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Random Perturbations of Non-Singular Transformations on $[0, 1]$

Yukiko Iwata$^1$, Tomohiro Ogihara$^2$

Abstract. We consider random perturbations of non-singular measurable transformations $S$ on $[0, 1]$. By using the spectral decomposition theorem of Komornik and Lasota, we prove that the existence of the invariant densities for random perturbations of $S$. Moreover the densities for random perturbations with small noise strongly converges to the density for Perron-Frobenius operator corresponding to $S$ with respect to $L^1([0, 1])$-norm.

AMS Classification: 34E10, 37A50, 37A30, 37H99, 60E05.
Key Words: random dynamical system, spectral decomposition theorem, random perturbations.

1. Introduction

It is known that every Markov process on a state space can be represented as a random dynamical system ([2]). There are many important Markov models in applications which are analysed as random dynamical systems. We focus on the following random dynamical system with additive noise: Let $S : X \to X$ be a non-singular measurable transformation on a measurable space $(X, \mathcal{B}, \lambda)$ and let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For a given random variable $X_0$ and an i.i.d. sequence $\{\xi_n\}_{n \geq 0}$ on $\Omega$ with values in $X$, we define the following Markov process $\{X_n\}_{n \geq 0}$ by

$$X_{n+1}(\omega) := S(X_n(\omega)) + \xi_n(\omega).$$

When $X = \mathbb{R}$, we call the above Markov process $\{X_n(\omega)\}_{n \geq 0}$ first-order nonlinear autoregressive model (NLAR(1)). On the other hand, if we let $Q(x, A)$ be a family of transition probabilities (from a point $x \in X$ to a Borel set $A \in \mathcal{B}$), then the Markov process on $X$ defined by the transition probabilities $Q(Sx, A)$ is called a random perturbation of the dynamical system $(X, S)$. In this paper, we consider NLAR(1) on $[0, 1]$, i.e. let $X = [0, 1]$ for (1) and we identify $X_n$ with $X_n - [X_n]$ for all $n \geq 0$, where $[x]$ is the largest integer less than or equal to $x$. Note that considering NLAR(1) on $[0, 1]$ is coincident with considering a random perturbation of the dynamical system $S$ on $[0, 1]$ in our case.

A stability property of NLAR(1) can be derived from contraction assumptions by Lasota and Mackey ([15]) by using the spectral decomposition theorem of Komorník and Lasota (Theorem 2.5). This theorem is our main method in this paper. Vu Kuok Fong [5] and independently Sine [18] have
showed that the generalization of this spectral decomposition theorem of Komornik-Lasota is a simple corollary of the Jacobs-de Leeuw-Glicksberg theorem. We prove that for any non-singular transformation \( S : [0, 1] \to [0, 1] \), there exists an invariant density of \( \{X_n\}_{n \geq 0} \) for NLAR(1) on \([0, 1] \) by using the spectral decomposition theorem of Komornik-Lasota.

In this paper, we also study the limiting distribution of NLAR(1) on \([0, 1] \) with small additive noise (or small perturbations of \(([0, 1], S)\)) given by

\[
X_{n+1}(\omega) := S(X_n(\omega)) + \varepsilon \xi_n(\omega) \mod 1, \tag{2}
\]

as \( \varepsilon \downarrow 0 \), where \( X_0 = X_0 \). Many authors observe the relation between deterministic dynamical systems and small perturbed random dynamical systems\([4],[6],[9],[11],[16]\). For example, in \[9\], Katok and Kifer considered small random perturbations, where \( S \) is an endomorphism of the interval \([0, 1]\) satisfying the conditions of Misiurewicz and small transition probabilities \( P^\varepsilon(x, A) = Q^\varepsilon(Sx, A) \) for sufficiently small \( \varepsilon > 0 \). They proved that the densities of \( X_n^\varepsilon \)-invariant measures \( \mu^\varepsilon \) converge weakly to a density of the invariant measure \( \mu_S \) corresponding to \( S \) as \( \varepsilon \to 0 \) in \( L^1 \) topology \([9]\).

In \[14\], Lasota and Mackey showed that the density functions of \( \{X_n^\varepsilon\}_{n \geq 0} \) for NLAR(1) (on \( \mathbb{R} \)) with small additive noise are given by

\[
P^\varepsilon f(x) := \int f(y)P_Sf(x - \varepsilon y)dy,
\]

where \( P_S \) is the Perron-Frobenius operator corresponding to \( S \), \( g \) is the density of \( \{\xi_n\}_{n \geq 0} \) and \( f \) is the density of \( X_0 \). They prove that

\[
\lim_{\varepsilon \to 0} \|P_\varepsilon f - P_S f\|_{L^1(\mathbb{R})} = 0 \tag{3}
\]

for all \( f \in L^1(\mathbb{R}) \) (see \([14]\)). We obtain the same result for NLAR(1) on \([0, 1] \). Moreover since the existence of the densities of \( X_n^\varepsilon \)-invariant measures are guaranteed by the spectral decomposition theorem of Komornik-Lasota, under certain conditions, we prove that if there exists the limit \( f_* \) of the densities of \( X_n^\varepsilon \)-invariant measures in \( L^1 \) as \( \varepsilon \downarrow 0 \) then the limit function \( f_* \) is an invariant density corresponding to \( S \). This implies that we gave the sufficient condition of the existence of an invariant density corresponding to \( S \). On the other hand, in the sense of weak convergence of invariant probability measures for small random perturbations of a dynamical system \( S \), the bounded variation case is first proved by Keller (see the condition S1 in \([10]\)). Afterwards, Young and Baladi considered random perturbations of piecewise \( C^2 \) expanding map \( S : [0, 1] \to [0, 1] \) for which there exists the unique invariant density \( f_* \). Indeed, in \([1]\), Young and Baladi proved that for any piecewise \( C^2 \) expanding map which has no periodic turning points, there exists invariant densities of small random perturbations and they converges to the invariant density \( f_* \) corresponding to \( S \) with respect to \( L^1 \)-norm as \( \varepsilon \to 0 \) (see also \([3]\)). In section 3, we can see that the spectral decomposition theorem of Komornik-Lasota and (3) hold for NLAR(1) on \([0, 1] \) defined by
2. Main theorems

2.1. Random perturbations of Dynamical systems. Let $\Omega, \mathcal{F}, \mu$ be a probability space, where $\mathcal{F}$ denotes a Borel $\sigma$-field and $\mu$ a probability measure. Let $x_0, \xi_0, \xi_1, \cdots$ be random variables on $\Omega$ with values in $[0, 1]$ and $S : [0, 1] \to [0, 1]$ be a non-singular measurable transformation (i.e. $\lambda(S^{-1}(A)) = 0$ for any Borel set $A \subset [0, 1]$ with $\lambda(A) = 0$, where $\lambda$ is the normalized Lebesgue measure on $[0, 1]$).

