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Author(s)	Kawaharada, Akane
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# Estimating topological entropy of multidimensional nonlinear cellular automata

Akane Kawaharada (Hokkaido University)\*

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## Abstract

Cellular automata are discrete dynamical systems whose configurations are determined by local rules acting on each cell in synchronous. Topological entropy is a tool for measuring the complexity of these dynamical systems. In this paper we estimate topological entropy of a two-dimensional nonlinear cellular automaton. The method we use is that a one-dimensional cellular automaton with positive topological entropy is “naturally” embedded into the two-dimensional cellular automaton. Hence we obtain a multidimensional cellular automaton with infinite topological entropy.

## 1. Introduction

Cellular automata (CA) are discrete dynamical systems whose configurations are determined by local rules acting on each single cell in synchronous. Von Neumann [18] used them as models of self-reproductions and Ulam [17] studied discrete universal models consisting of two-dimensional meshes of finite state machines. The famous CA are Conway’s Game of life by Gardner [13] and elementary CA by Wolfram [20]. Elementary CA are one-dimensional CA with states 0, 1 and the local rule defined by the nearest three neighborhood. Wolfram performed computer simulations and classified elementary CA into four classes according to the space-time diagrams of them [21]. CA can generate rich and complex behaviors, so often are used as models for complex systems in biology (tumor growth, excitement of muscular tissue), physics (spin glass systems, reaction-diffusion process, various phenomena in turbulence theory), computer science (pseudo-random number generation, image processing, cryptography) and so on. On the other hand, mathematical theory of CA was developed after Hedlund’s work [6] using symbolic dynamics. Cattaneo et al. [3] and Manzini and Margara [12], [11] investigate the topological behavior of CA and Blanchard et al. [2] study measure-theoretic properties of CA.

Topological entropy is a tool for measuring the complexity of dynamical systems. If a CA is equicontinuous then the behavior is simple and topological entropy is zero. For one-dimensional CA topological entropy are always zero or finite. In particular if expansiveness is satisfied then topological entropy is nonzero finite [4]. On the other hand, all multidimensional CA are non-expansive [16]. Moreover Hurd et al. [7] proved that topological entropy of one-dimensional CA cannot be algorithmically computed. For multidimensional ‘linear’ CA Morris and Ward [15] and D’amico et al. [4] proved that multidimensional linear CA

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\*e-mail: [aka@math.sci.hokudai.ac.jp](mailto:aka@math.sci.hokudai.ac.jp)

web: <http://www.math.sci.hokudai.ac.jp/~aka/>

are either equicontinuous in which case topological entropy is zero, or sensitive in which case topological entropy is infinity. Finally for multidimensional ‘non-linear’ CA there are a few results about topological entropy. Lakshtanov and Langvagen [10] showed that topological entropy of CA with a spaceship (a finite pattern, which reappears after some iterations in a different position) is infinity. Meyerovitch [14] insisted that there exists a multidimensional CA with nonzero finite topological entropy. An important tool in constructing this CA is a certain aperiodic set of tiles, associated to a substitution system for  $D = 2$  case, which is a generalization of Kari’s CA [8]. However Meyerovitch’s CA does not satisfy the usual definition of CA.

The main purpose of this paper is to show a new example of a multidimensional nonlinear CA  $(X, F)$  with infinite topological entropy. For this purpose we shall find a subsystem which is a one-dimensional CA  $(Y, G)$  with positive topological entropy.

This paper is organized as follows. In section 2 we give definitions of CA, some topological dynamics and measure-theoretical dynamical systems. In section 3 we show a relation between  $(X, F)$  and  $(Y, G)$ . We estimate topological entropy of  $(X, F)$ .

## 2. Preliminaries

### 2.1. Topological dynamical systems

Let  $(X, T)$  be a discrete dynamical system consisting of a compact metric space  $X$  with metric  $d$  and a continuous map  $T : X \rightarrow X$ . The  $n$ -th *iteration* of  $T$  is defined by  $T^n$ , so  $T^0$  is the identity map  $Id$  on  $X$ . If  $T$  is surjective,  $(X, T)$  is called an *endomorphism*. Let  $(Y, S)$  be an another dynamical system. A map  $\varphi : (X, T) \rightarrow (Y, S)$  is a *homeomorphism*, if  $\varphi$  is a bijective continuous map and  $\varphi^{-1}$  is also continuous. A map  $\varphi$  is a *conjugacy*, if  $\varphi$  is a homeomorphism such that  $\varphi \circ T = S \circ \varphi$ . Two topological dynamical systems  $(X, T)$  and  $(Y, S)$  are *conjugate*, if there exists a conjugacy map. A set  $A \subset X$  is *invariant* if  $T(A) \subset X$  and  $A$  is *strongly invariant* if  $T(A) = A$ . If  $A$  is invariant and closed,  $(T, A)$  is called a *subsystem* of  $(T, X)$ .

A point  $x \in X$  is a *periodic point* with period  $n \geq 1$  if  $T^n(x) = x$ . In particular if  $T(x) = x$ ,  $x$  is a *fixed point*. A point  $x \in X$  is a *eventually periodic point* if for some  $k \geq 0$   $T^k(x)$  is a periodic point.

**Definition 2.1.** A point  $x \in X$  is an *equicontinuous point* of  $(X, T)$ , if for all  $\epsilon > 0$  there exists  $\delta$  such that whenever  $y \in X$  satisfies  $d(x, y) < \delta$  then  $d(T^n x, T^n y) < \epsilon$  for all  $n > 0$ .

We consider some topological properties of dynamical systems.

**Definition 2.2.** (1) A dynamical system  $(X, T)$  is called *equicontinuous*, if every point of  $(X, T)$  is an equicontinuous point.

(2) A dynamical system  $(X, T)$  is *sensitive dependence on initial conditions*, if there exists  $\epsilon > 0$  such that for all  $x \in X$  and  $\delta > 0$ , there exists  $y$  with

$d(x, y) < \delta$  and  $d(T^n x, T^n y) \geq \epsilon$  for some  $n \geq 0$ . We will refer to this property simply as *sensitive*.

