ON THE PERIOD OF THE IKEDA LIFT FOR $U(m,m)$

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Abstract. Let $K = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$, and χ the Dirichlet character corresponding to the extension $K/\mathbf{Q}$. Let $m = 2n$ or $2n+1$ with n a positive integer. Let f be a primitive form of weight $2k+1$ and character χ for $\Gamma_0(D)$, or a primitive form of weight $2k$ for $SL_2(\mathbf{Z})$ according as $m = 2n$, or $m = 2n+1$. For such an f let $I_m(f)$ be the lift of f to the space of modular forms of weight $2k+2n$ and character $\det^{-k-n}$ for the Hermitian modular group $\Gamma_K^{(m)}$ constructed by Ikeda. We then express the period $\langle I_m(f), I_m(f)\rangle$ of $I_m(f)$ in terms of special values of the adjoint L-functions of f twisted by the character χ. This poves the conjecture concerning the period of the Ikeda lift proposed by Ikeda.

1. Introduction

In a joint paper with Kawamura [KK10b], we proved Ikeda's conjecture on the period of the Duke-Imamoglu-Ikeda lift. In this paper we prove a similar result for the period of the lift of an elliptic modular form to the space of Hermitian modular forms constructed by Ikeda. This also proves Ikeda's conjecture proposed in [Ike08] with some modification. Let $K = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$. Let $\mathcal{O}$ be the ring of integers in K, and χ the Kronecker character corresponding to the extension $K/\mathbf{Q}$. For a non-degenerate Hermitian matrix or alternating matrix T with entries in K, let $\mathcal{U}_T$ be the unitary group defined over $\mathbf{Q}$, whose group $\mathcal{U}_T(R)$ of R-valued points is given by

$$\mathcal{U}_T(R) = \{g \in GL_m(R \otimes K) \mid {}^t\overline{g}Tg = T\}$$

for any $\mathbf{Q}$-algebra R, where $\overline{g}$ denotes the automorphism of $M_n(R \otimes K)$ induced by the non-trivial automorphism of K over $\mathbf{Q}$. We also define the special unitary group $\mathcal{SU}_T$ over $\mathbf{Q}_p$ by $\mathcal{SU}_T = \mathcal{U}_T \cap R_{K/\mathbf{Q}}(SL_m)$, where $R_{K/\mathbf{Q}}$ is the Weil restriction. In particular we write $\mathcal{U}_T$ as $\mathcal{U}^{(m)}$ or $U(m,m)$ if $T = \left(\begin{smallmatrix} O & -1_m \\ 1_m & O \end{smallmatrix}\right)$. For a more precise description of $\mathcal{U}^{(m)}$ see Section 2. Put $\Gamma_K^{(m)} = U(m,m)(\mathbf{Q}) \cap GL_{2m}(\mathcal{O})$. Let k be a non-negative integer. Then for a primitive form $f \in \mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$ Ikeda

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[Ike08] constructed a lift $I_{2n}(f)$ of f to the space of modular forms of weight $2k+2n$ and a character $\det^{-k-n}$ for $\Gamma_K^{(2n)}$. This is a generalization of the Maass lift considered by Kojima [Koj82], Gritsenko [Gri90], Krieg [Kri91] and Sugano [Su95]. Similarly for a primitive form $f \in \mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ he constructed a lift $I_{2n+1}(f)$ of f to the space of modular forms of weight $2k+2n$ and a character $\det^{k+n}$ for $\Gamma_K^{(2n+1)}$. For the rest of this section, let $m=2n$ or $m=2n+1$. We then call $I_m(f)$ the Ikeda lift of f for $U(m,m)$ or the Hermitian Ikeda lift of degree m. Ikeda also showed that the automorphic form $Lift^{(m)}(f)$ on the adele group $\mathcal{U}^{(m)}(\mathbf{A})$ associated with $I_m(f)$ is a cuspidal Hecke eigenform whose standard L-function coincides with

$$\prod_{i=1}^{m} L(s+k+n-i+1/2, f)L(s+k+n-i+1/2, f, \chi),$$

where $L(s+k+n-i+1/2, f)$ is the Hecke L-function of f and $L(s+k+n-i+1/2, f, \chi)$ is its "modified twist" by χ. For the precise definition of $L(s+k+n-i+1/2, f, \chi)$ see Section 2. We also call $Lift^{(m)}(f)$ the adelic Ikeda lift of f for $U(m,m)$. Then our main result (Theorem 2.3) can be stated as follows:

The period $\langle I_m(f), I_m(f)\rangle$ of $I_m(f)$ is expressed as

$$L(1, f, \mathrm{Ad}) \prod_{i=2}^{m} L(i, f, \mathrm{Ad}, \chi^{i-1}) L(i, \chi^i)$$

up to elementary factor, where $L(s, f, \mathrm{Ad}, \chi^{i-1})$ is the "modified twist" of the adjoint L-function of f by χ^{i-1}, and $L(i, \chi^i)$ is the Dirichlet L-function for χ^i.

This result was already obtained in the case $m=2$, and was conjectured in general case by Ikeda [Ike08].

We note that $I_m(f)$ is not likely to be a theta lift except in the case $m=2$, and therefore the method in [Ral88] cannot be applied to prove our main result. The method we use is similar to that in the proof of the main result of [KK10b] and to give an explicit formula of the Dirichlet series of Rankin-Selberg type associated to $I_m(f)$, and compare their residues. We explain it more precisely. In Section 3, we consider the Dirichlet series $R(s, I_m(f))$ of Rankin Selberg type associated with $I_m(f)$. For the precise definition, see Section 3. This type of Dirichlet series was studied by Shimura [Sh00] for a classical Hermitian form F of weight $2k+2n$. In particular we can express its residue at $2k+2n$ in terms of the period of F (cf. Proposition 3.1.) Thus to prove Theorem 2.3, we have to get an explicit formula of $R(s, I_m(f))$ in terms of $L(s, f, \mathrm{Ad}, \chi^i)$. To get it, in Section 4, we reduce our computation to a computation of certain formal power series $H_{m,p}(d; X, Y, t)$ in t associated with local Siegel series similarly to [KK10b] (cf. Theorem 4.4).

Section 5 is devoted to the computation of them. This computation is similar to that in [KK10b], but we should be careful in dealing with the case where p is ramified in K. After such an elaborate computation, we can get explicit formulas of $H_{m,p}(d;X,Y,t)$ for all prime numbers p (cf. Theorems 5.5.2 and 5.5.3, and Corollary to Proposition 5.5.3.) In Section 6, by using explicit formulas for $H_{m,p}(d;X,Y,t)$, we immediately get an explicit formula of $R(s,I_m(f))$ (cf. Theorems 6.1 and 6.2) and by taking the residue of it at $2k+2n$ we prove the Theorem 2.3.

We note that we can give a similar period relation for the adelic Ikeda lift, and can apply it to a problem concerning congruence between the adelic Ikeda lifts and Hecke eigenforms not coming from the adelic Ikeda lifts. These will be discussed in subsequent papers. We also note that this type of result has been already given by [Bro07], [Kat08a], [Kat08b] in Siegel modular forms case.

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Notation. Let R be a commutative ring. We denote by $R^\times$ and R^* the semigroup of non-zero elements of R and the unit group of R, respectively. For a subset S of R we denote by $M_{mn}(S)$ the set of (m,n)-matrices with entries in S. In particular put $M_n(S)=M_{nn}(S)$. Put $GL_m(R)=\{A\in M_m(R)\mid \det A\in R^*\}$, where $\det A$ denotes the determinant of a square matrix A. Let K_0 be a field, and K a quadratic extension of K_0, or $K=K_0\oplus K_0$. In the latter case, we regard K_0 as a subring of K via the diagonal embedding. We also identify $M_{mn}(K)$ with $M_{mn}(K_0)\oplus M_{mn}(K_0)$ in this case. If K is a quadratic extension of K_0, let ρ be the non-trivial automorphism of K over K_0, and if $K=K_0\oplus K_0$, let ρ be the automorphism of K defined by $\rho(a,b)=(b,a)$ for $(a,b)\in K_0$. We sometimes write $\overline{x}$ instead of $\rho(x)$ for $x\in K$ in both cases. Let R be a subring of K. For an (m,n)-matrix $X=(x_{ij})_{m\times n}$ write $X^*=(\overline{x_{ji}})_{n\times m}$, and for an (m,m)-matrix A, we write $A[X]=X^*AX$. Let $\mathrm{Her}_n(R)$ denote the set of Hermitian matrices of degree n with entries in R, that is the subset of $M_n(R)$ consisting of matrices X such that $X^*=X$. Then a Hermitian matrix A of degree n with entries in K is said to be semi-integral over R if $\mathrm{tr}(AB)\in K_0\cap R$ for any $B\in\mathrm{Her}_n(R)$, where tr denotes the trace of a matrix. We denote by $\widehat{\mathrm{Her}}_n(R)$ the set of semi-integral matrices of degree n over R.

For a subset S of $M_n(R)$ we denote by $S^\times$ the subset of S consisting of non-degenerate matrices. If S is a subset of $\mathrm{Her}_n(\mathbf{C})$ with $\mathbf{C}$ the field of complex numbers, we denote by S^+ the subset of S consisting of positive definite matrices. The group $GL_n(R)$ acts on the set $\mathrm{Her}_n(R)$

in the following way:

$$GL_n(R) \times \mathrm{Her}_n(R) \ni (g, A) \longrightarrow g^*Ag \in \mathrm{Her}_n(R).$$

Let G be a subgroup of $GL_n(R)$. For a subset $\mathcal{B}$ of $\mathrm{Her}_n(R)$ stable under the action of G we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ with respect to $GL_n(R)$. We sometimes identify $\mathcal{B}/G$ with a complete set of representatives of $\mathcal{B}/G$. We abbreviate $\mathcal{B}/GL_n(R)$ as $\mathcal{B}/\sim$ if there is no fear of confusion. Two Hermitian matrices A and A' with entries in R are said to be G-equivalent and write $A \sim_G A'$ if there is an element X of G such that $A' = A[X]$. For square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

We put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbf{C}$, and for a prime number p we denote by $\mathbf{e}_p(*)$ the continuous additive character of $\mathbf{Q}_p$ such that $\mathbf{e}_p(x) = \mathbf{e}(x)$ for $x \in \mathbf{Q}$.

For a prime number p we denote by $\mathrm{ord}_p(*)$ the additive valuation of $\mathbf{Q}_p$ normalized so that $\mathrm{ord}_p(p) = 1$, and put $|x|_p = p^{-\mathrm{ord}_p(x)}$. Moreover we denote by $|x|_\infty$ the absolute value of $x \in \mathbf{C}$.

2. Period of the Ikeda lift for $U(m, m)$

Throughout the paper, we fix an imaginary quadratic extension K of $\mathbf{Q}$ with the discriminant $-D$, and denote by $\mathcal{O}$ the ring of integers in K. For such a K let $\mathcal{U}^{(m)} = U(m, m)$ be the unitary group defined in Section 1. Put $J_m = \begin{pmatrix} O_m & -1_m \\ 1_m & O_m \end{pmatrix}$, where 1_m denotes the unit matrix of degree m. Then

$$\mathcal{U}^{(m)}(\mathbf{Q}) = \{M \in GL_{2m}(K) \mid J_m[M] = J_m\}.$$

Put

$$\Gamma^{(m)} = \Gamma_K^{(m)} = \mathcal{U}^{(m)}(\mathbf{Q}) \cap GL_{2m}(\mathcal{O}).$$

Let $\mathfrak{H}_m$ be the Hermitian upper half-space defined by

$$\mathfrak{H}_m = \{Z \in M_m(\mathbf{C}) \mid \frac{1}{2\sqrt{-1}}(Z - Z^*) \text{ is positive definite}\}.$$

The group $\mathcal{U}^{(m)}(\mathbf{R})$ acts on $\mathfrak{H}_m$ by

$$g\langle Z\rangle = (AZ + B)(CZ + D)^{-1} \text{ for } g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{U}^{(m)}(\mathbf{R}), Z \in \mathfrak{H}_m.$$

We also put $j(g, Z) = CZ + D$ for such Z and g. Let l be an integer. For a subgroup Γ of $\Gamma^{(m)}$ and a character ψ of Γ, we denote by $\mathfrak{M}_l(\Gamma, \psi)$ the space of holomorphic modular forms of weight l with character ψ for Γ. We denote by $\mathfrak{S}_l(\Gamma, \psi)$ the subspace of $\mathfrak{M}_l(\Gamma, \psi)$ consisting of cusp forms. In particular, if ψ is the character of Γ defined by $\psi(\gamma) = (\det\gamma)^{-l}$ for $\gamma \in \Gamma$, we write $\mathfrak{M}_{2l}(\Gamma, \psi)$ as $\mathfrak{M}_{2l}(\Gamma, \det^{-l})$, and so on. Write the variable Z on $\mathfrak{H}_m$ as $Z = X + \sqrt{-1}Y$ with $X, Y \in \mathrm{Her}_m(\mathbf{C})$. We can identify $\mathrm{Her}_m(\mathbf{C})$ with $\mathbf{R}^{m^2}$ through the map $X = (x_{ij}) \longrightarrow$

$(x_{ii}, \mathrm{Re}(x_{ij}), \mathrm{Im}(x_{ij})\ (i<j))$, and define a measure dX on $\mathrm{Her}_m(\mathbf{C})$ by pulling back the standard measure on $\mathbf{R}^{m^2}$. Similarly we define a measure dY on $\mathrm{Her}_m(\mathbf{C})$ in the same way as above. For two elements F, G of $\mathfrak{S}_{2l}(\Gamma^{(m)}, \det^{-l})$, we define the Petersson scalar product $\langle F, G\rangle$ by

$$\langle F, G\rangle = \int_{\Gamma^{(m)}\backslash\mathfrak{H}_m} F(Z)\overline{G(Z)}\det(\mathrm{Im}(Z))^{l-2m}dXdY.$$

We call $\langle F, F\rangle$ the period of F. Let $\mathcal{U}^{(m)}(\mathbf{A})$ be the adelization of $\mathcal{U}^{(m)}$. We define the compact subgroup $\mathcal{K} = \mathcal{K}^{(m)}$ of $\mathcal{U}^{(m)}(\mathbf{A})$ by $\mathcal{U}^{(m)}(\mathbf{A}) \cap \prod_p GL_{2m}(\mathcal{O}_p)$, where p runs over all rational primes. Let $h = h_K$ be the class number of K. Then we have

$$\mathcal{U}^{(m)}(\mathbf{A}) = \bigsqcup_{i=1}^{h} \mathcal{U}^{(m)}(\mathbf{Q})\gamma_i\mathcal{K}\mathcal{U}^{(m)}(\mathbf{R})$$

with some subset $\{\gamma_1, ..., \gamma_h\}$ of $\mathcal{U}^{(m)}(\mathbf{A})$. We can take $\gamma_i\ (i = 1, ..., h)$ so that $\gamma_1 = 1_{2m}$ and $\gamma_i = \begin{pmatrix} t_i & 0 \\ 0 & \overline{t}_i^{-1} \end{pmatrix}$ with $t_i \in GL_m(K_A)$. Put $\Gamma_i = \mathcal{U}^{(m)}(\mathbf{Q}) \cap \gamma_i\mathcal{K}\gamma_i^{-1}\mathcal{U}^{(m)}(\mathbf{R})$. Then for an element $(F_1, ..., F_h) \in \bigoplus_{i=1}^h M_{2l}(\Gamma_i, \det^{-l})$, we define $(F_1, ..., F_h)^\sharp$ by

$$(F_1, ..., F_h)^\sharp(g) = F_i(x\langle\mathbf{i}\rangle)j(x, \mathbf{i})^{-2l}(\det x)^l$$

for $x = u\gamma_i x\kappa$ with $u \in \mathcal{U}^{(m)}(\mathbf{Q}), x \in \mathcal{U}^{(m)}(\mathbf{R}), \kappa \in \mathcal{K}$. We denote by $\mathcal{M}_l(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}), \det^{-l})$ the space of automorphic forms obtained in this way. We also put

$$\mathcal{S}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}), \det^{-l}) = \{(F_1, ..., F_h)^\sharp \mid F_i \in \mathfrak{S}_{2l}(\Gamma_i, \det^{-l})\}.$$

We can define the Hecke operators which act on the space $\mathcal{M}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}), \det^{-l})$. For the precise definition of them, see [Ike08].

Let $\widehat{\mathrm{Her}}_m(\mathcal{O})$ be the set of semi-integral Hermitian matrices over $\mathcal{O}$ of degree m as in the Notation. We note that A belongs to $\widehat{\mathrm{Her}}_m(\mathcal{O})$ if and only if its diagonal components are rational integers and $\sqrt{-D}A \in \mathrm{Her}_m(\mathcal{O})$. For a non-degenerate semi-integral matrix B over $\mathcal{O}$ of degree m, put $\gamma(B) = (-D)^{[m/2]}\det B$.

For a prime number p put $K_p = K \otimes \mathbf{Q}_p$, and $\mathcal{O}_p = \mathcal{O} \otimes \mathbf{Z}_p$. Then K_p is a quadratic extension of $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. In the former case, for $x \in K_p$, we denote by $\overline{x}$ the conjugate of x over $\mathbf{Q}_p$. In the latter case, for $x = (x_1, x_2)$ with $x_i \in \mathbf{Q}_p$, we put $\overline{x} = (x_2, x_1)$. For $x \in K_p$ we define the norm $N_{K_p/\mathbf{Q}_p}(x)$ by $N_{K_p/\mathbf{Q}_p}(x) = x\overline{x}$, and put $\nu_{K_p}(x) = \mathrm{ord}_p(N_{K_p/\mathbf{Q}_p}(x))$, and $|x|_{K_p} = |N_{K_p/\mathbf{Q}_p}(x)|_p$. Moreover put $|x|_{K_\infty} = |x\overline{x}|_\infty$ for $x \in \mathbf{C}$. Let $\widehat{\mathrm{Her}}_m(\mathcal{O}_p)$ be the set of semi-integral matrices over $\mathcal{O}_p$ of degree m as in the Notation. We put $\xi_p = 1, -1$, or 0 according as $K_p \cong \mathbf{Q}_p \oplus \mathbf{Q}_p, K_p$ is an unramified quadratic extension of $\mathbf{Q}_p$, or

K_p is a ramified quadratic extension of $\mathbf{Q}_p$. Now for $T \in \widehat{\mathrm{Her}}_m(\mathcal{O}_p)^\times$ we define the local Siegel series $b_p(T,s)$ by

$$b_p(T,s) = \sum_{R \in \mathrm{Her}_n(K_p)/\mathrm{Her}_n(\mathcal{O}_p)} \mathbf{e}_p(\mathrm{tr}(TR)) p^{-\mathrm{ord}_p(\mu_p(R))s},$$

where $\mu_p(R) = [R\mathcal{O}_p^m + \mathcal{O}_p^m : \mathcal{O}_p^m]$. We remark that there exists a unique polynomial $F_p(T,X)$ in X such that

$$b_p(T,s) = F_p(T,p^{-s}) \prod_{i=0}^{[(m-1)/2]} (1-p^{2i-s}) \prod_{i=1}^{[m/2]} (1-\xi_p p^{2i-1-s})$$

(cf. Shimura [Sh97].) We then define a Laurent polynomial $\widetilde{F}_p(T,X)$ as

$$\widetilde{F}_p(B,X) = X^{\mathrm{ord}_p(\gamma(T))} F_p(T, p^{-m}X^{-2}).$$

We remark that we have

$$\widetilde{F}_p(B,X^{-1}) = (-D,\gamma(B))_p \widetilde{F}_p(B,X) \qquad \text{if } m \text{ is even,}$$

$$\widetilde{F}_p(B,\xi_p X^{-1}) = \widetilde{F}_p(B,X) \qquad \text{if } m \text{ is even and } p \nmid D,$$

and

$$\widetilde{F}_p(B,X^{-1}) = \widetilde{F}_p(B,X) \qquad \text{if } m \text{ is odd}$$

(cf. [Ike08]). Here $(a,b)_p$ is the Hilbert symbol of $a,b \in \mathbf{Q}_p^\times$. Hence we have

$$\widetilde{F}_p(B,X) = (-D,\gamma(B))_p^{m-1} X^{-\mathrm{ord}_p(\gamma(T))} F_p(T, p^{-m}X^2).$$

Now we put

$$\widehat{\mathrm{Her}}_m(\mathcal{O})_i^+ = \{T \in \mathrm{Her}_m(K)^+ \mid t_{i,p}^* T t_{i,p} \in \widehat{\mathrm{Her}}_m(\mathcal{O}_p) \text{ for any } p\}.$$

Let k be a non-negative integer. First let $m=2n$ be a positive even integer and let

$$f(z) = \sum_{N=1}^{\infty} a(N)\mathbf{e}(Nz)$$

be a primitive form in $\mathfrak{S}_{2k+1}(\varGamma_0(D),\chi)$. For a prime number p not dividing D let $\alpha_p \in \mathbf{C}$ such that $\alpha_p + \chi(p)\alpha_p^{-1} = p^{-k}a(p)$, and for $p \mid D$ put $\alpha_p = p^{-k}a(p)$. Then for the Kronecker character χ we define Hecke's L-function $L(s,f,\chi^i)$ twisted by χ^i as

$$L(s,f,\chi^i) = \prod_{p \nmid D} \{(1-\alpha_p p^{-s+k}\chi(p)^i)(1-\alpha_p^{-1}p^{-s+k}\chi(p)^{i+1})\}^{-1}$$

$$\times \begin{cases} \prod_{p \mid D}(1-\alpha_p p^{-s+k})^{-1} & \text{if } i \text{ is even} \\ \prod_{p \mid D}(1-\alpha_p^{-1} p^{-s+k})^{-1} & \text{if } i \text{ is odd.} \end{cases}$$

In particular, if i is even, we sometimes write $L(s,f,\chi^i)$ as $L(s,f)$ as usual. Moreover for $i=1,...,h$ we define a Fourier series

$$I_m(f)_i(Z) = \sum_{T \in \widehat{\mathrm{Her}}_m(\mathcal{O})_i)^+} a_{I_m(f)_i}(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where

$$a_{I_{2n}(f)_i}(T) = |\gamma(T)|^k \prod_p |\det(t_{i,p}) \det(\overline{t_{i,p}})|_p^n \widetilde{F}_p(t_{i,p}^* T t_{i,p}, \alpha_p).$$

Next let $m = 2n+1$ be a positive odd integer and let

$$f(z) = \sum_{N=1}^{\infty} a(N)\mathbf{e}(Nz)$$

be a primitive form in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$. For a prime number p let $\alpha_p \in \mathbf{C}$ such that $\alpha_p + \alpha_p^{-1} = p^{-k+1/2}a(p)$. Then we define Hecke's L-function $L(s,f,\chi^i)$ twisted by χ^i as

$$L(s,f,\chi^i)$$
$$= \prod_p \{(1-\alpha_p p^{-s+k-1/2}\chi(p)^i)(1-\alpha_p^{-1}p^{-s+k-1/2}\chi(p)^i)\}^{-1}.$$

In particular, if i is even we write $L(s,f,\chi^i)$ as $L(s,f)$ as usual. Moreover for $i = 1,...,h$ we define a Fourier series

$$I_{2n+1}(f)_i(Z) = \sum_{T \in \widehat{\mathrm{Her}}_{2n+1}(\mathcal{O})_i^+} a_{I_{2n+1}(f)_i}(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where

$$a_{I_{2n+1}(f)_i}(T) = |\gamma(T)|^{k-1/2} \prod_p |\det(t_{i,p}) \det(\overline{t_{i,p}})|_p^{n+1/2} \widetilde{F}_p(t_{i,p}^* T t_{i,p}, \alpha_p).$$

Then Ikeda [Ike08] showed the following:

Theorem 2.1. *Let $m = 2n$ or $2n+1$. Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$ or in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ according as $m = 2n$ or $m = 2n+1$. Moreover let Γ_i be the subgroup of $\mathcal{U}^{(m)}$ defined as above. Then $I_m(f)_i(Z)$ is an element of $\mathfrak{S}_{2k+2n}(\Gamma_i, \det^{-k-n})$ for any i. In particular, $I_m(f) := I_m(f)_1$ is an element of $\mathfrak{S}_{2k+2n}(\Gamma^{(m)}, \det^{-k-n})$.*

This is a Hermitian analogue of the lifting constructed in [Ike01]. We call $I_m(f)$ the Ikeda lift of f for $\mathcal{U}^{(m)}$.