Consider the following stochastic process defined by

$$x_{n+1}(\omega) = S(x_n(\omega)) + \xi_n(\omega) \pmod{1}$$

for each $n \geq 0$.

**Definition 2.1.** We say that a random dynamical system \{\(x_n\)\}_{n \geq 0} generated by (4) with respect to $(\Omega, [0, 1], S, x_0, \{\xi_n\}_{n \geq 0})$ is first-order nonlinear autoregressive model on $[0, 1]$ (NLAR(1) on $[0, 1]$) if the following conditions C1-C3 hold:

- **C1:** $x_0, \xi_0, \xi_1, \cdots$ are independent random variables;
- **C2:** $x_0$ has the density function $f_0 \in D$ (i.e. $\mu(\{\omega : x_0(\omega) \in B\}) = \int_B f_0(x) dx$ for any Borel set $B \subset [0, 1]$), where $D := \{f \in L^1([0, 1]) : f \geq 0$ and $\int_{[0,1]} f(x) dx = 1\}$;
- **C3:** each $\xi_n$ has the same density function $g \in L^1(\mathbb{R})$ such that $g \geq 0$, $\text{supp}(g) := \{x \in [0,1] : g(x) \neq 0\} \subseteq [0,1]$ and $\int_{\mathbb{R}} g(x) dx = 1$.

Under conditions C1-C3, there exists a Markov operator $P : L^1([0, 1]) \to L^1([0, 1])$ such that

$$\mu_n(A) := \mu(\{\omega : x_n(\omega) \in A\}) = \int_A P^n f_0(x) dx$$

for all Borel set $A$ on $[0, 1]$ and $n \geq 0$.

**Proposition 2.2.** Let \{\(x_n\)\}_{n \geq 0} be a NLAR(1) on $[0, 1]$ generated by (4) with respect to $(\Omega, [0, 1], S, x_0, \{\xi_n\}_{n \geq 0})$. Then there exists a Markov operator $P : L^1([0, 1]) \to L^1([0, 1])$ defined by

$$Pf(x) = \int_{[0,1]} f(y) \left( \sum_{i=0}^{1} g(x - S(y) + i) \right) dy,$$

which satisfies (5).

In our paper, the spectral decomposition theorem of Komorník and Lasota [13] plays a central role. We introduce the sufficient condition for this theorem:
Definition 2.3. Let \((X, \mathcal{F}, \nu)\) be a finite measure space. A linear operator \(P : L^1(X, \nu) \to L^1(X, \nu)\) is constrictive if there exists \(\delta > 0\) and \(\kappa < 1\) such that for every \(f \in D\) there is an integer \(n_0(f)\) for which

\[
\int_E P^n f(x) \nu(dx) \leq \kappa \quad \text{for all } n \geq n_0(f) \text{ and } E \text{ with } \nu(E) \leq \delta.
\]

Remark 2.4. If the space \((X, \mathcal{F}, \mu)\) is \(\sigma\)-finite, we can substitute the above condition by the following:

there exists \(\delta > 0\), \(\kappa < 1\) and a measurable set \(B\) with \(\nu(B) < \infty\) such that for every \(f \in D\) there is an integer \(n_0(f)\) for which

\[
\int_{(X \setminus B) \cup E} P^n f(x) \nu(dx) \leq \kappa \quad \text{for all } n \geq n_0(f) \text{ and } E \text{ with } \nu(E) \leq \delta.
\]

It is easy to see that this condition reduces to that of Definition 2.3 when \(X\) is finite and let \(X = B\).

Theorem 2.5. (spectral decomposition theorem [13]) Let \(P : L^1(X, \mathcal{F}, \nu) \to L^1(X, \mathcal{F}, \nu)\) be a constrictive Markov operator. Then there is an integer \(r\), non-negative functions \(g_i \in D_0 := \{ f \in L^1(X, \mathcal{F}, \nu) : \| f \|_{L^1} = 1, f \geq 0 \}\) and \(k_i \in L^\infty(X, \mathcal{F}, \nu), i = 1, 2, \ldots, r\) and a operator \(Q : L^1(X, \mathcal{F}, \nu) \to L^1(X, \mathcal{F}, \nu)\) such that for every \(f \in L^1(X, \mathcal{F}, \nu)\), \(Pf\) is represented by the form

\[
Pf(x) = \sum_{i=1}^{r} \lambda_i(f) g_i(x) + Qf,
\]

where

\[
\lambda_i(f) = \int_X f(x) k_i(x) \nu(dx).
\]

Moreover the functions \(g_i\) and the operator \(Q\) have the following properties:

- \(g_i(x)g_j(x) = 0\) for all \(i \neq j\).
- For each integer \(i\), there exists an unique integer \(\sigma(i)\) such that \(Pg_i = g_{\sigma(i)}\). Further \(\sigma(i) \neq \sigma(j)\) for \(i \neq j\).
- \(\lim_{n \to \infty} \| P^n Qf \| = 0\) for every \(f \in L^1(X, \mathcal{F}, \nu)\).

Remark 2.6. The spectral decomposition theorem of Komornik and Lasota holds when the space \((X, \mathcal{F}, \nu)\) is \(\sigma\)-finite space and Markov operator is constrictive.

Remark 2.7. If Theorem 2.5 holds for a Markov operator \(P\), then there is an invariant density \(f^*\) defined by

\[
f^* = \frac{1}{r} \sum_{i=1}^{r} g_i.
\]
Indeed,

\[ Pf_* = \frac{1}{r} \sum_{i=1}^{r} Pg_i = \frac{1}{r} \sum_{i=1}^{r} g_i = f_* . \]

Therefore \( Pf_* = f_* \).

The following theorem is our main result.

**Theorem 2.8.** The Markov operator \( P : L^1([0,1]) \to L^1([0,1]) \) defined by (6) corresponding to a NLAR(1) on \([0,1]\) generated by (4) with respect to \((\Omega, [0,1], S, x_0, \{\xi_n\}_{n \geq 0})\) is constrictive, that is, theorem 2.5 holds for \( P \).

