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Instructions for use

L^1 MAXIMAL REGULARITY FOR THE LAPLACIAN AND APPLICATIONS

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ABSTRACT. Inter alia we prove L^1 maximal regularity for the Laplacian in the space of Fourier transformed finite Radon measures FM. This is remarkable, since FM is not a UMD space and by the fact that we obtain L^p maximal regularity for $p = 1$, which is not even true for the Laplacian in L^2. We apply our result in order to construct strong solutions to the Navier-Stokes equations for initial data in FM in a rotating frame. In particular, the obtained results are uniform in the angular velocity of rotation.

1. **Introduction and main results.** In [3], [5], and [6] the space of Fourier transformed finite Radon measures, denoted by FM (see next section for the precise definition, in particular (2)), is used to construct local and global solutions to the rotating Navier-Stokes equations. From both, the mathematical and the applied point of view here two requirements are essential: the ground space should contain a large class of nondecaying, such as almost periodic, functions, and the obtained results should be uniformly in Ω, i.e., in the angular velocity of rotation. We refer to [9], [3], [4], [5], [6] and to the literature cited therein for more information.

In [3], [5], and [6] it is indeed shown that in FM the two requirements can be satisfied. Hereby the uniformness in Ω relies on a multiplier result similar to the situation in L^2. In fact, merely boundedness and continuity of a symbol implies the associated operator to be bounded on FM (see Proposition 1). Moreover, by construction of FM all computations and estimations take place in Fourier space. This makes the approach very elementary and explicit.

The intention of this note at first is to display and discuss further interesting properties of $-\Delta$ in FM. In particular, we will prove the following result. For a rigorous definition and an introduction to L^p maximal regularity we refer to [1], [8].

Theorem 1.1. *The negative Laplacian* $-\Delta$ *with domain* $\mathscr{D}(-\Delta) = \mathrm{FM}^2(\mathbb{R}^n)$ *(see* (4) *for the definition) has* L^1 *maximal regularity on* $\mathrm{FM}(\mathbb{R}^n)$.

This is remarkable for two reasons. Firstly, by the fact that FM is not reflexive, and hence not a UMD space, and, secondly, since we obtain L^p-maximal regularity for $p = 1$. Observe that L^1 maximal regularity does not even hold for $-\Delta$ on $L^2(\mathbb{R}^n)$. An advantage of L^1 maximal regularity is that no higher regularity on the initial data is required. In fact, then the class of initial data for mild and strong solutions coincide, although strong solutions are more regular. To our knowledge a Banach space admitting such interesting functional analytic properties so far has not been listed in the available literature.

By utilizing the L^1 maximal regularity, in Section 3 we will construct strong solutions for the rotating Navier-Stokes equations. This system reads as

$$\begin{cases} \partial_t u - \nu \Delta u + \Omega e_3 \times u + (u \cdot \nabla)u &= -\nabla p & \text{in } (0, T) \times \mathbb{R}^3, \\ \operatorname{div} u &= 0 & \text{in } (0, T) \times \mathbb{R}^3, \\ u|_{t=0} &= u_0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $u = u(x, t)$ is the velocity vector field and $p = p(x, t)$ is the pressure. Here $\nu > 0$ is the kinematic viscosity coefficient and $\Omega e_3 \times u$ the Coriolis force with angular velocity of rotation $\Omega/2$. The coordinate system is chosen in a way such that the rotation is about the axis $e_3 = (0, 0, 1)^T$. For this system we will prove the following results. For the definition of the appearing subspaces of FM, see (3), (11), and (12).

Theorem 1.2. *For every* $u_0 \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^3)$ *there is a* $T_0 > 0$ *and a unique solution*

$$(u, p) \in \left(W^{1,1}((0, T_0), \mathrm{FM}_{0,\sigma}) \cap L^1((0, T_0), \mathrm{FM}_0^2) \right) \times L^1((0, T_0), \widehat{\mathrm{FM}}_0^1).$$

of (1). *Furthermore, For every* $T \in (0, \infty)$ *there is an* $\varepsilon = \varepsilon(T) > 0$ *such that for* $\|u_0\|_{\mathrm{FM}} < \varepsilon$ *this solution exists on* $(0, T)$. *In particular, the quantities* T_0 *and* ε *are uniformly in* $\Omega \in \mathbb{R}$.

Remark 1. (a) Note that the space $\mathrm{FM}_{0,\sigma}(\mathbb{R}^3)$ contains almost periodic functions of the form

$$u_0(x) = \sum_{j=1}^{\infty} a_j e^{i\lambda_j \cdot x}, \quad x \in \mathbb{R}^3, \ a_j \in \mathbb{R}^3, \ \lambda_j \in \mathbb{R}^3 \setminus \{0\},$$

whenever $\sum_{j=1}^{\infty} |a_j| < \infty$ and $\operatorname{div} u_0 = 0$.

