# Large-time Asymptotics for One-dimensional Dirichlet Problems for Hamilton-Jacobi Equations with Noncoercive Hamiltonians

Yoshikazu Giga[a], Qing Liu[a], Hiroyoshi Mitake[b]

[a] *Graduate School of Mathematical Sciences, University of Tokyo*
*3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914 Japan*
[b] *Department of Applied Mathematics, Graduate School of Engineering*
*Hiroshima University, Higashi-Hiroshima, 739-8517 Japan*

**Abstract**

We study large-time asymptotics for a class of noncoercive Hamilton-Jacobi equations with Dirichlet boundary condition in one space dimension. We prove that the average growth rate of a solution is constant only in a subset of the whole domain and give the asymptotic profile in the subset. We show that the large-time behavior for noncoercive problems may depend on the space variable in general, which is different from the usual results under the coercivity condition. This work is an extension with more rigorous analysis of a recent paper by E. Yokoyama, Y. Giga and P. Rybka, in which a growing crystal model is established and the asymptotic behavior described above is first discovered.

*Keywords:* Large-time Behavior, Boundary Value Problems, Noncoercive Hamilton-Jacobi Equations, Viscosity Solution.
*2010 AMS Subject Classification:* 35B40, 35F30, 35F31, 49L25.

## 1. Introduction

We are concerned with large-time behavior of solutions of a Hamilton-Jacobi equation in one space dimension with homogeneous Dirichlet bound-

*Email addresses:* `labgiga@ms.u-tokyo.ac.jp` (Yoshikazu Giga), `liuqing@ms.u-tokyo.ac.jp` (Qing Liu), `mitake@hiroshima-u.ac.jp` (Hiroyoshi Mitake)

ary condition:

$$\text{(CD)} \qquad \begin{cases} u_t + H(x, u_x) = 0 & \text{in } (0,\infty) \times (0,\infty), \\ u(0,t) = 0 & \text{for } t \in (0,\infty), \\ u(x,0) = u_0(x) & \text{for } x \in [0,\infty), \end{cases}$$

where $u_0$ is a locally bounded function, continuous at $x = 0$ with $u_0(0) = 0$. The continuous function $H : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is called a *Hamiltonian*. It is necessary to emphasize that our Dirichlet boundary condition is fulfilled in the classical sense instead of the viscosity sense.

We begin with a brief presentation of our main results in this paper. Suppose that $H$ has a limit at $|p| = \infty$; namely, for every $x \in [0, x_c)$

$$\lim_{|p|\to\infty} H(x,p) = c(x) \neq \pm\infty, \tag{1.1}$$

where $c : [0,\infty) \to \mathbb{R}$ is a continuous function which is assumed to have a unique zero $x_c$, dividing the domain into two parts. The function $c$ satisfies

$$c > 0 \text{ in } [0, x_c) \text{ and } c < 0 \text{ in } (x_c, \infty). \tag{1.2}$$

Assume for the moment that

$$H(x,0) \leq 0 \quad \text{for every } x \in (0, x_c), \tag{1.3}$$

and

$$\sup_{p\in\mathbb{R}} H(x,p) < 0 \quad \text{for all } x \in (x_c, \infty). \tag{1.4}$$

We assert that if the initial data $u_0$ fulfills a *compatibility condition*, the solution $u$ satisfies

$$u(x,t) \to v(x) \quad \text{for } x \in (0, x_c) \text{ locally uniformly; and} \tag{1.5a}$$

$$u(x,t) \to \infty \quad \text{for } x \in (x_c, \infty) \text{ locally uniformly,} \tag{1.5b}$$

as $t \to \infty$, where $v$ is a solution of the stationary equation

$$\begin{cases} H(x, v_x) = 0 & \text{in } (0, x_c), \\ v(0) = 0. \end{cases} \tag{1.6}$$

Here the compatibility condition is as follows.

$$
\begin{cases} u_0 \text{ is continuous at } x = 0 \text{ satisfying } u_0(0) = 0 \\ \text{and } u_0(x) \geq \displaystyle\int_0^x p_1(y)\,dy, \end{cases} \tag{1.7}
$$

where $p_1 \in C[0, x_c)$ determined by the Hamiltonian $H$; see (A5) below. The asymptotic profiles $v$ are the same for any $u_0$ satisfying (1.7). If the uniqueness of solutions of the stationary equation (1.6) holds, then we can prove (1.5a) relatively easily. However, it does not hold in general if they are considered in the usual sense. We need to adjust the definition of viscosity solutions and impose extra assumptions to guarantee the uniqueness. It therefore becomes another key issue in this work.

Since the Hamiltonian is noncoercive, the asymptotic behavior we described is very different from other results on this topic. We briefly recall the results on the large-time behavior of solutions of Hamilton-Jacobi equations in a compact manifold $\mathcal{M}$ (or in $\mathbb{R}^N$, mainly in the periodic case) under the assumption that the Hamiltonian is coercive, i.e.,

$$
\inf\{H(x, p) \mid x \in \mathcal{M}, |p| \geq r\} \to +\infty \text{ as } r \to \infty.
$$

Under the coercivity assumption it is well-known that there exists $(v, \lambda) \in W^{1,\infty}(\mathcal{M}) \times \mathbb{R}$, which are called an *ergodic function* and an *ergodic constant*, respectively, such that

$$
H(x, Dv(x)) = \lambda \quad \text{in } \mathcal{M}. \tag{1.8}
$$

This result was first proved in $\mathbb{R}^N$ for the periodic case by Lions, Papanicolaou and Varadhan [29]. By using this result and the comparison principle for the initial value problem, we can relatively easily get the convergence

$$
\frac{u(x, t)}{t} \to -\lambda \text{ uniformly on } \mathcal{M} \text{ as } t \to \infty.
$$

It is worth mentioning that whereas an ergodic constant is uniquely determined, ergodic functions may not be unique. We also mention that when we consider noncoercive Hamiltonians, we cannot expect the existence of bounded solutions of (1.8) in general, which is one of the important difference between the coercive case and noncoercive case.

The usual result of the large-time behavior of solutions of Hamilton-Jacobi equations is the convergence

$$u(x,t) - \lambda \to v(x) \text{ uniformly for all } x \in \mathcal{M} \tag{1.9}$$

as $t \to \infty$. It is generally rather difficult to show the above convergence, since, as we mentioned above, ergodic functions may not be unique. Since the works by Namah and Roquejoffre [33] and Fathi [15], it has been studied by many authors. We refer to [8, 12, 15, 33, 34] for Cauchy problems on the periodic setting and to [6, 26, 30, 31, 32, 34] for various boundary problems in a bounded domain.

In the case of non-periodic setting in the whole domain $\mathbb{R}^N$, the large time asymptotics is more complicated than that on the periodic setting or a compact manifold. We refer to [7, 19, 20, 21, 25] for details. In this case the ratio of large-time asymptotics of solutions, i.e., $\lambda$ in (1.9) may depend on the initial value. In [7] the authors pointed out the influence of $u_0$ at infinity on $\lambda$. In [19, 20, 21, 25], the authors clarified a class of initial values in order to obtain the large-time asymptotics (1.9) with $\lambda$ which is independent of the initial value.

To explain our results heuristically, we give two basic observations for the problem when the coercivity assumption is dropped. On one hand, there is possibility that the ergodic constant may still be determined provided that, roughly speaking, the $p$-dependence of $H$ is strong enough. For example, when the ranges of $x \mapsto H(x,\infty)$ and $x \mapsto H(x,0)$ have empty intersection, the Lipschitz continuity of solutions holds and thus through the usual methods the ergodic constant is still obtained.

On the other hand, if the $x$-dependence turns to be dominant, the situation can easily become different and one cannot expect the usual results to be true any more. A trivial example is the case that $H$ depends only on $x$. We give a less trivial but simple one-dimensional example to make this clear.

Consider the following noncoercive Hamilton-Jacobi equation

$$\begin{cases} u_t - \dfrac{1}{1+|u_x|} - x = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

When $u_0 \equiv 0$, the unique solution is clearly $u(x,t) = xt + \ln(1+t)$, which cannot have the behavior as in (1.9). In fact, one can prove that in this case the solution $u$ should satisfy

$$\lim_{t\to\infty} u(x,t)/t = x \quad \text{and} \quad \lim_{t\to\infty} (u(x,t) - xt)/\ln t = 1 \tag{1.10}$$

for any bounded and continuous initial data $u_0$. The *resonance* in this case is quite strong because the $x$-derivative of $H$ never vanishes.

Our result in this paper, as is given in (1.5), can be viewed as a combination of the two perspectives above. The assumptions (1.1), (1.2) and (1.3) reveal that $H$ varies considerably in $p$ for $x \in (0, x_c)$ in that the intersection of ranges $\{H(x,0) : x \in (0,x_c)\}$ and $\{H(x,\infty) : x \in (0,x_c)\}$ is empty. It follows that the growth velocity in $(0,x_c)$ is a constant. With the Dirichlet condition taken into consideration, the velocity must be 0 and we are consequently led to (1.5a). The condition (1.4), on the other hand, amounts to saying that once $x$ exceeds $x_c$, $H$ has different structure so that the unconventional asymptotic behavior as in (1.10) may take place. Of course since (1.4) is still rough, the behavior (1.5b) in the region $(x_c,\infty)$ can be much more complicated than (1.10), but we will not touch it in this paper.