- (3) A dynamical system  $(X, T)$  is *positively expansive*, if there exists  $\epsilon > 0$  such that for all  $x \neq y \in X$ ,  $d(T^n x, T^n y) \geq \epsilon$  for some  $n \geq 0$ .

We define topological entropy. It is a tool for measuring the randomness of dynamical systems introduced by Adler et al. in 1965 [1]. Let  $\mathcal{U}$  be an open cover of  $X$ .  $H(\mathcal{U}) = \inf \log \text{card}(\tilde{\mathcal{U}})$ , where the infimum is taken over the set of finite subcovers  $\tilde{\mathcal{U}}$  of  $\mathcal{U}$ . Let  $\mathcal{U}$  and  $\mathcal{V}$  be covers of  $X$ . We denote by  $\mathcal{U} \vee \mathcal{V}$  the cover made up of all intersections  $U \cap V$ , where  $U \in \mathcal{U}$  and  $V \in \mathcal{V}$ . *Topological entropy of the cover  $\mathcal{U}$ ,*

$$h(\mathcal{U}, T) = \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right).$$

**Definition 2.3.** *Topological entropy of  $(X, T)$  is defined by*

$$h_{\text{top}}(X, T) = \sup_{\mathcal{U}} h(\mathcal{U}, T),$$

where the supremum is taken over all finite open covers of  $X$ .

Both topological entropy of the cover  $\mathcal{U}$  and topological entropy of  $(X, T)$  are nonnegative. If two dynamical systems are topologically conjugate then topological entropy of them are same. But the inverse is not always true.

## 2.2. Measure-theoretical dynamics

We give a *measure preserving dynamical system*  $(X, \mathcal{B}, \mu, T)$ . It consists of a probability space  $(X, \mathcal{B}, \mu)$  and a collection of  $\mathcal{B}$ -measurable actions  $T$  which leaves the measure  $\mu$  invariant. An *invariant measure*  $\mu$  on  $(X, \mathcal{B}, \mu, T)$  is one such that  $T\mu = \mu = T^{-1}\mu$ .

We consider some entropies. Let  $\alpha = \{A_i \mid i \in I\}$  be a collection of measurable subsets of  $X$ . The collection  $\alpha$  is a  $\mu$ -*partition* of  $X$ , if  $\mu(A_i \cap A_j) = 0$  for  $i \neq j$ ,  $\mu(X \setminus \bigcup_{i \in I} A_i) = 0$  and  $\mu(A_i) > 0$  for all  $i \in I$ . For a finite or countable  $I$  the collection  $\alpha$  is a *partition* of  $X$ ,  $A_i \cap A_j = \emptyset$  for  $i \neq j$  and  $\bigcup_{i \in I} A_i = X$ . Let  $\beta$  be an another  $\mu$ -partition of  $X$ . The *common refinement* of  $\alpha$  and  $\beta$  is the  $\mu$ -partition  $\alpha \vee \beta = \{A \cap B \mid A \in \alpha, B \in \beta, \mu(A \cap B) > 0\}$ .

**Definition 2.4.** (1) The *entropy of  $\alpha$*  is defined by

$$H(\alpha) = - \sum_{A \in \alpha} \mu(A) \log \mu(A).$$

(2) The *metrical entropy of  $(X, \mathcal{B}, \mu, T)$  relative to  $\alpha$*  is defined by

$$h(\alpha, T) = \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right).$$

(3) The *entropy* of  $(X, \mathcal{B}, \mu, T)$  is defined by

$$h_\mu(T) = \sup_{\alpha \in \mathcal{P}} h(\alpha, T),$$

where  $\mathcal{P}$  is the set of all finite measurable partitions of  $X$ .

A partition  $\alpha$  of  $X$  is a *generating partition* for invertible  $T$  if  $H(\alpha) < \infty$  and  $\bigvee_{k=-\infty}^{\infty} T^k \alpha = \mathcal{B}$ .

**Proposition 2.1** (Kolmogorov-Sinai theorem [19]). If  $\alpha$  is a generating partition for  $(X, \mathcal{B}, \mu, T)$ , then  $h_\mu(T) = h(\alpha, T)$ .

In order to distinguish the entropy  $h_\mu(T)$  from other entropies this quantity is often called *Kolmogorov-Sinai entropy*.

**Proposition 2.2** (Variational Principle [5]). For topological dynamical system  $(X, T)$  and measure preserving dynamical system  $(X, \mathcal{B}, \mu, T)$

$$h_{top}(T) = \sup_{\mu \in \mathcal{M}(T)} h_\mu(T),$$

where  $\mathcal{M}(T)$  is the set of all  $T$ -invariant Borel probability measures on  $X$ .

A measure  $\mu$  for which  $h_{top}(T) = h_\mu(T)$  is called a *maximal measure*.

### 2.3. Symbolic dynamics and cellular automata

Let  $A = \{0, \dots, n-1\}$  ( $n \geq 1$ ) be a finite state set. The two-sided sequence space is defined by  $\Sigma = A^{\mathbb{Z}}$ . The element is denoted by  $x = (\dots, x_{-1}, x_0, x_1, x_2, \dots) \in \Sigma$  where each  $x_i \in A$ . We define metric  $d_1$  on  $\Sigma$ ,

$$d_1(x, y) = 2^{-k(x,y)}, \quad \text{where } k(x, y) = \min\{|i| \mid x_i \neq y_i\} \quad \text{for } x, y \in \Sigma.$$

The space  $\Sigma$  is compact, perfect and totally disconnected. A *cylinder set* is defined by

$$[a_0 \cdots a_l]_t^{t+l} = \{x \mid a_0 = x_t, \dots, a_l = x_{t+l}\}, \quad \text{where } t \in \mathbb{Z}, l \in \mathbb{N}.$$

Every cylinder set is closed and open, and every closed and open set is a finite union of cylinders. The *shift* map  $\sigma : \Sigma \rightarrow \Sigma$  is defined by  $\sigma(x)_i = x_{i+1}$ . The map is a homeomorphism and the dynamical system  $(\Sigma, \sigma)$  is called the *full shift*. If  $\Gamma$  is a closed subset of  $\Sigma$  with  $\sigma(\Gamma) = \Gamma$ , the dynamical system  $(\Gamma, \sigma)$  is called the *subshift*. Let  $(A, \tilde{\mathcal{B}}, m)$  be a probability space and  $(\Sigma, \mathcal{B}, \mu) = \prod_{-\infty}^{\infty} (A, \tilde{\mathcal{B}}, m)$ . Then the shift map  $\sigma$  on  $\Sigma$  is a measure preserving transformation and called the *Bernoulli shift* with  $(A, \tilde{\mathcal{B}}, m)$ . In particular a  $(\frac{1}{n}, \dots, \frac{1}{n})$ -shift with a probability vector  $(\frac{1}{n}, \dots, \frac{1}{n})$  is called *n-Bernoulli shift*.

