A comparison principle for singular diffusion equations with spatially inhomogeneous driving force for graphs

M.-H. Giga, Y. Giga and P. Rybka

August 4, 2011

1 Introduction

As a continuation of [GG98Ar], [GG01Ar] this paper studies a degenerate nonlinear parabolic equation (in one space dimension) whose diffusion effect is very strong at particular slopes of unknown functions. We are in particular interested in an equation, where the driving force term is spatially inhomogeneous. A typical example is a quasilinear equation

$$u_t = a(u_x)[W'(u_x)_x + \sigma(t,x)], \tag{1.1}$$

where W is a given convex function on $\mathbf{R}$ but may not be of class C^1 so that its derivative W' may have jump discontinuities; here a is a given non-negative continuous function and σ is a given smooth function depending on x and also on t, where u_t and u_x denote the time and the space derivative of $u = u(t,x)$.

As explained in detail in [GG98Ar] the equation is viewed as an evolution law of the graph of u moved by an anisotropic mean curvature flow $V = M(\mathbf{n})\ (\kappa_\gamma + \sigma)$ with a singular interfacial energy density γ, where κ_γ is a weighted curvature and M is mobility; V denotes the normal velocity of the evolving curve in the direction of $\mathbf{n}$. (The quantity κ_γ formally equals $(\gamma''+\gamma)\kappa$ with curvature κ and $\gamma = \gamma(\theta)$ is an interfacial density as a function of the argument θ of $\mathbf{n} = (\cos\theta, \sin\theta)$.)

Our eventual goal is to establish a kind of the theory of viscosity solutions for a class of equations including (1.1) as a particular example so that we are able to construct a global-in-time solution for example for periodic initial data. In this paper we give a new notion of viscosity solutions for (1.1) and establish a comparison principle.

If σ in (1.1) is independent of x, the theory of viscosity solutions has been already established in [GG98Ar], [GG01Ar]. Even in this simpler case the quality $(W'(u_x))_x$ turns to be nonlocal so conventional viscosity theory does not work. For example if $W(p) = |p|$, then $W''(p)$ is twice the delta function so that (1.1) becomes

$$u_t = a(u_x)[2\delta(u_x)u_{xx} + \sigma(t)] \tag{1.2}$$

which is of course, not a classical partial differential equation. If $u = u(t,x)$ has a flat part (called a facet) with zero slope, it is expected to move with speed $u_t = a(0)[2\chi/L + \sigma]$ provided that a facet persists and it does not break. Here L is the length of a facet (which is a nonlocal quantity) and $\chi = \pm 1, 0$ is a transition number of the facet depending upon local behavior of u near facet. For example, if u is 'concave' near the facet, then χ should be -1. When σ is spatially homogeneous, this hypothesis that a facet does not break is justified either by viscosity theory developed by [GG98Ar], [GG99] or by subdifferential theory [FG] (in the case $\sigma \equiv 0$), in the sense that such a solution is an appropriate limit of solutions to strictly parabolic problems. When W is piecewise linear and σ is independent of x, then (1.1) is analyzed in [T], [AG1] for a very restrictive class of piecewise linear unknown functions whose slopes belong to jump discontinuities of W. Their 'admissible' solution is actually a solution in viscosity sense [GG98Ar].

If σ depends on the space variable, the hypothesis that all facets do not break is no longer true. For example, if we postulate this hypothesis, then the speed u_t of a facet with slope zero of u of (1.2) equals $a(0)[2\chi/L + \fint \sigma dx]$, where $\fint$ denotes the average over the facet. As noticed in [GG98Pit] if we assigned the speed in this way the solution may not enjoy in general the comparison principle. This shows that such a 'solution' is not obtained as a limit of approximate problems satisfying the comparison principle. On the other hand if $|\sigma_x|$ is sufficiently small compared with the length of facets, such a solution is known to enjoy comparison principle [BGN].

If a is a constant, say $a \equiv 1$, and σ is independent of t, (1.1) can be viewed as a subdifferential formulation

$$u_t \in -\partial\varphi(u), \tag{1.3}$$

where φ is an energy which formally equals

$$\varphi(u) = \int_{\mathbf{T}} [W(u_x) - \sigma(x)u]dx;$$

for simplicity we assumed here a periodic boundary condition so that $\mathbf{T} = \mathbf{R}/\omega\mathbf{Z}$. As observed in [GG98DS] for (1.3) a general theory of subdifferential equations in a Hilbert space $L^2(\mathbf{T})$ provides not only unique existence of the solution but also the value of right derivative du^+/dt (of u as a function with value in H). A general theory further yields

$$d^+u/dt = -\partial^0\varphi(u),$$

where $\partial^0\varphi$ is the canonical restriction of a closed convex set $\partial\varphi\,(u(t))$, *i.e.*

$$\partial^0\varphi(u) = \text{ arg } \min\{||f||_H | f \in \partial\varphi(u) \subset H\}.$$

In [GG98DS] it is observed that $\partial^0\varphi$ can be calculated by solving an obstacle problem. Let us review their observation. Since the condition

$$f \in -\partial\varphi(u)$$

is equivalent to

$$f(x) = \eta_x(x) + \sigma(x), \eta(x) \in \partial W(u_x(x)), \text{ a.e. } x \in \mathbf{T},$$

the quantity

$$-\partial^0\varphi(u) = \begin{cases} (W'(u_x(x)))_x + \sigma(x) & \text{if } u_x \notin P \\ \eta_x^0(x) + \sigma(x) & \text{if } u_x \in P, \end{cases} \tag{1.4}$$

where P is the jump discontinuity of W' and u is assumed to be of class C^2 and P-faceted [GG98Ar]. Here, η^0 minimizes

$$\{\int_F |\eta_x + \sigma|^2 dx \ ; \ \eta \in \partial W\,(u_x(x))\} \tag{1.5}$$

under suitable boundary condition at the end of the facet F depending on whether u is 'convex' or 'concave' near F. This is a convex minimizing problem so a unique minimizer always exists. Moreover, if σ is independent of x, η_x must be constant and $\eta_x^0 + \sigma = \chi/L + \sigma$. If σ depends on x, $\eta_x^0 + \sigma$ may not be a constant over F and this is one reason why the speed may not be a constant on F when σ depends on x. The subdifferential equation (1.3) can be approximated by a smooth parabolic problem, so the solution is expected to enjoy the comparison principle. Thus, it is natural to guess that $\eta_x^0 + \sigma$ gives a candidate for the value of

$$\Lambda_W^\sigma(u)(x)) = (W'(u_x))_x + \sigma(x) \tag{1.6}$$

when W' has jump discontinuities. Note that this quantity agrees with the minimal velocity profile proposed by [Rs] as observed in [GG98DS].

Unfortunately, a general equation (1.1) cannot be viewed as a subdifferential equation (1.3). However, we still use the value (1.4) to define (1.6). We establish a notion of viscosity solutions by assigning the value Λ_W^σ by (1.4) for test functions which we call admissible. The class of test functions is the same as [GG98Ar] so a facet of a test function never vanishes or breaks. The idea of the proof of the comparison principle is similar to that of [GG98Ar] except for a simplification of handling end points of facets observed by [GG01Ar] and the use of continuity of $\Lambda^\sigma_W(u)$ under translation of a faceted region which is obvious when σ is constant. So we have to study an obstacle problem in this paper carefully. Let $\Lambda(F)(x)$ be a quantity defined by

$$\Lambda(F)(x) = \eta^0_x(x) + \sigma(x)\ ,\ x \in F,$$

where η^0 is the minimizer of (1.5). In particular, we prove that

$$\Lambda(F^\mu)\ (x-\mu) \to \Lambda(F)\ (x)$$

as $\mu \to 0$, where $F^\mu = F - \mu = \{x | x + \mu \in F\}$ provided that σ_x is bounded. Moreover, the convergence is uniform with respect to F provided that F is bounded. This problem can be viewed as a stability problem for (1.5) with respect to perturbations of σ. Since our obstacle problem is convex, it is not difficult to prove these facts. We also need comparison results (maximum principle) for Λ^σ_W so see that it behaves like curvature or usual second derivatives. It is often convenient to consider $\xi = \eta + \int^x \sigma$ as a variable, instead of η itself, so we shall use variable ξ. We warn the reader that in Section 5 we will use differently defined ξ.

To establish the comparison principle we argue by contradiction using the doubling variables technique. Let u be a subsolution and v be a supersolution. We are interested in maximizers of

$$u(t,x) - v(s,y) - B_\varepsilon(x-y) - (t-s)^2/\delta - \gamma/(T-t) - \gamma/(T-s)$$

for small $\varepsilon, \delta, \gamma > 0$. Here $B_\varepsilon = \epsilon B(x/\varepsilon)$, $B(x) \backsim x^2$ for large x and B is a (non-negative) faceted C^2 convex function with $B(0) = 0$. This choice of a test function B is different from [GG98Ar] and this choice simplifies the argument. We use sup-convolutions with a faceted function to regularize the problem as in [GG98Ar]. The quantity Λ^σ_W behaves like usual second derivative in the sense that it satisfies the maximum principle. At the final stage we have to compare $\Lambda(F^\mu)$ and $\Lambda(F)$ which is trivial when σ is constant, because it is independent of μ.

Although this paper focuses on comparison principle for (1.1), as observed in [GG01Ar], the method developed here is fundamental to establish a level set method for $V = M(\mathbf{n})\ (\kappa_\gamma + \sigma)$ when σ depends on x. For a standard level set method for smooth γ see [CGG1], [ES1], [G]. Also a stability result is expected [GG99] but we do not intend to include any progress in this direction in the present paper. A general existence result through Perron's method is almost the same as that in [GG98Ar], though we do not state it explicitly. Instead, we give a couple of examples of solutions.

Recently, besides examples in [GG98DS] several semi-explicit variational solutions are constructed for (1.1) for a special choice of M, σ and γ by solving a free boundary problem [GR1], [GR2], [GGR]. Their variational solutions are expected to be our viscosity solutions. In this paper we shall confirm this consistency at least for some typical examples.

We do not know much about surface evolutions. In surface evolving problems a facet may not stay as a facet even if $\sigma \equiv 0$ see e.g. [BNP], [BNP1], [BNP2] and [BNP3]. A notion of a generalized solution is established and a comparison principle is proved in [BN], see also [BGN]. However, the existence of solution is known only when initial surface is convex see [BCCN]; note that their problem is formulated for $V = \gamma\kappa_\gamma$ where mobility parallels the interfacial energy.

The bibliographies of review papers [GGK], [G04], [GG04], [GG10] include several articles dealing with anisotropic curvature flow equations with singular interfacial energy or singular diffusion equations. Here we only mention a few recent works related to this topic but not included there. In particular, we have in mind the approach developed by Mucha and Rybka, which is based on an original definition of composition of multivalued operators, see [MR1], [MR2]. So far it is restricted to one dimension by allows to study facet evolution for quite general data and regularity of solutions.

This paper is organized as follows. We first study an obstacle problem in Section 2. In Section 3, we establish a notion of viscosity solutions. In Section 4, we prove our main comparison theorem. In section 5 we shall prove that the semi-explicit solutions in [GR1] are indeed solutions in our viscosity sense.

2 Variational properties of nonlocal curvature with a nonuniform driving force term

We shall give a variational characterization of the quantity Λ_W^σ, which is formally defined by

$$\Lambda_W^\sigma(u)\ (x) = (W'(u_x))_x + \sigma(x), \tag{2.1}$$

by solving an obstacle problem. This characterization enables us to derive various important properties to establish the theory of viscosity solutions for singular diffusion equations.

2.1 An obstacle problem

Let Z be a real-valued C^2 (or $C^{1,1}$) function defined in a bounded interval $\overline{I}$, where $I = (a, b)$. For a given $\Delta > 0$ let $K_{\chi_l\chi_r}^Z$ be the set of all $\xi \in H^1(I)$ satisfying

$$Z(x) - \Delta/2 \le \xi(x) \le Z(x) + \Delta/2 \text{ for } x \in I \text{ (obstacle condition)} \tag{2.2}$$

and

$$\xi(a) = Z(a) - \chi_l\Delta/2,\ \xi(b) = Z(b) + \chi_r\Delta/2 \text{ (boundary condition)}. \tag{2.3}$$

Here, χ_l and χ_r take values ± 1. Let $J_{\chi_l\chi_r}^Z$ be the functional on $L^2(I)$ defined by

$$J_{\chi_l\chi_r}^Z(\xi) = \begin{cases} \int_a^b |\xi'(x)|^2\, dx, & \xi \in K_{\chi_l\chi_r}^Z \\ \infty, & \text{otherwise.} \end{cases}$$

In this subsection, we suppress the dependence with respect to Z since we fix Z. By the definition of $J_{\chi_l\chi_r}$, it is easy to see that $\inf J_{\chi_l\chi_r}$ is the H^1 distance from zero to convex closed set $K_{\chi_l\chi_r}$ in H^1. Thus, $J_{\chi_l\chi_r}$ admits a unique absolute minimizer denoted by $\xi_{\chi_l\chi_r}$. Evidently, $\xi_{\chi_l\chi_r} \in H^1(I) \subset C^{1/2}(\overline{I})$ by the Sobolev embedding. In fact, it is $C^{1,1}$ as proved in [[KS], Chap II Theorem 7.1]. (In [KS] the regularity of multidimensional obstacle problem is also discussed.) In our one-dimensional case it is easy to prove that $\xi_{\chi_l\chi_r}$ is $C^{1,1}$ since the obstacle is $C^{1,1}$ as described below.

For $\xi \in H^1(I)$ let $D_\pm(\xi)$ be the coincidence set defined by

$$D_\pm = D_\pm(\xi) = \{x \in \overline{I} \mid \xi(x) = Z(x) \pm \Delta/2\}.$$

We say that D_+ is the *upper coincidence* set while D_- is the *lower coincidence* set.

Definition 2.1. We say that $\xi \in K_{\chi_l\chi_r}$ satisfies the *concave-convex condition* if ξ is concave outside the upper coincidence set D_+ and convex outside the lower coincidence set D_-, i.e., $\xi'' \le 0$ outside D_+ and $\xi'' \ge 0$ outside D_-. In particular, ξ is $C^{1,1}$ in I and $\xi'' = 0$ outside $D_- \cup D_+$.

