Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **2010**

Website: http://AIMsciences.org

pp. **X–XX**

SHARP BLOW-UP FOR SEMILINEAR WAVE EQUATIONS WITH NON-COMPACTLY SUPPORTED DATA

Hiroyuki Takamura

Department of Complex and Intelligent Systems, Faculty of Systems Information Science,
Future University Hakodate,
116-2 Kamedanakano-cho, Hakodate, Hokkaido 041-8655, Japan.

Hiroshi Uesaka

Department of Mathematics, College of Science and Technology, Nihon University,
Chiyodaku Kanda Surugadai 1-8, Tokyo, 101-8308, Japan.

Kyouhei Wakasa

The 4th year of undergraduate, Department of Complex Systems,
School of Systems Information Science, Future University Hakodate,
116-2 Kamedanakano-cho, Hakodate, Hokkaido 041-8655, Japan.

(Communicated by Grozdena Todorova)

Abstract. This paper corrects Asakura's observation on semilinear wave equations with non-compactly supported data by showing a sharp blow-up theorem for classical solutions. We know that there is no global in time solution for any power nonlinearity if the spatial decay of the initial data is weak, in spite of finite propagation speed of the linear wave. Our theorem clarifies the final criterion on such a phenomenon.

1. **Introduction.** We consider the initial-value problem for semilinear wave equation

$$\left\{\begin{array}{l} u_{tt}-\Delta u=F(u) \quad \text{in} \quad \mathbf{R}^n\times[0,\infty) \\ u(x,0)=f(x),\ u_t(x,0)=g(x), \end{array}\right. \tag{1}$$

where $n\ge 2$ and $u=u(x,t)$ is a scalar unknown function of space-time variables. The assumptions on the nonlinear term F will be given precisely later, but at this moment we may assume that $F(u)=|u|^p$ or, $F(u)=|u|^{p-1}u$ with $p>1$.

In the case where the initial data (f,g) has compact support, we have the following Strauss' conjecture. There exists a critical number $p_0(n)$ such that (1) has a global in time solution for "small" data if $p>p_0(n)$ and has no global solution for "positive" data if $1<p\le p_0(n)$. As in Section 4 in Strauss [16], $p_0(n)$ is a positive root of the quadratic equation $(n-1)p^2-(n+1)p-2=0$.

This conjecture was first verified by John [6] for $n=3$ except for $p=p_0(3)$. Later, Glassey [4, 5] verified this for $n=2$ except for $p=p_0(2)$. Both critical cases were studied by Schaeffer [14]. In high dimensions, $n\ge 4$, the subcritical case was proved by Sideris [15] and the supercritical case was proved by Georgiev, Lindblad and Sogge [3]. Finally, the critical case in high dimensions was obtained

2000 *Mathematics Subject Classification.* Primary: 35L70; Secondary: 35B05, 35E15.
Key words and phrases. blow-up, semilinear wave equations.

by Yordanov and Zhang [23] or, Zhou [24] independently. We note that the blow-up results in high dimensions are available only for the positive nonlinear term, $F(u)=|u|^p$. All the cited works on this conjecture are summarized in the following table.

	$1<p<p_0(n)$	$p=p_0(n)$	$p>p_0(n)$
$n=2$	[4]	[14]	[5]
$n=3$	[6]	[14]	[6]
$n\geq 4$	[15]	[23], [24] indep.	[3]

On the contrary, if the support of the initial data (f,g) is non-compact, we may have no global solution even for the supercritical case. Actually we have the following Asakura's observation. There exists a critical decay κ_0 of the initial data such that (1) has no global solution provided (f,g) satisfies that

$$f(x)\equiv 0, \quad g(x)\geq \frac{C}{(1+|x|)^{1+\kappa}} \quad \text{with } 0<\kappa<\kappa_0 \tag{2}$$

for some constant $C>0$, and has a global solution provided (f,g) satisfies that

$$(1+|x|)^{1+\kappa}\left(\sum_{|\alpha|\leq[n/2]+2}|\nabla_x^\alpha f(x)|+\sum_{|\beta|\leq[n/2]+1}|\nabla_x^\beta g(x)|\right) \tag{3}$$

is sufficiently small with $\kappa\geq\kappa_0$ and $p>p_0(n)$.

