call  $\bar{\kappa}_{\pm}^h[\phi](p)$  and  $\kappa_h(p)$ , a  $\phi^{\pm}$ -hyperbolic principal curvature and a principal curvature of  $M^D = \mathbf{X}^d(U)$  at  $p = \mathbf{X}^d(u_0)$ , respectively. We give the following definitions of the curvatures of  $M^D = \mathbf{X}^d(U)$  at  $p = \mathbf{X}^d(u_0)$ :

$K_{\pm}^h[\phi](u_0) = \det S_{\pm}^h[\phi](p)$ ; the  $\phi^{\pm}$ -hyperbolic Gauss-Kronecker curvature,

$H_{\pm}^h[\phi](u_0) = \frac{1}{n-1} \text{Trace } S_{\pm}^h[\phi](p)$ ; the  $\phi^{\pm}$ -hyperbolic mean curvature.

We also define the hyperbolic Gauss-Kronecker curvature and the hyperbolic mean curvature of  $M^D = \mathbf{X}^d(U)$  at  $p = \mathbf{X}^d(u_0)$  by  $K^h(p) = \det A_h(p)$  and  $H^h(p) = \frac{1}{n-1} \text{Trace } A_h(p)$ , respectively. If  $n = 3$ , we have a relation  $K_{\pm}^h[\phi] = \cos^2\phi \mp 2\cos\phi H^h + K^h$ .

Since  $\mathbf{X}_{u_i}^d$  ( $i = 1, \dots, n-1$ ) are spacelike vectors, the induced Riemannian metric (the first fundamental form) on  $M^D = \mathbf{X}^d(U)$  is given by

$$ds^2 = \sum_{i,j=1}^{n-1} g_{ij}^D du_i du_j,$$

where  $g_{ij}^D(u) = \langle \mathbf{X}_{u_i}^d(u), \mathbf{X}_{u_j}^d(u) \rangle$  for any  $u \in U$ . We also define the  $\phi^{\pm}$ -hyperbolic second fundamental invariant by  $h_{\pm}^H[\phi]_{ij}(u) = \langle -(\mathbb{N}_{\pm}^h[\phi])_{u_i}(u), \mathbf{X}_{u_j}^d(u) \rangle$  for any  $u \in U$ . If we denote that  $h_{ij}(u) = \langle -\mathbf{X}_{u_i}^d(u), \mathbf{X}_{u_j}^h(u) \rangle$ , then we have the following relation:

$$h_{\pm}^H[\phi]_{ij}(u) = -\cos\phi g_{ij}^D(u) \pm h_{ij}(u).$$

**Proposition 9.2** *Under the above notations, we have the following  $\phi^{\pm}$ -hyperbolic Weingarten formula:*

$$(\mathbb{N}_{\pm}^h[\phi])_{u_i} = -\sum_{j=1}^{n-1} h_{\pm}^H[\phi]_i^j \mathbf{X}_{u_j}^d,$$

where  $(h_{\pm}^H[\phi]_i^j) = (h_{\pm}^H[\phi]_{ik}) ((g^D)^{kj})$  and  $((g^D)^{kj}) = (g_{kj}^D)^{-1}$ .

As a corollary of the above proposition, we obtain an explicit expression of the  $\phi^{\pm}$ -hyperbolic Gauss-Kronecker curvature by Riemannian metric and the  $\phi$ -hyperbolic second fundamental invariant.

**Corollary 9.3** *Under the same notations as in the above proposition, the  $\phi^{\pm}$ -hyperbolic Gauss-Kronecker curvature is given by*

$$K_{\pm}^h[\phi] = \frac{\det(h_{\pm}^H[\phi]_{ij})}{\det(g_{\alpha\beta}^D)}.$$

We say that a point  $u \in U$  or  $p = \mathbf{X}^d(u)$  is an *umbilic point* if  $S_{\pm}^h[\phi](p) = \bar{\kappa}_{\pm}^h[\phi](p) \text{id}_{T_p M^D}$ . We also say that  $M^D = \mathbf{X}^d(U)$  is *totally umbilic* if all points of  $M^D$  are umbilic. Here, we give a classification of totally umbilic spacelike hypersurfaces by using the  $\phi^{\pm}$ -hyperbolic principal curvature.

