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RANDOM PERTURBATIONS OF NON-SINGULAR TRANSFORMATIONS ON $[0, 1]$

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ABSTRACT. We consider random perturbations of non-singular measurable transformations S on $[0, 1]$. By using the spectral decomposition theorem of Komornik and Lasota, we prove that the existence of the invariant densities for random perturbations of S . Moreover the densities for random perturbations with small noise strongly converges to the density for Perron-Frobenius operator corresponding to S with respect to $L^1([0, 1])$ -norm.

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Key Words : random dynamical system, spectral decomposition theorem, random perturbations.

1. INTRODUCTION

It is known that every Markov process on a state space can be represented as a random dynamical system ([2]). There are many important Markov models in applications which are analysed as random dynamical systems. We focus on the following random dynamical system with additive noise : Let $S : X \rightarrow X$ be a non-singular measurable transformation on a measurable space $(X, \mathcal{B}, \lambda)$ and let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For a given random variable X_0 and an i.i.d. sequence $\{\xi_n\}_{n \geq 0}$ on Ω with values in X , we define the following Markov process $\{X_n\}_{n \geq 0}$ by

$$(1) \quad X_{n+1}(\omega) := S(X_n(\omega)) + \xi_n(\omega).$$

When $X = \mathbb{R}$, we call the above Markov process $\{X_n(\omega)\}_{n \geq 0}$ *first-order nonlinear autoregressive model (NLAR(1))*. On the other hand, if we let $Q(x, A)$ be a family of transition probabilities (from a point $x \in X$ to a Borel set $A \in \mathcal{B}$), then the Markov process on X defined by the transition probabilities $Q(Sx, A)$ is called a *random perturbation* of the dynamical system (X, S) . In this paper, we consider NLAR(1) on $[0, 1]$, i.e. let $X = [0, 1]$ for (1) and we identify X_n with $X_n - [X_n]$ for all $n \geq 0$, where $[x]$ is the largest integer less than or equal to x . Note that considering NLAR(1) on $[0, 1]$ is coincident with considering a random perturbation of the dynamical system S on $[0, 1]$ in our case.

A stability property of NLAR(1) can be derived from contraction assumptions by Lasota and Mackey ([15]) by using the spectral decomposition theorem of Komornik and Lasota (Theorem 2.5). This theorem is our main method in this paper. Vu Kuok Fong [5] and independently Sine [18] have

showed that the generalization of this spectral decomposition theorem of Komorník-Lasota is a simple corollary of the Jacobs-de Leeuw-Glicksberg theorem. We prove that for any non-singular transformation $S : [0, 1] \rightarrow [0, 1]$, there exists an invariant density of $\{X_n\}_{n \geq 0}$ for NLAR(1) on $[0, 1]$ by using the spectral decomposition theorem of Komorník-Lasota.

In this paper, we also study the limiting distribution of NLAR(1) on $[0, 1]$ with small additive noise (or small perturbations of $([0, 1], S)$) given by

$$(2) \quad X_{n+1}^\varepsilon(\omega) := S(X_n^\varepsilon(\omega)) + \varepsilon \xi_n(\omega) \pmod{1},$$

as $\varepsilon \downarrow 0$, where $X_0^\varepsilon = X_0$. Many authors observe the relation between deterministic dynamical systems and small perturbed random dynamical systems ([4],[6],[9],[11],[16]). For example, in [9], Katok and Kifer considered small random perturbations, where S is an endomorphism of the interval $[0, 1]$ satisfying the conditions of Misiurewicz and small transition probabilities $P^\varepsilon(x, A) = Q^\varepsilon(Sx, A)$ for sufficiently small $\varepsilon > 0$. They proved that the densities of X_n^ε -invariant measures μ^ε converge weakly to a density of the invariant measure μ_S corresponding to S as $\varepsilon \rightarrow 0$ in L^1 topology ([9]).

In [14], Lasota and Mackey showed that the density functions of $\{X_n^\varepsilon\}_{n \geq 0}$ for NLAR(1) (on \mathbb{R}) with small additive noise are given by

$$P_\varepsilon^n f(x) := \int_{\mathbb{R}} g(y) P_S f(x - \varepsilon y) dy,$$

where P_S is the Perron-Frobenius operator corresponding to S , g is the density of $\{\xi_n\}_{n \geq 0}$ and f is the density of X_0 . They prove that

$$(3) \quad \lim_{\varepsilon \rightarrow 0} \|P_\varepsilon f - P_S f\|_{L^1(\mathbb{R})} = 0$$

for all $f \in L^1(\mathbb{R})$ (see [14]). We obtain the same result for NLAR(1) on $[0, 1]$. Moreover since the existence of the densities of X_n^ε -invariant measures are guaranteed by the spectral decomposition theorem of Komorník-Lasota, under certain conditions, we prove that if there exists the limit f_* of the densities of X_n^ε -invariant measures in L^1 as $\varepsilon \downarrow 0$ then the limit function f_* is an invariant density corresponding to S . This implies that we gave the sufficient condition of the existence of an invariant density corresponding to S . On the other hand, in the sense of weak convergence of invariant probability measures for small random perturbations of a dynamical system S , the bounded variation case is first proved by Keller (see the condition S1 in [10]). Afterwards, Young and Baladi considered random perturbations of piecewise C^2 expanding map $S : [0, 1] \rightarrow [0, 1]$ for which there exists the unique invariant density f_* . Indeed, in [1], Young and Baladi proved that for any piecewise C^2 expanding map which has no periodic turning points, there exists invariant densities of small random perturbations and they converges to the invariant density f_* corresponding to S with respect to L^1 -norm as $\varepsilon \rightarrow 0$ (see also [3]). In section 3, we can see that the spectral decomposition theorem of Komorník-Lasota and (3) hold for NLAR(1) on $[0, 1]$ defined by

(1) with respect to intermittent maps S which have an infinite invariant density function.

2. MAIN THEOREMS

2.1. Random perturbations of Dynamical systems. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, where \mathcal{F} denotes a Borel σ -field and μ a probability measure. Let x_0, ξ_0, ξ_1, \dots be random variables on Ω with values in $[0, 1]$ and $S : [0, 1] \rightarrow [0, 1]$ be a non-singular measurable transformation (i.e. $\lambda(S^{-1}(A)) = 0$ for any Borel set $A \subset [0, 1]$ with $\lambda(A) = 0$, where λ is the normalized Lebesgue measure on $[0, 1]$).