(b) Note that in [3], [5], and [6] merely mild solutions are constructed, in [3] and [5] even for the same class of initial data. In this direction Theorem 1.2 represents an improvement of the results obtained there.

(c) In [3] also an explicit bound for T_0 from below by the square of the norm of initial data independent of Ω is given. Observe that we could derive a similar bound here, by estimating the nonlinear terms in Section 3 (see (14) and (15)) in a more detailed way.

The paper is organized as follows. After the introduction and the statement of the main results in Section 1, in Section 2 we first give a brief introduction to the theory of finite Radon measures. Then we prove the L^1 maximal regularity for the Laplacian and discuss some further consequences of the theory. In Section 3 we apply the obtained results for $-\Delta$ in order to prove Theorem 1.2.

2. **Radon measures and maximal regularity.** We introduce some notation. By $\mathscr{L}(X,Y)$ we denote the space of all bounded linear operators from the Banach space X to the Banach space Y. Its subalgebra of isomorphisms is denoted by $\mathscr{L}_{is}(X,Y)$. If $X = Y$, we write $\mathscr{L}(X)$, $\mathscr{L}_{is}(X)$ for short. For $G \subseteq \mathbb{R}^n$, $C(G,X)$ denotes the space of continuous functions and $\mathrm{BC}(G,X)$ the space of all bounded and continuous functions on G with values in X. The standard X-valued Lebesgue spaces as usual are denoted by $L^p(G,X)$ and the Sobolev spaces by $W^{k,p}(G,X)$. In this note we denote the Fourier transform by

$$\widehat{u}(\xi) = \mathcal{F}u(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) \mathrm{d}x, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of rapidly decreasing functions. Its extension on $\mathcal{S}'(\mathbb{R}^n)$ is defined as usual.

We recall briefly some properties of the spaces $\mathrm{FM}(\mathbb{R}^n)$ and $\mathrm{FM}_0(\mathbb{R}^n)$ from [3] and [6]. For a comprehensive introduction to vector measures we refer to [2].

Definition 2.1. Let $K^n \in \{\mathbb{R}^n, \mathbb{C}^n\}$ be equipped with the euklidean norm $|\cdot|$, and let $\mathscr{A}$ be a σ-algebra over $\mathbb{R}^n$. The set map $\mu : \mathscr{A} \to K^n$ is called a *finite K^n-valued (or more general vector valued) Radon measure*, if

(i) μ is a K^n-valued measure, i.e., it is σ-additive and $\mu(\emptyset) = 0$,

(ii) the variation of μ given by

$$|\mu|(\mathcal{O}) := \sup \left\{ \sum_{E \in \Pi(\mathcal{O})} |\mu(E)| : \ \Pi(\mathcal{O}) \subseteq \mathscr{A} \text{ finite decomposition of } \mathcal{O} \right\}$$

for $\mathcal{O} \in \mathscr{A}$ is a finite Radon measure. (Note that $\Pi(\mathcal{O}) \subseteq \mathscr{A}$ is a decomposition of $\mathcal{O} \in \mathscr{A}$, if $A \cap B = \emptyset$ for all $A, B \in \Pi$ with $A \neq B$ and $\bigcup_{A \in \Pi} A = \mathcal{O}$.)

We denote by $\mathrm{M}(\mathbb{R}^n)$ the space of all finite K^n-valued Radon measures.

Note that for $\mu \in \mathrm{M}(\mathbb{R}^n)$ and $f \in \mathrm{BC}(\mathbb{R}^n)$ the integral $\int_{\Omega} f \mathrm{d}\mu$ can be defined in the standard way by approximation via simple functions. Recall that $\eta : \mathscr{A} \to [0, \infty)$ is a finite Radon measure, if $\eta(\mathbb{R}^n) < \infty$ and if it is Borel regular, that is, if $\mathscr{B} \subseteq \mathscr{A}$ and if for each $A \subseteq \mathbb{R}^n$ there exists a $B \in \mathscr{B}$, $B \subset A$, such that $\eta^*(A) = \eta^*(B)$, where $\mathscr{B}$ denotes the Borel σ-algebra over $\mathbb{R}^n$ and η^* denotes the outer measure associated to η. Also observe that we indentify η by its outer measure, so that η is complete in the sense that all subsets B of a set $A \in \mathscr{A}$ satisfying $\eta(A) = 0$ belong to $\mathscr{A}$. By the Riesz representation theorem it is well-known that $\mathrm{M}(\mathbb{R}^n)$ can be regarded as the dual space of

$$C_{\infty}(\mathbb{R}^n) = \left\{ u \in C(\mathbb{R}^n) : \ \lim_{R \to \infty} \|u\|_{L^{\infty}(\mathbb{R}^n \setminus B_R)} = 0 \right\},$$

where B_R denotes the ball with center 0 and radius R (see [2]).