**Proposition 9.4** *Suppose that  $M^D = \mathbf{X}^d(U)$  is totally umbilic for fixed  $\phi \in [0, \frac{\pi}{2}]$ . Then  $\bar{\kappa}_{\pm}^h[\phi](p)$  is constant  $\bar{\kappa}_{\pm}^h[\phi]$ . Under this condition, we have the following classification:*

(1) *Suppose that  $\bar{\kappa}_{\pm}^h[\phi] \neq 0$ .*

(a) *If  $0 < |\kappa_h| = |\bar{\kappa}_{\pm}^h[\phi] + \cos \phi| < 1$ , then  $M^D$  is a part of an elliptic hyperquadric.*

(b) *If  $1 < |\kappa_h| = |\bar{\kappa}_{\pm}^h[\phi] + \cos \phi|$ , then  $M^D$  is a part of a hyperbolic hyperquadric.*

(c) *If  $|\kappa_h| = |\bar{\kappa}_{\pm}^h[\phi] + \cos \phi| = 1$ , then  $M^D$  is a part of a parabolic hyperquadric.*

(d) *If  $\kappa_h = \bar{\kappa}_{\pm}^h[\phi] + \cos \phi = 0$ , then  $M^D$  is a part of a flat elliptic hyperquadric.*

(2) *If  $\bar{\kappa}_{\pm}^h[\phi] = 0$ , then  $M^D$  is a part of a  $\phi$ -flat elliptic hyperquadric.*

In the above classification, the hyperquadric  $HL(\mathbf{c}, -(1 + \cos \phi))$  ( $\phi \in [0, \pi/2]$ ) with  $\bar{\kappa}^h[\phi] = 0$  is called a  $\phi$ -hyperbolic flat hyperquadric. We say that  $p = \mathbf{X}^d(u)$  is a  $\phi^{\pm}$ -hyperbolic parabolic point if  $K_{\pm}^h[\phi^{\pm}](u) = 0$  and a  $\phi^{\pm}$ -hyperbolic flat point if it is an umbilic point and  $K_{\pm}^h[\phi](u) = 0$  which are equivalent to the condition (2).

## 10 $\phi$ -hyperbolic height functions

We define a family of functions

$$H_{\phi}^H : U \times H^n(-\sin^2 \phi) \longrightarrow \mathbb{R}$$

by  $H_{\phi}^H(u, \mathbf{v}) = \langle \mathbf{X}^d(u), \mathbf{v} \rangle - \cos \phi$ . We call  $H_{\phi}^H$  a  $\phi$ -hyperbolic height function on  $M^D = \mathbf{X}^d(U)$ . We have the following proposition:

**Proposition 10.1** *Let  $H_{\phi}^H : U \times H^n(-\sin^2 \phi) \longrightarrow \mathbb{R}$  be a  $\phi$ -hyperbolic height function on  $\mathbf{X}^d : U \longrightarrow S_1^n$ . Then*

(1)  *$H_{\phi}^H(u, \mathbf{v}) = 0$  if and only if  $(\mathbf{v}, \mathbf{X}^d(u)) \in \Delta_{31}^+(\phi)$ .*

(2)  *$H_{\phi}^H(u, \mathbf{v}) = \frac{\partial H_{\phi}^H}{\partial u_i}(u, \mathbf{v}) = 0$  ( $i = 1, \dots, n-1$ ) if and only if  $\mathbf{v} = \mathbb{N}_{\pm}^h[\phi](u)$ .*

We denote the Hessian matrix of the  $\phi$ -hyperbolic height function  $h_{\phi, v_0}^H(u) = H_{\phi}^H(u, \mathbf{v}_0)$  at  $u_0$  by  $\text{Hess}(h_{\phi, v_0}^H)(u_0)$ .