Consider the following stochastic process defined by

$$(4) \quad x_{n+1}(\omega) = S(x_n(\omega)) + \xi_n(\omega) \pmod{1}$$

for each $n \geq 0$.

Definition 2.1. We say that a random dynamical system $\{x_n\}_{n \geq 0}$ generated by (4) with respect to $(\Omega, [0, 1], S, x_0, \{\xi_n\}_{n \geq 0})$ is first-order nonlinear autoregressive model on $[0, 1]$ (NLAR(1) on $[0, 1]$) if the following conditions C1-C3 hold :

- C1:** $x_0, \xi_0, \xi_1, \xi_2, \dots$ are independent random variables;
- C2:** x_0 has the density function $f_0 \in D$ (i.e. $\mu(\{\omega : x_0(\omega) \in B\}) = \int_B f_0(x) dx$ for any Borel set $B \subset [0, 1]$), where $D := \{f \in L^1([0, 1]) : f \geq 0 \text{ and } \int_{[0, 1]} f(x) dx = 1\}$;
- C3:** each ξ_n has the same density function $g \in L^1(\mathbb{R})$ such that $g \geq 0$,
 $\text{supp}(g) := \overline{\{x \in [0, 1] : g(x) \neq 0\}} \subseteq [0, 1]$ and $\int_{\mathbb{R}} g(x) dx = 1$.

Under conditions C1-C3, there exists a Markov operator $P : L^1([0, 1]) \rightarrow L^1([0, 1])$ such that

$$(5) \quad \mu_n(A) := \mu(\{\omega : x_n(\omega) \in A\}) = \int_A P^n f_0(x) dx$$

for all Borel set A on $[0, 1]$ and $n \geq 0$.

Proposition 2.2. Let $\{x_n\}_{n \geq 0}$ be a NLAR(1) on $[0, 1]$ generated by (4) with respect to $(\Omega, [0, 1], S, x_0, \{\xi_n\}_{n \geq 0})$. Then there exists a Markov operator $P : L^1([0, 1]) \rightarrow L^1([0, 1])$ defined by

$$(6) \quad Pf(x) = \int_{[0, 1]} f(y) \left(\sum_{i=0}^1 g(x - S(y) + i) \right) dy,$$

which satisfies (5).

In our paper, the spectral decomposition theorem of Komorník and Lasota [13] plays a central role. We introduce the sufficient condition for this theorem :

Definition 2.3. Let (X, \mathcal{F}, ν) be a finite measure space. A linear operator $P : L^1(X, \nu) \rightarrow L^1(X, \nu)$ is constrictive if there exists $\delta > 0$ and $\kappa < 1$ such that for every $f \in D$ there is an integer $n_0(f)$ for which

$$(7) \quad \int_E P^n f(x) \nu(dx) \leq \kappa \quad \text{for all } n \geq n_0(f) \text{ and } E \text{ with } \nu(E) \leq \delta.$$

Remark 2.4. If the space (X, \mathcal{F}, μ) is σ -finite, we can substitute the above condition by the following :

there exists $\delta > 0$, $\kappa < 1$ and a measurable set B with $\nu(B) < \infty$ such that for every $f \in D$ there is an integer $n_0(f)$ for which

$$(8) \quad \int_{(X \setminus B) \cup E} P^n f(x) \nu(dx) \leq \kappa \quad \text{for all } n \geq n_0(f) \text{ and } E \text{ with } \nu(E) \leq \delta.$$

It is easy to see that this condition reduces to that of Definition 2.3 when X is finite and let $X = B$.

Theorem 2.5. (*spectral decomposition theorem* [13]) Let $P : L^1(X, \mathcal{F}, \nu) \rightarrow L^1(X, \mathcal{F}, \nu)$ be a constrictive Markov operator. Then there is an integer r , non negative functions $g_i \in D_0 := \{f \in L^1(X, \mathcal{F}, \nu) : \|f\|_{L^1} = 1, f \geq 0\}$ and $k_i \in L^\infty(X, \mathcal{F}, \nu)$, $i = 1, 2, \dots, r$ and a operator $Q : L^1(X, \mathcal{F}, \nu) \rightarrow L^1(X, \mathcal{F}, \nu)$ such that for every $f \in L^1(X, \mathcal{F}, \nu)$, Pf is represented by the form

$$(9) \quad Pf(x) = \sum_{i=1}^r \lambda_i(f) g_i(x) + Qf,$$

where

$$\lambda_i(f) = \int_X f(x) k_i(x) \nu(dx).$$

Moreover the functions g_i and the operator Q have the following properties:

- $g_i(x) g_j(x) = 0$ for all $i \neq j$.
- For each integer i , there exists a unique integer $\sigma(i)$ such that $Pg_i = g_{\sigma(i)}$. Further $\sigma(i) \neq \sigma(j)$ for $i \neq j$.
- $\lim_{n \rightarrow \infty} \|P^n Qf\| = 0$ for every $f \in L^1(X, \mathcal{F}, \nu)$.

Remark 2.6. The spectral decomposition theorem of Komorník and Lasota holds when the space (X, \mathcal{F}, ν) is σ -finite space and Markov operator is constrictive.

Remark 2.7. If Theorem 2.5 holds for a Markov operator P , then there is an invariant density f^* defined by

$$f_* = \frac{1}{r} \sum_{i=1}^r g_i.$$

Indeed,

$$Pf_* = \frac{1}{r} \sum_{i=1}^r Pgi = \frac{1}{r} \sum_{i=1}^r gi = f_*.$$

Therefore $Pf_* = f_*$.

The following theorem is our main result.

Theorem 2.8. *The Markov operator $P : L^1([0, 1]) \rightarrow L^1([0, 1])$ defined by (6) corresponding to a NLAR(1) on $[0, 1]$ generated by (4) with respect to $(\Omega, [0, 1], S, x_0, \{\xi_n\}_{n \geq 0})$ is constrictive, that is, theorem 2.5 holds for P .*

Moreover when the density of noise $g(x)$ is not zero for all x , we have the following result.

