RANDOM PERTURBATIONS OF NON-SINGULAR
TRANSFORMATIONS ON $[0, 1]$

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**Abstract.** We consider random perturbations of non-singular measurable transformations $S$ on $[0, 1]$. By using the spectral decomposition theorem of Komornik and Lasota, we prove that the existence of the invariant densities for random perturbations of $S$. Moreover the densities for random perturbations with small noise strongly converges to the density for Perron-Frobenius operator corresponding to $S$ with respect to $L^1([0, 1])$-norm.

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1. Introduction

It is known that every Markov process on a state space can be represented as a random dynamical system ([2]). There are many important Markov models in applications which are analysed as random dynamical systems. We focus on the following random dynamical system with additive noise : Let $S : X \to X$ be a non-singular measurable transformation on a measurable space $(X, \mathcal{B}, \lambda)$ and let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For a given random variable $X_0$ and an i.i.d. sequence $\{\xi_n\}_{n\geq0}$ on $\Omega$ with values in $X$, we define the following Markov process $\{X_n\}_{n\geq0}$ by

$$X_{n+1}(\omega) := S(X_n(\omega)) + \xi_n(\omega).$$

When $X = \mathbb{R}$, we call the above Markov process $\{X_n(\omega)\}_{n\geq0}$ first-order nonlinear autoregressive model (NLAR(1)). On the other hand, if we let $Q(x, A)$ be a family of transition probabilities (from a point $x \in X$ to a Borel set $A \in \mathcal{B}$), then the Markov process on $X$ defined by the transition probabilities $Q(Sx, A)$ is called a random perturbation of the dynamical system $(X, S)$. In this paper, we consider NLAR(1) on $[0, 1]$, i.e. let $X = [0, 1]$ for (1) and we identify $X_n$ with $X_n - \lfloor X_n \rfloor$ for all $n \geq 0$, where $\lfloor x \rfloor$ is the largest integer less than or equal to $x$. Note that considering NLAR(1) on $[0, 1]$ is coincident with considering a random perturbation of the dynamical system $S$ on $[0, 1]$ in our case.

A stability property of NLAR(1) can be derived from contraction assumptions by Lasota and Mackey ([15]) by using the spectral decomposition theorem of Komornik and Lasota (Theorem 2.5). This theorem is our main method in this paper. Vu Kuok Fong [5] and independently Sine [18] have
showed that the generalization of this spectral decomposition theorem of Komornik-Lasota is a simple corollary of the Jacobs-de Leeuw-Glicksberg theorem. We prove that for any non-singular transformation \( S : [0, 1] \rightarrow [0, 1] \), there exists an invariant density of \( \{X_n\}_{n \geq 0} \) for NLAR(1) on \([0, 1]\) by using the spectral decomposition theorem of Komornik-Lasota.

In this paper, we also study the limiting distribution of NLAR(1) on \([0, 1]\) with small additive noise (or small perturbations of \(([0, 1], S)\)) given by

\[
X_{n+1}^\varepsilon(\omega) := S(X_n^\varepsilon(\omega)) + \varepsilon \xi_n(\omega) \pmod{1},
\]

as \( \varepsilon \downarrow 0 \), where \( X_0^\varepsilon = X_0 \). Many authors observe the relation between deterministic dynamical systems and small perturbed random dynamical systems (\([4],[6],[9],[11],[16]\)). For example, in \([9]\), Katok and Kifer considered small random perturbations, where \( S \) is an endomorphism of the interval \([0, 1]\) satisfying the conditions of Misiurewicz and small transition probabilities \( P^\varepsilon(x, A) = Q^\varepsilon(Sx, A) \) for sufficiently small \( \varepsilon > 0 \). They proved that the densities of \( X_n^\varepsilon \)-invariant measures \( \mu^\varepsilon \) converge weakly to a density of the invariant measure \( \mu_S \) corresponding to \( S \) as \( \varepsilon \rightarrow 0 \) in \( L^1 \) topology (\([9]\)).

In \([14]\), Lasota and Mackey showed that the density functions of \( \{X_n^\varepsilon\}_{n \geq 0} \) for NLAR(1) (on \( \mathbb{R} \)) with small additive noise are given by

\[
P^\varepsilon f(x) := \int_{\mathbb{R}} g(y) P_S f(x - \varepsilon y) dy,
\]

where \( P_S \) is the Perron-Frobenius operator corresponding to \( S \), \( g \) is the density of \( \{\xi_n\}_{n \geq 0} \) and \( f \) is the density of \( X_0 \). They prove that

\[
\lim_{\varepsilon \rightarrow 0} \| P^\varepsilon f - P_S f \|_{L^1(\mathbb{R})} = 0
\]

for all \( f \in L^1(\mathbb{R}) \) (see \([14]\)). We obtain the same result for NLAR(1) on \([0, 1]\). Moreover since the existence of the densities of \( X_n^\varepsilon \)-invariant measures are guaranteed by the spectral decomposition theorem of Komornik-Lasota, under certain conditions, we prove that if there exists the limit \( f_* \) of the densities of \( X_n^\varepsilon \)-invariant measures in \( L^1 \) as \( \varepsilon \downarrow 0 \) then the limit function \( f_* \) is an invariant density corresponding to \( S \). This implies that we gave the sufficient condition of the existence of an invariant density corresponding to \( S \). On the other hand, in the sense of weak convergence of invariant probability measures for small random perturbations of a dynamical system \( S \), the bounded variation case is first proved by Keller (see the condition S1 in \([10]\)). Afterwards, Young and Baladi considered random perturbations of piecewise \( C^2 \) expanding map \( S : [0, 1] \rightarrow [0, 1] \) for which there exists the unique invariant density \( f_* \). Indeed, in \([1]\), Young and Baladi proved that for any piecewise \( C^2 \) expanding map which has no periodic turning points, there exists invariant densities of small random perturbations and they converges to the invariant density \( f_* \) corresponding to \( S \) with respect to \( L^1 \)-norm as \( \varepsilon \rightarrow 0 \) (see also \([3]\)). In section 3, we can see that the spectral decomposition theorem of Komornik-Lasota and (3) hold for NLAR(1) on \([0, 1]\) defined by
(1) with respect to intermittent maps $S$ which have an infinite invariant density function.