This was first proved by Asakura [2] in $n=3$ except for the critical case clarifying

$$\kappa_0=\frac{2}{p-1}. \tag{4}$$

The critical case in $n=3$ was studied by Kubota [12] or, Tsutaya [21] independently. For $n=2$, the nonexistence part was verified by Agemi and Takamura [1], and the existence part was verified by Kubota [12] or, both parts by Tsutaya [19, 20] independently. In high dimensions, only the radially symmetric solution has been studied. The nonexistence part was proved by Takamura [17], and the existence part was proved by Kubo and Kubota [10, 11], and Kubo [9]. All the cited works on this observation are summarized in the following table.

	$0<\kappa<\kappa_0$	$\kappa=\kappa_0$	$\kappa>\kappa_0$
$n=2$	[1], [20] indep.	[12], [19] indep.	[12], [20] indep.
$n=3$	[2]	[12], [21] indep.	[2]
$n\geq 4$	[17]	[9]	[10] and [11]

It is remarkable that the critical decay κ_0 does not depend on space dimensions n. We also note that the nonlinear equation is invariant under a scaling $u(x,t)\to u_M(x,t)=M^{\kappa_0}u(Mx,Mt)$ $(M>0)$. For this rescaled solution, the data is of the form

$$u_M(x,0)=M^{\kappa_0}f(Mx), \quad (u_M)_t(x,0)=M^{1+\kappa_0}g(Mx). \tag{5}$$

In view of this fact, we can say that the blow-up of the subcritical decay is equivalent to the blow-up for large data at spatial infinity. As suggested by this fact, $(1+|x|)^{1+\kappa}$ in (3) cannot be replaced by, for example, $(1+|x|)^{1+\kappa_0}\log^{-l}(2+|x|)$ with any $l>0$. See Kurokawa and Takamura [13]. Moreover, we point out that (3) on $\alpha=0$ is strong.

In order to correct Asakura's observation, we need a blow-up result for the case where $f\not\equiv 0$ and $g\equiv 0$. The breakthrough was obtained by Uesaka [22] in low dimensions under some decaying assumption on f. Later, in our previous paper

[18], the result was extended to all space dimensions, and the relation between such an assumption and Asakura's observation was clarified. Finally we propose the following corrected criterion for the global existence of the solution.

(1) has no global solution provided there exists a constant $R > 0$ such that (f, g) satisfies

$$f(x) \equiv 0 \quad \text{and} \quad g(x) \geq \frac{\phi(|x|)}{(1+|x|)^{1+\kappa}}, \tag{6}$$

or

$$f(x) > 0,\ \Delta f(x) + F(f(x)) \geq \frac{\phi(|x|)}{(1+|x|)^{2+\kappa}} \quad \text{and} \quad g(x) \equiv 0, \tag{7}$$

for $|x| \geq R$ with

$$0 < \kappa < \kappa_0 \quad \text{and} \quad \phi(x) \equiv \text{positive const.}, \tag{8}$$

or

$$\kappa = \kappa_0, \quad \phi \text{ is positive, monotonously increasing and } \lim_{r\to\infty} \phi(r) = \infty. \tag{9}$$

On the other hand, (1) has a global solution provided (f, g) satisfies that

$$(1+|x|)^{1+\kappa}\left(\frac{|f(x)|}{1+|x|} + \sum_{0<|\alpha|\leq[n/2]+2} |\nabla_x^\alpha f(x)| + \sum_{|\beta|\leq[n/2]+1} |\nabla_x^\beta g(x)|\right) \tag{10}$$

with $\kappa \geq \kappa_0$ and $p > p_0(n)$ is sufficiently small.

In high dimensions we have to treat radially symmetric solutions, so that there is no loss of the decay of the initial data by the representation formula of the solution. Hence the existence part has been already obtained in [9, 10, 11]. In low dimensions, a non-symmetric assumption may make the loss of the decay, but easy modifications of [2, 12, 19, 20, 21] overcome it. See [18]. Therefore we summarize the results only on the blow-up part in the following table according to the combinations of the assumptions.

assumptions	(6)	(7)
(8)	[17]	[18]
(9)	[13]	this paper

Finally, we note that the assumption (7) in some special case can be rewritten in the form of

$$f(x) \geq \frac{\phi(|x|)}{(1+|x|)^{\kappa}}, \quad g(x) \equiv 0. \tag{11}$$

See [18] for details.

