On completely integrable implicit ordinary differential equations

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Abstract

For smooth explicit n-th order ordinary differential equations, there exists a unique solution with initial condition and hence exists an n-parameter family of solutions at least locally. On the other hand, for smooth implicit ordinary differential equations, existence and uniqueness for solutions with initial condition does not hold. In this paper, we give a necessary and sufficient condition for existence of an n-parameter family of geometric solutions in the smooth category. Moreover, we give a sufficient condition that implicit ordinary differential equations have a unique geometric solution with initial condition. As a consequence, we classify completely integrable first and second ordinary differential equations in detail.

1 Introduction

For a smooth explicit ordinary differential equation

$$\frac{d^n y}{dx^n}(x) = f\left(x, y(x), \frac{dy}{dx}(x), \cdots, \frac{d^{n-1}y}{dx^{n-1}}(x)\right), \tag{1}$$

it is well-known that there exists a unique smooth solution with initial condition for (1), where f is a smooth function (for instance, see [1, 2, 4]). It follows that there exists an n-parameter family of smooth solutions at least locally.

On the other hand, for a smooth implicit ordinary differential equation (briefly, an implicit ODE)

$$F(x, y, p_1, \ldots, p_n) = 0, \tag{2}$$

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existence for a local solution with initial condition does not hold, where F is a smooth function of the independent variable x, the function y and its i-th derivatives $p_i = d^i y/dx^i, i = 1, \ldots, n$.

A natural question is what conditions guarantee existence and uniqueness for a local solution around a point of an implicit ODE. In this paper we shall discuss a qualitative theory for implicit ODEs.

It is natural to consider (2) as being defined on a subset in the space of n-jets of smooth functions of one variable, $F : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O}$ is an open subset in $J^n(\mathbb{R}, \mathbb{R})$. Throughout this paper, we assume that 0 is a regular value of F. It follows that the set $F^{-1}(0)$ is a hypersurface in $J^n(\mathbb{R}, \mathbb{R})$. We call $F^{-1}(0)$ the *equation hypersurface*. Let $(x, y, p_1, \ldots, p_n)$ be a local coordinate on $J^n(\mathbb{R}, \mathbb{R})$ and $\xi \subset TJ^n(\mathbb{R}, \mathbb{R})$ be the canonical contact system on $J^n(\mathbb{R}, \mathbb{R})$ described by the vanishing of the 1-forms

$$\begin{cases} \alpha_1 &= dy - p_1 dx, \\ \alpha_2 &= dp_1 - p_2 dx, \\ & \vdots \\ \alpha_n &= dp_{n-1} - p_n dx. \end{cases}$$

We now define the notion of solutions. A *smooth solution* (or, a *classical solution*) of $F = 0$ passing through a point z_0 is a smooth function germ $y = f(x)$ at a point t_0 such that $(t_0, f(t_0), f'(t_0), \ldots, f^{(n)}(t_0)) = z_0$ and $F(x, f(x), f'(x), \ldots, f^{(n)}(x)) = 0$, where $f^{(i)}(x) = (d^i f/dx^i)(x)$. In other words, there exists a smooth function germ $f : (\mathbb{R}, t_0) \to \mathbb{R}$ such that the image of the n-jet extension, $j^n f : (\mathbb{R}, t_0) \to (J^n(\mathbb{R}, \mathbb{R}), z_0)$; $j^n f(x) = (x, f(x), f'(x), \ldots, f^{(n)}(x))$, is contained in the equation hypersurface. It is easy to see that the map $j^n f$ is an immersion germ with $(j^n f)^* \alpha_i = 0$ for $i = 1, \ldots, n$.

More generally, a *geometric solution* of $F = 0$ passing through a point z_0 is an integral immersion germ $\gamma : (\mathbb{R}, t_0) \to (J^n(\mathbb{R}, \mathbb{R}), z_0)$ such that the image of γ is contained in the equation hypersurface, namely, $\gamma' \neq 0$, $\gamma^* \alpha_i = 0$ for $i = 1, \ldots, n$ and $F(\gamma(t)) = 0$ for each $t \in (\mathbb{R}, t_0)$.

By the definition, a smooth solution is also a geometric solution. Conversely, it is easy to see that if $\gamma(t) = (x(t), y(t), p_1(t), \ldots, p_n(t))$ is a geometric solution of $F = 0$ and $x'(t_0) \neq 0$, then we can reparametrize $\gamma(t)$ as a smooth solution.

The following notions are basic in this paper (cf. [3, 15, 17]). By the definition of parametrized version for smoothness of the solutions (*i.e.*, smooth solutions), a *smooth complete solution of* (2) *at* z_0 is defined to be an n-parameter family of smooth function germs $y = f(t, \boldsymbol{c}) = f(t, c_1, \ldots, c_n)$ such that

$$F\left(t, f(t, \boldsymbol{c}), \frac{\partial f}{\partial t}(t, \boldsymbol{c}), \ldots, \frac{\partial^n f}{\partial t^n}(t, \boldsymbol{c})\right) = 0$$

and the map germ $j_1^n f : (\mathbb{R} \times \mathbb{R}^n, (t_0, \boldsymbol{c}_0)) \to (F^{-1}(0), z_0)$ defined by

$$j_1^n f(t, \boldsymbol{c}) = \left(t, f(t, \boldsymbol{c}), \frac{\partial f}{\partial t}(t, \boldsymbol{c}), \ldots, \frac{\partial^n f}{\partial t^n}(t, \boldsymbol{c})\right)$$

is an immersion. It follows that the equation hypersurface is foliated by an n-parameter family of smooth solutions.

On the other hand, we consider the corresponding definition of parametrized version for geometric solutions. Let $\Gamma : (\mathbb{R} \times \mathbb{R}^n, (t_0, \boldsymbol{c}_0)) \to (F^{-1}(0), z_0)$ be an n-parameter family of geometric solutions, *i.e.*, $\Gamma(\cdot, \boldsymbol{c})$ is a geometric solution of $F = 0$ for each $\boldsymbol{c} \in (\mathbb{R}^n, \boldsymbol{c}_0)$.

We call Γ a *complete solution of* (2) *at* z_0 if Γ is an immersion germ, namely,

$$\operatorname{rank}\begin{pmatrix} \partial x/\partial t & \partial y/\partial t & \partial p_1/\partial t & \cdots & \partial p_n/\partial t \\ \partial x/\partial \boldsymbol{c} & \partial y/\partial \boldsymbol{c} & \partial p_1/\partial \boldsymbol{c} & \cdots & \partial p_n/\partial \boldsymbol{c} \end{pmatrix}(t_0, \boldsymbol{c}_0) = n+1,$$

where $\Gamma(t, \boldsymbol{c}) = (x(t, \boldsymbol{c}), y(t, \boldsymbol{c}), p_1(t, \boldsymbol{c}), \dots, p_n(t, \boldsymbol{c}))$. It follows that the equation hypersurface is foliated by an n-parameter family of geometric solutions.

We say that an equation $F = 0$ is *smooth completely integrable* (respectively, *completely integrable*) at z_0 if there exists a smooth complete solution (respectively, a complete solution) of $F = 0$ at z_0.

In the study of implicit ODEs from the view point of singularity theory, there is a lot of research. For example, generic singularities and properties were given in [5, 6, 8, 16, 14] for the case of first order, in [12, 13] for the case of second order and in [7] for the case of any order etc. In this paper is focused on the theory of completely integrable implicit ODEs.

In §2, we give a necessary and sufficient condition for existence of complete solutions and smooth complete solutions at a point. We show that $F = 0$ is completely integrable at z_0 if and only if $F = 0$ is either of Clairaut type or of reduced type at z_0 (cf. Theorem 2.2). This result guarantees existence for a geometric solution in Proposition 3.1. In §3, we give a sufficient condition for existence and uniqueness for a geometric solution with initial condition. In §4 and §5, we classify completely integrable first order and second order ODEs respectively. We also give examples of completely integrable implicit ODEs.

All map germs and manifolds considered here are differential of class C^∞.

2 Existence and uniqueness for complete solutions

In this section, we consider existence and uniqueness conditions for a complete solution and a smooth complete solution of implicit ODEs. We denote a map $F_x + p_1 F_y + p_2 F_{p_1} + \cdots + p_n F_{p_{n-1}}$ by F_X. Here F_x (respectively, F_y, F_{p_i}) is the partial derivative of F with respect to x (respectively, with respect to y, p_i). We refer to the following Lemma. See in case of first order in [9, 16], and of second order in [3, Lemma 3.1].