Moreover when the density of noise \( g(x) \) is not zero for all \( x \), we have the following result.

**Proposition 2.9.** Let \( P : L^1([0,1]) \to L^1([0,1]) \) be the Markov operator defined by (6) corresponding to a NLAR(1) on \([0,1]\) generated by (4) with respect to \((\Omega, [0,1], S, x_0, \{\xi_n\}_{n \geq 0})\). If \( g(x) > 0 \) for all \( x \in [0,1] \), then there exists a unique \( f_* \in D \) such that \( Pf_* = f_* \) and

\[ \lim_{n \to \infty} \|P^n f - f_*\| = 0 \quad \text{for every} \quad f \in D. \]

**Remark 2.10.** A sequence \( \{P^n\}_{n \geq 1} \) satisfying (9) is called asymptotically periodic. Proposition 2.9 implies that \( r = 1 \) for (9). In this case, the sequence \( \{P^n\}_{n \geq 1} \) is called asymptotically stable.

### 2.2. Small random perturbations of dynamical systems.

In this section, we observe limiting behaviour of density functions of a NLAR(1) on \([0,1]\) generated by (4) with respect to \((\Omega, [0,1], S, x_0, \{\xi_n\}_{n \geq 0})\) parametrized by \( \varepsilon > 0 \) as \( \varepsilon \to 0 \).

We consider the following first-order nonlinear autoregressive model \( \{x^n_\varepsilon\}_{n \geq 0} \) on \([0,1]\) with respect to \((\Omega, [0,1], S, x_0, \{\xi_n\}_{n \geq 0})\) parametrized by \( \varepsilon > 0 \):

\[ x^n_\varepsilon + 1 (\omega) = S(x^n_\varepsilon (\omega)) + \varepsilon \xi_n (\omega) \quad \text{for} \quad 0 < \varepsilon < 1, \]

where \( x^n_0 = x_0 \).

Since random variables \( \varepsilon \xi_n \) have the same density \( \frac{1}{\varepsilon} g(\frac{1}{\varepsilon}) \), we have the Markov operator \( P_\varepsilon : L^1([0,1]) \to L^1([0,1]) \) defined by

\[ P_\varepsilon f(x) = \frac{1}{\varepsilon} \int_{[0,1]} f(y) \left( \sum_{i=0}^{1} g \left( \frac{x - S(y) + i}{\varepsilon} \right) \right) dy \]

which satisfies that \( f_{n+1}^\varepsilon = P_\varepsilon f_n^\varepsilon \), where \( \{f_n^\varepsilon\}_{n \geq 0} \) is the sequence of the density function of \( x_n^\varepsilon \). Since \( S \) is non-singular, there exists the Perron-Frobenius operator \( P_S : L^1([0,1]) \to L^1([0,1]) \) with respect to \( S : [0,1] \to \)
Hence, if we let \( g_{x,i,\varepsilon}(y) := g\left(\frac{x+i-y}{\varepsilon}\right) \), then we have that

\[
P_{\varepsilon}f(x) = \frac{1}{\varepsilon} \int_{[0,1]} f(y) \left( \sum_{i=0}^{1} g_{x,i,\varepsilon}(S(y)) \right) dy
\]

\[
= \frac{1}{\varepsilon} \int_{[0,1]} P_{S}f(y) \left( \sum_{i=0}^{1} g_{x,i,\varepsilon}(y) \right) dy
\]

\[
= \frac{1}{\varepsilon} \int_{[0,1]} g_{x,0,\varepsilon}(y) \left( x + \frac{1}{\varepsilon} \right) dy
\]

\[
= \sum_{i=0}^{1} \int_{[x+i-\varepsilon, x+i]} P_{S}f(x + i - \varepsilon y) g(y) dy
\]

by condition C3.

We should expect that in some sense \( \lim_{\varepsilon \to 0} P_{\varepsilon}f(x) = P_{S}f(x) \).

Let \( \|f\|_\infty := \inf \{ M : \|f(x)\| \leq M \text{ for } \lambda \text{-a.e. } x \in [0,1] \} \), where \( \lambda \) is the normalized Lebesgue measure on \( [0,1] \).

**Theorem 2.11.** Let \( S : [0,1] \to [0,1] \) be a non-singular measurable transformation and \( P_{\varepsilon} \) be the Markov operator defined by (11) corresponding to a NLAR(1) on \( [0,1] \) generated by (10) with respect to \( (\Omega, [0,1], S, x_0, \{\xi_n\}_{n \geq 0}) \).

Suppose that \( \|P_{S}f\|_\infty < \infty \) for any continuous function \( f \) on \( [0,1] \). Then we have that

\[
(12) \quad \lim_{\varepsilon \to 0} \|P_{\varepsilon}f - P_{S}f\|_{L^1([0,1])} = 0
\]

for all \( f \in L^1([0,1]) \).

**Remark 2.12.** There is a big class of dynamical systems \( S : [0,1] \to [0,1] \) satisfying \( \|P_{S}f\|_\infty < \infty \) for any continuous function \( f \) on \( [0,1] \). For example, piecewise monotonic maps (including unimodal maps) and piecewise convex maps satisfy the assumption of Theorem 2.11.

It is obviously that \( \{P_{\varepsilon}^n\}_{n \geq 1} \) defined by (11) is asymptotically periodic for each \( \varepsilon > 0 \). Hence the function \( f_{\varepsilon} \) defined by

\[
(13) \quad f_{\varepsilon}(x) = \frac{1}{r(\varepsilon)} \sum_{i=1}^{r(\varepsilon)} g_{i,\varepsilon}(x),
\]

where \( r(\varepsilon) \) is a positive integer and \( g_{i,\varepsilon}(x) \) are density functions depending only on \( \varepsilon \), satisfies that \( f_{\varepsilon} \in D \) and \( P_{\varepsilon}f_{\varepsilon} = f_{\varepsilon} \). This implies that for each \( \varepsilon > 0 \), Markov operator \( P_{\varepsilon} \) has at least one invariant density.