**Proposition 2.3** (Walters [19]). Let  $\sigma$  be a  $(p_0, \dots, p_{n-1})$ -shift. Then the entropy

$$h_\mu(\sigma) = - \sum_{i=0}^{n-1} p_i \log p_i.$$

Let  $A$  be a finite *state set*,  $\{0, \dots, n-1\}$ . A *word* is any finite sequence from  $A$ ,  $x = [x_m, x_{m+1}, \dots, x_{m+l-1}]$  for any  $m \in \mathbb{Z}$  and  $l \in \mathbb{N}$ . A *pattern* is a  $D$ -dimensional generalized word, a finite connected set of states from  $A$  on a subset of  $\mathbb{Z}^D$ . For example the following pattern is a two-dimensional pattern of size  $(l+1) \times (m+1)$  for some coordinate  $(i, j)$ .

$$x = \begin{bmatrix} x_{(i,j)} & x_{(i+1,j)} & \cdots & x_{(i+l,j)} \\ x_{(i,j+1)} & x_{(i+1,j+1)} & \cdots & x_{(i+l,j+1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(i,j+m)} & x_{(i+1,j+m)} & \cdots & x_{(i+l,j+m)} \end{bmatrix}.$$

A  $D$ -dimensional *configuration space* (full shift space)  $X$ , is defined by  $A^{\mathbb{Z}^D}$  and each element  $x \in X$  is called a *configuration*. For  $x \in A^{\mathbb{Z}^D}$  and a subset  $\Gamma \in \mathbb{Z}^D$ ,  $x|_{\Gamma}$  is a pattern restricting  $x$  to the coordinates  $\Gamma$ . A subset  $X|_{\Gamma}$  of  $A^{\mathbb{Z}^D}$  is a set of points such that  $x|_{\Gamma}$ .

**Definition 2.5.** Let  $v_1, \dots, v_t \in \mathbb{Z}^D$  be finite pairwise distinct vectors. Let  $T$  be a shift-commuting and continuous map on  $X$ . A CA map  $T : X \rightarrow X$  with *local rule*  $f : A^t \rightarrow A$  is defined by

$$(Tx)_{(i_1, \dots, i_D)} = f(x_{(i_1, \dots, i_D)+v_1}, \dots, x_{(i_1, \dots, i_D)+v_t}).$$

Then the discrete dynamical system  $(X, T)$  is a  $D$ -dimensional *cellular automaton* (CA) for  $D \geq 1$ .

We define a metric function  $d_D$  on  $X$  for  $x, y \in X$

$$d_D(x, y) = 2^{-k(x,y)},$$

where  $k(x, y) = \min\{\max\{|i_1|, \dots, |i_D|\} \mid x_{(i_1, \dots, i_D)} \neq y_{(i_1, \dots, i_D)}\}$ . Then  $(X, d_D)$  is a compact metric space which is perfect and totally disconnected.

The simplest nontrivial CA are *elementary cellular automata*. They are one-dimensional CA with two possible states, 0 or 1, for each cell and the local rules depend only on the nearest three neighbor cells. So there are  $2^3 = 8$  possible patterns. For example, Rule 150 is represented by 3-block maps,  $[111] \mapsto 1$ ,  $[110] \mapsto 0$ ,  $[101] \mapsto 0$ ,  $[100] \mapsto 1$ ,  $[011] \mapsto 0$ ,  $[010] \mapsto 1$ ,  $[001] \mapsto 1$  and  $[000] \mapsto 0$ . Then there exist  $2^8 = 256$  different elementary CA. The name for Rule 150, is from this constructing manner. List all results of the 3-block maps. Then 10010110 in binary notation is just 150 in the decimal notation.

There are classifications for one-dimensional CA according to the local behavior. In 1995 K urka remarked in [9] that a one-dimensional CA is sensitive if and only if it has no equicontinuous points, and accordingly introduced a purely topological classification.

**Proposition 2.4** ( K urka [9]). Any one-dimensional CA  $(A^{\mathbb{Z}}, T)$  belongs to exactly one of the following classes:

- (1)  $T$  is equicontinuous,

- (2)  $T$  has equicontinuous points but is not equicontinuous,
- (3)  $T$  is sensitive but not positively expansive and
- (4)  $T$  is positively expansive.

A linear CA is equicontinuous if and only if it has at least one equicontinuous point (Manzini [11]) and positively expansiveness is a property only for one-dimensional (Shereshevsky [16]). Therefore we get the classification.

**Corollary 2.1.** Any multidimensional linear CA  $(A^{\mathbb{Z}^D}, T)$  belongs to exactly one of the following classes:

- (1)  $T$  is equicontinuous,
- (2)  $T$  is sensitive.

### 3. Results

#### 3.1. The relation of two cellular automata

The main purpose of section 3.1 is to prove that a subsystem of a two-dimensional nonlinear CA  $(X, F)$  is topologically conjugate to a one-dimensional elementary CA  $(Y, G)$  which is called Rule 150. Figure 1 is the space-time diagram of  $\varphi \circ F$ , where  $\varphi$  is a map from a subset of  $X$  to  $Y$ . If a cell has 1 it is represented by a black dot, and if 0 by a white dot. The top row is an initial configuration and here we give the only center cell of the top row 1, and the others 0s. The space-time diagram of  $\varphi \circ F$  and that of  $G$  are the same, i.e., the two CA are the same. The detail can be found in Theorem 3.1. For a space-time diagram on  $\mathbb{Z}^2 \times \mathbb{N}$ , a vertical section is similar to that of  $G$ . Indeed we can show the vertical section is generated by twice iteration of  $G$ . For further information, see Proposition 3.1.