Lemma 2.1 *Let $F = 0$ be an implicit ODE at z_0. The equation $F = 0$ is completely integrable at $z_0 \in F^{-1}(0)$ if and only if there exist function germs $\alpha, \beta : (F^{-1}(0), z_0) \to \mathbb{R}$, which do not vanish simultaneously, such that*

$$\alpha \cdot F_X|_{F^{-1}(0)} + \beta \cdot F_{p_n}|_{F^{-1}(0)} \equiv 0.$$

Proof. Suppose that $F = 0$ is completely integrable at z_0 and let

$$\Gamma : (\mathbb{R} \times \mathbb{R}^n, (t_0, \boldsymbol{c}_0)) \to (F^{-1}(0), z_0)$$

be a complete solution of the implicit ODE at z_0. Then differentiating Γ with respect to t yields a vector field $Z : (F^{-1}(0), z_0) \to TF^{-1}(0)$ given by $Z(\Gamma(t, \boldsymbol{c})) = \Gamma_t(t, \boldsymbol{c})$. Since $Z(z)$ lies in the contact plane ξ_z for each $z \in (F^{-1}(0), z_0)$, it has the form $Z = (\alpha, p_1\alpha, \dots, p_n\alpha, \beta)$ for some function germs $\alpha, \beta : (F^{-1}(0), z_0) \to \mathbb{R}$ which do not vanish simultaneously. Besides $Z(z)$ also lies in $T_z F^{-1}(0)$. It follows that the identity

$$\alpha \cdot F_X|_{F^{-1}(0)} + \beta \cdot F_{p_n}|_{F^{-1}(0)} \equiv 0$$

holds. Reversing the argument yields the converse. □

We say that an equation $F=0$ is of *n-th order Clairaut type* (for short, *Clairaut type*) at z_0 if there exist smooth function germs $A, B : (J^n(\mathbb{R},\mathbb{R}), z_0) \to \mathbb{R}$ such that

$$F_X = A \cdot F + B \cdot F_{p_n},$$

and of *reduced type* at z_0 if there exist smooth function germs $A', B' : (J^n(\mathbb{R},\mathbb{R}), z_0) \to \mathbb{R}$ such that

$$F_{p_n} = A' \cdot F + B' \cdot F_X.$$

We give a necessary and sufficient condition for existence of a smooth complete solution and a complete solution of implicit ODEs.

Theorem 2.2 *Let $F=0$ be an implicit ODE at z_0.*

(1) *$F=0$ is smooth completely integrable at z_0 if and only if $F=0$ is of Clairaut type at z_0.*

(2) *$F=0$ is completely integrable at z_0 if and only if $F=0$ is of Clairaut type, or of reduced type at z_0.*

Proof. (1) The proof follows from a direct analogy of the proof for Theorem 2.2 in [10] or Theorem 3.1 in [15], so that we omit it.

(2) The result is a consequence of Lemma 2.1 and the fact that $F=0$ is regular. □

The uniqueness of the complete solution is the following.

Proposition 2.3 *Let $\Gamma_1 : (\mathbb{R}\times\mathbb{R}^n, (t_1, \boldsymbol{c}_1)) \to (F^{-1}(0), z_0)$ and $\Gamma_2 : (\mathbb{R}\times\mathbb{R}^n, (t_2, \boldsymbol{c}_2)) \to (F^{-1}(0), z_0)$ be complete solutions of $F=0$ at z_0. Then there exists a diffeomorphism germ $\Phi : (\mathbb{R}\times\mathbb{R}^n, (t_2, \boldsymbol{c}_2)) \to (\mathbb{R}\times\mathbb{R}^n, (t_1, \boldsymbol{c}_1))$ of the form $\Phi(t, \boldsymbol{c}) = (\phi_1(t, \boldsymbol{c}), \phi_2(\boldsymbol{c}))$ such that $\Gamma_1 \circ \Phi = \Gamma_2$.*

Proof. Suppose that the assertion does not hold. Since the complete solution is an n-parameter family of curves in $F^{-1}(0)$, then there exists a point $z_1 \in (F^{-1}(0), z_0)$ such that $\Gamma_1(\cdot, \boldsymbol{c}_1)$ and $\Gamma_2(\cdot, \boldsymbol{c}_2)$ are transversal near the point z_1. Then we can construct a map germ $\Gamma : (\mathbb{R}\times\mathbb{R}^n, 0) \to (F^{-1}(0), z_1)$ such that (at least) $\Gamma(t, \cdot, \cdot, c_3, \dots, c_n)$ is an immersion germ,

$$\frac{\partial y}{\partial c_1}(t, \boldsymbol{c}) = p_1(t, \boldsymbol{c})\frac{\partial x}{\partial c_1}(t, \boldsymbol{c}), \dots, \frac{\partial p_{n-1}}{\partial c_1}(t, \boldsymbol{c}) = p_n(t, \boldsymbol{c})\frac{\partial x}{\partial c_1}(t, \boldsymbol{c}) \tag{3}$$

and

$$\frac{\partial y}{\partial c_2}(t, \boldsymbol{c}) = p_1(t, \boldsymbol{c})\frac{\partial x}{\partial c_2}(t, \boldsymbol{c}), \dots, \frac{\partial p_{n-1}}{\partial c_2}(t, \boldsymbol{c}) = p_n(t, \boldsymbol{c})\frac{\partial x}{\partial c_2}(t, \boldsymbol{c}), \tag{4}$$

where $\Gamma(t, \boldsymbol{c}) = (x(t, \boldsymbol{c}), y(t, \boldsymbol{c}), p_1(t, \boldsymbol{c}), \dots, p_n(t, \boldsymbol{c}))$. If we calculate second order partial derivatives of the last equality for (3) with respect to c_2 and for (4) with respect to c_1, we get

$$\frac{\partial^2 p_{n-1}}{\partial c_2 \partial c_1} = \frac{\partial p_n}{\partial c_2} \cdot \frac{\partial x}{\partial c_1} + p_n \cdot \frac{\partial^2 x}{\partial c_2 \partial c_1} \text{ and } \frac{\partial^2 p_{n-1}}{\partial c_1 \partial c_2} = \frac{\partial p_n}{\partial c_1} \cdot \frac{\partial x}{\partial c_2} + p_n \cdot \frac{\partial^2 x}{\partial c_1 \partial c_2}.$$

Therefore we obtain the equality $(\partial p_n/\partial c_2)\cdot(\partial x/\partial c_1) = (\partial p_n/\partial c_1)\cdot(\partial x/\partial c_2)$. This contradicts the fact that $\Gamma(t, \cdot, \cdot, c_3, \dots, c_n)$ is an immersion germ. □

3 Existence and uniqueness for geometric solutions

In this section, we give an existence and uniqueness condition for a geometric solution with initial condition.

Let $F = 0$ be an implicit ODE at z_0. Consider a point $z \in F^{-1}(0)$ such that the contact plane ξ_z intersects $T_z F^{-1}(0)$ transversally. Then it is easy to see that a complete solution exists at z by integrating the line field $\xi \cap TF^{-1}(0)$ (see, Lemma 2.1). We call points where transversality fails *contact singular points* and denote the set of such points by $\Sigma_c = \Sigma_c(F)$. We call $\Sigma_c(F)$ the *contact singular set* of $F^{-1}(0)$. It is easy to check that the contact singular set is given by

$$\Sigma_c(F) = \{z \in F^{-1}(0) | \ F_X(z) = 0, F_{p_n}(z) = 0\}.$$

We say that a geometric solution $\gamma : (\mathbb{R}, 0) \to (F^{-1}(0), z_0)$ is a *singular solution* of $F = 0$ passing through z_0 if for any representative $\widetilde{\gamma} : I \to F^{-1}(0)$ of γ and any open subinterval $(a, b) \subset I$ at 0, $\widetilde{\gamma}|_{(a,b)}$ is never contained in a leaf of a complete solution (cf. [3, 7, 9, 11]). If a completely integrable implicit ODE has a singular solution, then uniqueness for geometric solutions break down at such points (see, Example 4.4).

Proposition 3.1 *Let $F = 0$ be an implicit ODE at z_0. If $z_0 \notin \Sigma_c(F)$, then there exists a unique geometric solution passing through z_0.*

By Proposition 3.1, a geometric solution $\gamma : (\mathbb{R}, t_0) \to (F^{-1}(0), z_0)$ is a singular solution only if it is contained in $\Sigma_c(F)$. If $z_0 \notin \Sigma_c(F)$, either $F_X \neq 0$ or $F_{p_n} \neq 0$ at z_0. The latter case, the consequence of Proposition 3.1 is follows from the classical results for existence and uniqueness of smooth solution of smooth explicit equations. Thus in order to prove Proposition 3.1, it is enough to show the former case. If $F_X(z_0) \neq 0$, then $F = 0$ has a unique geometric solution around z_0 by integrating the line field $\xi_z \cap T_z F^{-1}(0)$. Therefore we have the following result.

Lemma 3.2 *Let $F = 0$ be an implicit ODE at z_0. If $F_X(z_0) \neq 0$, then there exists the unique geometric solution passing through z_0.*

The above Lemma is a well-known result. However, in the Appendix, we shall prove Lemma 3.2 explicitly by an elementary argument like as explicit ODEs, since we use this method to prove in Theorem 5.6.

Now suppose that $F = 0$ is completely integrable at z_0 and $\Sigma_c(F)$ is an n-dimensional manifold around z_0. Remark that if $F = 0$ is completely integrable at z_0, the condition that $\Sigma_c(F)$ is an n-dimensional manifold around z_0 is a generic condition by Theorem 2.2.