Proposition 2.2 (A characterization of the minimizer). *The function $\xi \in K_{\chi_l\chi_r}$ is the minimizer of $J_{\chi_l\chi_r}$ if and only if ξ fulfills the concave-convex condition. In particular, $\xi_{\chi_l\chi_r}$ is $C^{1,1}$ in I and*

$$\sup_{x\in I} |\xi''_{\chi_l\chi_r}(x)| \le \sup_{x\in I} |Z''(x)|. \tag{2.4}$$

Proof. By convexity of $J_{\chi_l\chi_r}$ and the uniqueness of the minimizer, $\xi \in K_{\chi_l\chi_r}$ is the absolute minimizer if and only if ξ is a local minimizer of $J_{\chi_l\chi_r}$ i.e.,

$$\int_{D_+^c} \xi'\varphi' dx \ge 0, \ \int_{D_-^c} \xi'\varphi' dx \le 0$$

for all $\varphi \in H^1(I)$ satisfying $\varphi(a) = \varphi(b) = 0$ and $\varphi \ge 0$ in $D_+^c = \overline{I}\backslash D_+$ and for all $\varphi \in H^1(I)$ satisfying $\varphi(a) = \varphi(b) = 0$ and $\varphi \ge 0$ in $D_-^c = \overline{I}\backslash D_-$ by the obstacle condition (2.2) and the boundary condition (2.3). This is equivalent to the concave-convex condition; for equivalence of convexity in distribution sense and strong convexity see e.g., Schwartz [S] or Hörmander [H]. The remaining statement is a simple consequence of the concave-convexity condition. □

As a trivial application we give two cases, where the minimizer is explicitly written.

Corollary 2.3.

(i) If the concave hull Z_{cave} of Z in I is smaller than $Z + \Delta$, i.e., $Z_{\text{cave}} \le Z + \Delta$ in I, then $\xi_{+-} = Z_{\text{cave}} - \Delta/2$.

(ii) If the straight line function $\xi(x) = \xi(a)+(Z(b)-Z(a)+\Delta)\,(x-a)/(b-a)$ fulfills the obstacle condition (2.2), it is the minimizer of J_{++} provided that $\xi(a) = Z(a) - \Delta/2$ and $\xi(b) = Z(b) + \Delta/2$. Here, $J_{++} = J_{\chi_l\chi_r}$ when $\chi_l = \chi_r = 1$.

2.2 Comparison principle

So far we have fixed the interval I to define $\xi_{\chi_l\chi_r}$. We shall study the dependence of $\xi'_{\chi_l\chi_r}$ upon I. To clarify this we write $J_{\chi_l\chi_r,\ I}$ instead of $J^Z_{\chi_l\chi_r}$ and $\xi_{\chi_l\chi_r,\ I}$ instead of $\xi^Z_{\chi_l\chi_r}$. We set

$$\Lambda^{Z'}_{\chi_l\chi_r}(x,I)=\frac{d\xi_{\chi_l\chi_r,\ I}(x)}{dx}. \tag{2.5}$$

It is easy to observe that this quantity agrees with $\eta^0_x+\sigma$, when Z equals a primitive of σ. It is sufficient to take $\xi=\eta+Z$. The reason we write Z' instead of Z is that the derivative of $\xi^Z_{\chi_l\chi_r}$ depends on Z only through its derivative. We suppress Z' in (2.5) when we fix Z. We shall write Λ_{-+} etc. instead of writing $\Lambda_{\{-1\},\{+1\}}$.

Theorem 2.4 (Comparison principle). *Assume that I_1 and I_2 are bounded open intervals.*

(i) *If $I_2\subset I_1$, then*

$$\Lambda_{--}(x,I_2)\le\Lambda_{\pm\pm}(x,I_1)\le\Lambda_{++}(x,I_2)\ \text{for}\ x\in I_2. \tag{2.6}$$

(ii) *If $a\le c<b\le d$ for $I_1=(a,b)$, $I_2=(c,d)$, then for $x\in(c,b)$*

$$\Lambda_{\pm-}(x,I_1)\le\Lambda_{+\pm}(x,I_2),\ \ \Lambda_{-\pm}(x,I_2)\le\Lambda_{\pm+}(x,I_1). \tag{2.7}$$

This can be proved by a comparison principle for parabolic equations by an approximation as is done in Giga-Gurtin-Matias [GGM]. However, since the problem is one dimensional, we rather give an elementary proof.

Proof. It suffices to prove

(a) $\Lambda_{++}(x,I_1)\le\Lambda_{++}(x,I_2),\ \Lambda_{--}(x,I_2)\le\Lambda_{--}(x,I_1)$ for $x\in I_2\subset I_1$,

(b) $\Lambda_{-+}(x,I_1)\le\Lambda_{++}(x,I_1),\ \Lambda_{--}(x,I_1)\le\Lambda_{-+}(x,I_1)$ and

$\Lambda_{+-}(x,I_1)\le\Lambda_{++}(x,I_1),\ \Lambda_{--}(x,I_1)\le\Lambda_{+-}(x,I_1)$ for $x\in I_1$.

We begin with the proof of (a). Since the argument is symmetric, it suffices to prove the first inequality. We may assume that one of the end point of I_1 and I_2 is the same. By symmetry, it suffices to prove that

$$\Lambda_{++}(x,I_1)\le\Lambda_{++}(x,I_2),\ x\in I_2$$

with $I_1 = (a, c)$, $I_2 = (a, b)$ for c $(\geq b)$ sufficiently close to b. We divide the situation into two cases depending on the structure of the coincidence set of the minimizer $\xi_2 = \xi_{++,I_2}$.

Case 1. *There is $\delta > 0$ such that $(b-\delta, b)$ is not included in any coincidence set and the point $b - \delta$ is a point of lower coincidence set.*

In this situation the graph of ξ_2 is a straight line on $(b - \delta, b)$ and $\xi_2(b) = Z(b) + \Delta/2$. We extend $\xi_2(x)$ for $x \geq b$ such that the slope of ξ_2 is constant for $x \geq b - \delta$. The extension is still denoted ξ_2. If $Z(c) + \Delta/2 \geq \xi_2(c)$, it is clear that $\xi_1 = \xi_2$ on I_2, where $\xi_1 = \xi_{++,I_1}$. If $Z(c) + \Delta/2 < \xi_2(c)$ and c is close to b, the graph of ξ_1 is a straight line from $(b - \delta', c)$, $0 < \delta' < \delta$ and $\xi_1(b-\delta') = Z(b-\delta') - \Delta/2$ with $\xi_1'(b-\delta') = Z'(b-\delta')$. Moreover, ξ_1 $(\leq \xi_2)$ agrees with the concave hull of $Z(x) - \Delta/2$ in $(b - \delta, b - \delta')$. Thus, it is clear that $\xi_1' \leq \xi_2'$ for $x \in (b - \delta, b)$ where $\xi_i' = d\xi_i/dx$.

Case 2. *The minimizer ξ_2 agrees with the convex hull of $Z(x) + \Delta/2$ in $(b - \delta, b)$ for some (small) $\delta > 0$.*

If c is sufficiently close to b, then ξ_1 agrees with the convex hull of $Z(x) + \Delta/2$ in $(b - \delta, c)$. By comparison of slopes of the convex hull it is clear that $\xi_1' \leq \xi_2'$ on $(b - \delta, c)$. This completes the proof of (a).

We next prove (b). By symmetry it suffices to prove one of four inequalities. We shall prove that $\Lambda_{--}(x, I_1) \leq \Lambda_{-+}(x, I_1)$.

Let $\xi = \xi_{--,I_1}$ be the minimizer such that $\xi' = \Lambda_{--}(x, I_1)$ and η $(= \xi_{-+,I_2})$ be the minimizer such that $\eta' = \Lambda_{-+}(x, I_2)$. By the structure of minimizer (Proposition 2.2) if $\xi(x_0) = \eta(x_0)$ for $a < x_0 < b$, then $\xi = \eta$ on (a, x_0). Thus there exists the maximum $x_* \geq a$ such that $\xi = \eta$ on (a, x_*). If $\xi(x_*) = \eta(x_*) = Z(x_*) + \Delta/2$, then η is a convex hull of $Z(x) + \Delta/2$ in (x_*, b) while ξ is a concave hull of $\max(Z(x) - \Delta/2, \xi(x_*)I(x))$ where $I(x) = -\infty$ for $x \neq x_*$ and $I(x_*) = 1$. Thus it is easy to see that $\xi' \leq \eta'$ on I_1. A symmetric argument in the case of $\xi(x_*) = \eta(x_*) = Z(x_*) - \Delta/2$ yields $\xi' \leq \eta'$ on I_1. We have thus proved that $\Lambda_{--} \leq \Lambda_{-+}$.

2.3 Stability of curvature like quantity

Our goal in this section is to show that the curvature like quantity $\Lambda_{\chi_l \chi_r}(x, I)$ defined by (2.5) is 'continuous' with respect to change of the interval I. Stability result for Λ of the convex obstacle problem with respect to Z is essentially known in the literature e.g. [Rd, p.156, Chapter 5, Theorem 4.5 and Remark 4.6]. However, we rather give a proof for the reader's convenience since the situation is slightly different.

We recall several stability properties of $J_{\chi_l\chi_r}$. Let $\{Z^k\}_{k=1}^{\infty}$ be a sequence of real-valued C^2 (or $C^{1,1}$) functions in $\overline{I}$, where $I=(a,b)$. In this subsection we fix $\chi_l\chi_r$, so we often suppress its dependence and simply write $J^Z_{\chi_l\chi_r}$ for J and $J^{Z^k}_{\chi_l\chi_r}$ instead of J^k.

Proposition 2.5 (Lower semicontinuity). *Assume that Z^k uniformly converges to Z as $k\to\infty$, i.e., $Z^k\to Z$ in $C(\overline{I})$. Assume that ξ_k weakly converges to ξ in $L^2(I)$ as $k\to\infty$. Then $J(\xi)\le\liminf_{k\to\infty}J^k(\xi_k)$.*

Proof. We may assume that $\xi_k\in K^{Z^k}$. Since ξ^k-Z^k converges to $\xi-Z$ weakly in $L^2(I)$ and sign is conserved through weak limit, we observe that $\xi\in K^Z$. The desired conclusion now follows from the lower semicontinuity of H^1-norm with respect to L^2-weak convergence. □

Proposition 2.6 (Approximability). *Assume that Z^k converges to Z, with its first derivative, uniformly in $\overline{I}$ as $k\to\infty$, i.e., $Z^k\to Z$ in $C^1(\overline{I})$. Then for each $\xi\in L^2(I)$ there is a sequence $\xi_k\to\xi$ in $L^2(I)$ such that $J(\xi)=\lim_{k\to\infty}J^k(\xi_k)$.*

Proof. We may assume that $\xi\in K^Z$ since otherwise $\xi\notin K^{Z^k}$ for sufficiently large k. We set $\xi_k=\xi-Z+Z^k$ and observe that ξ_k is in K^{Z^k} by (2.3) and (2.4). Since $Z^k\to Z$, $(Z^k)'\to Z'$ uniformly in $\overline{I}$ as $k\to\infty$, the convergence $J(\xi_k)\to J(\xi)$ and $\xi_k\to\xi$ (as $k\to\infty$) in $L^2(I)$ is easily verified. □

These two above Propositions say that J^k converges to J in the sense of Mosco, i.e., both strong and weak Γ^- limits of J^k equal J. Thus we easily obtain the convergence of minimizers.

Proposition 2.7 (Convergence of minimizers). *Assume that $Z^k\to Z$ in $C^1(\overline{I})$ as $k\to\infty$. Let $\xi^k_{\chi_l\chi_r}$ be the minimizer of $J^k_{\chi_l\chi_r}$. Then $\xi^k_{\chi_l\chi_r}$ converges to $\xi_{\chi_l\chi_r}$ in $L^2(I)$ which is the minimizer of $J_{\chi_l\chi_r}$.*

Proof. Applying Proposition 2.6 to $\xi_{\chi_l\chi_r}$, we observe that $\{\min J^k\}_{k=1}^{\infty}$ is bounded. Since $H^1(I)$ is compactly embedded in $L^2(I)$, $\xi^k_{\chi_l\chi_r}$ subsequently converges to an element $\zeta\in L^2(I)$ as $k\to\infty$. By Proposition 2.5, we observe that

$$J(\zeta)\le\liminf_{k\to\infty}\min\ J^k.$$

For a given $\xi\in L^2(I)$, due to Proposition 2.6, there is always a sequence

$\xi_k \to \xi$ in $L^2(I)$ such that $J^k(\xi_k) \to J(\xi)$ as $k \to \infty$. Thus

$$J(\zeta) \geq \liminf_{k\to\infty} \min \ J^k.$$

Therefore, $J(\zeta) \leq J(\xi)$ so ζ must be the unique minimizer of J. Thus $\xi^k_{\chi_l\chi_r}$ converges $\xi_{\chi_l\chi_r}$ without taking a subsequence. $\square$

We define $\Lambda^k_{\chi_l\chi_r}(x,I)$ by (2.5) where Z is replaced by Z^k. We simply write $\Lambda^k_{\chi_l\chi_r}$ in place of $\Lambda^k_{\chi_l\chi_r}(x,I)$ and $\Lambda_{\chi_l\chi_r}$ instead of $\Lambda^{Z'}_{\chi_l\chi_r}(x,I)$ in the next Theorem.

Theorem 2.8 (Continuity with respect to Z'). *Assume that*

$$\sup_{k\geq 1} \sup_{x\in I} |(d/dx)^2 Z^k(x)| < \infty \text{ and } (Z^k)' \to Z' \text{ in } C(\overline{I}).$$

Then $\Lambda^k_{\chi_l\chi_r} \to \Lambda_{\chi_l\chi_r}$ in $C(\overline{I})$ as $k \to \infty$.

Proof. We may assume that $Z^k \to Z$ in $C^1(\overline{I})$ by adding a constant to fix a value at some point of I, for example $Z^k((a+b)/2) = 0,\ Z((a+b)/2) = 0$. By Proposition 2.7 we observe that $\xi^k_{\chi_l\chi_r} \to \xi_{\chi_l\chi_r}$ in $L^2(I)$. By Proposition 2.2 our assumption on the bound of the second derivative of Z^k implies that $|(d/dx)^2\xi^k_{\chi_l\chi_r}|$ is bounded by (2.4). Thus $\xi^k_{\chi_l\chi_r} \to \xi_{\chi_l\chi_r}$ in $C^1(\overline{I})$ so $\Lambda^k_{\chi_l\chi_r} \to \Lambda_{\chi_l\chi_r}$ in $C(\overline{I})$. $\square$

We are now in position to state continuity of $\Lambda_{\chi_l\chi_r}$ with respect to I. This notion will be explained below.