2. Main theorems

2.1. Random perturbations of Dynamical systems. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, where $\mathcal{F}$ denotes a Borel $\sigma$-field and $\mu$ a probability measure. Let $x_0, \xi_0, \xi_1, \cdots$ be random variables on $\Omega$ with values in $[0, 1]$ and $S : [0, 1] \to [0, 1]$ be a non-singular measurable transformation (i.e. $\lambda(S^{-1}(A)) = 0$ for any Borel set $A \subset [0, 1]$ with $\lambda(A) = 0$, where $\lambda$ is the normalized Lebesgue measure on $[0, 1]$).

Consider the following stochastic process defined by

$$x_{n+1}(\omega) = S(x_n(\omega)) + \xi_n(\omega) \pmod{1}$$

for each $n \geq 0$.

**Definition 2.1.** We say that a random dynamical system $\{x_n\}_{n \geq 0}$ generated by (4) with respect to $(\Omega, [0, 1], S, x_0, \{\xi_n\}_{n \geq 0})$ is first-order nonlinear autoregressive model on $[0, 1]$ (NLAR(1) on $[0, 1]$) if the following conditions C1-C3 hold:

- **C1:** $x_0, \xi_0, \xi_1, \cdots$ are independent random variables;
- **C2:** $x_0$ has the density function $f_0 \in D$ (i.e. $\mu(\{\omega : x_0(\omega) \in B\}) = \int_B f_0(x)dx$ for any Borel set $B \subset [0, 1]$), where $D := \{f \in L^1([0, 1]) : f \geq 0$ and $\int_{[0, 1]} f(x)dx = 1\}$;
- **C3:** each $\xi_n$ has the same density function $g \in L^1(\mathbb{R})$ such that $g \geq 0$, $\operatorname{supp}(g) := \{x \in [0, 1] : g(x) \neq 0\} \subset [0, 1]$ and $\int_{\mathbb{R}} g(x)dx = 1$.

Under conditions C1-C3, there exists a Markov operator $P : L^1([0, 1]) \to L^1([0, 1])$ such that

$$\mu_n(A) := \mu(\{\omega : x_n(\omega) \in A\}) = \int_A P^n f_0(x)dx$$

for all Borel set $A$ on $[0, 1]$ and $n \geq 0$.

**Proposition 2.2.** Let $\{x_n\}_{n \geq 0}$ be a NLAR(1) on $[0, 1]$ generated by (4) with respect to $(\Omega, [0, 1], S, x_0, \{\xi_n\}_{n \geq 0})$. Then there exists a Markov operator $P : L^1([0, 1]) \to L^1([0, 1])$ defined by

$$P f(x) = \int_{[0, 1]} f(y) \left(\sum_{i=0}^1 g(x - S(y) + i)\right) dy,$$

which satisfies (5).

In our paper, the spectral decomposition theorem of Komorník and Lasota [13] plays a central role. We introduce the sufficient condition for this theorem:
Definition 2.3. Let \((X, \mathcal{F}, \nu)\) be a finite measure space. A linear operator \(P : L^1(X, \nu) \to L^1(X, \nu)\) is constrictive if there exists \(\delta > 0\) and \(\kappa < 1\) such that for every \(f \in D\) there is an integer \(n_0(f)\) for which

\[(7) \quad \int_E P^n f(x) \nu(dx) \leq \kappa \quad \text{for all } n \geq n_0(f) \text{ and } E \text{ with } \nu(E) \leq \delta.\]

Remark 2.4. If the space \((X, \mathcal{F}, \mu)\) is \(\sigma\)-finite, we can substitute the above condition by the following:

there exists \(\delta > 0\), \(\kappa < 1\) and a measurable set \(B\) with \(\nu(B) < \infty\) such that for every \(f \in D\) there is an integer \(n_0(f)\) for which

\[(8) \quad \int_{(X \setminus B) \cup E} P^n f(x) \nu(dx) \leq \kappa \quad \text{for all } n \geq n_0(f) \text{ and } E \text{ with } \nu(E) \leq \delta.\]

It is easy to see that this condition reduces to that of Definition 2.3 when \(X\) is finite and let \(X = B\).

Theorem 2.5. (spectral decomposition theorem [13]) Let \(P : L^1(X, \mathcal{F}, \nu) \to L^1(X, \mathcal{F}, \nu)\) be a constrictive Markov operator. Then there is an integer \(r\), non negative functions \(g_i \in D_0 := \{f \in L^1(X, \mathcal{F}, \nu) : \|f\|_{L^1} = 1, f \geq 0\}\) and \(k_i \in L^\infty(X, \mathcal{F}, \nu), i = 1, 2, \ldots, r\) and an operator \(Q : L^1(X, \mathcal{F}, \nu) \to L^1(X, \mathcal{F}, \nu)\) such that for every \(f \in L^1(X, \mathcal{F}, \nu)\), \(Pf\) is represented by the form

\[(9) \quad Pf(x) = \sum_{i=1}^{r} \lambda_i(f) g_i(x) + Qf,\]

where

\[\lambda_i(f) = \int_X f(x) k_i(x) \nu(dx).\]

Moreover the functions \(g_i\) and the operator \(Q\) have the following properties:

- \(g_i(x)g_j(x) = 0\) for all \(i \neq j\).
- For each integer \(i\), there exists an unique integer \(\sigma(i)\) such that \(Pg_i = g_{\sigma(i)}\). Further \(\sigma(i) \neq \sigma(j)\) for \(i \neq j\).
- \(\lim_{n \to \infty} \|P^n Qf\| = 0\) for every \(f \in L^1(X, \mathcal{F}, \nu)\).

Remark 2.6. The spectral decomposition theorem of Komornik and Lasota holds when the space \((X, \mathcal{F}, \nu)\) is \(\sigma\)-finite space and Markov operator is constrictive.

Remark 2.7. If Theorem 2.5 holds for a Markov operator \(P\), then there is an invariant density \(f^*\) defined by

\[f^* = \frac{1}{r} \sum_{i=1}^{r} g_i.\]
Indeed,
\[ P f_* = \frac{1}{r} \sum_{i=1}^{r} P g_i = \frac{1}{r} \sum_{i=1}^{r} g_i = f_* \]
Therefore \( Pf_* = f_* \).

The following theorem is our main result.

**Theorem 2.8.** The Markov operator \( P : L^1([0,1]) \to L^1([0,1]) \) defined by (6) corresponding to a NLAR(1) on [0,1] generated by (4) with respect to \((\Omega, [0,1], S, x_0, \{\xi_n\}_{n \geq 0})\) is constrictive, that is, theorem 2.5 holds for \( P \).

Moreover when the density of noise \( g(x) \) is not zero for all \( x \), we have the following result.