We call a map germ $\Phi : (\mathbb{R} \times \mathbb{R}^{n-1}, (t_0, \boldsymbol{b}_0)) \to (\Sigma_c(F), z_0)$ a *complete solution of* $\Sigma_c(F)$ *at* z_0 if Φ is an immersion germ and $\Phi(\cdot, \boldsymbol{b})$ is a geometric solution for each $\boldsymbol{b} \in (\mathbb{R}^{n-1}, \boldsymbol{b}_0)$. Moreover, we call Φ a *complete singular solution of* $\Sigma_c(F)$ *at* z_0 if $\Phi(\cdot, \boldsymbol{b})$ is a singular solution for each $\boldsymbol{b} \in (\mathbb{R}^{n-1}, \boldsymbol{b}_0)$. If ξ_z intersects $T_z\Sigma_c(F)$ transversally in $T_z F^{-1}(0)$, then integrating the line field $\xi \cap T\Sigma_c(F)$ yields a complete solution of $\Sigma_c(F)$. We call a point where transversality does not hold a *second order contact singular point* and denote the set of such points by $\Sigma_{cc} = \Sigma_{cc}(F)$ (or, $\Sigma_{c^2} = \Sigma_{c^2}(F)$) (cf. [3, 17]). Inductively, if $\Sigma_{cc}(F)$ is an $(n-1)$-dimensional manifold around z_0, then we can define a *complete solution of* $\Sigma_{cc}(F)$ *at* z_0, a *complete singular solution of* $\Sigma_{cc}(F)$ *at* z_0 and third order contact singular set $\Sigma_{ccc} = \Sigma_{ccc}(F)$ (or, $\Sigma_{c^3} = \Sigma_{c^3}(F)$) etc. Therefore we have the following sequence when $\Sigma_{c^i}(F)$ are $(n-i+1)$-dimensional submanifolds, $i = 1, \ldots, n$

(cf. [7]):

$$\Sigma_{c^n}(F) \subset \Sigma_{c^{n-1}}(F) \subset \cdots \subset \Sigma_{c^2}(F) \subset \Sigma_c(F) \subset F^{-1}(0).$$

4 Completely integrable first order ordinary differential equations

In this section, we quickly review known results for the theory of completely integrable implicit first order ODEs

$$F(x, y, p) = 0, \ p = dy/dx.$$

For more detail, see [8, 9, 10, 11, 16]. In [9], it has shown the following results.

Theorem 4.1 *Let $F(x, y, p) = 0$ be an implicit first order ODE at z_0. $F = 0$ is completely integrable at z_0 if and only if $z_0 \notin \Sigma_c$ or Σ_c is an 1-dimensional manifold around z_0. Moreover, if Σ_c is an 1-dimensional manifold around z_0, then Σ_c is a singular solution of $F = 0$ passing through z_0.*

As a corollary of Theorem 4.1, the condition of Proposition 3.1 is a necessary and sufficient condition for uniqueness of geometric solutions of completely integrable implicit first order ODEs.

Corollary 4.2 *Let $F(x, y, p) = 0$ be a completely integrable implicit first order ODE at z_0. There exists a unique geometric solution passing through z_0 if and only if $z_0 \notin \Sigma_c$.*

Now suppose that $z_0 \in \Sigma_c$. Since $F = 0$ is regular, $F_y(z_0) \neq 0$. By the implicit function theorem, there exists a smooth function $f : U \to \mathbb{R}$, where U is an open set in $\mathbb{R}^2$, such that in a neighborhood of z_0, $(x, y, p) \in F^{-1}(0)$ if and only if $-y + f(x, p) = 0$. Thus we may assume without loss of generality that $F(x, y, p) = -y + f(x, p) = 0$. It follows that z_0 is a regular point of either $F_p|_{F^{-1}(0)}$ or $F_X|_{F^{-1}(0)}$.

Hence completely integrable implicit first order ODEs have four kinds of types (cf. [16]), see Table 1.

Conditions			Type	Name
$z_0 \notin \Sigma_c$	$F_p(z_0) \neq 0$		Clairaut type	C
	$F_X(z_0) \neq 0$		reduced type	R
$z_0 \in \Sigma_c$	$F_y(z_0) \neq 0$	regular point of $F_p\vert_{F^{-1}(0)}$	Clairaut type	RC_y
		regular point of $F_X\vert_{F^{-1}(0)}$	reduced type	RR_y

Table 1. Classifications of completely integrable implicit first order ODEs at z_0.

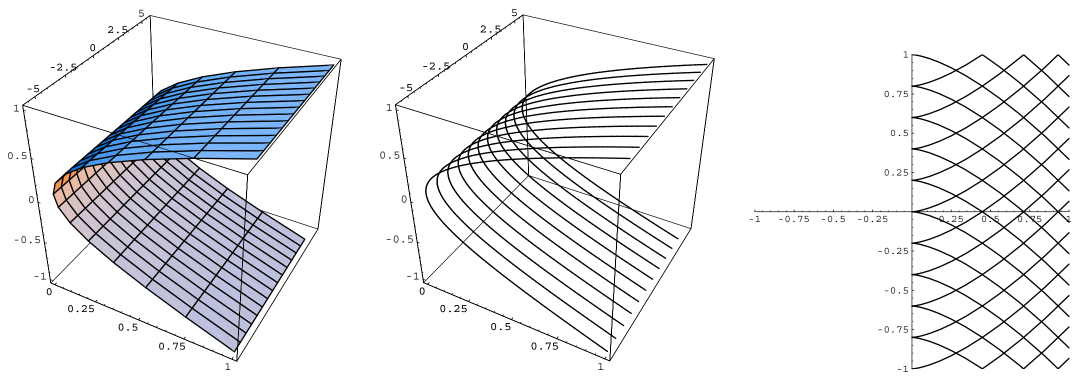
We now give two easy examples illustrating the notion of the complete solution and results. One is satisfied the condition of Proposition 3.1 and the other is not.

Example 4.3 (Type R). Let $F(x, y, p) = x - f(p) = 0$ with $f(0) = 0$. Then $F_X = F_x + pF_y = 1$ and $F_p = -f'(p)$. In this case, $\Sigma_c = \emptyset$ and $F = 0$ is of reduced type. By Theorem 2.2, this equation is completely integrable and a complete solution $\Gamma : (\mathbb{R} \times \mathbb{R}, 0) \to F^{-1}(0)$ is given by

$$\Gamma(t, c) = \left(f(t), \int t f'(t) dt + c, t \right).$$

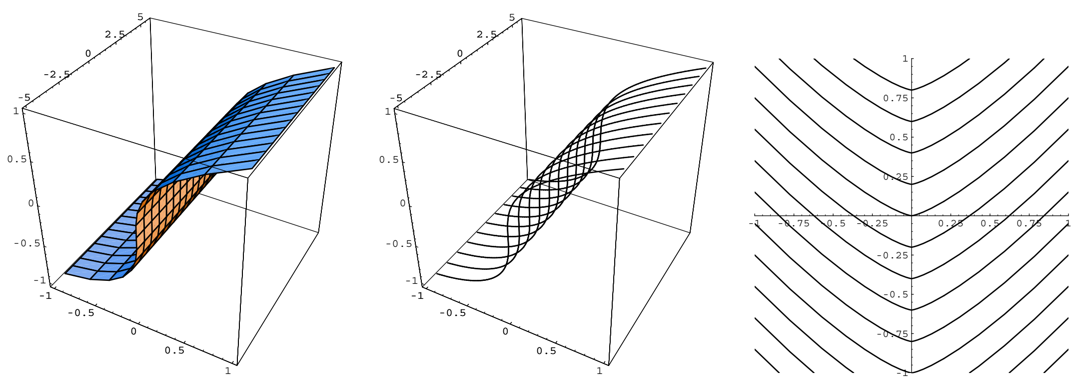
By Proposition 3.1 (or, Corollary 4.2), the geometric solution is unique passing through each point $z_0 \in F^{-1}(0)$.

For example, if we put $f(p) = p^2$ (respectively, $f(p) = p^3$), the complete solution is $\Gamma(t,c) = (t^2, (2/3)t^3 + c, t)$ (respectively, $\Gamma(t,c) = (t^3, (3/4)t^4 + c, t)$). We can draw pictures; the equation surface $F^{-1}(0)$, geometric solutions $\{\Gamma(t,c)\}_{c\in(\mathbb{R},0)}$ and the phase portrait $\{\pi \circ \Gamma(t,c)\}_{c\in(\mathbb{R},0)}$, where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the canonical projection $\pi(x,y,p) = (x,y)$. See Figures 1 and 2.



equation surface geometric solutions phase portrait

Figure 1. $F(x,y,p) = x - p^2 = 0$.



equation surface geometric solutions phase portrait

Figure 2. $F(x,y,p) = x - p^3 = 0$.

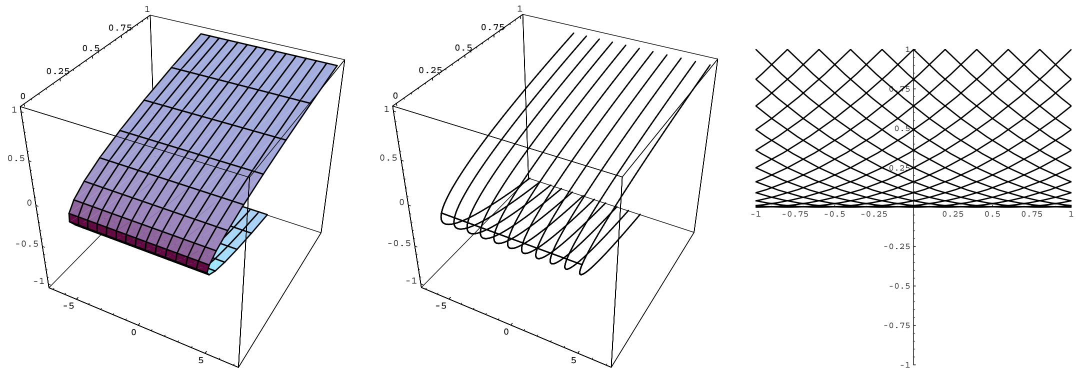
Example 4.4 (Type RR_y). Let $F(x,y,p) = y - f(p) = 0$ with $f(0) = 0$. If $f'(0) \neq 0$, then there exist not only a smooth complete solution, but also a unique smooth solution passing through each point of $F^{-1}(0)$.