Proposition 2.9. *Let $P : L^1([0, 1]) \rightarrow L^1([0, 1])$ be the Markov operator defined by (6) corresponding to a NLAR(1) on $[0, 1]$ generated by (4) with respect to $(\Omega, [0, 1], S, x_0, \{\xi_n\}_{n \geq 0})$. If $g(x) > 0$ for all $x \in [0, 1]$, then there exists a unique $f_* \in D$ such that $Pf_* = f_*$ and*

$$\lim_{n \rightarrow \infty} \|P^n f - f_*\| = 0 \quad \text{for every } f \in D.$$

Remark 2.10. *A sequence $\{P^n\}_{n \geq 1}$ satisfying (9) is called asymptotically periodic. Proposition 2.9 implies that $r = 1$ for (9). In this case, the sequence $\{P^n\}_{n \geq 1}$ is called asymptotically stable.*

2.2. Small random perturbations of dynamical systems. In this section, we observe limiting behaviour of density functions of a NLAR(1) on $[0, 1]$ generated by (4) with respect to $(\Omega, [0, 1], S, x_0, \{\xi_n\}_{n \geq 0})$ parametrized by $\varepsilon > 0$ as $\varepsilon \rightarrow 0$.

We consider the following first-order nonlinear autoregressive model $\{x_n^\varepsilon\}_{n \geq 0}$ on $[0, 1]$ with respect to $(\Omega, [0, 1], S, x_0, \{\xi_n\}_{n \geq 0})$ parametrized by $\varepsilon > 0$:

$$(10) \quad x_{n+1}^\varepsilon(\omega) = S(x_n^\varepsilon(\omega)) + \varepsilon \xi_n(\omega) \quad \text{for } 0 < \varepsilon < 1,$$

where $x_0^\varepsilon = x_0$.

Since random variables $\varepsilon \xi_n$ have the same density $\frac{1}{\varepsilon} g(\frac{\cdot}{\varepsilon})$, we have the Markov operator $P_\varepsilon : L^1([0, 1]) \rightarrow L^1([0, 1])$ defined by

$$(11) \quad P_\varepsilon f(x) = \frac{1}{\varepsilon} \int_{[0, 1]} f(y) \left(\sum_{i=0}^1 g\left(\frac{x - S(y) + i}{\varepsilon}\right) \right) dy$$

which satisfies that $f_{n+1}^\varepsilon = P_\varepsilon f_n^\varepsilon$, where $\{f_n^\varepsilon\}_{n \geq 0}$ is the sequence of the density function of x_n^ε . Since S is non-singular, there exists the Perron-Frobenius operator $P_S : L^1([0, 1]) \rightarrow L^1([0, 1])$ with respect to $S : [0, 1] \rightarrow$

$[0, 1]$. Hence, if we let $g_{x,i,\varepsilon}(y) := g\left(\frac{x+i-y}{\varepsilon}\right)$, then we have that

$$\begin{aligned} P_\varepsilon f(x) &= \frac{1}{\varepsilon} \int_{[0,1]} f(y) \left(\sum_{i=0}^1 g_{x,i,\varepsilon}(S(y)) \right) dy \\ &= \frac{1}{\varepsilon} \int_{[0,1]} P_S f(y) \left(\sum_{i=0}^1 g_{x,i,\varepsilon}(y) \right) dy \\ &= \frac{1}{\varepsilon} \int_{[0,1]} P_S f(y) \left(\sum_{i=0}^1 g\left(\frac{x+i-y}{\varepsilon}\right) \right) dy \\ &= \sum_{i=0}^1 \int_{\left[\frac{x+i-1}{\varepsilon}, \frac{x+i}{\varepsilon}\right] \cap [0,1]} P_S f(x+i-\varepsilon y) g(y) dy \end{aligned}$$

by condition C3.

We should expect that in some sense $\lim_{\varepsilon \rightarrow 0} P_\varepsilon f(x) = P_S f(x)$.

Let $\|f\|_\infty := \inf\{M : |f(x)| \leq M \text{ for } \lambda\text{-a.e. } x \in [0, 1]\}$, where λ is the normalized Lebesgue measure on $[0, 1]$.

Theorem 2.11. *Let $S : [0, 1] \rightarrow [0, 1]$ be a non-singular measurable transformation and P_ε be the Markov operator defined by (11) corresponding to a NLAR(1) on $[0, 1]$ generated by (10) with respect to $(\Omega, [0, 1], S, x_0, \{\xi_n\}_{n \geq 0})$. Suppose that $\|P_S f\|_\infty < \infty$ for any continuous function f on $[0, 1]$. Then we have that*

$$(12) \quad \lim_{\varepsilon \rightarrow 0} \|P_\varepsilon f - P_S f\|_{L^1([0,1])} = 0$$

for all $f \in L^1([0, 1])$.

Remark 2.12. *There is a big class of dynamical systems $S : [0, 1] \rightarrow [0, 1]$ satisfying $\|P_S f\|_\infty < \infty$ for any continuous function f on $[0, 1]$. For example, piecewise monotonic maps (including unimodal maps) and piecewise convex maps satisfy the assumption of Theorem 2.11.*

It is obviously that $\{P_\varepsilon^n\}_{n \geq 1}$ defined by (11) is asymptotically periodic for each $\varepsilon > 0$. Hence the function f_ε defined by

$$(13) \quad f_\varepsilon(x) = \frac{1}{r(\varepsilon)} \sum_{i=1}^{r(\varepsilon)} g_{i,\varepsilon}(x),$$

where $r(\varepsilon)$ is a positive integer and $g_{i,\varepsilon}(x)$ are density functions depending only on ε , satisfies that $f_\varepsilon \in D$ and $P_\varepsilon f_\varepsilon = f_\varepsilon$. This implies that for each $\varepsilon > 0$, Markov operator P_ε has at least one invariant density.

Corollary 2.13. *Let $S : [0, 1] \rightarrow [0, 1]$ be a non-singular measurable transformation, P_ε be the Markov operator defined by (11) corresponding to a NLAR(1) on $[0, 1]$ generated by (10) with respect to $(\Omega, [0, 1], S, x_0, \{\xi_n\}_{n \geq 0})$ and f_ε be an invariant density for P_ε defined by (13). Suppose that $\|P_S f\|_\infty <$*

∞ for any continuous function f on $[0, 1]$. If there exists an integrable function f_* on $[0, 1]$ such that

$$\lim_{\varepsilon \rightarrow 0} \|f_\varepsilon - f_*\|_{L^1([0,1])} = 0,$$

then f_* is an invariant density for P_S , that is $P_S f_* = f_*$.