Remark 2. (a) It can be shown that μ is a finite K^n-valued measure if and only if the variation $|\mu|$ is a finite nonnegative measure.

(b) Equipped with the norm $\|\mu\|_{\mathrm{M}} := \|\mu\|_{\mathrm{M}(\mathbb{R}^n)} := |\mu|(\mathbb{R}^n)$, $\mathrm{M}(\mathbb{R}^n)$ is a Banach space.

Since μ is $|\mu|$-absolutely continuous and $\mathbb{C}^n$ and $\mathbb{R}^n$ have the Radon-Nikodým property, we have

$$\mu(\mathcal{O}) = \int_{\mathcal{O}} \nu_{\mu} \mathrm{d}|\mu|, \quad \mathcal{O} \in \mathscr{B},$$

with a $\nu \in L^1(\mathbb{R}^n, |\mu|)$ such that $|\nu_\mu|(x) = 1$ $(x \in \mathbb{R}^n)$ (cf. [2]). Since each $\mu \in M(\mathbb{R}^n)$ is defined on $\mathscr{B}$, the expression

$$\mu \lfloor \psi(\mathcal{O}) := \int_{\mathcal{O}} \psi \nu_\mu \mathrm{d}|\mu|, \quad \mathcal{O} \in \mathscr{B},$$

is well-defined for every $\psi \in \mathrm{BC}(\mathbb{R}^n, \mathscr{L}(K^n, K^m))$. The proof of the following properties is straightforward.

Lemma 2.2. *Let* $K^n \in \{\mathbb{R}^n, \mathbb{C}^n\}$. *Let* $\mu \in M(\mathbb{R}^n)$ *and* $\psi, \phi \in \mathrm{BC}(\mathbb{R}^n, \mathscr{L}(K^n, K^m))$ *be given. Then we have*

(i) $|\mu \lfloor \psi| \leq |\mu| \lfloor |\psi|$, *(ii)* $\mu \lfloor \psi \in M(\mathbb{R}^n)$, *(iii)* $(\mu \lfloor \psi) \lfloor \phi = \mu \lfloor (\phi \psi)$.

In our applications to the Navier-Stokes equations we will frequently have $\psi = \sigma_P$, where $\sigma_P(\xi) = I - \xi \xi^T / |\xi|^2$ denotes the symbol of the Helmholtz projection on $\mathbb{R}^n$. However, σ_P is discontinuous at $\xi = 0$. This motivates the introduction of

$$\mathrm{M}_0(\mathbb{R}^n) := \{\mu \in M(\mathbb{R}^n) : \ \mu(\{0\}) = 0\},$$

which is a closed subspace of $M(\mathbb{R}^n)$. Next, note that by the identification

$$f \mapsto \lambda \lfloor f, \quad f \in L^1(\mathbb{R}^n),$$

where λ denotes the Lebesgue measure on $\mathbb{R}^n$, we see that every $f \in L^1(\mathbb{R}^n)$ has a unique representant in $M(\mathbb{R}^n)$. On the other hand, the identification

$$\mu \mapsto T_\mu, \ T_\mu f := \mu \lfloor f(\mathbb{R}^n), \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of rapidly decreasing functions, shows that each $\mu \in M(\mathbb{R}^n)$ can be regarded as a tempered distribution. Altogether we have

$$L^1(\mathbb{R}^n) \hookrightarrow \mathrm{M}_0(\mathbb{R}^n) \hookrightarrow M(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Thus, the Fourier transform on $M(\mathbb{R}^n)$ is well-defined and we have $\widehat{\mu}(\xi) = \mu \lfloor \varphi_\xi(\mathbb{R}^n)$ with $\varphi_\xi(x) = (2\pi)^{-n/2} e^{-ix \cdot \xi}$. This allows for the introduction of the spaces

$$\begin{align} \mathrm{FM}(\mathbb{R}^n) \quad &:= \quad \{\widehat{\mu} : \ \mu \in M(\mathbb{R}^n)\}, \tag{2} \\ \mathrm{FM}_0(\mathbb{R}^n) \quad &:= \quad \{\widehat{\mu} : \ \mu \in \mathrm{M}_0(\mathbb{R}^n)\}, \tag{3} \end{align}$$

which we equip with the canonical norm $\|u\|_{\mathrm{FM}} := \|\mathcal{F}^{-1} u\|_{\mathrm{M}}$. Observe that by definition we have $\|\mathcal{F} u\|_{\mathrm{M}} = \|\mathcal{F}^{-1} u\|_{\mathrm{M}}$. For $k \in \mathbb{N}_0$ we also set

$$\mathrm{FM}^k(\mathbb{R}^n) := \{u \in \mathrm{FM}_0(\mathbb{R}^n) : \ \partial^\alpha u \in \mathrm{FM}_0(\mathbb{R}^n) \ (|\alpha| \leq k)\}. \tag{4}$$