2. **Sharp blow-up theorem.**

For unknown functions $u = u(r,t)$, $r \in (0,\infty)$, $t \in [0,\infty)$, we consider the following radially symmetric version of (1).

$$\begin{cases} u_{tt} - \dfrac{n-1}{r}u_r - u_{rr} = F(u) & \text{in } (0,\infty)\times[0,\infty), \\ u(r,0) = f(r),\ u_t(r,0) = 0 & \text{for } r \in (0,\infty), \end{cases} \tag{12}$$

where we assume that $F \in C^1(\mathbf{R})$, $f(x) = f(|x|) \in C^3(0,\infty)$ and ϕ satisfy (7) with (9), and F' satisfies

$$F'(s) \geq pAs^{p-1} \text{ for } s \geq 0 \tag{13}$$

with $p > 1$ and $A > 0$.

Then we have the following theorem.

Theorem 2.1. *Let u be a C^3-solution of (12). Suppose that (13) is fulfilled. Then u cannot exist globally in time.*

Remark 1. As in [18], the simple example can be constructed by $F(u) = |u|^p$ or $|u|^{p-1}u$ with $p > 1$ and

$$f(r) = \frac{\log(1+r)}{r^{\kappa_0}}.$$

For this example, R in the assumption (7) is directly computed.

Remark 2. The same result for C^2-solution is available under the stronger assumption on f. See Sections 6 and 7 in [18] for details.

3. **Positive C^3 solution.** In order to prove Theorem 2.1, we need positivity of a solution of (12).

Lemma 3.1. *Assume that there exists a positive constant R such that $F \in C^1(\mathbf{R})$ and $f \in C^3(0,\infty)$ satisfy*

$$\begin{cases} F'(s) \geq 0 \ \textit{for} \ s \geq 0 \ \textit{and} \ f(r) > 0, \\ f''(r) + \dfrac{n-1}{r} f'(r) + F(f(r)) > 0 \ \textit{for} \ r \in [R,\infty). \end{cases} \tag{14}$$

Then there is a positive constant $\delta = \delta(n)$ such that a C^3-solution u of (12) satisfies

$$u_t > 0 \quad \textit{in} \ \Sigma = \left\{(r,t) \in (0,\infty)^2 \ : \ r - t \geq \max\{R, \delta t\} > 0\right\} \tag{15}$$

as far as u exists. Moreover, u in Σ satisfies

$$\begin{aligned} u_t(r,t) \geq \ & \frac{1}{8r^m}\int_{r-t}^{r+t} \lambda^m u_{tt}(\lambda,0)d\lambda \\ & + \frac{1}{8r^m}\int_0^t d\tau \int_{r-t+\tau}^{r+t-\tau} \lambda^m F'(u(\lambda,\tau))u_t(\lambda,\tau)d\lambda d\tau, \end{aligned} \tag{16}$$

where m is an integer part of $n/2$, namely $m = [n/2]$.

This is Lemma 4.1 in [18].

4. **Blow-up of C^3 solution.** Let u be a global in time C^3-solution of (12). We will see that this is false under the assumption of Theorem 2.1 by following up the basic iteration argument by John [6].

The assumption (13) on F and f enable us to make use of Lemma 3.1. Hence, cutting the domain of the integral, we have

$$u_t(r,t) \geq \frac{1}{8r^m}\int_r^{r+t} \lambda^m \frac{\phi(\lambda)}{(1+\lambda)^{\kappa_0+2}} d\lambda \geq \frac{t\phi(r)}{8(1+r+t)^{\kappa_0+2}}$$

in Σ by monotonicity of ϕ. This is the first step of our iteration.