We now suppose that $f'(0) = 0$. It follows that there exists a smooth function germ g such that $f(p) = p^2 g(p)$. Then $F_X = F_x + pF_y = p$ and $F_p = -f'(p) = -p(2g(p) + pg'(p))$. The contact singular set Σ_c is x-axis in $J^1(\mathbb{R},\mathbb{R})$ and $F = 0$ is of reduced type. By Theorem 2.2, this equation is completely integrable and a complete solution $\Gamma : (\mathbb{R}\times\mathbb{R}, 0) \to F^{-1}(0)$ is given by

$$\Gamma(t,c) = \left(\int (2g(t) + tg'(t))dt + c, f(t), t \right).$$

By Theorem 4.1, we have a singular solution $\gamma : (\mathbb{R}, 0) \to \Sigma_c \subset F^{-1}(0)$; $\gamma(t) = (t, 0, 0)$. Therefore it is easy to see that there are two geometric solutions of $F = 0$ passing through $z_0 \in \Sigma_c$.

For example, if we also put $f(p) = p^2$ (respectively, $f(p) = p^3$), the complete solution is $\Gamma(t,c) = (2t + c, t^2, t)$ (respectively, $\Gamma(t,c) = ((3/2)t^2 + c, t^3, t)$). We can draw pictures; the equation surface $F^{-1}(0)$, geometric solutions $\{\Gamma(t,c)\}_{c\in(\mathbb{R},0)}$ and the singular solution $\gamma(t)$, and the phase portrait $\{\pi\circ\Gamma(t,c)\}_{c\in(\mathbb{R},0)}$. See Figures 3 and 4.



equation surface | geometric solutions and the singular solution | phase portrait

Figure 3. $F(x,y,p) = y - p^2 = 0$.

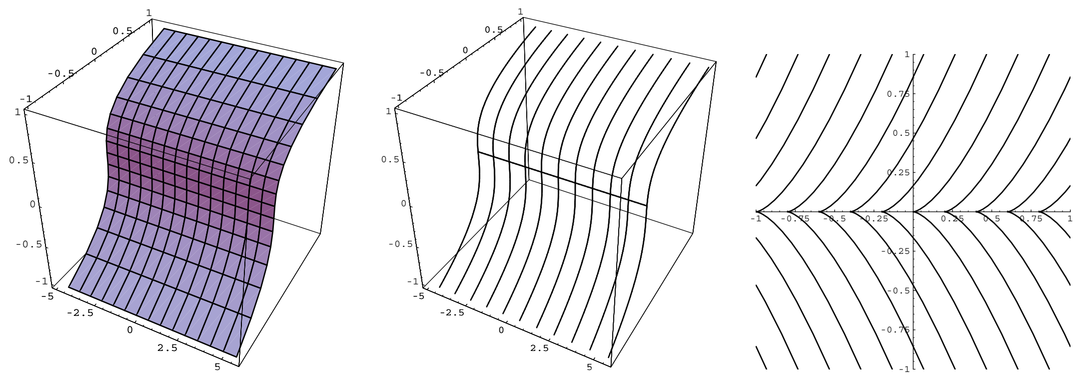


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Figure 4. $F(x,y,p) = y - p^3 = 0$.

5 Completely integrable second order ordinary differential equations

The condition of Proposition 3.1 is a necessary and sufficient condition for uniqueness for geometric solutions of completely integrable implicit first order ODEs. However, the result does not hold in the case of completely integrable implicit second order ODEs, that is, even if $z_0 \in \Sigma_c$ there is a unique geometric solution passing through z_0 (cf. Theorem 5.6). In this section, we analyse completely integrable implicit second order ODEs in detail. Let

$$F(x,y,p,q) = 0,\ p = dy/dx,\ q = d^2y/dx^2$$

be an implicit second order ODE at z_0. If $z_0 \notin \Sigma_c$, then $F = 0$ satisfies either $F_q(z_0) \neq 0$ or $F_X(z_0) \neq 0$.

First we assume that $F_q(z_0) \neq 0$. By the implicit function theorem, $F = 0$ can be represented by an explicit equation $q = f(x,y,p)$ where f is a function germ. In this case, $F = 0$ is of Clairaut type at z_0 and here we call this type C.

Next we assume that $F_X(z_0) \neq 0$. In this case, $F = 0$ is of reduced type at z_0 and we call this type R.

Both cases, there is a unique geometric solution passing through each point of $F^{-1}(0)$. It follows that there is a complete solution of $F^{-1}(0)$ and no singular solution.

By Theorem 2.2, a completely integrable ODE at z_0 is either of Clairaut type or reduced type at z_0. The main results in this paper are to classify the completely integrable implicit second order ODEs in detail and to characterize a complete (singular) solution of Σ_c for each type respectively. It is concluded that there are ten kinds of types, see Table 2.

Conditions				Type	Name
$z_0 \notin \Sigma_c$	$F_q(z_0) \neq 0$			Clairaut type	C
	$F_X(z_0) \neq 0$			reduced type	R
$z_0 \in \Sigma_c$	$F_p(z_0) \neq 0$	regular point of $F_q\vert_{F^{-1}(0)}$		Clairaut type	RC_p
		regular point of $F_X\vert_{F^{-1}(0)}$		reduced type	RR_p
	$F_y(z_0) \neq 0$	regular point of $F_q\vert_{F^{-1}(0)}$		Clairaut type	RC_y
	$F_p(z_0) = 0$	regular point of $F_X\vert_{F^{-1}(0)}$	$\Sigma_c = \Delta$	reduced type	RR_y^1
			$\Sigma_c \supsetneq \Delta = \Sigma_{cc}$	reduced type	RR_y^2
			$\Sigma_c \supsetneq \Delta \supsetneq \Sigma_{cc}$	reduced type	RR_y^3
		singular point of $F_q\vert_{F^{-1}(0)}$		Clairaut type	SC_y
		and $F_X\vert_{F^{-1}(0)}$		reduced type	SR_y

Table 2. Classifications of completely integrable implicit second order ODEs at z_0.

Here we consider the subset $\Delta = \Delta(F) \subset \Sigma_c$ which is defined to be the set of points $z \in \Sigma_c$ such that $T_z F^{-1}(0)$ coincides with the kernel of $\alpha_1(z)$. Explicitly, it is given by $\Delta = \{z \in \Sigma_c \mid F_p(z) = 0\}$.

We now review the previous results in [3, 15, 17]. Conditions for existence of a complete solution and a complete (singular) solution of Σ_c for implicit second order ODEs were given under a regularity condition.

Theorem 5.1 [3, Theorems 1.1, 1.2 and 1.3] *Suppose that* 0 *is a regular value of* $F_q|_{F^{-1}(0)}$.
(1) $F = 0$ *is completely integrable at* z_0 *if and only if* $z_0 \notin \Sigma_c$ *or* Σ_c *is a* 2*-dimensional manifold around* z_0.
(2) *Let* $F = 0$ *be completely integrable.*
(i) *The leaves of the complete solution which meet* Σ_c *away from* Δ *intersect* Σ_c *transversally.*
(ii) *The leaves of the complete solution which meet* Δ *are tangent to* Σ_c.
(3) *Let* $F = 0$ *be completely integrable and* $\Sigma_c \neq \emptyset$.
(i) $F = 0$ *admits a complete singular solution of* Σ_c *at* z_0 *if and only if* $z_0 \notin \Sigma_{cc}$ *or* Σ_{cc} *is a* 1*-dimensional manifold around* z_0.
(ii) *Suppose that* $F = 0$ *admits a complete singular solution of* Σ_c. *Then each leaf of the complete singular solution of* Σ_c *intersects* Σ_{cc} *transversally.*

Theorem 5.2 [17, Lemma 2.3, Propositions 2.4 and 3.8] *Suppose that* 0 *is a regular value of* $F_X|_{F^{-1}(0)}$.
(1) $F = 0$ *is completely integrable at* z_0 *if and only if* $z_0 \notin \Sigma_c$ *or* Σ_c *is a* 2*-dimensional manifold around* z_0.
(2) *Let* $F = 0$ *be completely integrable.*
(i) *The leaves of the complete solution which meet* Σ_c *away from* Δ *intersect* Σ_c *transversally.*

(ii) *The leaves of the complete solution which meet* Δ *are tangent to* Σ_c.

(3) *Let* $F = 0$ *be completely integrable and* $\Sigma_c \neq \emptyset$.

(i) $F = 0$ *admits a complete solution of* Σ_c *at* z_0 *if and only if* $z_0 \notin \Sigma_{cc}$ *or* Σ_{cc} *is a* 1-*dimensional manifold around* z_0.

(ii) *Suppose that* $F = 0$ *admits a complete solution of* Σ_c. *Then each leaf of the complete solution of* Σ_c *intersects* Σ_{cc} *transversally.*

Proposition 5.3 [15, Proposition 3.2] *Let* $F = 0$ *be completely integrable at* z_0 *and* $z_0 \in \Sigma_c$.
(1) *If* 0 *is a regular value of* $F_q|_{F^{-1}(0)}$, *then* $F = 0$ *is of Clairaut type at* z_0.
(2) *If* 0 *is a regular value of* $F_X|_{F^{-1}(0)}$, *then* $F = 0$ *is of reduced type at* z_0.

Proposition 5.4 [17, Propositions 2.7] *Let* $F = 0$ *be completely integrable at* z_0 *and* Σ_c *be a* 2-*dimensional manifold around* z_0. *Then* $\Sigma_{cc} \subset \Delta$.