It follows from Theorem 2.1 that we can define an element $(I_m(f)_1, ..., I_m(f)_h)^\sharp$ of $\mathcal{S}_{2k+2n}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash \mathcal{U}^{(m)}(\mathbf{A}), \det^{-k-n})$, which we write $Lift^{(m)}(f)$.

Theorem 2.2. *Let $m = 2n$ or $2n+1$. Assume that $I_m(f)$ is not identically zero. Then $Lift^{(m)}(f)$ is a Hecke eigenform in $\mathcal{S}_{2k+2n}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash \mathcal{U}^{(m)}(\mathbf{A}), \det^{-k-n})$ and its standard L-function $L(s, Lift^{(m)}(f), \mathrm{st})$ coincides with*

$$\prod_{i=1}^{m} L(s+k+n-i+1/2, f)L(s+k+n-i+1/2, f, \chi)$$

up to bad Euler factors.

We call $Lift^{(m)}(f)$ the adelic Ikeda lift of f for $\mathcal{U}^{(m)}$.

Remark. In [Ike08], Ikeda defined the standard L-function $L(s, F, \mathrm{st})$ of a Hecke eigenform F in the sense of automorphic representation theory, and therefore gave an explicit form of the Euler factor of the standard L-function of $Lift^{(m)}(f)$ only for good primes. However, we will define another L-function $\mathcal{L}(s, F, \mathrm{st})$ following [Sh00] in Subsection 5.2. It coincides with $L(s, F, \mathrm{st})$ up to Euler factors at ramified primes, and in particular we have

$$\mathcal{L}(s, Lift^{(m)}(f), \mathrm{st}) = \prod_{i=1}^{m} L(s+k+n-i+1/2, f)L(s+k+n-i+1/2, f, \chi).$$

To state our main result, put

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$$

and

$$\Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1).$$

We note that

$$\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

For an integer i let $L(s, \chi^i) = \zeta(s)$ or $L(s, \chi)$ according as i is even or odd, where $\zeta(s)$ and $L(s, \chi)$ are Riemann's zeta function, and Dirichlet L-function for χ, respectively, and put

$$\widetilde{\Lambda}(s, \chi^i) = \Gamma_{\mathbf{C}}(s)L(s, \chi^i).$$

For a primitive form f in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$, we define the adjoint L-function $L(s,\ f,\ \mathrm{Ad})$ and its twist $L(s,\ f,\ \mathrm{Ad}, \chi)$ by χ as

$$L(s, f, \mathrm{Ad}) = \prod_{p \nmid D}\{(1-\alpha_p^2\chi(p)p^{-s})(1-\alpha_p^{-2}\chi(p)p^{-s})(1-p^{-s})\}^{-1}\prod_{p \mid D}(1-p^{-s})^{-1},$$

and

$$L(s, f, \mathrm{Ad}, \chi) = \prod_{p \nmid D}\{(1-\alpha_p^2 p^{-s})(1-\alpha_p^{-2}p^{-s})(1-\chi(p)p^{-s})\}^{-1}$$
$$\times \prod_{p \mid D}\{(1-\alpha_p^2 p^{-s})(1-\alpha_p^{-2}p^{-s})\}^{-1}.$$

For a primitive form f in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$, we define the adjoint L-function $L(s,\ f,\ \mathrm{Ad})$ and its twist $L(s,\ f,\ \mathrm{Ad}, \chi)$ by χ as

$$L(s, f, \mathrm{Ad}) = \prod_{p}\{(1-\alpha_p^2 p^{-s})(1-\alpha_p^{-2}p^{-s})(1-p^{-s})\}^{-1},$$

and

$$L(s, f, \mathrm{Ad}, \chi) = \prod_{p}\{(1-\alpha_p^2\chi(p)p^{-s})(1-\alpha_p^{-2}\chi(p)p^{-s})(1-\chi(p)p^{-s})\}^{-1}.$$

Let f be a primitive form in $\mathfrak{S}_{2k+1}(\varGamma_0(D),\chi)$ or in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ according as $m=2n$ or $m=2n+1$. We then put

$$L(s,\,f,\,\mathrm{Ad},\chi^i)=\begin{cases} L(s,\,f,\mathrm{Ad}) & \text{if } i \text{ is even} \\ L(s,f,\mathrm{Ad},\chi) & \text{if } i \text{ is odd}\end{cases}$$

Moreover put

$$\widetilde{\Lambda}(s,\,f,\,\mathrm{Ad},\chi^i)=\Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s+l-1)L(s,\,f,\,\mathrm{Ad},\chi^i),$$

where $l=2k+1$ or $l=2k$ according as $f\in\mathfrak{S}_{2k+1}(\varGamma_0(D),\chi)$ or $f\in\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$. Let Q_D be the set of prime divisors of D. For each prime $q\in Q_D$, put $D_q=q^{\mathrm{ord}_q(D)}$. We define a Dirichlet character χ_q by

$$\chi_q(a)=\begin{cases}\chi(a') & \text{if } (a,q)=1 \\ 0 & \text{if } q|a\end{cases},$$

where a' is an integer such that

$$a'\equiv a \bmod D_q \quad \text{ and } a'\equiv 1 \bmod DD_q^{-1}.$$

For a subset Q of Q_D put $\chi_Q=\prod_{q\in Q}\chi_q$ and $\chi'_Q=\prod_{q\in Q_D,q\not\in Q}\chi_q$. Here we make the convention that $\chi_Q=1$ and $\chi'_Q=\chi$ if Q is the empty set. Let

$$f(z)=\sum_{m=1}^{\infty}c_f(m)\mathbf{e}(mz)$$

be a primitive form in $\mathfrak{S}_{2k+1}(\varGamma_0(D),\chi)$. Then there exists a primitive form

$$f_Q(z)=\sum_{m=1}^{\infty}c_{f_Q}(m)\mathbf{e}(mz)$$

such that

$$c_{f_Q}(p)=\chi_Q(p)c_f(p) \text{ for } p\not\in Q$$

and

$$c_{f_Q}(p)=\chi'_Q(p)\overline{c_f(p)} \text{ for } p\in Q.$$

Then our main result in this paper is:

Theorem 2.3. (1) *Let $m=2n$ be a positive even integer. For a primitive form f in $\mathfrak{S}_{2k+1}(\varGamma_0(D),\chi)$, we have*

$$\langle I_{2n}(f),\,I_{2n}(f)\rangle$$

$$=2^{-4nk-4n^2-4n+1}D^{2nk+5n^2-3n/2-1/2}\eta_n(f)\prod_{i=1}^{2n}\widetilde{\Lambda}(i,\,f,\,\mathrm{Ad},\chi^{i-1})\prod_{i=2}^{2n}\widetilde{\Lambda}(i,\chi^i),$$

where

$$\eta_n(f)=\sum_{\substack{Q\subset Q_D\\ f_Q=f}}\chi_Q((-1)^n).$$

(2) *Let $m=2n+1$ be a positive odd integer. For a primitive form f in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$, we have*

$$\langle I_{2n+1}(f),\, I_{2n+1}(f)\rangle$$

$$=2^{-2(2n+1)k-4n^2-6n-1}D^{2nk+5n^2+5n/2}\prod_{i=1}^{2n+1}\widetilde{\Lambda}(i,\,f,\,\mathrm{Ad},\chi^{i-1})\prod_{i=2}^{2n+1}\widetilde{\Lambda}(i,\chi^i).$$

Remark. In [Ike08] Ikeda showed that $I_m(f)$ is identically zero if and only if $m=2n$ and $\eta_n(f)=0$. Therefore the above theorem remains valid even if $I_m(f)$ is identically zero.

This type of result was conjectured by Ikeda [Ike08]. When $m=2$, by using the result of Sugano [Su95], Ikeda [Ike08] has been already proved that

$$\frac{\langle I_2(f),\, I_2(f)\rangle}{\langle f,\, f\rangle}=\eta_1(f)2^{-4k-7}D^{2k+3}\widetilde{\Lambda}(2)\widetilde{\Lambda}(1,f,\mathrm{Ad})\widetilde{\Lambda}(2,f,\mathrm{Ad},\chi).$$

His conjecture holds true up to a power of D. In fact, he conjectured that integer powers of D should appear on the right-hand sides of the above formulas. However, half-integer powers of D appear in some cases as shown in the above theorem.

Now put

$$\mathbf{L}(i,f,\mathrm{Ad},\chi^i)=\frac{\widetilde{\Lambda}(i,\,f,\,\mathrm{Ad},\chi^{i-1})}{\langle f,\, f\rangle}$$

for $i=1,...,m$

$$\mathbf{L}(2i,\chi^{2i})=\widetilde{\Lambda}(2i,\chi^{2i}),$$

and

$$\mathbf{L}(2i+1,\chi^{2i+1})=\widetilde{\Lambda}(2i+1,\chi^{2i+1})D^{2i+1/2}$$

for an integer $i\geq 1$. We note that

$$\mathbf{L}(1,f,\mathrm{Ad})=\begin{cases}2^{2k+1}\prod_{q|D}(1+q^{-1}) & \text{if } f\in\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)\\ 2^{2k} & \text{if } f\in\mathfrak{S}_{2k}(SL_2(\mathbf{Z})).\end{cases}$$

Moreover we have $\eta_n(f)=1,2$ or 4. Hence we obtain the following:

Theorem 2.4. *Let the notation be as above. Then we have*

$$\frac{\langle I_m(f),\, I_m(f)\rangle}{\langle f,\, f\rangle^m}=2^{\beta_{n,k}}\prod_{i=2}^{m}\mathbf{L}(i,\,f,\,\mathrm{Ad},\chi^{i+1})\mathbf{L}(i,\chi^i)$$

$$\times\begin{cases}D^{2nk+4n^2-n}\prod_{q|D}(1+q^{-1}) & \text{if } m=2n\\ D^{2nk+4n^2+n} & \text{if } m=2n+1,\end{cases}$$

where $\beta_{n,k}$ is an integer depending on n and k.

It is well known that $\mathbf{L}(i,\chi^i)$ is a rational number for any positive integer i. Moreover $\mathbf{L}(i,f,\mathrm{Ad},\chi^{i+1})$ is an algebraic number and belongs to the Hecke field $\mathbf{Q}(f)$ for $i = 2,....,k'$ where $k' = 2k$ or $2k-1$ according as if m is even or odd (cf. Shimura [Sh97], [Sh00].) Thus we have

Theorem 2.5. *In addition to the above notation and the assumption, assume that $m \le 2k$ or $m \le 2k-1$ according as m is even or odd. Then $\dfrac{\langle I_m(f),\, I_m(f)\rangle}{\langle f,\, f\rangle^m}$ is algebraic, and in particular it belongs to $\mathbf{Q}(f)$.*

3. Rankin-Selberg convolution product

To prove Theorem 2.3, we rewrite it in terms of the residue of the Rankin-Selberg convolution product of $I_m(f)$. Let

$$F(z) = \sum_{A \in \widehat{\mathrm{Her}}_m(\mathcal{O})^+} a_F(A)\mathbf{e}(\mathrm{tr}(Az)$$

be an element of $\mathfrak{S}_{2l}(\Gamma^{(m)},\det^{-l})$. We then define the Rankin-Selberg series $R(s,F)$ for F by

$$R(s,F) = \sum_{A \in \widehat{\mathrm{Her}}_m(\mathcal{O})^+/SL_n(\mathcal{O})} \frac{a_F(A)\overline{a_F(A)}}{(\det A)^s e^*(A)},$$

where $e^*(A) = \#(\{g \in SL_n(\mathcal{O}) \mid g^*Ag = A\})$.

Proposition 3.1. *Put*

$$R_m = \frac{2^{2lm+m-1}\prod_{i=2}^{m} L(i,\chi^{i+1})}{D^{m(m-1)/2}\prod_{i=0}^{m-1} L(2m-i,\chi^i)\prod_{i=1}^{m}\Gamma_{\mathbf{C}}(i)\Gamma_{\mathbf{C}}(2l-i+1)}.$$

Let $F \in \mathfrak{S}_{2l}(\Gamma^{(m)},\det^{-l})$. Then $R(s,F)$ is holomorphic in s for $\mathrm{Re}(s) > 2l$. Moreover it can be continued to a meromorphic function on the whole s-plane, and has a simple pole at $s = 2l$ with the residue $R_m\langle F,F\rangle$.

Proof. The assertion can be proved by a careful analysis of the proof of [[Sh00], Proposition 22.2]. However, for the convenience of the readers we here give an outline of the proof. We define another Rankin-Selberg series $\widetilde{R}(s,F)$ for F by

$$\widetilde{R}(s,F) = \sum_{A \in \widehat{\mathrm{Her}}_m(\mathcal{O})^+/GL_n(\mathcal{O})} \frac{a_F(A)\overline{a_F(A)}}{(\det A)^s e(A)},$$

where $e(A) = \#(\{g \in GL_n(\mathcal{O}) \mid g^*Ag = A\})$. Remark that

$$R(s,F) = \#(\mathcal{O}^*)\widetilde{R}(s,F).$$

We define the non-holomorphic Eisenstein series $E_{2l}(Z,s)$ for $\Gamma^{(m)}$ by

$$E_{2l}(Z,s) = (\det \mathrm{Im}(Z))^s \sum_{M \in \Gamma_\infty^{(m)} \backslash \Gamma^{(m)}} |j(M,Z)|^{-2s},$$

where $\Gamma_\infty^{(m)} = \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma^{(m)}\}$. Then by using the same argument as in Page 179 of [Sh00], we obtain

$$\widetilde{R}(s,F) = \frac{1}{\#(\mathcal{O}^*)\mathrm{vol}(\mathrm{Her}_m(\mathbf{C})/\mathrm{Her}_m(\mathcal{O}))\widetilde{\Gamma}_m(s)(4\pi)^{-ms}}$$

$$\times \int_{\Gamma^{(m)} \backslash \mathfrak{H}_m} F(Z)\overline{F(Z)}\mathrm{Im}(Z)^{2l}\overline{E_{2l}(Z, \bar{s} - 2l + m)}(\det Y)^{2l-2m} dX dY,$$

where $\mathrm{vol}(\mathrm{Her}_m(\mathbf{C})/\mathrm{Her}_m(\mathcal{O}))$ is the volume of $\mathrm{Her}_m(\mathbf{C})/\mathrm{Her}_m(\mathcal{O})$ with respect to the measure dX, and

$$\widetilde{\Gamma}_m(s) = \pi^{m(m-1)/2} \prod_{i=1}^m \Gamma(s-i).$$

By [[Sh97],Theorem 19.7], $E_{2l}(Z, s-2l+m)$ is holomorphic in s for $\mathrm{Re}(s) > 2l$. Moreover it has a meromorphic continuation to the whole s-plane, and has a simple pole at $s = 2l$ with the residue of the following form:

$$\pi^{m^2}\widetilde{\Gamma}_m(m)^{-1} \frac{2^{m(1-m)-1} \prod_{i=2}^m L(i, \chi^{i+1})}{\mathrm{vol}(\mathrm{Her}_m(\mathbf{C})/\mathrm{Her}_m(\mathcal{O})) \prod_{i=1}^{m-1} L(2m-i, \chi^i)}.$$

We note that

$$\mathrm{vol}(\mathrm{Her}_m(\mathbf{C})/\mathrm{Her}_m(\mathcal{O})) = 2^{m(1-m)/2}|D|^{m(m-1)/4}.$$

Thus we prove the assertion. □

4. Reduction to local computations

To prove our main result, we give an explicit formula for $R(s, I_m(f))$. To do this, we reduce the problem to local computations. Throughout the rest of this paper, let K_p be a quadratic extension of $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. In the former case let $\mathcal{O}_p$ be the ring of integers in K_p, and f_p the exponent of the conductor of $K_p/\mathbf{Q}_p$, and put $e_p = f_p - \delta_{2,p}$, where $\delta_{2,p}$ is Kronecker's delta. In the latter case, put $\mathcal{O}_p = \mathbf{Z}_p \oplus \mathbf{Z}_p$, and $e_p = f_p = 0$. Moreover put $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p) = p^{e_p}\widehat{\mathrm{Her}}_m(\mathcal{O}_p)$. We note that $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p) = \mathrm{Her}_m(\mathcal{O}_p)$ if K_p is not ramified over $\mathbf{Q}_p$. Let K be an imaginary quadratic extension of $\mathbf{Q}$ with the discriminant $-D$. We then put $\widetilde{D} = \prod_{p|D} p^{e_p}$, and $\widetilde{\mathrm{Her}}_m(\mathcal{O}) = \widetilde{D}\mathrm{Her}_m(\mathcal{O})$. Now let m and l be positive integers such that $m \ge l$. Then for an integer a and Hermitian matrices A and B of degree m and l respectively with entries in $\mathcal{O}_p$ put

$$\mathcal{A}_a(A,B) = \{X \in M_{ml}(\mathcal{O}_p)/p^a M_{ml}(\mathcal{O}_p) \mid A[X] - B \in p^a \widetilde{\mathrm{Her}}_l(\mathcal{O}_p)\},$$

and

$$\mathcal{B}_a(A,B) = \{X \in \mathcal{A}_a(A,B) \mid \mathrm{rank}_{\mathcal{O}_p/p\mathcal{O}_p} X = l\}.$$

Assume that A and B are non-degenerate. Then the number $p^{a(-2ml+l^2)}\#\mathcal{A}_a(A,B)$ is independent of a if a is sufficiently large. Hence we define the local density $\alpha_p(A,B)$ representing B by A as

$$\alpha_p(A,B) = \lim_{a\to\infty} p^{a(-2ml+l^2)}\#\mathcal{A}_a(A,B).$$

Similarly we can define the primitive local density $\beta_p(A,B)$ as

$$\beta_p(A,B) = \lim_{a\to\infty} p^{a(-2ml+l^2)}\#\mathcal{B}_a(A,B)$$

if A is non-degenerate. We remark that the primitive local density $\beta_p(A,B)$ can be defined even if B is not non-degenerate. In particular we write $\alpha_p(A) = \alpha_p(A,A)$. We also define $\upsilon_p(A)$ as

$$\upsilon_p(A) = \lim_{a\to\infty} p^{-am^2}\#(\Upsilon_a(A)),$$

where

$$\Upsilon_a(A) = \{X \in M_{ml}(\mathcal{O}_p)/p^a M_{ml}(\mathcal{O}_p) \mid A[X] - B \in p^a \mathrm{Her}_l(\mathcal{O}_p)\}.$$

The relation between $\alpha_p(A)$ and $\upsilon_p(A)$ is as follows:

Lemma 4.1. *Let $A \in \mathrm{Her}_m(\mathcal{O}_p)^\times$. Assume that K_p is ramified over $\mathbf{Q}_p$. Then we have*

$$\alpha_p(A) = p^{-m(m+1)f_p/2 + m^2\delta_{2,p}}\upsilon_p(A).$$

Otherwise, $\alpha_p(A) = \upsilon_p(A)$.

Proof. The proof is similar to that in [Kitaoka [Kit93], Lemma 5.6.5], and we here give an outline of the proof. The last assertion is trivial. Assume that K_p is ramified over $\mathbf{Q}_p$. Let $\{A_i\}_{i=1}^l$ be a complete set of representatives of $T' \in \mathrm{Her}_m(\mathcal{O}_p)$ such that $T' \equiv T \bmod p^{r+e_p}\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. Then it is easily seen that

$$l = [p^r\widetilde{\mathrm{Her}}_m(\mathcal{O}_p) : p^{r+e_p}\mathrm{Her}_m(\mathcal{O}_p)] = p^{m(m-1)f_p/2}.$$

Define a mapping

$$\phi : \bigsqcup_{i=1}^{l} \Upsilon_{r+e_p}(T_i) \longrightarrow \mathcal{A}_r(T,T)$$

by $\phi(X) = X \bmod p^r$. For $X \in \mathcal{A}_r(T,T)$ and $Y \in M_m(\mathcal{O}_p)$ we have

$$T[X + p^rY] \equiv T[X] \bmod p^r\widetilde{\mathrm{Her}}_m(\mathcal{O}_p).$$

Namely, $X + p^rY$ belongs to $\Upsilon_{r+e_p}(T_i)$ for some i and therefore ϕ is surjective. Moreover for $X \in \mathcal{A}_r(T,T)$ we have $\#(\phi^{-1}(X)) = p^{2m^2e_p}$.