**Corollary 2.13.** Let \( S : [0,1] \to [0,1] \) be a non-singular measurable transformation, \( P_{\varepsilon} \) be the Markov operator defined by (11) corresponding to a NLAR(1) on \( [0,1] \) generated by (10) with respect to \( (\Omega, [0,1], S, x_0, \{\xi_n\}_{n \geq 0}) \) and \( f_{\varepsilon} \) be an invariant density for \( P_{\varepsilon} \) defined by (13). Suppose that \( \|P_{S}f\|_\infty < \infty \), then we have that

\[
P_{\varepsilon}f(x) = \frac{1}{\varepsilon} \int_{[0,1]} f(y) \left( \sum_{i=0}^{1} g_{x,i,\varepsilon}(S(y)) \right) dy
\]

\[
= \frac{1}{\varepsilon} \int_{[0,1]} P_{S}f(y) \left( \sum_{i=0}^{1} g_{x,i,\varepsilon}(y) \right) dy
\]

\[
= \frac{1}{\varepsilon} \int_{[0,1]} g_{x,0,\varepsilon}(y) \left( x + \frac{1}{\varepsilon} \right) dy
\]

\[
= \sum_{i=0}^{1} \int_{[x+i-\varepsilon, x+i]} P_{S}f(x + i - \varepsilon y) g(y) dy
\]
\( \infty \) for any continuous function \( f \) on \([0, 1]\). If there exists an integrable function \( f_\ast \) on \([0, 1]\) such that
\[
\lim_{\varepsilon \to 0} \| f_\varepsilon - f_\ast \|_{L^1([0,1])} = 0,
\]
then \( f_\ast \) is an invariant density for \( P_S \), that is \( P_S f_\ast = f_\ast \).

**Remark 2.14.** Corollary 2.13 holds for any continuous piecewise \( C^2 \), piecewise expanding map \( S : [0, 1] \to [0, 1] \) which has no periodic turning points. Indeed, by Theorem 1.1 in [3] (and see Theorem 3 in [1]), there exists an unique absolutely continuous invariant probability measure \( \mu_0 = f_\ast dx \) which satisfies that
\[
\lim_{\varepsilon \to 0} \| f_\varepsilon - f_\ast \|_{L^1([0,1])} = 0.
\]

### 3. Examples

It is obviously that Theorem 2.8 holds for all non-singular transformations. We give some examples of non-singular transformations which also satisfy the assumptions of Theorem 2.11.

**1): \( m \)-adic transformation** [14].

Consider the transformation \( S : [0, 1] \to [0, 1] \) given by
\[
Sx = mx \pmod{1},
\]
where \( m \geq 1 \) is an integer. Thus the Perron-Frobenius operator \( P_S : L^1([0,1]) \to L^1([0,1]) \) corresponding to \( S \) is given by
\[
P_S f(x) = \frac{1}{m} \sum_{i=0}^{m-1} f \left( \frac{i + x}{m} \right).
\]
Since \( P_S 1 = 1 \), the Borel measure on \([0, 1]\) is invariant with respect to the \( m \)-adic transformation \( S \). Moreover it is obviously that for any continuous function \( f \) on \([0, 1]\), \( Pf(x) \) is equal to a continuous function, hence \( \| P_S f \|_{\infty} < \infty \).

**2): Maps with indifferent fixed points with infinite invariant measure** [19]

Let \( \alpha \in (0, \infty) \) be a real parameter and consider the one-parameter family of maps \( S_\alpha \) of the interval \([0, 1]\) onto itself defined by
\[
S_\alpha(x) := 2e^{\alpha x} - 1 \pmod{1}.
\]
For every \( \alpha > 0 \), \( S_\alpha \) is piecewise onto and \( C^\infty \)-class. When the parameter \( \alpha \) varies, the dynamics of the maps changes. Some properties of this family established in [17] are listed below:

1. For \( \alpha > 0 \) with \( |S'_\alpha(0)| > 1 \), \( S_\alpha \) is a piecewise expanding map (see Figure 1). Then there exists the unique absolutely continuous invariant probability measure with respect to the Lebesgue measure on \([0, 1]\) by the Lasota-Yorke theorem.
(2) For $\alpha > 0$ with $|S'_\alpha(0)| = 1$, $S_\alpha$ admits an indifferent fixed point 0 (see Figure 2). For these maps, there is NO finite absolutely continuous invariant measure. However there exists a $\sigma$-finite infinite absolutely continuous invariant measure.

(3) For $\alpha > 0$ with $|S'_\alpha(0)| < 1$, $S_\alpha$ admits a stable fixed point 0 (see Figure 3). For these maps, almost all points converge to 0 by using the symbolic dynamics with 4-symbols (see [17] more details.). Therefore there is no absolutely continuous invariant measure with respect to the Lebesgue measure.

Next, we shall apply our results (Theorem 2.11) to this family. Because $T_0(0) = 0$, $T_1(1) = 2$, where $T_\alpha(x) := 2^{e^{\alpha x} - 1}$ is monotonic continuous function for every $\alpha > 0$, there exists the unique point $x_\alpha \in (0, 1)$ such that $T_\alpha(x_\alpha) = 1$. Let $I_0 = [0, x_\alpha)$ and $I_1 = [x_\alpha, 1]$. Since $C^\infty$-extensions of the maps $S_\alpha|I_0 : I_0 \to [0, 1]$ and $S_\alpha|I_1 : I_1 \to [0, 1]$ are one-to-one and onto, there exist the local inverses $u_{\alpha,j} = (S_\alpha|I_j)^{-1}$ for $j = 0, 1$, we get

\begin{equation}
    u_{\alpha,j}(x) = \frac{1}{\alpha} \log(1 + \frac{\alpha - 1}{2} (x + j)).
\end{equation}

Thus the Perron-Frobenius operator corresponding to $S_\alpha$ is given by

\begin{equation}
    P_{S_\alpha}f = f \circ u_{\alpha,0} \circ u'_{\alpha,0} + f \circ u_{\alpha,1} \circ u'_{\alpha,1}.
\end{equation}

Therefore we have $\|P_{S_\alpha}f(x)\|_\infty < \infty$ for any continuous function $f$ on $[0, 1]$.

4. Proof

Proof. Proof of Proposition 2.2

We let the density of $x_n$ be denoted by $f_n \in D$ ($n \geq 1$) and desire a relation connecting $f_{n+1}$ and $f_n$.

We assume that $f_n$ exists for some $n \geq 0$.