**Proposition 2.9.** Let \( P : L^1([0,1]) \to L^1([0,1]) \) be the Markov operator defined by (6) corresponding to a NLAR(1) on [0,1] generated by (4) with respect to \((\Omega, [0,1], S, x_0, \{\xi_n\}_{n \geq 0})\). If \( g(x) > 0 \) for all \( x \in [0,1] \), then there exists a unique \( f_* \in D \) such that \( Pf_* = f_* \) and
\[ \lim_{n \to \infty} \| P^n f - f_* \| = 0 \quad \text{for every } f \in D. \]

**Remark 2.10.** A sequence \( \{P^n\}_{n \geq 1} \) satisfying (9) is called asymptotically periodic. Proposition 2.9 implies that \( r = 1 \) for (9). In this case, the sequence \( \{P^n\}_{n \geq 1} \) is called asymptotically stable.

### 2.2. Small random perturbations of dynamical systems.

In this section, we observe limiting behaviour of density functions of a NLAR(1) on [0,1] generated by (4) with respect to \((\Omega, [0,1], S, x_0, \{\xi_n\}_{n \geq 0})\) parametrized by \( \varepsilon > 0 \) as \( \varepsilon \to 0 \).

We consider the following first-order nonlinear autoregressive model \( \{x_n^\varepsilon\}_{n \geq 0} \) on [0,1] with respect to \((\Omega, [0,1], S, x_0, \{\xi_n\}_{n \geq 0})\) parametrized by \( \varepsilon > 0 \):

\[
x_{n+1}^\varepsilon(\omega) = S(x_n^\varepsilon(\omega)) + \varepsilon \xi_n(\omega) \quad \text{for } 0 < \varepsilon < 1,
\]
where \( x_0^\varepsilon = x_0 \).

Since random variables \( \varepsilon \xi_n \) have the same density \( \frac{1}{\varepsilon} g(\frac{1}{\varepsilon}) \), we have the Markov operator \( P_\varepsilon : L^1([0,1]) \to L^1([0,1]) \) defined by

\[
P_\varepsilon f(x) = \frac{1}{\varepsilon} \int_{[0,1]} f(y) \left( \sum_{i=0}^{1} g \left( \frac{x - S(y) + i}{\varepsilon} \right) \right) dy
\]
which satisfies that \( f_{n+1}^\varepsilon = P_\varepsilon f_n^\varepsilon \), where \( \{f_n^\varepsilon\}_{n \geq 0} \) is the sequence of the density function of \( x_n^\varepsilon \). Since \( S \) is non-singular, there exists the Perron-Frobenius operator \( P_S : L^1([0,1]) \to L^1([0,1]) \) with respect to \( S : [0,1] \to [0,1] \).
Hence, if we let $g_{x,i,\varepsilon}(y) := g\left(\frac{x+i-y}{\varepsilon}\right)$, then we have that

$$P_\varepsilon f(x) = \frac{1}{\varepsilon} \int_{[0,1]} f(y) \left(\sum_{i=0}^{1} g_{x,i,\varepsilon}(S(y))\right) dy$$

$$= \frac{1}{\varepsilon} \int_{[0,1]} P_\varepsilon f(y) \left(\sum_{i=0}^{1} g_{x,i}(y)\right) dy$$

$$= \frac{1}{\varepsilon} \int_{[0,1]} P_\varepsilon f(y) \left(\sum_{i=0}^{1} g\left(\frac{x+i-y}{\varepsilon}\right)\right) dy$$

$$= \sum_{i=0}^{1} \int_{\left[\frac{x+i-1}{\varepsilon}; \frac{x+i}{\varepsilon}\right]} P_\varepsilon f(x+i-\varepsilon) g(y) dy$$

by condition C3.

We should expect that in some sense $\lim_{\varepsilon \to 0} P_\varepsilon f(x) = P_\varepsilon f(x)$.

Let $\|f\|_\infty := \inf \{M : |f(x)| \leq M \text{ for } \lambda\text{-a.e. } x \in [0,1]\}$, where $\lambda$ is the normalized Lebesgue measure on $[0,1]$.

**Theorem 2.11.** Let $S : [0,1] \to [0,1]$ be a non-singular measurable transformation and $P_\varepsilon$ be the Markov operator defined by (11) corresponding to a NLAR(1) on $[0,1]$ generated by (10) with respect to $(\Omega, [0,1], S, x_0, \{\xi_n\}_{n\geq0})$. Suppose that $\|P_\varepsilon f\|_\infty < \infty$ for any continuous function $f$ on $[0,1]$. Then we have that

$$\lim_{\varepsilon \to 0} \|P_\varepsilon f - P_\varepsilon f\|_{L^1([0,1])} = 0$$

for all $f \in L^1([0,1])$.

**Remark 2.12.** There is a big class of dynamical systems $S : [0,1] \to [0,1]$ satisfying $\|P_\varepsilon f\|_\infty < \infty$ for any continuous function $f$ on $[0,1]$. For example, piecewise monotonic maps (including unimodal maps) and piecewise convex maps satisfy the assumption of Theorem 2.11.

It is obviously that $\{P_\varepsilon^n\}_{n\geq1}$ defined by (11) is asymptotically periodic for each $\varepsilon > 0$. Hence the function $f_\varepsilon$ defined by

$$f_\varepsilon(x) = \frac{1}{r(\varepsilon)} \sum_{i=1}^{r(\varepsilon)} g_{i,\varepsilon}(x),$$

where $r(\varepsilon)$ is a positive integer and $g_{i,\varepsilon}(x)$ are density functions depending only on $\varepsilon$, satisfies that $f_\varepsilon \in D$ and $P_\varepsilon f_\varepsilon = f_\varepsilon$. This implies that for each $\varepsilon > 0$, Markov operator $P_\varepsilon$ has at least one invariant density.

**Corollary 2.13.** Let $S : [0,1] \to [0,1]$ be a non-singular measurable transformation, $P_\varepsilon$ be the Markov operator defined by (11) corresponding to a NLAR(1) on $[0,1]$ generated by (10) with respect to $(\Omega, [0,1], S, x_0, \{\xi_n\}_{n\geq0})$ and $f_\varepsilon$ be an invariant density for $P_\varepsilon$ defined by (13). Suppose that $\|P_\varepsilon f\|_\infty < \infty$ for some continuous function $f$ on $[0,1]$.

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$$\lim_{\varepsilon \to 0} \|P_\varepsilon f - P_\varepsilon f\|_{L^1([0,1])} = 0$$

for all $f \in L^1([0,1])$.