For a sufficiently large integer r we have $\#\Upsilon_{r+e_p}(T_i) = \#\Upsilon_{r+e_p}(T)$ for any i. Hence

$$p^{m(m-1)f_p/2}\#\Upsilon_{r+e_p}(T) = \sum_{i=1}^{l} \#\Upsilon_{r+e_p}(T_i)$$

$$= p^{2m^2e_p}\#\mathcal{A}_r(T,T) = p^{m^2e_p}\#\mathcal{A}_{r+e_p}(T,T).$$

Recall that $e_p = f_p - \delta_{2,p}$. Hence

$$\#\Upsilon_{r+e_p}(T) = p^{m(m+1)f_p/2-m^2\delta_{2p}}\#\mathcal{A}_{r+e_p}(T,T).$$

This proves the assertion. □

For $T \in \mathrm{Her}_m(K)^+$, let $\mathcal{G}(T)$ denote the set of $SL_m(\mathcal{O})$-equivalence classes of positive definite Hermitian matrices T' such that T' is $SL_m(\mathcal{O}_p)$-equivalent to T for any prime number p. Moreover put

$$M^*(T) = \sum_{T' \in \mathcal{G}_i(T)} \frac{1}{e^*(T')}$$

for a positive definite Hermitian matrix T of degree m with entries in $\mathcal{O}$.

Let $\mathcal{U}_1$ be the unitary group defined in Section 1. Namely let

$$\mathcal{U}_1 = \{u \in R_{K/\mathbf{Q}}(GL_1) \mid \overline{u}u = 1\}.$$

For an element $T \in \mathrm{Her}_m(\mathcal{O}_p)$, let

$$\widetilde{U_{p,T}} = \{\det X \mid X \in \mathcal{U}_T(\mathcal{O}_p)\}.$$

Then $\widetilde{U_{p,T}}$ is a subgroup of $U_{1,p}$ of finite index. We then put $l_{p,T} = [U_{1,p} : \widetilde{U_{p,T}})]$.

Proposition 4.2. *Let $T \in \mathrm{Her}_m(\mathcal{O})^+$. Then*

$$M^*(T) = \frac{\det T^m \prod_{i=2}^m |D|^{i/2}\Gamma_{\mathbf{C}}(i)}{2^{m-1-\#(Q_D)}L(1,\chi)\prod_p l_{p,T}\upsilon_p(T)}.$$

Proof. The assertion is more or less well knwon. But for the sake of completeness we here give an outline of the proof. Let $M(m)$ be the affine space of all $m \times m$ matrices defined over K, and $\mathrm{Her}_m = \{X \in M(m) \mid X^* = X\}$, We then define a measuer ω' on $\mathcal{U}_T$ by

$$\omega'(x) = d\lambda_m(x)/d\sigma_m(x^*Tx),$$

where

$$d\lambda_m(x) = \wedge_{1\le i,j\le m} dx_{ij}, \quad x = (x_{ij}) \in R_{K/\mathbf{Q}}(M(m)),$$

and

$$d\sigma_m(s) = \wedge_{1\le i\le j\le m} ds_{ij}, \quad s = (s_{ij}) \in R_{K/\mathbf{Q}}(Her_m).$$

We also define a measure ω on $\mathcal{SU}_T$ by

$$\omega = \omega'/dt,$$

where dt is the measure on $\mathcal{U}_1$ normalized so that

$$\int_{\mathcal{U}_1(\mathcal{O}_p)} |dt|_p = u_p,$$

where

$$u_p = \begin{cases} 1+p^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is unramified} \\ 1-p^{-1} & \text{if } K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p \\ 2 & \text{if } K_p/\mathbf{Q}_p \text{ is ramified.} \end{cases}$$

We recall that the Tamagawa number of $\mathcal{SU}_T$ is 1 (cf. Weil [We82]). Hence by using the standard method we can prove the following mass formula for the $\mathcal{SU}_T$:

$$\int_{\mathcal{SU}_T(\mathcal{O}_\infty)} |\omega|_\infty \prod_{p<\infty} \int_{\mathcal{SU}_T(\mathcal{O}_p)} |\omega|_p = 1.$$

Then by the strong approximation theorem for SL_m, we have

$$\int_{\mathcal{SU}_T(\mathcal{O}_\infty)} |\omega|_\infty = \left(\frac{\det T^m \prod_{i=2}^m |D|^{i/2} \Gamma_{\mathbf{C}}(i)}{2^{m-1}} \right)^{-1} M^*(T).$$

On the other hand we have

$$\upsilon_p(T) = \int_{\mathcal{U}_T(\mathcal{O}_p)} |\omega'|_p = \int_{\mathcal{SU}_T(\mathcal{O}_p)} |\omega|_p \int_{\widetilde{U_{p,T}}} |dt|_p = u_p l_{p,T}^{-1} \int_{\mathcal{SU}_T(\mathcal{O}_p)} |\omega|_p.$$

We note that the infinite product $\prod_{p<\infty} u_p$ is conditional convergent and is equal to $2^{\#(Q_D)} L(1,\chi)^{-1}$. This completes the assertion.

□

Corollary. *Let $T \in \mathrm{Her}_m(\mathcal{O})^+$. Then*

$$M^*(T) = \frac{2^{c_D m^2} \det T^m \prod_{i=2}^m \Gamma_{\mathbf{C}}(i)}{2^{m-1-\#(Q_D)} L(1,\chi) D^{m(m+1)/4+1/2} \prod_p l_{p,T} \alpha_p(T)},$$

where $c_D = 1$ or 0 according as 2 divides D or not.

For a subset $\mathcal{T}$ of $\mathcal{O}_p$ put

$$\mathrm{Her}_m(\mathcal{T}) = \mathrm{Her}_m(\mathcal{O}_p) \cap M_m(\mathcal{T}),$$

and for a subset $\mathcal{S}$ of $\mathcal{O}_p$ put

$$\mathrm{Her}_m(\mathcal{S}, \mathcal{T}) = \{A \in \mathrm{Her}_m(\mathcal{T}) \mid \det A \in \mathcal{S}\},$$

and $\widetilde{\mathrm{Her}}_m(\mathcal{S}, \mathcal{T}) = \mathrm{Her}_m(\mathcal{S}, \mathcal{T}) \cap \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. In particular if $\mathcal{S}$ consists of a single element d we write $\mathrm{Her}_m(\mathcal{S}, \mathcal{T})$ as $\mathrm{Her}_m(d, \mathcal{T})$, and so on.

For $d \in \mathbf{Z}_{>0}$ we also define the set $\mathrm{Her}_m(d,\mathcal{O})^+$ in a similar way. For each $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)^\times$ put

$$F_p^{(0)}(T,X) = F_p(p^{-e_p}T,X)$$

and

$$\widetilde{F}_p^{(0)}(T,X) = \widetilde{F}_p(p^{-e_p}T,X).$$

We remark that

$$\widetilde{F}_p^{(0)}(T,X) = X^{-\mathrm{ord}_p(\det T)}X^{e_p m - f_p[m/2]}F_p^{(0)}(T,X).$$

For $d_0 \in \mathbf{Z}_p^\times$ put

$$\lambda_{m,p}(d_0,X,Y) = \sum_{A \in \widetilde{\mathrm{Her}}_m(d_0p^i,\mathcal{O}_p)/SL_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(A,X)\widetilde{F}_p^{(0)}(A,Y)}{l_{p,A}\alpha_p(A)}.$$

An explicit formula for $\lambda_{m,p}(d_0,X,Y)$ will be given in the next section for $d_0 \in \mathbf{Z}_p^*$.

Now let $\widetilde{\mathsf{Her}}_m = \prod_p(\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)/SL_m(\mathcal{O}_p))$. Then the diagonal embedding induces a mapping

$$\phi : \widetilde{\mathrm{Her}}_m(O)/\prod_p SL_m(\mathcal{O}_p) \longrightarrow \widetilde{\mathsf{Her}}_m.$$

Proposition 4.3. *In addition to the above notation and the assumption, for a positive integer d let*

$$\widetilde{\mathsf{Her}}_m(d) = \prod_p(\widetilde{\mathrm{Her}}_m(d,\mathcal{O}_p)/SL_m(\mathcal{O}_p)).$$

Then the mapping ϕ induces a bijection from $\widetilde{\mathrm{Her}}_m(d,O)/\prod_p SL_m(\mathcal{O}_p)$ to $\widetilde{\mathsf{Her}}_m(d)$, which will be denoted also by ϕ.

Proof. The proof is similar to that of [[IS95], Proposition 2.1], but we have to consider more carefully because the class number of K is not necessarily one. It is easily seen that ϕ is injective. Let $(x_p) \in \widetilde{\mathsf{Her}}_m(d)$. Then by the Hasse principle for Hermitian forms, there exists an element y in $\mathrm{Her}_m(K)^+$ such that $x_p = g_p^* y g_p$ with some $g_p \in GL_m(K_p)$ for any prime number p. For p not dividing Dd we may assume $g_p \in GL_m(O_p)$. Hence (g_p) defines an element of $R_{K/\mathbf{Q}}(GL_m)(\mathbf{A}) \cap \prod_{p<\infty} GL_m(K_p)$. Since we have $\det yd^{-1} \in \mathbf{Q}^\times \cap \prod_p N_{K_p/\mathbf{Q}_p}(K_p)$, we see that $\det yd^{-1} = N_{K/\mathbf{Q}}(u)$ with some $u \in K^\times$. Thus, by replacing y with $\begin{pmatrix} 1_{m-1} & O \\ O & \overline{u}^{-1} \end{pmatrix} y \begin{pmatrix} 1_{m-1} & O \\ O & u^{-1} \end{pmatrix}$, we may assume that $\det y = d$. Then we have $N_{K_p/\mathbf{Q}_p}(\det g_p) = 1$. It is easily seen that there exists an element $\delta_p \in GL_m(K_p)$ such that $\det \delta_p = \det g_p^{-1}$ and $\delta_p^* x_p \delta_p = x_p$. Thus we have $g_p\delta_p \in SL_m(K_p)$ and

$$x_p = (g_p\delta_p)^* y g_p \delta_p.$$

By the strong approximation theorem for SL_m there exists an element $\gamma \in SL_m(K)$ and $(\gamma_p) \in \prod_p SL_m(O_p)$ such that

$$(g_p \delta_p) = (\gamma_p)\gamma.$$

Put $x = \gamma^* y \gamma$. Then x belongs to $\widetilde{\mathrm{Her}}_m(d, \mathcal{O})^+$, and $\phi(x) = (x_p)$. This proves the surjectivity of ϕ. □

Theorem 4.4. *Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$ or in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ according as $m = 2n$ or $2n+1$. For such an f and a positive integer d_0 put*

$$a_m(f; d_0) = \prod_p \lambda_{m,p}(d_0, \alpha_p, \overline{\alpha}_p),$$

where α_p is the Satake p-parameter of f. Moreover put

$$\mu_{m,k,D} = D^{m(s-2k+l_0)+(2k-l_0)[m/2]-m(m+1)/4-1/2} 2^{-c_D m(s-2k-2n)-m+1+\#(Q_D)}$$

$$\times L(1,\chi)^{-1} \prod_{i=2}^{m} \Gamma_{\mathbf{C}}(i),$$

where $l_0 = 0$ or 1 according as m is even or odd. Then for $\mathrm{Re}(s) >> 0$, *we have*

$$R(s, I_m(f)) = \mu_{m,k,D} \sum_{d_0=1}^{\infty} a_m(f; d_0) d_0^{-s+2k+2n-1}.$$

Proof. We note that $R(s, I_m(f))$ can be rewritten as

$$R(s, I_m(f)) = \widetilde{D}^{ms} \sum_{T \in \widetilde{\mathrm{Her}}_m(\mathcal{O})^+/SL_m(\mathcal{O})} \frac{a_{I_m(f)}(\widetilde{D}^{-1}T)\overline{a_{I_m(f)}(\widetilde{D}^{-1}T)}}{e^*(T) \det T^s}.$$

For $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O})^+$ the Fourier coefficient $a_{I_m(f)}(\widetilde{D}^{-1}T)$ of $I_m(f)$ is uniquely determined by the genus to which T belongs, and can be expressed as

$$|a_{I_m(f)}(\widetilde{D}^{-1}T)|^2 = (D^{[m/2]} \widetilde{D}^{-m} \det T)^{2k-l_0} \prod_p \widetilde{F}_p^{(0)}(T, \alpha_p) \widetilde{F}_p^{(0)}(T, \overline{\alpha_p}).$$

Thus the assertion follows from Corollary to Proposition 4.2 and Proposition 4.3 similarly to [IS95]. □

5. Formal power series associated with local Siegel series

For $d_0 \in \mathbf{Z}_p^\times$ put

$$H_{m,p}(d_0,X,Y,t)=\sum_{i=0}^{\infty}\lambda_{m,p}^*(p^id_0,X,Y)t^i,$$

where for $d \in \mathbf{Z}_p^\times$ we define $\lambda_{m,p}^*(p^id_0,X,Y)$ as

$$\lambda_{m,p}^*(d,X,Y)=\sum_{A\in\widetilde{\mathrm{Her}}_m(dN_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*),\mathcal{O}_p)/GL_m(\mathcal{O}_p)}\frac{\widetilde{F}_p^{(0)}(A,X)\widetilde{F}_p^{(0)}(A,Y)}{\alpha_p(A)}.$$

We note that we have

$$\lambda_{m,p}^*(d,X,Y)=\lambda_{m,p}(d,X,Y)$$

for $d \in \mathbf{Z}_p^\times$ (cf. Proposition 5.5.1), and therefore

$$H_{m,p}(d_0,X,Y,t)=\sum_{i=0}^{\infty}\lambda_{m,p}(p^id_0,X,Y)t^i.$$

In this section, we give explicit formulas of $H_{m,p}(d_0,X,Y,t)$ for all prime numbers p (cf. Theorems 5.5.2 and 5.5.3.)

From now on we fix a prime number p. Throughout this section we simply write ord_p as ord and so on if the prime number p is clear from the context. We also write ν_{K_p} as ν. We also simply write $\widetilde{\mathrm{Her}}_{m,p}$ instead of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$, and so on.

5.1. **Preliminaries.**

Let m be a positive integer. Let K_p be a quadratic extension of $\mathbf{Q}_p$, and ϖ be a prime element of K_p. For a non-negative integer $i \le m$ let $\mathcal{D}_{m,i}=GL_m(\mathcal{O}_p)\begin{pmatrix}1_{m-i} & 0\\ 0 & \varpi 1_i\end{pmatrix}GL_m(\mathcal{O}_p)$, and for $W \in \mathcal{D}_{m,i}$, put $\pi_p(W)=(-1)^iN_{K_p/\mathbf{Q}_p}(\varpi)^{i(i-1)/2}$. Let $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then for a pair $i=(i_1,i_2)$ of non-negative integers such that $i_1,i_2\le m$, let $\mathcal{D}_{m,i}=GL_m(\mathcal{O}_p)\left(\begin{pmatrix}1_{m-i_1} & 0\\ 0 & p1_{i_1}\end{pmatrix},\begin{pmatrix}1_{m-i_2} & 0\\ 0 & p1_{i_2}\end{pmatrix}\right)GL_m(\mathcal{O}_p)$, and for $W\in\mathcal{D}_{m,i}$ put $\pi_p(W)=(-1)^{i_1+i_2}p^{i_1(i_1-1)/2+i_2(i_2-1)/2}$. In either case K_p is a quadratic extension of $\mathbf{Q}_p$, or $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$, we put $\pi_p(W)=0$ for $W\in M_n(\mathcal{O}_p^\times)\setminus\bigcup_{i=0}^m\mathcal{D}_{m,i}$.

For non-degenerate Hermitian matrices S and T of degree m, we put

$$\alpha_p(S,T;i)=2^{-1}\lim_{e\longrightarrow\infty}p^{-m^2e}\mathcal{A}_e(S,T;i),$$

where

$$\mathcal{A}_e(S,T;i)=\{\bar{X}\in M_m(\mathcal{O}_p)/p^eM_m(\mathcal{O}_p)\in\mathcal{A}_e(S,T)\mid X\in\mathcal{D}_{m,i}\}.$$

First we remark the following lemma, which can easily be proved by the usual Newton approximation method in $\mathcal{O}_p$:

Lemma 5.1.1. *Let $A, B \in \mathrm{Her}_m(\mathcal{O}_p)^\times$. Let e be an integer such that $p^e A^{-1} \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. Assume that $A \equiv B \bmod p^{e+1}\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. Then there exists a matrix $U \in GL_m(\mathcal{O}_p)$ such that $B = A[U]$.*

Lemma 5.1.2. *Let $S \in \mathrm{Her}_m(\mathcal{O}_p)^\times$ and $T \in \mathrm{Her}_n(\mathcal{O}_p)^\times$ with $m \geq n$. Then*

$$\alpha_p(S,T) = \sum_{W \in GL_n(\mathcal{O}_p)\backslash M_n(\mathcal{O}_p)^\times} p^{(n-m)\nu(\det W)} \beta_p(S, T[W^{-1}]).$$

Proof. The assertion can be proved by using the same argument as in the proof of [[Kit93], Theorem 5.6.1]. We here give an outline of the proof. For each $W \in M_n(\mathcal{O}_p)$, put

$$\mathcal{B}_e(S,T;W) = \{X \in \mathcal{A}_e(S,T) \mid XW^{-1} \text{ is primitive}\}.$$

Then we have

$$\mathcal{A}_e(S,T) = \bigsqcup_{W \in GL_n(\mathcal{O}_p)\backslash M_n(\mathcal{O}_p)^\times} \mathcal{B}_e(S,T;W).$$

Take a sufficiently large integer e, and for an element W of $M_n(\mathcal{O}_p)$, let $\{R_i\}_{i=1}^r$ be a complete set of representatives of $p^e\mathrm{Her}_m(\mathcal{O}_p)[W^{-1}]/p^e\mathrm{Her}_m(\mathcal{O}_p)$. Then we have $r = p^{\nu(\det W)n}$. Put

$$\widetilde{\mathcal{B}}_e(S,T;W) = \{X \in M_{mn}(\mathcal{O}_p)/p^e M_{mn}(\mathcal{O}_p)W \mid S[X] \equiv T \bmod p^e\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$$
$$\text{and } XW^{-1} \text{ is primitive}\}.$$

Then

$$\#(\widetilde{\mathcal{B}}_e(S,T)) = p^{\nu(\det W)m}\#(\mathcal{B}_e(S,T;W).$$

It is easily seen that

$$S[XW^{-1}] \equiv T[W^{-1}] + R_i \bmod p^e\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$$

for some i. Hence the mapping $X \mapsto XW^{-1}$ induces a bijection from $\widetilde{\mathcal{B}}_e(S,T;W)$ to $\bigsqcup_{i=1}^r \mathcal{B}_e(S, T[W^{-1}] + R_i)$. Recall that $\nu(W) \leq \mathrm{ord}(\det T)$. Hence

$$R_i \equiv O \bmod p^{[e/2]}\widetilde{\mathrm{Her}}_m(\mathcal{O}_p),$$

and therefore by Lemma 5.1.1,

$$T[W^{-1}] + R_i = T[W^{-1}][G]$$

for some $G \in GL_n(\mathcal{O}_p)$. Hence

$$\#(\widetilde{\mathcal{B}}_e(S,T;W)) = p^{\nu(\det W)n}\#(\mathcal{B}_e(S,T[W^{-1}]).$$

Hence

$$\alpha_p(S,T) = p^{-2mne+n^2e}\#(\mathcal{A}_e(S,T))$$

$$= p^{-2mne+n^2e} \sum_{W \in GL_n(\mathcal{O}_p)\backslash M_n(\mathcal{O}_p)^\times} p^{\nu(\det W)(-m+n)} \#(\mathcal{B}_e(S, T[W^{-1}]).$$

This proves the assertion.