**Proposition 10.2** *Let  $\mathbf{X}^d : U \longrightarrow S_1^n$  be a spacelike hypersurface in de Sitter space and  $\mathbf{v}_0 = \mathbb{N}_{\pm}^h[\phi](u_0)$ . Then we have the following:*

(1)  *$p = \mathbf{X}^d(u_0)$  is a parabolic point if and only if  $\det \text{Hess}(h_{\phi, v_0}^H)(u_0) = 0$ .*

(2)  *$p = \mathbf{X}^d(u_0)$  is a flat point if and only if  $\text{rank} \text{Hess}(h_{\phi, v_0}^H)(u_0) = 0$ .*

## 11 The $\phi$ -hyperbolic dual as a wave front

In this section, we naturally interpret the  $\phi^\pm$ -hyperbolic dual of a spacelike hypersurface in de Sitter space as a wave front set in the framework of contact geometry. We also need the theory of Legendrian singularities which has been reviewed in §5.

We consider a point  $\mathbf{v} = (v_0, v_1, \dots, v_n) \in H^n(-\sin^2 \phi)$ . Then we have the relation  $v_0 = \pm \sqrt{v_1^2 + \dots + v_n^2 + \sin^2 \phi}$ . Without the loss of generality, we choose the local coordinate neighbourhood  $V_+ = \{\mathbf{v} \in H^n(-\sin^2 \phi) \mid v_0 > 0\}$  and the canonical projection onto  $\mathbb{R}^n$  as a local coordinate system. We consider the projective cotangent bundle  $\pi : PT^*(H^n(-\sin^2 \phi)) \rightarrow H^n(-\sin^2 \phi)$  with the canonical contact structure. By using the above coordinate system, we have a trivialization as follows:

$$\Phi : PT^*(V_+) \cong V_+ \times P(\mathbb{R}^{n-1})^* ; \Phi([\sum_{i=1}^n \xi_i dv_i]) = ((v_0, v_1, \dots, v_n), [\xi_1 : \dots : \xi_n]).$$

On the other hand, we define the following mapping:

$$\Psi : \Delta_{31}^+(\phi) \rightarrow H^n(-\sin^2 \phi) \times P(\mathbb{R}^{n-1})^* ; \Psi(\mathbf{v}, \mathbf{w}) = (\mathbf{v}, [v_0 w_1 - v_1 w_0 : \dots : v_0 w_n - v_n w_0]).$$

By exactly the same calculations as those in §5, we have  $\Psi^* \theta = v_0 \theta[\phi]_{(31)1}^+$  for the canonical contact form  $\theta = \sum_{i=1}^n \xi_i dv_i$  on  $PT^*(V_+)$ , so that  $\Psi$  is a contact morphism. Then we can give the following proposition:

**Proposition 11.1** *The  $\phi$ -hyperbolic height function  $H_\phi^H : U \times H^n(-\sin^2 \phi) \rightarrow \mathbb{R}$  is a Morse family of hypersurfaces.*

We can also give the following theorem:

**Theorem 11.2** *For any spacelike hypersurface  $\mathbf{X}^d : U \rightarrow S_1^n$ , the  $\phi$ -hyperbolic height function  $H_\phi^H : U \times H^n(-\sin^2 \phi) \rightarrow \mathbb{R}$  of  $M^D = \mathbf{X}^d(U)$  is a generating family of the Legendrian immersion  $\mathcal{L}_{31}[\phi](U) \subset \Delta_{31}^+(\phi)$  with respect to the Legendrian fibration  $\pi[\phi]_{(31)1}^+ : \Delta_{31}^+(\phi) \rightarrow H^n(-\sin^2 \phi)$ .*

## 12 Contact with $\phi$ -hyperbolic flat hyperquadrics

In this section, we consider the contact of spacelike hypersurfaces in de Sitter space with  $\phi$ -hyperbolic flat hyperquadrics. We also use the theory of contact due to Montaldi [14].