We point out that noncoercive ergodicity and homogenization problems are also investigated in [1, 10] with *non-resonance conditions* and in [5] under partial coercivity assumption, which means $p \mapsto H(x,p)$ is coercive in several directions but not necessarily in the others. In contrast, we consider a "completely noncoercive" Hamiltonian and allow nonergodic part to exist at the same time. We impose the Dirichlet boundary condition only for our motivation from physics to be explained in a moment.

As we have mentioned, the uniqueness of solutions of (1.6) is nontrivial in general. One difficulty is about the seemingly missing boundary condition at $x = x_c$. An implicit fact is that our stationary solution satisfies $u_x \to \infty$ as $x \to x_c-$, which is a natural consequence due to the lack of coercivity, and in some cases, the *singular Neumann condition* escalates into a *singular Dirichlet* one. Such kind of singular boundary problems are studied by Lasry and Lions [28] for a class of viscous Hamilton-Jacobi equations and by some others in different contexts. For first-order equations, we point out that the work of Bardi [2], Evans and James [13] and also Bardi and Soravia [4] on singular Dirichlet boundary problems are closely related to ours. However, our method is different from theirs. In contrast to their characterization of free boundary problems to meet their applications in optimal control and differential games, we relax the problem in the whole domain $(0,\infty)$ and permits the solution to take infinity value, which facilitates us very much to handle not only the singular Dirichlet boundary condition but also the Neumann one at the boundary $x = x_c$ fixed by our assumptions.

Another difficulty comes from the abstract form of $H$. Usually, showing the comparison principle is difficult without homogeneity or convexity of $H$ in $p$ (see, e.g., [3, 24]), but because of (1.1) it is not convenient for us to add such assumptions. Another standard option is to consider special optimal control and game structure compatible with the use of *Kruzkov transform*, which is applied in [4, 13]. This transform however cannot be applied to our problems arising from crystal growth. In fact, in this paper we try to obtain results with minimal assumptions on the $p$-dependence of $H$. We assume neither convexity nor homogeneity of $H(x,p)$ with respect to the variable $p$. In the proof of our comparison theorem, we only use an extra monotonicity assumption on $x \mapsto H(x,p)$, which has physics background to be clarified later. Roughly speaking, we assume

$$x \mapsto H(x,p) \text{ is strictly decreasing in } (0,x_c) \text{ for all } p \in \mathbb{R} \tag{1.11}$$

so that we can scale the solutions with respect to $x$ without changing the gradient. See (A4) for the precise form. This assumption enables us to require no more assumptions on $p \mapsto H(x,p)$ or the optimal control and game characterization. One should also notice that while the results obtained in [2, 4, 13] hold only for positive solution, ours do not have this restriction.

Our present work of asymptotic behavior is more rigorous and general analysis of [35], in which a one-dimensional mathematical model for crystal growth is studied. We refer the reader to [35] and references therein for more details about crystal growth. The Hamiltonian $H$ of (CD) in practice usually has the form $H(x,p) = \sigma(x)m(p) - c$, where $\sigma : [0,\infty) \to [0,\infty)$, $m : \mathbb{R} \to [0,1)$ and $c > 0$ are given and denote respectively the *surface supersaturation*, *kinetic coefficient* and *speed of the step source*.

Typical choices of $\sigma$ and $m$ are

$$\sigma(x) = \sigma_0 \max\{(1-x^2), 0\} \text{ and } m(p) = p \tanh(1/p),$$

where $\sigma_0$ is a constant greater than $c$; see ([9, 11, 35]). The strict monotonicity of $\sigma(x)$ (and therefore $H(x,p)$) for $x \in (0,1)$, which agrees with our assumption (1.11), is based on *Berg's effect* and reflects the *nonuniformity of supersaturation*.

The main result of [35] says that the crystal grows in a stable manner only in a part of the domain while it hardly grows outside when the initial height $u_0 \equiv 0$. Our general large-time behavior here corresponds to this observation and our *effective domain* $(0, x_c)$ coincides with the stable region. In fact, we obtain more: the large-time behavior (1.5) holds only when the compatibility assumption on $u_0$ is fulfilled, as we mentioned above. This assumption is posed to keep our strict Dirichlet boundary condition satisfied for all $t > 0$. We will give an example to explain this more clearly in Section 3.4. If the strict Dirichlet boundary condition is broken, one has to consider the large-time behavior for a generalized Dirichlet boundary problem, which is another interesting topic. Consult [30, 31] for recent development at this aspect.

Our approach in this paper seems to be restricted only to one-dimensional cases (and to some higher dimensional cases with *spokewise monotonicity* of $H$ in space). It is worth mentioning that in our companion paper [18], we present further results related to this work. We show a similar large-time behavior of solutions for Cauchy problems in higher dimensions. To specify the asymptotic profile, we focus our attention there on the singular Neumann boundary problems and give another type of definitions of solutions of associated stationary problem following [28]. The equivalence of our two definitions in particular cases are also shown there.

We organize this paper in the following way. We first study the stationary problem in Section 2, giving a definition of viscosity solutions and proving the comparison principle. Several examples are also discussed at the end of this section. In Section 3, we present our main theorem about the large-time behavior of solutions of the Cauchy-Dirichlet problem and give its proof.

## 2. Stationary Problem

Our main purpose of this section is to establish a comparison result for solutions of

$$H(x, u_x) = 0 \text{ in } (0, \infty). \tag{2.1}$$

Let us start with definitions of viscosity solutions and related properties.

### *2.1. Definition and Properties*

Basic assumptions we need in this section are as follows.

(A1) $H : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist a constant $C > 0$ and a modulus $\omega$ such that

$$|H(x, p) - H(x, q)| \leq C(1 + |x|)|p - q|$$

and

$$|H(x,p) - H(y,p)| \leq \omega((1+|p|)|x-y|)$$

for all $x, y \in [0,\infty)$ and $p, q \in \mathbb{R}$.

(A2) $\lim_{|p|\to\infty} H(x,p) = c(x)$ locally uniformly in $x$, where $c(x) \in C([0,\infty))$ such that there is $x_c \in \mathbb{R}$ satisfying

$$c(x) \begin{cases} > 0 & \text{for } x \in [0, x_c); \\ = 0 & \text{for } x = x_c; \\ < 0 & \text{for } x \in (x_c, \infty). \end{cases}$$

We hereafter do not distinguish notations of $H(x, \pm\infty)$ and $c(x)$.

(A3) For any $x \in (x_c, \infty)$, $\sup_{p\in\mathbb{R}} H(x,p) < 0$.

Although our stationary equation looks established in an unbounded domain, we will later see that (A3) essentially turns it into a problem with the bounded domain $[0, x_c)$.

We next set, for any function $u : [0,\infty) \to \mathbb{R} \cup \{\pm\infty\}$,

$$u^+(x) = \begin{cases} +\infty & \text{if } u(x) = -\infty, \\ u(x) & \text{otherwise} \end{cases} \quad \text{and } u_-(x) = \begin{cases} -\infty & \text{if } u(x) = +\infty, \\ u(x) & \text{otherwise.} \end{cases}$$

We also denote $(u_-)^*$ by $\overline{u}$ and $(u^+)_*$ by $\underline{u}$, where $w^*$ (resp., $w_*$) stands for the usual *upper* (resp., *lower*) *semicontinuous envelope* of a function $w$.

**Definition 2.1** (Subsolution)**.** A function $u : [0,\infty) \to \mathbb{R} \cup \{\pm\infty\}$ is said to be a subsolution of (2.1) if it satisfies the followings:

(i) $\overline{u}(x) < \infty$ for all $x \in (0,\infty) \setminus \{x_c\}$;

(ii) whenever there exist $\varphi \in C^1([0,\infty) \setminus \{x_c\})$ and $\hat{x} \in (0,\infty) \setminus \{x_c\}$ satisfying

$$\max_{x\in[0,\infty)\setminus\{x_c\}} (\overline{u} - \varphi)(x) = (\overline{u} - \varphi)(\hat{x}),$$

then

$$H(\hat{x}, \varphi_x(\hat{x})) \leq 0. \tag{2.2}$$

(iii) if $\overline{u}(x_c) < \infty$, then for every $\varphi \in C^1([0,\infty))$ such that

$$\max_{x\in[0,\infty)} (\overline{u} - \varphi)(x) = (\overline{u} - \varphi)(x_c),$$

inequality (2.2) holds with $\hat{x} = x_c$, i.e.,

$$H(x_c, \varphi_x(x_c)) \leq 0.$$

**Definition 2.2** (Supersolution)**.** A function $u : [0, \infty) \to \mathbb{R} \cup \{\pm\infty\}$ is said to be a supersolution of (2.1) if it satisfies the followings:

(i) $\underline{u}(x) > -\infty$ for all $x \in (0, \infty) \setminus \{x_c\}$;

(ii) whenever there exist $\varphi \in C^1([0, \infty) \setminus \{x_c\})$ and $\hat{x} \in (0, \infty) \setminus \{x_c\}$ satisfying

$$\min_{x \in [0,\infty) \setminus \{x_c\}} (\underline{u} - \varphi)(x) = (\underline{u} - \varphi)(\hat{x}),$$

then

$$H(\hat{x}, \varphi_x(\hat{x})) \geq 0. \tag{2.3}$$

(iii) if $\underline{u}(x_c) > -\infty$, then for every $\varphi \in C^1([0, \infty))$ such that

$$\min_{x \in [0,\infty)} (\underline{u} - \varphi)(x) = (\underline{u} - \varphi)(x_c),$$

inequality (2.3) holds with $\hat{x} = x_c$, i.e.,

$$H(x_c, \varphi_x(x_c)) \geq 0.$$

**Definition 2.3** (Solution)**.** A function $u$ is said to be a solution if it is both a subsolution and a supersolution.