Remark 2.14. Corollary 2.13 holds for any continuous piecewise C^2 , piecewise expanding map $S : [0, 1] \rightarrow [0, 1]$ which has no periodic turning points. Indeed, by Theorem 1.1 in [3] (and see Theorem 3 in [1]), there exists a unique absolutely continuous invariant probability measure $\mu_0 = f_* dx$ which satisfies that

$$\lim_{\varepsilon \rightarrow 0} \|f_\varepsilon - f_*\|_{L^1([0,1])} = 0.$$

3. EXAMPLES

It is obviously that Theorem 2.8 holds for all non-singular transformations. We give some examples of non-singular transformations which also satisfy the assumptions of Theorem 2.11.

(1): m -adic transformation [14].

Consider the transformation $S : [0, 1] \rightarrow [0, 1]$ given by

$$Sx = mx \pmod{1},$$

where $m \geq 1$ is an integer. Thus the Perron-Frobenius operator $P_S : L^1([0, 1]) \rightarrow L^1([0, 1])$ corresponding to S is given by

$$P_S f(x) = \frac{1}{m} \sum_{i=0}^{m-1} f\left(\frac{i+x}{m}\right).$$

Since $P_S \mathbf{1} = \mathbf{1}$, the Borel measure on $[0, 1]$ is invariant with respect to the m -adic transformation S . Moreover it is obviously that for any continuous function f on $[0, 1]$, $Pf(x)$ is equal to a continuous function, hence $\|P_S f\|_\infty < \infty$.

(2): Maps with indifferent fixed points with infinite invariant measure [19]

Let $\alpha \in (0, \infty)$ be a real parameter and consider the one-parameter family of maps S_α of the interval $[0, 1]$ onto itself defined by

$$(14) \quad S_\alpha(x) := 2 \frac{e^{\alpha x} - 1}{e^\alpha - 1} \pmod{1}.$$

For every $\alpha > 0$, S_α is piecewise onto and C^∞ -class. When the parameter α varies, the dynamics of the maps changes. Some properties of this family established in [17] are listed below :

- (1) For $\alpha > 0$ with $|S'_\alpha(0)| > 1$, S_α is a piecewise expanding map (see Figure 1). Then there exists the unique absolutely continuous invariant probability measure with respect to the Lebesgue measure on $[0, 1]$ by the Lasota-Yorke theorem.

- (2) For $\alpha > 0$ with $|S'_\alpha(0)| = 1$, S_α admits an indifferent fixed point 0 (see Figure 2). For these maps, there is NO finite absolutely continuous invariant measure. However there exists a σ -finite infinite absolutely continuous invariant measure.
- (3) For $\alpha > 0$ with $|S'_\alpha(0)| < 1$, S_α admits a stable fixed point 0 (see Figure 3). For these maps, almost all points converge to 0 by using the symbolic dynamics with 4-symbols(see [17] more details.). Therefore there is no absolutely continuous invariant measure with respect to the Lebesgue measure.

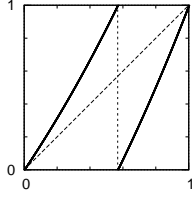


Figure 1. $|S'_\alpha(0)| = 1.5,$
 $\alpha \doteq 0.5502.$

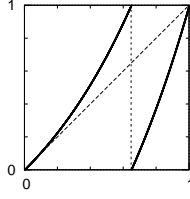


Figure 2. $|S'_\alpha(0)| = 1,$
 $\alpha \doteq 1.2564.$

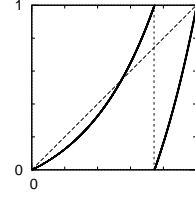


Figure 3. $|S'_\alpha(0)| = 0.5,$
 $\alpha \doteq 2.3366.$

Next, we shall apply our results (Theorem 2.11) to this family. Because $T_\alpha(0) = 0$, $T_\alpha(1) = 2$, where $T_\alpha(x) := 2\frac{e^{\alpha x} - 1}{e^\alpha - 1}$ is monotonic continuous function for every $\alpha > 0$, there exists the unique point $x_\alpha \in (0, 1)$ such that $T_\alpha(x_\alpha) = 1$. Let $I_0 = [0, x_\alpha]$ and $I_1 = [x_\alpha, 1]$. Since C^∞ -extensions of the maps $S_\alpha|_{I_0} : I_0 \rightarrow [0, 1]$ and $S_\alpha|_{I_1} : I_1 \rightarrow [0, 1]$ are one-to-one and onto, there exist the local inverses $u_{\alpha,j} = (S_\alpha|_{I_j})^{-1}$ for $j = 0, 1$, we get

$$(15) \quad u_{\alpha,j}(x) = \frac{1}{\alpha} \log\left(1 + \frac{e^\alpha - 1}{2}(x + j)\right).$$

Thus the Perron-Frobenius operator corresponding to S_α is given by

$$(16) \quad P_{S_\alpha} f = f \circ u_{\alpha,0} \cdot u'_{\alpha,0} + f \circ u_{\alpha,1} \cdot u'_{\alpha,1}.$$

Therefore we have $\|P_{S_\alpha} f(x)\|_\infty < \infty$ for any continuous function f on $[0, 1]$.

4. PROOF

Proof. Proof of Proposition 2.2

We let the density of x_n be denoted by $f_n \in D$ ($n \geq 1$) and desire a relation connecting f_{n+1} and f_n .

We assume that f_n exists for some $n \geq 0$.

Let $\bar{A} = A \setminus \{1\}$ for any Borel set $A \subset [0, 1]$. Note that since $x_{n+1}(\Omega) \subset [0, 1)$ and $S(x_n)$ and ξ_n are independent for all $n \geq 0$, we have that

$$\begin{aligned}
\text{(i): } & \mu(\{\omega \in \Omega : x_{n+1} \in A\}) = \mu(\{\omega \in \Omega : x_{n+1} \in \bar{A}\}), \\
\text{(ii): } & \bigcap_{i=0,1} \{\omega : S(x_n(\omega)) + \xi_n(\omega) \in \bar{A} + i\} \cap \{\omega : S(x_n(\omega)) + \xi_n(\omega) = 2\} = \phi, \\
\text{(iii): } & \\
& \mu(S(x_n(\omega)) + \xi_n(\omega) = 2) = \mu(S(x_n(\omega)) = 1 \text{ and } \xi_n(\omega) = 1) \\
& = \int_{S^{-1}(\{1\})} f_n(x) dx \int_{\{1\}} g(y) dy = 0.
\end{aligned}$$