Now, we consider a function  $\mathcal{H}_\phi^H : S_1^n \times H^n(-\sin^2 \phi) \rightarrow \mathbb{R}$  defined by  $\mathcal{H}_\phi^H(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle - \cos \phi$ . For any  $\mathbf{v}_0 \in H^n(-\sin^2 \phi)$ , we denote that  $(\mathfrak{h}_\phi^H)_{\mathbf{v}_0}(\mathbf{u}) = \mathcal{H}_\phi^H(\mathbf{u}, \mathbf{v}_0)$  and we have a  $\phi$ -hyperbolic flat hyperquadric  $(\mathfrak{h}_\phi^H)_{\mathbf{v}_0}^{-1}(0) = HP(\mathbf{v}_0, \cos \phi) \cap S_1^n = HQ^D(\mathbf{v}_0, \cos \phi)$ . For any  $u_0 \in U$ , we consider the timelike vector  $\mathbf{v}_0 = \mathbb{N}_\pm^h[\phi](u_0)$ . Then we have

$$(\mathfrak{h}_\phi^H)_{\mathbf{v}_0} \circ \mathbf{X}^d(u_0) = \mathcal{H}_\phi^H \circ (\mathbf{X}^d \times \text{id}_{H^n(-\sin^2 \phi)})(u_0, \mathbf{v}_0) = \mathcal{H}_\phi^H(u_0, \mathbb{N}_\pm^h[\phi](u_0)) = 0.$$

By Proposition 9.1, we also have the following relations for  $i = 1, \dots, n-1$ :

$$\frac{\partial ((\mathfrak{h}_\phi^H)_{\mathbf{v}_0} \circ \mathbf{X}^d)}{\partial u_i}(u_0) = \frac{\partial \mathcal{H}_\phi^H}{\partial u_i}(u_0, \mathbb{N}_\pm^h[\phi](u_0)) = 0.$$

This means that the  $\phi$ -hyperbolic flat hyperquadric

$$(\mathfrak{h}_\phi^H)^{-1}(0) = HQ^D(\mathbf{v}_0, \cos \phi)$$

is tangent to  $M^D = \mathbf{X}^d(U)$  at  $p = \mathbf{X}^d(u_0)$ . In this case, we call  $HQ^D(\mathbf{v}_0, \cos \phi)$  the *tangent  $\phi$ -hyperbolic flat hyperquadric* of  $M^D = \mathbf{X}^d(U)$  at  $p = \mathbf{X}^d(u_0)$  (or,  $u_0$ ), which we write  $T_{HHQ}[\phi](M^D, p)$  (or,  $T_{HHQ}[\phi](\mathbf{X}^d, u_0)$ ).

Eventually, we have tools for the study of the contact between spacelike hypersurfaces and  $\phi$ -hyperbolic flat hyperquadrics. Let  $(\mathbb{N}_\pm^h[\phi])_i : (U, u_i) \longrightarrow (H^n(-\sin^2 \phi), \mathbf{v}_i)$  ( $i = 1, 2$ ) be  $\phi$ -hyperbolic dual germs of spacelike hypersurface germs  $(\mathbf{X}^d)_i : (U, u_i) \longrightarrow (S_1^n, \mathbf{u}_i)$ . We can also understand the geometric meanings of the  $\mathcal{A}$ -equivalence among the  $\phi$ -hyperbolic dual germs as an application of the theory of Legendrian singularities [1, 16, 17].

If both of the regular sets of  $(\mathbb{N}_\pm^h[\phi])_i$  ( $i = 1, 2$ ) are dense in  $(U, u_i)$ , it follows from Proposition 6.2 that  $(\mathbb{N}_\pm^h[\phi])_1$  and  $(\mathbb{N}_\pm^h[\phi])_2$  are  $\mathcal{A}$ -equivalent if and only if the corresponding Legendrian immersion germs  $\mathcal{L}_{31}^1[\phi] : (U, u_1) \longrightarrow (\Delta_{31}^+(\phi), z_1)$  and  $\mathcal{L}_{31}^2[\phi] : (U, u_2) \longrightarrow (\Delta_{31}^+(\phi), z_2)$  are Legendrian equivalent. This condition is also equivalent to the condition that two generating families  $(H_\phi^H)_1$  and  $(H_\phi^H)_2$  are  $P$ - $\mathcal{K}$ -equivalent by Theorem 6.3. Here,  $(H_\phi^H)_i : (U \times H^n(-\sin^2 \phi), (u_i, \mathbf{v}_i)) \longrightarrow \mathbb{R}$  are the  $\phi$ -hyperbolic height function germs of  $(\mathbf{X}^d)_i$ .