In spite of our modification of values in $(x_c, \infty)$, we are unable to prevent a subsolution (resp., a supersolution) from attaining the value $+\infty$ (resp., $-\infty$) at $x = x_c$ a priori. This corresponds to the possibility that solutions of (2.1) with $u(0) = 0$ blow up at $x = x_c$, which we shall discuss in more detail in Section 2.3.

As usual comments on viscosity solutions, the maximum and minimum in Definitions 2.1 and 2.2 can both be replaced by a strict maximum and a strict minimum. One can also use the superdifferential $D^+\overline{u}$ and the subdifferential $D^-\underline{u}$ instead of the test function $\varphi$ to define the solutions. See [3, 17] for instance.

An immediate consequence of Definition 2.1 and assumption (A3) is that our supersolutions do take infinite value in $(x_c, \infty)$.

**Lemma 2.1** (Infinite value of supersolutions in the unstable region)**.** *Assume (A3). If $u$ is a supersolution of (2.1), then $\underline{u} = +\infty$ in $(x_c, \infty)$.*

*Proof.* We argue by contradiction. Suppose there exists $\hat{x} \in (x_c, \infty)$ such that $\underline{u}(\hat{x}) \neq +\infty$. It is obvious by Definition 2.1 that $\underline{u}$ is locally bounded from below; that is, there exist $M \in \mathbb{R}$ and an interval $[\hat{x} - r, \hat{x} + r]$ in which $\underline{u} \geq M$. Moreover, we may let $r$ be so small that $\hat{x} - 2r > x_c$. Now observe that

$$\underline{u}(x) - \underline{u}(\hat{x}) + \frac{1}{2\varepsilon}|x - \hat{x}|^2 \geq M - \underline{u}(\hat{x}) + \frac{1}{8\varepsilon}r^2 \text{ for } x = x_r \pm r$$

and

$$\min_{x \in [\hat{x}-r,\ \hat{x}+r]} \underline{u}(x) - \underline{u}(\hat{x}) + \frac{1}{2\varepsilon}|x - \hat{x}|^2 \leq 0.$$

Then a minimizer $x^\varepsilon$ above exists and lies in $(\hat{x}-r, \hat{x}+r)$ when $\varepsilon$ is sufficiently small. It follows from (A3) that

$$H(x^\varepsilon, \frac{\hat{x} - x^\varepsilon}{\varepsilon}) < 0,$$

which contradicts to the inequality in Definition 2.2. □

Let us study the regularity of subsolutions by using (A2).

**Lemma 2.2** (Local Lipschitz continuity of subsolutions)**.** *Assume (A2). Let $u$ be a subsolution of (2.1). Assume that for every $\varepsilon > 0$ there exists $M > 0$ such that $\overline{u} \leq M$ in $[0, x_c - \varepsilon]$. Then there exists $L > 0$ depending only on $\varepsilon$ such that*

$$|\overline{u}(x) - \overline{u}(y)| \leq L|x - y| \textit{ for all } x, y \in (0, x_c - \varepsilon].$$

*Proof.* We only show the case

$$\overline{u}(x) - \overline{u}(y) \geq -L|x - y| \tag{2.4}$$

for all $x \in [0, x_c - \varepsilon]$ and $y \in (0, x_c - \varepsilon]$. (The other half can be treated via a symmetric argument.) It suffices to show that for any fixed $\varepsilon > 0$ and $x \in [0, x_c - \varepsilon]$, there is $L > 0$ independent of $x$ such that (2.4) holds for all $y \in (0, x_c - \varepsilon]$.

We argue by contradiction. Assume that there are $\varepsilon > 0$ and $\hat{x} \in [0, x_c - \varepsilon]$ such that there always exists $y \in (0, x_c - \varepsilon]$ satisfying

$$\overline{u}(\hat{x}) - \overline{u}(y) < -L|\hat{x} - y|$$

no matter how large $L$ is. This means

$$\max_{y\in[0,x_c-\varepsilon]}(\overline{u}(y)-\overline{u}(\hat{x})-L|\hat{x}-y|)>0 \quad \text{for any } L>0.$$

Denote by $\hat{y}\in[0,x_c-\varepsilon]$ a maximizer above. Clearly $\hat{y}\neq\hat{x}$. Also, since $\overline{u}$ is bounded from above, we may let $L$ be sufficiently large without depending on $\hat{x}$ so that $\hat{y}\neq 0$ and $\hat{y}\neq x_c-\varepsilon$.

Put $\varphi(y):=\overline{u}(\hat{x})+L|\hat{x}-y|$. Now that $\overline{u}-\varphi$ attains at $\hat{y}\in(0,x_c-\varepsilon)$ a maximum, we deduce from Definition 2.1 that

$$H(\hat{y},-L\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|})\leq 0,$$

which cannot be true, in view of (A2), when $L$ is large. □

*2.2. Comparison Principle*

As stated in Introduction, we need an assumption on strict monotonicity in the space variable in order to prove a comparison principle for the stationary equation. A weaker and more precise assumption than (1.11) is as follows:

(A4) There exists $l>0$ such that, for all $x\in(0,x_c]$ and $p\in\mathbb{R}$ satisfying $H(x,p)=0$, we have

$$H(x_1,p)>H(x_2,p)$$

whenever $x_1,x_2\in(x-l,x+l)\cap[0,x_c]$ and $x_1<x_2$.

**Theorem 2.3** (Comparison theorem of (2.1))**.** *Assume (A1)–(A4) . Let $u$ and $v$ be respectively a subsolution and a supersolution of (2.1). Assume that there exists $M>0$ such that $\overline{u}\leq M$ and $\underline{v}\geq -M$. If $\overline{u}(0)\leq\underline{v}(0)$, then $\overline{u}\leq\underline{v}$ in $[0,x_c)$.*

*Remark* 1. We only conduct our comparison in $(0,x_c)$. It is obvious that $\overline{u}\leq\infty=\underline{v}$ in $(x_c,\infty)$. However, it is not necessarily true that $\underline{v}(x_c)\geq\overline{u}(x_c)$. In fact, through the Figure 1, which roughly shows the graph of a solution $u$, we easily observe that $u(x_c)$ can be modified to be $\infty$ or any value above $\limsup_{x\to x_c-} u(x)$.

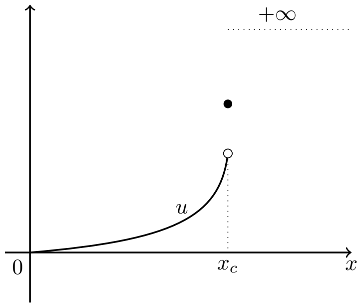


Figure 1: Graph of a solution $u$.

*Proof of Theorem 2.3.* Without loss of generality, let us assume $\overline{u}(0) \le \underline{v}(0) \le 0$. Indeed, we may establish our comparison for $u + C$ and $v + C$ with any constant $C \in \mathbb{R}$.

Take $0 < \lambda < 1$. Set $u_\lambda(x) = \frac{1}{\lambda}u(\lambda x)$ and $\overline{u}_\lambda = ((u_\lambda)_-)^*$. It is then obvious that $\overline{u}_\lambda(0) \le \underline{v}(0)$.

In view of the boundary condition $\overline{u}(0) \le \underline{v}(0)$, we assume by contradiction that there exist $\xi_0 \in (0, x_c)$ and $\mu > 0$ such that $\overline{u}(\xi_0) - \underline{v}(\xi_0) \ge 2\mu$. Our dilation preserves a positive maximum of $\overline{u} - \underline{v}$; that is,

$$\overline{u}_\lambda(\xi_0) - \underline{v}(\xi_0) \ge \mu \tag{2.5}$$

if we take $\lambda$ close to 1. Indeed, a direct calculation indicates

$$\overline{u}_\lambda(\xi_0) - \underline{v}(\xi) = \frac{1}{\lambda}\overline{u}(\lambda\xi_0) - \underline{v}(\xi_0) = \overline{u}(\xi_0) - \underline{v}(\xi_0) + \left(\frac{1}{\lambda}\overline{u}(\lambda\xi_0) - \overline{u}(\xi_0)\right),$$

which, together with the continuity of $\overline{u}$ obtained in Lemma 2.2, yields (2.5).