**Remark 2.12.** There is a big class of dynamical systems $S : [0,1] \to [0,1]$ satisfying $\|P_\varepsilon f\|_\infty < \infty$ for any continuous function $f$ on $[0,1]$. For example, piecewise monotonic maps (including unimodal maps) and piecewise convex maps satisfy the assumption of Theorem 2.11.

It is obviously that $\{P_\varepsilon^n\}_{n\geq1}$ defined by (11) is asymptotically periodic for each $\varepsilon > 0$. Hence the function $f_\varepsilon$ defined by

$$f_\varepsilon(x) = \frac{1}{r(\varepsilon)} \sum_{i=1}^{r(\varepsilon)} g_{i,\varepsilon}(x),$$

where $r(\varepsilon)$ is a positive integer and $g_{i,\varepsilon}(x)$ are density functions depending only on $\varepsilon$, satisfies that $f_\varepsilon \in D$ and $P_\varepsilon f_\varepsilon = f_\varepsilon$. This implies that for each $\varepsilon > 0$, Markov operator $P_\varepsilon$ has at least one invariant density.

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for any continuous function $f$ on $[0, 1]$. If there exists an integrable function $f_*$ on $[0, 1]$ such that
\[
\lim_{\varepsilon \to 0} \|f_\varepsilon - f_*\|_{L^1([0, 1])} = 0,
\]
then $f_*$ is an invariant density for $P_S$, that is $P_S f_\varepsilon = f_*$. 

Remark 2.14. Corollary 2.13 holds for any continuous piecewise $C^2$, piecewise expanding map $S : [0, 1] \to [0, 1]$ which has no periodic turning points. Indeed, by Theorem 1.1 in [3] (and see Theorem 3 in [1]), there exists an unique absolutely continuous invariant probability measure $\mu_0 = f_* dx$ which satisfies that
\[
\lim_{\varepsilon \to 0} \|f_\varepsilon - f_*\|_{L^1([0, 1])} = 0.
\]

3. Examples

It is obviously that Theorem 2.8 holds for all non-singular transformations. We give some examples of non-singular transformations which also satisfy the assumptions of Theorem 2.11.

(1): $m$-adic transformation [14].

Consider the transformation $S : [0, 1] \to [0, 1]$ given by
\[
Sx = mx \quad (\text{mod } 1),
\]
where $m \geq 1$ is an integer. Thus the Perron-Frobenius operator $P_S : L^1([0, 1]) \to L^1([0, 1])$ corresponding to $S$ is given by
\[
P_S f(x) = \frac{1}{m} \sum_{i=0}^{m-1} f\left(\frac{i + x}{m}\right).
\]
Since $P_S 1 = 1$, the Borel measure on $[0, 1]$ is invariant with respect to the $m$-adic transformation $S$. Moreover it is obviously that for any continuous function $f$ on $[0, 1]$, $Pf(x)$ is equal to a continuous function, hence $\|P_S f\|_\infty < \infty$.

(2): Maps with indifferent fixed points with infinite invariant measure [19]

Let $\alpha \in (0, \infty)$ be a real parameter and consider the one-parameter family of maps $S_\alpha$ of the interval $[0, 1]$ onto itself defined by
\[
S_\alpha(x) := 2^{e^{\alpha x} - 1} e^{\alpha x} - 1 \quad (\text{mod } 1).
\]
For every $\alpha > 0$, $S_\alpha$ is piecewise onto and $C^\infty$-class. When the parameter $\alpha$ varies, the dynamics of the maps changes. Some properties of this family established in [17] are listed below:

(1) For $\alpha > 0$ with $|S_\alpha'(0)| > 1$, $S_\alpha$ is a piecewise expanding map (see Figure 1). Then there exists the unique absolutely continuous invariant probability measure with respect to the Lebesgue measure on $[0, 1]$ by the Lasota-Yorke theorem.
(2) For $\alpha > 0$ with $|S_0'(0)| = 1$, $S_0$ admits an indifferent fixed point $0$ (see Figure 2). For these maps, there is NO finite absolutely continuous invariant measure. However there exists a $\sigma$-finite infinite absolutely continuous invariant measure.

(3) For $\alpha > 0$ with $|S_0'(0)| < 1$, $S_\alpha$ admits a stable fixed point $0$ (see Figure 3). For these maps, almost all points converge to $0$ by using the symbolic dynamics with 4-symbols (see [17] more details.). Therefore there is no absolutely continuous invariant measure with respect to the Lebesgue measure.

Next, we shall apply our results (Theorem 2.11) to this family. Because $T_\alpha(0) = 0$, $T_\alpha(1) = 2$, where $T_\alpha(x) := 2^{\alpha x - 1}$ is monotonic continuous function for every $\alpha > 0$, there exists the unique point $x_\alpha \in [0, 1)$ such that $T_\alpha(x_\alpha) = 1$. Let $I_0 = [0, x_\alpha)$ and $I_1 = [x_\alpha, 1]$. Since $C^\infty$-extensions of the maps $S_\alpha|_{I_0} : I_0 \to [0, 1]$ and $S_\alpha|_{I_1} : I_1 \to [0, 1]$ are one-to-one and onto, there exist the local inverses $u_{\alpha,j} = (S_\alpha|_{I_j})^{-1}$ for $j = 0, 1$, we get

$$u_{\alpha,j}(x) = \frac{1}{\alpha} \log(1 + \frac{\alpha - 1}{2}(x + j)).$$

Thus the Perron-Frobenius operator corresponding to $S_\alpha$ is given by

$$P_{S_\alpha} f = f \circ u_{\alpha,0} \cdot u_{\alpha,0}' + f \circ u_{\alpha,1} \cdot u_{\alpha,1}'. $$

Therefore we have $\|P_{S_\alpha} f(x)\|_{\infty} < \infty$ for any continuous function $f$ on $[0, 1]$.

4. Proof

**Proof. Proof of Proposition 2.2**

We let the density of $x_n$ be denoted by $f_n \in D$ ($n \geq 1$) and desire a relation connecting $f_{n+1}$ and $f_n$.

We assume that $f_n$ exists for some $n \geq 0$.