On the other hand, if we denote  $(h_{\phi, v_i}^H)(u) = (H_\phi^H)_i(u, \mathbf{v}_i)$ , then we have  $(h_{\phi, v_i}^H)(u) = (\mathfrak{h}_\phi^H)_{v_i} \circ (\mathbf{X}^d)_i(u)$ . By Theorem 6.1, for  $p_i = (\mathbf{X}^d)_i(u_i)$

$$K((M^D)_1, T_{HHQ}((M^D)_1, p_1), p_1) = K((M^D)_2, T_{HHQ}((M^D)_2, p_2), p_2)$$

if and only if  $(h_{\phi, v_1}^H)$  and  $(h_{\phi, v_2}^H)$  are  $\mathcal{K}$ -equivalent.

**Theorem 12.1** *Let  $(\mathbf{X}^d)_i : (U, u_i) \longrightarrow (S_1^n, p_i)$  ( $i = 1, 2$ ) be spacelike hypersurface germs such that the corresponding Legendrian map germs*

$$(\mathbb{N}_\pm^h[\phi])_i = \pi[\phi]_{(31)1}^+ \circ \mathcal{L}_{31}^i[\phi] : (U, u_i) \longrightarrow (H^n(-\sin^2 \phi), \mathbf{v}_i)$$

*are Legendrian stable. Then the following conditions are equivalent:*

- (1)  $(\mathbb{N}_\pm^h[\phi])_1$  and  $(\mathbb{N}_\pm^h[\phi])_2$  are  $\mathcal{A}$ -equivalent.
- (2)  $((\mathbb{N}_\pm^h[\phi])_1(U), z_1)$  and  $((\mathbb{N}_\pm^h[\phi])_2(U), z_2)$  are diffeomorphic as set germs.
- (3)  $\mathcal{L}_{31}^1[\phi] : (U, u_1) \longrightarrow (\Delta_{31}^+(\phi), z_1)$  and  $\mathcal{L}_{31}^2[\phi] : (U, u_2) \longrightarrow (\Delta_{31}^+(\phi), z_2)$  are Legendrian equivalent.
- (4)  $(H_\phi^H)_1$  and  $(H_\phi^H)_2$  are  $P$ - $\mathcal{K}$ -equivalent.
- (5)  $(h_{\phi, v_1}^H)$  and  $(h_{\phi, v_2}^H)$  are  $\mathcal{K}$ -equivalent.
- (6)  $K((M^D)_1, T_{HHQ}((M^D)_1, p_1), p_1) = K((M^D)_2, T_{HHQ}((M^D)_2, p_2), p_2)$ .

## 13 De Sitter Monge forms and examples

In [12], Kasedou introduced the notion of de Sitter Monge forms for spacelike hypersurfaces in de Sitter space.

We consider a function  $f(u_1, \dots, u_{n-1})$  with  $f(0) = 0$  and  $f_{u_i}(0) = 0$ . Then we have a spacelike hypersurface in  $S_1^n$  defined by

$$\mathbf{X}_f^d(u_1, \dots, u_{n-1}) = (-f(u_1, \dots, u_{n-1}), -\sqrt{f^2(u_1, \dots, u_{n-1}) + u_1^2 + \dots + u_{n-1}^2 + 1}, u_1, \dots, u_{n-1}).$$



We can easily calculate that  $\mathbf{X}_f^d(0) = (0, -1, 0, \dots, 0)$  and  $\mathbf{X}_f^h(0) = (1, 0, \dots, 0)$ , therefore  $\mathbb{N}_\pm^h[\phi](0) = (\pm 1, -\cos \phi, 0, \dots, 0)$ . We call  $\mathbf{X}_f^d$  a *spacelike de Sitter Monge form* (briefly, *space-like D-Monge form*). In [12], it has been shown that any spacelike hypersurface in  $S_1^n$  is locally given by the spacelike D-Monge form.

