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# On completely integrable implicit ordinary differential equations

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## Abstract

For smooth explicit  $n$ -th order ordinary differential equations, there exists a unique solution with initial condition and hence exists an  $n$ -parameter family of solutions at least locally. On the other hand, for smooth implicit ordinary differential equations, existence and uniqueness for solutions with initial condition does not hold. In this paper, we give a necessary and sufficient condition for existence of an  $n$ -parameter family of geometric solutions in the smooth category. Moreover, we give a sufficient condition that implicit ordinary differential equations have a unique geometric solution with initial condition. As a consequence, we classify completely integrable first and second ordinary differential equations in detail.

## 1 Introduction

For a smooth explicit ordinary differential equation

$$\frac{d^n y}{dx^n}(x) = f\left(x, y(x), \frac{dy}{dx}(x), \dots, \frac{d^{n-1}y}{dx^{n-1}}(x)\right), \quad (1)$$

it is well-known that there exists a unique smooth solution with initial condition for (1), where  $f$  is a smooth function (for instance, see [1, 2, 4]). It follows that there exists an  $n$ -parameter family of smooth solutions at least locally.

On the other hand, for a smooth implicit ordinary differential equation (briefly, an implicit ODE)

$$F(x, y, p_1, \dots, p_n) = 0, \quad (2)$$

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existence for a local solution with initial condition does not hold, where  $F$  is a smooth function of the independent variable  $x$ , the function  $y$  and its  $i$ -th derivatives  $p_i = d^i y/dx^i$ ,  $i = 1, \dots, n$ .

A natural question is what conditions guarantee existence and uniqueness for a local solution around a point of an implicit ODE. In this paper we shall discuss a qualitative theory for implicit ODEs.

It is natural to consider (2) as being defined on a subset in the space of  $n$ -jets of smooth functions of one variable,  $F : \mathcal{O} \rightarrow \mathbb{R}$  where  $\mathcal{O}$  is an open subset in  $J^n(\mathbb{R}, \mathbb{R})$ . Throughout this paper, we assume that 0 is a regular value of  $F$ . It follows that the set  $F^{-1}(0)$  is a hypersurface in  $J^n(\mathbb{R}, \mathbb{R})$ . We call  $F^{-1}(0)$  the *equation hypersurface*. Let  $(x, y, p_1, \dots, p_n)$  be a local coordinate on  $J^n(\mathbb{R}, \mathbb{R})$  and  $\xi \subset TJ^n(\mathbb{R}, \mathbb{R})$  be the canonical contact system on  $J^n(\mathbb{R}, \mathbb{R})$  described by the vanishing of the 1-forms

$$\begin{cases} \alpha_1 &= dy - p_1 dx, \\ \alpha_2 &= dp_1 - p_2 dx, \\ &\vdots \\ \alpha_n &= dp_{n-1} - p_n dx. \end{cases}$$

We now define the notion of solutions. A *smooth solution* (or, a *classical solution*) of  $F = 0$  passing through a point  $z_0$  is a smooth function germ  $y = f(x)$  at a point  $t_0$  such that  $(t_0, f(t_0), f'(t_0), \dots, f^{(n)}(t_0)) = z_0$  and  $F(x, f(x), f'(x), \dots, f^{(n)}(x)) = 0$ , where  $f^{(i)}(x) = (d^i f/dx^i)(x)$ . In other words, there exists a smooth function germ  $f : (\mathbb{R}, t_0) \rightarrow \mathbb{R}$  such that the image of the  $n$ -jet extension,  $j^n f : (\mathbb{R}, t_0) \rightarrow (J^n(\mathbb{R}, \mathbb{R}), z_0)$ ;  $j^n f(x) = (x, f(x), f'(x), \dots, f^{(n)}(x))$ , is contained in the equation hypersurface. It is easy to see that the map  $j^n f$  is an immersion germ with  $(j^n f)^* \alpha_i = 0$  for  $i = 1, \dots, n$ .

More generally, a *geometric solution* of  $F = 0$  passing through a point  $z_0$  is an integral immersion germ  $\gamma : (\mathbb{R}, t_0) \rightarrow (J^n(\mathbb{R}, \mathbb{R}), z_0)$  such that the image of  $\gamma$  is contained in the equation hypersurface, namely,  $\gamma' \neq 0$ ,  $\gamma^* \alpha_i = 0$  for  $i = 1, \dots, n$  and  $F(\gamma(t)) = 0$  for each  $t \in (\mathbb{R}, t_0)$ .

By the definition, a smooth solution is also a geometric solution. Conversely, it is easy to see that if  $\gamma(t) = (x(t), y(t), p_1(t), \dots, p_n(t))$  is a geometric solution of  $F = 0$  and  $x'(t_0) \neq 0$ , then we can reparametrize  $\gamma(t)$  as a smooth solution.

The following notions are basic in this paper (cf. [3, 15, 17]). By the definition of parametrized version for smoothness of the solutions (*i.e.*, smooth solutions), a *smooth complete solution of (2) at  $z_0$*  is defined to be an  $n$ -parameter family of smooth function germs  $y = f(t, \mathbf{c}) = f(t, c_1, \dots, c_n)$  such that

$$F \left( t, f(t, \mathbf{c}), \frac{\partial f}{\partial t}(t, \mathbf{c}), \dots, \frac{\partial^n f}{\partial t^n}(t, \mathbf{c}) \right) = 0$$

and the map germ  $j_1^n f : (\mathbb{R} \times \mathbb{R}^n, (t_0, \mathbf{c}_0)) \rightarrow (F^{-1}(0), z_0)$  defined by

$$j_1^n f(t, \mathbf{c}) = \left( t, f(t, \mathbf{c}), \frac{\partial f}{\partial t}(t, \mathbf{c}), \dots, \frac{\partial^n f}{\partial t^n}(t, \mathbf{c}) \right)$$

is an immersion. It follows that the equation hypersurface is foliated by an  $n$ -parameter family of smooth solutions.

On the other hand, we consider the corresponding definition of parametrized version for geometric solutions. Let  $\Gamma : (\mathbb{R} \times \mathbb{R}^n, (t_0, \mathbf{c}_0)) \rightarrow (F^{-1}(0), z_0)$  be an  $n$ -parameter family of geometric solutions, *i.e.*,  $\Gamma(\cdot, \mathbf{c})$  is a geometric solution of  $F = 0$  for each  $\mathbf{c} \in (\mathbb{R}^n, \mathbf{c}_0)$ .

We call  $\Gamma$  a *complete solution of (2) at  $z_0$*  if  $\Gamma$  is an immersion germ, namely,

$$\text{rank} \begin{pmatrix} \partial x/\partial t & \partial y/\partial t & \partial p_1/\partial t & \cdots & \partial p_n/\partial t \\ \partial x/\partial \mathbf{c} & \partial y/\partial \mathbf{c} & \partial p_1/\partial \mathbf{c} & \cdots & \partial p_n/\partial \mathbf{c} \end{pmatrix} (t_0, \mathbf{c}_0) = n + 1,$$

where  $\Gamma(t, \mathbf{c}) = (x(t, \mathbf{c}), y(t, \mathbf{c}), p_1(t, \mathbf{c}), \dots, p_n(t, \mathbf{c}))$ . It follows that the equation hypersurface is foliated by an  $n$ -parameter family of geometric solutions.

We say that an equation  $F = 0$  is *smooth completely integrable* (respectively, *completely integrable*) at  $z_0$  if there exists a smooth complete solution (respectively, a complete solution) of  $F = 0$  at  $z_0$ .

In the study of implicit ODEs from the view point of singularity theory, there is a lot of research. For example, generic singularities and properties were given in [5, 6, 8, 16, 14] for the case of first order, in [12, 13] for the case of second order and in [7] for the case of any order etc. In this paper is focused on the theory of completely integrable implicit ODEs.

In §2, we give a necessary and sufficient condition for existence of complete solutions and smooth complete solutions at a point. We show that  $F = 0$  is completely integrable at  $z_0$  if and only if  $F = 0$  is either of Clairaut type or of reduced type at  $z_0$  (cf. Theorem 2.2). This result guarantees existence for a geometric solution in Proposition 3.1. In §3, we give a sufficient condition for existence and uniqueness for a geometric solution with initial condition. In §4 and §5, we classify completely integrable first order and second order ODEs respectively. We also give examples of completely integrable implicit ODEs.

All map germs and manifolds considered here are differential of class  $C^\infty$ .

## 2 Existence and uniqueness for complete solutions

In this section, we consider existence and uniqueness conditions for a complete solution and a smooth complete solution of implicit ODEs. We denote a map  $F_x + p_1 F_y + p_2 F_{p_1} + \cdots + p_n F_{p_{n-1}}$  by  $F_X$ . Here  $F_x$  (respectively,  $F_y, F_{p_i}$ ) is the partial derivative of  $F$  with respect to  $x$  (respectively, with respect to  $y, p_i$ ). We refer to the following Lemma. See in case of first order in [9, 16], and of second order in [3, Lemma 3.1].

**Lemma 2.1** *Let  $F = 0$  be an implicit ODE at  $z_0$ . The equation  $F = 0$  is completely integrable at  $z_0 \in F^{-1}(0)$  if and only if there exist function germs  $\alpha, \beta : (F^{-1}(0), z_0) \rightarrow \mathbb{R}$ , which do not vanish simultaneously, such that*

$$\alpha \cdot F_X|_{F^{-1}(0)} + \beta \cdot F_{p_n}|_{F^{-1}(0)} \equiv 0.$$

*Proof.* Suppose that  $F = 0$  is completely integrable at  $z_0$  and let

$$\Gamma : (\mathbb{R} \times \mathbb{R}^n, (t_0, \mathbf{c}_0)) \rightarrow (F^{-1}(0), z_0)$$

be a complete solution of the implicit ODE at  $z_0$ . Then differentiating  $\Gamma$  with respect to  $t$  yields a vector field  $Z : (F^{-1}(0), z_0) \rightarrow TF^{-1}(0)$  given by  $Z(\Gamma(t, \mathbf{c})) = \Gamma_t(t, \mathbf{c})$ . Since  $Z(z)$  lies in the contact plane  $\xi_z$  for each  $z \in (F^{-1}(0), z_0)$ , it has the form  $Z = (\alpha, p_1\alpha, \dots, p_n\alpha, \beta)$  for some function germs  $\alpha, \beta : (F^{-1}(0), z_0) \rightarrow \mathbb{R}$  which do not vanish simultaneously. Besides  $Z(z)$  also lies in  $T_z F^{-1}(0)$ . It follows that the identity

$$\alpha \cdot F_X|_{F^{-1}(0)} + \beta \cdot F_{p_n}|_{F^{-1}(0)} \equiv 0$$

holds. Reversing the argument yields the converse.  $\square$

We say that an equation  $F = 0$  is of  $n$ -th order *Clairaut type* (for short, *Clairaut type*) at  $z_0$  if there exist smooth function germs  $A, B : (J^n(\mathbb{R}, \mathbb{R}), z_0) \rightarrow \mathbb{R}$  such that

$$F_X = A \cdot F + B \cdot F_{p_n},$$

and of *reduced type* at  $z_0$  if there exist smooth function germs  $A', B' : (J^n(\mathbb{R}, \mathbb{R}), z_0) \rightarrow \mathbb{R}$  such that

$$F_{p_n} = A' \cdot F + B' \cdot F_X.$$

We give a necessary and sufficient condition for existence of a smooth complete solution and a complete solution of implicit ODEs.