For every $\varepsilon > 0$, we double variables by setting an auxiliary function $\Phi_\varepsilon(x, y) = \overline{u}_\lambda(x) - \underline{v}(y) - \frac{1}{2\varepsilon}|x - y|^2$. It is clear that

$$\sup_{x,y\in[0,\infty)} \Phi_\varepsilon(x, y) \ge \mu.$$

We next claim that the maximum points $(\xi_\varepsilon, \eta_\varepsilon)$[1] of $\Phi_\varepsilon$ can be taken in a bounded interval. Usually, to do this, one needs another term to penalize at infinity when proving comparison theorems for an unbounded domain. We however do not need it in our case, thanks to Lemma 2.1. The term $\frac{1}{2\varepsilon}|x - y|^2$

will in essence play the role of penalizing at space infinity. Following the above idea, we see that Lemma 2.1 yields

$$\sup_{x,y\in[0,\infty)} \Phi_\varepsilon(x,y) = \sup_{\substack{x\in[0,\infty)\\ y\in[0,x_c]}} \Phi_\varepsilon(x,y).$$

Then since $\overline{u}$ and $-\underline{v}$ are bounded from above, the structure of $\Phi_\varepsilon$ refrains the supremum from being attained at a very large $x$ when $\varepsilon$ is sufficiently small, which implies the existence of $\xi_\varepsilon$ and $\eta_\varepsilon$. In addition, a usual argument gives

$$|\xi_\varepsilon - \eta_\varepsilon| \leq C_1\varepsilon^{\frac{1}{2}} \text{ with } C_1 > 0,$$

and then by taking a subsequence, still indexed by $\varepsilon$, we may let $\xi_\varepsilon$ and $\eta_\varepsilon$ converge to some $z \in [0, x_c]$. Since $\overline{u}_\lambda(0) \leq \underline{v}(0)$ and $\Phi_\varepsilon$ is upper semicontinuous in $[0,\infty)^2$, we must have $z > 0$.

Set $\varphi_1(x) := \lambda\underline{v}(\eta_\varepsilon) + \frac{1}{2\varepsilon\lambda}(x-\lambda\eta_\varepsilon)^2$ and $\varphi_2(y) := \overline{u}_\lambda(\xi_\varepsilon) - \frac{1}{2\varepsilon}|\xi_\varepsilon - y|^2$. The maximum of $\Phi$ at $(\xi_\varepsilon, \eta_\varepsilon)$ immediately implies that $\overline{u} - \varphi_1$ attains a maximum at $\lambda\xi_\varepsilon$ and $\underline{v} - \varphi_2$ attains a minimum at $\eta_\varepsilon$. We thus apply our definitions of subsolutions and supersolutions to get

$$H\left(\lambda\xi_\varepsilon, \frac{\xi_\varepsilon - \eta_\varepsilon}{\varepsilon}\right) \leq 0 \tag{2.6}$$

and

$$H\left(\eta_\varepsilon, \frac{\xi_\varepsilon - \eta_\varepsilon}{\varepsilon}\right) \geq 0. \tag{2.7}$$

The term $\frac{1}{\varepsilon}(\xi_\varepsilon - \eta_\varepsilon)$ must be bounded, for otherwise along the subsequence such that it diverges and the limit of (2.6) as $\varepsilon \to 0$ gives rise to a contradiction to (A2). So we discuss all the converging subsequences for $\frac{1}{\varepsilon}(\xi_\varepsilon - \eta_\varepsilon)$ as $\varepsilon \to 0$. Denote their limits, which depends on $\lambda$, by $q_\lambda$. By (A1), we then have

$$H(\lambda z, q_\lambda) \leq 0 \tag{2.8}$$

and

$$H(z, q_\lambda) \geq 0. \tag{2.9}$$

Since the Hamiltonian is continuous, there exists $x_\lambda \in [\lambda z, z]$ such that $H(x_\lambda, q_\lambda) = 0$. It is clear that $x_\lambda \to z$ as $\lambda \to 1$. Meanwhile, since $\lambda z$

and $z$ are lying in $(x_\lambda - l, x_\lambda + l)$, taking difference of (2.8) and (2.9) and using (A4) (with $x = x_\lambda$, $x_1 = \lambda z$, $x_2 = z$ and $p = q_\lambda$), we are led to a contradiction

$$0 < H(\lambda z, q_\lambda) - H(z, q_\lambda) \le 0.$$

□

**Corollary 2.4** (Uniqueness of solutions). *Assume (A1)–(A4). The solutions $u$ of (2.1) with $u(0) = 0$ are unique in the sense that if $u, v$ are solutions bounded in $[0, x_c)$, then $\overline{u} = \overline{v}$ and $\underline{u} = \underline{v}$ in $[0, x_c) \cup (x_c, \infty)$.*

*2.3. Comparison Principle for More Singular Solutions*

As we have mentioned, we can also deal with the stationary problem whose solutions blow up on the boundary of the effective domain. However, on this occasion the solutions as in Definition 2.3 may not be unique, as shown in the following simple example.

**Example 1.** When $H(x, p) = \arctan|p| - x$, we easily find that

$$\int_0^x \tan y \, dy \quad \text{and} \quad -\int_0^x \tan y \, dy \qquad (\text{for } x \in [0, \pi/2))$$

with extension of infinity value to $[\pi/2, \infty)$ are both solutions of the stationary problem under our present definition. We will see in a moment that the long time limit of our time-dependent problem is actually the former. We therefore choose to only consider the solutions bounded from below.

**Theorem 2.5.** *Assume* (A1)–(A4). *Let $u$ and $v$ be respectively a subsolution and a supersolution of (2.1). Assume in addition that $u$ is bounded from below and there exists $\gamma < 0$ such that $\overline{u}(x) - \beta|x - x_c|^\gamma$ is bounded from above for any $\beta > 0$. If $\overline{u}(0) \le \underline{v}(0)$, then $\overline{u} \le \underline{v}$ in $[0, x_c)$.*

*Remark* 2. The hypothesis that $\overline{u}(x) - \beta|x - x_c|^\gamma$ has an upper bound is just technical. We impose it for our convenience to deal with unbounded solutions.

*Proof of Theorem 2.5.* We only need to modify slightly the proof of Theorem 2.3. Assume as before that there exist $\xi_0 \in (0, x_c)$ and $\mu > 0$ such that

$$\overline{u}_\lambda(\xi_0) - \underline{v}(\xi_0) \ge \mu > 0, \quad (u_\lambda(x) = \frac{1}{\lambda} u(\lambda x), \quad 0 < \lambda < 1).$$

This time, we set

$$\Phi_\varepsilon(x,y) = \overline{u}_\lambda(x) - \underline{v}(y) - \frac{1}{2\varepsilon}|x-y|^2 - \beta|x - \frac{x_c}{\lambda}|^\gamma.$$

Then $\Phi_\varepsilon$ is bounded from above and greater than $\frac{\mu}{2}$ provided that $\lambda$ is close to 1 and $\beta > 0$ is small.

Fix such a $\beta > 0$. Let $(\xi_\varepsilon, \eta_\varepsilon)$ be the maximizers of $\Phi_\varepsilon$. It is clear, due to Lemma 2.1, that $\eta_\varepsilon \in [0, x_c]$. We stress that this is actually true for arbitrary $\beta > 0$. Since $\overline{u}_\lambda(x) - \underline{v}(y) - \beta|x - \frac{x_c}{\lambda}|^\gamma$ is bounded from above, we have

$$\xi_\varepsilon - \eta_\varepsilon \to 0 \quad \text{ as } \varepsilon \to 0.$$

Assume $\xi_\varepsilon, \eta_\varepsilon \to z \in [0, x_c]$. It is easily seen that $z \neq 0$ because of the comparison hypothesis on the boundary. Now we apply our definition of sub- and supersolutions. A direct computation gives

$$\frac{\xi_\varepsilon - \eta_\varepsilon}{\varepsilon} + X^\varepsilon_\beta \in D^+\overline{u}(\lambda\xi_\varepsilon) \text{ and } \frac{\xi_\varepsilon - \eta_\varepsilon}{\varepsilon} \in D^-\underline{v}(\eta_\varepsilon),$$

where $X^\varepsilon_\beta = \beta|\xi_\varepsilon - \frac{x_c}{\lambda}|^{\gamma-2}(\xi_\varepsilon - \frac{x_c}{\lambda})$. Hence we obtain

$$H(\lambda\xi_\varepsilon, \frac{\xi_\varepsilon - \eta_\varepsilon}{\varepsilon} + X^\varepsilon_\beta) \le 0 \text{ and } H(\eta_\varepsilon, \frac{\xi_\varepsilon - \eta_\varepsilon}{\varepsilon}) \ge 0.$$

Send $\varepsilon \to 0$, passing to subsequences if necessary. Then $X^\varepsilon_\beta \to X_\beta := \beta|z - \frac{x_c}{\lambda}|^{\gamma-2}(z - \frac{x_c}{\lambda})$. If $\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} \to \infty$, a contradiction from (A2) is clear. If otherwise the limit is $q_\lambda \in \mathbb{R}$, then

$$H(\lambda z, q_\lambda + X_\beta) \le 0 \text{ and } H(z, q_\lambda) \ge 0.$$

Noticing that $z - \frac{x_c}{\lambda} \le \frac{\lambda - 1}{\lambda} x_c$ uniformly in $\beta$, we have, with $\beta \to 0$,

$$H(\lambda z, q_\lambda) \le 0 \text{ and } H(z, q_\lambda) \ge 0,$$

which is a contradiction to (A4) when $\lambda$ approaches to 1 by the same argument as in the proof of Theorem 2.3. $\square$

**Corollary 2.6.** *Assume (A1)–(A4). The solutions $u$ of (2.1)) with $u(0) = 0$ are unique in the sense that if $u, v$ are solutions bounded from below, and there exists $\gamma < 0$ such that $\overline{u}(x) - \beta|x - x_c|^\gamma$ and $\overline{v}(x) - \beta|x - x_c|^\gamma$ are both bounded from above for any $\beta > 0$ in $[0, x_c)$, then $\overline{u} = \overline{v}$ and $\underline{u} = \underline{v}$ in $[0, x_c) \cup (x_c, \infty)$.*

*2.4. Examples*

We give several concrete examples for the stationary problem we study.