From (i)-(iii), we have that for any Borel set $A \subset [0, 1]$ and $n \geq 0$,

$$\begin{aligned}
& \mu(\{\omega \in \Omega : x_{n+1} \in A\}) = \mu(\{\omega \in \Omega : x_{n+1} \in \bar{A}\}) \\
& = \mu(\{\omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) \pmod{1} \in \bar{A}\}) \\
& = \mu(\{\omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) \in \bar{A}\}) + \mu(\{\omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) \in \bar{A} + 1\}) \\
& \quad \left(+ \mu(\{\omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) = 2\}) \quad \text{if } 0 \in A \right) \\
& = \int \int_{S(x)+y \in \bar{A}} f_n(x)g(y) dx dy + \int \int_{S(x)+y-1 \in \bar{A}} f_n(x)g(y) dx dy.
\end{aligned}$$

By a change of variables (see Lemma 5.2 in Appendix.), this can be written as

$$\begin{aligned}
\mu(\{\omega \in \Omega : x_{n+1} \in A\}) & = \int_{a \in \bar{A}} \left\{ \int_{B^0(a)} f_n(b)g(a - S(b)) db \right\} da \\
& \quad + \int_{a \in \bar{A}} \left\{ \int_{B^1(a)} f_n(b)g(a - S(b) + 1) db \right\} da,
\end{aligned}$$

where

$$B^0(a) := \{b \in [0, 1] : 0 \leq a - S(b) \leq 1\} = \{b \in [0, 1] : 0 \leq S(b) \leq a\}$$

and

$$B^1(a) := \{b \in [0, 1] : 0 \leq a - S(b) + 1 \leq 1\} = \{b \in [0, 1] : a \leq S(b) \leq 1\}$$

for each $a \in [0, 1]$. By condition C3, we have that

$$\begin{aligned}
g(x - S(y)) &= 0 & \text{for all } y \in \{b \in [0, 1] : x < S(b)\} = [0, 1] \setminus B^0(x) \\
g(x - S(y) + 1) &= 0 & \text{for all } y \in \{b \in [0, 1] : x > S(b)\} = [0, 1] \setminus B^1(x)
\end{aligned}$$

for each $x \in [0, 1]$. Hence we get that

$$\int_{[0,1] \setminus B^0(x)} f_n(y)g(x - S(y)) dy = 0 = \int_{[0,1] \setminus B^1(x)} f_n(y)g(x - S(y) + 1) dy$$

for each $x \in [0, 1]$. This implies that

$$\begin{aligned}
\int_{[0,1]} f_n(y)g(x - S(y)) dy &= \int_{B^0(x)} f_n(y)g(x - S(y)) dy \\
\int_{[0,1]} f_n(y)g(x - S(y) + 1) dy &= \int_{B^1(x)} f_n(y)g(x - S(y) + 1) dy.
\end{aligned}$$

Therefore we have that

$$\begin{aligned} \mu(\{\omega \in \Omega : x_{n+1} \in A\}) &= \int_{a \in \bar{A}} \int_{[0,1]} f_n(b)g(a - S(b))dbda \\ &\quad + \int_{a \in \bar{A}} \int_{[0,1]} f_n(b)g(a - S(b) + 1)dbda. \end{aligned}$$

Since $\{1\}$ is a 1-point set and $h(a) := \int_{b \in [0,1]} f_n(b)g(a - S(b) + i)db \in L^1([0, 1])$, we have that for $i = 0, 1$,

$$\int_{\{1\}} \left\{ \int_{[0,1]} f_n(b)g(a - S(b) + i)db \right\} da = \int_{\{1\}} h(a)da = 0.$$

Then we have that

$$\begin{aligned} \mu(\{\omega \in \Omega : x_{n+1} \in A\}) &= \sum_{i=0}^1 \int_{a \in \bar{A}} \int_{b \in [0,1]} f_n(b)g(a - S(b) + i)dbda \\ &= \sum_{i=0}^1 \int_{a \in A} \int_{b \in [0,1]} f_n(b)g(a - S(b) + i)dbda. \end{aligned}$$

Therefore using the fact that A was an arbitrary Borel set on $[0, 1]$, we get the density f_{n+1} of x_{n+1} defined by

$$f_{n+1}(x) = \sum_{i=0}^1 \int_{[0,1]} f_n(y)g(x - S(y) + i)dy \quad \text{a.e. } x \in [0, 1].$$

On the other hand, we get that

$$\int_{x \in [0,1]} \sum_{i=0}^1 g(x - S(y) + i)dx = \int_{[0,1]} g(x)dx = 1 \quad \text{for } \forall y \in [0, 1]$$

by condition C3. Then by Fubini's theorem, we have that

$$\begin{aligned} \int_{[0,1]} f_{n+1}(x)dx &= \sum_{i=0}^1 \int_{y \in [0,1]} \left\{ \int_{x \in [0,1]} f_n(y)g(x - S(y) + i)dx \right\} dy \\ &= \int_{y \in [0,1]} f_n(y)dy = 1. \end{aligned}$$

Moreover $f_{n+1} \geq 0$ because of the positivity of g and f_n . Therefore if x_n has the density $f_n \in D$, then f_{n+1} also have to exist in D .

From this fact, we can define a linear operator $P : L^1([0, 1]) \rightarrow L^1([0, 1])$ by

$$Pf(x) = \int_{y \in [0,1]} f(y) \left(\sum_{i=0}^1 g(x - S(y) + i) \right) dy$$

which satisfies that

$$f_{n+1} = Pf_n \quad \text{a.e.}$$

for all $n \geq 0$. Next we shall show that $P : L^1([0, 1]) \rightarrow L^1([0, 1])$ is a Markov operator, that is, P is a linear operator which satisfies that $Pf \geq 0$ and $\|Pf\|_{L^1([0,1])} = \|f\|_{L^1([0,1])}$ for any $f \in L^1([0, 1])$ with $f \geq 0$. It is easy to see that P is a positive linear operator on $L^1([0, 1])$ because g is positive. Moreover we have that for $f \in L^1([0, 1])$ with $f \geq 0$ by the Fubini's theorem,

$$\begin{aligned} \|Pf\|_{L^1([0,1])} &:= \int_{[0,1]} Pf(x)dx \\ &= \int_{x \in [0,1]} \int_{y \in [0,1]} f_n(y) \left(\sum_{i=0}^1 g(x - S(y) + i) \right) dy dx \\ &= \int_{x \in [0,1]} \sum_{i=0}^1 g(x - S(y) + i) \left\{ \int_{[0,1]} f(y) dy \right\} dx \\ &= \int_{[0,1]} f(y) dy = \|f\|_{L^1([0,1])}. \end{aligned}$$

Therefore P is a Markov operator. □

Proof. Proof of Theorem 2.8

From the spectral decomposition theorem by Komorník and Lasota [14], it is enough to show that P is constrictive : there exists a $\delta > 0$ and $\kappa < 1$ such that for every $f \in D$ there is an integer $n_0(f)$ for which

$$\int_B P^n f(x) dx \leq \kappa \quad \text{for all } n \geq n_0(f) \text{ and } B \subset [0, 1] \text{ with } \lambda(B) \leq \delta,$$

where λ is the normalized Lebesgue measure on $[0, 1]$.