Theorem 2.9.

(i) Let Z be a C^2 (or locally $C^{1,1}$) function on $\mathbf{R}$. Then $\Lambda^{Z'}_{\chi_l\chi_r}(x,I)$ is continuous with respect to I.

(ii) Assume furthermore that $|Z''(x)|$ is bounded in $\mathbf{R}$. Then for each $r > 0$

$$\lim_{\mu\to 0} \sup_{0<b-a<r} \sup_{a<x<b} |\Lambda^{Z'}_{\chi_l\chi_r}(x,(a,b)) - \Lambda^{Z'}_{\chi_l\chi_r}(x-\mu,(a-\mu,b-\mu))| = 0.$$

(The convergence is uniform in Z' for Z such that $|Z''| \leq M_0$ for a given constant $M_0 > 0$.)

We have to clarify the continuity with respect to I. For two bounded intervals $I = (a,b)$ and $J = (c,d)$ there is a unique affine map A: $x \mapsto y = \alpha x + \beta$ (dilation and translation) with $\alpha > 0$ such that $A(I) = J$. Assume that an open interval I^k converges to I as $k \to \infty$, i.e., the end points

a_k, b_k of $I_k = (a_k, b_k)$ tend to a and b, respectively. Let F be a mapping: $I \mapsto F(I) \in C(\overline{I})$. We say that F is continuous with respect to I if $F(I_k) \circ A_k$ converges to $F(I)$ in $C(I)$, as $k \to \infty$ for any $I^k \to I$, where A^k is the affine map which maps I to I^k.

Proof. These assertions easily follow from Theorem 2.8, once we compare $\Lambda^{Z'}_{\chi_l \chi_r}(x, I)$ with $\Lambda^{Z'}_{\chi_l \chi_r}(A^k(x), I_k)$, both defined on I, here A^k is the affine transformation mapping I to I^k, when $I^k \to I$. (In the assertion (ii) this affine map is just a translation.) $\square$

2.4 Nonlocal curvature with a nonuniform driving force term

In order to define the nonlocal curvature $\Lambda^\sigma_W(u)$ formally given by (2.1) we recall basic assumptions on W as in [GG98Ar] and a class of function u so that $\Lambda^\sigma_W(u)$ is well-defined.

(W) Let W be a convex function on $\mathbf{R}$ with values in $\mathbf{R}$. Assume that W is of class C^2 outside a closed discrete set P and that W'' is bounded in any compact set except all points in P.

We shall always assume (W) in this paper. By definition the set P is either a finite set or a countable set having no accumulation points in $\mathbf{R}$. If P is nonempty, P is of form $\{p_j\}_{j=1}^m$, $\{p_j\}_{j=-\infty}^\infty$, $\{r_j\}_{j=-\infty}^{-1}$ or $\{p_j\}_{j=1}^\infty$ with $\lim_{j\to\infty} p_j = \infty, \lim_{j\to-\infty} r_j = -\infty$, where the p_j's and r_j's are arranged in strictly increasing sequences $p_j < p_{j+1}$, $r_j < r_{j+1}$ and m is a positive integer.

We recall a notion of a faceted function. Let Ω be an open interval. A function f in $C(\Omega)$ is called *faceted* at x_0 with *slope* p on Ω (or p-*faceted* at x_0) if there is a closed nontrivial finite interval $I(\subset \Omega)$ containing x_0 such that f agrees with an affine function

$$\ell_p(x) = p(x - x_0) + f(x_0) \text{ in } I$$

and $f(x) \neq \ell_p(x)$ for all $x \in J \backslash I$ with some neighborhood $J(\subset \Omega)$ of I. The interval I is called a *faceted region* of f containing x_0 and is denoted by $R(f, x_0)$. A function f is called P-faceted at x_0 if it is p-faceted at x_0 for some $p \in P$.

We introduce the *left transition number* $\chi_l = \chi_l(f, x_0)$ and the *right transition number* $\chi_r = \chi_r(f, x_0)$ by

$$\chi_l = \begin{cases} +1 & \text{if } f \geq \ell_{p_i} \ \text{ in } \ \{x \in J | x \leq x_0\} \\ -1 & \text{if } f \leq \ell_{p_i} \ \text{ in } \ \{x \in J | x \leq x_0\} \end{cases}$$

$$\chi_r = \begin{cases} +1 & \text{if } f \geq \ell_{p_i} \ \text{ in } \ \{x \in J | x \geq x_0\} \\ -1 & \text{if } f \leq \ell_{p_i} \ \text{ in } \ \{x \in J | x \leq x_0\} \end{cases}$$

if f is p_i-faceted at x_0. The quantity $\chi = (\chi_l + \chi_r)/2$ is called the *transition number* describing the sign of Λ_W^σ when $\sigma \equiv 0$.

Definition 2.10. We assume that σ is a real-valued Lipschitz function on an open interval Ω and Z is its primitive, moreover, (W) holds. We assume that $f \in C(\Omega)$ p_i-faceted at $x_0 \in \Omega$ with $p_i \in P$. Then we define the *nonlocal curvature* Λ_W^σ by

$$\Lambda_W^\sigma(f)\ (x_0) = \Lambda_{\chi_l\chi_r}^{Z'}(x, I);$$

the right hand side is defined by (2.5) with $\Delta = W'(p_i + 0) - W'(p_i - 0)$ and I is the faceted region $R(f, x_0)$. If f is twice differentiable at x_0 and $f'(x_0) \notin P$, we set, as expected,

$$\Lambda_W^\sigma(f)\ (x_0) = W''\left(f'(x_0)\right) f''(x_0) + \sigma(x_0).$$

Remark 2.11. If σ is a constant, so that Z is an affine function, the minimizer $\xi_{\chi_l\chi_r}$ of $J_{\chi_l\chi_r}^Z$ is always a straight line function (cf. Corollary 2.3 for the case $\chi = 1$ or -1). Thus, it is easy to observe that

$$\Lambda_W^\sigma(f)\ (x_0) = \chi\Delta/L(f, x_0) + \sigma(x_0)$$

when f is p_i-faceted at x_0, where $L(f, x_0)$ is the length of the faceted region $R(f, x_0)$. In particular, our new quantity agrees with the weighted curvature $\Lambda_W(f, x_0)$, defined in [GG98Ar] when $\sigma \equiv 0$. Like $\Lambda_W(f, x_0)$, the quantity Λ_W^σ depends on W only through its second distributional derivative.

We conclude this section by rewriting Comparison Principle and Continuity with respect to translation in terms of Λ_W^σ. Let $C_P^2(\Omega)$ be the set of $f \in C^2(\Omega)$ such that f is P-faceted at x_0 whenever $f'(x_0) \in P$. For such a class of function the nonlocal curvature $\Lambda_W^\sigma(f)\ (x)$ is well-defined for all $x \in \Omega$ provided that σ is locally Lipschitz. The next two results are immediate consequences of Theorem 2.4 and Theorem 2.9, respectively.

Theorem 2.12 (Comparison). *Assume* (W) *and that σ is locally Lipschitz and in addition $f, g \in C^2_P(\Omega)$ and $x_0 \in \Omega$. If $\max_\Omega(f-g) = (f-g)\ (x_0)$, then $\Lambda^\sigma_W(f)\ (x_0) \le \Lambda^\sigma_W(g)\ (x_0)$.*

Theorem 2.13 (Continuity). *Let us suppose that the hypotheses of Theorem 2.12 concerning W and σ. We assume that $f \in C(\Omega)$ is p_i-faceted at $x_0 - \eta$ and g be p_i-faceted at $x_0 - \eta$ and $p_i \in P$. Assume moreover, $R(f, x_0) - \eta = R(g, x_0 - \eta)$. Then*

$$\Lambda^\sigma_W(g)\ (x_0 - \eta) \to \Lambda^\sigma_W(f)\ (x_0) \text{ as } |\eta| \to 0.$$

3 Definitions of generalized solutions

The goal of this section is to define a generalized solutions (in the viscosity sense) for evolution equations of the form

$$u_t + F\left(t, u_x, \Lambda^\sigma_W(u)\right) = 0 \tag{3.1}$$

when W is a singular interfacial energy. Such a notion is given when $\sigma \equiv 0$ in [GG98Ar]. Our definition will be a natural extension to the case when $\sigma \not\equiv 0$. In this section, we shall also give several equivalent definitions for later use.

3.1 Admissible functions and definition

We first recall a natural class of test function. Let us set $Q = (0, T) \times \Omega$, where Ω is an open interval and $T > 0$. Let $A_P(Q)$ be the set of all *admissible* functions ψ on Q in the sense of [GG98Ar] i.e., ψ is of the form

$$\psi(x, t) = f(x) + g(t),\ f \in C^2_P(\Omega),\ g \in C^1(0, T).$$

For our equation we often assume that

(F1) F is continuous in $[0, T] \times \mathbf{R} \times \mathbf{R}$ with values in $\mathbf{R}$,

(F2) (Monotonicity) $F(t, p, X) \le F(t, p, Y)$ for $X \ge Y,\ t \in [0, T],\ p \in \mathbf{R}$, (FL) (Lipschitz continuity.) There is a constant $C = C_{F,T}$ such that

$$|F(t,p,X) - F(t,p,Y)| \le C(1+|p|)\,|X-Y| \text{ for all } t \in [0,T],\ p, X, Y \in \mathbf{R}.$$

(FT) (Uniform continuity in curvature and time.) For each K the function $F(t,p,X)$ is uniformly continuous in $[0,T] \times [-K,K] \times \mathbf{R}$.

The third assumption in rather standard when $W \equiv 0$ and σ is Lipschitz so that $\Lambda_W^\sigma(u) = \sigma$. A typical example of (3.1) satisfying (F1), (F2), (FL) and (FT) is of the form

$$u_t - a(u_x)\ \Lambda_W^\sigma(u) - C(t) = 0 \tag{3.2}$$

where

$$F(t,p,X) = -a(t,p)X;$$

here $a \in C(\mathbf{R})$ satisfies $0 \le a(p) \le C(|p|+1)$ for all $p \in \mathbf{R}, C \in C[0,T]$. If $a(p) = (1+p^2)^{1/2}$ and $C \equiv 0$, then (3.2) says that the normal velocity V of the graph of u equals the nonlocal curvatures i.e., $V = \Lambda_W^\sigma$. The condition (FT) is redundant if F is independent of t since (FL) implies (FT).

The driving force term σ may depend on t. Here is an assumption we often use.

(S) The function $\sigma \in C\left([0,T] \times \overline{\Omega}\right)$ is Lipschitz in space uniformly in time, i.e. there is a constant L_T such that

$$|\sigma(t,x) - \sigma(t,y)| \le L_T|x-y|$$

for all $t \in [0,T],\ x, y \in \overline{\Omega}$.

We are now in position to give a notion of a generalized solution in the viscosity sense.

Definition 3.1. Assume (W), (S), (F1), (F2). A real-valued function u on Q is a (viscosity) *subsolution* of (3.1) in Q if the upper-semicontinuous envelope $u^* < \infty$ in $[0,T) \times \overline{\Omega}$ and

$$\psi_t(\hat{t},\hat{x}) + F\left(\hat{t}, \psi_x(\hat{t},\hat{x}),\ \Lambda_W^{\sigma(\hat{t},\cdot)}(\psi(\hat{t}))\ (\hat{x})\right) \le 0 \tag{3.3}$$

whenever $\left(\psi, (\hat{t},\hat{x})\right) \in A_P(Q) \times Q$ fulfills

$$\max_Q(u^* - \psi) = (u^* - \psi)\ (\hat{t},\hat{x}). \tag{3.4}$$

Here $\psi(\hat{t})$ is a function on Ω defined by $\psi(\hat{t}) = \psi(\hat{t}, \cdot)$ and u^* is defined by

$$u^*(t,x) = \lim_{\varepsilon \downarrow 0} \sup\{u(s,y) |\ |s-t| < \varepsilon,\ |x-y| < \varepsilon,\ (s,y) \in Q\}$$

for $(t,x) \in \overline{Q}$ and $u_* = (-u^*)$. A (viscosity) *supersolution* is defined by replacing $u^*(< \infty)$ by the lower-semicontinuous envelope $u_*(> -\infty)$, max by min in (3.4) and the inequality (3.3) by the opposite one. If u is both a sub- and supersolution, u is called a *viscosity solution* or a *generalized solution*. Hereafter we avoid using the word viscosity. Function ψ satisfying (3.4) is called a test function of u at $(\hat{t}, \hat{x})$. The monotonicity (F2) and the convexity (W) say that the equation is at least degenerate parabolic, so by comparison (Theorem 2.12) it is easy to see that $\psi \in A_P(Q)$ is a subsolution in Q if (and only if) ψ satisfies

$$\psi_t(t,x) + F\left(t, \psi_x(t,x),\ \Lambda_W^{\sigma(t,\cdot)}(\psi(t))(x)\right) \le 0$$

for all $(t,x) \in Q$.

3.2 An equivalent definition

To show comparison principle for sub- and supersolutions, it is convenient to recall equivalent definitions. One of them is regarded as an infinitesimal version. Such a definition is given in [GG98Ar] when $\sigma \equiv 0$. It is simplified by [GG01Ar]. We give a definition which is a natural extension of the one in [[GG01Ar], Theorem 4.3].