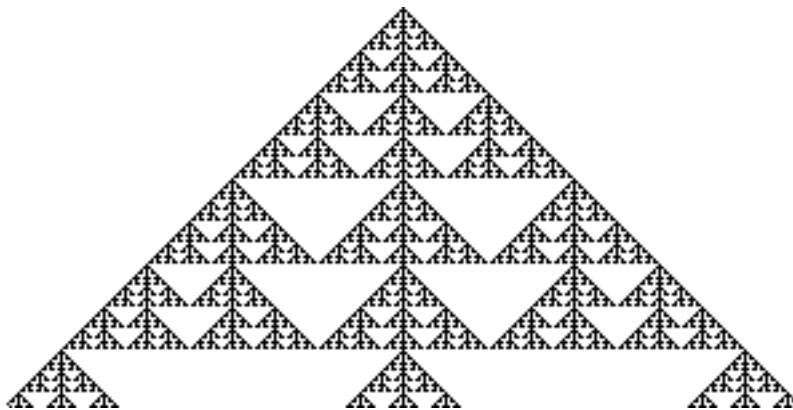


Figure 1: The space-time diagram of  $\varphi \circ F$  with the initial configuration consisting of a single 1 at the center of the top row (a black cell) and its surrounding 0s (white cells). In each step of time evolution, the result is drawn under the current row. Repeating these operations produce this interesting pattern. In fact it is the same as the space-time diagram of Rule 150.

Let  $X := \{0, 1\}^{\mathbb{Z}^2}$ . A two-dimensional nonlinear CA map,  $F: X \rightarrow X$  is defined by

$$(Fx)_{(i,j)} = x_{(i,j)} + x_{(i-1,j)}x_{(i+1,j)} + x_{(i,j-1)}x_{(i,j+1)} \pmod{2}.$$

This local rule means that the next state for a cell is determined by the states of the von Neumann neighborhood which is surrounding four cells and itself. We define a one-dimensional elementary CA  $(Y, G)$ . Let  $Y := \{0, 1\}^{\mathbb{Z}}$ . The CA map  $G: Y \rightarrow Y$  is defined by

$$(Gy)_l = y_l + y_{l-1} + y_{l+1} \pmod{2}.$$

The  $(Y, G)$  is called Rule 150 by Wolfram's numbering. Let  $\tilde{X}$  be a subset of  $X$ ,

$$\tilde{X} = \left\{ x \mid \begin{array}{l} x_{(i,j)} = 1 \text{ if } i+j = 0, 3 \\ x_{(i,j)} = 0 \text{ if } i+j = 4, 5, -1, -2 \end{array} \right\}$$

and let  $S := \{(i, j) \mid i+j = 1, 2\} \subset \mathbb{Z}^2$ . For the above two CA we can define a homeomorphism  $\varphi: \tilde{X}|_S \rightarrow Y$  as a conjugacy between  $(\tilde{X}|_S, F)$  and  $(Y, G)$ ,

$$y_l = (\varphi x)_l = \begin{cases} x_{(l,1-l)} & \text{if } l \text{ is odd,} \\ x_{(l,2-l)} & \text{if } l \text{ is even,} \end{cases}$$

for each  $l \in \mathbb{Z}$ . Each  $y_l$  is arranged in a zigzag path in  $\tilde{X}|_S$ .

**Theorem 3.1.** The CA  $(\tilde{X}|_S, F)$  and  $(Y, G)$  are topologically conjugate.

$$\begin{array}{ccc} \tilde{X}|_S & \xrightarrow{F} & \tilde{X}|_S \\ \varphi \downarrow & \circ & \downarrow \varphi \\ Y & \xrightarrow{G} & Y \end{array}$$

*Proof.* By the condition of  $\tilde{X}$  the set  $\tilde{X}|_S$  is strongly invariant for  $F$  and closed. Hence  $(\tilde{X}|_S, F)$  is a subsystem of  $(X, F)$ . To prove the above claim, it will be sufficient to say that  $\varphi$  commutes with  $F$  and  $G$ , and  $\varphi$  is continuous bijective map, that is to say a homeomorphism.

First we show  $\varphi \circ F = G \circ \varphi$ . The property of  $\tilde{X}$  implies if  $i+j = 1$  then  $(Fx)_{(i,j)} = x_{(i,j)} + x_{(i,j+1)} + x_{(i+1,j)}$  and if  $i+j = 2$  then  $(Fx)_{(i,j)} = x_{(i,j)} + x_{(i-1,j)} + x_{(i,j-1)}$ . For each  $l \in \mathbb{Z}$

$$(\varphi \circ F(x))_l = (Fx)_{(l,i+j-l)} = y_l + y_{l-1} + y_{l+1} = (Gy)_l = (G \circ \varphi(x))_l.$$

Next we show that  $\varphi$  is a homeomorphism between  $(\tilde{X}|_S, F)$  and  $(Y, G)$ . For the metric functions  $d_1$  on  $Y$  and  $d_2$  on  $\tilde{X}|_S$  we can see immediately

$$d_2(x, \tilde{x}) \leq d_1(\varphi(x), \varphi(\tilde{x})),$$

and if for any  $\epsilon$  we take  $\delta = \sqrt{\epsilon}$  such that  $d_2(x, \tilde{x}) \leq \delta$ , then

$$d_1(\varphi(x), \varphi(\tilde{x})) \leq \delta^2 = \epsilon.$$



So  $\varphi$  and  $\varphi^{-1}$  are continuous. For every  $y = (\dots, y_{l-1}, y_l, y_{l+1}, \dots) \in Y$ ,

$$y_l = (\varphi x)_l = \begin{cases} x_{(l,1-l)} & \text{if } l \text{ is odd,} \\ x_{(l,2-l)} & \text{if } l \text{ is even.} \end{cases}$$

If  $x \neq \tilde{x}$  for  $x, \tilde{x} \in X$ , we pick some  $l \in \mathbb{Z}$  such that  $x_{(l,m-l)} \neq \tilde{x}_{(l,m-l)}$  for  $m = 1, 2$ . Then

$$(\varphi x)_l = x_{(l,m-l)} \neq \tilde{x}_{(l,m-l)} = (\varphi \tilde{x})_l$$

Hence  $\varphi$  is bijective.