**Theorem 2.2** *Let  $F = 0$  be an implicit ODE at  $z_0$ .*

(1)  *$F = 0$  is smooth completely integrable at  $z_0$  if and only if  $F = 0$  is of Clairaut type at  $z_0$ .*

(2)  *$F = 0$  is completely integrable at  $z_0$  if and only if  $F = 0$  is of Clairaut type, or of reduced type at  $z_0$ .*

*Proof.* (1) The proof follows from a direct analogy of the proof for Theorem 2.2 in [10] or Theorem 3.1 in [15], so that we omit it.

(2) The result is a consequence of Lemma 2.1 and the fact that  $F = 0$  is regular.  $\square$

The uniqueness of the complete solution is the following.

**Proposition 2.3** *Let  $\Gamma_1 : (\mathbb{R} \times \mathbb{R}^n, (t_1, \mathbf{c}_1)) \rightarrow (F^{-1}(0), z_0)$  and  $\Gamma_2 : (\mathbb{R} \times \mathbb{R}^n, (t_2, \mathbf{c}_2)) \rightarrow (F^{-1}(0), z_0)$  be complete solutions of  $F = 0$  at  $z_0$ . Then there exists a diffeomorphism germ  $\Phi : (\mathbb{R} \times \mathbb{R}^n, (t_2, \mathbf{c}_2)) \rightarrow (\mathbb{R} \times \mathbb{R}^n, (t_1, \mathbf{c}_1))$  of the form  $\Phi(t, \mathbf{c}) = (\phi_1(t, \mathbf{c}), \phi_2(\mathbf{c}))$  such that  $\Gamma_1 \circ \Phi = \Gamma_2$ .*

*Proof.* Suppose that the assertion does not hold. Since the complete solution is an  $n$ -parameter family of curves in  $F^{-1}(0)$ , then there exists a point  $z_1 \in (F^{-1}(0), z_0)$  such that  $\Gamma_1(\cdot, \mathbf{c}_1)$  and  $\Gamma_2(\cdot, \mathbf{c}_2)$  are transversal near the point  $z_1$ . Then we can construct a map germ  $\Gamma : (\mathbb{R} \times \mathbb{R}^n, 0) \rightarrow (F^{-1}(0), z_1)$  such that (at least)  $\Gamma(t, \cdot, \cdot, c_3, \dots, c_n)$  is an immersion germ,

$$\frac{\partial y}{\partial c_1}(t, \mathbf{c}) = p_1(t, \mathbf{c}) \frac{\partial x}{\partial c_1}(t, \mathbf{c}), \dots, \frac{\partial p_{n-1}}{\partial c_1}(t, \mathbf{c}) = p_n(t, \mathbf{c}) \frac{\partial x}{\partial c_1}(t, \mathbf{c}) \quad (3)$$

and

$$\frac{\partial y}{\partial c_2}(t, \mathbf{c}) = p_1(t, \mathbf{c}) \frac{\partial x}{\partial c_2}(t, \mathbf{c}), \dots, \frac{\partial p_{n-1}}{\partial c_2}(t, \mathbf{c}) = p_n(t, \mathbf{c}) \frac{\partial x}{\partial c_2}(t, \mathbf{c}), \quad (4)$$

where  $\Gamma(t, \mathbf{c}) = (x(t, \mathbf{c}), y(t, \mathbf{c}), p_1(t, \mathbf{c}), \dots, p_n(t, \mathbf{c}))$ . If we calculate second order partial derivatives of the last equality for (3) with respect to  $c_2$  and for (4) with respect to  $c_1$ , we get

$$\frac{\partial^2 p_{n-1}}{\partial c_2 \partial c_1} = \frac{\partial p_n}{\partial c_2} \cdot \frac{\partial x}{\partial c_1} + p_n \cdot \frac{\partial^2 x}{\partial c_2 \partial c_1} \quad \text{and} \quad \frac{\partial^2 p_{n-1}}{\partial c_1 \partial c_2} = \frac{\partial p_n}{\partial c_1} \cdot \frac{\partial x}{\partial c_2} + p_n \cdot \frac{\partial^2 x}{\partial c_1 \partial c_2}.$$

Therefore we obtain the equality  $(\partial p_n / \partial c_2) \cdot (\partial x / \partial c_1) = (\partial p_n / \partial c_1) \cdot (\partial x / \partial c_2)$ . This contradicts the fact that  $\Gamma(t, \cdot, \cdot, c_3, \dots, c_n)$  is an immersion germ.  $\square$

### 3 Existence and uniqueness for geometric solutions

In this section, we give an existence and uniqueness condition for a geometric solution with initial condition.

Let  $F = 0$  be an implicit ODE at  $z_0$ . Consider a point  $z \in F^{-1}(0)$  such that the contact plane  $\xi_z$  intersects  $T_z F^{-1}(0)$  transversally. Then it is easy to see that a complete solution exists at  $z$  by integrating the line field  $\xi \cap T F^{-1}(0)$  (see, Lemma 2.1). We call points where transversality fails *contact singular points* and denote the set of such points by  $\Sigma_c = \Sigma_c(F)$ . We call  $\Sigma_c(F)$  the *contact singular set* of  $F^{-1}(0)$ . It is easy to check that the contact singular set is given by

$$\Sigma_c(F) = \{z \in F^{-1}(0) \mid F_X(z) = 0, F_{p_n}(z) = 0\}.$$

We say that a geometric solution  $\gamma : (\mathbb{R}, 0) \rightarrow (F^{-1}(0), z_0)$  is a *singular solution* of  $F = 0$  passing through  $z_0$  if for any representative  $\tilde{\gamma} : I \rightarrow F^{-1}(0)$  of  $\gamma$  and any open subinterval  $(a, b) \subset I$  at 0,  $\tilde{\gamma}|_{(a,b)}$  is never contained in a leaf of a complete solution (cf. [3, 7, 9, 11]). If a completely integrable implicit ODE has a singular solution, then uniqueness for geometric solutions break down at such points (see, Example 4.4).

**Proposition 3.1** *Let  $F = 0$  be an implicit ODE at  $z_0$ . If  $z_0 \notin \Sigma_c(F)$ , then there exists a unique geometric solution passing through  $z_0$ .*

By Proposition 3.1, a geometric solution  $\gamma : (\mathbb{R}, t_0) \rightarrow (F^{-1}(0), z_0)$  is a singular solution only if it is contained in  $\Sigma_c(F)$ . If  $z_0 \notin \Sigma_c(F)$ , either  $F_X \neq 0$  or  $F_{p_n} \neq 0$  at  $z_0$ . The latter case, the consequence of Proposition 3.1 is follows from the classical results for existence and uniqueness of smooth solution of smooth explicit equations. Thus in order to prove Proposition 3.1, it is enough to show the former case. If  $F_X(z_0) \neq 0$ , then  $F = 0$  has a unique geometric solution around  $z_0$  by integrating the line field  $\xi_z \cap T_z F^{-1}(0)$ . Therefore we have the following result.

**Lemma 3.2** *Let  $F = 0$  be an implicit ODE at  $z_0$ . If  $F_X(z_0) \neq 0$ , then there exists the unique geometric solution passing through  $z_0$ .*

The above Lemma is a well-known result. However, in the Appendix, we shall prove Lemma 3.2 explicitly by an elementary argument like as explicit ODEs, since we use this method to prove in Theorem 5.6.

Now suppose that  $F = 0$  is completely integrable at  $z_0$  and  $\Sigma_c(F)$  is an  $n$ -dimensional manifold around  $z_0$ . Remark that if  $F = 0$  is completely integrable at  $z_0$ , the condition that  $\Sigma_c(F)$  is an  $n$ -dimensional manifold around  $z_0$  is a generic condition by Theorem 2.2.

We call a map germ  $\Phi : (\mathbb{R} \times \mathbb{R}^{n-1}, (t_0, \mathbf{b}_0)) \rightarrow (\Sigma_c(F), z_0)$  a *complete solution* of  $\Sigma_c(F)$  at  $z_0$  if  $\Phi$  is an immersion germ and  $\Phi(\cdot, \mathbf{b})$  is a geometric solution for each  $\mathbf{b} \in (\mathbb{R}^{n-1}, \mathbf{b}_0)$ . Moreover, we call  $\Phi$  a *complete singular solution* of  $\Sigma_c(F)$  at  $z_0$  if  $\Phi(\cdot, \mathbf{b})$  is a singular solution for each  $\mathbf{b} \in (\mathbb{R}^{n-1}, \mathbf{b}_0)$ . If  $\xi_z$  intersects  $T_z \Sigma_c(F)$  transversally in  $T_z F^{-1}(0)$ , then integrating the line field  $\xi \cap T \Sigma_c(F)$  yields a complete solution of  $\Sigma_c(F)$ . We call a point where transversality does not hold a *second order contact singular point* and denote the set of such points by  $\Sigma_{cc} = \Sigma_{cc}(F)$  (or,  $\Sigma_{c^2} = \Sigma_{c^2}(F)$ ) (cf. [3, 17]). Inductively, if  $\Sigma_{cc}(F)$  is an  $(n-1)$ -dimensional manifold around  $z_0$ , then we can define a *complete solution* of  $\Sigma_{cc}(F)$  at  $z_0$ , a *complete singular solution* of  $\Sigma_{cc}(F)$  at  $z_0$  and third order contact singular set  $\Sigma_{ccc} = \Sigma_{ccc}(F)$  (or,  $\Sigma_{c^3} = \Sigma_{c^3}(F)$ ) etc. Therefore we have the following sequence when  $\Sigma_{c^i}(F)$  are  $(n-i+1)$ -dimensional submanifolds,  $i = 1, \dots, n$

(cf. [7]):

$$\Sigma_{c^n}(F) \subset \Sigma_{c^{n-1}}(F) \subset \cdots \subset \Sigma_{c^2}(F) \subset \Sigma_c(F) \subset F^{-1}(0).$$

## 4 Completely integrable first order ordinary differential equations

In this section, we quickly review known results for the theory of completely integrable implicit first order ODEs

$$F(x, y, p) = 0, \quad p = dy/dx.$$

For more detail, see [8, 9, 10, 11, 16]. In [9], it has shown the following results.