Let $\tilde{A} = A \setminus \{1\}$ for any Borel set $A \subset [0, 1]$. Note that since $x_{n+1}(\Omega) \subset [0, 1)$ and $S(x_n)$ and $\xi_n$ are independent for all $n \geq 0$, we have that...
for each $a$

By a change of variables (see Lemma 5.2 in Appendix.), this can be written

\[
\int_{\Omega : S_n(\omega) + \xi_n(\omega) = 2} g(x) dx dy = 0.
\]

From (i)-(iii), we have that for any Borel set $A \subset [0,1]$ and $n \geq 0,$

\[
\mu (\{ \omega \in \Omega : x_{n+1} \in A \}) = \mu (\{ \omega \in \Omega : x_{n+1} \in \bar{A} \})
\]

\[
= \mu (\{ \omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega)(\text{mod} 1) \in \bar{A} \})
\]

\[
= \mu (\{ \omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) \in \bar{A} \}) + \mu (\{ \omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) \in \bar{A} + 1 \})
\]

\[
(\mu (\{ \omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) = 1 \}) + 1) \text{ if } 0 \in A
\]

\[
= \int \int_{S(x) + y \in \bar{A}} f_n(x) g(y) dx dy + \int \int_{S(x) + y \in A} f_n(x) g(y) dx dy.
\]

By a change of variables (see Lemma 5.2 in Appendix.), this can be written as

\[
\mu (\{ \omega \in \Omega : x_{n+1} \in A \}) = \int a \in \bar{A} \left\{ \int_{B^0(a)} f_n(b) g(a - S(b)) db \right\} da
\]

\[
+ \int a \in \bar{A} \left\{ \int_{B^1(a)} f_n(b) g(a - S(b) + 1) db \right\} da,
\]

where

\[
B^0(a) := \{ b \in [0,1] : 0 \leq a - S(b) \leq 1 \} = \{ b \in [0,1] : 0 \leq S(b) \leq a \}
\]

and

\[
B^1(a) := \{ b \in [0,1] : 0 \leq a - S(b) + 1 \leq 1 \} = \{ b \in [0,1] : a \leq S(b) \leq 1 \}
\]

for each $a \in [0,1].$ By condition C3, we have that

\[
g(x - S(y)) = 0 \quad \text{for all } y \in \{ b \in [0,1] : x < S(b) \} = [0,1] \setminus B^0(x)
\]

\[
g(x - S(y) + 1) = 0 \quad \text{for all } y \in \{ b \in [0,1] : x > S(b) \} = [0,1] \setminus B^1(x)
\]

for each $x \in [0,1].$ Hence we get that

\[
\int_{[0,1] \setminus B^0(x)} f_n(y) g(x - S(y)) dy = 0 = \int_{[0,1] \setminus B^1(x)} f_n(y) g(x - S(y) + 1) dy
\]

for each $x \in [0,1].$ This implies that

\[
\int_{[0,1]} f_n(y) g(x - S(y)) dy = \int_{B^0(x)} f_n(y) g(x - S(y)) dy
\]

\[
\int_{[0,1]} f_n(y) g(x - S(y) + 1) dy = \int_{B^1(x)} f_n(y) g(x - S(y) + 1) dy.
\]
Therefore we have that

$$\mu (\{ \omega \in \Omega : x_{n+1} \in A \}) = \int_{a \in A} \int_{[0,1]} f_n(b)g(a - S(b))dbda$$

$$+ \int_{a \in A} \int_{[0,1]} f_n(b)g(a - S(b) + 1)dbda.$$

Since \( \{1\} \) is a 1-point set and \( h(a) := \int_{b \in [0,1]} f_n(b)g(a - S(b) + i)db \in L^1([0,1]) \), we have that for \( i = 0, 1 \),

$$\int_{\{1\}} \left\{ \int_{[0,1]} f_n(b)g(a - S(b) + i)db \right\} da = \int_{\{1\}} h(a)da = 0.$$

Then we have that

$$\mu (\{ \omega \in \Omega : x_{n+1} \in A \}) = \sum_{i=0}^{1} \int_{a \in A} \int_{b \in [0,1]} f_n(b)g(a - S(b) + i)dbda$$

$$= \sum_{i=0}^{1} \int_{a \in A} \int_{b \in [0,1]} f_n(b)g(a - S(b) + i)dbda.$$

Therefore using the fact that \( A \) was an arbitrary Borel set on \([0,1]\), we get the density \( f_{n+1} \) of \( x_{n+1} \) defined by

$$f_{n+1}(x) = \sum_{i=0}^{1} \int_{[0,1]} f_n(y)g(x - S(y) + i)dy \quad a.e. \ x \in [0,1].$$

On the other hand, we get that

$$\int_{x \in [0,1]} \sum_{i=0}^{1} g(x - S(y) + i)dx = \int_{[0,1]} g(x)dx = 1 \quad for \forall y \in [0,1]$$

by condition C3. Then by Fubini’s theorem, we have that

$$\int_{[0,1]} f_{n+1}(x)dx = \sum_{i=0}^{1} \int_{y \in [0,1]} \left\{ \int_{x \in [0,1]} f_n(y)g(x - S(y) + i)dx \right\} dy$$

$$= \int_{y \in [0,1]} f_n(y)dy = 1.$$

Moreover \( f_{n+1} \geq 0 \) because of the positivity of \( g \) and \( f_n \). Therefore if \( x_n \) has the density \( f_n \in D \), then \( f_{n+1} \) also have to exist in \( D \).

From this fact, we can define a linear operator \( P : L^1([0,1]) \to L^1([0,1]) \) by

$$Pf(x) = \int_{y \in [0,1]} f(y) \left( \sum_{i=0}^{1} g(x - S(y) + i) \right) dy$$

which satisfies that

$$f_{n+1} = Pf_n \quad a.e.$$
for all $n \geq 0$. Next we shall show that $P : L^1([0, 1]) \to L^1([0, 1])$ is a Markov operator, that is, $P$ is a linear operator which satisfies that $Pf \geq 0$ and $\|Pf\|_{L^1([0, 1])} = \|f\|_{L^1([0, 1])}$ for any $f \in L^1([0, 1])$ with $f \geq 0$. It is easy to see that $P$ is a positive linear operator on $L^1([0, 1])$ because $g$ is positive. Moreover we have that for $f \in L^1([0, 1])$ with $f \geq 0$ by the Fubini’s theorem,

$$\|Pf\|_{L^1([0, 1])} := \int_{[0,1]} Pf(x) dx \geq \int x \in [0,1] \int y \in [0,1] f_n(y) \left( \sum_{i=0}^{1} g(x - S(y) + i) \right) dy dx$$

$$= \int x \in [0,1] \sum_{i=0}^{1} g(x - S(y) + i) \left\{ \int_{[0,1]} f(y) dy \right\} dx$$

$$= \int_{[0,1]} f(y) dy = \|f\|_{L^1([0,1])}.$$

Therefore $P$ is a Markov operator.