We first recall upper time derivations on a faceted region. Let φ be a function on Q and $(\hat{t}, \hat{x}) \in Q$. Assume that $\varphi(\hat{t}, \cdot) \in C(\Omega)$ is p-faceted at $x \in \Omega$ with $p \in P$. We define

$$\begin{gathered}\mathcal{T}_P^+ \varphi(\hat{t}, \hat{x}) = \{\tau \in \mathbf{R} \mid \text{there are a modulus} \\ \omega \text{ and three positive numbers } \delta, \delta_+, \delta_- \text{ such that} \\ \varphi(t,x) - \varphi(\hat{t}, \hat{x}) \le \tau(t - \hat{t}) + p(x - \hat{x}) + \omega(|\hat{t} - t|)\ |t - \hat{t}| \\ \text{for } (t,x) \in (\hat{t} - \delta,\ \hat{t} + \delta) \times \tilde{N}^{-1}(\varphi(\hat{t}, \cdot), \hat{t}; \delta_+, \delta_-)\},\end{gathered}$$

where $\tilde{N}^{-1}$ denotes a semineighborhood of $\mathbf{R}(\varphi(\hat{t}, \cdot), \hat{x})$ defined in [GG98Ar]; by a modulus ω we mean that $\omega : [0, \infty) \to [0, \infty)$ is nondecreasing, continuous with $\omega(0) = 0$. For the reader's convenience, we recall the definition of

$\tilde{N}^{-1}$. Let $f \in C(\Omega)$ be p-faceted at $x_0 \in \Omega$ with $p \in P$. We set

$$N_{\chi_r}(f, x_0; \delta_+) = \begin{cases} \{x \in \Omega | \sup R(f, x_0) < x \le \sup R(f, x_0) + \delta_+\} & \text{if } \chi_r(f, x_0) = -1, \\ \emptyset & \text{if } \chi_r(f, x_0) = 1 \end{cases}$$

$$N_{\chi_l}(f, x_0; \delta_-) = \begin{cases} \{x \in \Omega | \inf R(f, x_0) - \delta_- \le x < \inf R(f, x_0)\} & \text{if } \chi_l(f, x_0) = -1, \\ \emptyset & \text{if } \chi_l(f, x_0) = 1 \end{cases}$$

and the set $\tilde{N}^{-1}$ is defined by

$$\tilde{N}^{-1}(f, x_0; \delta_-, \delta_+) = R(f, x_0) \cup N_{\chi_r}(f, x_0; \delta_+) \cup N_{\chi_l}(f, x_0; \delta_-).$$

The set $\tilde{N}^{+1}$ is defined by

$$\tilde{N}^{+1}(f, x_0; \delta_-, \delta_+) = \tilde{N}^{-1}(-f, x_0; \delta_-, \delta_+).$$

An element of $\mathcal{T}_P^+\varphi(\hat{t}, \hat{x})$ is an upper time derivative at $(\hat{t}, \hat{x})$. The set of lower time derivative defined by

$$\mathcal{T}_P^-\varphi(\hat{t}, \hat{x}) = -\mathcal{T}_{-P}^+(-\varphi)\ (\hat{t}, \hat{x}).$$

We next recall a class of functions (not necessarily admissible) for which upper time derivative is well-defined on a faceted region. The following definition is an improved one in [GG01Ar] not the original one in [GG98Ar]. In [GG01Ar] Q may not be noncylindrical but here we consider a simple case $Q = (0, T) \times \Omega$.

Definition 3.2. Let $\varphi : \Omega \to \mathbf{R}$ be an upper-semicontinuous function. For $(\hat{t}, \hat{x}) \in Q$ assume that $\varphi(t, \cdot) \in C(\Omega)$ for t near $\hat{t}$. We say that φ is an (*infinitesimally*) *admissible superfunction* at $(\hat{t}, \hat{x})$ in Q if one of following three conditions holds.

(A) The function $\varphi(\hat{t}, \cdot)$ is P-faceted (in Ω) at $\hat{x} \in$ int $R(\varphi(\hat{t}, \cdot), \hat{x})$. The set $\mathcal{T}_P^+\varphi(\hat{t}, \hat{x})$ is nonempty.

(B) There is $(\tau, p, X) \in \mathcal{P}^+\varphi(\hat{t}, \hat{x})$ with $p \notin P$, where $\mathcal{P}^+$ denotes the set of parabolic semijets in Q [CIL], [GG98Ar].

(C) The function $\varphi(\hat{t}, \cdot)$ is P-faceted at $\hat{x}$ but $\hat{x} \in \partial R(\varphi(\hat{t}, \cdot), \hat{x})$. There is an element $(\tau, p, 0) \in (\mathcal{P}^+\varphi(\hat{t}, \hat{x}))$ for some $\tau \in R$.

We say that φ is an *admissible subfunction* at $(\hat{t},\hat{x})$ in Q if φ is an admissible superfunction with P replaced by $-P$. We implicitly assume that $R\left(\varphi(\hat{t},\cdot),\hat{x}\right)$ does not touch the boundary of Ω. We are now in position to give a definition of subsolution in the infinitesimal sense.

Definition 3.3. Assume (W), (S), (F1), (F2). A real-valued function u on Q is a *subsolution in the infinitesimal sense* of (3.1) (in Q) if $u^* < \infty$ in $[0,T)\times\overline{\Omega}$ and the following conditions are fulfilled. For $(\hat{t},\hat{x})$ let φ be an admissible superfunction at $(\hat{t},\hat{x})$ in Q such that φ is a test function of u at $(\hat{t},\hat{x})$, i.e., (3.4) holds. Then

(i) $\tau + F\left(\hat{t},\varphi_x(\hat{t},\hat{x}),\Lambda_W^{\sigma(\hat{t},\cdot)}(\varphi(\hat{t},\cdot))(\hat{x})\right) \le 0$ for all $\tau\in\mathcal{T}_P^+\varphi(\hat{t},\hat{x})$ if (A) in Definition 3.2 holds;

(ii) $\tau + F(\hat{t},p,W''(p)X+\sigma(\hat{t},\hat{x})) \le 0$ for all $(\tau,p,X)\in\mathcal{P}^+\varphi(\hat{t},\hat{x})$ if (B) in Definition 3.2 holds;

(iii) $\tau + F(\hat{t},p,\sigma(\hat{t},\hat{x})) \le 0$ for all $(\tau,p,0)\in\mathcal{P}^+\varphi(\hat{t},\hat{x})$ if (C) in Definition 3.2 holds and
$$(u^*-\varphi)\ (\hat{t},x) < \max_Q (u^*-\varphi)$$
for all $x\in R\left(\varphi(\hat{t},\cdot),\hat{x}\right)\setminus\{\hat{x}\}$ near $\hat{x}$.

The definition of *supersolution in the infinitesimal sense* is given by replacing $u^*(<\infty)$ by $u_*(>-\infty)$, max by min in (3.4), supersolution by subfunction, $\mathcal{T}_P^+$ by $\mathcal{T}_P^-$, $\mathcal{P}^+$ by $\mathcal{P}^-$ and the inequalities in (i), (ii), (iii) by the opposite ones. It turns out that Definition 3.1 and Definition 3.3 are equivalent.

Theorem 3.4 (Equivalence). *Assume (W), (S), (F1), (F2). A real-valued function u on Q is a subsolution (resp. supersolution) of (3.1) in Q if and only if u is a subsolution (resp. supersolution) of (3.1) in Q in the infinitesimal sense.*

The proof essentially parallels that of [[GG98Ar], Theorem 6.9] and [[GG01Ar], Theorem 4.3]. In the proof of the 'only part', (iii) follows from the zero-curvature lemma [[GG01Ar], Lemma 4.2] with a trivial modification. We give a modified version of this lemma for reader's convenience. We do not repeat the tedious detail of the proof of the 'only if' part. The proof of the 'if' part is easier and written in the proof of [[GG01Ar], Theorem 4.3]; of course we need trivial modifications for example $\Lambda_W\left(\psi(\hat{t},\cdot),\hat{x}\right) < 0$ should be replaced by $\chi\left(\psi(\hat{t},\cdot),\hat{x}\right) < 0$.

Lemma 3.5 (Zero curvature). Let u be a subsolution of (3.1) in Q. Assume that $\varphi \in A_P(Q)$ and that

$$\max_Q(u^* - \varphi) = (u^* - \varphi)\ (\hat{t},\hat{x})$$

for $(\hat{t},\hat{x}) \in Q$. If $\hat{x}$ is an end point of a faceted region $R\left(\varphi(\hat{t},\cdot),\hat{x}\right)$ with $\varphi_x(\hat{t},\hat{x}) \in P$ and $(u^* - \varphi)\ (\hat{t},x) < (u^* - \varphi)\ (\hat{t},\hat{x})$ for all $x \in R\left(\varphi(\hat{t},\cdot),\hat{x}\right)$ near $\hat{x}$, then

$$\varphi_t(\hat{t},\hat{x}) + F\left(\hat{t},\varphi_x(\hat{t},\hat{x}),\sigma(\hat{t},\hat{x})\right) \le 0.$$

4 Comparison principle

We state our main comparison result for equation (3.1).

Theorem 4.1 (Comparison). *Assume that condition (W), (S), (F1), (F2), (FL) and (FT) hold. Assume that P is a finite set. Let u and v be respectively sub- and supersolutions of (3.1) in $Q = (0,T)\times\Omega$, where Ω is a bounded open interval. If $u^* \le v_*$ on the parabolic boundary $\partial_p Q (= [0,T)\times\partial\Omega \cup \{0\}\times\overline{\Omega})$ of Q, then $u^* \le v_*$ in Q.*

The proof will be given in the remaining part of this section. The basic strategy is in finding suitable test functions of u and v to obtain a contradiction after having assumed that the conclusion $u^* \le v_*$ had been false. This basic strategy is the same as in [GG98Ar]. However, the nonlocal curvature may depend on x even if x is in a faceted region. So one should be careful on this issue. This is a new aspect of the problem. On the other hand since the infinitesimal version of definitions of sub- and supersolutions are simplified compared with [GG98Ar], we need not to avoid to handle the case where functions take a maximum value at end points of faceted regions. In fact, it is mentioned in [GG01Ar] that the proof of [GG98Ar] is simplified.

4.1 Doubling variables

As usual we double the variables. For $z = (t,x),\ z' = (s,y) \in Q$, we set

$$w(z,z') = u(z) - v(z').$$

We take a barrier function which is different from the one in [GG98Ar]. Let $B \in C_P^2(\mathbf{R})$ be a function such that B is convex, $xB'(x) \geq 0$ for all $x \in \mathbf{R}$ with $B(0) = 0$ and

$$0 < \underline{\lim}_{|x|\to\infty} B'(x)/x,\ \overline{\lim}_{|x|\to\infty} B'(x)/x < \infty.$$

Moreover, the length of all faceted regions is the same. It is easy to find the derivative B' of such a function by modifying $y = x$, so that B is obtained as its primitive. We consider its rescaled version: $B_\varepsilon(x) = \varepsilon B(x/\varepsilon)$ for $\varepsilon > 0$. Clearly, $B_\varepsilon \in C_P^2(\mathbf{R})$ and satisfies the same properties as B's. We consider 'barrier functions' of the diagonal $z = z'$:

$$\Psi(z, z'; \varepsilon, \delta, \gamma, \gamma') = B_\varepsilon(x - y) + S(t, s; \delta, \gamma, \gamma')$$

$$S(t, s, \delta, \gamma, \gamma') = (t - s)^2/\delta + \gamma/(T - t) + \gamma'/(T - s)$$

for positive parameters $\varepsilon, \delta, \gamma, \gamma'$. (In [GG98Ar] we use $|x-y-\zeta|^2/\varepsilon^2$ instead of $B_\varepsilon(x - y)$, where ζ is an extra shift parameter used to avoid the situation when a point we are dealing with is an end point of faceted regions.) We often write $\Psi(z, z')$ and $S(t, s)$ instead of showing the dependence on all positive parameters. As usual, we shall analyze maximizers of

$$\Phi(z, z') = w(z, z') - \Psi(z, z').$$

4.2 Choice of parameters

We shall choose $\varepsilon, \delta, \gamma, \gamma'$ sufficiently small as usual. The next statement for behavior of maximizer of Φ is rather standard in the process of doubling variables; see e.g., [GGIS], [[GG98Ar] Proposition 7.1], [G].

Proposition 4.2. *Assume that u and $-v$ are upper-semicontinuous in $[0, T) \times \overline{\Omega}$ with values in $\mathbf{R} \cup \{-\infty\}$ and $u = u^*$, $v = v_*$ including $\{T\} \times \overline{\Omega}$, where Ω is an open set in $\mathbf{R}$. Assume that $m_0 = \sup_{z\in Q} w(z, z) > 0$.*

- *(i) For each m_0', $(0 < m_0' < m_0)$, there are $\gamma_0, \gamma_0' > 0$ such that $\sup_{Q\times Q} \Phi > m_0'$ for all $\varepsilon > 0,\ \delta > 0,\ \gamma_0 > \gamma > 0,\ \gamma_0' > \gamma' > 0$.*

- *(ii) (Behavior of a maximizer) Let $(\hat{z}, \hat{z}') = (\hat{t}, \hat{x}, \hat{s}, \hat{y})$ be a maximizer of Φ over $\overline{Q} \times \overline{Q}$. Then*

  $$|\hat{t} - \hat{s}| \leq M\delta^{1/2},\ B_\varepsilon(\hat{x} - \hat{y}) \leq M$$

  with $M = \sup_{\overline{Q}\times\overline{Q}} w$ for all $\varepsilon > 0,\ \delta > 0,\ \gamma_0 > \gamma > 0$ and $\gamma_0' > \gamma' > 0$. Moreover, $|\hat{t}-\hat{s}|^2/\delta \to 0,\ B_\varepsilon(\hat{x}-\hat{y}) \to 0$ since $M \to m$ as $\varepsilon \to 0,\ \delta \to 0$.

(iii) *(Effect of boundary condition) Assume that $u \le v$ on $\overline{\partial}_p Q (= \overline{\partial_p Q})$ and that Ω is a bounded open interval. Then, there are ε_0, δ_0 such that $(\hat{z}, \hat{z}')$ is an (interior) point of $Q \times Q$ for all $0 < \varepsilon < \varepsilon_0,\ 0 < \delta < \delta_0,\ 0 < \gamma < \gamma_0,\ 0 < \gamma' < \gamma_0'$.*

Remark 4.3. Since w is upper-semicontinuous, we may assume in (iii) that for each $\xi > 0$

$$w(z, z') \le \xi \text{ for } (z, z') \in \overline{\partial}_p Q \times \overline{Q} \cup \overline{Q} \times \overline{\partial}_p Q$$

satisfying $B_\varepsilon(x - y) < M,\ |t - s|^2/\delta < M$ with $z = (t, s),\ z' = (s, y)$.

In the sequel, we assume that $m_0 > 0$ with $\xi = \frac{1}{4} m_0,\ m_0' = m_0 - \xi/2$ and we fix $\varepsilon_0, \delta_0, \gamma_0, \gamma_0'$ so that all properties (i)-(iii) and these in Remark 4.3 hold.

4.3 Maximizers in a faceted region of test functions

We shall consider three cases depending on the location of maximizers $(\hat{z}, \hat{z}') = (\hat{t}, \hat{x}, \hat{s}, \hat{y})$ of Φ over $\overline{Q} \times \overline{Q}$.

Case A: $\hat{p} = B'(\hat{x} - \hat{y}) \in P$ and $\hat{x} - \hat{y} \in \text{ int } R(B_\varepsilon, \hat{x} - \hat{y})$.

Case B: $\hat{p} = B'(\hat{x} - \hat{y}) \notin P$.

