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<th>STREAMLINES CONCENTRATION AND APPLICATION TO THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS</th>
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1. INTRODUCTION

In this paper we consider divergence-free smooth vector fields \( u \in C^1(D \setminus \{0\}, \mathbb{R}^3) \) defined on a domain \( D \) of \( \mathbb{R}^3 \) containing the origin which may have a singular point at the origin. We give a definition based on streamline concentration towards the eventual singularity, and we show that if there is sufficient streamline concentration, then the vector field cannot be an \( L^2 \) function\(^1\). Therefore, this result rules out a certain geometric situation (streamline concentration) at a possible singular time for incompressible fluid equations such as the 3D Navier-Stokes equations. Before going any further, let us briefly recall a few results about the 3D Navier-Stokes equations on \( \mathbb{R}^3 \). The equations ruling the flow of an incompressible viscous fluid on \( \mathbb{R}^3 \) are

\[
\begin{aligned}
\partial_t v - \Delta v + \text{div}(v \otimes v) + \nabla p &= 0, \\
\text{div}(v) &= 0, \\
v|_{t=0} &= v_0
\end{aligned}
\]

(1.1)

in which

\( v \) is a vector-valued function representing the velocity of the fluid, and \( p \) is the pressure.

The initial value problem of the above equation is endowed with the condition that \( v(0, \cdot) = v_0 \in L^2(\mathbb{R}^3) \).

A finite energy weak solution to the Navier-Stokes equations (1.1) over a time interval \((0, T)\) is a pair \((v, p)\) satisfying

(1) equation (1.1) in the distributional sense,

(2) \((v, p) \in L^\infty([0, T], L^2) \cap L^2([0, T], H^1) \times L^2_{\text{loc}}((0, T) \times \mathbb{R}^3)\)

---

\( ^1 \)we define this situation precisely in the next section.
(3) the energy inequality, for $0 < t < T$

\[ \|v(t, \cdot)\|_{L^2}^2 + 2 \int_0^t \|\nabla v(t', \cdot)\|_{L^2}^2 \, dt' \leq \|v(0, \cdot)\|_{L^2}^2. \]

For a divergence free initial data $v_0 \in (L^2(\mathbb{R}^3))^3$, the existence of global in time and finite energy weak solutions to the Navier-Stokes equations is due to the pioneer works of J. Leray [13] in the case $D = \mathbb{R}^3$ and E. Hopf [10] in the case of the torus. Moreover, neither the uniqueness nor the global regularity are known. These questions are the outstanding problems of regularity for solutions to the Navier-Stokes equations. Recall that the space-time singular set $S(u)$ of $u$ is defined as follows.

**Definition 1.1.** A point $(x_0, t_0) \notin S(u)$ if there exists a parabolic cylinder $Q_{(x_0, t_0)}(r) := \{ |x - x_0| < r \} \times (t_0 - r^2, t_0)$ about $(x_0, t_0)$ such that the solution $u \in L^\infty(Q_{(x_0, t_0)}(r))$.

Modern regularity theory for solutions to equation (1.1) began with the works of Prodi [14], Serrin [16], Ladyzhenskaya [12] implying that if

\[ u \in L^p_t(L^q_r)(Q_{(x_0, t_0)}(r)), \quad \text{for } \frac{3}{q} + \frac{2}{p} < 1, \]

then $\partial^\alpha_t u \in C^\alpha((Q_{(x_0, t_0)}(r/2)))$ for some $0 < \alpha < 1$ and therefore $u$ is regular. Later on, M. Struwe [17] extended this to the case (of scaling invariant pair) i.e. $\frac{2}{q} + \frac{2}{p} = 1$, and recently this was extended to the limit case $u \in L^\infty_t(L^2_r)$ by L. Escauriaza, G. Seregin, and V. Sverak (see their famous work [8]). After the appearance of the Prodi-Serrin-Ladyzhenskaya criterion, many different regularity criteria and Liouville type theorem of solutions to (1.1) were established (see [1], [2], [6] and [11]).

We would like to mention a regularity criterion in [18] by A. Vasseur (see also [4]). He gave a regularity criterion for solutions $u$ to (1.1) in terms of the integral condition $\text{div}(\frac{u}{|u|}) \in L^p(0, \infty; L^q(\mathbb{R}^3))$ with $\frac{2}{p} + \frac{3}{q} \leq \frac{1}{2}$ imposed on the scalar quantity $F = \text{div}(\frac{u}{|u|})$. Note that the case $(p, q) = (6, \infty)$ is included.