□

Now by the inversion formula for π_p we obtain

Corollary. *Under the same notation as above, we have*

$$\beta_p(S,T) = \sum_{W \in GL_n(\mathcal{O}_p)\backslash M_n(\mathcal{O}_p)^\times} p^{(n-m)\nu(\det W)} \pi_p(W) \alpha_p(S, T[W^{-1}]).$$

For two elements $A, A' \in \mathrm{Her}_m(\mathcal{O}_p)$ we simply write $A \sim_{GL_m(\mathcal{O}_p)} A'$ as $A \sim A'$ if there is no fear of confusion. For a variables U and q put

$$(U,q)_m = \prod_{i-1}^{m} (1 - q^{i-1}U), \qquad \phi_m(q) = (q,q)_m.$$

We note that $\phi_m(q) = \prod_{i=1}^m (1-q^i)$. Moreover for a prime number p put

$$\phi_{m,p}(q) = \begin{cases} \phi_m(q^2) & \text{if } K_p/\mathbf{Q}_p \text{ is unramified} \\ \phi_m(q)^2 & \text{if } K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p \\ \phi_m(q) & \text{if } K_p/\mathbf{Q}_p \text{ is ramified} \end{cases}$$

Lemma 5.1.3. (1) *Let $\Omega(S,T) = \{w \in M_m(\mathcal{O}_p) \mid S[w] \sim T\}$, and $\Omega(S,T;i) = \Omega(S,T) \cap \mathcal{D}_{m,i}$. Then we have*

$$\frac{\alpha_p(S,T)}{\alpha_p(T)} = \#(\Omega(S,T)/GL_m(\mathcal{O}_p)) p^{-m(\mathrm{ord}(\det T) - \mathrm{ord}(\det S))},$$

and

$$\frac{\alpha_p(S,T;i)}{\alpha_p(T)} = \#(\Omega(S,T;i)/GL_m(\mathcal{O}_p)) p^{-m(\mathrm{ord}(\det T) - \mathrm{ord}(\det S))}.$$

(2) *Let $\widetilde{\Omega}(S,T) = \{w \in M_m(\mathbf{Z}) \mid S \sim T[w^{-1}]\}$, and $\widetilde{\Omega}(S,T;i) = \widetilde{\Omega}(S,T) \cap \mathcal{D}_{m,i}$. Then we have*

$$\frac{\alpha_p(S,T)}{\alpha_p(S)} = \#(GL_m(\mathcal{O}_p)\backslash \widetilde{\Omega}(S,T)),$$

and

$$\frac{\alpha_p(S,T;i)}{\alpha_p(S)} = \#(GL_m(\mathcal{O}_p)\backslash \widetilde{\Omega}(S,T;i)).$$

Proof. (1) The proof is similar to that of Lemma 2.2 of [BS87]. First we prove

$$\int_{\Omega(S,T)} |dx| = \phi_{m,p}(p^{-1})\frac{\alpha_p(S,T)}{\alpha_p(T)},$$

where $|dx|$ is the Haar measure on $M_m(K_p)$ normalized so that

$$\int_{M_m(\mathcal{O}_p)} |dx| = 1.$$

To prove this, for a positive integer e let $T_1, ..., T_l$ be a complete set of representatives of $\{T[\gamma] \bmod p^e \mid \gamma \in GL_m(\mathcal{O}_p)\}$. Then it is easy to see that

$$\int_{\Omega(S,T)} |dx| = p^{-2m^2e}\sum_{i=1}^{l} \#(\mathcal{A}_e(S,T_i))$$

and, by Lemma 5.1.1, T_i is $GL_m(\mathcal{O}_p)$-equivalent to T if e is sufficiently large. Hence we have

$$\#(\mathcal{A}_e(S,T_i)) = \#(\mathcal{A}_e(S,T))$$

for any i. Moreover we have

$$l = \#(GL_m(\mathcal{O}_p/p^e\mathcal{O}_p))/\#(\mathcal{A}_e(T,T)) = p^{m^2e}\phi_{m,p}(p^{-1})/\alpha_p(T).$$

Hence

$$\int_{\Omega(S,T)} |dx| = lp^{-2m^2e}\#(\mathcal{A}_e(S,T)) = \phi_{m,p}(p^{-1})\frac{\alpha_p(S,T)}{\alpha_p(T)},$$

which proves the above equality. Now we have

$$\int_{\Omega(S,T)} |dx| = \sum_{W\in\Omega(S,T)/GL_m(\mathcal{O}_p)} |\det W|_{K_p}^m = \sum_{W\in\Omega(S,T)/GL_m(\mathcal{O}_p)} |\det W\overline{\det W}|_p^m.$$

Remark that for any $W \in \Omega(S,T)/GL_m(\mathcal{O}_p)$ we have $|\det W\overline{\det W}|_p = p^{-m(\mathrm{ord}(\det T)-\mathrm{ord}(\det S))}$. Thus the assertion has been proved.

(2)By Lemma 5.1.2 we have

$$\alpha_p(S,T) = \sum_{W\in GL_m(\mathcal{O}_p)\backslash\widetilde{\Omega}(S,T)} \beta_p(S,T[W^{-1}]).$$

Then we have $\beta_p(S,T[W^{-1}]) = \alpha_p(S)$ or 0 according as $S \sim T[W^{-1}]$ or not. Thus the assertion (2) holds. □

We define a reduced matrix. A non-degenerate square matrix $W = (d_{ij})_{m\times m}$ with entries in $\mathbf{Z}_p$ is said to be reduced if $d_{ii} = p^{e_i}$ with e_i a non-negative integer, d_{ij} is a non-negative integer such that $d_{ij} \leq p^{e_j} - 1$ for $i < j$, and $d_{ij} = 0$ for $i > j$. Let $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then an element $W = (W_1, W_2)$ of $M_m(\mathcal{O}_p)^\times$ with $W_1, W_2 \in M_m(\mathbf{Z}_p)^\times$ is said to be reduced if W_1 and W_2 are reduced. Let K_p be an unramified quadratic extension of $\mathbf{Q}_p$, and θ be an element of $\mathcal{O}_p$ such that $\mathcal{O}_p = \mathbf{Z}_p + \mathbf{Z}_p\theta$. Then a non-degenerate square matrix $W = (d_{ij})_{m\times m}$ with entries in $\mathcal{O}_p$ is said to be reduced if $d_{ii} = p^{e_i}$ with e_i a non-negative integer, $d_{ij} = d_{ij}^{(1)} + d_{ij}^{(2)}\theta$ with $d_{ij}^{(1)}, d_{ij}^{(2)}$ non-negative integers such that $d_{ij}^{(1)}, d_{ij}^{(2)} \le p^{e_j} - 1$ for $i < j$, and $d_{ij} = 0$ for $i > j$. Let K_p be a ramified quadratic extension of $\mathbf{Q}_p$, and ϖ be a prime element of K_p. Then a non-degenerate square matrix $W = (d_{ij})_{m\times m}$ with entries in $\mathcal{O}_p$ is said to be reduced if $d_{ii} = \varpi^{e_i}$ with e_i a non-negative integer, $d_{ij} = d_{ij}^{(1)} + d_{ij}^{(2)}\varpi$ with $d_{ij}^{(1)}, d_{ij}^{(2)}$ non-negative integers sucth that $d_{ij}^{(1)} \le p^{[(e_j+1)/2]} - 1, 0 \le d_{ij}^{(2)} \le p^{[(e_j-1)/2]} - 1$ for $i < j$, and $d_{ij} = 0$ for $i > j$. In any case, we can take the set of all reduced matrices as a complete set of representatives of $GL_m(\mathcal{O}_p)\backslash M_m(\mathcal{O}_p)^\times$. Let m be an integer. For $B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$ put

$$\widetilde{\Omega}(B) = \{W \in GL_m(K_p) \cap M_m(\mathcal{O}_p) \mid B[W^{-1}] \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)\}.$$

Moreover put $\widetilde{\Omega}(B,i) = \widetilde{\Omega}(B) \cap \mathcal{D}_{m.i}$. Let $r \le m$, and $\psi_{r,m}$ be the mapping from $GL_r(K_p)$ into $GL_m(K_p)$ defined by $\psi_{r,m}(W) = 1_{m-r}\perp W$.

It is well known that we can take the set of all reduced matrices as a complete set of representatives of $GL_m(\mathcal{O}_p)\backslash M_m(\mathcal{O}_p)^\times$. Let m be an integer. For $B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$ put

$$\widetilde{\Omega}(B) = \{W \in GL_m(K_p) \cap M_m(\mathcal{O}_p) \mid B[W^{-1}] \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)\}.$$

Moreover put $\widetilde{\Omega}(B,i) = \widetilde{\Omega}(B) \cap \mathcal{D}_{m.i}$. Let $r \le m$, and $\psi_{r,m}$ be the mapping from $GL_r(K_p)$ into $GL_m(K_p)$ defined by $\psi_{r,m}(W) = 1_{m-r}\perp W$.

For a subset $\mathcal{T}$ of $\mathcal{O}_p$, we put

$$\mathrm{Her}_m(\mathcal{T})_k = \{A = (a_{ij}) \in \mathrm{Her}_m(\mathcal{T}) \mid a_{ii} \in p^k\mathbf{Z}_p\}.$$

From now on put

$$\mathrm{Her}_{m,*}(\mathcal{O}_p) = \begin{cases} \mathrm{Her}_m(\mathcal{O}_p)_1 & \text{if } p=2 \text{ and } f_p = 3, \\ \mathrm{Her}_m(\varpi\mathcal{O}_p)_1 & \text{if } p=2 \text{ and } f_p = 2 \\ \mathrm{Her}_m(\mathcal{O}_p) & \text{otherwise,} \end{cases}$$

where ϖ is a prime element of K_p, and $i_p = 0$ or 1 according as $p = 2$ and $f_2 = 2$, or not. Assume that $K_p/\mathbf{Q}_p$ is unramified or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then an element B of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$ can be expressed as $B \sim_{GL_m(\mathcal{O}_p)} 1_r\perp pB_2$ with some integer r and $B_2 \in \mathrm{Her}_{m-r,*}(\mathcal{O}_p)$. Assume that $K_p/\mathbf{Q}_p$ is ramified. For an even positive integer r define Θ_r by

$$\Theta_r = \overbrace{\begin{pmatrix} 0 & \varpi^{i_p} \\ \overline{\varpi}^{i_p} & 0 \end{pmatrix} \perp \ldots \perp \begin{pmatrix} 0 & \varpi^{i_p} \\ \overline{\varpi}^{i_p} & 0 \end{pmatrix}}^{r/2},$$

where $\overline{\varpi}$ is the conjugate of ϖ over $\mathbf{Q}_p$. Then an element B of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$ is expressed as $B \sim_{GL_m(\mathcal{O}_p)} \Theta_r \bot p^{i_p} B_2$ with some even integer r and $B_2 \in \mathrm{Her}_{m-r,*}(\mathcal{O}_p)$. For these results, see Jacobowitz [Jac62].

Lemma 5.1.4. (1) *Assume that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Let $B_1 \in \mathrm{Her}_{m-n_0}(\mathcal{O}_p)$. Then $\psi_{m-n_0,m}$ induces a bijection from $GL_{m-n_0}(\mathcal{O}_p)\backslash\widetilde{\Omega}(pB_1)$ to $GL_m(\mathcal{O}_p)\backslash\widetilde{\Omega}(1_{n_0}\bot pB_1)$, which will be also denoted by $\psi_{m-n_0,m}$.*
(2) *Assume that K_p is ramified over $\mathbf{Q}_p$ and that n_0 is even. Let $B_1 \in \mathrm{Her}_{m-n_0}(\mathcal{O}_p)$. Then $\psi_{m-n_0,m}$ induces a bijection from $GL_{m-n_0}(\mathcal{O}_p)\backslash\widetilde{\Omega}(p^{i_p}B_1)$ to $GL_m(\mathcal{O}_p)\backslash\widetilde{\Omega}(\Theta_{n_0}\bot p^{i_p}B_1)$, which will be also denoted by $\psi_{m-n_0,m}$. Here i_p is the integer defined above.*

Proof. (1) Clearly $\psi_{m-n_0,m}$ is injective. To prove the surjectivity, take a representative D of an element of $GL_m(\mathcal{O}_p)\backslash\widetilde{\Omega}(1_{n_0}\bot pB_1)$. Without loss of generality we may assume that W is a reduced matrix. Since we have $(1_{n_0}\bot pB_1)[W^{-1}] \in \widetilde{\mathrm{Her}_m}(\mathcal{O}_p)$, we have $W = \begin{pmatrix} 1_{n_0} & 0 \\ 0 & W_1 \end{pmatrix}$ with $W_1 \in \widetilde{\Omega}(pB_1)$. This proves the assertion.

(2) The assertion can be proved in the same manner as (1). $\square$

Lemma 5.1.5. *Let $B \in \mathrm{Her}_m(\mathcal{O}_p)^\times$. Then we have*

$$\alpha_p(p^r dB) = p^{rm^2}\alpha_p(B)$$

for any non-negative integer r and $d \in \mathbf{Z}_p^*$.

Proof. The assertion can be proved by using the same argument as in the proof of (a) of Theorem 5.6.4 of Kitaoka [Ki2]. $\square$

Now to prove an induction formula for local densities different from Lemma 5.1.2, we use the terms of Hermitian modules. Let M be $\mathcal{O}_p$ free module, and let b be a mapping from $M \times M$ to K_p such that

$$b(\lambda_1 u + \lambda_2 u_2, v) = \lambda_1 b(u_1, v) + \lambda_2 b(u_2, v)$$

for $u, v \in M$ and $\lambda_1, \lambda_2 \in \mathcal{O}_p$, and

$$b(u,v) = \overline{b(v,u)} \text{ for } u, v \in M.$$

We call such an M a Hermitian module with a Hermitian inner product b. We set $q(u) = b(u,u)$ for $u \in M$. Take an $\mathcal{O}_p$-basis $\{u_i\}_{i=1}^m$ of M, and put $T_M = (b(u_i,u_j))_{1\le i,j\le m}$. Then T_M is a Hermitian matrix, and its determinant is uniquely determined, up to $N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*)$, by M. We say M is non-degenerate if $\det T_M \neq 0$. Conversely for a Hermitian matrix T of degree m, we can define a Hermitian module M_T so that

$$M_T = \mathcal{O}_p u_1 + \mathcal{O}_p u_2 \cdots + \mathcal{O}_p u_m$$

with $(b(u_i,u_j))_{1\le i,j\le m}=T$. Let M_1 and M_2 be submodules of M. We then write $M=M_1\perp M_2$ if $M=M_1+M_2$, and $b(u,v)=0$ for any $u\in M_1, v\in M_2$. Let M and N be Hermitian modules. Then a homomorphism $\sigma : N \longrightarrow M$ is said to be an isometry if σ is injective and $b(\sigma(u),\sigma(v))=b(u,v)$ for any $u,v\in N$. In particular M is said to be isometric to N if σ is an isomorphism. We denote by U'_M the group of isometries of M to M itself. For Hermitian modules M and N over $\mathcal{O}_p$ of rank m and n respectively, put

$$\mathcal{A}'_a(N,M)=\{\sigma : N \longrightarrow M/p^aM \mid q(\sigma(u))\equiv q(u) \bmod p^{e_p+a}\},$$

and

$$\mathcal{B}'_a(N,M):=\{\sigma\in\mathcal{A}'_a(N,M) \mid \sigma \text{ is primitive}\}.$$

Here a homomorphism $\sigma : N \longrightarrow M$ is said to be primitive if ϕ induces an injective mapping from N/pN to M/pM. Then we can define the local density $\alpha'_p(N,M)$ as

$$\alpha'_p(N,M)=\lim_{a\to\infty} p^{-a(2mn-n^2)}\#(\mathcal{A}'_a(N,M))$$

if M and N are non-degenerate, and the primitive local density $\beta'_p(N,M)$ as

$$\beta'_p(N,M)=\lim_{a\to\infty} p^{-a(2mn-n^2)}\#(\mathcal{B}'_a(N,M))$$

if M is non-degenerate as in the matrix case. It is easily seen that

$$\alpha_p(S,T)=\alpha'_p(M_T,M_S),$$

and

$$\beta_p(S,T)=\beta'_p(M_T,M_S).$$

Now let L be a submodule of M isometric to N. Take a basis $\{v_i\}_{i=1}^n$ of N, and put

$$\mathcal{A}'_a(N,M;L)=\{\sigma\in\mathcal{A}'_a(N,M) \mid \sigma \text{ satisfies the condition } (*L)\},$$

where the condition $(*L)$ is
$(*L)$: there exists an isometry $\eta : M \longrightarrow M$ and $v'_1,...,v'_n\in M$ such that

$$\eta(L)=\mathcal{O}_pv'_1+\cdots+\mathcal{O}_pv'_n$$

with

$$v'_i\equiv\sigma(v_i) \bmod p^aM.$$

We note that the condition $(*L)$ is independent of the choice of $v'_1,...,v'_n$ if a is sufficiently large. Moreover the number $p^{-a(2mn-n^2)}\#(\mathcal{A}'_a(N,M;L))$ is independent of a if a is sufficiently large. Hence we define $\alpha'_p(N,M;L)$ as

$$\alpha'_p(N,M;L)=\lim_{a\longrightarrow\infty} p^{-a(2mn-n^2)}\#(\mathcal{A}'_a(N,M;L)).$$

Lemma 5.1.6. *Let M and N be non-degenerate Hermitian modules of rank m and n respectively. Assume that $N = N_1 \perp N_2$ with N_1, N_2 submodules of N of rank n_1 and n_2, respectively, and let $\{L_i\}_{i=1}^{l}$ be a complete set of submodules of M isometric to N_1 which are not transformed into each other by isometries of M. Moreover for a submodule L of M put*

$$\mathcal{D}'_a(N_2, M; L) = \{\sigma \in \mathcal{A}'_a(N_2, M) \mid b(L, \sigma(N_2)) \equiv 0 \bmod p^a \varpi^{i_p}\}.$$

Then

$$\#(\mathcal{A}'_a(N, M)) = \sum_{i=1}^{l} \#(\mathcal{A}'_a(N_1, M; L_i)) \#(\mathcal{D}'_a(N_2, M; L_i))$$

if a is sufficiently large.

Proof. For a sufficiently large e, we can define a mapping $\phi : \mathcal{A}'_e(N_1, M) \ni \sigma_1 \mapsto \phi(\sigma_1) \in U'_M$ such that

$$\phi(\sigma_1)(\sigma_1(N_1)) = L_i$$

for some i as follows: For $\sigma_1 \in \mathcal{A}'_e(N_1, M)$, we can take a homomorphism $\widetilde{\sigma}_1 : N_1 \longrightarrow M$ such that $\widetilde{\sigma}_1 \bmod p^e = \sigma_1$. Then N_1 is isometric to $\widetilde{\sigma}_1(N_1)$ when e is suffiently large. Hence there is an element $\alpha \in U'_M$ such that $\alpha(\sigma_1(N_1)) = L_i$ for some i. Put $\phi(\sigma_1) = \alpha$. Then ϕ satisfies the required property. Now for $\sigma \in \mathcal{A}_e(N, M)$, put

$$\sigma_1 = \sigma | N_1 \in \mathcal{A}'_e(N_1, M)$$

and

$$\sigma_2 = \phi(\sigma_1)(\sigma | N_2) \in \mathcal{A}'_e(N_2, M).$$

Then the fact that $b(N_1, N_2) = 0$ implies

$$b(\sigma(N_1), \sigma(N_2)) \equiv 0 \bmod p^r \varpi^{i_p},$$

and hence

$$b(\phi(\sigma_1)(\sigma(N_1)), \phi(\sigma_1)(\sigma(N_2))) \equiv 0 \bmod p^r \varpi^{i_p}.$$

Remarking that $\phi(\sigma_1)(\sigma(N_1)) = L_i$ for some i we obtain

$$b(L_i, \sigma_2(N_2)) \equiv 0 \bmod p^r \varpi^{i_p}.$$

We then define a mapping

$$\eta : \mathcal{A}'_e(N, M) \longrightarrow \bigsqcup_{i=1}^{s} (\mathcal{A}'_e(N_1, M; L_i) \times \mathcal{D}'_e(N_2, M; L_i))$$

by $\eta(\sigma) = (\sigma_1, \sigma_2)$ as above. Clearly η is injective. For a given $(\sigma_1, \sigma_2) \in \mathcal{A}'_e(N_1, M; L_i) \times \mathcal{D}'_e(N_2, M; L_i)$, we define $\sigma \in \mathcal{A}'_e(N, M)$ by $\sigma | N_1 = \sigma_1$ and $\sigma | N_2 = \phi(\sigma_1)^{-1} \sigma_2$. Then $\eta(\sigma) = (\sigma_1, \sigma_2)$, and hence η is bijective. Thus the assertion has been proved. $\square$

Lemma 5.1.7. *Let N_1 and M be non-degenerate Hermitian modules, and $\sigma \in \mathrm{Hom}_{\mathcal{O}_p}(N_1, M)$. Suppose that $T_{N_1} = \Theta_{n_1}$ with n_1 even, or $T_{N_1} = 1_{n_1}$ according as K_p is ramified over $\mathbf{Q}_p$, or not, and that*

$$q(\sigma(x)) \equiv q(x) \bmod p^{e_p+a}\mathcal{O}_p \text{ for } x \in N_1$$

for a sufficiently large a. Then there exists an isometry $\eta : N_1 \longrightarrow M$ such that

$$\eta(N_1) = \sigma(N_1) \text{ and } \eta(x) \equiv \sigma(x) \bmod p^{e_p+a}\sigma(N_1) \text{ for } x \in N_1.$$

Proof. The assertion can be proved by using the same argument as in the proof of [[Kit93], Corollary 5.4.2]. □

Lemma 5.1.8. *Let $N = N_1 \perp N_2$ and $M = M_1 \perp M_2$ where N_1, N_2, M_1, M_2 are non-degenerate Hermitian modules. Suppose that $T_{N_1} = \Theta_{n_1}$ with n_1 even, or $T_{N_1} = 1_{n_1}$ according as K_p is ramified over $\mathbf{Q}_p$, or not. Moreover suppose that N_1 is isometric to M_1. Then*

$$\#(\mathcal{A}'_a(N, M)) = \#(\mathcal{A}'_a(N_1, M))\#(\mathcal{A}'_a(N_2, M_2))$$

if a is sufficiently large.