Since g is the integrable function on \mathbb{R} supported in $[0, 1]$, for any $\varepsilon > 0$, there exists $0 < \delta(\varepsilon) \leq 1$ such that whenever $\lambda(A) \leq \delta(\varepsilon)$,

$$\int_A g(x) dx \leq \varepsilon.$$

Take arbitrary $0 < \varepsilon < 1$, hence there exists $\delta(\varepsilon) > 0$ which satisfies $\int_A g(x) dx \leq \frac{\varepsilon}{2}$ for any Borel set $A \subset [0, 1]$ with $\lambda(A) \leq \delta(\varepsilon)$. Thus we have that for each $f \in D$ and $n \geq 1$,

$$\begin{aligned} \int_A P^n f(x) dx &= \int_A \int_{[0,1]} P^{n-1} f(y) \left(\sum_{i=0}^1 g(x - S(y) + i) \right) dy dx \\ &= \int_{[0,1]} \left\{ \sum_{i=0}^1 \int_{A-S(y)+i} g(x) dx \right\} P^{n-1} f(y) dy. \end{aligned}$$

Let $\bar{\lambda}$ be the Lebesgue measure on \mathbb{R} . Since $\bar{\lambda}(A - S(y) + i) = \bar{\lambda}(A) = \lambda(A) \leq \delta(\varepsilon)$ for each $y \in [0, 1]$ and $i = 0, 1$, we obtain that

$$(17) \quad \int_A P^n f(x) dx \leq \varepsilon \int_{[0,1]} P^{n-1} f(y) dy = \varepsilon \quad \text{for all } n \geq 1,$$

which implies that P is constrictive. □

Proof. Proof of Proposition 2.9

From the theorem 5.6.1 in [14], it is enough to show that there exists a set $A \subset [0, 1]$ of nonzero measure $\lambda(A) > 0$ with the property that for every $f \in D$, there is an integer $n_0(f)$ such that

$$(18) \quad P^n f(x) > 0 \quad \text{for a.e. } x \in A \quad \text{and} \quad \text{for all } n \geq n_0(f).$$

Let $f \in D$ be arbitrary. From the assumption about g , there exists a positive number $0 < \varepsilon < 1$ which satisfies that there exists $\delta(\varepsilon) > 0$ such that for all $\lambda(A) \leq \delta(\varepsilon)$, $\int_A g(x) dx \leq \frac{\varepsilon}{2}$. Take an arbitrarily $0 < \delta < 1$ with $1 - \delta < \delta(\varepsilon)$. Since $\lambda((\delta - S(y) + i, 1 - S(y) + i]) = 1 - \delta \leq \delta(\varepsilon)$ for each $y \in [0, 1]$ and $i = 0, 1$, we have that

$$\begin{aligned} & \int_{\delta < x \leq 1} P^n f(x) dx \\ &= \int_{[0, 1]} \left\{ \sum_{i=0}^1 \int_{(\delta - S(y) + i, 1 - S(y) + i]} g(x) dx \right\} P^{n-1} f(y) dy \leq \varepsilon \end{aligned}$$

for all $n \geq 1$. From this inequality, we have that

$$(19) \quad \begin{aligned} \int_{0 \leq y \leq \delta} P^n f(y) dy &= \int_{[0, 1]} P^n f(y) dy - \int_{\delta < y \leq 1} P^n f(y) dy \\ &\geq 1 - \varepsilon > 0 \end{aligned}$$

for all $n \geq 1$.

On the other hand, we have that

$$(20) \quad \begin{aligned} P^{n+1} f(x) &= \int_{[0, 1]} P^n f(y) \left(\sum_{i=0}^1 g(x - S(y) + i) \right) dy \\ &\geq \int_{0 \leq y \leq \delta} P^n f(y) \left(\sum_{i=0}^1 g(x - S(y) + i) \right) dy. \end{aligned}$$

From the assumption about g , we have that

$$(21) \quad (g(x - S(y)) + g(x - S(y) + 1)) > 0 \quad \text{for all } x \in [0, 1] \text{ and } 0 \leq y \leq \delta.$$

From (19) and (21), we have that for a.e. $x \in [0, 1]$,

$$P^n f(y) \left(\sum_{i=0}^1 g(x - S(y) + i) \right) \quad \text{for } n \geq 1$$

as a function of y , does not vanish in $\{0 \leq y \leq \delta\}$. As a consequence, inequality (20) implies (18) with respect to the set $[0, 1]$, thus completing the proof of the proposition. □

Proof. Proof of Theorem 2.11

Since the set of continuous functions on $[0, 1]$ is dense in $L^1([0, 1])$ and P_ε, P_S are Markov operators, it is enough to prove the theorem for continuous functions on $[0, 1]$. Indeed, for any $f \in L^1([0, 1])$ and $\eta > 0$, there exists a continuous function f_η on $[0, 1]$ such that $\|f - f_\eta\|_{L^1([0,1])} \leq \eta$. Thus if we have that $\lim_{\varepsilon \rightarrow 0} \|P_\varepsilon f_\eta - P_S f_\eta\|_{L^1([0,1])} = 0$, then we have that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \|P_\varepsilon f - P_S f\|_{L^1([0,1])} &= \lim_{\varepsilon \rightarrow 0} \|P_\varepsilon(f - f_\eta) - P_S(f - f_\eta) + P_\varepsilon f_\eta - P_S f_\eta\|_{L^1([0,1])} \\ &\leq 2\|f - f_\eta\|_{L^1([0,1])} + \lim_{\varepsilon \rightarrow 0} \|P_\varepsilon f_\eta - P_S f_\eta\|_{L^1([0,1])} \\ &\leq 2\eta. \end{aligned}$$

From the fact that η was an arbitrary positive number, we have that $\lim_{\varepsilon \rightarrow 0} \|P_\varepsilon f - P_S f\|_{L^1([0,1])} = 0$.