Remark 5.5 *There is an important difference between the case where* 0 *is a regular value of* $F_q|_{F^{-1}(0)}$ *and where it is a regular value of* $F_X|_{F^{-1}(0)}$. *Namely, if* 0 *is a regular value of* $F_q|_{F^{-1}(0)}$ *and* $z_0 \in \Delta$, *then* Δ *is a* 1-*dimensional manifold around* z_0 *by Proposition 3.6 in [3]. However,* Δ *is not necessarily a* 1-*dimensional manifold even if* 0 *is a regular value of* $F_X|_{F^{-1}(0)}$.

Now suppose that $F = 0$ is completely integrable at $z_0 \in \Sigma_c$. Since F is regular at z_0, $F = 0$ satisfies either $F_y(z_0) \neq 0$ or $F_p(z_0) \neq 0$.

5.1 On the types RC_p and RR_p

If $F_p(z_0) \neq 0$, by the implicit function theorem, there exists a smooth function $g : V \to \mathbb{R}$, where V is an open set in $\mathbb{R}^3$, such that in a neighborhood of z_0, $(x, y, p, q) \in F^{-1}(0)$ if and only if $-p + g(x, y, q) = 0$. Thus we may assume without loss of generality that $F(x, y, p, q) = -p + g(x, y, q) = 0$. Under this notations, $F_q = g_q$ and $F_X = g_x + g \cdot g_y - q$. It follows that z_0 is a regular point of either $F_q|_{F^{-1}(0)}$ or $F_X|_{F^{-1}(0)}$.

If z_0 is a regular point of $F_q|_{F^{-1}(0)}$, then $F = 0$ is of Clairaut type at z_0 and Σ_c is a 2-dimensional manifold around z_0 by Proposition 5.3 and Theorem 5.1. We call this type RC_p. By $z_0 \notin \Delta$ and Proposition 5.4, $z_0 \notin \Sigma_{cc}$. Hence $F = 0$ has a complete singular solution of Σ_c at z_0 automatically.

On the other hand, suppose that z_0 is a regular point of $F_X|_{F^{-1}(0)}$. By Proposition 5.3 and Theorem 5.2, $F = 0$ is of reduced type at z_0 and Σ_c is a 2-dimensional manifold around z_0. We call this type RR_p. By $z_0 \notin \Delta$ and Proposition 5.4, $z_0 \notin \Sigma_{cc}$. Since the leaves of the complete solution which meet Σ_c away from Δ intersect Σ_c transversally, $F = 0$ has a complete singular solution of Σ_c at z_0 automatically.

5.2 On the type RC_y

If $F_y(z_0) \neq 0$, again by the implicit function theorem, there exists a smooth function $f : U \to \mathbb{R}$, where U is an open set in $\mathbb{R}^3$, such that in a neighborhood of z_0, $(x, y, p, q) \in F^{-1}(0)$ if and only if $-y + f(x, p, q) = 0$. Thus we may assume without loss of generality that $F(x, y, p, q) = -y + f(x, p, q) = 0$. Define the diffeomorphism $\phi : U \to F^{-1}(0), (x, p, q) \mapsto (x, f(x, p, q), p, q)$ and $u_0 = \phi^{-1}(z_0)$. Below, if $F_y(z_0) \neq 0$, we keep the notations of the above.

Suppose that z_0 is a regular point of $F_q|_{F^{-1}(0)}$. By Proposition 5.3 and Theorem 5.1, $F=0$ is of Clairaut type at z_0 and Σ_c is a 2-dimensional manifold around z_0. We call this type RC_y. Moreover, $F=0$ has a complete singular solution at $z_0 \in \Sigma_c$ if and only if $z_0 \notin \Sigma_{cc}$ or Σ_{cc} is a 1-dimensional manifold around z_0 by Theorem 5.1.

Remark that if Σ_{cc} is a 1-dimensional manifold around z_0, then $\Delta = \Sigma_{cc}$ (cf. Remark 5.5) and Σ_{cc} is an isolated singular solution passing through z_0 (see, [3, Proposition 1.4]).

5.3 On the type RR_y^1

Let $F_y(z_0) \neq 0$. Suppose that z_0 is a regular point of $F_X|_{F^{-1}(0)}$. By Proposition 5.3 and Theorem 5.2, $F=0$ is of reduced type at z_0 and Σ_c is a 2-dimensional manifold around z_0. In this case, there are three types. First case is $\Sigma_c = \Delta$ around z_0 (type RF_y^1), second is $\Sigma_c \supsetneqq \Delta = \Sigma_{cc}$ around z_0 (type RF_y^2), and the last is $\Sigma_c \supsetneqq \Delta \supsetneqq \Sigma_{cc}$ around z_0 (type RF_y^3). We may assume that $F_p(z_0)=0$, namely, $z_0 \in \Delta$, and $F=0$ is not of Clairaut type at z_0.

We now consider $F=0$ is the type RF_y^1 at z_0. By Theorem 5.2, $F=0$ has a complete solution of Σ_c at z_0 if and only if $z_0 \notin \Sigma_{cc}$ or Σ_{cc} is a 1-dimensional manifold around z_0. In this type, we have the following result.

Theorem 5.6 *Let $F=0$ be type RF_y^1 at $z_0 \in \Delta$. If $z_0 \notin \Sigma_{cc}$, then there exists a unique geometric solution passing through z_0.*

Proof. We denote that $F(x,y,p,q) = -y + f(x,p,q) = 0$. Since $F=0$ is of reduced type at z_0, there exists a smooth function germ $\alpha : (F^{-1}(0), z_0) \to (\mathbb{R}, 0)$ such that

$$f_q = \alpha \cdot (f_x - p + q f_p). \tag{5}$$

A complete solution, $\Gamma : (\mathbb{R} \times \mathbb{R}^2, 0) \to (F^{-1}(0), z_0)$, is given by integrating the vector field $\phi_* X$, where $X : U \to TU$ is given by

$$X = (-\alpha, -\alpha \cdot q, 1)$$

(cf. [3, Lemma 3.1]). By (5), we have

$$(f_x - p + qf_p)_q = (\alpha_x + q\alpha_p) \cdot (f_x - p + qf_p) + \alpha \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p) + f_p \tag{6}$$

It follows from the assumption $\Sigma_c = \Delta$ that

$$(f_x - p + qf_p)_q|_{\phi^{-1}(\Sigma_c)} = \alpha|_{\phi^{-1}(\Sigma_c)} \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)|_{\phi^{-1}(\Sigma_c)}.$$

In this case, a complete solution of Σ_c, $\Phi : (\mathbb{R} \times \mathbb{R}, 0) \to (\Sigma_c, z_0)$, is given by integrating the vector field $\phi_* Y$, where $Y : \phi^{-1}(\Sigma_c) \to T\phi^{-1}(\Sigma_c)$ is given by

$$Y = (-\alpha|_{\phi^{-1}(\Sigma_c)}, (-\alpha \cdot q)|_{\phi^{-1}(\Sigma_c)}, 1)$$

(cf. [17, Lemma 3.5]). This means that $\Gamma|_{\phi^{-1}(\Sigma_c)} = \Phi$. Therefore there is a geometric solution on Σ_c. Let $\gamma : (\mathbb{R}, t_0) \to (\Sigma_c, z_0); \gamma(t) = (x(t), y(t), p(t), q(t))$ be a geometric solution passing through z_0. By the definitions of Σ_c and Σ_{cc}, we have

$$\phi^{-1}(\Sigma_c) = (f_x - p + qf_p)^{-1}(0)$$

and

$$\phi^{-1}(\Sigma_{cc}) = (f_x - p + qf_p)^{-1}(0) \cap ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)^{-1}(0).$$

By differentiating the equality $(f_x - p + qf_p)(x(t), p(t), q(t)) = 0$ with respect to t, we have

$$((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)(x(t), p(t), q(t)) \cdot x'(t) + (f_x - p + qf_p)_q(x(t), p(t), q(t)) \cdot q'(t) = 0$$

On the other hand, by differentiating (5) with respect to x and p, we have

$$f_{xq} = \alpha_x \cdot (f_x - p + qf_p) + \alpha \cdot (f_x - p + qf_p)_x$$

and

$$f_{pq} = \alpha_p \cdot (f_x - p + qf_p) + \alpha \cdot (f_x - p + qf_p)_p$$

It follows that

$$(x'(t) + \alpha \cdot q'(t)) \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)(x(t), p(t), q(t)) = 0.$$

Since $z_0 \notin \Sigma_{cc}$, $(f_x - p + qf_p)_x + q(f_x - p + qf_p)_p \neq 0$ at u_0. Then we have $x'(t) + \alpha \cdot q'(t) = 0$ at t_0, and hence $q'(t_0) \neq 0$. It follows that we can reparametrize $\gamma(t)$ as $(x(t), y(t), p(t), t)$. By the analogous way in the proof of Lemma 3.2 in the Appendix, we can show the uniqueness for geometric solution passing through z_0. In fact, we may apply $x'(t) = -\alpha(x(t), y(t), p(t), t)$, $y'(t) = p(t)x'(t)$ and $p'(t) = tx'(t)$ to the case of $n = 2$ in the proof of Lemma 3.2. This completes the proof of Theorem 5.6. $\square$

Proposition 5.7 *Let $F = 0$ be type RF_y^1 at $z_0 \in \Delta$. If Σ_{cc} is a 1-dimensional manifold around z_0, then Σ_{cc} is a singular solution passing through z_0.*