**Theorem 4.1** *Let  $F(x, y, p) = 0$  be an implicit first order ODE at  $z_0$ .  $F = 0$  is completely integrable at  $z_0$  if and only if  $z_0 \notin \Sigma_c$  or  $\Sigma_c$  is an 1-dimensional manifold around  $z_0$ . Moreover, if  $\Sigma_c$  is an 1-dimensional manifold around  $z_0$ , then  $\Sigma_c$  is a singular solution of  $F = 0$  passing through  $z_0$ .*

As a corollary of Theorem 4.1, the condition of Proposition 3.1 is a necessary and sufficient condition for uniqueness of geometric solutions of completely integrable implicit first order ODEs.

**Corollary 4.2** *Let  $F(x, y, p) = 0$  be a completely integrable implicit first order ODE at  $z_0$ . There exists a unique geometric solution passing through  $z_0$  if and only if  $z_0 \notin \Sigma_c$ .*

Now suppose that  $z_0 \in \Sigma_c$ . Since  $F = 0$  is regular,  $F_y(z_0) \neq 0$ . By the implicit function theorem, there exists a smooth function  $f : U \rightarrow \mathbb{R}$ , where  $U$  is an open set in  $\mathbb{R}^2$ , such that in a neighborhood of  $z_0$ ,  $(x, y, p) \in F^{-1}(0)$  if and only if  $-y + f(x, p) = 0$ . Thus we may assume without loss of generality that  $F(x, y, p) = -y + f(x, p) = 0$ . It follows that  $z_0$  is a regular point of either  $F_p|_{F^{-1}(0)}$  or  $F_X|_{F^{-1}(0)}$ .

Hence completely integrable implicit first order ODEs have four kinds of types (cf. [16]), see Table 1.

Conditions		Type	Name
$z_0 \notin \Sigma_c$	$F_p(z_0) \neq 0$	Clairaut type	$C$
	$F_X(z_0) \neq 0$	reduced type	$R$
$z_0 \in \Sigma_c$	$F_y(z_0) \neq 0$	regular point of $F_p _{F^{-1}(0)}$	Clairaut type
		regular point of $F_X _{F^{-1}(0)}$	reduced type

Table 1. Classifications of completely integrable implicit first order ODEs at  $z_0$ .

We now give two easy examples illustrating the notion of the complete solution and results. One is satisfied the condition of Proposition 3.1 and the other is not.

**Example 4.3** (Type  $R$ ). Let  $F(x, y, p) = x - f(p) = 0$  with  $f(0) = 0$ . Then  $F_X = F_x + pF_y = 1$  and  $F_p = -f'(p)$ . In this case,  $\Sigma_c = \emptyset$  and  $F = 0$  is of reduced type. By Theorem 2.2, this equation is completely integrable and a complete solution  $\Gamma : (\mathbb{R} \times \mathbb{R}, 0) \rightarrow F^{-1}(0)$  is given by

$$\Gamma(t, c) = \left( f(t), \int tf'(t)dt + c, t \right).$$

By Proposition 3.1 (or, Corollary 4.2), the geometric solution is unique passing through each point  $z_0 \in F^{-1}(0)$ .

For example, if we put  $f(p) = p^2$  (respectively,  $f(p) = p^3$ ), the complete solution is  $\Gamma(t, c) = (t^2, (2/3)t^3 + c, t)$  (respectively,  $\Gamma(t, c) = (t^3, (3/4)t^4 + c, t)$ ). We can draw pictures; the equation surface  $F^{-1}(0)$ , geometric solutions  $\{\Gamma(t, c)\}_{c \in (\mathbb{R}, 0)}$  and the phase portrait  $\{\pi \circ \Gamma(t, c)\}_{c \in (\mathbb{R}, 0)}$ , where  $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is the canonical projection  $\pi(x, y, p) = (x, y)$ . See Figures 1 and 2.

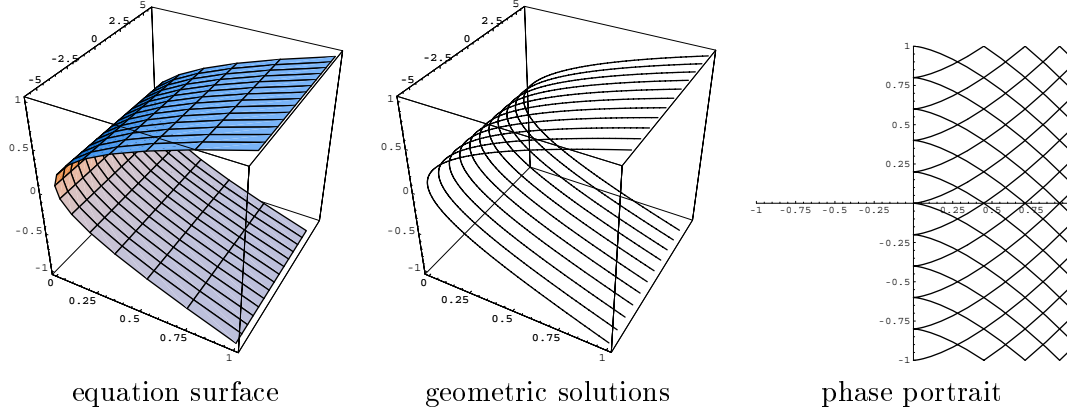


Figure 1.  $F(x, y, p) = x - p^2 = 0$ .

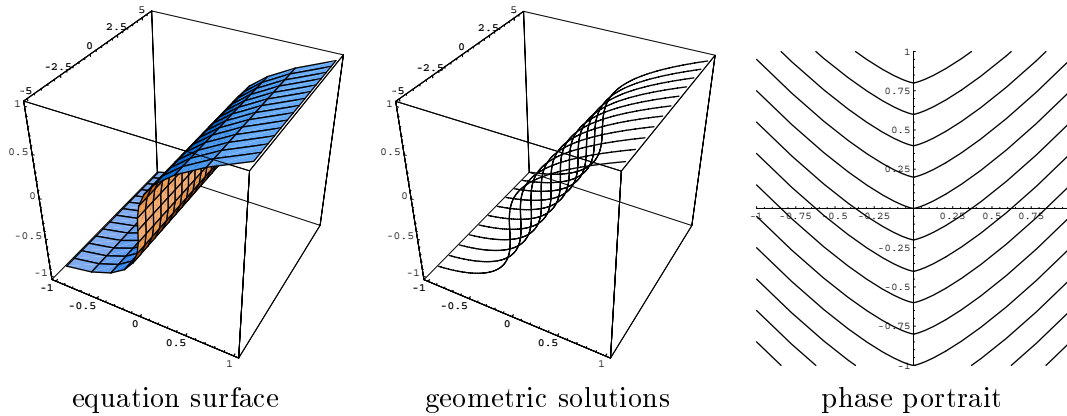


Figure 2.  $F(x, y, p) = x - p^3 = 0$ .

**Example 4.4** (Type  $RR_y$ ). Let  $F(x, y, p) = y - f(p) = 0$  with  $f(0) = 0$ . If  $f'(0) \neq 0$ , then there exist not only a smooth complete solution, but also a unique smooth solution passing through each point of  $F^{-1}(0)$ .

We now suppose that  $f'(0) = 0$ . It follows that there exists a smooth function germ  $g$  such that  $f(p) = p^2g(p)$ . Then  $F_x = F_x + pF_y = p$  and  $F_p = -f'(p) = -p(2g(p) + pg'(p))$ . The contact singular set  $\Sigma_c$  is  $x$ -axis in  $J^1(\mathbb{R}, \mathbb{R})$  and  $F = 0$  is of reduced type. By Theorem 2.2, this equation is completely integrable and a complete solution  $\Gamma : (\mathbb{R} \times \mathbb{R}, 0) \rightarrow F^{-1}(0)$  is given by

$$\Gamma(t, c) = \left( \int (2g(t) + tg'(t))dt + c, f(t), t \right).$$

By Theorem 4.1, we have a singular solution  $\gamma : (\mathbb{R}, 0) \rightarrow \Sigma_c \subset F^{-1}(0)$ ;  $\gamma(t) = (t, 0, 0)$ . Therefore it is easy to see that there are two geometric solutions of  $F = 0$  passing through  $z_0 \in \Sigma_c$ .

For example, if we also put  $f(p) = p^2$  (respectively,  $f(p) = p^3$ ), the complete solution is  $\Gamma(t, c) = (2t + c, t^2, t)$  (respectively,  $\Gamma(t, c) = ((3/2)t^2 + c, t^3, t)$ ). We can draw pictures; the



equation surface  $F^{-1}(0)$ , geometric solutions  $\{\Gamma(t, c)\}_{c \in (\mathbb{R}, 0)}$  and the singular solution  $\gamma(t)$ , and the phase portrait  $\{\pi \circ \Gamma(t, c)\}_{c \in (\mathbb{R}, 0)}$ . See Figures 3 and 4.