Fix an arbitrarily continuous function f on $[0, 1]$. We split the integral into two parts,

$$\begin{aligned} \|P_\varepsilon f - P_S f\|_{L^1([0,1])} &= \int_{[0,\varepsilon]} |P_\varepsilon f - P_S f| dx + \int_{(\varepsilon,1]} |P_\varepsilon f - P_S f| dx \\ &= C_1(\varepsilon) + C_2(\varepsilon) \quad \text{for } 0 < \varepsilon < 1. \end{aligned}$$

Firstly, we consider $C_1(\varepsilon)$. Let $H_i(x, y) := P_S f(x+i-\varepsilon y)g(y)\mathbf{1}_{[\frac{x+i-1}{\varepsilon}, \frac{x+i}{\varepsilon}]}(y)$ for $i = 0, 1$. Note that the essential supremum of $|P_S f|$ is finite (i.e. $\|P_S f\|_\infty < \infty$) from the assumption about $P_S f$. Fix an arbitrarily point $x_0 \in [0, 1]$. Since

$$0 \leq x_0 + i - \varepsilon y \leq 1 \quad \text{for all } y \in \left[\frac{x_0 + i - 1}{\varepsilon}, \frac{x_0 + i}{\varepsilon} \right],$$

we have that for each $i = 0, 1$,

$$|P_S f(x_0 + i - \varepsilon y)| \leq \|P_S f\|_\infty \quad \text{for } \lambda\text{-a.e. } y \in \left[\frac{x_0 + i - 1}{\varepsilon}, \frac{x_0 + i}{\varepsilon} \right].$$

Moreover we have that

$$[0, 1] \subset \bigcup_{i=\{0,1\}} \left[\frac{x_0 + i - 1}{\varepsilon}, \frac{x_0 + i}{\varepsilon} \right] = \left[\frac{x_0 - 1}{\varepsilon}, \frac{x_0}{\varepsilon} \right] \cup \left[\frac{x_0}{\varepsilon}, \frac{x_0 + 1}{\varepsilon} \right]$$

for all $0 < \varepsilon < 1$. Then we have that,

$$\begin{aligned}
\left| \sum_{i=\{0,1\}} \int_{[0,1]} H_i(x_0, y) dy \right| &\leq \sum_{i=\{0,1\}} \int_{[0,1]} |P_S f(x_0 + i - \varepsilon y)| g(y) \mathbf{1}_{[\frac{x_0+i-1}{\varepsilon}, \frac{x_0+i}{\varepsilon}]}(y) dy \\
&\leq \sum_{i=\{0,1\}} \|P_S f\|_\infty \int_{[0,1]} g(y) \mathbf{1}_{[\frac{x_0+i-1}{\varepsilon}, \frac{x_0+i}{\varepsilon}]}(y) dy \\
&= \|P_S f\|_\infty \left\{ \int_{\bigcup_{i=0}^1 [\frac{x_0+i-1}{\varepsilon}, \frac{x_0+i}{\varepsilon}] \cap [0,1]} g(y) dy \right\} \\
(22) \qquad \qquad \qquad &= \|P_S f\|_\infty \left\{ \int_{[0,1]} g(y) dy \right\} = \|P_S f\|_\infty
\end{aligned}$$

by condition C3. Since x_0 was an arbitrary point in $[0, 1]$, we have that

$$\begin{aligned}
\|P_\varepsilon f\|_{L^2([0,1])} &= \left(\int_{[0,1]} \left| \sum_{i=0}^1 \int_{[0,1]} H_i(x, y) dy \right|^2 dx \right)^{1/2} \\
&\leq \|P_S f\|_\infty < \infty.
\end{aligned}$$

This implies that the family $\{P_\varepsilon f, 0 < \varepsilon < 1\}$ is uniformly integrable. Then we have that

$$(23) \qquad \lim_{\varepsilon \rightarrow 0} \sup_{0 < \eta < 1} \int_{[0, \varepsilon]} |P_\eta f| dx = 0$$

by Lemma 4.10 in [8]. Since

$$\int_{[0, \varepsilon]} |P_\varepsilon f| dx \leq \sup_{0 < \eta < 1} \int_{[0, \varepsilon]} |P_\eta f| dx \quad \text{for } 0 < \varepsilon < 1,$$

we have that

$$0 \leq \underline{\lim}_{\varepsilon \rightarrow 0} \int_{[0, \varepsilon]} |P_\varepsilon f| dx \leq \overline{\lim}_{\varepsilon \rightarrow 0} \int_{[0, \varepsilon]} |P_\varepsilon f| dx \leq \overline{\lim}_{\varepsilon \rightarrow 0} \sup_{0 < \eta < 1} \int_{[0, \varepsilon]} |P_\eta f| dx = 0$$

by (23). Therefore we have that $\lim_{\varepsilon \rightarrow 0} \int_{[0, \varepsilon]} |P_\varepsilon f| dx = 0$. Moreover since the family $\{P_S f\}$ consisting of only one function $P_S f$ is obviously uniformly integrable, we also have that

$$\lim_{\varepsilon \rightarrow 0} \int_{[0, \varepsilon]} |P_S f| dx = 0.$$

Therefore we have that

$$(24) \qquad \lim_{\varepsilon \rightarrow 0} C_1(\varepsilon) \leq \lim_{\varepsilon \rightarrow 0} \int_{[0, \varepsilon]} |P_\varepsilon f| dx + \lim_{\varepsilon \rightarrow 0} \int_{[0, \varepsilon]} |P_S f| dx = 0.$$

Note that $[0, 1] \subset [\frac{x-1}{\varepsilon}, \frac{x}{\varepsilon}]$ and $[\frac{x}{\varepsilon}, \frac{x+1}{\varepsilon}] \subset (1, \infty)$ for each $x \in (\varepsilon, 1]$. Hence we have that

$$\begin{aligned} P_\varepsilon f(x) &= \sum_{i=0}^1 \int_{[\frac{x+i-1}{\varepsilon}, \frac{x+i}{\varepsilon}]} P_S f(x+i-\varepsilon y) g(y) dy \\ &= \int_{[0,1]} P_S f(x-\varepsilon y) g(y) dy. \end{aligned}$$