First we assume that u_t has an estimate

$$u_t(r,t) \geq \frac{ct^a\phi(r)^d}{(1+r+t)^b} \quad \text{in } \Sigma, \tag{17}$$

where all a, b, c are positive constants. This is true with $a = 1, b = l, c = 1/8, d = 1$ as we see. Integrating this inequality with respect to t, we obtain, by $u(r,0) = f(r) > 0$, that

$$u(r,t) \geq \frac{ct^{a+1}\phi(r)^d}{(a+1)(1+r+t)^b} \quad \text{in } \Sigma. \tag{18}$$

Then we can put (17) and (18) into the second term in the right hand side of (16) because its domain of the integral is included in Σ. Hence, neglecting the first term by positivity, we have by (13) that

$$\begin{aligned}
&u_t(r,t)\\
&\geq \frac{pA}{8r^m}\int_0^t d\tau \int_r^{r+t-\tau}\lambda^m\left(\frac{c\tau^{a+1}\phi(\lambda)^d}{(a+1)(1+\lambda+\tau)^b}\right)^{p-1}\frac{c\tau^a\phi(\lambda)^d}{(1+\lambda+\tau)^b}d\lambda\\
&\geq \frac{pAc^p}{8(a+1)^{p-1}r^m(1+r+t)^{pb}}\int_0^t \tau^{p(a+1)-1}d\tau\int_r^{r+t-\tau}\lambda^m\phi(\lambda)^{pd}d\lambda\\
&\geq \frac{pAc^p\phi(r)^{pd}}{8(a+1)^{p-1}(1+r+t)^{pb}}\int_0^t \tau^{p(a+1)-1}(t-\tau)d\tau.
\end{aligned}$$

That is

$$u_t(r,t)\geq \frac{Ac^p}{8(a+1)^p\{p(a+1)+1\}}\cdot\frac{t^{p(a+1)+1}\phi(r)^{pd}}{(1+r+t)^{pb}}\quad \text{in } \Sigma. \tag{19}$$

In order to repeat this procedure infinitely many times, one should compare (17) with (19) and define sequences $\{a_j\},\{b_j\},\{c_j\},\{d_j\}$ by

$$\begin{aligned}
&a_j = p(a_{j-1}+1)+1,\ a_0=1,\\
&b_j = pb_{j-1},\ b_0=\kappa_0+2,\\
&c_j = \frac{Ac_{j-1}^p}{8(a_{j-1}+1)^p\{p(a_{j-1}+1)+1\}},\ c_0=\frac{1}{8},\\
&d_j = pd_{j-1},\ d_0=1.
\end{aligned}$$

Therefore we have

$$a_j=\left(1+\frac{p+1}{p-1}\right)p^j-\frac{p+1}{p-1},\ b_j=(\kappa_0+2)p^j, d_j=p^j. \tag{20}$$

This implies

$$c_j>\frac{Bc_{j-1}^p}{p^{(p+1)j}},\ \text{where } B=\frac{A}{8p}\left(\frac{p-1}{2p}\right)^{p+1}>0.$$

So one can get inductively

$$c_j>B^{(p^j-1)/(p-1)}\frac{c_0^{p^j}}{p^{(p+1)s_j}},\ \text{where } s_j=p^j\sum_{k=1}^{j}\frac{k}{p^k}. \tag{21}$$

Summing up (17), (20) and (21), we obtain

$$u_t(r,t)>B^{-1/(p-1)}t^{-(p+1)/(p-1)}\exp\left(p^jK(r,t)\right)\quad \text{in } \Sigma,$$

where

$$\begin{aligned}
K(r,t)= \quad &\log\left(B^{1/(p-1)}c_0\phi(r)\right)-(p+1)\sum_{k=1}^{\infty}\frac{k}{p^k}\log p\\
&+\left(1+\frac{p+1}{p-1}\right)\log t-(\kappa_0+2)\log(1+r+t).
\end{aligned} \tag{22}$$

It is easy to find a point $(r_0,t_0)\in\Sigma$ such that $K(r_0,t_0)>0$ because of the monotonicity of ϕ and $\kappa_0+2=2p/(p-1)$. Therefore, letting $j\to\infty$, we get a contradiction $u_t(r_0,t_0)\to\infty$. The proof is now completed.

REFERENCES

[1] R.Agemi and H.Takamura, *The lifespan of classical solutions to nonlinear wave equations in two space dimensions*, Hokkaido Math. J., **21**(1992), 517-542.

[2] F.Asakura, *Existence of a global solution to a semi-linear wave equation with slowly decaying initial data in three space dimensions*, Comm. in Partial Differential Equations, **11(13)**(1986), 1459-1487.