Let $\bar{A} = A \setminus \{1\}$ for any Borel set $A \subset [0, 1]$. Note that since $x_{n+1}(\Omega) \subset [0, 1)$ and $S(x_n)$ and $\xi_n$ are independent for all $n \geq 0$, we have that
for each $a$

as By a change of variables (see Lemma 5.2 in Appendix.), this can be written

and $g$

for each $a$

$\int \int \int S(x_n(\omega)) + \xi_n(\omega) \in \bar{A} + i \cap \{ \omega : S(x_n(\omega)) + \xi_n(\omega) = 2 \} = \phi,$

(iii):

\[ \mu (S(x_n(\omega)) + \xi_n(\omega) = 2) = \mu (S(x_n(\omega)) = 1 \text{ and } \xi_n(\omega) = 1) \]

\[ = \int_{S^{-1}(\{1\})} f_n(x) dx \int_{\{1\}} g(y) dy = 0. \]

From (i)-(iii), we have that for any Borel set $A \subset [0, 1]$ and $n \geq 0,$

\[ \mu (\{ \omega \in \Omega : x_{n+1} \in A \}) = \mu (\{ \omega \in \Omega : x_{n+1} \in A \}) \]

\[ = \mu (\{ \omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) (\text{mod } 1) \in \bar{A} \}) \]

\[ = \mu (\{ \omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) \in \bar{A} \}) + \mu (\{ \omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) \in \bar{A} + i \} \]

\[ \begin{cases} + \mu (\{ \omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) = 2 \}) & \text{if } 0 \notin \bar{A} \end{cases} \]

\[ = \int \int_{S(x)+y \in \bar{A}} f_n(x) g(y) dx dy + \int \int_{S(x)+y-1 \in \bar{A}} f_n(x) g(y) dx dy. \]

By a change of variables (see Lemma 5.2 in Appendix.), this can be written as

\[ \mu (\{ \omega \in \Omega : x_{n+1} \in A \}) = \int_{\bar{A}} \left\{ \int_{B^0(a)} f_n(b) g(a-S(b)) db \right\} da \]

\[ + \int_{\bar{A}} \left\{ \int_{B^1(a)} f_n(b) g(a-S(b)+1) db \right\} da, \]

where

\[ B^0(a) := \{ b \in [0, 1] : 0 \leq a - S(b) \leq 1 \} = \{ b \in [0, 1] : 0 \leq S(b) \leq a \} \]

and

\[ B^1(a) := \{ b \in [0, 1] : 0 \leq a - S(b) + 1 \leq 1 \} = \{ b \in [0, 1] : a \leq S(b) \leq 1 \} \]

for each $a \in [0, 1].$ By condition C3, we have that

\[ g(x-S(y)) = 0 \quad \text{for all } y \in \{ b \in [0, 1] : x < S(b) \} = [0, 1] \setminus B^0(x) \]

\[ g(x-S(y)+1) = 0 \quad \text{for all } y \in \{ b \in [0, 1] : x > S(b) \} = [0, 1] \setminus B^1(x) \]

for each $x \in [0, 1].$ Hence we get that

\[ \int_{[0,1]\setminus B^0(x)} f_n(y)g(x-S(y))dy = 0 = \int_{[0,1]\setminus B^1(x)} f_n(y)g(x-S(y)+1)dy \]

for each $x \in [0, 1].$ This implies that

\[ \int_{[0,1]} f_n(y)g(x-S(y))dy = \int_{B^0(x)} f_n(y)g(x-S(y))dy \]

\[ \int_{[0,1]} f_n(y)g(x-S(y)+1)dy = \int_{B^1(x)} f_n(y)g(x-S(y)+1)dy. \]
Therefore we have that
\[
\mu \left( \{ \omega \in \Omega : x_{n+1} \in A \} \right) = \int_{a \in A} \int_{[0,1]} f_n(b)g(a - S(b))dbda + \int_{a \in A} \int_{[0,1]} f_n(b)g(a - S(b) + 1)dbda.
\]
Since \(\{1\}\) is a 1-point set and \(h(a) := \int_{b \in [0,1]} f_n(b)g(a - S(b) + i)db \in L^1([0,1])\), we have that for \(i = 0, 1, \)
\[
\int_{\{1\}} \left\{ \int_{[0,1]} f_n(b)g(a - S(b) + i)db \right\} da = \int_{\{1\}} h(a)da = 0.
\]
Then we have that
\[
\mu \left( \{ \omega \in \Omega : x_{n+1} \in A \} \right) = \sum_{i=0}^{1} \int_{a \in A} \int_{b \in [0,1]} f_n(b)g(a - S(b) + i)dbda = \sum_{i=0}^{1} \int_{a \in A} \int_{b \in [0,1]} f_n(b)g(a - S(b) + i)dbda.
\]
Therefore using the fact that \(A\) was an arbitrary Borel set on \([0,1]\), we get the density \(f_{n+1}\) of \(x_{n+1}\) defined by
\[
f_{n+1}(x) = \sum_{i=0}^{1} \int_{[0,1]} f_n(y)g(x - S(y) + i)dy \quad \text{a.e.} \ x \in [0,1].
\]
On the other hand, we get that
\[
\int_{x \in [0,1]} \sum_{i=0}^{1} g(x - S(y) + i)dx = \int_{[0,1]} g(x)dx = 1 \quad \text{for} \ y \in [0,1]
\]
by condition C3. Then by Fubini’s theorem, we have that
\[
\int_{[0,1]} f_{n+1}(x)dx = \sum_{i=0}^{1} \int_{y \in [0,1]} \left\{ \int_{x \in [0,1]} f_n(y)g(x - S(y) + i)dx \right\} dy = \int_{y \in [0,1]} f_n(y)dy = 1.
\]
Moreover \(f_{n+1} \geq 0\) because of the positivity of \(g\) and \(f_n\). Therefore if \(x_n\) has the density \(f_n \in D\), then \(f_{n+1}\) also have to exist in \(D\).
From this fact, we can define a linear operator \(P : L^1([0,1]) \to L^1([0,1])\) by
\[
Pf(x) = \int_{y \in [0,1]} f(y) \left( \sum_{i=0}^{1} g(x - S(y) + i) \right) dy
\]
which satisfies that
\[
f_{n+1} = Pf_n \quad \text{a.e.}
\]
for all \( n \geq 0 \). Next we shall show that \( P : L^1([0, 1]) \to L^1([0, 1]) \) is a Markov operator, that is, \( P \) is a linear operator which satisfies that \( Pf \geq 0 \) and \( \|Pf\|_{L^1([0, 1])} = \|f\|_{L^1([0, 1])} \) for any \( f \in L^1([0, 1]) \) with \( f \geq 0 \). It is easy to see that \( P \) is a positive linear operator on \( L^1([0, 1]) \) because \( g \) is positive. Moreover we have that for \( f \in L^1([0, 1]) \) with \( f \geq 0 \) by the Fubini’s theorem,

\[
\|Pf\|_{L^1([0, 1])} := \int_{[0, 1]} Pf(x) \, dx
= \int_{x \in [0, 1]} \int_{y \in [0, 1]} f_n(y) \left( \sum_{i=0}^{1} g(x - S(y) + i) \right) \, dy \, dx
= \int_{x \in [0, 1]} \sum_{i=0}^{1} g(x - S(y) + i) \left\{ \int_{[0, 1]} f(y) \, dy \right\} \, dx
= \int_{[0, 1]} f(y) \, dy = \|f\|_{L^1([0, 1])}.
\]

Therefore \( P \) is a Markov operator.