Proof. By assumption there exists at least one submodule M_1 of M isometric to N_1. Let L be a submodule of M isometric to N_1. Then we have $M = L \perp L^{\perp}$, and M_2 is isometric to $L^{\perp}$. Hence there exits an isometry $\widetilde{\eta} : M \longrightarrow M$ such that $\widetilde{\eta}|M_1 = L$. Hence by Lemma 5.1.6.

$$\#(\mathcal{A}'_a(N, M)) = \#(\mathcal{A}'_a(N_1, M; M_1))\#(\mathcal{A}'_a(N_2, M_2)).$$

First we will show

$$\#(\mathcal{A}'_a(N_1, M; M_1)) = \#(\mathcal{A}'_a(N_1, M))$$

if a is sufficiently large. To prove this, let $\{u_i\}_{i=1}^r$ be a basis of N_1. Let $\sigma \in \mathcal{A}'_a(N_1, M)$. Then, by Lemma 5.1.7, there exits an isometry $\widetilde{\sigma} : N_1 \longrightarrow M$ such that $\sigma(N_1) = \widetilde{\sigma}(N_1)$, and we have

$$\widetilde{\sigma}(N_1) = \mathcal{O}_p v'_1 + ... + \mathcal{O}_p v'_r,$$

where v'_i is an element of M such that $v'_i \equiv \sigma(u_i) \bmod p^a N$. On the other hand, there exists an isometry $\eta : M \longrightarrow M$ such that $\eta(M_1) = \widetilde{\sigma}(N_1)$. This implies that σ belongs to $\mathcal{A}'_a(N_1, M; M_1)$, and thus we prove the claim. Next we will show that

$$\#(\mathcal{D}'_a(N_2, M; M_1)) = \#(\mathcal{A}'_a(N_2, M_2))$$

if a is sufficiently large. To prove this, first recall that $M = M_1 \perp M_2$, and remark that

$$\{x \in K_p \otimes M_1 \mid b(x, M_1) \in \mathcal{O}_p\} = \varpi^{i_p} M_1.$$

Hence we easily see that

$$\{x \in M \mid b(M_1, x) \equiv 0 \bmod p^a \varpi^{i_p}\} = p^a M_1 \perp M_2.$$

Hence we have

$$\mathcal{D}'_a(N_2, M; M_1)$$
$$= \{\sigma_2 : N_2 \longrightarrow (p^a M_1 \perp M_2)/p^a M \mid q(\sigma_2(x)) \equiv q(x) \bmod p^{e_p+a} \text{ for } x \in N_2\}.$$

Hence we have

$$\#(\mathcal{D}'_a(N_2, M; M_1))$$
$$= \#(\{\sigma_2 : N_2 \longrightarrow M_2/p^a M_2 \mid q(\sigma_2(x)) \equiv q(x) \bmod p^{e_p+a} \text{ for } x \in N_2\})$$
$$= \#(\mathcal{A}'_a(N_2, M_2)).$$

This proves the assertion. □

Lemma 5.1.9. (1) *Assume that K_p is unramified over $\mathbf{Q}_p$. Let $B \in \mathrm{Her}_m(\mathcal{O}_p)$. Then we have*

$$\beta_p(1_{2k}, pB) = \prod_{i=0}^{2m-1} (1 - (-1)^i p^{-2k+i})$$

(2) *Let $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Let $B \in \mathrm{Her}_m(\mathcal{O}_p)$. Then we have*

$$\beta_p(1_{2k}, pB) = \prod_{i=0}^{2m-1} (1 - p^{-2k+i})$$

(3) *Assume that K_p is ramified over $\mathbf{Q}_p$. Let n_0 be even. Let $B \in \mathrm{Her}_{m,*}(\mathcal{O}_p)$. Then we have*

$$\beta_p(\Theta_{2k}, p^{i_p} B) = \prod_{i=0}^{m-1} (1 - p^{-2k+2i}).$$

Lemma 5.1.10. (1) *Assume that K_p is unramified over $\mathbf{Q}_p$. Then we have*

$$\alpha_p(1_{2k}, 1_m) = \prod_{i=0}^{m-1} (1 - (-1)^i p^{-2k+i})$$

(2) *Let $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then we have*

$$\alpha_p(1_{2k}, 1_m) = \prod_{i=0}^{m-1} (1 - p^{-2k+i})$$

(3) *Assume that K_p is ramified over $\mathbf{Q}_p$. Let m be even. Then we have*

$$\alpha_p(\Theta_{2k}, \Theta_m) = \prod_{i=0}^{m/2-1} (1 - p^{-2k+2i}).$$

5.2. **Formal power series of Andrianov type.**

For an element $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$, we define a polynomial $\widetilde{G}_p(T,X,t)$ in X and t by

$$\widetilde{G}_p(T,X,t) = \sum_{i=0}^{m} \sum_{W \in GL_m(\mathcal{O}_p)\backslash \mathcal{D}_{m,i}} \pi_p(W) t^{\nu(\det W)} \widetilde{F}_p^{(0)}(T[W^{-1}],X).$$

We also define a polynomial $G_p(T,X)$ in X by

$$G_p(T,X) = \sum_{i=0}^{m} \sum_{W \in GL_m(\mathcal{O}_p)\backslash \mathcal{D}_{m,i}} (Xp^m)^{\nu(\det W)} \pi_p(W) F_p^{(0)}(T[W^{-1}],X).$$

Moreover for an element $T \in \widetilde{\mathrm{Her}}_{m,p}$ we define a polynomial $B_p(T,t)$ in t by

$$B_p(T,t) = \frac{\prod_{i=0}^{m-1}(1-\tau_p^{m+i} p^{m+i} t^2)}{G_p(T,t^2)},$$

where $\tau_p^i = 1$ or ξ_p according as i is even or odd. We note that

$$\widetilde{G}_p(T,X,1) = X^{-\mathrm{ord}(\det T)} X^{e_p m - f_p[m/2]} G_p(T, Xp^{-m}).$$

Lemma 5.2.1. (1) *Assume that K_p is unramified over $\mathbf{Q}_p$. Let $B_1 \in \mathrm{Her}_{m-n_0}(\mathcal{O}_p)$. Then we have*

$$\alpha_p(1_{n_0} \perp pB_1) = \prod_{i=1}^{n_0} (1-(-p)^{-i}) \alpha_p(pB_1)$$

(2) *Let $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Let $B_1 \in \mathrm{Her}_{m-n_0}(\mathcal{O}_p)$. Then we have*

$$\alpha_p(1_{n_0} \perp pB_1) = \prod_{i=1}^{n_0} (1-p^{-i}) \alpha_p(pB_1)$$

(3) *Assume that K_p is ramified over $\mathbf{Q}_p$. Let n_0 be even. Let $B_1 \in \mathrm{Her}_{m-n_0,*}(\mathcal{O}_p)$. Then we have*

$$\alpha_p(\Theta_{n_0} \perp p^{i_p} B_1) = \prod_{i=1}^{n_0/2} (1-p^{-2i}) \alpha_p(p^{i_p} B_1).$$

Proof. (1) By Lemma 5.1.8, we have

$$\alpha_p(1_{n_0} \perp pB_1) = \alpha_p(1_{n_0})_p \alpha_p(pB_1).$$

By Lemma 5.1.10, we have

$$\alpha_p(1_{n_0}) = \alpha_p(1_{n_0}, 1_{n_0}) = \prod_{i=1}^{n_0} (1-(-p)^{-i}).$$

This proves the assertion. Similarly the assertions (2) and (3) are proved. $\square$

Lemma 5.2.2. (1) *Assume that K_p is unramified over $\mathbf{Q}_p$. Let $T = 1_{m-r} \perp pB_1$ with $B_1 \in \mathrm{Her}_r(\mathcal{O}_p)$. Then*

$$\beta_p(1_{2k}, T) = \prod_{i=0}^{m+r-1} (1 - p^{-2k+i}(-1)^i).$$

(2) *Assume that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Let $T = 1_{m-r} \perp pB_1$ with $B_1 \in \mathrm{Her}_r(\mathcal{O}_p)$. Then*

$$\beta_p(1_{2k}, T) = \prod_{i=0}^{m+r-1} (1 - p^{-2k+i}).$$

(3) *Assume that K_p is ramified over $\mathbf{Q}_p$. Assume that $m-r$ is even. Let $T = \Theta_{m-r} \perp p^{i_p} B_1$ with $B_1 \in \mathrm{Her}_{r,*}(\mathcal{O}_p)$. Then*

$$\beta_p(\Theta_{2k}, T) = \prod_{i=0}^{(m+r-2)/2} (1 - p^{-2k+2i}).$$

Proof. (1) By Lemma 5.1.8, we have

$$\alpha_p(1_{2k}, T) = \alpha_p(1_{2k}, 1_{m-r})\alpha_p(1_{2k-m+r}, pB_1).$$

We have

$$\beta_p(1_{2k}, T) = \alpha_p(1_{2k}, 1_{m-r})\beta_p(1_{2k-m+r}, pB_1).$$

Hence the assertion can be proved by Lemmas 5.1.9 and 5.1.10. Similarly the assertions (2) and (3) can be proved. $\square$

Corollary. (1) *Assume that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Let $T = \Theta_{m-r} \perp pB_1$ with $B_1 \in \mathrm{Her}_r(\mathcal{O}_p)$. Then we have*

$$G_p(T, Y) = \prod_{i=0}^{r-1} (1 - \xi_p^{i-1} p^{m+i} Y^2).$$

(2) *Assume that K_p is ramified over $\mathbf{Q}_p$. Let $T = \Theta_{m-r} \perp p^{i_p} B_1$ with $B_1 \in \mathrm{Her}_{r,*}(\mathcal{O}_p)$. Assume that $m-r$ is even. Then*

$$G_p(T, Y) = \prod_{i=0}^{[(r-2)/2]} (1 - 2^{2i+2[(m+1)/2]} Y^2).$$

Proof. We have

$$\beta_p(\Theta_{2k}, T) = G_p(T, p^{-2k}) \prod_{i=0}^{[(m-1)/2]} (1 - p^{2i-2k}) \prod_{i=1}^{[m/2]} (1 - \xi_p p^{2i-1-2k}).$$

Thus the assertion follows from Lemma 5.2.2. $\square$

Lemma 5.2.3. *Let $B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. Then we have*

$$\widetilde{F}^{(0)}(B,X) = X^{e_p m - f_p[m/2]} \sum_{B' \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)/GL_m(\mathcal{O}_p)} X^{-\mathrm{ord}(\det B')} \frac{\alpha_p(B',B)}{\alpha_p(B')}$$
$$\times G(B', p^{-m}X^2) X^{\mathrm{ord}(\det B) - \mathrm{ord}(\det B')}.$$

Proof. We have

$$\widetilde{F}^{(0)}(B,X) = X^{e_p m - f_p[m/2]} X^{-\mathrm{ord}(\det B)} F^{(0)}(B,X)$$
$$= X^{e_p m - f_p[m/2]} \sum_{W \in GL_m(\mathcal{O}_p) \backslash \widetilde{\Omega}(B)} X^{-\mathrm{ord}(\det B)} G(B[W^{-1}], p^{-m}X^2)(X^2)^{\nu(\det W)}$$
$$= X^{e_p m - f_p[m/2]}$$
$$\times \sum_{B' \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)/GL_m(\mathcal{O}_p)} \sum_{W \in GL_m(\mathcal{O}_p) \backslash \widetilde{\Omega}(B',B)} X^{-\mathrm{ord}(\det B)} G(B', p^{-m}X^2)(X^2)^{\nu(\det W)}$$
$$= X^{e_p m - f_p[m/2]} \sum_{B' \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)/GL_m(\mathcal{O}_p)} X^{-\mathrm{ord}(\det B')} \#(GL_m(\mathcal{O}_p) \backslash \widetilde{\Omega}(B',B))$$
$$\times p^{\mathrm{ord}(\det B) - \mathrm{ord}(\det B')} G(B', p^{-m}X^2) X^{\mathrm{ord}(\det B) - \mathrm{ord}(\det B')}.$$

Thus the assertion follows from (2) of Lemma 5.1.3. □

Lemma 5.2.4. (1) *Assume that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Let $T = 1_{m-r} \bot pB_1$ with $B_1 \in \mathrm{Her}_r(\mathcal{O}_p)$. Then we have*

$$B_p(T,t) = \prod_{i=r}^{m-1} (1 - \xi_p^{m+i} p^{m+i} t^2).$$

(2) *Assume that K_p is ramified over $\mathbf{Q}_p$. Let $T = \Theta_{m-r} \bot p^{i_p} B_1$ with $B_1 \in \mathrm{Her}_{r,*}(\mathcal{O}_p)$. Then*

$$B_p(T,t) = \prod_{i=[(r-1)/2]+1}^{[(m-2)/2]} (1 - 2^{2i+2[(m+1)/2]} t^2).$$

For a non-degenerate semi-integral matrix T over $\mathcal{O}_p$ of degree n, put

$$S_p(T,X,t) = \sum_w \widetilde{F}_p^{(0)}(T[w],X) t^{\nu(\det w)}.$$

This type of formal power series was first introduced by Andrianov [A] to study the standard L-functions of Siegel modular forms of integral weight. Thus we call it the formal power series of Andrianov type. (See also [Böc86], [KK10a].) The following proposition can easily be proved by (1) of Lemma 5.1.3.

Proposition 5.2.5. *Let $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. Then we have*

$$\sum_{B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(B,X)\alpha_p(T,B)}{\alpha_p(B)} t^{\mathrm{ord}(\det B)} = t^{\mathrm{ord}(\det T)} S_p(T,X,p^{-m}t).$$

Let $\mathcal{H}(\mathcal{U}^{(m)}(\mathbf{A}),\Gamma^{(m)})$ be the Hecke ring associated with the Hecke pair $(\mathcal{U}^{(m)}(\mathbf{A}),\Gamma^{(m)})$. Then $\mathcal{H}(\mathcal{U}^{(m)}(\mathbf{A}),\Gamma^{(m)})$ acts on $\mathcal{M}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}),\det^{-l})$ as in [Ike08]. We call an element F of $\mathcal{M}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}),\det^{-l})$ a Hecke eigen form if it is a common eigefunction of all Hecke operators T in $\mathcal{H}(\mathcal{U}^{(m)}(\mathbf{A}),\Gamma^{(m)})$. Then for each $r \in GL_m(\mathbf{A}) \cap \prod_p M_m(\mathcal{O}_p)$, let $\lambda_F(r)$ be the eigenvalue of $\mathcal{K}\begin{pmatrix} r^{-1} & 0 \\ 0 & r^* \end{pmatrix}\mathcal{K}$ with respect to F, and define a Dirichlet series $\mathfrak{L}(s,F)$ by

$$\mathfrak{L}(s,F) = \sum_{r \in \mathcal{K}(GL_m(\mathbf{A})\cap\prod_p M_m(\mathcal{O}_p))\mathcal{K}} \lambda_F(r)|\det r|_{\mathbf{A}}^s,$$

where $|\det r|_{\mathbf{A}} = \prod_p |\det r_p|_{K_p}$ for $r = (r_p) \in GL_m(\mathbf{A}) \cap \prod_p M_m(\mathcal{O}_p)$. Then there exists an Euler product $\mathcal{Z}(s,F)$ such that

$$\mathfrak{L}(s,F) = \prod_{i=1}^m L(2s-i+1,\chi^{i-1})\mathcal{Z}(s,F).$$

We then put

$$L(s,F,\mathrm{st}) = \mathcal{Z}(s+m-1/2,F),$$

and call it the standard L-function of F in the sense of Shimura. We note that our standard L-function coincides with that in [Ike08] up to Euler factors at ramified primes.

Now we define the Eisenstein series on $\mathcal{U}^{(m)}(\mathbf{A})$ and consider its standard L-function in the sense of Shimura. Let $\mathcal{P}$ be the maximal parabolic subgroup of $\mathcal{U}^{(m,m)}$ defined by

$$\mathcal{P}(R) = \{\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathcal{U}^{(m,m)}(R)\}$$

for any $\mathbf{Q}$-algebra R. Write an element $g = (g_v) \in \mathcal{U}^{(m)}(\mathbf{A})$ as

$$(g_p)_{p<\infty} = \left(\left(\begin{smallmatrix} a_p & b_p \\ 0 & d_p \end{smallmatrix}\right)\right)_{p<\infty} (\kappa_p)_{p<\infty}$$

with $\left(\left(\begin{smallmatrix} a_p & b_p \\ 0 & d_p \end{smallmatrix}\right)\right)_{p<\infty} \in \prod_{p<\infty}\mathcal{P}(\mathbf{Q}_p)$ and $(\kappa_p)_{p<\infty} \in \mathcal{K}$, and define the function on $\mathcal{U}^{(m)}(\mathbf{A})$ by

$$\mathbf{f}_{2l}(g) = \prod_p |\det(d_p\overline{d}_p)|_p^{-l} j(g_\infty,\mathbf{i})^{-2l}(\det g_\infty)^l.$$

We then define the normalized Eisenstein series as

$$\mathbf{E}_{2l}^{(m)}(g) = 2^{-m}\prod_{i=1}^m L(i-2l,\chi^{i-1}) \sum_{\gamma \in \mathcal{P}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{Q})} \mathbf{f}_{2l}(\gamma g).$$

We note that $\mathbf{E}_{2l}^{(m)}$ is written as

$$\mathbf{E}_{2l}^{(m)} = (\mathcal{E}_{2l,m}^{(1)}, \mathcal{E}_{2l,m}^{(2)}, ..., \mathcal{E}_{2l,m}^{(h)})^{\sharp},$$

where

$$\mathcal{E}_{2l,m}^{(a)}(Z) = 2^{-m}\prod_{i=1}^{m} L(j-2l, \chi^{i-1}) \sum_{g \in (\Gamma_a \cap \mathcal{P}(\mathbf{Q}))\backslash \Gamma_a} (\det g)^l j(g,Z)^{-2l}$$

for $a = 1, ..., h$. Now put

$$\mathcal{L}_{m,p}(X,t)$$
$$= \begin{cases} \prod_{i=1}^{m}\{(1-p^{-m+2i-1}X^2t^2)(1-p^{-m+2i-1}X^{-2}t^2)\}^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is unramified} \\ \prod_{i=1}^{m}\{(1-p^{-m/2+i-1/2}Xt)^2(1-p^{-m+i-1/2}X^{-1}t)^2\}^{-1} & \text{if } K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p \\ \prod_{i=1}^{m}\{(1-p^{-m/2+i-1/2}Xt)(1-p^{-m/2+i-1/2}X^{-1}t)\}^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is ramified} \end{cases}$$

Then by a careful analysis of the proof of [Proposition 13.5, [Ike08]], we obtain

Proposition 5.2.6. *Then $\mathbf{E}_{2l}^{(m)}$ is a Hecke eigenform in $\mathcal{M}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}), \det^{-l})$, and its standard L-function $\mathcal{L}(s, \mathbf{E}_{2l}^{(m)}, \mathrm{st})$ in the sense of Shimura is given by*

$$\mathcal{L}(s, \mathbf{E}_{2l}^{(m)}, \mathrm{st}) = \prod_{q} \mathcal{L}_{m,q}(q^{-l+m/2}, q^{-s}).$$

Remark. By Proposition 5.2.6 combined with the argument as in the proof of [Thorem 18.1, [Ike08]], we see that $Lift^{(m)}(f)$ is a Hecke eigenform and that its standard L-function in the sense of Shimura is given by

$$\prod_{i=1}^{m} L(s+k+n-i+1/2, f)L(s+k+n-i+1/2, f, \chi).$$

Theorem 5.2.7. *Let T be an element of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)^{\times}$. Then we have*

$$S_p(T,X,t) = B_p(T, p^{-m/2}t)\widetilde{G}_p(T,X,t)\mathcal{L}_{m,p}(X, p^{m/2-1/2}t).$$

Proof. Take an element $\widetilde{T} \in \widetilde{\mathrm{Her}}_m(\mathcal{O})^{+}$ such that $\widetilde{T} \sim_{GL_m(\mathcal{O}_p)} T$. Then we have

$$S_p(\widetilde{T},X,t) = S_p(T,X,t)$$

and

$$B_p(\widetilde{T}, p^{-m/2}t)\widetilde{G}_p(\widetilde{T},X,t) = B_p(T, p^{-m/2}t)\widetilde{G}_p(T,X,t).$$

Write $S_p(\widetilde{T},X,t)$ and $B_p(\widetilde{T},p^{-m/2}t)\widetilde{G}_p(\widetilde{T},X,t)\mathcal{L}_{m,p}(X,p^{m/2-1/2}t)$ as

$$S_p(\widetilde{T},X,t)=\sum_{i=0}^{\infty} r_i(X)t^i,$$

and

$$B_p(\widetilde{T},p^{-m/2}t)\widetilde{G}_p(\widetilde{T},X,t)\mathcal{L}_{m,p}(X,p^{m/2-1/2}t)=\sum_{i=0}^{\infty} s_i(X)t^i.$$

Then $r_i(X)$ and $s_i(X)$ are polynomials in X and X^{-1}. For a positive integer l and $A\in\widehat{\mathrm{Her}}_m(\mathcal{O})^+$, let $c_{2l,m}(A)$ denote the A-th Fourier coefficients of $\mathcal{E}^{(1)}_{2l,m}(Z)$, and

$$\widetilde{G}_{2l,m}(A,s)=\sum_{W\in GL_m(\mathcal{O})\backslash M_m(\mathcal{O})^\times}\prod_q \pi_q(W)c_{2l,m}(A[W^{-1}])(\det W)^{2l}(N_{K/\mathbf{Q}_p}(\det W))^{-s}.$$

Then by Proposition 5.2.6 and [Theorem 20.7, Shimura [Sh00]], we obtain

$$\sum_{V\in M_m(\mathcal{O})^\times/GL_m(\mathcal{O})} c_{2l,m}(\widetilde{D}^{-1}\widetilde{T}[V])(\det V)^{-2l}(N_{K/\mathbf{Q}}(\det W))^{-s+m}$$
$$=\widetilde{G}_{2l,m}(\widetilde{D}^{-1}\widetilde{T},s)\prod_q B_q(\widetilde{T},q^{-s})\mathcal{L}_{m,q}(q^{-l+m/2},q^{m/2-1/2-s})$$

for infinitely many positive integers l. As stated in [Ike08], $c_{2l,m}(A)$ is given by

$$c_{2l,m}(A)=|\gamma(A)|^{l-m/2}\prod_q \widetilde{F}_q(A,p^{l-m/2})$$

for $A\in\widehat{\mathrm{Her}}_m(\mathcal{O})^+$. Hence putting $t=p^{-s+m/2}$, and comparing the p-factors on the both hand-sides of the above formula, we obtain

$$S_p(\widetilde{T},p^{-l+m/2},t)=B_p(\widetilde{T},p^{-m/2}t)\widetilde{G}_p(\widetilde{T},p^{-l+m/2},t)\mathcal{L}_m(p^{-l+m/2},p^{m/2-1/2}t)$$

for infinitely many positive integers l. This implies that $r_i(p^{-l+m/2})=s_i(p^{-l+m/2})$ for infinitely many positive integers l. Hence we have $r_i(X)=s_i(X)$. $\square$

Now by Theorem 5.2.7, we can rewrite $H_{m,p}(d_0,X,Y,t)$ in terms of $G_p(B',p^{-(n+1)/2}Y)$, $B_p(T,p^{n/2-1}t^2)$ and $\widetilde{G}_p(T,X,t^2)$ in the following way: For $d_0\in\mathbf{Z}_p^\times$ put

$$\widetilde{\mathcal{F}}_{m,p}(d_0)=\bigcup_{i=0}^{\infty}\widetilde{\mathrm{Her}}_m(p^i d_0 N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*),\mathcal{O}_p),$$

and define a formal power series $R_m(d_0,X,Y,t)$ in t by

$$R_m(d_0,X,Y,t)=\sum_{B'\in\widetilde{\mathcal{F}}_{m,p}(d_0)}\frac{\widetilde{G}_p(B',X,p^{-m}Yt)}{\alpha_p(B')}$$

$$\times (tY^{-1})^{\mathrm{ord}(\det B')} B_p(B', p^{-3m/2}Yt) G_p(B', p^{-m}Y^2).$$

Theorem 5.2.8. *We have*

$$H_{m,p}(d_0, X, Y, t) = Y^{e_p m - f_p [m/2]} R_{m,p}(d_0, X, Y, t) \mathcal{L}_{m,p}(X, tYp^{-m/2-1/2})$$

for $d_0 \in \mathbf{Z}_p^{\times}$.