**Example 2.** Take $p_0 \in \mathbb{R}$ and $\alpha > 0$. It is not hard to verify that

$$H(x,p) = \max\{2-x, 0\}\frac{|p-p_0|^\alpha}{1+|p-p_0|^\alpha} - 1$$

satisfies (A1)–(A4) with $x_c = 1$. The unique solution, bounded from below, of (2.1) with Dirichlet condition $u(0) = 0$ is

$$u(x) = \begin{cases} p_0 x - \ln(1-x), & \text{if } \alpha = 1; \\ p_0 x - \frac{\alpha}{\alpha-1}[(1-x)^{1-1/\alpha} - 1], & \text{if } \alpha \neq 1. \end{cases}$$

for $x \in (0,1)$. It is worth noticing that this example contains two different cases. When $0 < \alpha \leq 1$, the solution blows up at $x = 1$. When $\alpha > 1$, the solution is continuous and bounded in $[0,1]$ but its derivative tends to infinity as $x \to 1$. The uniqueness of solutions is guaranteed either by Theorem 2.3 or by Theorem 2.5.

The following example indicates that even $H(x,p)$ is not bounded, our conclusion can still be true.

**Example 3.** For $H(x,p) = (1-x)^\alpha |p| - 1$, which is coercive only locally away from $x = 1$, if for instance $\alpha = \frac{1}{3}$ or 3, then the unique solution (bounded from below) in the effective domain $(0,1)$ is

$$u(x) = -\frac{1}{1-\alpha}(1-x)^{1-\alpha} + \frac{1}{1-\alpha} \quad \text{for } x \in (0,1).$$

We do not study this kind of Hamilton-Jacobi equations in this paper, but such generalization is possible.

We next show that (A4) is necessary for the uniqueness.

**Example 4.** Consider the following Hamiltonian

$$H(x,p) = -\frac{1}{1+|p|}(x-1)^2 + \frac{|p|}{1+|p|}(1-x)^\alpha.$$

We take $\alpha = 1$ and then we get two solutions, which in $(0,1)$ are

$$u(x) = \pm\left(\frac{1}{2}x^2 - x\right).$$

The same situation takes place for other choices of $\alpha$ like $7/3$ and $3$ as well. The reason for nonuniqueness is that $H$ does not satisfy (A4). Note that $x \mapsto H(x, 0)$ is not strictly decreasing at $x = 1$ while $H(1, p) = 0$ for all $p \in \mathbb{R}$.

**Example 5.** Another example of Hamiltonian to show the necessity of (A4) is

$$H(x,p) = \max\left\{ \left(\arctan|p|^2 - x\right), 0\right\} + \min\left\{ \left(\frac{\pi}{2} - x\right), 0\right\}.$$

It is not difficult to see that $x_c = \pi/2$ and there are infinitely many bounded solutions including

$$u(x) \equiv 0 \text{ and } u(x) = \pm \int_0^x (\tan y)^{\frac{1}{2}}\, dy$$

in $(0, \pi/2)$. All of the assumptions except (A4) are satisfied.

## 3. Large-time Behavior

We discuss in this section the asymptotic behavior of the solution of (CD) as $t \to \infty$. Let us impose further assumptions:

(A5) There exists $p_1 \in C([0, x_c)) \cap L^1(0, x_c)$ such that $H(x, p_1(x)) \le 0$ for all $x \in [0, x_c)$.

(A6) There exist $\gamma_0 < 0$ and $C_0 > 0$ such that $H(x, p) \ge 0$ for all $x \in [0, x_c)$ and $p \ge C_0|x - x_c|^{\gamma_0 - 1}$.

These assumptions enable us to construct subsolutions and supersolutions for the Cauchy problem and thus to specify the long time profile of the solutions. Another viewpoint is that (A5) and (A6) give a subsolution and a supersolution for the stationary problem (2.1) with homogeneous Dirichlet condition at $x = 0$ and by Perron's method the existence of stationary solutions, missing in the last section, follows immediately.

### *3.1. Large-time Asymptotics*

We adapt our analysis to the setting of solutions which are not necessarily continuous for the following two reasons: (a) to find asymptotic behavior for a discontinuous solution itself is interesting; (b) constructing semicontinuous subsolutions and supersolutions is comparatively easier when we are looking for precise bounds of the solutions of (CD).

Our definition of the solutions of Cauchy-Dirichlet problem is conventional, without the infinity value being involved. For the reader's convenience, we provide in what follows the definition of solutions of

$$u_t + H(x, u_x) = 0 \quad \text{in } (0,\infty)\times(0,\infty), \tag{3.1a}$$

$$u(0,t) = 0 \quad \text{for } t \in (0,\infty), \tag{3.1b}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in [0,\infty). \tag{3.1c}$$

This version of definitions is essentially due to Ishii [23].

**Definition 3.1.** Given any locally bounded function $u_0 : [0,\infty) \to \mathbb{R}$, a locally bounded function $u$ on $([0,\infty)\times[0,\infty))$ is called a subsolution (resp., supersolution) of (3.1a)–(3.1b) if

(i) $u^*(0,t) \le 0$ for all $t \in [0,\infty)$
(resp., $u_*(0,t) \ge 0$ for all $t \in [0,\infty)$);

(ii) whenever there exist $\varphi \in C^1([0,\infty))$ and $(\hat{x},\hat{t}) \in (0,\infty)\times(0,\infty)$ satisfying

$$\max_{(x,t)\in[0,\infty)\times[0,\infty)} (u^* - \varphi)(x,t) = (u^* - \varphi)(\hat{x},\hat{t})$$

$$\left(\text{resp., } \min_{(x,t)\in[0,\infty)\times[0,\infty)} (u_* - \varphi)(x,t) = (u_* - \varphi)(\hat{x},\hat{t})\right),$$

then

$$\varphi_t(\hat{x},\hat{t}) + H(\hat{x}, \varphi_x(\hat{x},\hat{t})) \le 0 \tag{3.2}$$

$$\left(\text{resp., } \varphi_t(\hat{x},\hat{t}) + H(\hat{x}, \varphi_x(\hat{x},\hat{t})) \ge 0\right). \tag{3.3}$$

A locally bounded function $u : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is called a solution if $u$ is both a subsolution and a supersolution.

**Definition 3.2.** We call $u$ a solution of (CD) provided that $u$ is a solution in the sense of Definition 3.1 and ${u_0}_*(x) \le u_*(x,0) \le u^*(x,0) \le {u_0}^*(x)$ for all $x \in [0,\infty)$.

Our main characterization of the large-time behavior is as follows.

**Theorem 3.1** (Main theorem). *Assume (A1)–(A6). Assume that $u_0$ is a locally bounded function which satisfies (1.7) with the function $p_1$ given in (A6). Let $u$ be a solution of (CD). Then for all $x \in [0, x_c) \cup (x_c, \infty)$, $u(x,t) \to v(x)$ as $t \to \infty$, where $v$ is the unique solution of (2.1) in the sense of Definition 2.3 with $v(0) = 0$ and takes value $+\infty$ in $(x_c, \infty)$. Moreover, the convergence is locally uniform respectively in $[0, x_c)$ and in $(x_c, \infty)$.*

*Remark* 3. The theorem states that the asymptotic profile $v$ is independent of $u_0$. It is continuous in $[0, x_c)$ even if $u_0$ is not continuous in $(0, \infty)$. In addition, when $u_0 \in C([0, \infty))$, the solutions $u$ are unique and continuous as well. Theorem 3.1 then reduces to a result of long time behavior for the continuous solution as usual.

The proof, which is by now standard, will be given in the next subsections. The point is to build subsolutions and supersolutions, which both have different behavior in $(0, x_c)$ and $(x_c, \infty)$.