Proof. Proof of Theorem 2.8
From the spectral decomposition theorem by Komornik and Lasota [14], it is enough to show that \( P \) is constrictive : there exists a \( \delta > 0 \) and \( \kappa < 1 \) such that for every \( f \in D \) there is an integer \( n_0(f) \) for which

\[
\int_B P^n f(x) \, dx \leq \kappa \quad \text{for all } n \geq n_0(f) \text{ and } B \subset [0, 1] \text{ with } \lambda(B) \leq \delta,
\]

where \( \lambda \) is the normalized Lebesgue measure on \([0, 1]\).

Since \( g \) is the integrable function on \( \mathbb{R} \) supported in \([0, 1]\), for any \( \varepsilon > 0 \), there exists \( 0 < \delta(\varepsilon) \leq 1 \) such that whenever \( \lambda(A) \leq \delta(\varepsilon) \),

\[
\int_A g(x) \, dx \leq \varepsilon.
\]

Take arbitrary \( 0 < \varepsilon < 1 \), hence there exists \( \delta(\varepsilon) > 0 \) which satisfies \( \int_A g(x) \, dx \leq \frac{\varepsilon}{2} \) for any Borel set \( A \subset [0, 1] \) with \( \lambda(A) \leq \delta(\varepsilon) \). Thus we have that for each \( f \in D \) and \( n \geq 1 \),

\[
\int_A P^n f(x) \, dx = \int_A \int_{[0, 1]} P^{n-1} f(y) \left( \sum_{i=0}^{1} g(x - S(y) + i) \right) \, dy \, dx
= \int_{[0, 1]} \left\{ \sum_{i=0}^{1} \int_{A-S(y)+i} g(x) \, dx \right\} P^{n-1} f(y) \, dy.
\]

Let \( \lambda \) be the Lebesgue measure on \( \mathbb{R} \). Since \( \lambda(A - S(y) + i) = \lambda(A) = \lambda(A) \leq \delta(\varepsilon) \) for each \( y \in [0, 1] \) and \( i = 0, 1 \), we obtain that

\[
\int_A P^n f(x) \, dx \leq \varepsilon \int_{[0, 1]} P^{n-1} f(y) \, dy = \varepsilon \quad \text{for all } n \geq 1,
\]

(17)
which implies that $P$ is constrictive.

Proof. Proof of Proposition 2.9
From the theorem 5.6.1 in [14], it is enough to show that there exists a set $A \subset [0, 1]$ of nonzero measure $\lambda(A) > 0$ with the property that for every $f \in D$, there is an integer $n_0(f)$ such that

$$P^n f(x) > 0 \quad \text{for a.e.} \, x \in A \quad \text{and for all} \, n \geq n_0(f). \quad (18)$$

Let $f \in D$ be arbitrary. From the assumption about $g$, there exists a positive number $0 < \varepsilon < 1$ which satisfies that there exists $\delta(\varepsilon) > 0$ such that for all $\lambda(A) \leq \delta(\varepsilon)$, $\int_A g(x) dx \leq \frac{\varepsilon}{2}$. Take an arbitrarily $0 < \delta < 1$ with $1 - \delta < \delta(\varepsilon)$. Since $\lambda((\delta - S(y) + i, 1 - S(y) + i)) = 1 - \delta \leq \delta(\varepsilon)$ for each $y \in [0, 1]$ and $i = 0, 1$, we have that

$$\int_{\delta < x \leq 1} P^n f(x) dx = \int_{[0, 1]} \left\{ \sum_{i=0}^{1} \int_{(\delta - S(y) + i, 1 - S(y) + i)} g(x) dx \right\} P^{n-1} f(y) dy \leq \varepsilon$$

for all $n \geq 1$. From this inequality, we have that

$$\int_{0 \leq y \leq \delta} P^n f(y) dy = \int_{[0, 1]} P^n f(y) dy - \int_{\delta < y \leq 1} P^n f(y) dy \geq 1 - \varepsilon > 0$$

(19) for all $n \geq 1$.

On the other hand, we have that

$$P^{n+1} f(x) = \int_{[0, 1]} P^n f(y) \left( \sum_{i=0}^{1} g(x - S(y) + i) \right) dy \geq \int_{0 \leq y \leq \delta} P^n f(y) \left( \sum_{i=0}^{1} g(x - S(y) + i) \right) dy.$$ 

(20)

From the assumption about $g$, we have that

$$\left( \sum_{0 \leq y \leq \delta} g(x - S(y) + i) \right) + g(x - S(y) + 1) > 0 \quad \text{for all} \, x \in [0, 1] \quad \text{and} \, 0 \leq y \leq \delta.$$

From (19) and (21), we have that for a.e. $x \in [0, 1]$,

$$P^n f(y) \left( \sum_{i=0}^{1} g(x - S(y) + i) \right) \quad \text{for} \, n \geq 1$$

as a function of $y$, does not vanish in $\{0 \leq y \leq \delta\}$. As a consequence, inequality (20) implies (18) with respect to the set $[0, 1]$, thus completing the proof of the proposition.

\qed
Proof. Proof of Theorem 2.11
Since the set of continuous functions on \([0, 1]\) is dense in \(L^1([0, 1])\) and \(P_\varepsilon,\ P_S\) are Markov operators, it is enough to prove the theorem for continuous functions on \([0, 1]\). Indeed, for any \(f \in L^1([0, 1])\) and \(\eta > 0\), there exists a continuous function \(f_\eta\) on \([0, 1]\) such that \(\|f - f_\eta\|_{L^1([0, 1])} \leq \eta\). Thus if we have that \(\lim_{\varepsilon \to 0} \|P_\varepsilon f_\eta - P_S f_\eta\|_{L^1([0, 1])} = 0\), then we have that

\[
\lim_{\varepsilon \to 0} \|P_\varepsilon f - P_S f\|_{L^1([0, 1])} = \lim_{\varepsilon \to 0} \|P_\varepsilon (f - f_\eta) - P_S (f - f_\eta) + P_\varepsilon f_\eta - P_S f_\eta\|_{L^1([0, 1])}
\leq 2\|f - f_\eta\|_{L^1([0, 1])} + \lim_{\varepsilon \to 0} \|P_\varepsilon f_\eta - P_S f_\eta\|_{L^1([0, 1])}
\leq 2\eta.
\]

From the fact that \(\eta\) was an arbitrary positive number, we have that \(\lim_{\varepsilon \to 0} \|P_\varepsilon f - P_S f\|_{L^1([0, 1])} = 0\).