Concerning the analysis of the singular set $S(u)$, we recall the following facts: First, by definition, the set $S(u)$ is closed, and thanks to the result of C. Foias and R. Temam [9], the $\frac{1}{2}$-dimensional Hausdorff measure of the set of singular times $\tau(u) := \text{proj}_t S(u)^2$ is zero. Next, V. Scheffer [15] and then L. Caffarelli, R. Kohn and L. Nirenberg [3] showed the best result concerning partial regularity of suitable weak solutions\(^3\) of the Navier-Stokes equations stating that the parabolic one-dimensional Hausdorff measure of $S(u)$ is zero. Finally, a consequence of the latter result is a bound on the spatial singular set for each time slice $S_T := S(u) \cap \{ t = T \}$ which has at most one-dimensional Hausdorff measure.

In this paper, we focus on the vector field at a possible singular time $T \in \tau(u)$, and examine the geometry of its streamlines. Recall that in [5], C-H. Chan and the third author proposed a possible scenario for an isolated space singularity at a possible blow-up time by using the energy inequality and regularity criterions especially [8] and [18]. They constructed a divergence free velocity field $u$ within a streamtube segment with increasing twisting (i.e., increasing swirl).

The construction of such a vector field $u$ demonstrates the way in which excessive increase of twisting of streamlines can result in the blow up of the quantities $\|u\|_{L^\infty(\mathbb{R}^3)}$ (for some $2 < \alpha < 3$) and $\|\text{div}(\frac{u}{|u|})\|_{L^p(\mathbb{R}^3)}$ while at the same time preserving the finite energy property $u \in L^2(\mathbb{R}^3)$ of the fluid. Note that the increasing swirl streamtube is not included in the sufficient concentration streamlines case. The device of streamtube has already proposed as the vortex-tube (see[7]).

In this work, we show that if “enough” streamlines of a smooth and divergence free vector field

\[^2\text{the map } (x, t) \to t\]

\[^3\text{roughly, these are weak solutions satisfying the local energy inequality instead of the global one (1.2).}\]
field concentrate towards a possible isolated singular point, then the vector field cannot be an $L^2$ function. The main idea is to construct an appropriate “streamline flux tube” and apply Stokes’ Theorem.

2. A CLASSIFICATION OF DIVERGENCE VECTOR FIELDS

**Definition 2.1.** (Streamline) Let $D$ be a smooth domain containing the origin and $u : D \setminus \{0\} \to \mathbb{R}^3$ be a smooth vector field. For a starting point $\eta \in D$, we define a streamline $\gamma_{\eta}(s) : [0, \infty) \to \mathbb{R}^3$ as the curve solving

$$\partial_s \gamma_{\eta}(s) = u(\gamma_{\eta}(s)) \quad \text{for} \quad s > 0 \quad \text{with} \quad \gamma_{\eta}(0) = \eta.$$ 

One may assume that streamlines are global, because otherwise, they go towards the possible singular point at the origin.

The following definition is the key to classify the divergence-free vector field with a possible isolated singularity at the origin. Let $B_\alpha$ be the open ball with radius $\alpha$ centered at the origin.

**Definition 2.2.** For $\alpha > r$ let

$$A^\alpha_r = \{ \eta \in \partial B_\alpha : \gamma_{\eta}(s) \in B_r \text{ for some } s > 0, \text{ and } \gamma_{\eta}(s') \in B_\alpha \text{ for } 0 < s' < s \}.$$ 

The above definition excludes the streamlines entering the ball $B_\alpha$ infinitely many times before entering $B_r$. If it happens and a streamline enters $B_\alpha$ finitely many times before getting into $B_r$, then one can re-parametrize the time so that its last entrance occurs at time $s = 0$.

**Remark 2.3.** For streamlines from $A^\alpha_r$ we have the following properties

- $|A^\alpha_r|$ is monotone decreasing with respect to $\alpha$ and increasing with respect to $r$. Indeed, $|A^\alpha_r| \geq |A^\alpha_{r'}|$ for $r > r'$, $|A^\alpha_r| \geq |A^{\alpha'}_r|$ for $\alpha < \alpha'$.
- Without loss of generality, we can assume that streamlines from $A^\alpha_r$ are globally defined.
- From definition of $A^\alpha_r$ we cannot have stagnation points of the fluid (i.e. $u(\gamma_{\eta}(s)) = 0$ for some $s > 0$).

**Definition 2.4.** (Stream-surface & flux-tube) Let $D \subset \mathbb{R}^3$ be a surface and $s$ be such that $\gamma_{\eta}(s)$ is defined for each $\eta \in D$.

- A stream-surface $S^D(s)$ is defined as $S^D(s) = \bigcup_{\eta \in D} \gamma_{\eta}(s)$.
- A flux-tube $T^D(s)$ is given by $T^D(s) = \bigcup_{0 \leq s' \leq s} S^D(s')$.
- The mantle of the flux-tube $T^D(s)$ is $\partial T^D(s)$.

For $|x| \neq 0$ denote by $\hat{n}(x) = x/|x|$. Smoothness and membership in $C^1$ are used interchangeably. The main result reads as follows.