Proof. We note that $H_{m,p}(d_0, X, Y, t)$ can be written as

$$H_{m,p}(d_0, X, Y, t) = \sum_{B \in \widetilde{\mathcal{F}}_{m,p}(d_0)} t^{\mathrm{ord}(\det B)} \frac{\widetilde{F}_p^{(0)}(B, X) \widetilde{F}_p^{(0)}(B, Y)}{\alpha_p(B)}.$$

Hence by Lemma 5.2.3 and Proposition 5.2.5, we have

$$H_{m,p}(d_0, X, Y, t) = Y^{e_p m - f_p [m/2]} \sum_{B \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{t^{\mathrm{ord}(\det B)} \widetilde{F}_p^{(0)}(B, X)}{\alpha_p(B)}$$

$$\times \sum_{B' \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)} \frac{Y^{-\mathrm{ord}(\det B')} G_p(B', p^{-m}Y^2) \alpha_p(B', B)}{\alpha_p(B')} Y^{\mathrm{ord}(\det B) - \mathrm{ord}(\det B')}.$$

Let $B, B' \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$, and assume that $\alpha_p(B', B) \neq 0$. Then we note that $B \in \widetilde{\mathcal{F}}_{m,p}(d_0)$ if and only if $B' \in \widetilde{\mathcal{F}}_{m,p}(d_0)$. Hence by Theorem 5.2.7 we have

$$Y^{-e_p m + f_p [m/2]} H_{m,p}(d_0, X, Y, t) = \sum_{B' \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{G_p(B', p^{-m}Y^2) Y^{-2\mathrm{ord}(\det B')}}{\alpha_p(B')}$$

$$\times \sum_{B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(B, X) \alpha_p(B', B)}{\alpha_p(B)} (tY)^{\mathrm{ord}(\det B)}$$

$$= \sum_{B' \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{G_p(B', p^{-m}Y^2) Y^{-2\mathrm{ord}(\det B')}}{\alpha_p(B')} (tY)^{\mathrm{ord}(\det B')} S_p(B', X, tYp^{-m})$$

$$= \sum_{B' \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{\widetilde{G}_p(B', X, p^{-m}Yt)}{\alpha_p(B')} (tY^{-1})^{\mathrm{ord}(\det B')}$$

$$\times B_p(B', p^{-3m/2}Yt) G_p(B', p^{-m}Y^2) \mathcal{L}_{m,p}(tYp^{-m/2-1/2}, X).$$

□

5.3. **Formal power series of modified Koecher-Maass type.**

Let r be a positive integer, and $d_0 \in \mathbf{Z}_p^*$. We then define a formal power series $P_r(d_0, X, t)$ in t by

$$P_r(d_0, X, t) = \sum_{B \in \widetilde{\mathcal{F}}_{r,p}(d_0)} \frac{\widetilde{F}_p^{(0)}(B, X)}{\alpha_p(B)} t^{\mathrm{ord}(\det B)}.$$

This type of formal power series appears in an explicit formula of the Koecher-Maass series associated with the Siegel Eisenstein series and the Ikeda lift (cf. [IK04], [IK06].) Thus we call this the formal power series of Koecher-Maass type. To prove Theorems 5.5.2 and 5.5.3, the main results of Section 5, we define a formal power series $\widetilde{P}_r(d_0, X, Y, t)$ in t by

$$\widetilde{P}_r(d_0, X, Y, t) = \sum_{B' \in \widetilde{\mathcal{F}}_{r,p}(d_0)} \frac{\widetilde{G}_p(B', X, tY)}{\alpha_p(B')} (tY^{-1})^{\mathrm{ord}(\det B')}.$$

The relation between $\widetilde{P}_r(d_0, X, Y, t)$ and $P_r(d_0, X, t)$ will be given in the following proposition:

Proposition 5.3.1. (1) *Assume that K_p is unramified over $\mathbf{Q}_p$. Then*

$$\widetilde{P}_r(d_0, X, Y, t) = P_r(d_0, X, tY^{-1}) \prod_{i=1}^{r} (1 - t^4 p^{-2r-2+2i}).$$

(2) *Assume that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$\widetilde{P}_r(d_0, X, Y, t) = P_r(d_0, X, tY^{-1}) \prod_{i=1}^{r} (1 - t^2 p^{-r-1+i})^2.$$

(3) *Assume that K_p is ramified over $\mathbf{Q}_p$. Then*

$$\widetilde{P}_r(d_0, X, Y, t) = P_r(d_0, X, tY^{-1}) \prod_{i=1}^{r} (1 - t^2 p^{-r-1+i}).$$

Proof. First assume that K_p is a quadratic extension of $\mathbf{Q}_p$. For each non-negative integer $i \le r$ put

$$P_{r,i}(d_0, X, t) = \sum_{B \in \widetilde{\mathcal{F}}_{r,p}(d_0)} \sum_{W \in \mathcal{D}_{r,i}} \frac{\widetilde{F}_p^{(0)}(B[W^{-1}], X)}{\alpha_p(B)} t^{\mathrm{ord}(\det B)}.$$

Then by (2) of Lemma 5.1.3 we have

$$P_{r,i}(d_0, X, t) = \sum_{B \in \widetilde{\mathcal{F}}_{r,p}(d_0)} \frac{1}{\alpha_p(B)} \sum_{B' \in \widetilde{\mathrm{Her}}_r(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(X, B') \alpha_p(B', B; i)}{\alpha_p(B')}$$

$$\times p^{-(\mathrm{ord}(\det B)-\mathrm{ord}(\det B'))} t^{\mathrm{ord}(\det B)}.$$

Let $B, B' \in \widetilde{\mathrm{Her}}_r(\mathcal{O}_p)$, and assume that $\alpha_p(B', B; i) \neq 0$. Then we note that $B \in \widetilde{\mathcal{F}}_{r,p}(d_0)$ if and only if $B' \in \widetilde{\mathcal{F}}_{r,p}(d_0)$. Thus by (1) of Lemma 5.1.3 we have

$$P_{r,i}(d_0, X, t)$$

$$= \sum_{B' \in \widetilde{\mathcal{F}}_{r,p}(d_0)} \frac{\widetilde{F}_p^{(0)}(B', X)}{\alpha_p(B')} p^{\mathrm{ord}(\det B')} \sum_{B \in \widetilde{\mathrm{Her}}_r(\mathcal{O}_p)} t^{\mathrm{ord}(\det B)} \frac{\alpha_p(B', B; i)}{\alpha_p(B)}$$

$$= \sum_{B' \in \widetilde{\mathcal{F}}_{r,p}(d_0)} \frac{\widetilde{F}_p^{(0)}(B', X)}{\alpha_p(B')} p^{\mathrm{ord}(\det B')} t^{\mathrm{ord}(\det B')} \# \mathcal{D}_{r,i} (tp^{-r})^{ei},$$

where $e = 2$ or 1 according as $K_p/\mathbf{Q}_p$ is unramified or ramified. By using the same argument as in the proof of Lemma 3.2.18 of Andrianov [A], we have

$$\# \mathcal{D}_{r,i} = \frac{\phi_r(p^e)}{\phi_i(p^e)\phi_{r-i}(p^e)}.$$

Hence we have

$$P_{r,i}(d_0, X, t)$$

$$= \sum_{B' \in \widetilde{\mathcal{F}}_{r,p}(d_0)} \frac{\widetilde{F}_p^{(0)}(B', X)}{\alpha_p(B')} t^{\mathrm{ord}(\det B')} \frac{\phi_r(p^e)}{\phi_i(p^e)\phi_{r-i}(p^e)} (tp^{-r})^{ei}$$

$$= \frac{\phi_r(p^e)}{\phi_i(p^e)\phi_{r-i}(p^e)} P_r(d_0, X, t)(tp^{-r})^{ei}.$$

Then we have

$$\widetilde{P}_r(d_0, X, Y, t)$$

$$= \sum_{i=0}^{r} (-1)^i p^{i(i-1)e/2} (tY)^{ei} P_{r,i}(d_0, X, tY^{-1}).$$

Hence we have

$$\widetilde{P}_r(d_0, X, Y, t)$$

$$= \sum_{i=0}^{r} (-1)^i p^{i(i+1)e/2} (p^{e(-r-1)} t^{2e})^i \frac{\phi_r(p^e)}{\phi_i(p^e)\phi_{r-i}(p^e)} P_r(d_0, X, tY^{-1})$$

$$= P_r(d_0, X, tY^{-1}) \prod_{i=1}^{r} (1 - t^{2e} p^{e(-r-1+i)}).$$

Next assume that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. For a pair $i = (i_1, i_2)$ of non-negative integers such that $i_1, i_2 \leq r$, put

$$P_{r,i}(d_0, X, t) = \sum_{B \in \widetilde{\mathcal{F}}_{r,p}(d_0)} \sum_{W \in \mathcal{D}_{r,i}} \frac{\widetilde{F}_p^{(0)}(B[W^{-1}], X)}{\alpha_p(B)} t^{\mathrm{ord}(\det B)}.$$

Then by using the same argument as above we can prove that

$$P_{r,i}(d_0,X,t)=\frac{\phi_r(p)}{\phi_{i_1}(p)\phi_{r-i_1}(p)}\frac{\phi_r(p)}{\phi_{i_2}(p)\phi_{r-i_2}(p)}P_r(d_0,X,t)(tp^{-r})^{i_1+i_2}.$$

Hence we have

$$\begin{aligned}&\widetilde{P}_r(d_0,X,Y,t)\\ &=\sum_{i_1=0}^{r}\sum_{i_2=0}^{r}(-1)^{i_1+i_2}p^{i_1(i_1+1)/2+i_2(i_2+1)/2}(p^{-r-1}t^2)^{i_1+i_2}\\ &\times\frac{\phi_r(p)}{\phi_{i_1}(p)\phi_{r-i_1}(p)}\frac{\phi_r(p)}{\phi_{i_2}(p)\phi_{r-i_2}(p)}P_r(d_0,X,tY^{-1})\\ &=P_r(d_0,X,tY^{-1})\prod_{i=1}^{r}(1-t^2p^{-r-1+i})^2.\end{aligned}$$

This proves the assertion. $\square$

Let

$$\mathcal{F}_{m,p,*}(d_0)=\bigcup_{i=0}^{\infty}(\widetilde{\mathrm{Her}}_m(p^id_0N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*),\mathcal{O}_p)\cap\mathrm{Her}_{m,*}(\mathcal{O}_p)).$$

Now we consider a partial series of $\widetilde{P}_r(d_0,X,Y,t)$. We put

$$\begin{aligned}&Q_r(d_0,X,Y,t)\\ &=\sum_{B'\in p^{-i_p}\widetilde{\mathcal{F}}_{r,p}(d_0)\cap\mathrm{Her}_{r,*}(\mathcal{O}_p)}\frac{\widetilde{G}_p(p^{i_p}B',\xi,X,tY)}{\alpha_p(p^{i_p}B')}(tY^{-1})^{\mathrm{ord}(\det pB')}.\end{aligned}$$

To consider the relation between $\widetilde{P}_r(d_0,X,Y,t)$ and $Q_r(d_0,X,Y,t)$, and to express $R_m(d_0,X,Y,t)$ in terms of $\widetilde{P}_r(d_0,X,Y,t)$, we provide some more preliminary results.

First assume that K_p is unramified over $\mathbf{Q}_p$ or $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Let H_m be a function on $\mathrm{Her}_m(\mathcal{O}_p)^\times$ satisfying the following condition:

$H_m(1_{m-r}\perp pB)=H_r(pB)$ for any $B\in\mathrm{Her}_r(\mathcal{O}_p)$.

Let $d_0\in\mathbf{Z}_p^*$. Then we put

$$Q(d_0,H_m,r,t)=\sum_{B\in p^{-1}\mathcal{F}_{r,p}(d_0)\cap\mathrm{Her}_r(\mathcal{O}_p)}\frac{H_m(1_{m-r}\perp pB)}{\alpha_p(1_{m-r}\perp pB)}t^{\mathrm{ord}(\det(pB))}.$$

Next assume that K_p is ramified over $\mathbf{Q}_p$. Let H_m be a function on $\mathrm{Her}_m(\mathcal{O}_p)^\times$ satisfying the following condition:

$H_m(\Theta_{m-r}\perp p^{i_p}B)=H_r(p^{i_p}B)$ for any $B\in\mathrm{Her}_{r,*}(\mathcal{O}_p)$ if $m-r$ is even.

Let $d_0 \in \mathbf{Z}_p^*$ and $m-r$ be even. Then we put

$$Q(d_0, H_m, r, t) = \sum_{B \in p^{-i_p}\widetilde{\mathcal{F}}_{r,p}(d_0) \cap \mathrm{Her}_{r,*}(\mathcal{O}_p)} \frac{H_m(\Theta_{m-r} \perp p^{i_p} B)}{\alpha_p(\Theta_{m-r} \perp pB)} t^{\mathrm{ord}(\det(p^{i_p}B))}.$$

Tnen by using the same argument as in the proof of [[KK10b], Propositions 5.3.4 and 5.3.5], we obtain

Proposition 5.3.2. (1) *Assume that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then for any $d_0 \in \mathbf{Z}_p^*$ and a non-negative integer r we have*

$$Q(d_0, H_m, r, t) = \frac{Q(d_0, H_r, r, t)}{\phi_{m-r}(\xi_p p^{-1})}.$$

(2) *Assume that K_p is ramified over $\mathbf{Q}_p$. Then for any $d_0 \in \mathbf{Z}_p^*$ and a non-negative integer r such that $m-r$ is even, we have*

$$Q(d_0, H_m, r, t) = \frac{Q(d_0, H_r, r, t)}{\phi_{(m-r)/2}(p^{-2})}.$$

Now to apply Proposition 5.3.2 to the formal power series $R_m(d_0, X, Y, t)$ and $Q_r(d_0, X, Y, t)$ we give some more lemmas.

Lemma 5.3.3. *Let m be an integer.*
(1) *Assume that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then for any integer such that $r \leq m$, and $B' \in \mathrm{Her}_r(\mathcal{O}_p)$ we have*

$$\widetilde{G}_p(1_{m-r} \perp pB', X, t) = \widetilde{G}_p(pB', X, t).$$

(2) *Assume that K_p is ramified over $\mathbf{Q}_p$. Then for any non-negative integer r such that $m-r$ is even, and $B' \in \mathrm{Her}_{r,*}(\mathcal{O}_p)$, we have*

$$\widetilde{G}_p(\Theta_{m-r} \perp p^{i_p} B', X, t) = \widetilde{G}_p(p^{i_p} B', X, t).$$

Proof. We have

$$G_p(1_{m-r,d} \perp pB', X) = G_p(pB', X)$$

for $B' \in \mathrm{Her}_r(\mathcal{O}_p)$. Hence by Corollary to Lemma 5.2.2 we have

$$\widetilde{F}_p^{(0)}(1_{m-r} \perp pB', X) = \widetilde{F}_p^{(0)}(pB', X)$$

for $B' \in \mathrm{Her}_r(\mathcal{O}_p)$. Thus the assertion (1) follows from (1) of Lemma 5.1.4. The assertion (2) can be proved in a similar way. □

Let $R_m(d_0, X, Y, t)$ be the formal power series defined at the beginning of Section 5. We express $R_m(d_0, X, Y, t)$ in terms of $Q_r(d_0, X, Y, t)$.

Theorem 5.3.4. *Let $d_0 \in \mathbf{Z}_p^*$.*
(1) *Assume that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$R_m(d_0,X,Y,t) = \sum_{r=0}^{m} \frac{\prod_{i=1}^{r-1}(1-\tau_p^i p^i Y^2)\prod_{i=r}^{m-1}(1-p^{-2m+i}Y^2t^2)}{\phi_{m-r-1}(\xi_p p^{-1})}$$

$$\times Q_r(d_0,X,p^{-m/2}Y,p^{-m/2}t).$$

(2) *Assume that K_p is ramified over $\mathbf{Q}_p$.*
(2.1) *Let m be odd. Then*

$$R_m(d_0,X,Y,t) = \sum_{r=0}^{(m-1)/2} \frac{\prod_{i=1}^{r-1}(1-p^{2i}Y^2)\prod_{i=r}^{(m-1/2}(1-p^{-2m+2i}Y^2t^2)}{\phi_{(m-2r-1)/2}(p^{-2})}$$

$$\times Q_{2r+1}(d_0,X,p^{-m/2}Y,p^{-m/2}t).$$

(2.2) *Let m be even. Then*

$$R_m(d_0,X,Y,t) = \sum_{r=0}^{m/2} \frac{\prod_{i=1}^{r-1}(1-p^{2i}Y^2)\prod_{i=r}^{m/2}(1-p^{-2m+2i}Y^2t^2)}{\phi_{(m-2r)/2}(p^{-2})}$$

$$\times Q_{2r}(d_0,X,p^{-m/2}Y,p^{-m/2}t).$$

Proof. (1) Let B be an element of $\widetilde{\mathrm{Her}}_r(\mathcal{O}_p)$. Then we note that $1_{m-r}\perp pB$ belongs to $\widetilde{\mathcal{F}}_{m,p}(d_0)$ if and only if $B \in p^{-1}\widetilde{\mathcal{F}}_{r,p}(d_0)\cap\widetilde{\mathrm{Her}}_r(\mathcal{O}_p)$. Thus the assertion (1) follows from Lemmas 5.2.2, 5.2.4, and 5.3.4, and Proposition 5.3.3.

(2) Let B be an element of $\widetilde{\mathrm{Her}}_r(\mathcal{O}_p)$. Let $m-r$ be even. Then we note that $\Theta_{m-r}\perp p^{i_p}B$ belongs to $\widetilde{\mathcal{F}}_{m,p}(d_0)$ if and only if $B \in p^{-i_p}\widetilde{\mathcal{F}}_{r,p}(d_0)\cap \mathrm{Her}_{r,*}(\mathcal{O}_p)$. Thus the assertion (2) can be proved similarly to (1).

Similarly the assertion (2) can be proved. □

Now to rewrite the above theorem, first we express $\widetilde{P}_m(d_0,X,Y,t)$ in terms of $Q_r(d_0,X,Y,t)$.

Proposition 5.3.5. *Let $d_0 \in \mathbf{Z}_p^*$.*
(1) *Assume that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$\widetilde{P}_m(d_0,X,Y,t) = \sum_{r=0}^{m} \frac{1}{\phi_{m-r}(\xi_p p^{-1})} Q_r(d_0,X,Y,t).$$

(2) *Assume that K_p is ramified over $\mathbf{Q}_p$.*
(2.1) *Let m be odd. Then*

$$\widetilde{P}_m(d_0,X,Y,t) = \sum_{r=0}^{(m-1)/2} \frac{1}{\phi_{(m-2r-1)/2}(p^{-2})} Q_{2r+1}(d_0,X,Y,t).$$

(2.2) *Let m be even. Then*

$$\widetilde{P}_m(d_0,X,Y,t)=\sum_{r=0}^{m/2}\frac{1}{\phi_{(m-2r)/2}(p^{-2})}Q_{2r}(d_0,X,Y,t).$$

Proof. The assertion can be proved in a way similar to Theorem 5.3.4. □

Corollary. *Let d_0 be an element of $\mathbf{Z}_p^*$.*
(1) *Assume that K_p is unramified over $\mathbf{Q}_p$ or $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then*

$$Q_r(d_0,X,Y,t)=\sum_{m=0}^{r}\frac{(-1)^m p^{(m-m^2)/2}}{\phi_m(\xi_p p^{-1})}\widetilde{P}_{r-m}(d_0,X,Y,t).$$

(2) *Assume that K_p is ramified over $\mathbf{Q}_p$. Then*

$$Q_{2r+1}(d_0,X,Y,t)=\sum_{m=0}^{r}\frac{(-1)^m p^{m-m^2}}{\phi_m(p^{-2})}\widetilde{P}_{2r+1-2m}(d_0,X,Y,t),$$

and

$$Q_{2r}(d_0,X,Y,t)=\sum_{m=0}^{r}\frac{(-1)^m p^{m-m^2}}{\phi_m(p^{-2})}\widetilde{P}_{2r-2m}(d_0,X,Y,t).$$

Proof. We can prove the assertions by induction on r. □

The following lemma follows from [[IK06], Lemma 3.4]

Lemma 5.3.6. *Let l be a positive integer. Then we have the following identity on the three variables q,U and Q :*

$$\prod_{i=1}^{l}(1-U^{-1}Qq^{-i+1})U^l$$

$$=\sum_{m=0}^{l}\frac{\phi_l(q^{-1})}{\phi_{l-m}(q^{-1})\phi_m(q^{-1})}\prod_{i=1}^{l-m}(1-Qq^{-i+1})\prod_{i=1}^{m}(1-Uq^{i-1})(-1)^m q^{(m-m^2)/2}.$$

Theorem 5.3.7. *Let the notation be as in Theorem 5.3.5.*
(1) *Assume that K_p is unramified over $\mathbf{Q}_p$ or $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then*

$$R_m(d_0,X,Y,t)$$

$$=\sum_{l=0}^{m}(p^lY^2)^{m-l}\widetilde{P}_l(d_0,X,p^{-m/2}Y,p^{-m/2}t)\frac{\prod_{i=1}^{m-l}(1-(\xi_p p)^{-l-m-i}t^2)\prod_{i=0}^{l}(1-(\xi_p p)^iY^2)}{\phi_{m-l}(\xi_p p^{-1})}.$$

(2) *Assume that K_p is ramified over $\mathbf{Q}_p$.*
(2.1) *Let m be odd. Then*

$$R_m(d_0,X,Y,t)=\sum_{l=0}^{(m-1)/2}\widetilde{P}_{2l+1}(d_0,X,p^{-m/2}Y,p^{-m/2}t)$$

$$\times \frac{(p^{2l+1}Y^2)^{(m-2l-1)/2}\prod_{i=0}^{l}(1-p^{2i}Y^2)\prod_{i=1}^{(m-2l-1)/2}(1-p^{-2l-m-2i+1}t^2)}{\phi_{(m-2l-1)/2}(p^{-2})}.$$

(2.2) *Let m be even. Then*

$$R_m(d_0,X,Y,t)=\sum_{l=0}^{m/2}\widetilde{P}_{2l}(d_0,X,p^{-m/2}Y,p^{-m/2}t)$$

$$\times \frac{(p^{2l}Y^2)^{(m-2l)/2}\prod_{i=0}^{l}(1-p^{2i}Y^2)\prod_{i=1}^{(m-2l)/2}(1-p^{-2l-m-2i}t^2)}{\phi_{(m-2l)/2}(p^{-2})}.$$

Proof. (1) By Theorem 5.3.4 and Corollary to Proposition 5.3.5, we have

$$R_m(d_0,X,Y,t)$$

$$=\sum_{r=0}^{m}\frac{\prod_{i=0}^{r-1}(1-(\xi_p p)^{2i}Y^2)\prod_{i=0}^{m-r-1}(1-(\xi_p p)^{-m-1-i}Y^2t^2)}{\phi_{m-r}((\xi_p p)^{-1})}$$

$$\times\sum_{j=0}^{r}\frac{(-1)^j(\xi_p p)^{(j-j^2)/2}}{\phi_j((\xi_p p)^{-1})}\widetilde{P}_{r-j}(d_0,X,p^{-m/2}Y,p^{-m/2}t)$$

$$=\sum_{l=0}^{m/2}\widetilde{P}_l(d_0,X,p^{-m/2}Y,p^{-m/2}t)$$

$$\times\sum_{j=0}^{m-l}(-1)^j(\xi_p p)^{(j-j^2)/2}\frac{\prod_{i=0}^{l+j-1}(1-p^{2i}Y^2)\prod_{i=0}^{m-l-j-1}(1-(\xi_p p)^{-m-i-1}Y^2t^2)}{\phi_j(\xi_p p^{-1})\phi_{m-j-l}(\xi_p p^{-1})}.$$

Then the assertion (1) follows from Lemma 5.3.6.