Case C: $\hat{p} = B'(\hat{x} - \hat{y}) \in P$ and $\hat{x} - \hat{y} \in \partial R(B_\varepsilon, \hat{x} - \hat{y})$.

Proposition 4.4. *Assume the conditions of Case A for $(\hat{z}, \hat{z}') = (\hat{t}, \hat{x}, \hat{s}, \hat{y}) \in Q \times Q$. Let u_0 and v_0 denote*

$$u_0(t, x) = u(t, x) - \hat{p}x,\ \ v_0(s, y) = v(s, y) - \hat{p}y$$

with $\hat{p} = B'(\hat{x} - \hat{y})$. Then $u_0(\hat{t}, \cdot),\ -v_0(\hat{s}, \cdot)$ take their local maxima at $\hat{x}$ and $\hat{y}$ respectively. Moreover,

$$u_0(t, x) - v_0(s, y) - S(t, s) \le u_0(\hat{t}, \hat{x}) - v_0(\hat{s}, \hat{y}) - S(\hat{t}, \hat{s})$$

for all $(x, y) \in \Sigma_\kappa,\ t, s, \in [0, T]$ for sufficiently small $\kappa > 0$ where

$$\Sigma_\kappa = \{(x, y) \in \overline{\Omega} \times \overline{\Omega} \mid |x - y - (\hat{x} - \hat{y})| < \kappa\}.$$

This follows from definition since B_ε is a P-faceted function. (We even do not invoke Proposition 4.2.)

Proposition 4.5 (No touching of faceted region on the boundary). *Assume the conditions of Case A for* $(\hat{z}, \hat{z}')$ *and choose parameters* $\varepsilon_0, \delta_0, \gamma_0, \gamma_0'$ *as in Remark 4.3. Assume that* $0 < \varepsilon < \varepsilon_0,\ 0 < \delta < \delta_0, 0 < \gamma < \gamma_0,\ 0 < \gamma' < \gamma_0'$. *Let* Ω *denote* $\Omega = (a, b)$. *Then there is* $x_1 \in (\hat{x}, b_1)$ *or* $y_1 \in (\hat{y}, b_2)$ *such that*

$$u_0(\hat{t}, x_1) < u_0(\hat{t}, \hat{x}) \text{ or } v_0(\hat{s}, y_1) > v_0(\hat{s}, \hat{y})$$

with $\eta = \hat{x} - \hat{y},\ b_1 = \min(b, b + \eta), b_2 = \min(b, b - \eta)$. *The same assertion is valid if* $(\hat{x}, b_1)$ *and* $(\hat{y}, b_2)$ *are replaced by* $(a, \hat{x})$ *and* $(a_2, \hat{y})$ *respectively, with* $a_1 = \max(a, a + \eta),\ a_2 = \max(a, a - \eta)$.

For the proof, we invoke Remark 4.3. The proof depends on the boundary condition (Proposition 4.2. (iii)) and it parallels that of [[GG98Ar], Proposition 7.10].

4.4 Existence of admissible superfunctions

Unfortunately, functions u_0 and v_0 may not be faceted at $\hat{x}$ and $\hat{y}$. We have to regularize them by taking sup-convolution with faceted functions. For $\rho > 0$ let $\vartheta(x, \rho)$ denote

$$\vartheta(x, \rho) = \begin{cases} (x - \rho)^2/\rho, & x > \rho, \\ 0 & |x| \le \rho, \\ (x + \rho)^2/\rho & x < -\rho. \end{cases}$$

We consider sup-convolutions of u_0 and $-v_0$ by ϑ. For $\alpha > 0$ let u_0^α be the sup-convolution of u_0 in the x-direction, i.e.,

$$u_0^\alpha(t, x) = (u_0(t, \cdot))^\alpha = \sup\{u_0(t, \xi) - \vartheta(\xi - x, \alpha); \xi \in \mathbf{R}\}$$

where we use the convention that $u_0 = -\infty$ if $\xi \notin \Omega$. The inf-convolution of v_0 is defined by $v_{0\rho} = -(-v_0)^\beta$ for $\beta > 0$. Functions $u_0^\alpha, v_{0\beta}$ are defined in $[0, T] \times \mathbf{R}$. Basing on these regularizations and the maximum principle for faceted sub- and supersolutions, the desired admissible super- and subfunctions are constructed. The proof is essentially the same as in [[GG98Ar], Proposition 7.12-7.15]. Although it is highly nontrivial, we do not repeat the proof.

Theorem 4.6. *Assume the condition of Case A and choose parameters* $\varepsilon_0, \delta_0, \gamma_0, \gamma_0'$ *as in Remark 4.3. Let* $0 < \varepsilon < \varepsilon_0,\ 0 < \delta < \delta_0,\ 0 < \gamma < \gamma_0$

and $0 < \gamma' < \gamma_0'$. *Then, there exists an admissible superfunction* U *at* $(\hat{t}, \hat{x})$ *in* Q *and an admissible subfunction* V *at* $(\hat{s}, \hat{y})$ *in* Q *satisfying the following properties.*

(i) U *and* V *are test functions of* u *and* v *at* $(\hat{t}, \hat{x})$ *and* $(\hat{s}, \hat{y})$ *respectively. In fact,*

$$\max_Q (u - U) = (u - U)(\hat{t}, \hat{x}) = 0, \ \min_Q (v - V) = (v - V)(\hat{s}, \hat{y}) = 0.$$

(ii) $U(\hat{t}, \cdot)$ *is* $\hat{p}$*-faceted at* $\hat{x} \in \operatorname{int} R\left(U(\hat{t}, \cdot), \hat{x}\right)$ *and* $\mathcal{T}_P^+ U(\hat{t}, \hat{x}) \ni S_t(\hat{t}, \hat{s})$*;* $V(\hat{s}, \cdot)$ *is* $\hat{p}$*-faceted at* $\hat{y} \in \operatorname{int} R\left(V(\hat{s}, \cdot), \hat{y}\right)$ *and* $\mathcal{T}_P^- V(\hat{s}, \hat{y}) \ni S_s(\hat{t}, \hat{s})$.

(iii) $R\left((U(\hat{t}, \cdot), \hat{x}\right) = R\left(V(\hat{s}, \cdot), \hat{y}\right) + (\hat{x} - \hat{y})$. *In particular,* $L\left(U(\hat{t}, \cdot), \hat{x}\right) = L\left(V(\hat{s}, \cdot), \hat{y}\right)$.

(iv) $\chi\left(U(\hat{t}, \cdot), \hat{x}\right) + \chi\left(-V(\hat{s}, \cdot), \hat{y}\right) \leq 0$.

The function $u_0^\alpha + p_0 x$ is essentially an admissible superfunction so we are tempted to set $U = u_0^\alpha + p_0 x$. However, faceted region may contain the boundary point of $\partial\Omega$. Since

$$u_0^\alpha(t, x) - v_{0\alpha}(s, y) \leq u_0^\alpha(\hat{t}, \hat{x}) - v_{0\alpha}(\hat{s}, \hat{y}) + \vartheta\left(x - y - \eta, \frac{\lambda_0}{2}\right) + S(t, s) - S(\hat{t}, \hat{s})$$

on $([0, T] \times \mathbf{R})^2$ for sufficiently small α as observed in [[GG98Ar], Proposition 7.13] we are able to apply the maximum principle for faceted functions [[GG98Ar], Corollary 4.6] to construct U. The properties (ii)–(iv) are obtained by the comparison principle for Λ_W^σ (Theorem 2.4, Theorem 2.12).

4.5 Proof of comparison theorem

We are now in position to prove Theorem 4.1. Suppose that the conclusion were false. We may assume that u and v satisfy the assumptions of Proposition 4.2 by considering u^* and v_* on $\overline{Q}$. In particular, we may assume $m_0 > 0$. We shall fix $\varepsilon_0, \delta_0, \gamma_0, \gamma_0'$ as in Remark 4.3 and assume that $0 < \varepsilon < \varepsilon_0$, $0 < \delta < \delta_0$, $0 < \gamma < \gamma_0$ and $0 < \gamma' < \gamma_0'$. Since $\overline{Q}$ is compact and u and $-v$ are upper-semicontinuous, there is always a maximizer $(\hat{z}, \hat{z}') = (\hat{t}, \hat{x}, \hat{y}, \hat{s})$ of Φ over $\overline{Q} \times \overline{Q}$ and it is in $Q \times Q$ by the choice of parameters (Proposition 4.2 (iii) and Remark 4.3). We shall fix γ and γ'. We divide the situations into three cases.

Case I. For sufficiently small $\varepsilon, \delta(>0)$ say $\varepsilon < \varepsilon_1(<\varepsilon_0), \delta < \delta_1(<\delta_0)$ there is a maximizer $(\hat{z}, \hat{z}')$ such that Case A occurs (for $\hat{x}$ and $\hat{y}$).

Case II. There is a sequence $\varepsilon_j \to 0,\ \delta_j \to 0$ such that there is a maximizer $(\hat{z}, \hat{z}')$ such that Case B occurs.

Case III. There is a sequence $\varepsilon_j \to 0,\ \delta_j \to 0$ such that there is a maximizer $(\hat{z}, \hat{z}')$ such that Case C occurs and there is no maximizer $(\hat{z}, \hat{z}')$ such that either Case A or Case B occurs.

In the Case I we invoke Theorem 4.6. Since U is an admissible superfunction at $(\hat{t}, \hat{x})$ in Q and since u is a subsolution we have, by Definition 3.2 and Theorem 4.6 (i), (ii).

$$S_t(\hat{t}, \hat{s}) + F\left(\hat{t}, \hat{p}, \Lambda_W^{\sigma(\hat{t},\cdot)}(U(\hat{t},\cdot))(\hat{x})\right) \leq 0 \tag{4.1}$$

Similarly,

$$-S_s(\hat{t}, \hat{s}) + F\left(\hat{s}, \hat{p}, \Lambda_W^{\sigma(\hat{s},\cdot)}(V(\hat{s},\cdot))(\hat{y})\right) \geq 0. \tag{4.2}$$

By Theorem 4.6 (iv) we have

$$\Lambda_W^{\sigma(\hat{t},\cdot)}(U(\hat{t},\cdot))(\hat{x}) = \Lambda_{\chi_l^U \chi_r^U}^{\sigma(\hat{t},\cdot)}(\hat{x}, I_U) \leq \Lambda_{\chi_l^V \chi_r^V}^{\sigma(\hat{t},\cdot)}(\hat{x}, I_U), \tag{4.3}$$

$$I_U = R(U(\hat{t},\cdot), \hat{x})$$

where χ_l^U and χ_r^U denote the transition numbers of $U(\hat{t},\cdot)$ on I_U and χ_l^V and χ_r^V denote the transition numbers of $V(\hat{s},\cdot)$ on $I_V = R(V(\hat{s},\cdot), \hat{y})$. Since we have assumed that P is a finite set, there is K such that $P \subset [-K, K]$. Thus, by (FT) and (F2), inequalities (4.1) and (4.3) yield

$$S_t(\hat{t}, \hat{s}) + F\left(\hat{s}, \hat{p}, \Lambda_{\chi_l^V \chi_r^V}^{\sigma(\hat{t},\cdot)}(\hat{x}, I_U)\right) - \omega_K(\hat{t} - \hat{s}) \leq 0 \tag{4.4}$$

with some modulus ω_K. By definition inequality (4.2) can be rewritten as

$$-S_s(\hat{t}, \hat{s}) + F\left(\hat{s}, \hat{p}, \Lambda_{\chi_l^V \chi_r^V}^{\sigma(\hat{s},\cdot)}(\hat{y}, I_V)\right) \geq 0. \tag{4.5}$$

Subtracting (4.5) from (4.4) yields

$$\frac{\gamma}{(T-\hat{t})^2} + \frac{\gamma'}{(T-\hat{s})^2} + F\left(\hat{s}, \hat{p}, \Lambda_{\chi_l^V \chi_r^V}^{\sigma(\hat{t},\cdot)}(\hat{x}, I_U)\right) - F\left(\hat{s}, \hat{p}, \Lambda_{\chi_l^V \chi_r^V}^{\sigma(\hat{s},\cdot)}(\hat{y}, I_V)\right) \leq \omega_K(|\hat{t} - \hat{s}|).$$

This implies

$$(\gamma+\gamma')/T^2 \le C(1+K)|\Lambda^{\sigma(\hat{t},\cdot)}_{\chi_l^V \chi_r^V}(\hat{x}, I_U) - \Lambda^{\sigma(\hat{s},\cdot)}_{\chi_l^V \chi_r^V}(\hat{y}, I_V)| + \omega_K(|\hat{t}-\hat{s}|) \quad (4.6)$$

by (FL). By Theorem 4.6(iii) we know $I_U = I_V + \hat{x} - \hat{y}$. Sending ε to zero we observe that $\hat{x} - \hat{y} \to 0$ by Proposition 4.2 (ii). By (S) we know that $\sigma_x(s, \cdot)$ is uniformly bounded. We now invoke continuity results (Theorem 2.8 and Theorem 2.9 (ii)) to get

$$\begin{aligned} &\Lambda^{\sigma(\hat{t},\cdot)}_{\chi_l^V \chi_r^V}(\hat{x}, I_U) \to \Lambda^{\sigma(\bar{t},\cdot)}_{\chi_l^V \chi_r^V}(\overline{x}, I), \\ &\Lambda^{\sigma(\hat{s},\cdot)}_{\chi_l^V \chi_r^V}(\hat{y}, I_V) \to \Lambda^{\sigma(\overline{s},\cdot)}_{\chi_l^V \chi_r^V}(\overline{x}, I) \end{aligned} \quad (4.7)$$

as $\varepsilon \to 0$, where $\overline{x}(=\overline{y}), \bar{t}, \overline{s}$ is a subsequent limit of $\hat{x}, \hat{y}, \hat{t}, \hat{s}$ as $\varepsilon \to 0$ and I is a subsequent limit of I_U which is the same as the limit of I_V. Note that U and V depend ε, so do I_U and I_V. However, the convergence is uniform with respect to the interval and σ, so we are able to obtain (4.7). Applying Theorem 2.8 and Theorem 2.9(ii) again to (4.7), we let $\delta \to 0$ and observe that the right hand sides of (4.7) converge to the same value. We now send $\varepsilon \to 0$ and then $\delta \to 0$ in (4.6) to get $(\gamma+\gamma')/T^2 \le 0$, which is a contradiction.