The spaces $\mathrm{FM}_0^k(\mathbb{R}^n)$ are defined accordingly. Since in this note we only consider measures and transformed measures on $\mathbb{R}^n$, from now on we simply write M, FM, FM_0, and so on. Finally, we define the convolution of finite Radon measures by

$$\eta * \mu(\mathcal{O}) := \int_{\mathbb{R}^n} \eta(\mathcal{O} - x) \cdot \nu_\mu(x) \mathrm{d}|\mu|(x), \quad \mathcal{O} \in \mathscr{B}.$$

It is not difficult to check that $x \mapsto \eta(\mathcal{O} - x)$ is bounded $|\mu|$-a.e., hence $\eta * \mu(\mathcal{O})$ is well-defined. The following properties are straightforward consequences of the definitions.

Lemma 2.3. *We have*

(i) $\mathcal{F}(\eta * \mu) = (2\pi)^{n/2} \widehat{\eta} \cdot \widehat{\mu}$, $\eta, \mu \in \mathrm{M}$,

(ii) $\|uv\|_{\mathrm{FM}} \leq (2\pi)^{-n/2} \|u\|_{\mathrm{FM}} \|v\|_{\mathrm{FM}}$,

(iii) $\mathcal{F}L^1 \hookrightarrow \mathrm{FM}_0 \hookrightarrow \mathrm{FM} \hookrightarrow \mathrm{BUC}$.

The next result, see also [3, Lemma 2.2], is essential for the uniformness of our results in the Coriolis parameter Ω. It is obtained as a consequence of Lemma 2.2.

Proposition 1. *Let $K^n \in \{\mathbb{R}^n, \mathbb{C}^n\}$ and $\sigma \in \mathrm{BC}(\mathbb{R}^n \setminus \{0\}, \mathscr{L}(K^n, K^m))$. Then, we have*

$$Op(\sigma) := \mathcal{F}^{-1}\sigma\mathcal{F} \in \mathscr{L}(\mathrm{FM}_0),$$

$$\|Op(\sigma)\|_{\mathscr{L}(\mathrm{FM}_0)} = \|\sigma\|_{L^\infty(\mathbb{R}^n \setminus \{0\}, \mathscr{L}(K^n, K^m))}.$$

If σ is also continuous at the origin, then $Op(\sigma) \in \mathscr{L}(\mathrm{FM})$ with the corresponding equality for the operator norm.

We also prepare the following general result on convolution.

Lemma 2.4. *Let X, Y be Banach spaces, $1 \le p \le \infty$, $T \in (0, \infty]$, and set $J = (0, T)$. For $g \in \mathscr{L}(X, L^p(J, Y))$ and $f \in L^1(J, X)$ we have*

$$\left(t \mapsto g \star f := \int_0^t g(t - s)f(s)\mathrm{d}s\right) \in L^p(J, Y)$$

and

$$\|g \star f\|_{L^p(J,Y)} \le \|g\|_{\mathscr{L}(X, L^p(J,Y))}\|f\|_{L^1(J,X)}.$$

Proof. For a function h defined on J we denote by $\tilde{h}$ its trivial extension on $\mathbb{R}$. Pick $g \in \mathscr{L}(X, L^p(J, Y))$ and $f \in L^1(J, X)$. By assumption we have for a.e. $s \in J$ that $(t \mapsto \tilde{g}(t - s)\tilde{f}(s)) \in L^p(\mathbb{R}, Y)$ and that

$$
\begin{aligned}
\|\tilde{g}(t - s)\tilde{f}(s)\|_{L^p(\mathbb{R},Y)} &= \left(\int_s^T \|g(t - s)f(s)\|_Y^p \mathrm{d}t\right)^{1/p} \\
&\le \left(\int_0^T \|g(r)f(s)\|_Y^p \mathrm{d}r\right)^{1/p} \\
&\le \|g\|_{\mathscr{L}(X, L^p(J,Y))}\|f(s)\|_X. \quad (5)
\end{aligned}
$$

For $s \in \mathbb{R} \setminus J$ this estimate is trivially true. This yields

$$(s \mapsto \|\tilde{g}(t - s)\tilde{f}(s)\|_{L^p(\mathbb{R},Y)}) \in L^1(\mathbb{R}).$$

Hence $\int_{\mathbb{R}} \tilde{g}(t - s)f(s)\mathrm{d}s$, and therefore also $g \star f(t)$, exists as a Bochner integral with values in $L^p(J, Y)$. Thanks to (5) we also obtain

$$
\begin{aligned}
\|g \star f\|_{L^p(J,Y)} &= \|\int_R \tilde{g}(t - s)\tilde{f}(s)\mathrm{d}s\|_{L^p(R,Y)} \\
&\le \|g\|_{\mathscr{L}(X, L^p(J,Y))}\|f\|_{L^1(J,X)}.
\end{aligned}
$$

$\square$

For $s \ge 0$ we set

$$\mathrm{FM}^s := \{u \in \mathrm{FM} : |\xi|^s \mathcal{F}u \in \mathrm{M}\}.$$

Observe that in view of Proposition 1 this definition coincides with definition (4) for $s = k \in \mathbb{N}_0$. For $-\Delta$ with domain FM^2 we have the following maximal regularity estimates in FM.