Therefore we arrive at the conclusion that  $\varphi$  is a conjugacy map between  $F$  on  $\tilde{X}|_S$  and  $G$  on  $Y$ .  $\square$

Since  $\varphi$  is a homeomorphism,  $\varphi^{-1}: Y \rightarrow \tilde{X}|_S$  can be represented by

$$(\varphi^{-1}y)_{(l,m-l)} = y_l \quad \text{if } m = 1, 2.$$

The actions of  $\varphi$  and  $\varphi^{-1}$  are regarded as just relabelling on  $X$  and  $Y$ . We consider new subsets of  $X$ . Let  $\tilde{X}_1$  and  $\tilde{X}_2$  be

$$\tilde{X}_1 = \left\{ x \mid \begin{array}{l} x_{(i,j)} = 1 \quad \text{if } i+j = 0, 3 \\ x_{(i,j)} = 0 \quad \text{if } i+j = 2, 4, 5, -1, -2 \end{array} \right\},$$

$$\tilde{X}_2 = \left\{ x \mid \begin{array}{l} x_{(i,j)} = 1 \quad \text{if } i+j = 0, 3 \\ x_{(i,j)} = 0 \quad \text{if } i+j = 1, 4, 5, -1, -2 \end{array} \right\}.$$

We define new conjugacy maps  $\varphi_m: \tilde{X}_m \rightarrow Y$  for  $m = 1, 2$ ,

$$y_l = (\varphi_m x)_l = x_{(l,m-l)} \quad \text{for each } l \in \mathbb{Z}.$$

Figures 2 and 3 are the space-time diagrams of  $\varphi_1 \circ F^2$  and  $\varphi_2 \circ F^2$  respectively and the right charts consisting of 0s and 1s are magnified the left structures. They are analogous to figure 1 with reason that  $F^2$  are the square root of  $F$ . Every other row keeps the CA  $(Y, G)$ . However figures 2 and 3 are subtly different. The pattern of  $\varphi_1 \circ F^2$  starts with a single 1 but  $\varphi_2 \circ F^2$  starts with a row of all 0s and virtually  $\varphi_2 \circ F^2$  starts with two 1s in the second row. In addition on figure 2 every odd row and the next even row are the same. On figure 3 there are rows generated by  $G$  and rows with all cells 0 alternately. Of course  $(\dots, 0, 0, 0, \dots)$  is invariant for  $G$ , so it can be considered acting on  $G$ .

Let  $S_m := \{(i, j) \mid i + j = m\} \subset \mathbb{Z}^2$  for  $m = 1, 2$ .

**Proposition 3.1.** The CA  $(\tilde{X}_m|_{S_m}, F^2)$  and  $(Y, G)$  are topologically conjugate for  $m = 1, 2$ .

$$\begin{array}{ccc} \tilde{X}_m|_{S_m} & \xrightarrow{F^2} & \tilde{X}_m|_{S_m} \\ \varphi_m \downarrow & \circlearrowleft & \downarrow \varphi_m \\ Y & \xrightarrow{G} & Y \end{array}$$

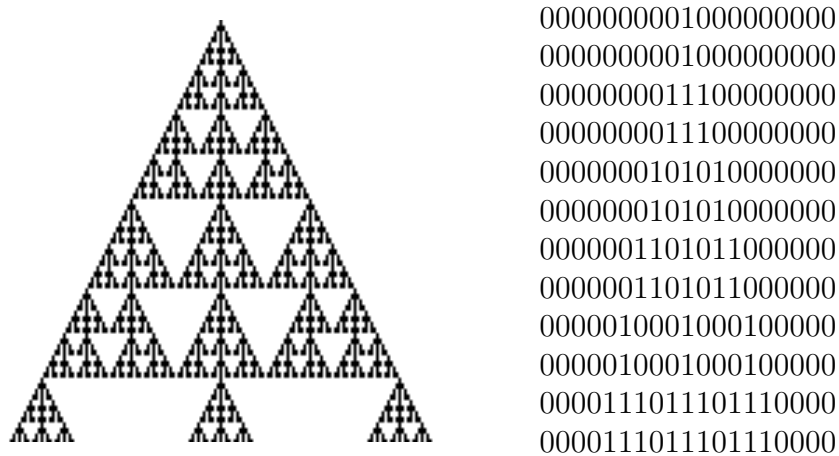


Figure 2: The space-time diagram of  $\varphi_1 \circ F^2$  with the initial configuration consisting a single 1 (a single black dot). Seeing the right chart (magnified the left pattern), we can find that every other row keeps Rule 150, besides an odd row and the next row are the same.

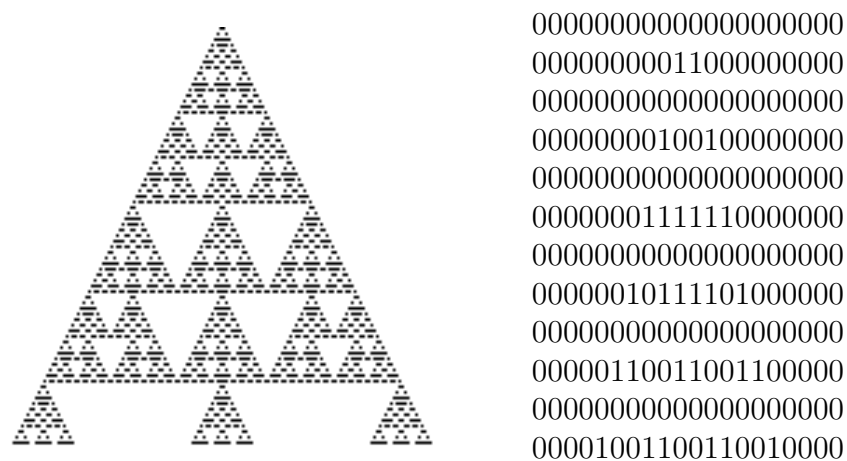


Figure 3: The space-time diagram of  $\varphi_2 \circ F^2$  with the initial configuration consisting of two 1s (two black dots). Seeing the right chart (magnified the left pattern), we find that every other row keeps Rule 150, besides the odd rows are all 0s.