Proof. By the assumption, it is easy to see that Σ_{cc} is a geometric solution passing through z_0. As before, $\phi^{-1}(\Sigma_c) = (f_x - p + qf_p)^{-1}(0)$ and $\phi^{-1}(\Sigma_{cc}) = (f_x - p + qf_p)^{-1}(0) \cap ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)^{-1}(0)$. To show that Σ_{cc} is not a leaf of the complete solution of $F^{-1}(0)$ (and Σ_c) at z_0, it is sufficient to check that the scalar product of $\mathrm{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)$ and the vector field X is nonzero at u_0. Now

$$\begin{aligned}
&\langle \mathrm{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p), (-\alpha, -\alpha \cdot q, 1)\rangle \\
&= -\alpha \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_x - \alpha \cdot q((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_p \\
&\quad + ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_q.
\end{aligned} \tag{7}$$

By (5), (7) is equal to $2(f_{xp} + qf_{pp}) - 1$ at u_0. It follows from the assumption $\Sigma_c = \Delta$ that there exists a smooth function germ β such that $f_p = \beta \cdot (f_x - p + qf_p)$. Differentiating this equality with respect to x and p, then

$$f_{xp} = \beta_x \cdot (f_x - p + qf_p) + \beta \cdot (f_x - p + qf_p)_x$$

and

$$f_{pp} = \beta_p \cdot (f_x - p + qf_p) + \beta \cdot (f_x - p + qf_p)_p.$$

It follows that (7) is nonzero at u_0. $\square$

5.4 On the type RR^2_y

Suppose that $F=0$ is the type RR^2_y at z_0. Then $\Sigma_c \supsetneq \Delta = \Sigma_{cc}$ around z_0. By Theorem 5.2, $F=0$ has a complete solution of Σ_c at z_0 if and only if Σ_{cc} is a 1-dimensional manifold around z_0. However, we have the following.

Theorem 5.8 *Let $F=0$ be type RR^2_y at $z_0 \in \Delta$. $F=0$ has a complete singular solution of Σ_c at z_0 if and only if Σ_{cc} is a 1-dimensional manifold around z_0.*

Proof. By Theorem 5.2, each leaf of the complete solution of $F^{-1}(0)$ which meet Σ_c away from Σ_{cc} intersect Σ_c transversally, and each leaf of the complete solution of Σ_c intersects Σ_{cc} transversally. Therefore the complete solution of Σ_c is the complete singular solution of Σ_c. □

By the definition of Σ_{cc},

$$(f_x - p + qf_p)_x + q(f_x - p + qf_p)_p = 0,\ (f_x - p + qf_p)_q = 0$$

at $z_0 \in \Sigma_{cc}$ (cf. [17]). Since $f_x - p + qf_p$ is regular at z_0, $(f_x - p + qf_p)_p \neq 0$ at z_0. Hence $F=0$ satisfies either (i) $((f_x-p+qf_p)_x+q(f_x-p+qf_p)_p)_q \neq 0$ or (ii) $((f_x-p+qf_p)_x+q(f_x-p+qf_p)_p)_q = 0$ at z_0. It follows that z_0 is a regular point of $(f_x-p+qf_p)_x+q(f_x-p+qf_p)_p$, or of $(f_x-p+qf_p)_q$.

Proposition 5.9 *Let $F=0$ be type RR^2_y at $z_0 \in \Delta$. Suppose that Σ_{cc} is a 1-dimensional manifold around z_0.*

(1) *If $F=0$ satisfies the condition* (i), *then each leaf of the complete solution of $F^{-1}(0)$ is intersects Σ_{cc} transversally and hence Σ_{cc} is a singular solution passing through z_0.*

(2) *If $F=0$ satisfy the condition* (ii) *and $F_{pq}|_{\Sigma_{cc}} \equiv 0$ around z_0, then each leaf of the complete solution of $F^{-1}(0)$ is tangent to Σ_{cc}. If $\gamma(t)=(x(t),y(t),p(t),q(t)) \in \Sigma_{cc}$ is a geometric solution, $\gamma(t)$ is represented by the form (a,b,c,t) where $a,b,c \in \mathbb{R}$. Moreover, $\gamma(t)$ is a leaf of the complete solution of $F^{-1}(0)$.*

Proof. (1) Since $\phi^{-1}(\Sigma_{cc}) = (f_x-p+qf_p)^{-1}(0) \cap ((f_x-p+qf_p)_x + q(f_x-p+qf_p)_p)^{-1}(0)$, it is sufficient to check that the scalar product of $\mathrm{grad}((f_x-p+qf_p)_x + q(f_x-p+qf_p)_p)$ and the vector field X is nonzero at u_0. By the same calculations in Proposition 5.7,

$$\langle \mathrm{grad}((f_x-p+qf_p)_x + q(f_x-p+qf_p)_p), (-\alpha, -\alpha \cdot q, 1)\rangle = 2(f_{xp}+qf_{pp}) - 1$$

at u_0. The condition (i) guarantees that $2(f_{xp}+qf_{pp}) - 1 \neq 0$ at u_0. Therefore each leaf of the complete solution of $F^{-1}(0)$ is intersects Σ_{cc} transversally and hence Σ_{cc} is a singular solution passing through z_0.

(2) Since $\phi^{-1}(\Sigma_{cc}) = (f_x-p+qf_p)^{-1}(0) \cap ((f_x-p+qf_p)_q)^{-1}(0)$, it is sufficient to check that the scalar product of $\mathrm{grad}(f_x-p+qf_p)_q$ and the vector field X is zero. By the direct calculations, the consequence follows from the condition $F_{pq}|_{\Sigma_{cc}} \equiv 0$ around z_0.

Let $\gamma(t)=(x(t),y(t),p(t),q(t)) \in \Sigma_{cc}$ be a geometric solution passing through z_0. By differentiating $f_p(x(t),p(t),q(t))=0$ with respect to t,

$$(f_{xp}+qf_{pp})(x(t),p(t),q(t)) \cdot x'(t) + f_{pq}(x(t),p(t),q(t)) \cdot q'(t) = 0.$$

By the condition (ii), we have $f_{xp}+qf_{pp} = 1/2$ at u_0 and hence $x'(t) \equiv 0$. This means that $x(t)$ is constant on Σ_{cc} around z_0.

Differentiating (5) with respect to p, we have $f_{pq} = \alpha_p \cdot (f_x-p+qf_p) + \alpha \cdot (f_x-p+qf_p)_p$. It follows that $\alpha|_{\Sigma_{cc}} \equiv 0$ around z_0. By the form of the vector field X (see, in the proof of Theorem 5.6), $\Gamma|_{\Gamma^{-1}(\Sigma_{cc})} = \gamma$. □

5.5 On the type RR_y^3

Suppose that $F = 0$ is the type RR_y^3 at z_0. Then $\Sigma_c \supsetneq \Delta \supsetneq \Sigma_{cc}$ around z_0. In this subsection, we assume that Δ is a 1-dimensional manifold around z_0 and $z_0 \notin \Sigma_{cc}$. By Theorem 5.2, $F = 0$ has a complete solution of Σ_c at z_0. If Δ is not a geometric solution passing through z_0, the complete solution of Σ_c is the complete singular solution of Σ_c. On the other hand, if Δ is a geometric solution passing through z_0, we have the following.

Proposition 5.10 *Let $F = 0$ be type RR_y^3 at $z_0 \in \Delta \setminus \Sigma_{cc}$. If $\gamma(t) = (x(t), y(t), p(t), q(t)) \in \Delta$ is a geometric solution passing through z_0, then $\gamma(t)$ is represented by the form (a, b, c, t) where $a, b, c \in \mathbb{R}$. Moreover, $\gamma(t)$ is a leaf of both complete solutions of $F^{-1}(0)$ and Σ_c.*

Proof. Since $z_0 \notin \Sigma_{cc}$ and (6), we have $(f_x - p + qf_p)_x + q(f_x - p + qf_p)_p \neq 0$ at u_0. Differentiating equalities $(f_x - p + qf_p)(x(t), p(t), q(t)) = 0$ and $f_p(x(t), p(t), q(t)) = 0$ with respect to t, we have

$$\begin{pmatrix} (f_x - p + qf_p)_x + q(f_x - p + qf_p)_p & (f_x - p + qf_p)_q \\ f_{xp} + qf_{pp} & f_{pq} \end{pmatrix} \begin{pmatrix} x'(t) \\ q'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $\gamma(t)$ is a geometric solution, $(x'(t), q'(t)) \neq (0, 0)$ on Δ. Thus

$$\det \begin{pmatrix} (f_x - p + qf_p)_x + q(f_x - p + qf_p)_p & (f_x - p + qf_p)_q \\ f_{xp} + qf_{pp} & f_{pq} \end{pmatrix} = 0$$

on Δ. It follows from (5) that $\alpha|_\Delta \equiv 0$ and hence $x'(t) \equiv 0$. This means that $x(t)$ is constant on Δ around z_0.

Since $(f_x - p + qf_p)_q|_\Delta = 0$ and the forms of the vector fields X for a complete solution of $F^{-1}(0)$ and Y for a complete solution of Σ_c (cf. Proof of Theorem 5.6), $\Gamma|_{\Gamma^{-1}(\Delta)} = \Phi|_{\Phi^{-1}(\Delta)} = \gamma$. $\square$

5.6 On the type SC_y

Suppose that $F = 0$ is of Clairaut type at z_0 and z_0 is a singular point of $F_q|_{F^{-1}(0)}$. We call this type SC_y.