Theorem 2.5. *Let $1 \le p \le \infty$. For $T(t) := e^{t\nu\Delta}$ we have*

(i) $\|\Delta Tu_0\|_{L^p(\mathbb{R}_+, \mathrm{FM})} \le \frac{1}{(p\nu)^{1/p}}\|u_0\|_{\mathrm{FM}^{2-2/p}}$ $(u_0 \in \mathrm{FM}^{2-2/p})$,

(ii) $\|\Delta T \star f\|_{L^1(\mathbb{R}_+, \mathrm{FM})} \le \frac{1}{\nu}\|f\|_{L^1(\mathbb{R}_+, \mathrm{FM})}$ $(f \in L^1(\mathbb{R}_+, \mathrm{FM}))$.

Proof. (i) From Lemma 3.1(i) we infer that

$$\|\Delta e^{\nu t\Delta}u_0\|_{\mathrm{FM}} \leq \int_{\mathbb{R}^n} |\xi|^2 e^{-\nu t|\xi|^2} \mathrm{d}|\widehat{u_0}|(\xi). \tag{6}$$

This implies directly that

$$\|\Delta Tu_0\|_{L^p(\mathbb{R}_+,\mathrm{FM})} \leq \int_{\mathbb{R}^n} |\xi|^2 \|e^{-\nu(\cdot)|\xi|^2}\|_{L^p(\mathbb{R}_+)} \mathrm{d}|\widehat{u_0}|(\xi) \leq \frac{1}{(p\nu)^{1/p}} \|u_0\|_{\mathrm{FM}^{2-2/p}}.$$

(ii) For $p = 1$ part (i) gives us

$$\Delta T \in \mathscr{L}(\mathrm{FM},\, L^1(\mathbb{R}_+,\mathrm{FM})),$$

Setting $X = Y = \mathrm{FM}$, $p = 1$, and $g := \Delta T_\Omega$, assertion (ii) follows from Lemma 2.4. $\qquad\square$

Let us discuss some consequences of the obtained results. For a comprehensive approach to and relations between the appearing notions such as L^p maximal regularity, $\mathcal{R}$-boundedness, and so on, we refer to the booklets [1], and [8]. First observe that Theorem 2.5(ii) implies Theorem 1.1. In [1] it is shown that L^p maximal regularity for one $p \in [1,\infty)$ implies $\mathcal{R}$-sectoriality. This is even true for arbitrary Banach spaces. Hence we have

Corollary 1. *The negative Laplacian* $-\Delta$ *with domain* $\mathscr{D}(-\Delta) = \mathrm{FM}^2$ *is* $\mathcal{R}$-*sectorial on* FM *with* $\mathcal{R}$-*angle* $\phi^{\mathcal{R}}(-\Delta) = 0$.

The importance of the concept of $\mathcal{R}$-boundedness is underlined by the fact that for UMD spaces the converse is also true. To be precise, if A is $\mathcal{R}$-sectorial on a UMD space X, then A has L^p maximal regularity on X for $p \in (1,\infty)$. However, since FM is not a UMD space, here this argument does not apply. In fact, in this note the question whether or not $-\Delta$ has L^p maximal regularity on FM for $p \in (1,\infty)$ remains open. At least, for $p \in (1,\infty)$, we have the estimate in Theorem 2.5(i).

By applying directly Proposition 1 we obtain even a much stronger result than Corollary 1. For $h \in \mathrm{BUC}(\mathbb{R}_+)$ we can set

$$h(-\Delta) := \mathcal{F}^{-1}[h(|\xi|^2)]\mathcal{F}. \tag{7}$$

Thanks to Proposition 1, $h(-\Delta)$ is well-defined on FM and we obtain

$$\|h(-\Delta)\|_{\mathscr{L}(\mathrm{FM})} \leq \|h\|_\infty. \tag{8}$$

This implies in particular the following result.

Corollary 2. *The negative Laplacian* $-\Delta$ *admits a bounded* $\mathcal{H}^\infty$-*calculus on* FM *with* $\mathcal{H}^\infty$-*angle* $\phi^\infty(-\Delta) = 0$.