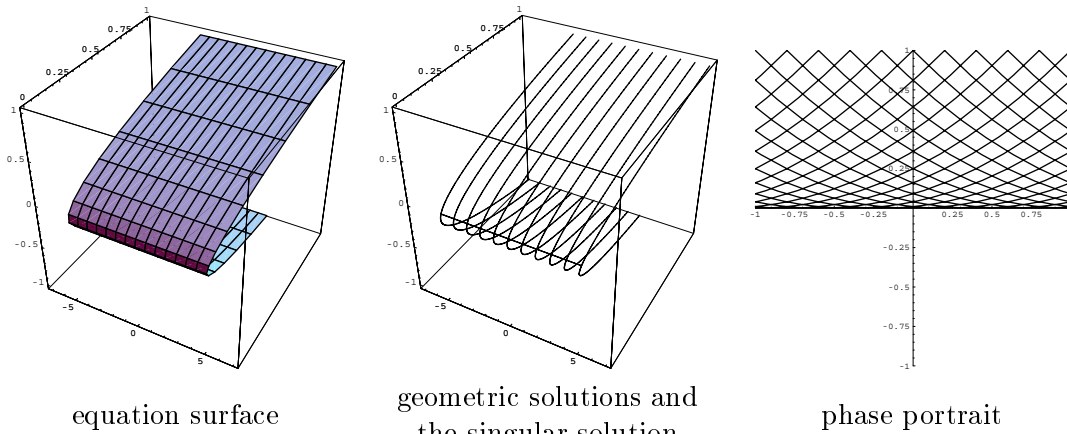


Figure 3.  $F(x, y, p) = y - p^2 = 0$ .

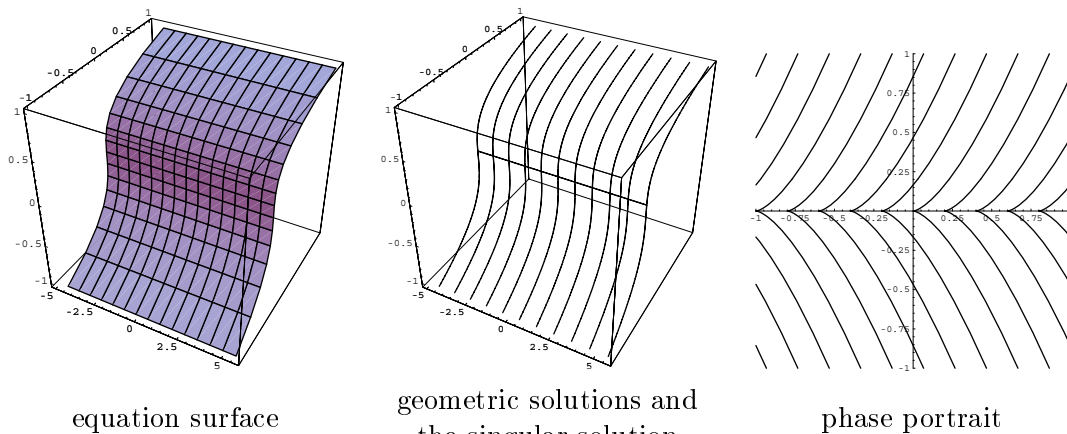


Figure 4.  $F(x, y, p) = y - p^3 = 0$ .

## 5 Completely integrable second order ordinary differential equations

The condition of Proposition 3.1 is a necessary and sufficient condition for uniqueness for geometric solutions of completely integrable implicit first order ODEs. However, the result does not hold in the case of completely integrable implicit second order ODEs, that is, even if  $z_0 \in \Sigma_c$  there is a unique geometric solution passing through  $z_0$  (cf. Theorem 5.6). In this section, we analyse completely integrable implicit second order ODEs in detail. Let

$$F(x, y, p, q) = 0, \quad p = dy/dx, \quad q = d^2y/dx^2$$

be an implicit second order ODE at  $z_0$ . If  $z_0 \notin \Sigma_c$ , then  $F = 0$  satisfies either  $F_q(z_0) \neq 0$  or  $F_x(z_0) \neq 0$ .

First we assume that  $F_q(z_0) \neq 0$ . By the implicit function theorem,  $F = 0$  can be represented by an explicit equation  $q = f(x, y, p)$  where  $f$  is a function germ. In this case,  $F = 0$  is of Clairaut type at  $z_0$  and here we call this type  $C$ .

Next we assume that  $F_X(z_0) \neq 0$ . In this case,  $F = 0$  is of reduced type at  $z_0$  and we call this type  $R$ .

Both cases, there is a unique geometric solution passing through each point of  $F^{-1}(0)$ . It follows that there is a complete solution of  $F^{-1}(0)$  and no singular solution.

By Theorem 2.2, a completely integrable ODE at  $z_0$  is either of Clairaut type or reduced type at  $z_0$ . The main results in this paper are to classify the completely integrable implicit second order ODEs in detail and to characterize a complete (singular) solution of  $\Sigma_c$  for each type respectively. It is concluded that there are ten kinds of types, see Table 2.

Conditions			Type	Name	
$z_0 \notin \Sigma_c$	$F_q(z_0) \neq 0$		Clairaut type	$C$	
	$F_X(z_0) \neq 0$		reduced type	$R$	
$z_0 \in \Sigma_c$	$F_p(z_0) \neq 0$	regular point of $F_q _{F^{-1}(0)}$	Clairaut type	$RC_p$	
		regular point of $F_X _{F^{-1}(0)}$	reduced type	$RR_p$	
	$F_y(z_0) \neq 0$ $F_p(z_0) = 0$	regular point of $F_q _{F^{-1}(0)}$		Clairaut type	$RC_y$
		regular point of $F_X _{F^{-1}(0)}$	$\Sigma_c = \Delta$	reduced type	$RR_y^1$
			$\Sigma_c \supsetneq \Delta = \Sigma_{cc}$	reduced type	$RR_y^2$
			$\Sigma_c \supsetneq \Delta \supsetneq \Sigma_{cc}$	reduced type	$RR_y^3$
		singular point of $F_q _{F^{-1}(0)}$ and $F_X _{F^{-1}(0)}$		Clairaut type	$SC_y$
			reduced type	$SR_y$	

Table 2. Classifications of completely integrable implicit second order ODEs at  $z_0$ .

Here we consider the subset  $\Delta = \Delta(F) \subset \Sigma_c$  which is defined to be the set of points  $z \in \Sigma_c$  such that  $T_z F^{-1}(0)$  coincides with the kernel of  $\alpha_1(z)$ . Explicitly, it is given by  $\Delta = \{z \in \Sigma_c \mid F_p(z) = 0\}$ .

We now review the previous results in [3, 15, 17]. Conditions for existence of a complete solution and a complete (singular) solution of  $\Sigma_c$  for implicit second order ODEs were given under a regularity condition.

**Theorem 5.1** [3, Theorems 1.1, 1.2 and 1.3] *Suppose that 0 is a regular value of  $F_q|_{F^{-1}(0)}$ .*

(1)  $F = 0$  is completely integrable at  $z_0$  if and only if  $z_0 \notin \Sigma_c$  or  $\Sigma_c$  is a 2-dimensional manifold around  $z_0$ .

(2) Let  $F = 0$  be completely integrable.

(i) The leaves of the complete solution which meet  $\Sigma_c$  away from  $\Delta$  intersect  $\Sigma_c$  transversally.

(ii) The leaves of the complete solution which meet  $\Delta$  are tangent to  $\Sigma_c$ .

(3) Let  $F = 0$  be completely integrable and  $\Sigma_c \neq \emptyset$ .

(i)  $F = 0$  admits a complete singular solution of  $\Sigma_c$  at  $z_0$  if and only if  $z_0 \notin \Sigma_{cc}$  or  $\Sigma_{cc}$  is a 1-dimensional manifold around  $z_0$ .

(ii) Suppose that  $F = 0$  admits a complete singular solution of  $\Sigma_c$ . Then each leaf of the complete singular solution of  $\Sigma_c$  intersects  $\Sigma_{cc}$  transversally.

**Theorem 5.2** [17, Lemma 2.3, Propositions 2.4 and 3.8] *Suppose that 0 is a regular value of  $F_X|_{F^{-1}(0)}$ .*

(1)  $F = 0$  is completely integrable at  $z_0$  if and only if  $z_0 \notin \Sigma_c$  or  $\Sigma_c$  is a 2-dimensional manifold around  $z_0$ .

(2) Let  $F = 0$  be completely integrable.

(i) The leaves of the complete solution which meet  $\Sigma_c$  away from  $\Delta$  intersect  $\Sigma_c$  transversally.

- (ii) The leaves of the complete solution which meet  $\Delta$  are tangent to  $\Sigma_c$ .
- (3) Let  $F = 0$  be completely integrable and  $\Sigma_c \neq \emptyset$ .
- (i)  $F = 0$  admits a complete solution of  $\Sigma_c$  at  $z_0$  if and only if  $z_0 \notin \Sigma_{cc}$  or  $\Sigma_{cc}$  is a 1-dimensional manifold around  $z_0$ .
- (ii) Suppose that  $F = 0$  admits a complete solution of  $\Sigma_c$ . Then each leaf of the complete solution of  $\Sigma_c$  intersects  $\Sigma_{cc}$  transversally.

**Proposition 5.3** [15, Proposition 3.2] *Let  $F = 0$  be completely integrable at  $z_0$  and  $z_0 \in \Sigma_c$ .*

- (1) *If 0 is a regular value of  $F_q|_{F^{-1}(0)}$ , then  $F = 0$  is of Clairaut type at  $z_0$ .*
- (2) *If 0 is a regular value of  $F_X|_{F^{-1}(0)}$ , then  $F = 0$  is of reduced type at  $z_0$ .*

**Proposition 5.4** [17, Propositions 2.7] *Let  $F = 0$  be completely integrable at  $z_0$  and  $\Sigma_c$  be a 2-dimensional manifold around  $z_0$ . Then  $\Sigma_{cc} \subset \Delta$ .*

**Remark 5.5** *There is an important difference between the case where 0 is a regular value of  $F_q|_{F^{-1}(0)}$  and where it is a regular value of  $F_X|_{F^{-1}(0)}$ . Namely, if 0 is a regular value of  $F_q|_{F^{-1}(0)}$  and  $z_0 \in \Delta$ , then  $\Delta$  is a 1-dimensional manifold around  $z_0$  by Proposition 3.6 in [3]. However,  $\Delta$  is not necessarily a 1-dimensional manifold even if 0 is a regular value of  $F_X|_{F^{-1}(0)}$ .*

Now suppose that  $F = 0$  is completely integrable at  $z_0 \in \Sigma_c$ . Since  $F$  is regular at  $z_0$ ,  $F = 0$  satisfies either  $F_y(z_0) \neq 0$  or  $F_p(z_0) \neq 0$ .