Proposition 5.11 *Let $F = 0$ be type SC_y at z_0. If Σ_c is a 2-dimensional manifold around z_0, then $z_0 \notin \Sigma_{cc}$.*

Proof. Let $F(x, y, p, q) = -y + f(x, p, q) = 0$. Since $F = 0$ is of Clairaut type at z_0, there is a function germ $\alpha : (F^{-1}(0), z_0) \to \mathbb{R}$ such that

$$f_x - p + qf_p = \alpha \cdot f_q. \tag{8}$$

By differentiating (8) with respect to p, $f_{xp} - 1 + qf_{pp} = \alpha_p \cdot f_q + \alpha \cdot f_{pq}$. Hence $f_{xp} + qf_{pp} = 1$ at u_0. By a direct calculation,

$$(f_x - p + qf_p)_{xq} + q(f_x - p + qf_p)_{pq} = (f_{xq} + qf_{pq})_x + q(f_{xq} + qf_{pq})_p + f_{xp} + qf_{pp} \tag{9}$$

On the other hand, by (8), we have

$$\begin{aligned} &(f_x - p + qf_p)_{xq} + q(f_x - p + qf_p)_{pq} \\ &= (\alpha_{xq} + q\alpha_{pq}) \cdot f_q + \alpha_q \cdot (f_{qx} + qf_{pq}) + (\alpha_x + q\alpha_p) \cdot f_{qq} + \alpha \cdot (f_{xqq} + qf_{pqq}) \end{aligned} \tag{10}$$

By definition, $\phi^{-1}(\Sigma_c) = f_q^{-1}(0)$. Since Σ_c is a 2-dimensional manifold around z_0, there is a regular function germ $g : (U, u_0) \to \mathbb{R}$ and a function germ $k : (U, u_0) \to (\mathbb{R}, 0)$ such that $\phi^{-1}(\Sigma_c) = g^{-1}(0)$ and $f_q = k \cdot g$ at least locally. By a direct calculation, the right hand of (9) is given by

$$((k_x+qk_p)_x+q(k_x+k_p)_p)\cdot g+2(k_x+qk_p)\cdot(g_x+qg_p)+k\cdot((g_x+qg_p)_x+q(g_x+qg_p)_p)+f_{xp}+qf_{pp}.$$

Also the right hand of (10) is given by

$$\begin{aligned}&(\alpha_{xq}+q\alpha_{pq})\cdot k\cdot g+\alpha_q\cdot((k_x+qk_p)\cdot g+k\cdot(g_x+qg_p))+(\alpha_x+q\alpha_p)\cdot(k_q\cdot g+k\cdot g_q)\\ &\quad+\alpha\cdot((k_{xq}+qk_{pq})\cdot g+k_q\cdot(g_x+qg_p)+(k_x+qk_p)\cdot g_q+k\cdot(g_{xq}+qg_{pq}))\end{aligned}$$

If $z_0 \in \Sigma_{cc}$, then $g = g_x + qg_p = g_q = 0$ at u_0. This contradicts the fact that (9) = (10), namely 1=0 at u_0. □

Under the assumption of Proposition 5.11, it follows from $z_0 \notin \Sigma_{cc}$ that there is a complete solution of Σ_c at z_0. According to Theorem 5.15 in below, a geometric solution passing through z_0 on Σ_c is a singular solution. Hence there is a complete singular solution of Σ_c at z_0 automatically.

5.7 On the type SR_y

Suppose that $F = 0$ is of reduced type at z_0 and z_0 is a singular point of $(F_x+pF_y+qF_p)|_{F^{-1}}(0)$. We call this type SR_y. We can prove the following result by using the same arguments in the proof of Proposition 5.11, so we omit it.

Proposition 5.12 *Let $F = 0$ be type SR_y at z_0. If Σ_c is a 2-dimensional manifold around z_0, then $z_0 \notin \Sigma_{cc}$.*

Moreover, we have the following result.

Proposition 5.13 *Let $F = 0$ be type SF_y and not of Clairaut type at z_0. If Σ_c is a 2-dimensional manifold around z_0, then Δ is a 1-dimensional manifold around z_0. Moreover, Δ is not a geometric solution passing through z_0.*

Proof. By (5), $f_q = \alpha \cdot (f_x - p + qf_p)$ with $\alpha(z_0) = 0$. Since $\phi^{-1}(\Sigma_c) = (f_x - p + qf_p)^{-1}(0)$ is a 2-dimensional manifold around z_0, there exist a regular function germ $g : (U, u_0) \to (\mathbb{R}, 0)$ and a function germ $k : (U, u_0) \to (\mathbb{R}, 0)$ such that $f_x - p + qf_p = k \cdot g$ and $k^{-1}(0) \subset g^{-1}(0)$ at least locally. By a direct calculation, we have

$$(f_x - p + qf_p)_{xq} + q(f_x - p + qf_p)_{pq} = 1$$

at u_0. On the other hand,

$$(f_x - p + qf_p)_{xq} + q(f_x - p + qf_p)_{pq} = k_q \cdot (g_x + qg_p) + (k_x + qk_p) \cdot g_q$$

at u_0. Hence $k_q \cdot (g_x + qg_p) + (k_x + qk_p) \cdot g_q = 1$ at u_0. If $g_q(u_0) = 0$, then $k_q(u_0) \neq 0$. It follows that k is represented by $\lambda(x,p,q)\cdot(q-\mu(x,p))$ at least locally, where λ and μ are function germs with $\lambda(u_0) \neq 0$. Since $k^{-1}(0) \subset g^{-1}(0)$, $g(x, p, \mu(x, p)) = 0$. By differentiating this equality with respect to x and p, we have

$$g_x(x, p, \mu(x, p)) + \mu_x(x, p)g_q(x, p, \mu(x, p)) = 0$$

and

$$g_p(x,p,\mu(x,p)) + \mu_p(x,p)g_q(x,p,\mu(x,p)) = 0.$$

This contradicts the fact that g is regular at u_0. Therefore we have $g_q \neq 0$ at u_0.

By the definition of Δ, $\phi^{-1}(\Delta) = g^{-1}(0) \cap f_p^{-1}(0)$. To show that Δ is a 1-dimensional manifold around z_0, it is sufficient to show that the matrix

$$A = \begin{pmatrix} g_x & g_p & g_q \\ f_{xp} & f_{pp} & f_{pq} \end{pmatrix}$$

has rank 2 at u_0. Since $f_x - p + qf_p$ and f_q are singular at u_0, $f_{xp} + qf_{pp} = 1$ and $f_{pq} = 0$ at u_0. Therefore rank$A = 2$ at u_0.

Next suppose that $\gamma : (\mathbb{R}, t_0) \to (\Delta, z_0); \gamma(t) = (x(t), y(t), p(t), q(t))$ is a geometric solution passing through z_0. By differentiating the equalities $g(x(t),p(t),q(t)) = 0$ and $f_p(x(t),p(t),q(t)) = 0$ with respect to t, we have

$$\begin{pmatrix} (g_x + qg_p)(x(t),p(t),q(t)) & g_q(x(t),p(t),q(t)) \\ (f_{xp} + qf_{pp})(x(t),p(t),q(t)) & f_{pq}(x(t),p(t),q(t)) \end{pmatrix} \begin{pmatrix} x'(t) \\ q'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since the determinant of the matrix $\begin{pmatrix} g_x + qg_p & g_q \\ f_{xp} + qf_{pp} & f_{pq} \end{pmatrix}$ is not vanish at t_0, $(x'(t), q'(t)) = (0,0)$ at t_0. This contradicts the fact that $\gamma(t)$ is a geometric solution passing through z_0. $\square$

As a conclusion, if $F = 0$ is of type SF_y and not of Clairaut type at z_0, then there is a complete singular solution of Σ_c at z_0 automatically by Propositions 5.12 and 5.13.

Finally in this section, we give an important difference between Clairaut type and reduced type.

Lemma 5.14 *Let $F = 0$ be type RC_y at z_0. If $z_0 \in \Delta \setminus \Sigma_{cc}$, then Δ is not a geometric solution passing through z_0.*

Proof. By remark 5.5, Δ is a 1-dimensional manifold around z_0. We assume that $\gamma : (\mathbb{R}, t_0) \to (\Delta, z_0); \gamma(t) = (x(t), y(t), p(t), q(t))$ is a geometric solution passing through z_0. By (8), $f_x - p + qf_p = \alpha \cdot f_q$. Differentiating $f_p(x(t),p(t),q(t)) = 0$ and $f_q(x(t),p(t),q(t)) = 0$ with respect to t, we have

$$\begin{pmatrix} (f_{xp} + qf_{pp})(x(t),p(t),q(t)) & f_{pq}(x(t),p(t),q(t)) \\ (f_{xq} + qf_{pq}(x(t),p(t),q(t)) & f_{qq}(x(t),p(t),q(t)) \end{pmatrix} \begin{pmatrix} x'(t) \\ q'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $f_{xp} - 1 + qf_{pp} = \alpha_p \cdot f_q + \alpha \cdot f_{pq}$ and $f_{xq} + f_p + qf_{pq} = \alpha_q \cdot f_q + \alpha \cdot f_{qq}$,

$$\det \begin{pmatrix} (f_{xp} + qf_{pp})(x(t),p(t),q(t)) & f_{pq}(x(t),p(t),q(t)) \\ (f_{xq} + qf_{pq})(x(t),p(t),q(t)) & f_{qq}(x(t),p(t),q(t)) \end{pmatrix} = f_{qq}(x(t),p(t),q(t)).$$

The condition $z_0 \notin \Sigma_{cc}$ guarantees that $f_{qq} \neq 0$ at u_0. It follows that $(x'(t), q'(t)) = (0,0)$ at t_0. This contradicts the fact that $\gamma(t)$ is a geometric solution passing through z_0. $\square$

Theorem 5.15 *Let $F = 0$ be of Clairaut type at z_0. If $\gamma(t) = (x(t), y(t), p(t), q(t)) \in \Sigma_c$ is a geometric solution passing through z_0, then $\gamma(t)$ is the singular solution.*

Proof. First we assume that z_0 is a regular point of $F_q|_{F^{-1}(0)}$. If $z_0 \notin \Delta$, then $\gamma(t)$ is a singular solution passing through z_0 by Theorem 5.2. Hence we may assume that $\gamma(t) \subset \Delta$. Also if $z_0 \notin \Sigma_{cc}$, then $\gamma(t)$ is not a geometric solution passing through z_0 by Lemma 5.14. We may assume that $\gamma(t) \subset \Sigma_{cc}$. Then we can conclude that $\gamma(t)$ is a singular solution passing through z_0 (cf. Section 5.2).