*Proof.* We prove that a map  $\varphi_m$  is a conjugacy map of  $\tilde{X}_m|_{S_m}$  and  $Y$ . The set  $\tilde{X}_m|_{S_m}$  is strongly invariant for  $F$  and closed and hence  $(\tilde{X}_m|_{S_m}, F)$  is a subsystem of  $(X, F)$ . we show  $\varphi_m \circ F^2 = G \circ \varphi_m$ . Recall the definition of  $F$  and consider twice iteration

$$(F^2x)_{(i,j)} = (Fx)_{(i,j)} + (Fx)_{(i-1,j)}(Fx)_{(i+1,j)} + (Fx)_{(i,j-1)}(Fx)_{(i,j+1)} \pmod{2}.$$

In the case of  $m = 1$  for  $x \in \tilde{X}_1|_{S_1}$  and besides if  $i + j = 1$ ,

$$\begin{aligned} (F^2x)_{(i,j)} &= x_{(i,j)} + x_{(i,j)} + x_{(i,j)} + x_{(i+1,j-1)} + x_{(i-1,j+1)} \\ &= x_{(i,j)} + x_{(i-1,j+1)} + x_{(i+1,j-1)}, \end{aligned}$$

because of mod 2. If  $i + j \neq 1$ ,  $(F^2x)_{(i,j)} = x_{(i,j)}$ . According to the property of  $\tilde{X}_1|_{S_1}$ , we can see  $x_{(i,j)} = x_{(i-1,j+1)} = x_{(i+1,j-1)}$  when  $i + j \neq 1$ . So for all  $x_{(i,j)} \in x \in \tilde{X}_1|_{S_1}$ ,

$$(F^2x)_{(i,j)} = x_{(i,j)} + x_{(i-1,j+1)} + x_{(i+1,j-1)}.$$

In addition this, the case  $m = 2$  is the same above. Therefore in  $\tilde{X}_1|_{S_1}$  and  $\tilde{X}_2|_{S_2}$  for each  $l \in \mathbb{Z}$ , we have

$$\begin{aligned} (\varphi_m \circ F^2(x))_l &= (\varphi_m(F^2x))_l \\ &= (F^2x)_{(l,m-l)} \\ &= x_{(l,m-l)} + x_{(l-1,(m+1)-l)} + x_{(l+1,(m-1)-l)} \\ &= y_l + y_{l-1} + y_{l+1} \\ &= (Gy)_l \\ &= (G(\varphi_mx))_l \\ &= (G \circ \varphi_m(x))_l. \end{aligned}$$

Next we show that  $\varphi_m$  is a homeomorphism between  $F^2$  on  $\tilde{X}_m|_{S_m}$  and  $G$  on  $Y$ . Recall the metric functions  $d_2$  on  $\tilde{X}_m|_{S_m}$  and  $d_1$  on  $Y$ . We can see immediately

$$d_2(x, \tilde{x}) \leq d_1(\varphi_m(x), \varphi_m(\tilde{x})),$$

and if for any  $\epsilon$  we take  $\delta = \sqrt{\epsilon}$  such that  $d_2(x, \tilde{x}) \leq \delta$ , then

$$d_1(\varphi_m(x), \varphi_m(\tilde{x})) \leq \delta^2 = \epsilon.$$

Hence  $\varphi_m$  and  $\varphi_m^{-1}$  are continuous. For any  $y \in Y$

$$y_l = x_{(l,m-l)} = (\varphi_mx)_l \quad \text{for each } l \in \mathbb{Z}.$$

If  $x \neq \tilde{x}$  for  $x, \tilde{x} \in X_m$ , there exists  $l \in \mathbb{Z}$  such that  $x_{(l,m-l)} \neq \tilde{x}_{(l,m-l)}$ . Then

$$(\varphi_mx)_l = x_{(l,m-l)} \neq \tilde{x}_{(l,m-l)} = (\varphi_m\tilde{x})_l.$$

Hence  $\varphi_m$  is bijective, thereby completing the proof.  $\square$

Proposition 3.1 explains  $\varphi_m^{-1}: Y \rightarrow X_m$  for  $m = 1, 2$  exists, because  $\varphi_m$  is a homeomorphism.

### 3.2. Topological entropy of cellular automata

We discuss topological entropy of CA which is a tool for measuring the complexity of dynamical systems. Every equicontinuous CA has zero topological entropy in any dimension (see Remark 3.1). When dimension  $D = 1$  topological entropy is always finite. Moreover D'amico et al. [4] showed that expansive cellular automata have nonzero finite topological entropy. In case  $D \geq 2$  Morris and Ward [15] and D'amico et al. [4] proved that topological entropy of any linear CA is either zero or infinity. Lakshtanov and Langvagen [10] showed that if there exists a spaceship, then topological entropy is infinity. In 2008 Meyerovitch [14] showed that there exists a multidimensional CA with nonzero finite topological entropy, but this does not satisfy the usual definition of CA.

Now we prove that the multidimensional nonlinear CA  $(X, F)$  has infinite topological entropy by using the relation between  $(X, F)$  and  $(Y, G)$  in Theorem 3.1.