Fix an arbitrarily continuous function \(f\) on \([0, 1]\). We split the integral into two parts,

\[
\|P_\varepsilon f - P_S f\|_{L^1([0, 1])} = \int_{[0, \varepsilon]} |P_\varepsilon f - P_S f| \, dx + \int_{(\varepsilon, 1]} |P_\varepsilon f - P_S f| \, dx
= C_1(\varepsilon) + C_2(\varepsilon) \quad \text{for } 0 < \varepsilon < 1.
\]

Firstly, we consider \(C_1(\varepsilon)\). Let \(H_i(x, y) := P_S f(x + i - \varepsilon y)g(y)1_{\left[\frac{x + i - 1}{\varepsilon}, \frac{x + i}{\varepsilon}\right]}(y)\) for \(i = 0, 1\). Note that the essential supremum of \(|P_S f|\) is finite (i.e. \(\|P_S f\|_\infty < \infty\)) from the assumption about \(P_S f\). Fix an arbitrarily point \(x_0 \in [0, 1]\). Since

\[
0 \leq x_0 + i - \varepsilon y \leq 1 \quad \text{for all } y \in \left[\frac{x_0 + i - 1}{\varepsilon}, \frac{x_0 + i}{\varepsilon}\right],
\]

we have that for each \(i = 0, 1\),

\[
|P_S f(x_0 + i - \varepsilon y)| \leq \|P_S f\|_\infty \quad \text{for } \lambda\text{-a.e. } y \in \left[\frac{x_0 + i - 1}{\varepsilon}, \frac{x_0 + i}{\varepsilon}\right].
\]

Moreover we have that

\[
[0, 1] \subset \bigcup_{i = 0, 1} \left[\frac{x_0 + i - 1}{\varepsilon}, \frac{x_0 + i}{\varepsilon}\right] = \left[\frac{x_0 - 1}{\varepsilon}, \frac{x_0}{\varepsilon}\right] \cup \left[\frac{x_0}{\varepsilon}, \frac{x_0 + 1}{\varepsilon}\right].
\]
for all $0 < \varepsilon < 1$. Then we have that,

$$\left| \sum_{i=0}^{1} \int_{[0,1]} H_i(x_0, y)dy \right| \leq \sum_{i=0}^{1} \int_{[0,1]} |P_S f(x_0 + i \varepsilon y)| g(y) \mathbf{1}_{\left[\frac{z_n+i-1}{\varepsilon}, \frac{z_n+i}{\varepsilon}\right]}(y)dy$$

$$\leq \sum_{i=0}^{1} \|P_S f\|_{\infty} \int_{[0,1]} g(y) \mathbf{1}_{\left[\frac{z_n+i-1}{\varepsilon}, \frac{z_n+i}{\varepsilon}\right]}(y)dy$$

$$= \|P_S f\|_{\infty} \left\{ \int_{0}^{1} \left[ \frac{1}{2} \sum_{i=0}^{1} \int_{[0,1]} H_i(x, y)dy \right] \right\}^{1/2}$$

(22)

$$= \|P_S f\|_{\infty} \left\{ \int_{[0,1]} g(y)dy \right\} = \|P_S f\|_{\infty}$$

by condition C3. Since $x_0$ was an arbitrary point in $[0,1]$, we have that

$$\|P_\varepsilon f\|_{L^2([0,1])} = \left( \int_{[0,1]} \left[ \sum_{i=0}^{1} \int_{[0,1]} H_i(x, y)dy \right]^{2} \right)^{1/2} \leq \|P_S f\|_{\infty} < \infty.$$ 

This implies that the family $\{P_\varepsilon f, \ 0 < \varepsilon < 1\}$ is uniformly integrable. Then we have that

$$\lim_{\varepsilon \to 0} \sup_{0 < \eta < 1} \int_{[0,\varepsilon]} |P_\eta f|dx = 0$$

by Lemma 4.10 in [8]. Since

$$\int_{[0,\varepsilon]} |P_\varepsilon f|dx \leq \sup_{0 < \eta < 1} \int_{[0,\varepsilon]} |P_\eta f|dx \quad \text{for} \ 0 < \varepsilon < 1,$$

we have that

$$0 \leq \lim_{\varepsilon \to 0} \int_{[0,\varepsilon]} |P_\varepsilon f|dx \leq \lim_{\varepsilon \to 0} \int_{[0,\varepsilon]} |P_\varepsilon f|dx \leq \lim_{\varepsilon \to 0} \sup_{0 < \eta < 1} \int_{[0,\varepsilon]} |P_\eta f|dx = 0$$

by (23). Therefore we have that $\lim_{\varepsilon \to 0} \int_{[0,\varepsilon]} |P_\varepsilon f|dx = 0$. Moreover since the family $\{P_S f\}$ consisting of only one function $P_S f$ is obviously uniformly integrable, we also have that

$$\lim_{\varepsilon \to 0} \int_{[0,\varepsilon]} |P_S f|dx = 0.$$

Therefore we have that

$$\lim_{\varepsilon \to 0} C_1(\varepsilon) \leq \lim_{\varepsilon \to 0} \int_{[0,\varepsilon]} |P_\varepsilon f|dx + \lim_{\varepsilon \to 0} \int_{[0,\varepsilon]} |P_S f|dx = 0.$$
Note that \([0, 1] \subset \left[\frac{x-1}{\varepsilon}, \frac{x+1}{\varepsilon}\right]\) and \([\frac{x}{\varepsilon}, \frac{x+1}{\varepsilon}] \subset (1, \infty)\) for each \(x \in (\varepsilon, 1]\). Hence we have that

\[
P_\varepsilon f(x) = \sum_{i=0}^{1} \int_{\left[\frac{x+i-1}{\varepsilon}, \frac{x+i}{\varepsilon}\right]} P_S f(x + i - \varepsilon) g(y) dy
\]