Proof. Proof of Theorem 2.8
From the spectral decomposition theorem by Komornik and Lasota [14], it is enough to show that $P$ is constrictive: there exists a $\delta > 0$ and $\kappa < 1$ such that for every $f \in D$ there is an integer $n_0(f)$ for which

$$\int_B P^n f(x) dx \leq \kappa \quad \text{for all } n \geq n_0(f) \text{ and } B \subset [0, 1] \text{ with } \lambda(B) \leq \delta,$$

where $\lambda$ is the normalized Lebesgue measure on $[0, 1]$. Since $g$ is the integrable function on $\mathbb{R}$ supported in $[0, 1]$, for any $\varepsilon > 0$, there exists $0 < \delta(\varepsilon) \leq 1$ such that whenever $\lambda(A) \leq \delta(\varepsilon)$,

$$\int_A g(x) dx \leq \varepsilon.$$

Take arbitrary $0 < \varepsilon < 1$, hence there exists $\delta(\varepsilon) > 0$ which satisfies

$$\int_A g(x) dx \leq \frac{\delta(\varepsilon)}{2}$$

for any Borel set $A \subset [0, 1]$ with $\lambda(A) \leq \delta(\varepsilon)$. Thus we have that for each $f \in D$ and $n \geq 1$,

$$\int_A P^n f(x) dx = \int_A \int_{[0,1]} P^{n-1} f(y) \left( \sum_{i=0}^{1} g(x - S(y) + i) \right) dy dx$$

$$= \int_{[0,1]} \left\{ \sum_{i=0}^{1} \int_{A-S(y)+i} g(x) dx \right\} P^{n-1} f(y) dy.$$

Let $\tilde{\lambda}$ be the Lebesgue measure on $\mathbb{R}$. Since $\tilde{\lambda}(A - S(y) + i) = \tilde{\lambda}(A) = \lambda(A) \leq \delta(\varepsilon)$ for each $y \in [0, 1]$ and $i = 0, 1$, we obtain that

$$\int_A P^n f(x) dx \leq \varepsilon \int_{[0,1]} P^{n-1} f(y) dy = \varepsilon \quad \text{for all } n \geq 1,$$

(17)
which implies that $P$ is constrictive.

\[ P^n f(x) > 0 \quad \text{for a.e. } x \in A \quad \text{and for all } n \geq n_0(f). \]

Let $f \in D$ be arbitrary. From the assumption about $g$, there exists a positive number $0 < \varepsilon < 1$ which satisfies that there exists $\delta(\varepsilon) > 0$ such that for all $\lambda(A) \leq \delta(\varepsilon)$, $\int_A g(x)dx \leq \frac{\varepsilon}{2}$. Take an arbitrarily $0 < \delta < 1$ with $1 - \delta < \delta(\varepsilon)$. Since $\lambda((\delta - S(y) + i, 1 - S(y) + i)) = 1 - \delta \leq \delta(\varepsilon)$ for each $y \in [0, 1]$ and $i = 0, 1$, we have that

\[
\int_{\delta < x \leq 1} P^n f(x)dx = \int_{[0, 1]} \left\{ \sum_{i=0}^{1} \int_{(\delta - S(y) + i, 1 - S(y) + i)} g(x)dx \right\} P^{n-1} f(y)dy \leq \varepsilon
\]

for all $n \geq 1$. From this inequality, we have that

\[
\int_{0 \leq y \leq \delta} P^n f(y)dy = \int_{[0, 1]} P^n f(y)dy - \int_{\delta < y \leq 1} P^n f(y)dy \geq 1 - \varepsilon > 0
\]

for all $n \geq 1$.

On the other hand, we have that

\[ P^{n+1} f(x) = \int_{[0, 1]} P^n f(y) \left( \sum_{i=0}^{1} g(x - S(y) + i) \right) dy \geq \int_{0 \leq y \leq \delta} P^n f(y) \left( \sum_{i=0}^{1} g(x - S(y) + i) \right) dy, \]

From the assumption about $g$, we have that

\[ (2\delta) - S(y) + g(x - S(y) + 1) > 0 \quad \text{for all } x \in [0, 1] \text{ and } 0 \leq y \leq \delta. \]

From (19) and (21), we have that for a.e. $x \in [0, 1]$,

\[ P^n f(y) \left( \sum_{i=0}^{1} g(x - S(y) + i) \right) \quad \text{for } n \geq 1 \]

as a function of $y$, does not vanish in $\{0 \leq y \leq \delta\}$. As a consequence, inequality (20) implies (18) with respect to the set $[0, 1]$, thus completing the proof of the proposition.
Proof. Proof of Theorem 2.11

Since the set of continuous functions on \([0, 1]\) is dense in \(L^1([0, 1])\) and \(P_\varepsilon, P_S\) are Markov operators, it is enough to prove the theorem for continuous functions on \([0, 1]\). Indeed, for any \(f \in L^1([0, 1])\) and \(\eta > 0\), there exists a continuous function \(f_\eta\) on \([0, 1]\) such that \(\|f - f_\eta\|_{L^1([0, 1])} \leq \eta\). Thus if we have that \(\lim_{\varepsilon \to 0} \|P_\varepsilon f_\eta - P_S f_\eta\|_{L^1([0, 1])} = 0\), then we have that

\[
\lim_{\varepsilon \to 0} \|P_\varepsilon f - P_S f\|_{L^1([0, 1])} = \lim_{\varepsilon \to 0} \|P_\varepsilon(f - f_\eta) - P_S(f - f_\eta) + P_\varepsilon f_\eta - P_S f_\eta\|_{L^1([0, 1])} \\
\leq 2\|f - f_\eta\|_{L^1([0, 1])} + \lim_{\varepsilon \to 0} \|P_\varepsilon f_\eta - P_S f_\eta\|_{L^1([0, 1])} \\
\leq 2\eta.
\]

From the fact that \(\eta\) was an arbitrary positive number, we have that \(\lim_{\varepsilon \to 0} \|P_\varepsilon f - P_S f\|_{L^1([0, 1])} = 0\).