We remark that the functional calculus introduced through (7) is stronger than an $\mathcal{H}^\infty$-calculus, since then estimate (8) is only required with a right hand side $C\|h\|_\infty$ for a $C > 0$ and for bounded holomorphic functions h on a complex sector containing $\mathbb{R}_+$. Furthermore, it is not difficult to check that the space FM has property α. By Corollary 2 and [7, Theorem 5.3] it then follows that $-\Delta$ automatically has an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus. This, in turn, again implies the $\mathcal{R}$-sectoriality of $-\Delta$, i.e., Corollary 1.

As an application of Corollary 2 we can determine precisely complex interpolation spaces between FM and FM^2. Indeed, [10, Theorem 1.15.3] implies that

$$[\mathrm{FM},\mathrm{FM}^2]_s = \mathscr{D}((-\Delta)^s) = \mathrm{FM}^s.$$

In particular, for $s = 1/2$ we obtain

$$[\mathrm{FM}, \mathrm{FM}^2]_{1/2} = \mathrm{FM}^1 = \{u \in \mathrm{FM} : \ \nabla u \in \mathrm{FM}\}. \tag{9}$$

Note that it is not difficult to check that all results above remain true for the space FM_0.

3. **Strong solutions for system (1).** From now on we assume that $n = 3$. We first consider the linear situation. Let $\sigma_P(\xi) = I - \xi\xi^T/|\xi|^2$ be the symbol of the Helmholtz projection P (projection on divergence free vector fields in $\mathbb{R}^3$). Next, set $S := PJP$ with

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

We call S the Poincaré-Riesz operator and denote its symbol by $\sigma_S = \sigma_P J \sigma_P$. Note that the solenoidal part of the Corolis force then can be represented as $\Omega P\mathbf{e}_3 \times u = \Omega Su$. Observe that σ_P is orthogonal, σ_S is skew-symmetric, and thus $\sigma(\mathrm{e}^{tS})$ is unitary on $\mathbb{C}^3$. The symbol of this operator can be expressed in terms of classical Riesz operators. We refer to [3] for additional information. As a consequence of Proposition 1, we therefore obtain the following estimates (see [3, Lemma 2.5, Lemma 2.9]).

Lemma 3.1. *The operators P, S, and e^{tS} are bounded on FM_0. In particular, we have*

$$\begin{aligned} \|Pf\|_{\mathrm{FM}} &\leq& \|f\|_{\mathrm{FM}} \quad (f \in \mathrm{FM}_0), \\ \|\mathrm{e}^{tS}f\|_{\mathrm{FM}} &\leq& \|f\|_{\mathrm{FM}} \quad (t \in \mathbb{R}, \ f \in \mathrm{FM}_0). \end{aligned}$$

Observe that by the first estimate in Lemma 3.1 we easily obtain the Helmholtz decomposition

$$\mathrm{FM}_0 = \mathrm{FM}_{0,\sigma} \oplus \mathrm{G}_{FM},$$

where

$$\mathrm{FM}_{0,\sigma} := P\mathrm{FM}_0 = \{u \in \mathrm{FM}_0 : \ \mathrm{div}\, u = 0\}, \tag{11}$$

$\mathrm{G}_{\mathrm{FM}} = \{\nabla p : p \in \widehat{\mathrm{FM}}_0^1\}$, and where

$$\widehat{\mathrm{FM}}_0^1 = \{p \in \mathcal{S}'(\mathbb{R}^3) : \ \nabla p \in \mathrm{FM}_0\}/\mathbb{C}. \tag{12}$$

Next we consider the semigroup $(e^{-tA})_{t\geq 0}$ generated by the Stokes-Poincaré-Riesz operator which is given by

$$A = -\nu\Delta + \Omega S$$

and defined in $\mathrm{FM}_{0,\sigma}$ with domain $\mathscr{D}(A) = \{u \in \mathrm{FM}_{0,\sigma} : \ \partial^\alpha u \in \mathrm{FM}_0 \ (|\alpha| \leq 2)\}$. Since $\mathrm{FM}_{0,\sigma}$ is a closed subspace of FM and thanks to Lemma 3.1, it is not difficult to generalize Theorem 2.5 to $(e^{-tA})_{t\geq 0}$. To be precise, we obtain

Lemma 3.2. *Let $1 \leq p \leq \infty$. For $T_\Omega(t) := e^{t(\nu\Delta - \Omega S)}$ we have*

(i) $\|\Delta T_\Omega u_0\|_{L^p(\mathbb{R}_+, \mathrm{FM})} \leq \frac{1}{(p\nu)^{1/p}}\|u_0\|_{\mathrm{FM}^{2-2/p}} \quad (u_0 \in \mathrm{FM}_0^{2-2/p} \cap \mathrm{FM}_{0,\sigma})$,

(ii) $\|\Delta T_\Omega \star f\|_{L^1(\mathbb{R}_+, \mathrm{FM})} \leq \frac{1}{\nu}\|f\|_{L^1(\mathbb{R}_+, \mathrm{FM})} \quad (f \in L^1((\mathbb{R}_+, \mathrm{FM}_{0,\sigma}))$.