For the timelike vector  $\mathbf{v}_0^\pm[\phi] = (\pm 1, -\cos \phi, 0, \dots, 0)$ , we have the spacelike D-Monge form of the  $\phi$ -flat de Sitter hyperquadric  $HQ^D(\mathbf{v}_0^\pm[\phi], \cos \phi)$ :

$$\mathbf{X}_\pm^{HQ^D}[\phi](u_1, \dots, u_{n-1}) = \left( \begin{array}{c} \mp \cos \phi \left( 1 - \sqrt{1 - \sin^2 \phi (u_1^2 + \dots + u_{n-1}^2)} \right) \\ \frac{\sin^2 \phi}{1 - \sin^2 \phi - \sqrt{1 - \sin^2 \phi (u_1^2 + \dots + u_{n-1}^2)}}, u_1, \dots, u_{n-1} \end{array} \right),$$

for  $\phi \in (0, \pi/2]$ .

On the other hand,  $\mathbf{X}_\pm^{HQ^D}[\phi](0) = (0, -1, 0, \dots, 0) = p$  and  $(\mathbf{X}_\pm^{HQ^D}[\phi])_{u_i}(0) = (\mathbf{X}_f^d)_{u_i}(0)$ , for  $i = 1, 2, \dots, n-1$ . This means that  $T_p M^D = T_p(\mathbf{X}_\pm^{HQ^D}[\phi](U))$ . Therefore,  $\mathbf{X}_\pm^{HQ^D}[\phi](U) = HQ^D(\mathbf{v}_0^\pm[\phi], \cos \phi)$  is the tangent  $\phi$ -hyperbolic flat elliptic hyperquadric of  $M^D = \mathbf{X}_f^d(U)$  at  $p = \mathbf{X}_f^d(0)$ . It follows from this fact that the tangent  $\phi$ -flat hyperquadric indicatrix of the spacelike D-Monge form germ  $(\mathbf{X}_f^d, 0)$  is given as follows:

$$\begin{aligned} (\mathbf{X}_f^d)^{-1}(HQ^D(\mathbf{v}_0^\pm[\phi], \cos \phi)) &= \{(u_1, \dots, u_{n-1}) \in U \mid \pm \sin^2 \phi f(u_1, \dots, u_{n-1}) \\ &= \cos \phi \left( 1 - \sqrt{1 - \sin^2 \phi (u_1^2 + \dots + u_{n-1}^2)} \right)\}. \end{aligned}$$

Since the  $\phi$ -hyperbolic height function of  $\mathbf{X}_f^d$  at  $\mathbf{v}_0^\pm[\phi]$  is

$$h_{\mathbf{v}_0^\pm[\phi]}^H(u) = \cos \phi \sqrt{f^2(u_1, \dots, u_{n-1}) - u_1^2 - \dots - u_{n-1}^2 + 1} \pm f(u_1, \dots, u_{n-1}) - \cos \phi,$$

we can calculate the Hessian matrix. Then we have  $\text{Hess}(h_{\mathbf{v}_0^\pm[\phi]}^H(0)) = \pm \text{Hess}(f)(0) - \cos \phi I_{n-1}$ .

On the other hand, since  $f(0) = f_{u_i}(0) = 0$ , we may write

$$f(u_1, \dots, u_{n-1}) = \frac{\kappa_1}{2} u_1^2 + \dots + \frac{\kappa_{n-1}}{2} u_{n-1}^2 + g(u_1, \dots, u_{n-1}),$$

where  $g \in \mathfrak{M}_{n-1}^3$  and  $\kappa_1, \dots, \kappa_{n-1}$  are eigenvalues of  $\text{Hess}(f)(0)$ . Under this representation, we can easily calculate that  $\mathbf{X}_{f, u_i u_j}^d(0) = (-f_{u_i u_j}(0), \delta_{ij}, 0, \dots, 0)$ . It follows from this fact that

$$h_\pm^H[\phi]_{ij}(0) = \delta_{ij}(-\cos \phi \pm \kappa_i) \text{ and } g_{ij}^D(0) = \delta_{ij}.$$