Table 1: Topological entropy  $h_{top}$  of cellular automata

$h_{top}$	0	finite	$\infty$
$D = 1$	equicontinuous [1]	expansive [4]	$\phi$
$D \geq 2$ &linear	equicontinuous [1],[15]	$\phi$ [15]	sensitive [15]
$D \geq 2$ &nonlinear	equicontinuous [1]	(Meyerovitch's CA [14])	spaceship [10]

**Remark 3.1.** Topological entropy of a CA  $(X, T)$  is zero if it is equicontinuous. The map  $T$  is an isometry with respect to  $\tilde{d}_D$  defined by  $\sup_{-\infty < n < \infty} d_D(F^n x, F^n y)$  which is equivalent to  $d_D$ . Let  $\beta$  be a partition consisting of  $D$ -dimensional cubes, and then  $\beta$  is generating under  $T$ . Therefore topological entropy of  $(X, T)$  is zero. (see [1] for detail)

**Proposition 3.2.** The two-dimensional nonlinear CA  $(F, X)$  has both an equicontinuous point and a non-equicontinuous point. In particular all the points of  $\tilde{X}(\subset X)$  are non-equicontinuous points.

*Proof.* First we show there exists an equicontinuous point  $x$ . Let  $x$  be a configuration of all  $x_{(i,j)} = 0$  for every  $(i, j) \in \mathbb{Z}^2$ . We know that a 0s pattern of the size  $2 \times 2$ ,  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  is the smallest blocking pattern of this CA  $(F, X)$ . It is obvious if  $d_2(x, \tilde{x}) < 2^{-k(x, \tilde{x})}$ , then  $\tilde{x}$  has an blocking pattern consisting of  $k \times k$  0s. Hence for all  $n \in \mathbb{N}$   $d_2(F^n x, F^n \tilde{x}) < 2^{-k(x, \tilde{x})}$  is valid. Therefore  $x$  is an equicontinuous point. Next let  $y$  be a configuration of all  $y_{(i,j)} = 1$  for every  $(i, j) \in \mathbb{Z}^2$ .

Since  $d_2(y, \tilde{y}) < 2^{-k(y, \tilde{y})}$  implies  $d_2(Fy, F\tilde{y}) < 2^{-k(y, \tilde{y})+1}$ ,  $y$  is a non-equicontinuous point. For every  $z \in \tilde{X}$  there exists  $\tilde{z} (\neq z)$  such that  $d_2(z, \tilde{z}) < 2^{-k(z, \tilde{z})}$ . Because  $z_{(l, m-l)} \neq \tilde{z}_{(l, m-l)}$  for  $m = 1$  or  $2$ , we have  $d_2(F^n z, F^n \tilde{z}) = 1$  for some  $n \in \mathbb{N}$ . We complete the proof.  $\square$

**Remark 3.2.** K urka [9] proved that for one-dimensional CA if there exists an equicontinuous point then the set of equicontinuous points is a residual set. For our two-dimensional CA  $(X, F)$  there exists an equicontinuous point but the set of equicontinuous points does not seem to be a residual set.

In order to prove our main theorem we prepare two lemmas. The first lemma allows us to calculate topological entropy of  $(Y, G)$ . The second lemma gives a sufficient condition for CA having infinite topological entropy.

**Lemma 3.1.** For the one-dimensional elementary CA  $(Y, G)$   $h_{top}(G) = 2 \log 2$ .

*Proof.* Let  $(Y, \mathcal{B}, \mu, G)$  be a measure preserving dynamical system with Bernoulli measure  $\mu$ . Take a finite partition  $\alpha = \{[00]_0^1, [01]_0^1, [10]_0^1, [11]_0^1\}$ . Inverse images of four elements are

$$\begin{aligned} G^{-1}([00]_0^1) &= [0000]_{-1}^2 \cup [1011]_{-1}^2 \cup [1101]_{-1}^2 \cup [0110]_{-1}^2, \\ G^{-1}([01]_0^1) &= [0001]_{-1}^2 \cup [1010]_{-1}^2 \cup [1100]_{-1}^2 \cup [0111]_{-1}^2, \\ G^{-1}([10]_0^1) &= [1000]_{-1}^2 \cup [0011]_{-1}^2 \cup [0101]_{-1}^2 \cup [1110]_{-1}^2, \\ G^{-1}([11]_0^1) &= [1001]_{-1}^2 \cup [0010]_{-1}^2 \cup [0100]_{-1}^2 \cup [1111]_{-1}^2. \end{aligned}$$

The refinement of  $\alpha$  and  $G^{-1}\alpha$  form all varieties of  $[a_{-1}a_0a_1a_2]_{-1}^2$ ,

$$\alpha \vee G^{-1}\alpha = \{[a_{-1}a_0a_1a_2]_{-1}^2 \mid a_i \in \{0, 1\} \text{ for each } i \in \mathbb{Z}\}.$$

Similarly, we investigate up to  $(n - 1)$ -th inverse images,

$$\bigvee_{i=0}^{n-1} G^{-i}\alpha = \{[a_{-(n-1)} \cdots a_n]_{-(n-1)}^n \mid a_m \in \{0, 1\} \text{ for each } m \in \mathbb{Z}\}.$$

As  $n$  to infinity,  $\bigvee_{i=0}^{\infty} G^{-i}\alpha = \mathcal{B}$ , and hence the partition  $\alpha$  is generating. Here

$$H\left(\bigvee_{k=0}^{n-1} G^{-k}\alpha\right) = -2^{2n} \mu([i_{-n+1} \cdots i_n]_{-n+1}^n) \log \mu([i_{-n+1} \cdots i_n]_{-n+1}^n) = 2n \log 2.$$

By Kolmogorov-Sinai Theorem (Proposition 2.1),

$$h_G(\mu) = h(\alpha, G) = \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{k=0}^{n-1} G^{-k}\alpha\right) = 2 \log 2.$$

The maps  $G$  and 4-Bernoulli shift are topologically conjugate, and so topological entropy of them are the same. In fact the partition  $\alpha = \{[00]_0^1, [01]_0^1, [10]_0^1, [11]_0^1\}$  is Bernoulli partition. Hence we obtain  $h_{top}(G) = 2 \log 2$ .  $\square$

**Lemma 3.2.** Let  $(X, T)$  be a  $D$ -dimensional CA with  $D \geq 2$  and for every subset  $A_m$  with  $\bigcup_{m=1}^{\infty} A_m \subset X$  let  $(A_m, T)$  be a subsystem of  $(X, T)$ . Suppose that a  $(D-k)$ -dimensional CA  $(Y, S)$  for  $1 \leq k \leq D-1$  satisfies the following properties:

1. topological entropy of  $(Y, S)$  is positive and
2.  $(A_m, T)$  and  $(Y, S)$  are topologically conjugate for each  $m \in \mathbb{N}$ .