## 5.1 On the types $RC_p$ and $RR_p$

If  $F_p(z_0) \neq 0$ , by the implicit function theorem, there exists a smooth function  $g : V \rightarrow \mathbb{R}$ , where  $V$  is an open set in  $\mathbb{R}^3$ , such that in a neighborhood of  $z_0$ ,  $(x, y, p, q) \in F^{-1}(0)$  if and only if  $-p + g(x, y, q) = 0$ . Thus we may assume without loss of generality that  $F(x, y, p, q) = -p + g(x, y, q) = 0$ . Under this notations,  $F_q = g_q$  and  $F_X = g_x + g \cdot g_y - q$ . It follows that  $z_0$  is a regular point of either  $F_q|_{F^{-1}(0)}$  or  $F_X|_{F^{-1}(0)}$ .

If  $z_0$  is a regular point of  $F_q|_{F^{-1}(0)}$ , then  $F = 0$  is of Clairaut type at  $z_0$  and  $\Sigma_c$  is a 2-dimensional manifold around  $z_0$  by Proposition 5.3 and Theorem 5.1. We call this type  $RC_p$ . By  $z_0 \notin \Delta$  and Proposition 5.4,  $z_0 \notin \Sigma_{cc}$ . Hence  $F = 0$  has a complete singular solution of  $\Sigma_c$  at  $z_0$  automatically.

On the other hand, suppose that  $z_0$  is a regular point of  $F_X|_{F^{-1}(0)}$ . By Proposition 5.3 and Theorem 5.2,  $F = 0$  is of reduced type at  $z_0$  and  $\Sigma_c$  is a 2-dimensional manifold around  $z_0$ . We call this type  $RR_p$ . By  $z_0 \notin \Delta$  and Proposition 5.4,  $z_0 \notin \Sigma_{cc}$ . Since the leaves of the complete solution which meet  $\Sigma_c$  away from  $\Delta$  intersect  $\Sigma_c$  transversally,  $F = 0$  has a complete singular solution of  $\Sigma_c$  at  $z_0$  automatically.

## 5.2 On the type $RC_y$

If  $F_y(z_0) \neq 0$ , again by the implicit function theorem, there exists a smooth function  $f : U \rightarrow \mathbb{R}$ , where  $U$  is an open set in  $\mathbb{R}^3$ , such that in a neighborhood of  $z_0$ ,  $(x, y, p, q) \in F^{-1}(0)$  if and only if  $-y + f(x, p, q) = 0$ . Thus we may assume without loss of generality that  $F(x, y, p, q) = -y + f(x, p, q) = 0$ . Define the diffeomorphism  $\phi : U \rightarrow F^{-1}(0)$ ,  $(x, p, q) \mapsto (x, f(x, p, q), p, q)$  and  $u_0 = \phi^{-1}(z_0)$ . Below, if  $F_y(z_0) \neq 0$ , we keep the notations of the above.

Suppose that  $z_0$  is a regular point of  $F_q|_{F^{-1}(0)}$ . By Proposition 5.3 and Theorem 5.1,  $F = 0$  is of Clairaut type at  $z_0$  and  $\Sigma_c$  is a 2-dimensional manifold around  $z_0$ . We call this type  $RC_y$ . Moreover,  $F = 0$  has a complete singular solution at  $z_0 \in \Sigma_c$  if and only if  $z_0 \notin \Sigma_{cc}$  or  $\Sigma_{cc}$  is a 1-dimensional manifold around  $z_0$  by Theorem 5.1.

Remark that if  $\Sigma_{cc}$  is a 1-dimensional manifold around  $z_0$ , then  $\Delta = \Sigma_{cc}$  (cf. Remark 5.5) and  $\Sigma_{cc}$  is an isolated singular solution passing through  $z_0$  (see, [3, Proposition 1.4]).

### 5.3 On the type $RR_y^1$

Let  $F_y(z_0) \neq 0$ . Suppose that  $z_0$  is a regular point of  $F_X|_{F^{-1}(0)}$ . By Proposition 5.3 and Theorem 5.2,  $F = 0$  is of reduced type at  $z_0$  and  $\Sigma_c$  is a 2-dimensional manifold around  $z_0$ . In this case, there are three types. First case is  $\Sigma_c = \Delta$  around  $z_0$  (type  $RF_y^1$ ), second is  $\Sigma_c \supsetneq \Delta = \Sigma_{cc}$  around  $z_0$  (type  $RF_y^2$ ), and the last is  $\Sigma_c \supsetneq \Delta \supsetneq \Sigma_{cc}$  around  $z_0$  (type  $RF_y^3$ ). We may assume that  $F_p(z_0) = 0$ , namely,  $z_0 \in \Delta$ , and  $F = 0$  is not of Clairaut type at  $z_0$ .

We now consider  $F = 0$  is the type  $RF_y^1$  at  $z_0$ . By Theorem 5.2,  $F = 0$  has a complete solution of  $\Sigma_c$  at  $z_0$  if and only if  $z_0 \notin \Sigma_{cc}$  or  $\Sigma_{cc}$  is a 1-dimensional manifold around  $z_0$ . In this type, we have the following result.

**Theorem 5.6** *Let  $F = 0$  be type  $RF_y^1$  at  $z_0 \in \Delta$ . If  $z_0 \notin \Sigma_{cc}$ , then there exists a unique geometric solution passing through  $z_0$ .*

*Proof.* We denote that  $F(x, y, p, q) = -y + f(x, p, q) = 0$ . Since  $F = 0$  is of reduced type at  $z_0$ , there exists a smooth function germ  $\alpha : (F^{-1}(0), z_0) \rightarrow (\mathbb{R}, 0)$  such that

$$f_q = \alpha \cdot (f_x - p + qf_p). \quad (5)$$

A complete solution,  $\Gamma : (\mathbb{R} \times \mathbb{R}^2, 0) \rightarrow (F^{-1}(0), z_0)$ , is given by integrating the vector field  $\phi_*X$ , where  $X : U \rightarrow TU$  is given by

$$X = (-\alpha, -\alpha \cdot q, 1)$$

(cf. [3, Lemma 3.1]). By (5), we have

$$(f_x - p + qf_p)_q = (\alpha_x + q\alpha_p) \cdot (f_x - p + qf_p) + \alpha \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p) + f_p \quad (6)$$

It follows from the assumption  $\Sigma_c = \Delta$  that

$$(f_x - p + qf_p)_q|_{\phi^{-1}(\Sigma_c)} = \alpha|_{\phi^{-1}(\Sigma_c)} \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)|_{\phi^{-1}(\Sigma_c)}.$$

In this case, a complete solution of  $\Sigma_c$ ,  $\Phi : (\mathbb{R} \times \mathbb{R}, 0) \rightarrow (\Sigma_c, z_0)$ , is given by integrating the vector field  $\phi_*Y$ , where  $Y : \phi^{-1}(\Sigma_c) \rightarrow T\phi^{-1}(\Sigma_c)$  is given by

$$Y = (-\alpha|_{\phi^{-1}(\Sigma_c)}, (-\alpha \cdot q)|_{\phi^{-1}(\Sigma_c)}, 1)$$

(cf. [17, Lemma 3.5]). This means that  $\Gamma|_{\phi^{-1}(\Sigma_c)} = \Phi$ . Therefore there is a geometric solution on  $\Sigma_c$ . Let  $\gamma : (\mathbb{R}, t_0) \rightarrow (\Sigma_c, z_0)$ ;  $\gamma(t) = (x(t), y(t), p(t), q(t))$  be a geometric solution passing through  $z_0$ . By the definitions of  $\Sigma_c$  and  $\Sigma_{cc}$ , we have

$$\phi^{-1}(\Sigma_c) = (f_x - p + qf_p)^{-1}(0)$$

and

$$\phi^{-1}(\Sigma_{cc}) = (f_x - p + qf_p)^{-1}(0) \cap ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)^{-1}(0).$$

By differentiating the equality  $(f_x - p + qf_p)(x(t), p(t), q(t)) = 0$  with respect to  $t$ , we have

$$((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)(x(t), p(t), q(t)) \cdot x'(t) + (f_x - p + qf_p)_q(x(t), p(t), q(t)) \cdot q'(t) = 0$$

On the other hand, by differentiating (5) with respect to  $x$  and  $p$ , we have

$$f_{xq} = \alpha_x \cdot (f_x - p + qf_p) + \alpha \cdot (f_x - p + qf_p)_x$$

and

$$f_{pq} = \alpha_p \cdot (f_x - p + qf_p) + \alpha \cdot (f_x - p + qf_p)_p$$

It follows that

$$(x'(t) + \alpha \cdot q'(t)) \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)(x(t), p(t), q(t)) = 0.$$

Since  $z_0 \notin \Sigma_{cc}$ ,  $(f_x - p + qf_p)_x + q(f_x - p + qf_p)_p \neq 0$  at  $u_0$ . Then we have  $x'(t) + \alpha \cdot q'(t) = 0$  at  $t_0$ , and hence  $q'(t_0) \neq 0$ . It follows that we can reparametrize  $\gamma(t)$  as  $(x(t), y(t), p(t), t)$ . By the analogous way in the proof of Lemma 3.2 in the Appendix, we can show the uniqueness for geometric solution passing through  $z_0$ . In fact, we may apply  $x'(t) = -\alpha(x(t), y(t), p(t), t)$ ,  $y'(t) = p(t)x'(t)$  and  $p'(t) = tx'(t)$  to the case of  $n = 2$  in the proof of Lemma 3.2. This completes the proof of Theorem 5.6.  $\square$

**Proposition 5.7** *Let  $F = 0$  be type  $RF_y^1$  at  $z_0 \in \Delta$ . If  $\Sigma_{cc}$  is a 1-dimensional manifold around  $z_0$ , then  $\Sigma_{cc}$  is a singular solution passing through  $z_0$ .*