**Example 6.** Let us consider a simple example:

$$\begin{cases} u_t + \arctan u_x^2 - x = 0 & \text{in } (0,\infty) \times (0,\infty), \\ u(0,t) = 0 & \text{for all } t \in (0,\infty), \\ u(x,0) = u_0(x) & \text{for } x \in (0,\infty). \end{cases} \tag{3.4}$$

In this example $x_c = \pi/2$ and the explicit solution of the stationary equation is

$$v(x) = \int_0^{|x|} (\tan y)^{\frac{1}{2}}\, dy$$

for all $x \in [0, \pi/2)$. Theorem 3.1 gives the long time behavior that $u(x,t) \to v(x)$ for $x \in [0, \pi/2]$ and $u(x,t) \to \infty$ for $x > \pi/2$ as $t \to \infty$. For this result we require that the initial data satisfy $u_0(0) = 0$ and the compatibility condition

$$u_0(x) \geq -\int_0^x (\tan y)^{\frac{1}{2}}\, dy \quad \text{in } (0, \pi/2).$$

*3.2. Construction of Sub- and Supersolutions*

We construct a subsolution and a supersolution of (CD) to bound the solution. Our choice of the lower bound is an upper semicontinuous function in the form of

$$w_-(x,t) = \begin{cases} \int_0^x p_1(z)\, dz, & \text{if } 0 \leq x \leq x_c; \\ W_1(x) - h_1(x)t, & \text{if } x > x_c, \end{cases} \tag{3.5}$$

where $h_1(x) := \sup_{p\in\mathbb{R}} H(x,p) < 0$ for all $x \in (x_c,\infty)$ by (A3) and $W_1 \in USC([0,\infty))$ will be determined later. On the other hand, an upper bound is written as

$$w_+^\delta(x,t) = \begin{cases} W_2(x) & \text{if } 0 \le x \le x_c - \delta; \\ W_2(x) - h_2(x)t, & \text{if } x > x_c - \delta, \end{cases} \tag{3.6}$$

where $\delta > 0$ is taken small, $h_2(x) := \inf_{p\in\mathbb{R}} H(x,p) < 0$ for all $x \in (x_c,\infty)$ and $W_2$ is also to be determined in terms of the initial data $u_0$.

The construction of a subsolution is simpler.

**Lemma 3.2** (A lower bound)**.** *Assume (A5). For any locally bounded lower semicontinuous function $f : [0,\infty) \to \mathbb{R}$ which satisfies*

$$f(x) \ge \int_0^x p_1(z)\,dz \quad \textit{for all } 0 \le x \le x_c,$$

*where $p_1$ is the function given in (A5), let $W_1 \in C([0,\infty)) \cap C^1((0,\infty))$ be such that $W_1(x) = \int_0^x p_1(z)\,dz$ and $W_1 \le f$ in $(x_c,\infty)$. Then $w_- \in USC([0,\infty) \times [0,\infty))$ as defined in* (3.5) *is a subsolution of* (3.1a) *with $w_-(0,t) = 0$ and $w_-(x,0) \le f(x)$ for all $x,t \in [0,\infty)$.*

*Proof.* It is clear that $w_-(x,0) \le f(x)$ and $w_-$ is continuous except at $x = x_c$. By the definition of $p_1$, we obtain with great ease that

$$(w_-)_t(x,t) + H(x,(w_-)_x(x,t)) = H(x,p_1(x)) \le 0 \text{ for } x \in (0,x_c).$$

For every $x \in [x_c,\infty)$, whenever there exist a test function $\varphi \in C^1([0,\infty)^2)$ and $t > 0$ satisfying

$$(w_- - \varphi)(x,t) = \max_{(y,s)\in(0,\infty)^2} (w_- - \varphi)(y,s),$$

we immediately have $\varphi_t(x,t) = -h_1(x)$. It then follows that

$$\varphi_t(x,t) + H(x,\varphi_x(x,t)) \le -h_1(x) + \sup_{p\in\mathbb{R}} H(x,p) = 0.$$

□

**Lemma 3.3** (An upper bound)**.** *Assume (A6). Then for any locally bounded upper semicontinuous function $f : [0,\infty) \to \mathbb{R}$ with $f(0) \le 0$, there exists $W_2 \in LSC([0,\infty))$ such that for every small $\delta > 0$, $w_+^\delta \in LSC([0,\infty) \times [0,\infty))$ is a supersolution of* (3.1a) *and satisfies $w_+^\delta(0,t) = 0$ and $w_+^\delta(x,0) \ge f(x)$ for all $x,t \in [0,\infty)$.*

*Proof.* Take a constant $\delta < r < x_c$. We build $W_2$ in a proper way according to $f$ so that $w_+^\delta$ has jumps merely on $x = x_c - r$ and $x = x_c - \delta$.

In terms of the assumptions given, it is possible to choose a nondecreasing function $f_1 \in C([0, \infty))$ satisfying $f_1(0) = 0$ and $f_1 \geq f$ as well as

$$f_1(x_c - r) \geq \max_{[x_c - r, x_c]} f. \tag{3.7}$$

We construct $W_1$ in three intervals.

(i) For $x \in (0, x_c - r]$, we use a variation of the method to regularize a modulus. Consult [17, Lemma 2.1.9] for more details. Set

$$f_2(x) = \left( \max_{y \in [x, x_c - r]} \frac{f_1(y)}{y} \right) \quad \text{for all } x \in (0, x_c - r]$$

and $F(x) = \int_{x/2}^{x} 2 f_2(y)\, dy$. Then it is easy to show that $F \in C^1((0, x_c - r]) \cap C([0, x_c - r])$ with $F(0) = 0$ and $F(x) \geq f_1(x)$ for $x \in [0, x_c - r]$. However $F$ may not be a stationary supersolution in $(0, x_c - r) \times (0, \infty)$. To overcome this, notice in (A6) that there is $\hat{p} > 0$ such that $\max_{x \in [0, x_c - \delta]} H(x, p) \geq 0$ for all $p \geq \hat{p}$. We next only need to set

$$W_2(x) := \int_0^x \max \left\{ \frac{dF}{dy}(y), \hat{p} \right\} dy$$

and it is then obvious that $w_+^\delta$ is a supersolution in $(0, x_c - r) \times (0, \infty)$.

(ii) For $x \in (x_c - r, x_c - \delta]$, set $W_2(x) := C_1 |x - x_c|^{\gamma_0} + C_2$. It then follows by (A6) that

$$H(x, \frac{d}{dx} W_2(x)) \geq 0 \quad \text{for } x \in (x_c - r, x_c - \delta), \tag{3.8}$$

and $\lim_{x \to (x_c - r)+} W_2(x) \geq \lim_{x \to (x_c - r)-} W_2(x_c - r)$ when $C_1, C_2$ are sufficiently large without depending on $\delta$. In addition, in view of (3.7) we have

$$w_+^\delta(x, 0) \geq f_1(x_c - r) \geq f(x) \quad \text{for } x \in [x_c - r, x_c].$$

(iii) For $x \in (x_c - \delta, \infty]$, we extend $W_2$ so that $W_2(x) \geq f(x)$ and then get

$$(w_+^\delta)_t + H(x, (w_+^\delta)_x) \geq -h_2(x) + \inf_{p \in \mathbb{R}} H(x, p) = 0$$

for $(x, t) \in (x_c - \delta, \infty) \times (0, \infty)$.

It remains to show $w_+^\delta$ is a subsolution at $x = x_c - r$ and $x = x_c - \delta$, where the jumps of value take place. Suppose there is a test function $\varphi$ touching $w_+^\delta$ from below at $x = x_c - r$, then our construction of $W_2$ yields $\varphi_x(x_c - r, t) \geq \hat{p}$ and hence by (A6)

$$\varphi_t(x_c - r, t) + H(x_c - r, \varphi_x(x_c - r, t)) \geq 0.$$

The same argument, together with an application of (A6), works for the verification at $x = x_c - \delta$ as well. □

We remark that the existence of solutions of (CD) follows easily since we have constructed the subsolution and supersolution associated with the classical boundary condition. More precisely, we can show that the functions

$$b_-(x,t) = w_-(x,t) \vee (u_0(x) - h_1(x)t) \tag{3.9}$$

and

$$b_+(x,t) = w_+^\delta(x,t) \wedge (u_0(x) - h_2(x)t) \tag{3.10}$$

are respectively a subsolution and a supersolution of (CD). Our assumptions (A2) and (A5) yield $b_- \leq b_+$, which enables us to show the existence of solutions of (CD) by Perron's method [23].

Before proving our main theorem, we need a comparison theorem for our Cauchy-Dirichlet problem.

**Theorem 3.4** (Comparison principle of (CD))**.** *Assume (A1). Let $u_1$ and $u_2$ be respectively a subsolution and a supersolution of (3.1a)–(3.1b). If ${u_1}^*(x,0) \leq u_{2*}(x,0)$ for all $x \in [0,\infty)$, then ${u_1}^* \leq u_{2*}$ in $[0,\infty) \times [0,\infty)$.*

This theorem works for unbounded solutions. It is analogous to the following theorem for Cauchy problem presented in [22], which is reproduced in [3]. We apply their idea to the setting of semicontinuous solutions.