Then topological entropy of  $(X, T)$  is infinity.

*Proof.* By Theorem 3 in [1] we have for two dynamical systems  $(A_1, T)$  and  $(A_2, T)$  satisfying the above conditions,

$$h_{top}(T|_{A_1} \times T|_{A_2}) = h_{top}(S \times S) = h_{top}(S) + h_{top}(S).$$

Using the above formula gives Therefore we estimates topological entoropy of  $(T, X)$ ,

$$h_{top}(T) \geq h_{top}(T|_{A_1} \times \cdots \times T|_{A_m} \times \cdots) = h_{top}(S \times \cdots \times S \times \cdots) = \sum_{m=1}^{\infty} h_{top}(S) = \infty.$$

We complete the proof. □

Now we estimate topological entoropy of our two-dimensional nonlinear CA  $(X, F)$ .

**Theorem 3.2.** For the two-dimensional nonlinear CA  $(X, F)$

$$h_{top}(F) = \infty.$$

*Proof.* By Theorem 3.1 the CA  $(Y, G)$  is a subsystem of  $(X, F)$  and by Lemma 3.1 topological entropy of  $(Y, G)$  is  $2 \log 2 > 0$ . Then the conditions of Lemma 3.2 are satisfied and hence topological entropy of  $(X, F)$  is infinite. □

## 4. Concluding remarks

Our result suggest that topological entoropy of multidimensional CA diverges easily. For this reason, topological entoropy seems to be inappropriate to classify CA to the complexity classes. Therefore we should find another tool for measuring the complexity of CA.

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## References

- [1] R. L. Adler, A. G. Konheim, and M. H. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.*, 114:309–319, 1965.
- [2] François Blanchard, Petr Kůrka, and Alejandro Maass. Topological and measure-theoretic properties of one-dimensional cellular automata. *Phys. D*, 103(1-4):86–99, 1997. Lattice dynamics (Paris, 1995).
- [3] Gianpiero Cattaneo, Enrico Formenti, Giovanni Manzini, and Luciano Margara. Ergodicity, transitivity, and regularity for linear cellular automata over  $Z_m$ . *Theoret. Comput. Sci.*, 233(1-2):147–164, 2000.
- [4] Michele D’amico, Giovanni Manzini, and Luciano Margara. On computing the entropy of cellular automata. *Theoret. Comput. Sci.*, 290(3):1629–1646, 2003.
- [5] Manfred Denker, Christian Grillenberger, and Karl Sigmund. *Ergodic theory on compact spaces*. Lecture Notes in Mathematics, Vol. 527. Springer-Verlag, Berlin, 1976.
- [6] G. A. Hedlund. Endomorphisms and automorphisms of the shift dynamical system. *Math. Systems Theory*, 3:320–375, 1969.
- [7] Lyman P. Hurd, Jarkko Kari, and Karel Culik. The topological entropy of cellular automata is uncomputable. *Ergodic Theory Dynam. Systems*, 12(2):255–265, 1992.
- [8] Jarkko Kari. Reversibility and surjectivity problems of cellular automata. *J. Comput. System Sci.*, 48(1):149–182, 1994.
- [9] Petr Kůrka. Languages, equicontinuity and attractors in cellular automata. *Ergodic Theory Dynam. Systems*, 17(2):417–433, 1997.
- [10] E. L. Lakshtanov and E. S. Langvagen. A criterion for the infinity of the topological entropy of multidimensional cellular automata. *Problemy Peredachi Informatsii*, 40(2):70–72, 2004.
- [11] Giovanni Manzini and Luciano Margara. A complete and efficiently computable topological classification of  $D$ -dimensional linear cellular automata over  $Z_m$ . In *Automata, languages and programming (Bologna, 1997)*, volume 1256 of *Lecture Notes in Comput. Sci.*, pages 794–804. Springer, Berlin, 1997.
- [12] Giovanni Manzini and Luciano Margara. Attractors of  $D$ -dimensional linear cellular automata. In *STACS 98 (Paris, 1998)*, volume 1373 of *Lecture Notes in Comput. Sci.*, pages 128–138. Springer, Berlin, 1998.
- [13] Gardner Martin. Mathematical games. *Scientific American*, 223:120–123, 1970.
- [14] Tom Meyerovitch. Finite entropy for multidimensional cellular automata. *Ergodic Theory Dynam. Systems*, 28(4):1243–1260, 2008.
- [15] G. Morris and T. Ward. Entropy bounds for endomorphisms commuting with  $K$  actions. *Israel J. Math.*, 106:1–11, 1998.
- [16] Mark A. Shereshevsky. Expansiveness, entropy and polynomial growth for groups acting on subshifts by automorphisms. *Indag. Math. (N.S.)*, 4(2):203–210, 1993.
- [17] S. Ulam. Random processes and transformations. In *Proceedings of the International Congress of Mathematicians, Vol. 2, Cambridge, Mass., 1950*, pages 264–275, Providence, R. I., 1952. Amer. Math. Soc.
- [18] John von Neumann. The general and logical theory of automata. In *Cerebral Mechanisms in Behavior. The Hixon Symposium*, pages 1–31; discussion, pp. 32–

41. John Wiley & Sons Inc., New York, N. Y., 1951.
- [19] Peter Walters. *Ergodic theory—introductory lectures*. Lecture Notes in Mathematics, Vol. 458. Springer-Verlag, Berlin, 1975.
- [20] Stephen Wolfram. Statistical mechanics of cellular automata. *Rev. Modern Phys.*, 55(3):601–644, 1983.
- [21] Stephen Wolfram. Universality and complexity in cellular automata. *Phys. D*, 10(1-2):1–35, 1984. Cellular automata (Los Alamos, N.M., 1983).