Fix an arbitrarily continuous function \(f\) on \([0, 1]\). We split the integral into two parts,

\[
\|P_\varepsilon f - P_S f\|_{L^1([0, 1])} = \int_{[0,\varepsilon]} |P_\varepsilon f - P_S f| dx + \int_{(\varepsilon, 1]} |P_\varepsilon f - P_S f| dx = C_1(\varepsilon) + C_2(\varepsilon) \quad \text{for } 0 < \varepsilon < 1.
\]

Firstly, we consider \(C_1(\varepsilon)\). Let \(H_i(x, y) := P_S f(x + i - \varepsilon y)g(y)1_{[x + i - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}]}(y)\) for \(i = 0, 1\). Note that the essential supremum of \(|P_S f|\) is finite (i.e. \(\|P_S f\|_\infty < \infty\)) from the assumption about \(P_S f\). Fix an arbitrarily point \(x_0 \in [0, 1]\). Since

\[
0 \leq x_0 + i - \varepsilon y \leq 1 \quad \text{for all } y \in \left[\frac{x_0 + i - 1}{\varepsilon}, \frac{x_0 + i}{\varepsilon}\right],
\]

we have that for each \(i = 0, 1\),

\[
|P_S f(x_0 + i - \varepsilon y)| \leq \|P_S f\|_\infty \quad \text{for } \lambda\text{-a.e. } y \in \left[\frac{x_0 + i - 1}{\varepsilon}, \frac{x_0 + i}{\varepsilon}\right].
\]

Moreover we have that

\[
[0, 1] \subset \bigcup_{i=(0,1)} \left[\frac{x_0 + i - 1}{\varepsilon}, \frac{x_0 + i}{\varepsilon}\right] = \left[\frac{x_0 - 1}{\varepsilon}, \frac{x_0}{\varepsilon}\right] \cup \left[\frac{x_0}{\varepsilon}, \frac{x_0 + 1}{\varepsilon}\right].
\]
for all $0 < \varepsilon < 1$. Then we have that,
\[
\left| \sum_{i=0}^{1} \int_{[0,1]} H_i(x_0, y) dy \right| \leq \sum_{i=0}^{1} \int_{[0,1]} |P_S f (x_0 + i - \varepsilon y)| g(y) 1_{\frac{2n+i-1}{\varepsilon}, \frac{2n+i}{\varepsilon}}(y) dy
\]
\[
\leq \sum_{i=0}^{1} \|P_S f\|_{\infty} \int_{[0,1]} g(y) 1_{\frac{2n+i-1}{\varepsilon}, \frac{2n+i}{\varepsilon}}(y) dy
\]
\[
= \|P_S f\|_{\infty} \left\{ \int_{[0,1]} g(y) dy \right\} = \|P_S f\|_{\infty} (22)
\]
by condition C3. Since $x_0$ was an arbitrary point in $[0, 1]$, we have that
\[
\|P_\varepsilon f\|_{L^2([0,1])} = \left( \int_{[0,1]} \left\{ \sum_{i=0}^{1} \int_{[0,1]} H_i(x, y) dy \right\}^2 dx \right)^{1/2}
\]
\[
\leq \|P_S f\|_{\infty} < \infty.
\]
This implies that the family $\{P_\varepsilon f, 0 < \varepsilon < 1\}$ is uniformly integrable. Then we have that
\[
(23) \lim_{\varepsilon \to 0} \sup_{0 < \eta < 1} \int_{[0,\varepsilon]} |P_\eta f| dx = 0
\]
by Lemma 4.10 in [8]. Since
\[
\int_{[0,\varepsilon]} |P_\varepsilon f| dx \leq \sup_{0 < \eta < 1} \int_{[0,\varepsilon]} |P_\eta f| dx \quad \text{for } 0 < \varepsilon < 1,
\]
we have that
\[
0 \leq \lim_{\varepsilon \to 0} \int_{[0,\varepsilon]} |P_\varepsilon f| dx \leq \lim_{\varepsilon \to 0} \int_{[0,\varepsilon]} |P_\varepsilon f| dx \leq \lim_{\varepsilon \to 0} \sup_{0 < \eta < 1} \int_{[0,\varepsilon]} |P_\eta f| dx = 0
\]
by (23). Therefore we have that $\lim_{\varepsilon \to 0} \int_{[0,\varepsilon]} |P_\varepsilon f| dx = 0$. Moreover since the family $\{P_S f\}$ consisting of only one function $P_S f$ is obviously uniformly integrable, we also have that
\[
\lim_{\varepsilon \to 0} \int_{[0,\varepsilon]} |P_S f| dx = 0.
\]
Therefore we have that
\[
(24) \lim_{\varepsilon \to 0} C_1(\varepsilon) \leq \lim_{\varepsilon \to 0} \int_{[0,\varepsilon]} |P_\varepsilon f| dx + \lim_{\varepsilon \to 0} \int_{[0,\varepsilon]} |P_S f| dx = 0.
\]
Note that \([0, 1] \subset [\frac{x-1}{\varepsilon}, \frac{x}{\varepsilon}]\) and \([\frac{x}{\varepsilon}, \frac{x+1}{\varepsilon}] \subset (1, \infty)\) for each \(x \in (\varepsilon, 1]\). Hence we have that

\[
P_{\varepsilon} f(x) = \sum_{i=0}^{1} \int_{[\frac{x+i-1}{\varepsilon}, \frac{x+i}{\varepsilon}]} P_{S} f(x + i - \varepsilon) g(y) dy
= \int_{[0, 1]} P_{S} f(x - \varepsilon) g(y) dy.
\]

Thus we have that with respect to \(C_{2}(\varepsilon)\),

\[
C_{2}(\varepsilon) = \int_{(\varepsilon, 1]} \left| \int_{[0, 1]} P_{S} f(x - \varepsilon) g(y) dy - P_{S} f(x) \right| dx
= \int_{(\varepsilon, 1]} \left| \int_{[0, 1]} [P_{S} f(x - \varepsilon) - P_{S} f(x)] g(y) dy \right| dx.
\]

Without loss of generality, we can assume that \(P_{S} f(x) = 0\) for all \(x \notin [0, 1]\) (for example set \(S(x) = x, f(x) = 0\) for all \(x \notin [0, 1]\)). Since \(P_{S} f\) is an integrable function and the set \(\{P_{S} f\}\) is compact in \(L^{1}(\mathbb{R})\), we have that for an arbitrarily small \(\delta > 0\), there exists \(\varepsilon_{0}\) such that for all \(\varepsilon \leq \varepsilon_{0}\),

\[
\int_{[0, 1]} |P_{S} f(x - \varepsilon) - P_{S} f(x)| dx \leq \delta
\]

for each \(y \in [0, 1]\). Thus we have that

\[
C_{2}(\varepsilon) \leq \int_{[0, 1]} \int_{[0, 1]} |P_{S} f(x - \varepsilon) - P_{S} f(x)| g(y) dy dx
\leq \delta \int_{[0, 1]} g(y) dy = \delta.
\]

Therefore \(\lim_{\varepsilon \to 0} C_{2}(\varepsilon) = 0\). Then theorem is proved.