**Theorem 3.5** ([3, Theorem 3.16])**.** *Let $H : \mathbb{R}^{2N} \to \mathbb{R}$ be continuous. Assume that there exists a constant $K > 0$ such that*

$$|H(x,p) - H(x,q)| \leq K(|x|+1)|p-q|$$

*for all $x, p, q \in \mathbb{R}^N$. Assume that $H$ also satisfies*

$$|H(x,p) - H(y,p)| \leq \omega(|x-y|, R) + \omega(|x-y||p|, R)$$

*for all $p \in \mathbb{R}^N$, $x, y \in B_R(0), R > 0$. Here $\omega(\cdot, R)$ is a modulus of continuity for any $R > 0$. If $u_1, u_2 \in C(\mathbb{R}^N \times [0,T])$ are respectively a viscosity sub- and supersolution of*

$$u_t + H(x, \nabla u) = 0 \quad \text{in } \mathbb{R}^N \times (0,T)$$

*and $u_1(x,0) \le u_2(x,0)$ for all $x \in \mathbb{R}^N$, then $u_1 \le u_2$ in $\mathbb{R}^N \times [0,T]$.*

The proof of Theorem 3.5 relies on "a cone of dependence" type argument. To prove Theorem 3.4, it consequently suffices to show the comparison theorem in any cone relative to our domain; namely, for any fixed $T > 0$ and $x_0 \ge 0$, let

$$\mathcal{C}_D = \{|x - x_0| \le C|T - t| \text{ and } x \ge 0\}.$$

We denote the parts of its boundary lying on axes by

$$\Gamma_1 = \{(x,t) \in [0,\infty) \times \{0\} : |x - x_0| \le CT\}$$

and

$$\Gamma_2 = \{(x,t) \in \{0\} \times [0,\infty) : t \le T - x_0/C\},$$

where $C > 0$ is the constant given in (A1).

**Lemma 3.6** (Local comparison in cones)**.** *Assume (A1). Let $u_1$ and $u_2$ be respectively a viscosity subsolution and supersolution of (3.1a)–(3.1b). If ${u_1}^*(x,t) \le {u_2}_*(x,t)$ for all $(x,t) \in \Gamma_1 \cup \Gamma_2$, then ${u_1}^*(x,t) \le {u_2}_*(x,t)$ in $\mathcal{C}_D$.*

*Sketch of proof.* Assume by contradiction that there exist $0 < \delta < T$ and $(\tilde{x}, \tilde{t})$ such that

$$(u_1 - u_2)(\tilde{x}, \tilde{t}) = \delta \text{ and } |\tilde{x} - x_0| \le C(T - \tilde{t}) - 2\delta$$

and $\sup_{(x,t,y,s)\in\mathcal{C}_D^2}(u_1(x,t) - u_2(y,s)) = N > 0$. Doubling the variables, we define a function $\Phi$ by

$$\begin{aligned}\Phi(x,y,t,s) =& u_1(x,t) - u_2(y,s) - \frac{|x-y|^2 + |t-s|^2}{2\varepsilon} - k(t+s) \\ &+ g(\langle x - x_0\rangle_\beta - C(T-t)) + g(\langle y - x_0\rangle_\beta - C(T-s)),\end{aligned}$$

where $\varepsilon > 0$, $k > 0$, $\langle x\rangle_\beta := (x^2 + \beta^2)^{\frac{1}{2}}$ for $\beta > 0$ and $g \in C^1(\mathbb{R})$ is such that $g' \le 0$ in $\mathbb{R}$, $g(x) = 0$ for $x \le -\delta$ and $g(x) = -3N$ for $x \ge 0$.

The penalization with the utilization of $g$ and comparison conditions along $x = 0$ and $t = 0$ eliminate the possibility of finding maximizers of $\Phi$ on any edge of $\mathcal{C}_D$ when $k$ and $\beta$ are sufficiently small. Then the standard arguments come into play such as showing convergence of maximizers and taking difference of two viscosity inequalities. $\square$

*3.3. Proof of Main Theorem*

We take the relaxed limits

$$U_1 = \operatorname*{limsup^*}_{t\to\infty} u \text{ and } U_2 = \operatorname*{liminf_*}_{t\to\infty} u$$

and show that they have the following properties.

**Proposition 3.7** (Properties of half relaxed limits)**.** *Assume that $u_0$ satisfies (1.7). Then $U_1$ and $U_2$ have the following properties:*

*(i)* $U_1$ *and* $U_2$ *are both bounded from below in* $[0, x_c]$*;*

*(ii) If* $\gamma < \gamma_0$*, then* $U_1 - \beta|x - x_c|^\gamma$ *and* $U_2 - \beta|x - x_c|^\gamma$ *are both bounded from above in* $[0, x_c)$ *for any* $\beta > 0$*;*

*(iii)* $U_1(0) \le 0 \le U_2(0)$*.*

*(iv)* $U_1 = U_2 = +\infty$ *in* $(x_c, \infty)$*.*

*Proof.* Take $f = u_{0*}$ in Lemma 3.2 and $f = {u_0}^*$ in Lemma 3.3 and we therefore note that $u_0(0) = w_-(0, t) = w_+(0, t)$ for all $t \in [0, \infty)$ and

$$w_-(x, 0) \le u_{0*}(x) \le {u_0}^*(x) \le w_+(x, 0) \quad \text{for all } x \in [0, \infty).$$

Hence, by Theorem 3.4, we have

$$w_- \le u \le w_+ \text{ in } [0, \infty) \times [0, \infty).$$

All of the statements thus follow easily by our choices of $w_-$ and $w_+$. To prove (ii) for example, one can actually show $U_1, U_2 \le C_1|x - x_c|^{\gamma_0} + C_2$ for any $x \in (0, x_c)$ due to our concrete form of $w_+^\delta$ in Lemma 3.3. Then the assertion becomes clear. □

**Proposition 3.8** (Viscosity inequalities for half limits)**.** *$U_1$ and $U_2$ are respectively a subsolution and a supersolution of (2.1).*

*Proof.* Let us prove the part for $U_1$ first. Since Proposition 3.7(iv) holds, there is no need to apply Definition 2.2 in $(x_c, \infty)$. We thus turn our attention to the interval $(0, x_c]$. By definition, we need to take test functions for $\overline{U_1}$.

For any $\phi \in C^1([0, \infty))$ such that $\overline{U_1}(x) - \phi(x)$ attains a strict minimum at $x = x_c$, by (A1) and (A3), we have

$$H(x_c, \phi_x(x_c)) \le 0,$$

which means $U_1$ is a subsolution at $x = x_c$.

We finally handle the case when $\overline{U_1}$ is tested at $x \in (0, x_c)$. Notice that $U_1(x) = \overline{U_1}(x)$ in this case. Put $u^\varepsilon(x,t) = u(x, \frac{t}{\varepsilon})$ for every $x \in [0,\infty)$ and $t \in [0,\infty)$ and then further let

$$u_1(x,t) := (\limsup_{\varepsilon\to 0}{}^* u^\varepsilon)(x,t) \text{ and } u_2(x,t) := (\liminf_{\varepsilon\to 0}{}_* u^\varepsilon)(x,t).$$

It is easily seen that $u^\varepsilon$ solves

$$\varepsilon u^\varepsilon_t + H(x, u^\varepsilon_x) = 0 \quad \text{in } (0, x_c) \times (0,\infty),$$

so the stability of viscosity solutions guarantees

$$H(x, (u_1)_x) \le 0 \quad \text{in } (0, x_c) \times (0,\infty)$$

in the viscosity sense. Indeed, for any $\phi \in C^1([0,\infty) \times [0,\infty))$ such that $u_1 - \phi$ attains a strict maximum at $(\hat{x}, \hat{t}) \in (0, x_c) \times (0,\infty)$, there exist a sequence $\varepsilon_n \to 0$ and $(x_n, t_n) \to (\hat{x}, \hat{t})$ as $n \to \infty$ satisfying

$$\begin{aligned} &(u^{\varepsilon_n})^*(x_n, t_n) \to u_2(\hat{x}, \hat{t}); \\ &\left((u^{\varepsilon_n})^* - \phi\right)(x_n, t_n) = \max_{[0,x_c)\times[0,\infty)} \left((u^{\varepsilon_n})^* - \phi\right). \end{aligned}$$

(Refer to [17, Lemma 2.2.5] for a very general version of this maximizer convergence result.) Then taking into account the fact that $u$ is a solution of the Cauchy-Dirichlet problem, we obtain

$$\varepsilon_n \phi_t(x_n, t_n) + H(x, \phi_x(x_n, t_n)) \le 0.$$

We conclude by passing to the limit $n \to \infty$ and noticing that $u_1 = U_1$ is independent of the variable $t$.

The other part of our statements about $U_2$ can be shown more easily thanks to $U_2 = \underline{U_2}$. □

*Proof of Theorem 3.1.* Combining Theorem 2.5, Propositions 3.7 and 3.8, we are led to

$$U_1 \le U_2 \text{ in } [0, x_c),$$

which obviously implies that $U_1(x) = U_2(x) = v(x)$ for $x \in [0, x_c)$ and thus $u(\cdot, t) \to v$ locally uniformly in $[0, x_c)$ as $t \to \infty$.