Let $T \in (0, \infty)$. To simplify notation we define the classes of maximal regularity for short as

$$
\begin{aligned}
\mathbb{E}_T^1 &:= W^{1,1}((0,T), \mathrm{FM}_{0,\sigma}) \cap L^1((0,T), \mathscr{D}(A)) \\
\mathbb{E}_T^2 &:= L^1((0,T), \widehat{\mathrm{FM}}_0^1) \\
\mathbb{E}_T &:= \mathbb{E}_T^1 \times \mathbb{E}_T^2 \\
\mathbb{F}_T &:= L^1((0,T), \mathrm{FM}_0).
\end{aligned}
$$

We define by $_0\mathbb{E}_T^1$ the corresponding space with zero time trace at $t = 0$ and set $_0\mathbb{E}_T := {}_0\mathbb{E}_T^1 \times \mathbb{E}_T^2$. Furthermore, we set

$$
L_\Omega(u,p) := \begin{pmatrix} u_t - \Delta u + \Omega e_3 \times u + \nabla p \\ u|_{t=0} \end{pmatrix}.
$$

Let $(f, u_0) \in \mathbb{F}_T \times \mathrm{FM}_{0,\sigma}$ with f not necessarily being divergence free. Theorem 1.1 then implies the existence of unique $u \in \mathbb{E}_T^1$ solving $(\partial_t + A)u = Pf$, $u(0) = u_0$. Setting $\nabla p = (I - P)(f - \Omega e_3 \times u) \in L^1((0,T), \mathrm{FM}_0)$, we see that $(u,p) \in \mathbb{E}_T$ solves $L_\Omega(u,p) = (f, u_0)$ uniquely. Thus, we have

$$
L_\Omega \in \mathscr{L}_{is}(\mathbb{E}_T, \mathbb{F}_T \times \mathrm{FM}_{0,\sigma}) \quad (T \in (0, \infty)). \tag{13}
$$

In particular, the uniformness in Ω of the estimates in Lemma 3.2 implies also the norm of L_Ω^{-1} to be bounded uniformly in Ω. On the other hand, we remark that the norm of L_Ω^{-1} is increasing T. This is due to the fact that the global estimates in Lemma 3.2 are merely available for highest order derivatives, but not for T_Ω itself. So, relation (13) does not hold for $T = \infty$.

Now we are in position to prove the existence of local-in-time strong solutions, i.e., the first statement in Theorem 1.2. The full nonlinear problem (1) can be formulated as

$$
L_\Omega(u,p) = (F(u), u_0)
$$

with $F(u) = (u \cdot \nabla)u$. In order to be able to use Sobolev embeddings with constants independent of the time interval, we reduce this system to a problem with zero time traces. To this end, let $(\tilde{u}, \tilde{p}) := L_\Omega^{-1}(0, u_0)$ be the solution of the linear problem with fixed initial data $u_0 \in \mathrm{FM}_{0,\sigma}$. Setting $(w, \pi) = (u - \tilde{u}, p - \tilde{p})$ we can rephrase the problem as the fixed point equation

$$
(w, \pi) = H(w, \pi) := L_\Omega^{-1}(F(w + \tilde{u}), 0).
$$

Let $\mathbb{B}(T, r) := \{(w, \pi) \in {}_0\mathbb{E}_T : \|(w,\pi)\|_{\mathbb{E}_T} \leq r\}$ denote the closed ball of radius $r > 0$ centered at 0. It remains to prove that for suitable $r, T > 0$, $H : \mathbb{B}(T, r) \to \mathbb{B}(T, r)$ is contractive. First note that due to $(u(t) \cdot \nabla)v(t) = \mathrm{div}\, u(t)v(t)^T \in \mathrm{FM}_0$ for $u, v \in \mathbb{E}_T^1$ and $t > 0$, the application of L_Ω^{-1} is well-defined. Next, for $u \in \mathbb{B}(T, r)$, by virtue of Lemma 2.3(ii) we can estimate

$$
\begin{aligned}
\|H(w, \pi)\|_{\mathbb{E}_T} &\leq \|L_\Omega^{-1}\|_{\mathscr{L}(\mathbb{F}_T, \mathbb{E}_T)} \|F(w + \tilde{u})\|_{\mathbb{F}_T} \\
&\leq C\Big(\|w\|_{L^\infty((0,T), \mathrm{FM})} \|\nabla w\|_{\mathbb{F}_T} + \|w\|_{L^\infty((0,T), \mathrm{FM})} \|\nabla \tilde{u}\|_{\mathbb{F}_T} \tag{14} \\
&\qquad + \|\tilde{u}\|_{L^{q'}((0,T), \mathrm{FM})} \|\nabla w\|_{L^q((0,T), \mathrm{FM})} + \|(\tilde{u} \cdot \nabla)\tilde{u}\|_{\mathbb{F}_T} \Big). \tag{15}
\end{aligned}
$$