*Proof.* By the assumption, it is easy to see that  $\Sigma_{cc}$  is a geometric solution passing through  $z_0$ . As before,  $\phi^{-1}(\Sigma_c) = (f_x - p + qf_p)^{-1}(0)$  and  $\phi^{-1}(\Sigma_{cc}) = (f_x - p + qf_p)^{-1}(0) \cap ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)^{-1}(0)$ . To show that  $\Sigma_{cc}$  is not a leaf of the complete solution of  $F^{-1}(0)$  (and  $\Sigma_c$ ) at  $z_0$ , it is sufficient to check that the scalar product of  $\text{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)$  and the vector field  $X$  is nonzero at  $u_0$ . Now

$$\begin{aligned} & \langle \text{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p), (-\alpha, -\alpha \cdot q, 1) \rangle \\ &= -\alpha \cdot ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_x - \alpha \cdot q((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_p \\ & \quad + ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_q. \end{aligned} \tag{7}$$

By (5), (7) is equal to  $2(f_{xp} + qf_{pp}) - 1$  at  $u_0$ . It follows from the assumption  $\Sigma_c = \Delta$  that there exists a smooth function germ  $\beta$  such that  $f_p = \beta \cdot (f_x - p + qf_p)$ . Differentiating this equality with respect to  $x$  and  $p$ , then

$$f_{xp} = \beta_x \cdot (f_x - p + qf_p) + \beta \cdot (f_x - p + qf_p)_x$$

and

$$f_{pp} = \beta_p \cdot (f_x - p + qf_p) + \beta \cdot (f_x - p + qf_p)_p.$$

It follows that (7) is nonzero at  $u_0$ .  $\square$

## 5.4 On the type $RR_y^2$

Suppose that  $F = 0$  is the type  $RR_y^2$  at  $z_0$ . Then  $\Sigma_c \supsetneq \Delta = \Sigma_{cc}$  around  $z_0$ . By Theorem 5.2,  $F = 0$  has a complete solution of  $\Sigma_c$  at  $z_0$  if and only if  $\Sigma_{cc}$  is a 1-dimensional manifold around  $z_0$ . However, we have the following.

**Theorem 5.8** *Let  $F = 0$  be type  $RR_y^2$  at  $z_0 \in \Delta$ .  $F = 0$  has a complete singular solution of  $\Sigma_c$  at  $z_0$  if and only if  $\Sigma_{cc}$  is a 1-dimensional manifold around  $z_0$ .*

*Proof.* By Theorem 5.2, each leaf of the complete solution of  $F^{-1}(0)$  which meet  $\Sigma_c$  away from  $\Sigma_{cc}$  intersect  $\Sigma_c$  transversally, and each leaf of the complete solution of  $\Sigma_c$  intersects  $\Sigma_{cc}$  transversally. Therefore the complete solution of  $\Sigma_c$  is the complete singular solution of  $\Sigma_c$ .  $\square$

By the definition of  $\Sigma_{cc}$ ,

$$(f_x - p + qf_p)_x + q(f_x - p + qf_p)_p = 0, (f_x - p + qf_p)_q = 0$$

at  $z_0 \in \Sigma_{cc}$  (cf. [17]). Since  $f_x - p + qf_p$  is regular at  $z_0$ ,  $(f_x - p + qf_p)_p \neq 0$  at  $z_0$ . Hence  $F = 0$  satisfies either (i)  $((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_q \neq 0$  or (ii)  $((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)_q = 0$  at  $z_0$ . It follows that  $z_0$  is a regular point of  $(f_x - p + qf_p)_x + q(f_x - p + qf_p)_p$ , or of  $(f_x - p + qf_p)_q$ .

**Proposition 5.9** *Let  $F = 0$  be type  $RR_y^2$  at  $z_0 \in \Delta$ . Suppose that  $\Sigma_{cc}$  is a 1-dimensional manifold around  $z_0$ .*

(1) *If  $F = 0$  satisfies the condition (i), then each leaf of the complete solution of  $F^{-1}(0)$  is intersects  $\Sigma_{cc}$  transversally and hence  $\Sigma_{cc}$  is a singular solution passing through  $z_0$ .*

(2) *If  $F = 0$  satisfy the condition (ii) and  $F_{pq}|_{\Sigma_{cc}} \equiv 0$  around  $z_0$ , then each leaf of the complete solution of  $F^{-1}(0)$  is tangent to  $\Sigma_{cc}$ . If  $\gamma(t) = (x(t), y(t), p(t), q(t)) \in \Sigma_{cc}$  is a geometric solution,  $\gamma(t)$  is represented by the form  $(a, b, c, t)$  where  $a, b, c \in \mathbb{R}$ . Moreover,  $\gamma(t)$  is a leaf of the complete solution of  $F^{-1}(0)$ .*

*Proof.* (1) Since  $\phi^{-1}(\Sigma_{cc}) = (f_x - p + qf_p)^{-1}(0) \cap ((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)^{-1}(0)$ , it is sufficient to check that the scalar product of  $\text{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p)$  and the vector field  $X$  is nonzero at  $u_0$ . By the same calculations in Proposition 5.7,

$$\langle \text{grad}((f_x - p + qf_p)_x + q(f_x - p + qf_p)_p), (-\alpha, -\alpha \cdot q, 1) \rangle = 2(f_{xp} + qf_{pp}) - 1$$

at  $u_0$ . The condition (i) guarantees that  $2(f_{xp} + qf_{pp}) - 1 \neq 0$  at  $u_0$ . Therefore each leaf of the complete solution of  $F^{-1}(0)$  is intersects  $\Sigma_{cc}$  transversally and hence  $\Sigma_{cc}$  is a singular solution passing through  $z_0$ .

(2) Since  $\phi^{-1}(\Sigma_{cc}) = (f_x - p + qf_p)^{-1}(0) \cap ((f_x - p + qf_p)_q)^{-1}(0)$ , it is sufficient to check that the scalar product of  $\text{grad}(f_x - p + qf_p)_q$  and the vector field  $X$  is zero. By the direct calculations, the consequence follows from the condition  $F_{pq}|_{\Sigma_{cc}} \equiv 0$  around  $z_0$ .

Let  $\gamma(t) = (x(t), y(t), p(t), q(t)) \in \Sigma_{cc}$  be a geometric solution passing through  $z_0$ . By differentiating  $f_p(x(t), p(t), q(t)) = 0$  with respect to  $t$ ,

$$(f_{xp} + qf_{pp})(x(t), p(t), q(t)) \cdot x'(t) + f_{pq}(x(t), p(t), q(t)) \cdot q'(t) = 0.$$

By the condition (ii), we have  $f_{xp} + qf_{pp} = 1/2$  at  $u_0$  and hence  $x'(t) \equiv 0$ . This means that  $x(t)$  is constant on  $\Sigma_{cc}$  around  $z_0$ .

Differentiating (5) with respect to  $p$ , we have  $f_{pq} = \alpha_p \cdot (f_x - p + qf_p) + \alpha \cdot (f_x - p + qf_p)_p$ . It follows that  $\alpha|_{\Sigma_{cc}} \equiv 0$  around  $z_0$ . By the form of the vector field  $X$  (see, in the proof of Theorem 5.6),  $\Gamma|_{\Gamma^{-1}(\Sigma_{cc})} = \gamma$ .  $\square$

## 5.5 On the type $RR_y^3$

Suppose that  $F = 0$  is the type  $RR_y^3$  at  $z_0$ . Then  $\Sigma_c \supsetneq \Delta \supsetneq \Sigma_{cc}$  around  $z_0$ . In this subsection, we assume that  $\Delta$  is a 1-dimensional manifold around  $z_0$  and  $z_0 \notin \Sigma_{cc}$ . By Theorem 5.2,  $F = 0$  has a complete solution of  $\Sigma_c$  at  $z_0$ . If  $\Delta$  is not a geometric solution passing through  $z_0$ , the complete solution of  $\Sigma_c$  is the complete singular solution of  $\Sigma_c$ . On the other hand, if  $\Delta$  is a geometric solution passing through  $z_0$ , we have the following.

**Proposition 5.10** *Let  $F = 0$  be type  $RR_y^3$  at  $z_0 \in \Delta \setminus \Sigma_{cc}$ . If  $\gamma(t) = (x(t), y(t), p(t), q(t)) \in \Delta$  is a geometric solution passing through  $z_0$ , then  $\gamma(t)$  is represented by the form  $(a, b, c, t)$  where  $a, b, c \in \mathbb{R}$ . Moreover,  $\gamma(t)$  is a leaf of both complete solutions of  $F^{-1}(0)$  and  $\Sigma_c$ .*

*Proof.* Since  $z_0 \notin \Sigma_{cc}$  and (6), we have  $(f_x - p + qf_p)_x + q(f_x - p + qf_p)_p \neq 0$  at  $u_0$ . Differentiating equalities  $(f_x - p + qf_p)(x(t), p(t), q(t)) = 0$  and  $f_p(x(t), p(t), q(t)) = 0$  with respect to  $t$ , we have

$$\begin{pmatrix} (f_x - p + qf_p)_x + q(f_x - p + qf_p)_p & (f_x - p + qf_p)_q \\ f_{xp} + qf_{pp} & f_{pq} \end{pmatrix} \begin{pmatrix} x'(t) \\ q'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since  $\gamma(t)$  is a geometric solution,  $(x'(t), q'(t)) \neq (0, 0)$  on  $\Delta$ . Thus

$$\det \begin{pmatrix} (f_x - p + qf_p)_x + q(f_x - p + qf_p)_p & (f_x - p + qf_p)_q \\ f_{xp} + qf_{pp} & f_{pq} \end{pmatrix} = 0$$

on  $\Delta$ . It follows from (5) that  $\alpha|_{\Delta} \equiv 0$  and hence  $x'(t) \equiv 0$ . This means that  $x(t)$  is constant on  $\Delta$  around  $z_0$ .

Since  $(f_x - p + qf_p)_q|_{\Delta} = 0$  and the forms of the vector fields  $X$  for a complete solution of  $F^{-1}(0)$  and  $Y$  for a complete solution of  $\Sigma_c$  (cf. Proof of Theorem 5.6),  $\Gamma|_{\Gamma^{-1}(\Delta)} = \Phi|_{\Phi^{-1}(\Delta)} = \gamma$ .  $\square$

## 5.6 On the type $SC_y$

Suppose that  $F = 0$  is of Clairaut type at  $z_0$  and  $z_0$  is a singular point of  $F_q|_{F^{-1}(0)}$ . We call this type  $SC_y$ .