Case II is rather standard [GG98Ar], [CIL], [G]. The assumptions (FL) and (S) are useful in this step. Case III is essentially the same as Case I (or even easier) if one admits the zero curvature lemma (Lemma 3.5). □

4.6 Periodic version

As noted in [GG98Ar] a similar argument yields the comparison principle under spatially periodic boundary conditions. In fact, the argument is even simpler because there is no lateral boundary of $Q = (0, T) \times \mathbf{T}$, $\mathbf{T} = \mathbf{R}/\omega\mathbf{Z}$, $\omega > 0$. For the reader's convenience we state the comparison principle for the periodic boundary condition.

Theorem 4.7 (Comparison). *Let us assume that the conditions (W), (S), (F1), (F2), (FL) and (FT) hold and in addition set P is finite. Let u and v be respectively sub- and supersolutions of (3.1) in $Q = (0, T) \times \mathbf{T}$, $\mathbf{T} = \mathbf{R}/\omega\mathbf{Z}$ with period ω. If $u^* \le v_*$ at $t = 0$, then $u^* \le v_*$ in Q.*

Remark 4.8. As usual Theorem 4.1 and 4.7 can be extended to the case when $F = F(u, t, p, X)$ depends also on the value of u explicitly, provided that $u \mapsto F(u,t,p,X) + ku =: \tilde{F}$ is nondecreasing for some $k \geq 0$ and $\tilde{F}$ is continuous as a function of (u,t,p,X). Of course, the assumptions (FL) and (FT) should be uniform for all u with $|u| \leq K$ for a given K. If $k = 0$, the proof is the same except the trivial modification of the way of comparing (4.4) and (4.5). If $k > 0$, we have to introduce a new variable $\tilde{u} = u \exp(-kt)$ and reduce the problem to the case $k = 0$. Note that, differently from the standard case [G], when the singularity set P is empty, our singular set (jump discontinuity) for $\tilde{u}_x$ depends on time which apparently yields an extra difficulty. However, we are able to circumvent this difficulty by using old variables to calculate Λ and the slope, while using new variable $\tilde{u}$ and $\tilde{v}$ to find maximizer of Φ.

5 Examples of solutions

In [GR1], [GR2], [GGR] we constructed variational solutions to

$$\beta V - \kappa_\gamma = \sigma, \tag{5.1}$$

while increasing generality of the setting, where $\beta = M^{-1}$ is the kinetic coefficient. We considered graphs, possibly satisfying additional boundary condition, and simple closed Lipschitz curves we called bent rectangles. We will show that the variational solutions to (5.1) for evolution of graphs are viscosity solutions in the sense of the present paper. For the sake of illustration the theory we will not consider the general case of [GGR] but only simple ones presented in [GR1]. To be precise, we dealt with a simplification of the case studied in [GR1], where we investigated graphs of functions defined over a finite interval J. We considered solutions having exactly three facets and two of them touched the boundary at the right angle. Here, we study a graph over $\mathbf{R}$, with some restrictions on the data.

We expect that the results of the present paper may be applied to closed curves, but we will not elaborate upon this.

An advantage of studying graphs in the parametric approach is that the set of parameters is independent of time. Thus, the main difficulty is interpreting (5.1) in a local coordinate system. We present the setting after [GR1].

We specify the surface energy density (or anisotropy function) by formula

$$\gamma(p_1, p_2) = |p_1|\gamma_\Lambda + |p_2|\gamma_T, \ \ \gamma_\Lambda, \gamma_T > 0; \tag{5.2}$$

we assume a simplifying form of the kinetic coefficient $\beta = 1/M$

$$\beta(n_1, n_2) = \frac{1}{\max(|n_1|, |n_2|)}, \tag{5.3}$$

for $n_1^2 + n_2^2 = 1$. Subsequently, β is extended by 1-homogeneity to $\mathbf{R}^2$.

5.1 Graphs over R

We consider evolution of a graph $\Gamma(t) = \{(x, y) \in \mathbf{R}^2 : \ y = d(t, x)\}$, where $d(t, \cdot) : \mathbf{R} \to \mathbf{R}_+$. For the sake of simplicity we assume that function $d(t, \cdot)$ is admissible (in x) for all $t \geq 0$. We shall say that a function d is *admissible* provided that:
(a) d is Lipschitz continuous;
(b) d is even;
(c) it is bounded;
(d) $(\lambda_0, +\infty) \ni x \mapsto d(x)$ is strictly increasing, for a positive λ_0;
(e) $\{d_x = 0\} = (-\lambda_0, \lambda_0)$.

The last condition means that we consider a simple yet nontrivial case when d has exactly one faceted region. We stress, however, that the facet $(-l_0, l_0)$ may be strictly included in $(-\lambda_0, \lambda_0)$.

We have to explain the definition of κ_γ. Formally,

$$\kappa_\gamma = -\mathrm{div}_S(\nabla_\zeta \gamma(\mathbf{n})), \tag{5.4}$$

where $\mathbf{n}$ is the outer normal to Γ and for γ, given by (5.4), we have,

$$\nabla \gamma(p_1, p_2) = (\gamma_\Lambda \mathrm{sgn}(p_1), \gamma_T \mathrm{sgn}(p_2)).$$

In the present case $\mathbf{n} = (-d_x, 1)/\sqrt{1 + d_x^2}$. Thus, we immediately obtain

$$\frac{\beta(\mathbf{n}) d_t}{\sqrt{1 + d_x^2}} = \sigma + \gamma_\Lambda \frac{\partial}{\partial x} \left(\frac{d}{dp_1} |d_x| \right). \tag{5.5}$$

This is exactly equation (1.1) with $W(p_1) = \gamma_\Lambda |p_1|$ and $a(p_1) = \max\{|p_1|, 1\}$, hence our theory applies.

In [GR1], we interpreted (5.1) differently. Namely, we replaced gradient, $\nabla_\zeta \gamma$, which is defined only almost everywhere by the subdifferential, $\partial_\zeta \gamma$, which is well defined for *all* $p \in \mathbf{R}^2$, because γ is convex. However, we had to consider sections ξ of the subdifferential, i.e. $\xi(x) \in \partial_\zeta \gamma(\mathbf{n}(x))$. That is here, where we change notation as compared with the Introduction and Section 2. In the Introduction our present ξ was denoted by η. On the other hand, writing $\xi(x) \in \partial_\zeta \gamma(\mathbf{n}(x))$ is consistent with the papers being the source of our examples. $\xi(x) \in \partial_\zeta \gamma(\mathbf{n}(x))$.

As a result, we ended up with

$$\frac{\beta(\mathbf{n}) d_t}{\sqrt{1+d_x^2}} = \sigma - \tau \cdot \frac{\partial \xi}{\partial \tau}, \tag{5.6}$$

where τ is a unit tangent, (see [GR1, eq. (2.3)]).

In order to select ξ we introduced a functional

$$\mathcal{E}(\xi) = \frac{1}{2} \int_{\Gamma(t)} |\sigma - \mathrm{div}_S \xi|^2 \, d\mathcal{H}^1 \tag{5.7}$$

defined over $\mathcal{D}$,

$$\mathcal{D} = \{\xi \in L^\infty(\Gamma) : \ \xi(x) \in \partial\gamma(\mathbf{n}(x)), \ \mathrm{div}_S \xi \in L^2(\Gamma)\}. \tag{5.8}$$

The graph of $\Gamma(t)$ has infinite one-dimensional Hausdorff measure. But the condition $\mathrm{div}_S \xi \in L^2(\Gamma)$ does not introduce additional unexpected restrictions, because outside of the facets we have $\xi = \nabla\gamma(\mathbf{n})$, where $\mathbf{n} \neq \mathbf{n}_\Lambda, \mathbf{n}_R$ and $\mathbf{n}_\Lambda = (1,0)$, $\mathbf{n}_R = (0,1)$.

We call a couple (Γ, ξ) a *variational solution* to (5.1) provided that Γ is the graph of an admissible function d, as described above, and at each time instant t, the vector field $\xi(t,\cdot) : \Gamma \to \mathbf{R}^2$ is a minimizer of $\mathcal{E}$, i.e.

$$\mathcal{E}(\xi) = \min\{\mathcal{E}(\zeta) : \ \zeta \in \mathcal{D}\}. \tag{5.10}$$

We can show that under natural conditions on σ, equation (5.1) takes a form suitable for analysis.

We notice that if ξ is a solution to (5.10), then the boundary of the coincidence set $\pm l_0$ need not coincide with boundary of the flat region $\pm\lambda_0$ postulated by the definition of the admissible function, thus $l_0 \leq \lambda$. For the sake of simplicity of notation we shall write

$$R_0 := d|_{(-l_0, l_0)}.$$

Once we settle the notation we establish the following fact.

Proposition 5.1 *We assume that $\sigma, \sigma_x, \in C(\mathbf{R}_+ \times \mathbf{R})$ and σ satisfies the following conditions:*

$$\sigma(t,-x) = \sigma(t,x), \quad x\frac{\partial \sigma}{\partial x}(t,x) > 0, \quad \text{for } x \neq 0. \tag{5.11}$$

Let us suppose that (Γ, ξ) is a variational solution to (5.1), where $\Gamma = \Gamma(d)$ is the graph of d, such that at each time instant $t \geq 0$ $d(t,\cdot)$ has exactly one faceted region, $(-l_0, l_0)$. Furthermore, for all $t \geq 0$ function $d(t,\cdot)$ is piecewise C^1. Then,

(a) We have the following formula for ξ_1 for each time $t \geq 0$

$$\xi_1(t,x) = \begin{cases} x\left(⨍_0^x \sigma(t,s)\,ds - ⨍_0^{l_0} \sigma(t,s)\,ds\right) - \dfrac{x}{l_0}\gamma(\mathbf{n}_\Lambda) & \text{for } x \in [0, l_0); \\ -\gamma(\mathbf{n}_\Lambda) & \text{for } x \in [l_0, \infty); \end{cases} \tag{5.12}$$

where we write $⨍_A f\,d\mu = \frac{1}{\mu(A)} \int_A f\,d\mu$. In addition, $\dot{R}_0 > 0$.

(b) Equation (5.1) (and hence (5.6)) takes the following form,

$$\begin{aligned} \dot{R}_0 = ⨍_0^{l_0} \sigma(t,s)\,ds + \frac{\gamma(\mathbf{n}_\Lambda)}{l_0} \qquad & \text{on} \quad (-l_0, l_0); \\ d_t = \sigma \qquad & \text{on} \quad [l_0, \infty). \end{aligned} \tag{5.13}$$

Remark 5.2. The above result is based upon [GR1, Proposition 2.5], [GR2, Proposition 3.2] derived for graphs over $[-L, L]$ having three facets, two of them touching the boundary of $[-L, L]$. In the absence of the additional facets the argument gets simpler than in [GR1] and [GR2] and it is omitted.

Let us warn the reader that we use the notion 'faceted region' in the sense defined in the present paper. In [GR1], [GR2] its meaning is different.

It turns out that $l_0(\cdot)$ is a genuine free boundary. We obviously need information about its behavior. Without it the above system is not closed.

Let us suppose that $t \geq 0$, the necessary and sufficient condition for continuity of the function given below

$$\chi_{[0,l_0(t)]} R_0(t) + \chi_{(l_0(t),\infty)} d(t,x)$$

is the following *matching condition*

$$R_0(t) = d(t, l_0). \tag{5.14}$$

In addition, since we have a faceted region, the coincidence set of the obstacle problem (5.10) may not be empty. By definition, $\pm l_0$ form its boundary, i.e., $l_0 \le \lambda_0$, then at such a point

$$\frac{\partial \xi}{\partial x}(l_0) = 0. \tag{5.15}$$

We shall say that (Γ, ξ) satisfies the *tangency condition at* l_0.

However, if $d_x^+(l_0(t), t) > 0$, then we just have a boundary condition at this point and (5.15) does not hold.

We have the following two existence results.

Theorem 5.3 *Let us assume (5.3) and consider system (5.13) augmented with initial condition* (Γ_0, ξ_0), *where*

$$\Gamma_0 = \{(x, y) \in \mathbf{R}^2 : \; x \in \mathbf{R}, \; y = d_0(x)\},$$

d_0 *is an admissible function, satisfying* $|d_{0,x}(x)| < 1$ *for all* $x \in \mathbf{R}$. *In particular, the real, positive numbers* l_{00} $R_{00} = d|_{(-l_{00}, l_{00})}$, *are given. We assume that* σ *satisfies (5.11) Moreover, we impose the following conditions:*

(a) $d_0 \in C^1(\mathbf{R} \setminus (-l_{00}, l_{00}))$ *and for all* $x \in \mathbf{R} \setminus (-l_{00}, l_{00})$ *the derivative* $d_{0,x}$ *is different from zero;*

(b) *there is exactly one faceted region of* d_0, *where* $\Gamma(0) = \Gamma(d_0)$, *namely it is* $(-l_{00}, l_{00})$;

(c) *the matching condition (5.14) holds at* $t = 0$, *i.e.* $R_{00} = d_0(l_{00})$;

(d) *the tangency condition (5.15) is satisfied at* $t = 0$, *i.e.*

$$\sigma(0, l_{00}) = \fint_0^{l_{00}} \sigma(0, s)\, ds + \frac{\gamma(\mathbf{n}_\Lambda)}{l_{00}},$$

(e)

$$\Sigma_0 = \fint_0^{l_{00}} \sigma_t(0, y)\, dy - \sigma_t(0, l_{00}) < 0.$$

Then,

(i) There exists a unique local in time solution to (5.13), R_0 and $d(t,\cdot) \in C^1((-\infty,-l_0] \cup [l_0,\infty))$ and $d(t,\cdot)$ is strictly increasing, whose derivative $d_x(t,x)$ never vanishes for $x \in \mathbf{R} \setminus (-l_0(t), l_0(t))$;

(ii) The matching (5.14) and tangency (5.15) conditions hold for all times $t > 0$, that is if we extend $d(t,\cdot)$ to $\mathbf{R}$ by

$$\bar{d}(t,x) = \begin{cases} d(t,x) & \text{if } |x| \in [l_0,\infty); \\ R_0(t) & \text{if } |x| \in [0,l_0), \end{cases} \tag{5.16}$$

then $\bar{d}(t,\cdot)$ is Lipschitz continuous on $\mathbf{R}$. (Subsequently we drop the bar over the extension.)

(iii) If $\xi_1(t,x)$ is given by formula (5.12) for $x > 0$ and we set $\xi_1(t,x) = -\xi_1(t,-x)$ for $x < 0$, then $(\Gamma(d(t,\cdot)), \xi(t,\cdot))_{t\in[0,T)}$ is a variational solution to (5.1), provided that $\xi(t,\cdot) = (\xi_1(t,\cdot), \gamma(\mathbf{n}_R))$.