Thus we have that with respect to $C_2(\varepsilon)$,

$$\begin{aligned} C_2(\varepsilon) &= \int_{(\varepsilon,1]} \left| \int_{[0,1]} P_S f(x-\varepsilon y) g(y) dy - P_S f(x) \right| dx \\ &= \int_{(\varepsilon,1]} \left| \int_{[0,1]} [P_S f(x-\varepsilon y) - P_S f(x)] g(y) dy \right| dx. \end{aligned}$$

Without loss of generality, we can assume that $P_S f(x) = 0$ for all $x \notin [0, 1]$ (for example set $S(x) = x$, $f(x) = 0$ for all $x \notin [0, 1]$). Since $P_S f$ is an integrable function and the set $\{P_S f\}$ is compact in $L^1(\mathbb{R})$, we have that for an arbitrarily small $\delta > 0$, there exists ε_0 such that for all $\varepsilon \leq \varepsilon_0$,

$$\int_{[0,1]} |P_S f(x-\varepsilon y) - P_S f(x)| dx \leq \delta$$

for each $y \in [0, 1]$. Thus we have that

$$\begin{aligned} C_2(\varepsilon) &\leq \int_{[0,1]} \int_{[0,1]} |P_S f(x-\varepsilon y) - P_S f(x)| g(y) dy dx \\ &\leq \delta \int_{[0,1]} g(y) dy = \delta. \end{aligned}$$

Therefore $\lim_{\varepsilon \rightarrow 0} C_2(\varepsilon) = 0$. Then theorem is proved. \square

Proof. Proof of Corollary 2.13

Since P_ε is the Markov operator, we have that

$$\|P_\varepsilon(f_* - f_\varepsilon)\|_{L^1([0,1])} \leq \|f_* - f_\varepsilon\|_{L^1([0,1])}.$$

Hence we have that

$$\begin{aligned} \|P_\varepsilon f_* - f_*\|_{L^1([0,1])} &= \|f_\varepsilon + P_\varepsilon(f_* - f_\varepsilon) - f_*\|_{L^1([0,1])} \\ &\leq \|f_\varepsilon - f_*\|_{L^1([0,1])} + \|P_\varepsilon(f_* - f_\varepsilon)\|_{L^1([0,1])} \\ &\leq 2\|f_\varepsilon - f_*\|_{L^1([0,1])} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Thus $P_\varepsilon f_*$ converges to f_* in $L^1([0, 1])$ -norm. On the other hand, from Theorem 2.11, $P_\varepsilon f_*$ converges to $P_S f_*$ in $L^1([0, 1])$ -norm. Therefore $P_S f_* = f_*$. \square

5. APPENDIX

In this section, we give a supplementary explanation of the change of variables theorem for the Lebesgue integral on \mathbb{R} which is applied in the proof of Proposition 2.2.

Lemma 5.1. ([7]) *If $h(t) \geq 0$ is an integrable function on $[\alpha, \beta]$ such that there exists a increasing function $H(t)$ satisfying $H(t) = \int_c^t h(t)dt$, where c is a constant. Let $a = H(\alpha), b = H(\beta)$. Then we have that*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(H(t))h(t)dt$$

for all integrable function f defined on $[a, b]$.

By using Lemm 5.1, we prove the following lemma.

Lemma 5.2. *Let X and Y are independent random variables on a probability space $(\Omega, \mathcal{F}, \mu)$ with values in $[0, 1]$ which satisfy the followings:*

- (1) *X has the density function $f : [0, 1] \rightarrow \mathbb{R}$ with $f \geq 0$ such that*

$$\int_{[0,1]} f(x)dx = 1,$$

- (2) *Y has the density function $g : \mathbb{R} \rightarrow \mathbb{R}$ with $g \geq 0$ such that*

$$\text{supp}(g) := \overline{\{x \in \mathbb{R} : g(x) \neq 0\}} \subset [0, 1] \quad \text{and} \quad \int_{[0,1]} g(x)dx = 1.$$

Then we have that for any Borel set $A \subset [0, 1]$,

$$\mu(\{\omega \in \Omega : X(\omega) + Y(\omega) \in A\}) = \int_{x \in A} \int_{y \in B(x)} f(y)g(x-y)dydx,$$

where $B(x) = \{y \in [0, 1] : 0 \leq x - y \leq 1\}$ for each $x \in [0, 1]$.

Proof. Since X and Y are independent,

$$\mu(\{\omega \in \Omega : X(\omega) + Y(\omega) \in A\}) = \int \int_{\{(x,y) \in [0,1] \times [0,1] : x+y \in A\}} f(x)g(y)dx dy.$$

Since f and g are positive integrable functions on $[0, 1]$, we have

$$\int \int_{\{(x,y) \in [0,1] \times [0,1] : x+y \in A\}} f(x)g(y)dx dy < \infty,$$

so, we can apply the Fubini's theorem to this integral. Indeed, we have that

$$\int \int_{\{(x,y) \in [0,1] \times [0,1] : x+y \in A\}} f(x)g(y)dx dy = \int_{x \in [0,1]} \int_{\{y \in [0,1] : x+y \in A\}} f(x)g(y)dy dx.$$

Let $a := x + y$ and $Z(a) := a - x$ for fixed $x \in [0, 1]$. Since $Z(a)$ is absolutely continuous (i.e. $Z(a) = \int_x^a \mathbf{1}(t)dt$), we have that by Lemma 5.1 and Fubini's theorem, we have that

$$\begin{aligned} & \int_{x \in [0, 1]} \int_{\{y \in [0, 1]: x+y \in A\}} f(x)g(y)dydx \\ &= \int_{x \in [0, 1]} \int_{\{a \in A: 0 \leq a-x \leq 1\}} f(x)g(a-x)dadx \quad (\text{change of variables}) \\ &= \int_{x \in [0, 1]} \int_{\{a \in [0, 1]: 0 \leq a-x \leq 1\}} f(x)g(a-x)\mathbf{1}_A(a)dadx \\ &= \int_{a \in [0, 1]} \int_{\{x \in [0, 1]: 0 \leq a-x \leq 1\}} f(x)g(a-x)\mathbf{1}_A(a)dxda \quad (\text{Fubini's theorem}) \\ &= \int_{a \in A} \int_{x \in B(a)} f(x)g(a-x)dxda. \end{aligned}$$

Therefore we have that

$$\mu(\{\omega \in \Omega : X(\omega) + Y(\omega) \in A\}) = \int_{a \in A} \int_{x \in B(a)} f(x)g(a-x)dxda.$$

□

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