**Proposition 5.11** *Let  $F = 0$  be type  $SC_y$  at  $z_0$ . If  $\Sigma_c$  is a 2-dimensional manifold around  $z_0$ , then  $z_0 \notin \Sigma_{cc}$ .*

*Proof.* Let  $F(x, y, p, q) = -y + f(x, p, q) = 0$ . Since  $F = 0$  is of Clairaut type at  $z_0$ , there is a function germ  $\alpha : (F^{-1}(0), z_0) \rightarrow \mathbb{R}$  such that

$$f_x - p + qf_p = \alpha \cdot f_q. \quad (8)$$

By differentiating (8) with respect to  $p$ ,  $f_{xp} - 1 + qf_{pp} = \alpha_p \cdot f_q + \alpha \cdot f_{pq}$ . Hence  $f_{xp} + qf_{pp} = 1$  at  $u_0$ . By a direct calculation,

$$(f_x - p + qf_p)_{xq} + q(f_x - p + qf_p)_{pq} = (f_{xq} + qf_{pq})_x + q(f_{xq} + qf_{pq})_p + f_{xp} + qf_{pp} \quad (9)$$

On the other hand, by (8), we have

$$\begin{aligned} & (f_x - p + qf_p)_{xq} + q(f_x - p + qf_p)_{pq} \\ &= (\alpha_{xq} + q\alpha_{pq}) \cdot f_q + \alpha_q \cdot (f_{qx} + qf_{pq}) + (\alpha_x + q\alpha_p) \cdot f_{qy} + \alpha \cdot (f_{xqq} + qf_{pqq}) \end{aligned} \quad (10)$$

By definition,  $\phi^{-1}(\Sigma_c) = f_q^{-1}(0)$ . Since  $\Sigma_c$  is a 2-dimensional manifold around  $z_0$ , there is a regular function germ  $g : (U, u_0) \rightarrow \mathbb{R}$  and a function germ  $k : (U, u_0) \rightarrow (\mathbb{R}, 0)$  such that  $\phi^{-1}(\Sigma_c) = g^{-1}(0)$  and  $f_q = k \cdot g$  at least locally. By a direct calculation, the right hand of (9) is given by

$$((k_x + qk_p)_x + q(k_x + k_p)_p) \cdot g + 2(k_x + qk_p) \cdot (g_x + qg_p) + k \cdot ((g_x + qg_p)_x + q(g_x + qg_p)_p) + f_{xp} + qf_{pp}.$$

Also the right hand of (10) is given by

$$\begin{aligned} & (\alpha_{xq} + q\alpha_{pq}) \cdot k \cdot g + \alpha_q \cdot ((k_x + qk_p) \cdot g + k \cdot (g_x + qg_p)) + (\alpha_x + q\alpha_p) \cdot (k_q \cdot g + k \cdot g_q) \\ & + \alpha \cdot ((k_{xq} + qk_{pq}) \cdot g + k_q \cdot (g_x + qg_p) + (k_x + qk_p) \cdot g_q + k \cdot (g_{xq} + qg_{pq})) \end{aligned}$$

If  $z_0 \in \Sigma_{cc}$ , then  $g = g_x + qg_p = g_q = 0$  at  $u_0$ . This contradicts the fact that (9) = (10), namely  $1=0$  at  $u_0$ .  $\square$

Under the assumption of Proposition 5.11, it follows from  $z_0 \notin \Sigma_{cc}$  that there is a complete solution of  $\Sigma_c$  at  $z_0$ . According to Theorem 5.15 in below, a geometric solution passing through  $z_0$  on  $\Sigma_c$  is a singular solution. Hence there is a complete singular solution of  $\Sigma_c$  at  $z_0$  automatically.

## 5.7 On the type $SR_y$

Suppose that  $F = 0$  is of reduced type at  $z_0$  and  $z_0$  is a singular point of  $(F_x + pF_y + qF_p)|_{F^{-1}(0)}$ . We call this type  $SR_y$ . We can prove the following result by using the same arguments in the proof of Proposition 5.11, so we omit it.

**Proposition 5.12** *Let  $F = 0$  be type  $SR_y$  at  $z_0$ . If  $\Sigma_c$  is a 2-dimensional manifold around  $z_0$ , then  $z_0 \notin \Sigma_{cc}$ .*

Moreover, we have the following result.

**Proposition 5.13** *Let  $F = 0$  be type  $SF_y$  and not of Clairaut type at  $z_0$ . If  $\Sigma_c$  is a 2-dimensional manifold around  $z_0$ , then  $\Delta$  is a 1-dimensional manifold around  $z_0$ . Moreover,  $\Delta$  is not a geometric solution passing through  $z_0$ .*

*Proof.* By (5),  $f_q = \alpha \cdot (f_x - p + qf_p)$  with  $\alpha(z_0) = 0$ . Since  $\phi^{-1}(\Sigma_c) = (f_x - p + qf_p)^{-1}(0)$  is a 2-dimensional manifold around  $z_0$ , there exist a regular function germ  $g : (U, u_0) \rightarrow (\mathbb{R}, 0)$  and a function germ  $k : (U, u_0) \rightarrow (\mathbb{R}, 0)$  such that  $f_x - p + qf_p = k \cdot g$  and  $k^{-1}(0) \subset g^{-1}(0)$  at least locally. By a direct calculation, we have

$$(f_x - p + qf_p)_{xq} + q(f_x - p + qf_p)_{pq} = 1$$

at  $u_0$ . On the other hand,

$$(f_x - p + qf_p)_{xq} + q(f_x - p + qf_p)_{pq} = k_q \cdot (g_x + qg_p) + (k_x + qk_p) \cdot g_q$$

at  $u_0$ . Hence  $k_q \cdot (g_x + qg_p) + (k_x + qk_p) \cdot g_q = 1$  at  $u_0$ . If  $g_q(u_0) = 0$ , then  $k_q(u_0) \neq 0$ . It follows that  $k$  is represented by  $\lambda(x, p, q) \cdot (q - \mu(x, p))$  at least locally, where  $\lambda$  and  $\mu$  are function germs with  $\lambda(u_0) \neq 0$ . Since  $k^{-1}(0) \subset g^{-1}(0)$ ,  $g(x, p, \mu(x, p)) = 0$ . By differentiating this equality with respect to  $x$  and  $p$ , we have

$$g_x(x, p, \mu(x, p)) + \mu_x(x, p)g_q(x, p, \mu(x, p)) = 0$$



and

$$g_p(x, p, \mu(x, p)) + \mu_p(x, p)g_q(x, p, \mu(x, p)) = 0.$$

This contradicts the fact that  $g$  is regular at  $u_0$ . Therefore we have  $g_q \neq 0$  at  $u_0$ .

By the definition of  $\Delta$ ,  $\phi^{-1}(\Delta) = g^{-1}(0) \cap f_p^{-1}(0)$ . To show that  $\Delta$  is a 1-dimensional manifold around  $z_0$ , it is sufficient to show that the matrix

$$A = \begin{pmatrix} g_x & g_p & g_q \\ f_{xp} & f_{pp} & f_{pq} \end{pmatrix}$$

has rank 2 at  $u_0$ . Since  $f_x - p + qf_p$  and  $f_q$  are singular at  $u_0$ ,  $f_{xp} + qf_{pp} = 1$  and  $f_{pq} = 0$  at  $u_0$ . Therefore  $\text{rank} A = 2$  at  $u_0$ .

Next suppose that  $\gamma : (\mathbb{R}, t_0) \rightarrow (\Delta, z_0); \gamma(t) = (x(t), y(t), p(t), q(t))$  is a geometric solution passing through  $z_0$ . By differentiating the equalities  $g(x(t), p(t), q(t)) = 0$  and  $f_p(x(t), p(t), q(t)) = 0$  with respect to  $t$ , we have

$$\begin{pmatrix} (g_x + qg_p)(x(t), p(t), q(t)) & g_q(x(t), p(t), q(t)) \\ (f_{xp} + qf_{pp})(x(t), p(t), q(t)) & f_{pq}(x(t), p(t), q(t)) \end{pmatrix} \begin{pmatrix} x'(t) \\ q'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since the determinant of the matrix  $\begin{pmatrix} g_x + qg_p & g_q \\ f_{xp} + qf_{pp} & f_{pq} \end{pmatrix}$  is not vanish at  $t_0$ ,  $(x'(t), q'(t)) = (0, 0)$  at  $t_0$ . This contradicts the fact that  $\gamma(t)$  is a geometric solution passing through  $z_0$ .  $\square$

As a conclusion, if  $F = 0$  is of type  $SF_y$  and not of Clairaut type at  $z_0$ , then there is a complete singular solution of  $\Sigma_c$  at  $z_0$  automatically by Propositions 5.12 and 5.13.

Finally in this section, we give an important difference between Clairaut type and reduced type.