Remark 5.4. Let us stress again that l_{00} is defined as the boundary of the coincidence set

$$\{x : \ |\xi_1(x)| = \gamma_\Lambda\},$$

where ξ is a solutions to the variational problem (5.10). We note, that in general

$$[-l_0, l_0] \subset \{x : \ d_x(t,x) = 0\}$$

and the inclusion may be strict.

Theorem 5.5 *Let us suppose that all the assumptions of Theorem 5.2 hold, except (d) i.e. the tangency condition (5.15) and the inequality sign in (e) is reversed, i.e. we have*

$$\Sigma_0 > 0.$$

Instead of (5.15) the following inequality is satisfied

$$\sigma(0,l_{00}) - \fint_0^{l_{00}} \sigma(0,s)\,ds + \frac{\gamma(\mathbf{n}_\Lambda)}{l_{00}} < 0.$$

Moreover, we assume that $d_0 \in C^{1,1}([l_{00},\infty))$, the right derivative $d_x^+(0,l_{00})$ is positive and $\sigma \in C^{1,1}$. Then, there is a unique local in time solution *to (5.13), such that at no time $t > 0$ the tangency condition (5.15) holds. Subsequently, if $\xi(t, \cdot)$ is defined as in Theorem 5.3, (iii), then $(\Gamma(d(t, \cdot), \xi(t, \cdot))$ is a variational solution to (5.1).*

Remark 5.6. We note that l_0 is a genuine free boundary, its behavior is determined by σ. For instance if σ is independent of time and $\sigma = \sigma(x)$, then $l_0(t) = l_{00}$. The type of behavior of the interfacial curve is determined by Σ_0, this quantity is defined by [GR2, eq. (3.14)] and the properties of l_0 are presented in [GR2, Proposition 3.4].

These two Theorems are based upon [GR1, Theorem 2.10] and the analysis of [GR2, Section 3.1]. The present statements are easier than the original ones in [GR1, Theorem 2.10] and in [GR2, Section 3.1], because we deal with a single facet for a graph of an admissible function, but the main difference is that here we have an unbounded domain. For the sake of completeness, we offer a sketch of the proof in the Appendix.

5.2 Variational solutions are viscosity solutions and they are unique

Here, we shall see that our variational solution over $\mathbf{R}$ can be regarded as the viscosity solutions. Hence, they will be unique. The comparison principle has been shown for equation on a bounded domain, but our sub- and supersolutions are fully determined for large values of $|x|$, thus a comparison principle for bounded $|x|$ is sufficient. We will explain it in Corollary 5.8 following Theorem 5.7.

Theorem 5.7. *Under the conditions specified above the variational solutions constructed in Theorem 5.3 and in Theorem 5.5 are viscosity solutions in the sense of the present paper, as long as $|d_x| \leq 1$.*

Proof. Of course equation (5.5) augmented with the initial condition may be written as

$$\begin{cases} d_t + a(d_x)\Lambda^\sigma_W(d) = 0, \\ d(x, 0) = d_0(x) \end{cases}$$

where $\Lambda^\sigma_W(d) = \frac{d}{dx}\zeta_{\chi_l\chi_r}$ is given by (2.5) and the signs of χ_r, χ_l depend upon the point we consider. We will show that if $(\Gamma(d), \xi)$ is a variational solution, then $\Lambda^\sigma_W(d) = \sigma - \frac{\partial \xi}{\partial x}$, where ξ is given by (5.12) in Proposition 5.1. The interval $(-l_0, l_0)$ is the inverse image of a faceted region of Γ in the language of [GR1], [GR2], it is the faceted region in the present paper sense. If I is any interval containing $(-l_0, l_0)$, then $\bar{\xi} = \xi|_I$ is a solution to the minimization problem,

$$\min\{\mathcal{E}_I(\zeta) : \ \zeta \in \mathcal{D}_I\}. \tag{5.17}$$

We write, $\Gamma_I(t) = \{(x, y) \in \Gamma(t) : \ x \in I\}$ and

$$\mathcal{E}_I(\zeta) = \frac{1}{2}\int_{\Gamma_I(t)} |\sigma - \mathrm{div}_S\zeta|^2 \, d\mathcal{H}^1,$$

$$\mathcal{D}_I = \{\zeta \in L^\infty(\Gamma_I) : \ \zeta(x) \in \partial\gamma(\mathbf{n}(x)), \ \mathrm{div}_S\zeta \in L^2(\Gamma_I), \ \zeta = \xi|_{\partial I}\}.$$

The pictures below illustrate the cases $l_0 = l_1$ and $l_0 < l_1$ appearing in the coincidence set, where I is of the form $(-i_0, i_0)$ containing $(-l_1, l_1)$.

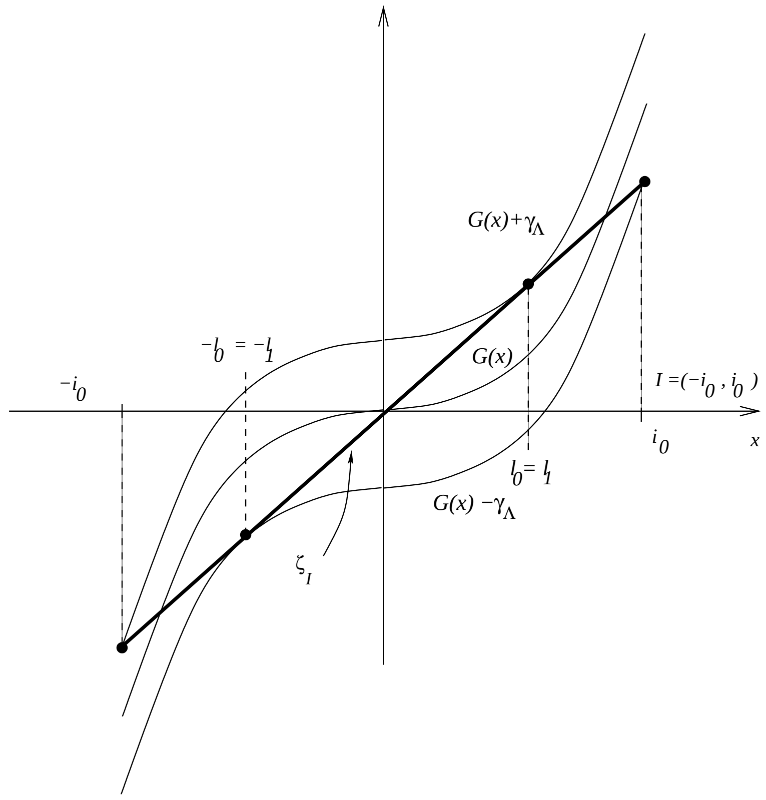

Figure 1: Graph of ζ_I (case $l_0 = l_1$)

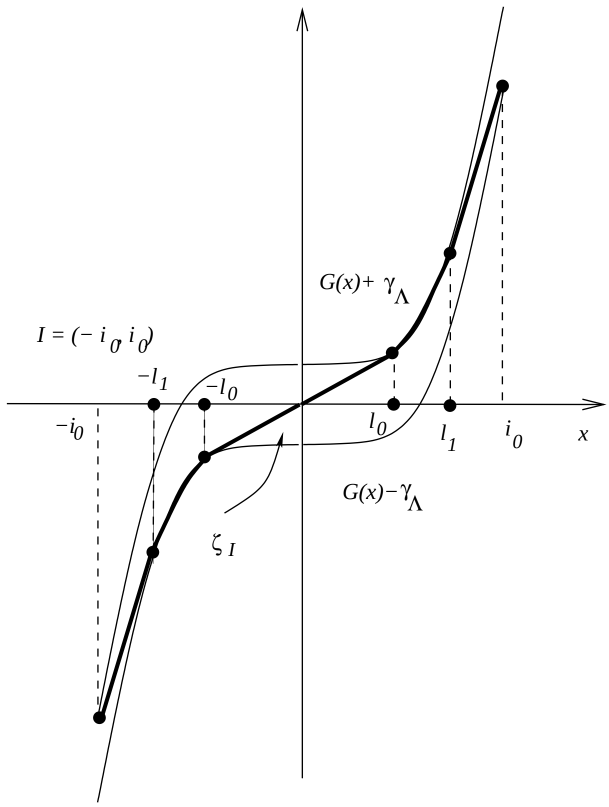

Figure 2: Graph of ζ_I (case $l_0 < l_1$)

Indeed, if there existed ζ_I, a solution to (5.17) such that $\mathcal{E}_I(\zeta_I) < \mathcal{E}_I(\xi_I)$, then this indicates that ξ is not a solution to (5.10), which is not possible.

We have to justify possibility of taking the boundary conditions in the definition of $\mathcal{D}_I$. We know that ξ is a solution to the obstacle problem (5.10) and $(-\infty, -l_0) \cup (l_0, \infty)$ is the coincidence set. Using the argument of the proof of [GR1, Proposition 2.5], [GR2, Proposition 3.2] one can show that $\xi|_{(-\infty,-l_0]} = \gamma_\Lambda$ and $\xi|_{[l_0,\infty)} = -\gamma_\Lambda$. Thus, ξ restricted to each connected component of the $I \setminus [-l_0, l_0]$ is constant.

Let us now calculate Λ^σ_W. For points of the coincidence set, it is clear that $\Lambda^\sigma_W = \sigma$ as desired. Let us consider interval $[-l_0, l_0]$. By the definition, see (2.5), $\Lambda^\sigma_W = \frac{d}{dx}\zeta_{\chi_l\chi_r,I}$, where $\zeta_{\chi_l\chi_r,I}$ is a solution to the following obstacle problem,

$$\min\{J^Z_{\chi_l\chi_r}(\omega, I) :\ \omega \in K^Z_{\chi_l\chi_r}\}, \tag{5.18}$$

where $Z(x) = \int_0^x \sigma(t,s)\,ds$ and for $[-l_0, l_0]$, we have $\chi_l = +1 = \chi_r$,

$K^Z_{++} = \{\omega \in H^1 :\ Z(x) - \gamma_\Lambda \le \omega(x) \le Z(x) + \gamma_\Lambda, x \in [-l_0, l_0],\ \omega(\pm l_0) = Z(\pm l_0) \pm \gamma_\Lambda\}.$

Since the boundary conditions in K^Z_{++} are that of $\mathcal{D}_{[-l_0,l_0]}$, we immediately conclude by previous considerations that ζ defined by $Z - \xi$ is the solution to (5.18). Hence, $\Lambda^\sigma_W = \sigma - \frac{\partial \xi}{\partial x}$.

After these preparation, we may check that a variational solution is a viscosity solution. First, we shall see that d is a supersolution. For this purpose we take a test function $\varphi \in A_P(Q)$ such that $d - \varphi$ attains a minimum at (x_0, t_0), where $t_0 \in (0, T)$. We have to show that

$$\varphi_t - \Lambda^\sigma_W \ge 0. \tag{5.19}$$

Inequality (5.19) (and (5.22) below) is to be checked at each point. We have to consider two cases for the interfacial curves: (a) the free boundary l_0 is a tangency curve (b) the free boundary l_0 is a matching curve and the tangency condition is violated.

In the course of proving (5.19) we will consider three cases separately: (i) $|x_0| > l_0(t_0)$, (ii) $|x_0| \in [0, l_0(t_0))$, (iii) $|x_0| = l_0(t_0)$.

We begin with (i). Since we assumed that $d_0 \in C^1$, we know (see Theorem 5.3 or Theorem 5.5) that at (x_0, t_0) function d is differentiable. Hence for $\varphi(x,t) = f(x) + g(t)$ with $d - \varphi \ge 0$ in a neighborhood of (x_0, t_0) we have

$$d_x(x_0, t_0) = f'(x_0), \qquad d_t(x_0, t_0) = g'(t_0).$$

Due to Definition 2.10 we have $\Lambda^\sigma_W(\varphi) = \sigma = \Lambda^\sigma_W(d)$. As a result,

$$0 = d_t - \sigma = g' - \Lambda^\sigma_W(\varphi) = \varphi_t - \Lambda^\sigma_W(\varphi),$$

as desired.

Now we look at (ii). The argument depends on the type of the interfacial curve l_0. Let us first assume that l_0 is tangency curve.

In the considered case d is also differentiable at (x_0, t_0). If φ is a test function such that $d - \varphi$ attains its minimum at (x_0, t_0), then

$$d_x(x_0, t_0) = 0 = f'(x_0), \qquad d_t(x_0, t_0) = g'(t_0).$$

Since $f \in C^2_P(\Omega)$, we immediately see that $I = R(f, x_0)$, the faceted region of φ at (x_0, t_0), must contain $[-l_0, l_0]$. Let us suppose that ξ_I is the solution to

$$\min\{\mathcal{E}_I(\omega) :\ \omega \in \mathcal{D}_I\}.$$

By the geometric interpretation of the obstacle problem (5.10), [GR1, Proposition 2.3], the coincidence set is $I \setminus (-l_0, l_0)$. This is the place, where we use the fact that the tangency condition holds at x_0.

As a result of the above observation, we have $\Lambda^\sigma_W(d) = \Lambda^\sigma_W(\varphi)$. Moreover,

$$\begin{aligned}
\Lambda^\sigma_W(d) &= \sigma - \frac{\partial \xi}{\partial x} \\
&= \fint_0^{l_0} \sigma(t,s)\, ds + \frac{\gamma(\mathbf{n}_\Lambda)}{l_0}.
\end{aligned}$$

Thus, by (5.13)

$$0 = \dot{R}_0 - \fint_0^{l_0(t_0)} \sigma(s)\, ds - \frac{\gamma(\mathbf{n}_\Lambda)}{l_0(t_0)} = d_t - \Lambda^\sigma_W(d) = \varphi_t - \Lambda^\sigma_W(\varphi),$$

as desired.

Let us note that this argument works well for $(x_0, t_0) = (l_0(t_0), t_0)$ if the tangency condition holds, so (iii) holds in this case.

We continue our analysis of case (ii). We have to consider the situation when l_0 is a matching curve. We will have to compare $\Lambda^\sigma_W(d)$ and $\Lambda^\sigma_W(\varphi)$. One way is to invoke Theorem 2.12, but we think it is instructive to check it directly.