Next we assume that z_0 is a singular point of $F_q|_{F^{-1}(0)}$. Also we may assume that $\gamma(t) \subset \Delta$. By differentiating $f_p(x(t), p(t), q(t)) = 0$ with respect to t,

$$(f_{xp} + qf_{pp})(x(t), p(t), q(t)) \cdot x'(t) + f_{pq}(x(t), p(t), q(t)) \cdot q'(t) = 0.$$

Since $f_{xp} - 1 + qf_{pp} = \alpha_p \cdot f_q + \alpha_p \cdot f_{pq}$, we have

$$(1 + \alpha \cdot f_{pq}(x(t), p(t), q(t))) \cdot x'(t) + f_{pq}(x(t), p(t), q(t)) \cdot q'(t) = 0.$$

By the assumption, $f_{pq}(u_0) = 0$. Hence $x'(t_0) = 0$ and $q'(t_0) \neq 0$. It follows from the form of smooth complete solution, $\gamma(t)$ is the singular solution passing through z_0. This completes the proof of Theorem 5.15. $\square$

As a consequence, if $F = 0$ is of Clairaut type and there exists a geometric solution on the contact singular set, then uniqueness for geometric solutions does not hold.

6 Appendix: Proof of Lemma 3.2

In this appendix, we prove Lemma 3.2 by an elementary argument like as explicit ODEs.

Proof of Lemma 3.2. Suppose that $F_X(z_0) \neq 0$. It follows that $F = 0$ is of reduced type at z_0. By Theorem 2.2, there exists a complete solution at z_0 and hence there exists a geometric solution passing through z_0.

We assume that $F_X \equiv 1$ on the equation hypersurface, if necessary, we may consider $F/F_X = 0$ as $F = 0$. Moreover if $\gamma : (\mathbb{R}, t_0) \to (F^{-1}(0), z_0); \gamma(t) = (x(t), y(t), p_1(t), \ldots, p_n(t))$ is a geometric solution passing through z_0, then $p_n'(t_0) \neq 0$. Hence we can reparametrize $\gamma(t)$ as $(x(t), y(t), p_1(t), \ldots, p_{n-1}(t), t)$.

Let $\gamma(t) = (x(t), y(t), p_1(t), \ldots, p_{n-1}(t), t)$ and $\widetilde{\gamma}(t) = (\widetilde{x}(t), \widetilde{y}(t), \widetilde{p}_1(t), \ldots, \widetilde{p}_{n-1}(t), t)$ be geometric solutions passing through z_0, that is, $\gamma(t_0) = \widetilde{\gamma}(t_0) = z_0$. It is enough to show that $\gamma(t) = \widetilde{\gamma}(t)$ for $t_0 \leq t \leq t_0 + \varepsilon$, where ε is a small positive real number.

If we differentiate the equality $F(\gamma(t)) = F(x(t), y(t), p_1(t), \ldots, p_{n-1}(t), t) = 0$ with respect to t, then we get $x'(t) = -F_{p_n}(\gamma(t))$. By integrating this equality,

$$x(t) = x(t_0) - \int_{t_0}^{t} F_{p_n}(\gamma(t))dt.$$

Since $\gamma(t)$ is a geometric solution, namely, $y'(t) = p_1(t)x'(t)$, $p_i'(t) = p_{i+1}(t)x'(t)$ $(i = 1, \ldots, n-2)$ and $p_{n-1}'(t) = tx'(t)$, we have

$$y(t) = y(t_0) + \int_{t_0}^{t} p_1(t)x'(t)dt, \; p_i(t) = p_i(t_0) + \int_{t_0}^{t} p_{i+1}(t)x'(t)dt$$

$(i = 1, \ldots, n-2)$ and

$$p_{n-1}(t) = p_{n-1}(t_0) + \int_{t_0}^{t} tx'(t)dt.$$

It follows that

$$x(t)-\widetilde{x}(t)=\int_{t_0}^{t}\left(-F_{p_n}(\gamma(t))+F_{p_n}(\widetilde{\gamma}(t))\right)dt, \tag{11}$$

$$\begin{aligned} y(t)-\widetilde{y}(t) &= \int_{t_0}^{t}\left(p_1(t)x'(t)-\widetilde{p}_1(t)\widetilde{x}'(t)\right)dt \\ &= \int_{t_0}^{t} p_1(t)\left(x'(t)-\widetilde{x}'(t)\right)dt+\int_{t_0}^{t}\widetilde{x}'(t)\left(p_1(t)-\widetilde{p}_1(t)\right)dt \end{aligned} \tag{12}$$

$$\begin{aligned} p_i(t)-\widetilde{p}_i(t) &= \int_{t_0}^{t}\left(p_{i+1}(t)x'(t)-\widetilde{p}_{i+1}(t)\widetilde{x}'(t)\right)dt \\ &= \int_{t_0}^{t} p_{i+1}(t)\left(x'(t)-\widetilde{x}'(t)\right)dt+\int_{t_0}^{t}\widetilde{x}'(t)\left(p_{i+1}(t)-\widetilde{p}_{i+1}(t)\right)dt \end{aligned} \tag{13}$$

$(i=1,\ldots,n-2)$ and

$$\begin{aligned} p_{n-1}(t)-\widetilde{p}_{n-1}(t) &= \int_{t_0}^{t} t\left(x'(t)-\widetilde{x}'(t)\right)dt. \\ &= \int_{t_0}^{t} t\left(-F_{p_n}(\gamma(t))+F_{p_n}(\widetilde{\gamma}(t))\right)dt. \end{aligned} \tag{14}$$

Since F is a smooth mapping, there exists some number K such that

$$|-F_{p_n}(\gamma(t))+F_{p_n}(\widetilde{\gamma}(t))|\leq K|\gamma(t)-\widetilde{\gamma}(t)|\leq K\alpha(t),$$

where $t_0\leq t\leq t_0+\varepsilon$ and

$$\alpha(t)=|x(t)-\widetilde{x}(t)|+|y(t)-\widetilde{y}(t)|+|p_1(t)-\widetilde{p}_1(t)|+\cdots+|p_{n-1}(t)-\widetilde{p}_{n-1}(t)|.$$

Moreover, since γ and $\widetilde{\gamma}'$ are smooth mappings, we put

$$a_i=\max_{t_0\leq t\leq t_0+\varepsilon}\{|p_i(t)|\}\ (i=1,\ldots,n-1),\ \ a_n=\max\{|t_0|,|t_0+\varepsilon|\},\ \ b=\max_{t_0\leq t\leq t_0+\varepsilon}\{|\widetilde{x}'(t)|\}.$$

We denote an integration repeated i-times $\displaystyle\int_{t_0}^{t}\left(\cdots\left(\int_{t_0}^{t}\alpha(t)dt\right)\cdots\right)dt$ by $\displaystyle\left(\int_{t_0}^{t}\right)^{i}\alpha(t)(dt)^i$. It follows from (11), (12), (13) and (14) that

$$\begin{aligned} \alpha(t)\leq{}&(1+a_1+\cdots+a_n)K\int_{t_0}^{t}\alpha(t)dt+(a_2+\cdots+a_n)bK\left(\int_{t_0}^{t}\right)^{2}\alpha(t)(dt)^2+ \\ &\cdots+(a_i+\cdots+a_n)b^{i-1}K\left(\int_{t_0}^{t}\right)^{i}\alpha(t)(dt)^i+\cdots+a_nb^{n-1}K\left(\int_{t_0}^{t}\right)^{n}\alpha(t)(dt)^n. \end{aligned} \tag{15}$$

Moreover we put

$$L=\max\{1+a_1+\cdots+a_n,(a_2+\cdots+a_n)b,\cdots,a_nb^{n-1}\},\ \ M=\max_{t_0\leq t\leq t_0+\varepsilon}\{\alpha(t)\}.$$

By (15), we have

$$M \leq LKM \sum_{i=1}^{n} \frac{1}{i!}(t-t_0)^i \leq LKM \sum_{i=1}^{n} \frac{1}{i!}\varepsilon^i \tag{16}$$

We now consider a function $f(x) = \sum_{i=1}^{n}(1/i!)x^i - 1/LK$. If $x = 0$, then $f(x)$ is negative, and if x is sufficient large, then $f(x)$ is positive. By the means value theorem, there exists a positive real number t_n such that $f(x) < 0$ on $0 < x < t_n$.

If we take a small real number $\varepsilon > 0$ which satisfies $0 < \varepsilon < t_n$, then $f(\varepsilon)M \geq 0$ by (16). It follows that $M = 0$ and concludes that $\gamma(t) = \widetilde{\gamma}(t)$ for $t_0 \leq t \leq t_0 + \varepsilon$. This completes the proof of Lemma 3.2. □

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