On the other hand, observing the concrete form of $w_-$ and $w_+^\delta$, we get $u(\cdot, t) \to +\infty$ locally uniformly in $(x_c, \infty)$ as $t \to \infty$. □

*3.4. Remark on the Dirichlet Condition*

Since in this paper we only study the Dirichlet boundary condition in the strict sense, several assumptions should be viewed as the compatibility condition on the boundary. It is certainly interesting to understand the long time behavior when these assumptions are dropped. Let us repeat our assumption on the initial data in Theorem 3.1. We only treat $u_0$ which satisfies the conditions below.

(a) $u_0$ is locally bounded in $[0, \infty)$ and is continuous at $x = 0$ with $u_0(0) = 0$.
(b) $u_{0*}(x) \geq \int_0^x p_1(z)\, dz$ for all $x \in [0, x_c)$, where $p_1$ is given in (A5).

The condition (b) plays an important role and can hardly be relaxed especially when $p_1$ is taken minimal. To see this, we give a simple example in the following.

**Example 7.** Let us revisit Example 6. Suppose $H(x, p) = \arctan(p^2) - x$. Then all assumptions (A1)–(A6) are satisfied. In particular, $x_c = \pi/2$ and $p_1(x) = -(\tan(x))^{\frac{1}{2}}$ in this case. Since such a Hamiltonian is smooth, the equation of its characteristics writes

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{2p}{1+p^4} \\ \dfrac{dp}{dt} = 1 \end{cases}$$

with prescribed initial data

$$x(0) = x_0 \text{ and } p(0) = p_0.$$

Its solution could be explicitly calculated as

$$x(t) = x_0 + \arctan(p_0 + t)^2 - \arctan p_0^2 \quad \text{and} \quad p(t) = p_0 + t,$$

which evidently demonstrates that the trajectory $x(t)$ starting from $x_0 \in (0, \pi/2)$ will hit the boundary $x = 0$ before time $t = -p_0$ whenever $p_0 < p_1(x_0)$. On this occasion, the solution $u$ must violate the classical Dirichlet boundary condition.

The seemingly particular value $p_1(0) = 0$ in the example above does not actually cause any loss of generality. One may observe analogous examples such as $H(x, p) = \arctan(p^2 - a^2)^2 - x$, where $p_1(0) = -|a|$ with $a \in \mathbb{R}$ arbitrarily chosen.

**Acknowledgments**

The authors are grateful to Professor Etsuro Yokoyama and Professor Hitoshi Ishii for fruitful discussions.

The work of the first author was partly supported by Grant-in-aid for Scientific Research, No. 21224001(Kiban S), the Japan Society for the Promotion of Science (JSPS). The work of the second and third authors was supported by Grants-in-aid for Scientific Research of JSPS Fellows No. 21-7428 (Q. Liu) and No. 22-1725 (H. Mitake).

This work was completed while the authors visited the Department of Mathematics and the Research Center for Integrative Mathematics, Hokkaido University for a minisemester on evolution of interfaces, Sapporo 2010. Its hospitality is gratefully acknowledged.

## References

[1] M. Arisawa and P. L. Lions, *On ergodic stochastic control*, Comm. Partial Differential Equations, 23 (1998), 2187-2217.

[2] M. Bardi, *A boundary value problem for the minimum-time function*, SIAM J. Control Optim., 27 (1989), 776–785.

[3] M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Systems and Control: Foundations and Applications, Birkhäuser Boston, Boston, 1997.

[4] M. Bardi and P. Soravia, *Hamilton-Jacobi equations with singular boundary conditions on a free boundary and applications to differential games*, Trans. Amer. Math. Soc., 325 (1991), 205–229.

[5] G. Barles, *Some homogenization results for non-coercive Hamilton-Jacobi equations*, Calc. Var. Partial Differential Equations, 30 (2007), 449–466.

[6] G. Barles and H. Mitake, *A PDE approach to large-time asymptotics for boundary-value problems for Hamilton-Jacobi equations*, preprint.

[7] G. Barles and J. M. Roquejoffre, *Ergodic type problems and large time behavior of unbounded solutions of Hamilton-Jacobi equations*, Comm. Partial Differential Equations, 31 (2006), 1209–1225.

[8] G. Barles and P. E. Souganidis, *On the large time behavior of solutions of Hamilton-Jacobi equations*, SIAM J. Math. Anal., 31 (2000), 925–939.

[9] W. K. Burton, N. Cabrera and F. C. Frank, *The growth of crystals and the equilibrium structure of their surfaces*, Phils. Tran. Roy. Soc. London A, 243 (1951), 299–358

[10] P. Cardaliaguet, *Ergodicity of Hamilton-Jacobi equations with a non coercive non convex Hamilton in $\mathbb{R}^2/\mathbb{Z}^2$*, preprint.

[11] A. A. Chernov, *Stability of faceted shapes*, J. Cryst. Growth, 24/25 (1974), 11–31

[12] A. Davini and A. Siconolfi, *A generalized dynamical approach to the large time behavior of solutions of Hamilton-Jacobi equations*, SIAM J. Math. Anal., 38 (2006), 478–502.

[13] L. C. Evans and M. R. James, *The Hamiltonian-Jacobi-Bellman equation for time-optimal control*, SIAM J. Control Optim., 6 (1989), 1477–1489.

[14] A. Fathi, *Theoreme KAM faible et theorie de Mather sur les systemes lagrantiens*, C. R. Acad. Sci. Paris Ser. I Math., 324 (1997), 1043–1046.

[15] A. Fathi, *Sur la convergence du semi-groupe de Lax-Oleinik*, C. R. Acad. Sci. Paris Sér. I Math., 327 (1998), 267–270.

[16] A. Fathi and A. Siconolfi, *PDE aspects of Aubry-Mather theory for quasiconvex Hamiltonians*, Calc. Var. Partial Differential Equations, 22 (2005), 185–228.

[17] Y. Giga, *Surface evolution equations, a level set approach*, Monographs in Mathematics 99, Birkhäuser Verlag, Basel, 2006.

[18] Y. Giga, Q. Liu and H. Mitake, *Singular Neumann problems and large-time behavior of solutions of noncoercive Hamilton-Jacobi equations*, preprint.

[19] N. Ichihara and H. Ishii, *Asymptotic solutions of Hamilton-Jacobi equations with semi- periodic Hamiltonians*, Comm. Partial Differential Equations 33 (2008), 784–807.

[20] N. Ichihara and H. Ishii, *The large-time behavior of solutions of Hamilton-Jacobi equations on the real line*, Methods Appl. Anal., 15 (2008), 223–242.

[21] N. Ichihara and H. Ishii, *Long-time behavior of solutions of Hamilton-Jacobi equations with convex and coercive Hamiltonians*, Arch. Ration. Mech. Anal. 194 (2009), 383–419.

[22] H. Ishii, *Uniqueness of unbounded viscosity solution of Hamilton-Jacobi equations*, Indiana Univ. Math. J., 33 (1984), 721–748.

[23] H. Ishii, *Perron's method for Hamilton-Jacobi equations*, Duke Math. J., 55 (1987), 369–384.

[24] H. Ishii, *A simple, direct proof of uniqueness for solutions of the Hamilton-Jacobi equations of Eikonal type*, Proc. Amer. Math. Soc., 100 (1987), 247–251.

[25] H. Ishii, *Asymptotic solutions for large time of Hamilton-Jacobi equations in Euclidean n space*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 25 (2008), 231–266.

[26] H. Ishii, *Long-time asymptotic solutions of convex Hamilton-Jacobi equations with Neumann type boundary conditions*, preprint.

[27] H. Ishii and H. Mitake, *Representation formulae for solutions of Hamilton-Jacobi equations with convex Hamiltonians*, Indiana Univ. Math. J. 56 (2007), 2159–2183.

[28] J. M. Lasry and P. L. Lions, *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints*, Math. Ann. 283 (1989), 583–630.

[29] P. L. Lions, G. Papanicolaou and S. R. S. Varadhan, *Homogenization of Hamilton-Jacobi equations*, preprint.

[30] H. Mitake, *Asymptotic solutions of Hamilton-Jacobi equations with state constraints*, Appl. Math. Optim. 58 (2008), 393–410.

[31] H. Mitake, *The large-time behavior of solutions of the Cauchy-Dirichlet problem for Hamilton-Jacobi equations*, NoDEA Nonlinear Differential Equations App., 15 (2008), 347–362.

[32] H. Mitake, *Large time behavior of solutions of Hamilton-Jacobi equations with periodic boundary data*, Nonlinear Anal., 71 (2009), 5392–5405.

[33] G. Namah and J. M. Roquejoffre, *Remarks on the long time behavior of the solutions of Hamilton-Jacobi equations*, Comm. Partial Differential Equations, 24 (1999), 883–893.

[34] J. M. Roquejoffre, *Convergence to steady states or periodic solutions in a class of Hamilton-Jacobi equations*, J. Math. Pures Appl. 80 (2001), 85–104.

[35] E. Yokoyama, Y. Giga, P. Rybka, *A microscopic time scale approximation to the behavior of the local slope on the faceted surface under a nonuniformity in supersaturation*, Physica D, 237 (2008), 2845–2855.