[3] V.Georgiev, H.Lindblad and C.Sogge, *Weighted Strichartz estimates and global existence for semilinear wave equations*, Amer. J. Math., **119**(1997), 1291-1319.

[4] R.Glassey, *Finite-time blow-up for solutions of nonlinear wave equations*, Math. Z., **177**(1981), 323-340.

[5] R.Glassey, *Existence in the large for $\Box u = f(u)$ in two space dimensions*, Math. Z, **178**(1981), 233-261.

[6] F.John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math., **28**(1979), 235-268.

[7] F.John, "Nonlinear Wave Equations, Formation of Singularities", Pitcher Lectures in Mathematical Sciences, Lehigh University, AMS, 1990.

[8] J.B.Keller, *On solutions of nonlinear wave equations*, Comm. Pure and Appl. Math., **X**(1957), 523-530.

[9] H.Kubo, *On the critical decay and power for semilinear wave equations in odd space dimensions*, Discrete Contin. Dynam. Systems, **2**(1996), 173-190.

[10] H.Kubo and K.Kubota, *Asymptotic behavior of radially symmetric solutions of $\Box u = |u|^p$ for super critical values p in odd space dimensions*, Hokkaido Math. J., **24**(1995), 287-336.

[11] H.Kubo and K.Kubota, *Asymptotic behavior of radially symmetric solutions of $\Box u = |u|^p$ for super critical values p in even space dimensions*, Japan J. Math., **24**(1998), 191-256.

[12] K.Kubota, *Existence of a global solution to a semi-linear wave equation with initial data of non-compact support in low space dimensions*, Hokkaido Math. J., **22**(1993), 123-180.

[13] Y.Kurokawa and H.Takamura, *Blow-up for semilinear wave equations with a data of the critical decay having a small loss*, Rend. Instit. Mat. Trieste, **XXXV**(2003), 165-193.

[14] J.Schaeffer, *The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p*, Proc. Roy. Soc. Edinburgh, **101A**(1985), 31-44.

[15] T.C.Sideris, *Nonexistence of global solutions to semilinear wave equations in high dimensions*, J. Differential Equations, **52**(1984), 378-406.

[16] W.A.Strauss, "Nonlinear Wave equations", Conference board of the mathematical sciences, Regional conference series in mathematics 73, AMS, 1989.

[17] H.Takamura, *Blow-up for semilinear wave equations with slowly decaying data in high dimensions*, Differential and Integral Equations, **8**(1995), 647-661.

[18] H.Takamura, H.Uesaka and K.Wakasa, *Blow-up theorem for semilinear wave equations with non-zero initial position*, J. Differential Equations, **249**(2010), 914-930, doi:10.1016/j.jde.2010.01.010.

[19] K.Tsutaya, *A global existence theorem for semilinear wave equations with data of non compact support in two space dimensions*, Comm. Partial Differential Equations, **17(11**&**12)**(1992), 1925-1954.

[20] K.Tsutaya, *Global existence theorem for semilinear wave equations with non-compact data in two space dimensions*, J. Differential Equations, **104**(1993), 332-360.

[21] K.Tsutaya, *Global existence and the lifespan of solutions of semilinear wave equations with data of non compact support in three space dimensions*, Funkcialaj Ekvacioj, **37**(1994), 1-18.

[22] H.Uesaka, *Non-negative solutions of the Cauchy problem for semilinear wave equations and non-existence of global non-negative solutions*, More Progress in Analysis (ed. H.G.W.Begehr and F.Nicolosi), Proceedings of the 5th International ISAAC Congress (Catania, Italy, 25-30 July 2005), 571-579, World Scientific, 2009.

[23] B.Yordanov and Q.S.Zhang, *Finite time blow up for critical wave equations in high dimensions*, J. Funct. Anal., **231**(2006), 361-374.

[24] Y.Zhou, *Blow up of solutions to semilinear wave equations with critical exponent in high dimensions*, Chin. Ann. Math. Ser. B, **28**(2007), 205-212.

Received xxxx 2010.

E-mail address: takamura@fun.ac.jp
E-mail address: uesaka@math.cst.nihon-u.ac.jp
E-mail address: b1007206@fun.ac.jp