(2) The assertion can be proved in the same manner as above.

□

5.4. **Explicit formulas of formal power series of Koecher-Maass type.**

In this section we give an explicit formula for $P_m(d_0,X,t)$.

Theorem 5.4.1. *Let m be even, and $d_0\in\mathbf{Z}_p^*$.*

(1) *Assume that K_p is unramified over $\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t)=\frac{1}{\phi_m(-p^{-1})\prod_{i=1}^{m}(1-t(-p)^{-i}X)(1-t(-p)^{-i}X^{-1})}.$$

(2) *Assume that $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t)=\frac{1}{\phi_m(p^{-1})\prod_{i=1}^{m}(1-tp^{-i}X)(1-tp^{-i}X^{-1})}.$$

(3) *Assume that K_p is ramified over $\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t)=\frac{t^{mi_p/2}}{2\phi_{m/2}(p^{-2})}$$

$$\times \left\{ \frac{1}{\prod_{i=1}^{m/2}(1-tp^{-2i+1}X)(1-tp^{-2i}X^{-1})} + \frac{\chi_{K_p}((-1)^{m/2}d_0)}{\prod_{i=1}^{m/2}(1-tp^{-2i}X)(1-tp^{-2i+1}X^{-1})} \right\}.$$

Theorem 5.4.2. *Let m be odd, and $d_0 \in \mathbf{Z}_p^*$.*
(1) *Assume that K_p is unramified over $\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t) = \frac{1}{\phi_m(-p^{-1})\prod_{i=1}^{m}(1+t(-p)^{-i}X)(1+t(-p)^{-i}X^{-1})}.$$

(2) *Assume that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$P_m(d_0,X,t) = \frac{1}{\phi_m(p^{-1})\prod_{i=1}^{m}(1-tp^{-i}X)(1-tp^{-i}X^{-1})}.$$

(3) *Assume that K_p is ramified over $\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t) = \frac{t^{(m+1)i_p/2+\delta_{2p}}}{2\phi_{(m-1)/2}(p^{-2})\prod_{i=1}^{(m+1)/2}(1-tp^{-2i+1}X)(1-tp^{-2i+1}X^{-1})}.$$

To prove Theorems 5.4.1 and 5.4.2, put

$$K_m(d_0,X,t) = \sum_{B' \in \widetilde{\mathcal{F}}_{r,p}(d_0)} \frac{G_p(B',p^{-m}X^2)}{\alpha_p(B')}(tX^{-1})^{\mathrm{ord}(\det B')}.$$

Proposition 5.4.3. *Let m and d_0 be as above. Then we have*

$$P_m(d_0,X,t) = X^{me_p-[m/2]f_p}K_m(d_0,X,t)$$
$$\times \begin{cases} \prod_{i=1}^{m}(1-t^2X^2p^{2i-2-2m})^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is unramified} \\ \prod_{i=1}^{m}(1-tXp^{i-1-m})^{-2} & \text{if } K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p \\ \prod_{i=1}^{m}(1-tXp^{i-1-m})^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is ramified.} \end{cases}$$

Proof. We note that B' belongs to $\widetilde{\mathrm{Her}}_{m,p}(d_0)$ if B belongs to $\widetilde{\mathrm{Her}}_{m-l,p}(d_0)$ and $\alpha_p(B',B) \neq 0$. Hence by Lemma 5.2.3 we have

$$P_m(d_0,X,t)$$
$$= X^{me_p-[m/2]f_p} \sum_{B \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{1}{\alpha_p(B)} \sum_{B'} \frac{G_p(B',p^{-m}X^2)X^{-\mathrm{ord}(B')}\alpha_p(B',B)}{\alpha_p(B')}$$
$$\times X^{\mathrm{ord}(\det B)-\mathrm{ord}(\det B')}t^{\mathrm{ord}(\det B)}$$
$$= X^{me_p-[m/2]f_p} \sum_{B' \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{G_p(B',p^{-m}X^2)}{\alpha_p(B')}(tX^{-1})^{\mathrm{ord}(B')}$$
$$\times \sum_{B \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{\alpha_p(B',B)}{\alpha_p(B)}(tX)^{\mathrm{ord}(\det B)-\mathrm{ord}(\det B')}.$$

Hence by using the same argument as in the proof of [[BS87], Theorem 5], and by (1) of Lemma 5.1.3, we have

$$\sum_{B \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{\alpha_p(B',B)}{\alpha_p(B)}(tX)^{\mathrm{ord}(\det B)-\mathrm{ord}(\det B')}$$

$$= \sum_{W \in M_m(\mathcal{O}_p)^\times / GL_m(\mathcal{O}_p)} (tXp^{-m})^{\nu(\det W)}$$

$$= \begin{cases} \prod_{i=1}^m (1 - t^2X^2p^{2i-2-2m})^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is unramified} \\ \prod_{i=1}^m (1 - tXp^{i-1-m})^{-2} & \text{if } K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p \\ \prod_{i=1}^m (1 - tXp^{i-1-m})^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is ramified.} \end{cases}$$

Thus the assertion holds. □

In order to prove Theorems 5.4.2, we introduce some notation. For a positive integer r and $d_0 \in \mathbf{Z}_p^\times$ let

$$\zeta_m(d_0, t) = \sum_{T \in \mathcal{F}_{m,p,*}(d_0)} \frac{1}{\alpha_p(T)} t^{\mathrm{ord}(\det T)}.$$

We make the convention that $\zeta_0(d_0, t) = 1$ or 0 according as $d_0 \in \mathbf{Z}_p^*$ or not. To obtain an explicit formula of $\zeta_m(d_0, t)$ let $Z_m(u, d)$ be the integral defined as

$$Z_{m,*}(u, d) = \int_{\mathcal{F}_{m,p,*}(d_0)} |\det x|^{s-m} dx,$$

where $u = p^{-s}$, and dx is the measure on $\mathrm{Her}_m(K_p)$ so that the volume of $\mathrm{Her}_m(\mathcal{O}_p)$ is 1. Then by Theorem 4.2 of [Sa97] we obtain:

Proposition 5.4.4. *Let $d_0 \in \mathbf{Z}_p^*$.*
(1) *Assume that K_p is unramified over $\mathbf{Q}_p$. Then*

$$Z_{m,*}(u, d_0) = \frac{(p^{-1}, p^{-2})_{[(m+1)/2]}(-p^{-2}, p^{-2})_{[m/2]}}{\prod_{i=1}^{[m/2]}(1 - (-1)^{m+i}p^{i-1}u)}.$$

(2) *Assume that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$Z_{m,*}(u, d_0) = \frac{\phi_m(p^{-1})}{\prod_{i=1}^{[m/2]}(1 - p^{i-1}u)}.$$

(3) *Assume that K_p is ramified over $\mathbf{Q}_p$.*
(3.1) *Let $p \neq 2$. Then*

$$Z_{m,*}(u, d_0) = \frac{1}{2}(p^{-1}, p^{-2})_{[(m+1)/2]}$$

$$\times \begin{cases} \frac{1}{\prod_{i=1}^{(m+1)/2}(1-p^{2i-2}u)} & \text{if } m \text{ is odd,} \\ \left(\frac{1}{\prod_{i=1}^{m/2}(1-p^{2i-1}u)} + \frac{\chi_{K_p}((-1)^{m/2}d_0)p^{-m/2}}{\prod_{i=1}^{m/2}(1-p^{2i-2}u)} \right) & \text{if } m \text{ is even.} \end{cases}$$

(3.2) *Let $p = 2$ and $f_2 = 2$. Then*

$$Z_{m,*}(u, d_0) = \frac{1}{2}(p^{-1}, p^{-2})_{[(m+1)/2]}$$

$$\times \begin{cases} \frac{u^{(m+1)/2}}{\prod_{i=1}^{(m+1)/2}(1-p^{2i-2}u)} & \text{if } m \text{ is odd,} \\ u^{m/2}p^{-m/2}\Big(\frac{1}{\prod_{i=1}^{m/2}(1-p^{2i-1}u)} + \frac{\chi_{K_p}((-1)^{m/2}d_0)p^{-m/2}}{\prod_{i=1}^{m/2}(1-p^{2i-2}u)}\Big) & \text{if } m \text{ is even.} \end{cases}$$

(3.3) *Let $p=2$ and $f_2=3$. Then*

$$Z_{m,*}(u,d_0) = \frac{1}{2}(p^{-1},p^{-2})_{[(m+1)/2]}$$

$$\times \begin{cases} \frac{u}{\prod_{i=1}^{(m+1)/2}(1-p^{2i-2}u)} & \text{if } m \text{ is odd,} \\ p^{-m}\Big(\frac{1}{\prod_{i=1}^{m/2}(1-p^{2i-1}u)} + \frac{\chi_{K_p}((-1)^{m/2}d_0)p^{-m/2}}{\prod_{i=1}^{m/2}(1-p^{2i-2}u)}\Big) & \text{if } m \text{ is even.} \end{cases}$$

Proof. First assume that K_p is unramified over $\mathbf{Q}_p$, $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$, or K_p is ramified over $\mathbf{Q}_p$ and $p \neq 2$. Then $Z_{m,*}(u,d_0)$ coincides with $Z_m(u,d_0)$ in [[Sa97], Theorem 4.2]. Hence the assertion follows from (1) and (2) and the former half of (3) of [Loc. cit]. Next assume that $p=2$ and $f_2=2$. Then $Z_{m,*}(u,d_0)$ is not treated in [Loc. cit]. but we can prove the assertion (3.2) using the same argument as in the proof of the latter half of (3) of [Loc. cit]. Similarly we can prove (3.3) by using the same argument as in the proof of the former half of (3) of [Loc. cit]. □

Corollary. *Let $d_0 \in \mathbf{Z}_p^*$.*
(1) *Assume that K_p is unramified over $\mathbf{Q}_p$. Then*

$$\zeta_m(d_0,t) = \frac{1}{\phi_m(-p^{-1})}\frac{1}{\prod_{i=1}^m(1+(-1)^ip^{-i}t)}.$$

(2) *Assume that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$\zeta_m(d_0,t) = \frac{1}{\phi_m(p^{-1})}\frac{1}{\prod_{i=1}^m(1-p^{-i}t)}.$$

(3) *Assume that K_p is ramified over $\mathbf{Q}_p$.*
(3.1) *Let m be even. Then*

$$\zeta_m(d_0,t) = \frac{p^{m(m+1)f_p/2-m^2\delta_{2,p}}\kappa_p(t)}{2\phi_{m/2}(p^{-2})}$$

$$\times\{\frac{1}{\prod_{i=1}^{m/2}(1-p^{-2i-1}t)} + \frac{\chi_{K_p}((-1)^{m/2}d_0)p^{-pm/2}}{\prod_{i=1}^{m/2}(1-p^{-2i}t)}\},$$

where

$$\kappa_p(t) = \begin{cases} 1 & \text{if } p \neq 2 \\ t^{m/2}p^{-m(m+1)/2} & \text{if } p=2 \text{ and } f_2=2 \\ p^{-m} & \text{if } p=2 \text{ and } f_2=3 \end{cases}$$

(3.2) *Let m be odd. Then*

$$\zeta_m(d_0,t) = \frac{p^{m(m+1)f_p/2-m^2\delta_{2,p}}\kappa_p(t)}{2\phi_{(m-1)/2}(p^{-2})}\frac{1}{\prod_{i=1}^{(m+1)/2}(1-p^{-2i+1}t)},$$

where

$$\kappa_p(t)=\begin{cases}1 & \text{if } p\neq 2\\ t^{(m+1)/2}p^{-m(m+1)/2} & \text{if } p=2 \text{ and } f_2=2\\ tp^{-m} & \text{if } p=2 \text{ and } f_2=3\end{cases}$$

Proof. First assume that K_p is unramified over $\mathbf{Q}_p$. Then by a simple computation we have

$$\zeta_m(d_0,t)=\frac{Z_{m,*}(p^{-m}t,d_0)}{\phi_m(p^{-2})}.$$

Next assume that $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then similarly to above

$$\zeta_m(d_0,t)=\frac{Z_{m,*}(p^{-m}t,d_0)}{\phi_m(p^{-1})^2}.$$

Finally assume that K_p is ramified over $\mathbf{Q}_p$. Then by a simple computation and Lemma 4.1

$$\zeta_m(d_0,t)=\frac{p^{m(m+1)f_p/2-m^2\delta_{2,p}}Z_{m,*}(p^{-m}t,d_0)}{\phi_m(p^{-1})}.$$

Thus the assertions follow from Proposition 4.4. □

Proposition 5.4.5. *Let* $d_0\in\mathbf{Z}_p^*$.
(1) *Assume that* K_p *is unramified over* $\mathbf{Q}_p$. *Then*

$$K_m(d_0,X,t)$$
$$=\sum_{r=0}^{m}\frac{p^{-r^2}(-tX^{-1})^r\prod_{i=0}^{r-1}(1-(-p)^{i-1}X^2)}{\phi_{m-r}(-p^{-1})}\zeta_r(d_0,tX^{-1}).$$

(2) *Assume that* $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. *Then*

$$K_m(d_0,X,t)$$
$$=\sum_{r=0}^{m}\frac{p^{-r^2}(tX^{-1})^r\prod_{i=0}^{r-1}(1-p^{i-1}X^2)}{\phi_{m-r}(p^{-1})}\zeta_r(d_0,tX^{-1}).$$

(3) *Assume that* K_p *is ramified over* $\mathbf{Q}_p$. *Then*

$$K_m(d_0,X,t)$$
$$=\sum_{r=0}^{m/2}\frac{p^{-4i_pr^2}(tX^{-1})^{(m/2+r)i_p}\prod_{i=0}^{r-1}(1-p^{2i-2}X^2)}{\phi_{(m-2r)/2}(p^{-2})}\zeta_{2r}((-1)^{m/2-r}d_0,tX^{-1})$$

if m *is even, and*

$$K_m(d_0,X,t)$$
$$=\sum_{r=0}^{(m-1)/2}\frac{p^{-(2r+1)^2i_p}(tX^{-1})^{((m+1)/2+r)i_p}\prod_{i=0}^{r-1}(1-p^{2i-2}X^2)}{\phi_{(m-2r-1)/2}(p^{-2})}\zeta_{2r+1}((-1)^{(m-2r-1)/2}d_0,tX^{-1})$$

if m *is odd.*

Proof. The assertions can be proved by using the same argument as in the proof of [[KK10b], Proposition 5.4.4]. □

It is well known that $\#(\mathbf{Z}_p^*/N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*)) = 2$ if $K_p/\mathbf{Q}_p$ is ramified. Hence we can take a complete set $\mathcal{N}_p$ of representatives of $\mathbf{Z}_p^*/N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*)$ so that $\mathcal{N}_p = \{1, \xi_0\}$ with $\chi_{K_p}(\xi_0) = -1$.

Proof of Theorem 5.4.1. (1) By Corollary to Proposition 5.4.4 and Proposition 5.4.5, we have

$$K_m(d_0, X, t) = \frac{1}{\phi_m(-p^{-1})} \frac{L_m(d_0, X, t)}{\prod_{i=1}^m (1 + (-1)^i p^{-i} X^{-1} t)},$$

where $L_m(d_0, X, t)$ is a polynomial in t of degree m. Hence

$$P_m(d_0, X, t) = \frac{1}{\phi_m(-p^{-1})} \frac{L_m(d_0, X, t)}{\prod_{i=1}^m (1 + (-1)^i p^{-i} X^{-1} t) \prod_{i=1}^m (1 - p^{-2i} X^2 t^2)}.$$

We have

$$\widetilde{F}(B, -X^{-1}) = \widetilde{F}(B, X)$$

for any $B \in \widetilde{F}_p^{(0)}(B, X)$. Hence we have

$$P_m(d_0, -X^{-1}, t) = P_m(d_0, X, t),$$

and therefore the denominator of the rational function $P_m(d_0, -X^{-1}, t)$ in t is at most

$$\prod_{i=1}^m (1 + (-1)^i p^{-i} X^{-1} t) \prod_{i=1}^m (1 + (-1)^i p^{-i} X t).$$

Thus

$$P_m(d_0, X, t) = \frac{a}{\phi_m(-p^{-1}) \prod_{i=1}^m (1 + (-1)^i p^{-i} X^{-1} t) \prod_{i=1}^m (1 - (-1)^i p^{-i} X t)},$$

with some constant a. It is easily seen that we have $a = 1$. This proves the assertion.

(2) The assertion can be proved by using the same argument as above.

(3) By Corollary to Proposition 5.4.4 and Proposition 5.4.5, we have

$$K_m(d, X, t)$$

$$= \sum_{r=0}^{m/2} \frac{\prod_{i=0}^{r-1} (1 - p^{2i-2} X^2)}{\phi_{m/2-r}(p^{-2})} p^{-4i_p r^2} (X^{-1}t)^{(m/2+r)i_p} \zeta_{2r}((-1)^{m/2-r} d, X^{-1}t)$$

$$= \sum_{r=0}^{m/2} \frac{1}{2\phi_{m/2-r}(p^{-2})} p^{-2r^2+r} (X^{-1}t)^r$$

$$\times \left\{ \frac{1}{\prod_{i=1}^r (1 - p^{-2i+1} X^{-1} t)} + \frac{\chi_{K_p}((-1)^{m/2} d_0) p^{-pr}}{\prod_{i=1}^r (1 - p^{-2i} X^{-1} t)} \right\}.$$

Thus we have

$$P_m(d,X,t)=\sum_{r=0}^{m/2}\frac{1}{2\phi_{m/2-r}(p^{-2})}p^{-2r^2+r}(X^{-1}t)^r$$

$$\times\left\{\frac{1}{\prod_{i=1}^{r}(1-p^{-2i+1}X^{-1}t)\prod_{i=1}^{r}(1-p^{-i}Xt)}+\frac{\chi_{K_p}((-1)^{m/2}d_0)p^{-pr}}{\prod_{i=1}^{r}(1-p^{-2i}X^{-1}t)\prod_{i=1}^{r}(1-p^{-i}Xt)}.\right\}$$

For $l=0,1$ put

$$\widetilde{P}_m^{(l)}(X,t)=\frac{1}{2}\sum_{d\in\mathcal{N}_p}\chi_{K_p}((-1)^{m/2}d)^l P_m(d,X,t).$$

Then

$$\widetilde{P}_m^{(0)}(X,t)=\frac{L^{(0)}(X,t)}{2\phi_{m/2}(p^{-2})}\frac{1}{\prod_{i=1}^{m/2}(1-p^{-2i+1}X^{-1}t)\prod_{i=1}^{m}(1-p^{-i}Xt)},$$

and

$$\widetilde{P}_m^{(1)}(X,t)=\frac{L^{(1)}(X,t)}{2\phi_{m/2}(p^{-2})}\frac{1}{\prod_{i=1}^{m/2}(1-p^{-2i}X^{-1}t)\prod_{i=1}^{m}(1-p^{-i}Xt)}$$

with some polynomials $L^{(0)}(X,t)$ and $L^{(1)}(X,t)$ in t of degrees at most m. Then by the functional equation of Siegel series we have

$$P_m(d,X^{-1},t)=\chi_{K_p}((-1)^{m/2}d)P_m(d,X,t)$$

for any $d\in\mathcal{N}_p$. Hence we have

$$\widetilde{P}_m^{(0)}(X^{-1},t)=\widetilde{P}_m^{(1)}(X,t).$$

Hence the reduced denominator of the rational function $\widetilde{P}_m^{(0)}(X,t)$ in t is at most

$$\prod_{i=1}^{m/2}(1-p^{-2i+1}X^{-1}t)\prod_{i=1}^{m/2}(1-p^{-2i}Xt),$$

and similarly to (1) we have

$$\widetilde{P}_m^{(0)}(X,t)=\frac{1}{2\phi_{m/2}(p^{-2})\prod_{i=1}^{m/2}(1-p^{-2i+1}X^{-1}t)\prod_{i=1}^{m/2}(1-p^{-2i}Xt)}.$$

Similarly

$$\widetilde{P}_m^{(1)}(X,t)=\frac{1}{2\phi_{m/2}(p^{-2})\prod_{i=1}^{m/2}(1-p^{-2i}X^{-1}t)\prod_{i=1}^{m/2}(1-p^{-2i+1}Xt)}.$$

We have

$$P_m(d_0,X,t)=\widetilde{P}_m^{(0)}(X,t)+\chi_{K_p}((-1)^{m/2}d_0)\widetilde{P}_m^{(1)}(X,t).$$

This proves the assertion.