**Lemma 5.14** *Let  $F = 0$  be type  $RC_y$  at  $z_0$ . If  $z_0 \in \Delta \setminus \Sigma_{cc}$ , then  $\Delta$  is not a geometric solution passing through  $z_0$ .*

*Proof.* By remark 5.5,  $\Delta$  is a 1-dimensional manifold around  $z_0$ . We assume that  $\gamma : (\mathbb{R}, t_0) \rightarrow (\Delta, z_0); \gamma(t) = (x(t), y(t), p(t), q(t))$  is a geometric solution passing through  $z_0$ . By (8),  $f_x - p + qf_p = \alpha \cdot f_q$ . Differentiating  $f_p(x(t), p(t), q(t)) = 0$  and  $f_q(x(t), p(t), q(t)) = 0$  with respect to  $t$ , we have

$$\begin{pmatrix} (f_{xp} + qf_{pp})(x(t), p(t), q(t)) & f_{pq}(x(t), p(t), q(t)) \\ (f_{xq} + qf_{pq})(x(t), p(t), q(t)) & f_{qq}(x(t), p(t), q(t)) \end{pmatrix} \begin{pmatrix} x'(t) \\ q'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since  $f_{xp} - 1 + qf_{pp} = \alpha_p \cdot f_q + \alpha \cdot f_{pq}$  and  $f_{xq} + f_p + qf_{pq} = \alpha_q \cdot f_q + \alpha \cdot f_{qq}$ ,

$$\det \begin{pmatrix} (f_{xp} + qf_{pp})(x(t), p(t), q(t)) & f_{pq}(x(t), p(t), q(t)) \\ (f_{xq} + qf_{pq})(x(t), p(t), q(t)) & f_{qq}(x(t), p(t), q(t)) \end{pmatrix} = f_{qq}(x(t), p(t), q(t)).$$

The condition  $z_0 \notin \Sigma_{cc}$  guarantees that  $f_{qq} \neq 0$  at  $u_0$ . It follows that  $(x'(t), q'(t)) = (0, 0)$  at  $t_0$ . This contradicts the fact that  $\gamma(t)$  is a geometric solution passing through  $z_0$ .  $\square$

**Theorem 5.15** *Let  $F = 0$  be of Clairaut type at  $z_0$ . If  $\gamma(t) = (x(t), y(t), p(t), q(t)) \in \Sigma_c$  is a geometric solution passing through  $z_0$ , then  $\gamma(t)$  is the singular solution.*

*Proof.* First we assume that  $z_0$  is a regular point of  $F_q|_{F^{-1}(0)}$ . If  $z_0 \notin \Delta$ , then  $\gamma(t)$  is a singular solution passing through  $z_0$  by Theorem 5.2. Hence we may assume that  $\gamma(t) \subset \Delta$ . Also if  $z_0 \notin \Sigma_{cc}$ , then  $\gamma(t)$  is not a geometric solution passing through  $z_0$  by Lemma 5.14. We may assume that  $\gamma(t) \subset \Sigma_{cc}$ . Then we can conclude that  $\gamma(t)$  is a singular solution passing through  $z_0$  (cf. Section 5.2).

Next we assume that  $z_0$  is a singular point of  $F_q|_{F^{-1}(0)}$ . Also we may assume that  $\gamma(t) \subset \Delta$ . By differentiating  $f_p(x(t), p(t), q(t)) = 0$  with respect to  $t$ ,

$$(f_{xp} + qf_{pp})(x(t), p(t), q(t)) \cdot x'(t) + f_{pq}(x(t), p(t), q(t)) \cdot q'(t) = 0.$$

Since  $f_{xp} - 1 + qf_{pp} = \alpha_p \cdot f_q + \alpha_p \cdot f_{pq}$ , we have

$$(1 + \alpha \cdot f_{pq}(x(t), p(t), q(t))) \cdot x'(t) + f_{pq}(x(t), p(t), q(t)) \cdot q'(t) = 0.$$

By the assumption,  $f_{pq}(u_0) = 0$ . Hence  $x'(t_0) = 0$  and  $q'(t_0) \neq 0$ . It follows from the form of smooth complete solution,  $\gamma(t)$  is the singular solution passing through  $z_0$ . This completes the proof of Theorem 5.15.  $\square$

As a consequence, if  $F = 0$  is of Clairaut type and there exists a geometric solution on the contact singular set, then uniqueness for geometric solutions does not hold.

## 6 Appendix: Proof of Lemma 3.2

In this appendix, we prove Lemma 3.2 by an elementary argument like as explicit ODEs.

*Proof of Lemma 3.2.* Suppose that  $F_X(z_0) \neq 0$ . It follows that  $F = 0$  is of reduced type at  $z_0$ . By Theorem 2.2, there exists a complete solution at  $z_0$  and hence there exists a geometric solution passing through  $z_0$ .

We assume that  $F_X \equiv 1$  on the equation hypersurface, if necessary, we may consider  $F/F_X = 0$  as  $F = 0$ . Moreover if  $\gamma : (\mathbb{R}, t_0) \rightarrow (F^{-1}(0), z_0); \gamma(t) = (x(t), y(t), p_1(t), \dots, p_n(t))$  is a geometric solution passing through  $z_0$ , then  $p'_n(t_0) \neq 0$ . Hence we can reparametrize  $\gamma(t)$  as  $(x(t), y(t), p_1(t), \dots, p_{n-1}(t), t)$ .

Let  $\gamma(t) = (x(t), y(t), p_1(t), \dots, p_{n-1}(t), t)$  and  $\tilde{\gamma}(t) = (\tilde{x}(t), \tilde{y}(t), \tilde{p}_1(t), \dots, \tilde{p}_{n-1}(t), t)$  be geometric solutions passing through  $z_0$ , that is,  $\gamma(t_0) = \tilde{\gamma}(t_0) = z_0$ . It is enough to show that  $\gamma(t) = \tilde{\gamma}(t)$  for  $t_0 \leq t \leq t_0 + \varepsilon$ , where  $\varepsilon$  is a small positive real number.

If we differentiate the equality  $F(\gamma(t)) = F(x(t), y(t), p_1(t), \dots, p_{n-1}(t), t) = 0$  with respect to  $t$ , then we get  $x'(t) = -F_{p_n}(\gamma(t))$ . By integrating this equality,

$$x(t) = x(t_0) - \int_{t_0}^t F_{p_n}(\gamma(t)) dt.$$

Since  $\gamma(t)$  is a geometric solution, namely,  $y'(t) = p_1(t)x'(t)$ ,  $p'_i(t) = p_{i+1}(t)x'(t)$  ( $i = 1, \dots, n-2$ ) and  $p'_{n-1}(t) = tx'(t)$ , we have

$$y(t) = y(t_0) + \int_{t_0}^t p_1(t)x'(t) dt, \quad p_i(t) = p_i(t_0) + \int_{t_0}^t p_{i+1}(t)x'(t) dt$$

( $i = 1, \dots, n-2$ ) and

$$p_{n-1}(t) = p_{n-1}(t_0) + \int_{t_0}^t tx'(t) dt.$$

It follows that

$$x(t) - \tilde{x}(t) = \int_{t_0}^t (-F_{p_n}(\gamma(t)) + F_{p_n}(\tilde{\gamma}(t))) dt, \quad (11)$$

$$\begin{aligned} y(t) - \tilde{y}(t) &= \int_{t_0}^t (p_1(t)x'(t) - \tilde{p}_1(t)\tilde{x}'(t)) dt \\ &= \int_{t_0}^t p_1(t)(x'(t) - \tilde{x}'(t)) dt + \int_{t_0}^t \tilde{x}'(t)(p_1(t) - \tilde{p}_1(t)) dt \end{aligned} \quad (12)$$

$$\begin{aligned} p_i(t) - \tilde{p}_i(t) &= \int_{t_0}^t (p_{i+1}(t)x'(t) - \tilde{p}_{i+1}(t)\tilde{x}'(t)) dt \\ &= \int_{t_0}^t p_{i+1}(t)(x'(t) - \tilde{x}'(t)) dt + \int_{t_0}^t \tilde{x}'(t)(p_{i+1}(t) - \tilde{p}_{i+1}(t)) dt \end{aligned} \quad (13)$$

( $i = 1, \dots, n-2$ ) and

$$\begin{aligned} p_{n-1}(t) - \tilde{p}_{n-1}(t) &= \int_{t_0}^t t(x'(t) - \tilde{x}'(t)) dt. \\ &= \int_{t_0}^t t(-F_{p_n}(\gamma(t)) + F_{p_n}(\tilde{\gamma}(t))) dt. \end{aligned} \quad (14)$$

Since  $F$  is a smooth mapping, there exists some number  $K$  such that

$$|-F_{p_n}(\gamma(t)) + F_{p_n}(\tilde{\gamma}(t))| \leq K|\gamma(t) - \tilde{\gamma}(t)| \leq K\alpha(t),$$

where  $t_0 \leq t \leq t_0 + \varepsilon$  and

$$\alpha(t) = |x(t) - \tilde{x}(t)| + |y(t) - \tilde{y}(t)| + |p_1(t) - \tilde{p}_1(t)| + \dots + |p_{n-1}(t) - \tilde{p}_{n-1}(t)|.$$

Moreover, since  $\gamma$  and  $\tilde{\gamma}'$  are smooth mappings, we put

$$a_i = \max_{t_0 \leq t \leq t_0 + \varepsilon} \{|p_i(t)|\} \quad (i = 1, \dots, n-1), \quad a_n = \max\{|t_0|, |t_0 + \varepsilon|\}, \quad b = \max_{t_0 \leq t \leq t_0 + \varepsilon} \{|\tilde{x}'(t)|\}.$$

We denote an integration repeated  $i$ -times  $\int_{t_0}^t \left( \dots \left( \int_{t_0}^t \alpha(t) dt \right) \dots \right) dt$  by  $\left( \int_{t_0}^t \right)^i \alpha(t)(dt)^i$ .

It follows from (11), (12), (13) and (14) that

$$\begin{aligned} \alpha(t) &\leq (1 + a_1 + \dots + a_n)K \int_{t_0}^t \alpha(t) dt + (a_2 + \dots + a_n)bK \left( \int_{t_0}^t \right)^2 \alpha(t)(dt)^2 + \\ &\dots + (a_i + \dots + a_n)b^{i-1}K \left( \int_{t_0}^t \right)^i \alpha(t)(dt)^i + \dots + a_n b^{n-1}K \left( \int_{t_0}^t \right)^n \alpha(t)(dt)^n. \end{aligned} \quad (15)$$

Moreover we put

$$L = \max\{1 + a_1 + \dots + a_n, (a_2 + \dots + a_n)b, \dots, a_n b^{n-1}\}, \quad M = \max_{t_0 \leq t \leq t_0 + \varepsilon} \{\alpha(t)\}.$$

By (15), we have

$$M \leq LKM \sum_{i=1}^n \frac{1}{i!} (t - t_0)^i \leq LKM \sum_{i=1}^n \frac{1}{i!} \varepsilon^i \quad (16)$$

We now consider a function  $f(x) = \sum_{i=1}^n (1/i!)x^i - 1/LK$ . If  $x = 0$ , then  $f(x)$  is negative, and if  $x$  is sufficient large, then  $f(x)$  is positive. By the means value theorem, there exists a positive real number  $t_n$  such that  $f(x) < 0$  on  $0 < x < t_n$ .

If we take a small real number  $\varepsilon > 0$  which satisfies  $0 < \varepsilon < t_n$ , then  $f(\varepsilon)M \geq 0$  by (16). It follows that  $M = 0$  and concludes that  $\gamma(t) = \tilde{\gamma}(t)$  for  $t_0 \leq t \leq t_0 + \varepsilon$ . This completes the proof of Lemma 3.2.  $\square$

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