Therefore, we have  $\bar{\kappa}^\pm[\phi]_i(0) = -\cos \phi \pm \kappa_i$  and

$$K_\pm^h[\phi](0) = \prod_{i=1}^{n-1} \bar{\kappa}^\pm[\phi]_i(0) = \prod_{i=1}^{n-1} (-\cos \phi \pm \kappa_i).$$

The tangent  $\phi^\pm$ -flat elliptic hyperquadric indicatrix is given by

$$\begin{aligned} (\mathbf{X}_f^d)^{-1}(HQ^D(\mathbf{v}_0^\pm[\phi], \cos \phi)) &= \{(u_1, \dots, u_{n-1}) \mid \sin^2 \phi f^2(u_1, \dots, u_{n-1}) \\ &= \cos \phi \left( \sum_{i=1}^{n-1} \bar{\kappa}^\pm[\phi]_i(0) u_i^2 \pm 2g(u_1, \dots, u_{n-1}) \right)\}. \end{aligned}$$

In the last part of this section, we give some examples for the case  $n = 3$ . By the same reason as in §7, we can show that the  $\phi^\pm$ -hyperbolic dual  $\mathbb{N}_\pm^h[\phi]$  is the cuspidal edge or the swallowtail for a generic spacelike surface  $M^D \subset S_1^3$ . The corresponding tangent  $\phi^\pm$ -flat elliptic hyperquadric indicatrices are the ordinary cusp for the cupidal edge and the tachnodal for the swallowtail.

**Example 13.1** *Suppose that*

$$f(u_1, u_2) = \frac{\cos \phi}{\sin^2 \phi} \left( 1 - \sqrt{1 - \sin^2 \phi (u_1^2 + u_2^2)} \right) + \frac{2}{3} u_1^3 - u_2^2.$$

Then  $\kappa_1 = \cos \phi$  and  $\kappa_2 = -\cos \phi - 2$ , so that we have  $K_+^h[\phi](0) = 0$  and  $K_-^h[\phi](0) = 4 \cos \phi (\cos \phi - 1)$ . Therefore, the origin is a  $\phi^+$ -hyperbolic parabolic point for any  $\phi \in [0, \pi/2]$ . However,  $\bar{\kappa}_1^+(0) = -1$  and  $\bar{\kappa}_2^+(0) = 0$ . The positive tangent  $\phi$ -flat elliptic hyperquadric indicatrix is the ordinary cusp  $u_2^2 = \frac{2}{3} u_1^3$ . Consequently, the positive  $\phi$ -hyperbolic dual  $\mathbb{N}_+^h[\phi]$  is a cuspidal edge at the origin. However, the negative  $\phi$ -hyperbolic dual  $\mathbb{N}_-^h[\phi]$  is non-singular at the origin for  $\phi \neq \pi/2$ . Since the tangent small elliptic hyperquadric ( $\pi/2$ -flat elliptic quadric) is unique, the negative  $\pi/2$ -hyperbolic dual  $\mathbb{N}_-^h[\pi/2]$  is also a cuspidal edge at the origin.

**Example 13.2** *Consider a function*

$$f(u_1, u_2) = \frac{\cos \phi}{\sin^2 \phi} \left( 1 - \sqrt{1 - \sin^2 \phi (u_1^2 + u_2^2)} \right) + u_1^4 - u_2^2.$$

Then  $\kappa_1 = \cos \phi$  and  $\kappa_2 = \cos \phi - 2$ . By the same reason as the previous example the origin is a  $\phi^+$ -hyperbolic parabolic point for any  $\phi \in (0, \pi/2]$ .

The positive  $\phi$ -flat elliptic hyperquadric indicatrix is the tachnode  $u_2^2 = u_1^4$ . So, the positive  $\phi$ -hyperbolic dual  $\mathbb{N}_+^h[\phi]$  is a swallowtail at the origin. However, the negative  $\phi$ -hyperbolic dual  $\mathbb{N}_-^h[\phi]$  is non-singular at the origin for  $\phi \neq \pi/2$ . By the same reason as the previous example, the negative  $\pi/2$ -hyperbolic dual  $\mathbb{N}_-^h[\pi/2]$  is a swallowtail at the origin.

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