Let us suppose that $I = [-a, b]$ is the faceted region of φ containing (x_0, t_0). We consider the minimization problem (5.18) defining ζ_I on that interval. Without loss of generality, we may restrict our attention to a subinterval $[\mu_0, \mu_1] \subset [-a, b]$ such that $\frac{d\zeta_I}{dx}$ is constant on $[\mu_0, \mu_1]$. Let us first consider that situation when $\mu_0 = -\mu_1$. We have to compare velocities $\frac{d\zeta_I}{dx}$ and $\frac{d\xi}{dx}$ on $[-l_0, l_0]$. Since the tangency condition is violated at l_0, then there is a possibility of bigger faceted regions containing $[-l_0, l_0]$. Moreover, $\frac{d\zeta_I}{dx}$ is a slope of a line connecting 0 and $Z(\mu_1) + \gamma_\Lambda$, while $\frac{d\xi}{dx}$ is a slope of a line connecting 0 and $Z(l_0) + \gamma_\Lambda$. Since Z is strictly increasing, we deduce that $\frac{d\zeta_I}{dx} < \frac{d\xi}{dx}$. The same observation applies when we want to compare slopes of minimizers to (5.18) on $[-a, b]$ and $[-\mu_1, \mu_1]$ and $a = \mu_1$ or $b = \mu_1$ but $[-a, b] \supset [-\mu_1, \mu_1]$. Thus, we have

$$\begin{aligned}
\varphi_t - \Lambda^\sigma_W(\varphi) &\geq d_t - \Lambda^\sigma_W(d) \\
&= \dot{R}_0 - \fint_0^{l_0} \sigma(t,s)\, ds - \frac{\gamma(\mathbf{n}_\Lambda)}{l_0} \qquad (5.20) \\
&= 0.
\end{aligned}$$

(iii) In order to complete the discussion of the facet we have to consider the case when at the interfacial point the tangency condition is violated. Let us suppose that this happens at $x_0 = l_0$, (the case $x_0 = -l_0$ is analogous). At this point $d(t_0, x_0)$ need not be differentiable with respect to x. Hence, if φ is a test function such that $d - \varphi$ attains its minimum, then $d_x^-(l_0(t_0), t_0) = 0$ and $d_x^+(l_0(t_0), t_0) \geq 0$.

The point $(l_0(t_0), t_0)$ belongs to the faceted region of d, hence it belongs to the faceted region of the test function φ. As a result, the above consideration on $\Lambda_W^\sigma(\varphi)$ is valid. Hence, the series of inequalities (5.20) is valid too.

We also have to check that d is a subsolution. Similarly to the above considerations, for the purpose of checking that d is a subsolution, we take a test function $\varphi \in A_P(Q)$ such that

$$\max(d - \varphi) = d(t_0, x_0) - \varphi(t_0, x_0). \tag{5.21}$$

We shall show that

$$\varphi_t - \Lambda_W^\sigma \leq 0. \tag{5.22}$$

We consider the same three cases. They are handled in an analogous way, we exploit the fact that $d(t, \cdot)$ is a C^1 function on $(-l_0, l_0)$ and on $\mathbf{R} \setminus [-l_0, l_0]$.

The case (i) is handled as before, because of differentiability of d and φ at (x_0, t_0).

(ii) If $|x_0| < l_0(t)$, then the faceted region of φ is contained in $[-l_0(t_0), l_0(t_0)]$. By the previous analysis, we conclude that $\Lambda_W^\sigma(\varphi) \geq \Lambda_W^\sigma(d)$. Hence,

$$\varphi_t - \Lambda_W^\sigma(\varphi) \leq d_t - \Lambda_W^\sigma(d) = 0.$$

Case (iii) is handled in a completely analogous way as before. We omit the details. $\square$

Corollary 5.8. *Let us suppose that the assumption of Theorem 5.7 hold. The variational solutions constructed in Theorem 5.3 and 5.5 are unique, as long as $|d_x| \leq 1$ and the initial condition d_0 is strictly increasing on $[l_{00}, \infty)$.*

Proof. Let us suppose that $(\Gamma(d^i), \xi^i)$ are two variational solutions, with initial data $\Gamma(d_0)$, where d_0 is admissible. We notice that it is sufficient to show that $d^1 = d^2$.

Let us set $A = \max_{t\in[0,T)} l_0(t) + 1$. Due to (5.11) by formula (6.2) we conclude that $d_x^i(t, x) \neq 0$ for all $(t, x) \in (0, T) \times (A, \infty)$. Since we solve an ODE for $|x| > A$, by inspection of equation (6.1) we immediately see that if $v := d^1$ is a supersolution and $u := d^2$ is a subsolution to (6.1), then $v \geq u$. Subsequently, by interchanging the roles of d^1 and d^2 we conclude that $d^1 = d^2$ for $(t, x) \in [0, T) \times \mathbf{R} \setminus (-A, A)$. As a result, we can see that an application of the Comparison Principle on $(-A, A)$ yields that $d^1 = d^2$ for all $(t, x) \in [0, T) \times \mathbf{R}$. □

6 Appendix

Here we give a sketch of proof of Theorems 5.3 and 5.5 by pointing to the main differences with [GR1, Theorem 2.10] and [GR2, Section 3.1].

In [GR1] we considered equation (5.6) on a bounded interval J. The initial condition, hence the solution had three facets, two of them touching the endpoint of J. Here, we consider (5.6) on $\mathbf{R}$ and the data d_0 has a single facet, hence the same will hold for the solution. We have to check the existence of a solution for all $x \in \mathbf{R}$ for all $t \in [0, T]$. Here, the limitations arise from the constructions of the free boundary l_0 performed in [GR2, Section 3.1]. We have already mentioned that the construction essentially depends upon the sign of Σ_0, but it is local in the sense that it uses the data from a neighborhood of l_{00}.

Thus, we have to make sure that we can solve $(5.6)_2$, i.e.,

$$d_t(t, x) = \sigma(t, x), \quad d(0, x) = d_0(x) \tag{6.1}$$

for all large x, e.g. $x > A > l_{00}$ for a constant A and all $t \in [0, T]$. This problem can be solved for all $x \geq l_{00}$ uniformly in $t > 0$,

$$d(t, x) = \int_0^t \sigma(s, x)\, ds + d_0(x), \tag{6.2}$$

since we assumed that $\sigma_x \in C(\mathbf{R}_+ \times \mathbf{R})$. Moreover, the solution will be Lipschitz continuous if for all $t \geq 0$ we have that $\mathrm{Lip}\,(\sigma(t, \cdot)) \leq L$.

We notice that for all $t > 0$ function $d(t, \cdot)$ is not only strictly increasing in x, but also the derivative $d_x(t, x)$ is positive for all $x > l_{00}$.

We also have to check that the Cahn-Hoffman vector ξ specified in the statements of Theorems 5.3 and 5.5 is a unique minimizer of $\mathcal{E}$. This easy task is left to the reader. Hence, $(\Gamma(d(t, \cdot), \xi(t, \cdot))_{t \in [0,T)}$ is a variational solutions.

Remark 6.1. We notice that the same kind of argument shows that Theorem 5.3 and 5.5 are valid also if $\sigma = \sigma(x_1, x_2)$ and it satisfies and extension of condition (5.11), i.e.

$$\sigma(\pm x_1, \pm x_2) = \sigma(x_1, x_2), \qquad \frac{\partial \sigma}{\partial x_i}(x_1, x_2)x_i > 0 \quad \text{for } x_i \neq 0.$$

Moreover, by Remark 4.8 the Comparison Principle (Theorem 4.1) holds too in this case.

Acknowledgment. The second author was in part supported by the Japan Society for the Promotion of Science (JSPS) through grant for scientific research 21224001.

The third author was in part supported by the Polish Ministry of Science grant N N201 268935. During the preparation of this manuscript PR enjoyed hospitality of Hokkaido University, which is gratefully acknowledged.

References

[AG1] S.B. Angenent and M.E. Gurtin, Multiphase thermomechanics with interfacial structure 2 Evolution of an isothermal interface, Arch. Rational Mech. Anal., **108** (1989), 323-391.

[BCCN] G. Bellettini, V. Caselles, A. Chambolle and M. Novaga, Crystalline mean curvature flow of convex sets, Arch. Ration. Mech. Anal. **179** (2006), 109-152.

[BGN] G. Bellettini, R. Goglione and M. Novaga, Approximation to driven motion by crystalline curvature in two dimensions, Adv. Math. Sci. Appl., **10** (2000), 467-493.

[BN] G. Bellettini and M. Novaga, Approximation and comparison for nonsmooth anisotropic motion by mean curvature in $\mathbf{R}^N$, Math. Mod. Meth. Appl. Sc. **10** (2000), 1-10.

[BNP] G. Bellettini, M. Novaga and M. Paolini, Facet-breaking for three-dimensional crystals evolving by mean curvature, Interfaces Free Bound., **1** (1999), 39-55.

[BNP1] G. Bellettini, M. Novaga and M. Paolini, Characterization of facet breaking for nonsmooth mean curvature flow in the convex case, Interfaces and Free Boundaries, **3** (2001), 415-446.

[BNP2] G. Bellettini, M. Novaga and M. Paolini, On a crystalline variational problem, part I: first variation and global L^∞-regularity, Arch. Ration. Mech. Anal. **157** (2001), 165-191.

[BNP3] G, Bellettini, M. Novaga and M. Paolini, On a crystalline variational problem, part II: BV regularity and structure of minimizers on facets, Arch. Rational Mech. Anal., **157** (2001), 193-217.

[CGG1] Y.-G. Chen, Y. Giga and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Differential Geom., **33** (1991), 749-786.

[CIL] M. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc., **27** (1992), 1-67.

[EGS] C.M. Elliott, A. Gardiner and R. Schätzle, Crystalline curvature flow of a graph in a variational setting, Adv. Math. Sci. Appl., **8** (1998), 425-460.

[ES1] L.C. Evans and J. Spruck, Motion of level sets by mean curvature, I, J. Differential Geom., **33** (1991), 635-681.

[FG] T. Fukui and Y. Giga, Motion of a graph by nonsmooth weighted curvature, In: World Congress of Nonlinear Analysts '92 (ed. V. Lakshmikantham), Walter de Gruyter, Berlin **I** (1996), 47-56.

[GG98Ar] M.-H. Giga and Y. Giga, Evolving graphs by singular weighted curvature, Arch. Rational Mech. Anal., **141** (1998), 117-198.

[GG98DS] M.-H. Giga and Y. Giga, A subdifferential interpretation of crystalline motion under nonuniform driving force, In: Proc. of the International Conference in Dynamical Systems and Differential Equations, Springfield Missouri, (1996), Dynamical Systems and Differential Equations (eds. W.-X. Chen and S.-C. Hu), Southwest Missouri Univ. **1** (1998), pp.276-287.

[GG98Pit] M.-H. Giga and Y. Giga, Remarks on convergence of evolving graphs by nonlocal curvature, In: Progress in Partial Differential Equations, Vol. 1 (eds. H. Amann, C. Bandle, M. Chipot, F. Conrad and

L. Shafrir), Pitman Research Notes in Mathematics Series, **383** (1998) pp.99-116, Longman, Essex, England.

[GG99] M.-H. Giga and Y. Giga, Stability for evolving graphs by nonlocal weighted curvature, Commun. in Partial Differential Equations, **24** (1999), 109-184.

[GG01Ar] M.-H. Giga and Y. Giga, Generalized motion by nonlocal curvature in the plane, Arch. Rational Mech. Anal., **159** (2001), 295-333.

[GG04] M.-H. Giga and Y. Giga, A PDE approach for motion of phase-boundaries by a singular interfacical energy. In: Stochastic Analysis on Large Scale Interacting Systems (eds. F. Funaki and H. Osada), Advanced Studies in Pure Math. **39**, pp. 212-232 (2004), Mathematical Society of Japan.

[GG10] M.-H. Giga and Y. Giga, Very singular diffusion equations: second and fourth order problems, Japan J. Indust. Appl. Math. **27** (2010), 323-345.

[GGK] M.-H. Giga, Y. Giga and R. Kobayashi, Very singular diffusion equations, In: Taniguchi Conference on Mathematics, Nara '98. (eds. M. Maruyama and T. Sunada) Adv. Studies in Pure Math. **31**, pp. 93-125 (2001), Mathematical Society of Japan.

[G04] Y. Giga, Singular diffusivity - facets, shocks and more. In: Applied Math Entering the 21st Century (eds. J. M. Hill and R. Moore). ICIAM 2003 Sydney, pp. 121-138 (2004), SIAM, Philadelphia.

[G] Y. Giga, Surface Evolution Equations – a level set approach, Birkhäuser, Basel, (2006).

[GGIS] Y. Giga, S. Goto, H. Ishii and M.-H. Sato, Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains. Indiana Univ. Math. J. **40** (1991), 443-470.

[GGM] Y. Giga, M.-E. Gurtin and J. Matias, On the dynamics of crystalline motions, Japanese J. Ind. Appl. Math., **15** (1998), 7-50.

[GGR] Y. Giga, P. Górka and P. Rybka, Nonlocal spatially inhomogeneous Hamilton-Jacobi equation with unusual free boundary, Discrete Contin. Dyn. Syst. **26** (2010), 493-519.

[GR1] Y. Giga and P. Rybka, Facet bending in the driven crystalline curvature flow in the plane, J. Geom. Anal., **18** (2008), 109-147.

[GR2] Y. Giga and P. Rybka, Facet bending driven by the planar crystalline curvature with a generic nonuniform forcing term, J. Differential Equations, **246** (2009), 2264-2303.

[H] L. Hörmander, Notions of convexity, Birkhäuser, Boston (1994).

[KS] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and their Applications, Academic Press, New York (1980)

[MR1] P.B. Mucha and P.Rybka, A caricature of a singular curvature flow in the plane, Nonlinearity, **21**, (2008), 2281-2316.

[MR2] K. Kielak, P.B. Mucha, P.Rybka, Almost classical solutions to the total variation flow, submitted.

[Rd] J. F. Rodrigues, Obstacle problems in Mathematical physics, Elsevier, Amsterdam (1987).

[Rs] A. R. Roosen, Crystalline curvature and flat flow in a diffusion field, unpublished note (1994).

[S] L. Schwartz, Théorie des distributions, Publications de l'Institut de Mathématique de l'Université de Strasbourg, Hermann, Paris (1966).

[T] J. Taylor, Constructions and conjectures in crystalline nondifferential geometry, In: Differential geometry, (eds. B. Lawson and K. Tanenblat), Proceedings of the Conference on Differential Geometry, Rio de Janeiro, Pitman Monographs in Pure and Applied Math., **52** (1991), pp. 321-336, Pitman, London.