Thus we have that

\[
C_2(\varepsilon) = \int_{\left(\varepsilon, 1\right]} \left| \int_{[0,1]} P_S f(x - \varepsilon y) g(y) dy - P_S f(x) \right| dx
\]

Without loss of generality, we can assume that \(P_S f(x) = 0\) for all \(x \notin [0, 1]\)
(for example set \(S(x) = x, f(x) = 0\) for all \(x \notin [0, 1]\)). Since \(P_S f\) is an integrable function and the set \(\{P_S f\}\) is compact in \(L^1(\mathbb{R})\), we have that for an arbitrarily small \(\delta > 0\), there exists \(\varepsilon_0\) such that for all \(\varepsilon \leq \varepsilon_0\),

\[
\int_{[0,1]} |P_S f(x - \varepsilon y) - P_S f(x)| \, dx \leq \delta
\]

for each \(y \in [0, 1]\). Thus we have that

\[
C_2(\varepsilon) \leq \int_{[0,1]} \int_{[0,1]} |P_S f(x - \varepsilon y) - P_S f(x)| g(y) dy dx
\]

\[
\leq \delta \int_{[0,1]} g(y) dy = \delta.
\]

Therefore \(\lim_{\varepsilon \to 0} C_2(\varepsilon) = 0\). Then theorem is proved.

\[\square\]

**Proof. Proof of Corollary 2.13**

Since \(P_\varepsilon\) is the Markov operator, we have that

\[
\|P_\varepsilon (f_* - f_\varepsilon)\|_{L^1([0,1])} \leq \|f_* - f_\varepsilon\|_{L^1([0,1])}.
\]

Hence we have that

\[
\|P_\varepsilon f_* - f_*\|_{L^1([0,1])} = \|f_* + P_\varepsilon (f_* - f_\varepsilon) - f_*\|_{L^1([0,1])}
\]

\[
\leq \| f_* - f_\varepsilon\|_{L^1([0,1])} + \|P_\varepsilon (f_* - f_\varepsilon)\|_{L^1([0,1])}
\]

\[
\leq 2 \| f_* - f_\varepsilon\|_{L^1([0,1])} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

Thus \(P_\varepsilon f_*\) converges to \(f_*\) in \(L^1([0,1])\)-norm. On the other hand, from Theorem 2.11, \(P_\varepsilon f_*\) converges to \(P_S f_*\) in \(L^1([0,1])\)-norm. Therefore \(P_S f_* = f_*\).

\[\square\]
5. Appendix

In this section, we give a supplementary explanation of the change of variables theorem for the Lebesgue integral on \( \mathbb{R} \) which is applied in the proof of Proposition 2.2.

**Lemma 5.1.** ([7]) If \( h(t) \geq 0 \) is an integrable function on \([\alpha, \beta]\) such that there exists a increasing function \( H(t) \) satisfying \( H(t) = \int_{c}^{t} h(t) dt \), where \( c \) is a constant. Let \( a = H(\alpha) \), \( b = H(\beta) \). Then we have that

\[
\int_{a}^{b} f(x) dx = \int_{\alpha}^{\beta} f(H(t)) h(t) dt
\]

for all integrable function \( f \) defined on \([a, b]\).

By using Lemm 5.1, we prove the following lemma.

**Lemma 5.2.** Let \( X \) and \( Y \) are independent random variables on a probability space \((\Omega, \mathcal{F}, \mu)\) with values in \([0, 1]\) which satisfy the followings:

1. \( X \) has the density function \( f : [0, 1] \to \mathbb{R} \) with \( f \geq 0 \) such that
   \[
   \int_{[0, 1]} f(x) dx = 1,
   \]
2. \( Y \) has the density function \( g : \mathbb{R} \to \mathbb{R} \) with \( g \geq 0 \) such that
   \[
   \text{supp}(g) := \{ x \in \mathbb{R} : g(x) \neq 0 \} \subset [0, 1] \quad \text{and} \quad \int_{[0, 1]} g(x) dx = 1.
   \]

Then we have that for any Borel set \( A \subset [0, 1] \),

\[
\mu(\{ \omega \in \Omega : X(\omega) + Y(\omega) \in A \}) = \int_{x \in A} \int_{y \in B(x)} f(y) g(x - y) dy dx,
\]

where \( B(x) = \{ y \in [0, 1] : 0 \leq x - y \leq 1 \} \) for each \( x \in [0, 1] \).

**Proof.** Since \( X \) and \( Y \) are independent,

\[
\mu(\{ \omega \in \Omega : X(\omega) + Y(\omega) \in A \}) = \int \int_{\{(x,y)\in[0,1]\times[0,1]: x+y\in A\}} f(x) g(y) dx dy.
\]

Since \( f \) and \( g \) are positive integrable functions on \([0, 1]\), we have

\[
\int \int_{\{(x,y)\in[0,1]\times[0,1]: x+y\in A\}} f(x) g(y) dx dy < \infty,
\]

so, we can apply the Fubini’s theorem to this integral. Indeed, we have that

\[
\int \int_{\{(x,y)\in[0,1]\times[0,1]: x+y\in A\}} f(x) g(y) dy dx = \int_{x \in [0, 1]} \int_{y \in [0, 1]} f(x) g(y) dy dx.
\]
Let $a := x + y$ and $Z(a) := a - x$ for fixed $x \in [0, 1]$. Since $Z(a)$ is absolutely continuous (i.e. $Z(a) = \int_x^a 1(t)\,dt$), we have that by Lemma 5.1 and Fubini’s theorem, we have that

$$
\int_{x \in [0, 1]} \int \{y \in [0, 1] : x + y \in A\} f(x)g(y)\,dy\,dx
= \int_{x \in [0, 1]} \int \{a \in A : 0 \leq a - x \leq 1\} f(x)g(a - x)dadx \quad \text{(change of variables)}
= \int_{x \in [0, 1]} \int \{a \in [0, 1] : 0 \leq a - x \leq 1\} f(x)g(a - x)\mathbf{1}_A(a)dadx
= \int_{a \in [0, 1]} \int \{x \in [0, 1] : 0 \leq a - x \leq 1\} f(x)g(a - x)\mathbf{1}_A(a)dxdx \quad \text{(Fubini’s theorem)}
= \int_{a \in A} \int_{x \in B(a)} f(x)g(a - x)dxdx.
$$

Therefore we have that

$$
\mu (\{\omega \in \Omega : X(\omega) + Y(\omega) \in A\}) = \int_{a \in A} \int_{x \in B(a)} f(x)g(a - x)dxdx.
$$

\[\square\]

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