□

Proof of Theorem 5.4.2. The assertion can also be proved by using the same argument as above. □

5.5. Explicit formulas of formal power series of Rankin-Selberg type.

We give an explicit formula for $H_m(d,X,Y,t)$. First we remark the following.

Proposition 5.5.1. *Let $d \in \mathbf{Z}_p^{\times}$. Then we have*

$$\lambda^*_{m,p}(d,X,Y) = \lambda_{m,p}(d,X,Y).$$

Proof. Let I be the left-hand side of the above equation. Let

$$GL_m(\mathcal{O}_p)_1 = \{X \in GL_m(\mathcal{O}_p) \mid \overline{\det X} \det X = 1\}.$$

Then there exists a bijection from $\widetilde{\mathrm{Her}}_m(d,\mathcal{O}_p)/GL_m(\mathcal{O}_p)_1$ to $\widetilde{\mathrm{Her}}_m(dN_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*),\mathcal{O}_p)/GL_m(\mathcal{O}_p)$. Hence

$$I = \sum_{A \in \widetilde{\mathrm{Her}}_m(d,\mathcal{O}_p)/GL_m(\mathcal{O}_p)_1} \frac{\widetilde{F}_p^{(0)}(A,X)\widetilde{F}_p^{(0)}(A,Y)}{\alpha_p(A)}.$$

Now for $T \in \widetilde{\mathrm{Her}}_m(d,\mathcal{O}_p)$, let l be the number of $SL_m(\mathcal{O}_p)$-equivalence classes in $\widetilde{\mathrm{Her}}_m(d,\mathcal{O}_p)$ which are $GL_m(\mathcal{O}_p)$-equivalent to T. Then it can easily be shown that $l = l_{p,T}$. Hence the assertion holds. □

Theorem 5.5.2. *Let $m = 2n$ be even, and $d_0 \in \mathbf{Z}_p^*$.*
(1) *Assume that K_p is unramified over $\mathbf{Q}_p$. Then*

$$H_{2n}(d_0,X,Y,t) = \frac{\prod_{i=1}^{2n}(1-(-p)^{-2n-i}t^2)}{\phi_{2n}(-p^{-1})}$$
$$\times \frac{1}{\prod_{i=1}^{2n}(1+(-p)^{-2n+i-1}XYt)(1-(-p)^{-2n+i-1}XY^{-1}t)}$$
$$\times \frac{1}{\prod_{i=1}^{2n}(1-(-p)^{-2n+i-1}X^{-1}Yt)(1+(-p)^{-2n+i-1}X^{-1}Y^{-1}t)}.$$

(2) *Assume that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$H_{2n}(d_0,X,Y,t) = \frac{\prod_{i=1}^{2n}(1-p^{-2n-i}t^2)}{\phi_{2n}(p^{-1})}$$
$$\times \frac{1}{\prod_{i=1}^{2n}(1-p^{-2n+i-1}XYt)(1-p^{-2n+i-1}XY^{-1}t)}$$
$$\times \frac{1}{\prod_{i=1}^{2n}(1-p^{-2n+i-1}X^{-1}Yt)(1-p^{-2n+i-1}X^{-1}Y^{-1}t)}.$$

(3) *Assume that K_p is ramified over $\mathbf{Q}_p$. Then*

$$H_{2n}(d_0,X,Y,t) = \frac{t^{ni_p}}{2}\frac{\prod_{i=1}^{n}(1-p^{-2n-2i}t^2)}{\phi_n(p^{-2})}$$

$$\times\{\frac{1}{\prod_{i=1}^{n}(1-p^{-2n-2i+1}XYt)(1-p^{-2n-2i+1}X^{-1}Y^{-1}t)}$$
$$+\chi_{K_p}((-1)^n d_0)\frac{1}{\prod_{i=1}^{n}(1-p^{-2n-2i}X^{-1}Yt)(1-p^{-2n-2i}XY^{-1}t)}\}.$$

Proof. First we prove (1). By Proposition 5.3.1 and Theorems 5.3.7 and 5.4.1, we have

$$R_{2n}(d_0,X,Y,t)=\sum_{l=0}^{2n}\prod_{i=1}^{2n-l}(1-(-p)^{-2n-l-i}t^2)\prod_{i=0}^{l}(1-(-p)^{i}Y^2)$$
$$\times\frac{\prod_{i=1}^{l}(1-t^4p^{-2n-2l-2+2i})}{\phi_l(-p)\prod_{i=1}^{l}(1-t(-p)^{-i}XY^{-1})(1+t(-p)^{-i}X^{-1}Y^{-1})}.$$

Hence $R_{2n}(d_0,X,Y,t)$ can be expressed as

$$R_{2n}(d_0,X,Y,t)$$
$$=\frac{\prod_{i=1}^{2n}(1-(-p)^{-2n-i}t^2)S(X,Y,t)}{\phi_{2n}(-p)\prod_{i=1}^{2n}(1-t(-p)^{-2n+i-1}XY^{-1})(1+t(-p)^{-2n+i-1}X^{-1}Y^{-1})},$$

where $S(X,Y,t)$ is a polynomial in t of degree at most $4n$. Then by Theorem 5.2.8, we have

$$H_{2n}(d_0,X,Y,t)$$
$$=\frac{\prod_{i=1}^{2n}(1-(-p)^{-2n-i}t^2)S(X,Y,t)}{\phi_{2n}(-p)\prod_{i=1}^{2n}(1-t(-p)^{-2n+i-1}XY^{-1})(1+t(-p)^{-2n+i-1}X^{-1}Y^{-1})}$$
$$\times\frac{1}{\prod_{i=1}^{2n}(1-t^2p^{-4n+2i-2}X^2Y^2)(1-t^2p^{-4n+2i-2}X^{-2}Y^2)}.$$

Recall that we have the following functional equation

$$H_{2n}(d_0,X,Y^{-1},t)=H_{2n}(d_0,X,-Y,t).$$

Hence the reduced denominator of the rational function $H_{2n}(d_0,X,Y^{-1},t)$ in t is at most

$$\prod_{i=1}^{2n}\{(1-t(-p)^{-2n+i-1}XY^{-1})(1+t(-p)^{-2n+i-1}X^{-1}Y^{-1})$$
$$\times(1+t(-p)^{-2n+i-1}XY)(1-t(-p)^{-2n+i-1}X^{-1}Y)\},$$

and therefore we have

$$H_{2n}(d_0,X,Y,t)=\frac{c\prod_{i=1}^{2n}(1-(-p)^{-2n-i}t^2)}{\phi_{2n}(-p)}$$
$$\times\frac{1}{\prod_{i=1}^{2n}(1-t(-p)^{-2n+i}XY^{-1})(1+t(-p)^{-2n+i}X^{-1}Y^{-1})}$$
$$\times\frac{1}{\prod_{i=1}^{2n}(1+t(-p)^{-2n+i-1}XY)(1-t(-p)^{-2n+i-1}X^{-1}Y)}$$

with some constant c. We easily see that we have $c=1$. This proves the assertion (1). Similarly the assertions (2) and (3)can be proved. $\square$

Corollary. *Let $m=2n$ be even. Assume that K_p is ramified over $\mathbf{Q}_p$. Then we have*

$$H_{2n}(d,X,Y,t)=\frac{1}{2}(H_{2n}^{(0)}(X,Y,t)+\chi_{K_p}((-1)^n d)H_{2n}^{(1)}(X,Y,t)),$$

where

$$H_{2n}^{(0)}(X,Y,t)=t^{ni_p}\frac{\prod_{i=1}^{n}(1-p^{-2n-2i}t^2)}{\phi_n(p^{-2})}$$
$$\times\frac{1}{\prod_{i=1}^{n}(1-p^{-2n+2i-1}XYt)(1-p^{-2n+2i-1}X^{-1}Y^{-1}t)},$$

and

$$H_{2n}^{(1)}(X,Y,t)=t^{ni_p}\frac{\prod_{i=1}^{n}(1-p^{-2n-2i}t^2)}{\phi_n(p^{-2})}$$
$$\times\frac{1}{\prod_{i=1}^{n}(1-p^{-2n-2i}X^{-1}Yt)(1-p^{-2n-2i}XY^{-1}t)}.$$

Similarly to Theorem 5.5.3, we have

Theorem 5.5.3. (1) *Assume that K_p is unramified over $\mathbf{Q}_p$. Then*

$$H_{2n+1}(d_0,X,Y,t)=\frac{\prod_{i=1}^{2n+1}(1-(-p)^{-2n-i-1}t^2)}{\phi_{2n+1}(-p^{-1})}$$
$$\times\frac{1}{\prod_{i=1}^{2n+1}(1+(-p)^{-2n+i-2}XYt)(1+(-p)^{-2n+i-1}XY^{-1}t)}$$
$$\times\frac{1}{\prod_{i=1}^{2n}(1+(-p)^{-2n+i-2}X^{-1}Yt)(1+(-p)^{-2n+i-2}X^{-1}Y^{-1}t)}.$$

(2) *Assume that $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then*

$$H_{2n+1}(d_0,X,Y,t)=\frac{1}{2}\frac{\prod_{i=1}^{2n+1}(1-p^{-2n-i-1}t^2)}{\phi_{2n+1}(p^{-1})}$$
$$\times\frac{1}{\prod_{i=1}^{2n+1}(1-p^{-2n+i-2}XYt)(1-p^{-2n+i-2}XY^{-1}t)}$$
$$\times\frac{1}{\prod_{i=1}^{2n+1}(1-p^{-2n+i-2}X^{-1}Yt)(1-p^{-2n+i-2}X^{-1}Y^{-1}t)}.$$

(3) *Assume that K_p is ramified over $\mathbf{Q}_p$. Then*

$$H_{2n+1}(d_0,X,Y,t)=t^{(n+1)i_p+\delta_{2p}}\frac{\prod_{i=1}^{n+1}(1-p^{-2n-2i}t^2)}{\phi_n(p^{-2})}$$
$$\times\frac{1}{\prod_{i=1}^{n+1}(1-p^{-2n+2i-3}XYt)(1-p^{-2n+2i-3}X^{-1}Y^{-1}t)}$$
$$\times\frac{1}{(1-p^{-2n+2i-3}X^{-1}Yt)(1-p^{-2n+2i-3}XY^{-1}t)}.$$

6. Proof of the main theorem

Theorem 6.1. *Let k and n be positive integers. Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$. For a subset Q of Q_D and a positive integer i put*

$$M(s,f,\mathrm{Ad},\chi^{i-1},\chi_Q)$$

$$=\{\prod_{p\notin Q}(1-\alpha_p^2\chi(p)^i\chi_Q(p)p^{-s})(1-\alpha_p^{-2}\chi(p)^i\chi_Q(p)p^{-s})(1-\chi^{i-1}(p)\chi_Q(p)p^{-s})^2$$

$$\times\prod_{p\in Q}(1-\alpha_p^2\chi(p)^ip^{-s})(1-\alpha_p^{-2}\chi(p)^ip^{-s})(1-\chi^{i-1}(p)p^{-s})^2\}^{-1}.$$

Then, we have

$$R(s,I_{2n}(f))=D^{2n(s-2k)+2nk-n(2n+1)/2-1/2}2^{-2n}$$

$$\times D^{-n(s-2k-2n)}\prod_{i=2}^{2n}\widetilde{\Lambda}(i,\chi^i)\prod_{i=0}^{2n-1}L(2s-4k-i,\chi^i)^{-1}$$

$$\times\sum_{Q\in Q_D}\chi_Q((-1)^n)\prod_{i=1}^{2n}M(s-2k-2n+i,f,\mathrm{Ad},\chi^{i-1},\chi_Q)\}.$$

Proof. For a while $\lambda_p(d)=\lambda_{m,p}(d,\alpha_p,\overline{\alpha}_p)$. Then by Theorem 4.4, we have

$$R(s,I_{2n}(f))=\mu_{2n,k,D}\sum_d\prod_p\lambda_p(d)d^{-s+2k+2n-1}.$$

Then by (1) and (2) of Theorem 5.5.2, $\lambda_p(d)$ depends only on $p^{\mathrm{ord}_p(d)}$ if $p\nmid D$. Hence we write $\lambda_p(d)$ as $\widetilde{\lambda}_p(p^{\mathrm{ord}_p(d)})$. On the other hand, if $p\mid D$, by Corollary to Theorem 5.5.2, $\lambda_p(d)$ can be expressed as

$$\lambda_p(d)=\lambda_p^{(0)}(d)+\chi_{K_p}((-1)^ndp^{-\mathrm{ord}_p(d)})\lambda_p^{(1)}(d),$$

where $\lambda_p^{(l)}(d)$ is a rational number depending only on $p^{\mathrm{ord}_p(d)}$ for $l=0,1$. Hence we write $\lambda_p^{(l)}(d)$ as $\widetilde{\lambda}_p^{(l)}(p^{\mathrm{ord}_p(d)})$. We note that we have

$$a_m(f;d)=\sum_{Q\in Q_D}\prod_{p|d,p\nmid D}\widetilde{\lambda}_p(p^{\mathrm{ord}_p(d)})\prod_{q\in Q}\chi_{K_q}(p^{\mathrm{ord}_p(d)})$$

$$\times\prod_{p|d,p|D,p\notin Q}\widetilde{\lambda}_p^{(0)}(p^{\mathrm{ord}_p(d)})\prod_{q\in Q}\chi_{K_q}(p^{\mathrm{ord}_p(d)})\prod_{q|d,q\in Q}\widetilde{\lambda}_q^{(1)}(q^{\mathrm{ord}_q(d)})\chi_{K_q}((-1)^n)$$

for a positive integer d. We also note that for a subset Q of Q_D we have

$$\chi_Q(m)=\prod_{q\in Q}\chi_{K_q}(m)$$

for an integer m coprime to any $q\in Q$. Hence we have

$$R(s,I_{2n}(f))=\mu_{2n,k,D}\sum_{Q\subset Q_D}\prod_{p\nmid D}\sum_{i=0}^{\infty}\widetilde{\lambda}_p(p^i)\chi_Q(p^i)p^{(-s+2k+2n)i}$$

$$\times \prod_{p|D, p \notin Q} \sum_{i=0}^{\infty} \widetilde{\lambda}_p^{(0)}(p^i)\chi_Q(p^i)p^{(-s+2k+2n)i}\chi_Q((-1)^n)\prod_{p\in Q}\sum_{i=0}^{\infty}\widetilde{\lambda}_p^{(1)}(p^i)p^{(-s+2k+2n)i}.$$

$$= \mu_{2n,k,D}2^{-\#(Q_D)}\sum_{Q\subset Q_D}\chi_Q((-1)^n)\prod_{p\nmid D}H_{2n,p}(1,\alpha_p,\overline{\alpha_p},\chi_Q(p)p^{-s+2k+2n})$$

$$\times \prod_{p|D,p\notin Q} H_{2n,p}^{(1)}(\alpha_p,\overline{\alpha_p},\chi_Q(p)p^{-s+2k+2n})\prod_{p\in Q}H_{2n,p}^{(0)}(\alpha_p,\overline{\alpha_p},p^{-s+2k+2n}).$$

Now for $l=0,1$ write $H_{2n,p}^{(l)}(X,Y,t)$ as

$$H_{2n,p}^{(l)}(X,Y,t) = t^{ni_p}\widetilde{H}_{2n,p}^{(l)}(X,Y,t),$$

where $i_p = 0$ or 1 according as $4||D$ or not. Hence we have

$$R(s,I_{2n}(f)) = \mu_{2n,k,D}2^{-\#(Q_D)}\prod_{p\in Q_D'}p^{(-s+2k+2n)n}\sum_{Q\subset Q_D}\chi_Q((-1)^n)$$

$$\times\prod_{p\nmid D}H_{2n,p}(1,\alpha_p,\overline{\alpha_p},\chi_Q(p)p^{-s+2k+2n})$$

$$\times \prod_{p|D,p\notin Q} \widetilde{H}_{2n,p}^{(1)}(\alpha_p,\overline{\alpha_p},\chi_Q(p)p^{-s+2k+2n})\prod_{p\in Q}\widetilde{H}_{2n,p}^{(0)}(\alpha_p,\overline{\alpha_p},p^{-s+2k+2n}),$$

where $Q_D' = Q_D\backslash\{2\}$ or Q_D according as $4||D$ or not. Note that $2^{c_D n(-s+2k+2n)}\prod_{p\in Q_D'}p^{(-s+2k+2n)n} = D^{(-s+2k+2n)n}$. Thus the assertion follows from Theorem 5.5.2. □

Theorem 6.2. *Let k and n be positive integers. Given a primitive form $f \in \mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$. Then, we have*

$$R(s,I_{2n+1}(f)) = D^{(2n+1)(s-2k+1)+(2k-1)n-(2n+1)(n+1)/2-1/2}2^{-2n-1}$$

$$\times D^{-(n+1)(s-2k-2n)}\prod_{i=2}^{2n+1}\widetilde{\Lambda}(i,\chi^i)\prod_{i=1}^{2n}L(2s-4k-i,\chi^i)^{-1}$$

$$\times\prod_{i=1}^{2n+1}L(2s-2k-2n+i,f,\mathrm{Ad},\chi^{i-1})L(2s-2k-2n+i,\chi^{i-1}).$$

Proof. The assertion follows directly from Theorems 4.4 and 5.5.3. □

Lemma 6.3. *Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$. Suppose that $f_Q = f$ for $Q\subset Q_D$. Then for a positive integer i we have*

$$M(s,f,\mathrm{Ad},\chi^{i-1},\chi_Q) = L(s,f,\mathrm{Ad},\chi^{i-1})L(s,\chi^{i-1}).$$

Proof. For a prime number p let $M_p(s)$ and $L_p(s)$ be the p-Euler factor of $M(s,f,\mathrm{Ad},\chi^{i-1},\chi_Q)$ and $L(s,f,\mathrm{Ad},\chi^{i-1})L(s,\chi^{i-1})$, respectively. Then by definition, $M_p(s) = L_p(s)$ if $p\in Q$. Moreover, $M_p(s)=L_p(s)$ if $p\notin Q$ and $\chi_Q(p)=1$. By the assumption we have

$$\chi_Q(p)c_f(p) = c_f(p).$$

Since f is a primitive form, we have $c_f(p) \neq 0$ for $p|D$. Hence we have $M_p(s) = L_p(s)$ if $p \notin Q$ and $p|D$. Suppose $p \nmid D$ and $\chi_Q(p) = -1$. Then $c_f(p) = 0$ and hence $\alpha_p + \chi(p)\alpha_p^{-1} = 0$. Then by a simple computation we have

$$M_p(s) = (1-p^{-2s})^{-2}.$$

Similarly we have

$$L_p(s) = (1-p^{-2s})^{-2}.$$

This completes the assertion. □

Proposition 6.4. (1) *Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$, and Q be a subset of Q_D. Then for a positive integer $i \geq 2$ the Euler product $M(s,f,\mathrm{Ad},\chi^{i-1},\chi_Q)$ is holomorphic at $s=i$. Moreover $M(s,f,\mathrm{Ad},1,\chi_Q)$ has a simple pole at $s=1$ if and only if $f=f_Q$. In this case the residue of $M(s,f,\mathrm{Ad},1,\chi_Q)$ at $s=1$ is $L(1,f,\mathrm{Ad})$.*

(2) *Let f be a primitive form in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ and χ be a primitive quadratic odd character. Then for a positive integer $i \geq 2$ the Euler product $L(s,f,\mathrm{Ad},\chi^{i-1})L(s,\chi^{i-1})$ is holomorphic at $s=i$, and $L(s,f,\mathrm{Ad},1)L(s,1)$ has a simple pole at $s=1$ with the residue $L(1,f,\mathrm{Ad})$.*

Proof. The assertion can be proved by Theorem 2 of [Sh75]. □

Proof of Theorem 2.3.

(1) By Theorem 6.1 and Lemma 6.3, we have

$$R(s,I_m(f)) = \prod_{i=1}^{2n}\widetilde{\Lambda}(i,\chi^i)\prod_{i=0}^{2n-1}L(2s-4k-i,\chi^i)^{-1}$$

$$\times\{\eta_m(f)\prod_{i=1}^{2n}L(s-2k-2n+i,f,\mathrm{Ad},\chi^{i-1})L(s-2k-2n+i,\chi^{i-1})$$

$$+\sum_{\substack{Q\in Q_D\\ f_Q\neq f}}\chi_Q((-1)^n)\prod_{i=1}^{2n}M(s-2k-2n+i,f,\mathrm{Ad},\chi^{i-1},\chi_Q)\}.$$

By (1) of Lemma 6.4, the term

$$\prod_{i=0}^{2n-1}L(2s-4k-i,\chi^i)^{-1}\prod_{i=1}^{2n}M(2s-2k+i,f,\mathrm{Ad},\chi^{i-1},\chi_Q)$$

is holomorphic at $s=2k+2n$ if $f_Q \neq f$. On the other hand, the term

$$\prod_{i=0}^{2n-1}L(2s-4k-i,\chi^i)^{-1}\prod_{i=1}^{2n}L(s-2k-2n+i,f,\mathrm{Ad},\chi^{i-1})L(s-2k-2n+i,\chi^{i-1})$$

has a simple pole at $s=2k+2n$ with the residue

$$\prod_{i=0}^{2n-1}L(4n-i,\chi^i)^{-1}\prod_{i=1}^{2n}L(i,f,\mathrm{Ad},\chi^{i-1})\prod_{i=2}^{2n}L(i,\chi^{i-1}).$$

Hence $R(s, I_m(f))$ has a simple at $s = 2k+2n$ with the residue

$$\prod_{i=2}^{2n} \widetilde{\Lambda}(i,\chi^i) \prod_{i=0}^{2n-1} L(4n-i,\chi^i)^{-1} \prod_{i=1}^{2n} L(i,f,\mathrm{Ad},\chi^{i-1}) \prod_{i=2}^{2n} L(i,\chi^{i-1}).$$

Thus the assertion can be proved by comparing the above result with Proposition 3.1.

(2) The assertion holds if $m = 1$. In the case $m \geq 3$, the assertion can be proved by Theorem 6.2, (2) of Lemma 6.4, and Proposition 3.1 in the same manner as above.

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