\(\square\)

**Proof. Proof of Corollary 2.13**

Since \(P_{\varepsilon}\) is the Markov operator, we have that

\[
\|P_{\varepsilon} (f - f_{\varepsilon})\|_{L^{1}([0, 1])} \leq \|f - f_{\varepsilon}\|_{L^{1}([0, 1])}.
\]

Hence we have that

\[
\|P_{\varepsilon} f_{\varepsilon} - f_{\varepsilon}\|_{L^{1}([0, 1])} = \|f_{\varepsilon} + P_{\varepsilon} (f_{\varepsilon} - f_{\varepsilon}) - f_{\varepsilon}\|_{L^{1}([0, 1])}
\leq \|f_{\varepsilon} - f_{\varepsilon}\|_{L^{1}([0, 1])} + \|P_{\varepsilon} (f_{\varepsilon} - f_{\varepsilon})\|_{L^{1}([0, 1])}
\leq 2 \|f_{\varepsilon} - f_{\varepsilon}\|_{L^{1}([0, 1])} \to 0\quad \text{as} \quad \varepsilon \to 0.
\]

Thus \(P_{\varepsilon} f_{\varepsilon}\) converges to \(f_{\varepsilon}\) in \(L^{1}([0, 1])\)-norm. On the other hand, from Theorem 2.11, \(P_{\varepsilon} f_{\varepsilon}\) converges to \(P_{S} f_{\varepsilon}\) in \(L^{1}([0, 1])\)-norm. Therefore \(P_{S} f_{\varepsilon} = f_{\varepsilon}\).

\(\square\)
5. Appendix

In this section, we give a supplementary explanation of the change of variables theorem for the Lebesgue integral on $\mathbb{R}$ which is applied in the proof of Proposition 2.2.

**Lemma 5.1.** ([7]) If $h(t) \geq 0$ is an integrable function on $[\alpha, \beta]$ such that there exists a increasing function $H(t)$ satisfying $H(t) = \int_{c}^{t} h(t)dt$, where $c$ is a constant. Let $a = H(\alpha), b = H(\beta)$. Then we have that

$$\int_{a}^{b} f(x)dx = \int_{\alpha}^{\beta} f(H(t))h(t)dt$$

for all integrable function $f$ defined on $[a, b]$.

By using Lemm 5.1, we prove the following lemma.

**Lemma 5.2.** Let $X$ and $Y$ are independent random variables on a probability space $(\Omega, F, \mu)$ with values in $[0, 1]$ which satisfy the followings:

1. $X$ has the density function $f : [0, 1] \rightarrow \mathbb{R}$ with $f \geq 0$ such that $\int_{[0,1]} f(x)dx = 1$,

2. $Y$ has the density function $g : \mathbb{R} \rightarrow \mathbb{R}$ with $g \geq 0$ such that $\text{supp}(g) := \{x \in \mathbb{R} : g(x) \neq 0\} \subset [0, 1]$ and $\int_{[0,1]} g(x)dx = 1$.

Then we have that for any Borel set $A \subset [0, 1],$

$$\mu(\{\omega \in \Omega : X(\omega) + Y(\omega) \in A\}) = \int_{x \in A} \int_{y \in B(x)} f(y)g(x-y)dydx,$$

where $B(x) = \{y \in [0, 1] : 0 \leq x - y \leq 1\}$ for each $x \in [0, 1]$.

**Proof.** Since $X$ and $Y$ are independent,

$$\mu(\{\omega \in \Omega : X(\omega) + Y(\omega) \in A\}) = \int \int_{\{(x,y)\in[0,1] \times [0,1] : x+y \in A\}} f(x)g(y)dxdy.$$

Since $f$ and $g$ are positive integrable functions on $[0, 1]$, we have

$$\int \int_{\{(x,y)\in[0,1] \times [0,1] : x+y \in A\}} f(x)g(y)dxdy < \infty,$$

so, we can apply the Fubini’s theorem to this integral. Indeed, we have that

$$\int \int_{\{(x,y)\in[0,1] \times [0,1] : x+y \in A\}} f(x)g(y)dxdy = \int_{x\in[0,1]} \int_{\{y\in[0,1] : x+y \in A\}} f(x)g(y)dydx.$$

Let $a := x + y$ and $Z(a) := a - x$ for fixed $x \in [0, 1]$. Since $Z(a)$ is absolutely continuous (i.e. $Z(a) = \int_x^a 1(t)dt$), we have that by Lemma 5.1 and Fubini’s theorem, we have that

$$\int_{x \in [0, 1]} \int_{y \in [0, 1]} f(x)g(y)dydx = \int_{x \in [0, 1]} \int_{\{a \in A : 0 \leq a - x \leq 1\}} f(x)g(a - x)dadx$$

(change of variables)

$$= \int_{x \in [0, 1]} \int_{\{a \in [0, 1] : 0 \leq a - x \leq 1\}} f(x)g(a - x)1_A(a)dadx$$

$$= \int_{a \in [0, 1]} \int_{\{x \in [0, 1] : 0 \leq a - x \leq 1\}} f(x)g(a - x)1_A(a)dxdx$$

(Fubini’s theorem)

$$= \int_{a \in A} \int_{x \in B(a)} f(x)g(a - x)dxda.$$

Therefore we have that

$$\mu(\{\omega \in \Omega : X(\omega) + Y(\omega) \in A\}) = \int_{a \in A} \int_{x \in B(a)} f(x)g(a - x)dxda.$$

\[\square\]

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