The Sobolev embedding yields

$$
\|w\|_{L^\infty((0,T), \mathrm{FM})} \leq C\|(w, \pi)\|_{\mathbb{E}_T} \quad (T > 0).
$$

The estimate

$$\|(\tilde{u} \cdot \nabla)\tilde{u}\|_{\mathbb{F}_T} \leq \|\tilde{u}\|_{L^\infty((0,T),\mathrm{FM})}\|\nabla\tilde{u}\|_{\mathbb{F}_T} \leq \|\tilde{u}\|_{\mathbb{E}_T^1}^2$$

shows that $(\tilde{u} \cdot \nabla)\tilde{u} \in \mathbb{F}_T$. Next, fix $q \in (1,2)$ and put $s := q/(2-q) \in (0,\infty)$. From the general result [10, Theorem 1.18.4] and from (9) we obtain

$$
\begin{aligned}
\left[L^s((0,T),\mathrm{FM}_{0,\sigma}), L^1((0,T),\mathscr{D}(A))\right]_{1/2} &= L^q\left((0,T),[\mathrm{FM}_{0,\sigma},\mathscr{D}(A)]_{1/2}\right) \\
&\hookrightarrow L^q\left((0,T),[\mathrm{FM},\mathrm{FM}^2]_{1/2}\right) \\
&= L^q((0,T),\mathrm{FM}^1).
\end{aligned}
$$

The fact that $_0\mathbb{E}_T^1 \hookrightarrow L^s((0,T),\mathrm{FM}_{0,\sigma})$ uniformly in $T > 0$ then gives

$$\|\nabla w\|_{L^q((0,T),\mathrm{FM})} \leq C\|(w,\pi)\|_{\mathbb{E}_T} \quad (T > 0).$$

Since $\tilde{u}$ is a fixed function, we also can obtain the appearing norms of $\tilde{u}$ in (14) and (15) to be smaller than every $\delta > 0$ by choosing T sufficiently small. (Note that this does also not increase the norm of L_Ω^{-1}, since we work in zero trace spaces.) Summarizing, we achieve that

$$\|H(w,\pi)\|_{\mathbb{F}_T} \leq C(r^2 + r\delta + \delta^2).$$

Thus, by choosing r, T small enough we see that $H(\mathbb{B}(T,r)) \subset \mathbb{B}(T,r)$. Since the constant C in (14) is uniformly in Ω, it is also clear that the size of T can be chosen uniformly in $\Omega \in \mathbb{R}$. In a very similar way we can prove that H is contractive uniformly in Ω. The contraction mapping principle then yields the assertion.

For solutions existing on arbitrary intervals $(0,T)$ with fixed $0 < T < \infty$ we directly consider

$$(u,p) = L_\Omega^{-1}(F(u), u_0) =: G(u,p).$$

By the fact that $F \in C^1(\mathbb{E}_T^1, \mathbb{F}_T)$ such that $F(u) = DF(u) = 0$ (DF Fréchet derivative), we can choose small $r > 0$ so that

$$\sup_{(w,\pi)\in\mathbb{B}(T,r)} \|DF(w)\|_{\mathscr{L}(\mathbb{E}_T^1,\mathbb{F}_T)} \leq 1/2\|L_\Omega^{-1}\|_{\mathscr{L}(\mathbb{F}_T,\mathbb{E}_T)}.$$

We also set $\varepsilon := r/2\|L_\Omega^{-1}\|_{\mathscr{L}(\mathbb{F}_T,\mathbb{E}_T)}$. From this we see that ε is indeed uniformly in Ω. For $\|u_0\|_{\mathrm{FM}} < \varepsilon$ then we can estimate

$$
\begin{aligned}
&\|G(u,p)\|_{\mathbb{E}_T} \\
&\leq \|L_\Omega^{-1}\|_{\mathscr{L}(\mathbb{F}_T,\mathbb{E}_T)}(\|F(u) - F(0)\|_{\mathbb{F}_T} + \|u_0\|_{\mathrm{FM}}) \\
&\leq \|L_\Omega^{-1}\|_{\mathscr{L}(\mathbb{F}_T,\mathbb{E}_T)}\left(\sup_{(w,\pi)\in\mathbb{B}(T,r)}\|DF(w)\|_{\mathscr{L}(\mathbb{E}_T^1,\mathbb{F}_T)}\|(u,p)\|_{E_T} + \|u_0\|_{\mathrm{FM}}\right) \\
&\leq r/2 + r/2 \leq r \quad ((u,p) \in \mathbb{B}(T,r)).
\end{aligned}
$$

Also here in a very similar way we can show that G is contractive. Hence, the contraction mapping principle yields the assertion. The proof of Theorem 1.2 is now complete.

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