**Theorem 2.5.** If for some $\alpha > 0$ and for some $C > 0$ independent of $r$, $|\int_{A^\alpha_r} u \cdot \hat{n} d\sigma| \geq Cr^{1/2}$ as $r \to 0$, then $u \notin L^2(\mathbb{R}^3)$.

The following special case is worth noting. See Figure 1.

**Corollary 2.6.** Suppose for some $\alpha > 0$ and for $A \subset \partial B_\alpha$ that $\int_A u \cdot \hat{n} d\sigma \neq 0$ and $A^\alpha_r \supset A$ for $0 < r < \alpha$. Then $u \notin L^2(\mathbb{R}^3)$.

**Proof.** It follows from the definition of $A^\alpha_r$ that $u \cdot \hat{n}$ has constant (negative) sign on $A^\alpha_r$. Let $C = \int_A u \cdot \hat{n} d\sigma > 0$, then for $0 < |r| < \min\{1, \alpha\}$, $|\int_{A^\alpha_r} u \cdot \hat{n} d\sigma| \geq |\int_A u \cdot \hat{n} d\sigma| \geq Cr^{1/2}$. □

---

4note that such singular set has a zero one-dimensional Hausdorff measure.
The proof of Theorem 2.5 proceeds in a few steps. First of all suppose that
\[ \int_{\partial B_r} |u \cdot \hat{n}| d\sigma \geq \left| \int_{A_r^\alpha} u \cdot \hat{n} d\sigma \right| \]
for each \( r \) (this is proved in a moment). Then, Jensen’s inequality gives
\[ (2.2) \quad \frac{1}{|\partial B_r|} \int_{\partial B_r} |u|^2 d\sigma \geq \left( \frac{1}{|\partial B_r|} \int_{\partial B_r} |u| d\sigma \right)^2 \]
or
\[ (2.3) \quad \int_{\partial B_r} |u|^2 d\sigma \geq \left( \frac{1}{|\partial B_r|} \int_{\partial B_r} |u| d\sigma \right)^2 \]
and by assumption
\[ \left( \frac{1}{|\partial B_r|} \int_{\partial B_r} |u| d\sigma \right)^2 \geq \frac{1}{4\pi r^2} \left| \int_{A_r^\alpha} u \cdot \hat{n} d\sigma \right|^2 \geq \frac{1}{4\pi r^2} C r = \frac{C}{4\pi r} \]
from which it follows that
\[ \|u\|_{L^2} \geq \left( \int_0^\epsilon \int_{\partial B_r} |u|^2 d\sigma dr \right)^{1/2} \geq \left( \int_0^\epsilon \frac{C}{4\pi r} \right)^{1/2} = \infty \]
where \( \epsilon > 0 \) is such that \( |\int_{A_r^\alpha} u \cdot \hat{n} d\sigma| \geq C r^{1/2} \) for \( 0 < r \leq \epsilon \).

Now, to prove that \( \int_{\partial B_r} |u \cdot \hat{n}| d\sigma \geq \left| \int_{A_r^\alpha} u \cdot \hat{n} d\sigma \right| \) observe first of all that \( \int_{A_r^\alpha} u \cdot \hat{n} d\sigma = \int_{\text{reg} \ A_r^\alpha} u \cdot \hat{n} d\sigma \) where \( \text{reg} \ A_r^\alpha = \{ \eta \in A_r^\alpha : (u \cdot \hat{n})(\eta) \neq 0 \} \). Since \( \alpha \) is fixed, let \( A_r \) denote \( \text{reg} \ A_r^\alpha \). From the definition of \( A_r^\alpha \) it follows that \((u \cdot \hat{n})(\eta) < 0 \) for \( \eta \in A_r \).

**Lemma 2.7.** Let \( D \subset \partial B_\alpha \) have piecewise smooth boundary and \((u \cdot \hat{n})(\eta) < 0 \) for \( \eta \in D \). Suppose that \( S^D(s) \subset B_r \) for some \( s > 0 \) and that \( S^D(s') \subset B_\alpha \) for \( 0 < s' \leq s \). Then
\[ \int_D u \cdot \hat{n} d\sigma = \int_{D^r} u \cdot \hat{n} d\sigma \]
where \( D^* \equiv T^D(s) \cap \partial D_r \). Also, if \( D_1 \) and \( D_2 \) are two such sets with \( D_1 \cap D_2 = \emptyset \), then \( D_1^* \cap D_2^* = \emptyset \).

**Proof.** The function \( \gamma_\eta : D \times [0,s] \rightarrow T^D(s) \) is onto and it follows from the theory of ordinary differential equations and from \( u \in C^1 \) that \( \gamma_\eta \in C^1 \). Also, \( \gamma_\eta \) is injective, which follows from uniqueness of solutions and from the fact that for each \( \eta \in D \), \( \gamma_\eta(s) \notin D \) for \( s > 0 \). From these properties it can be shown that \( \partial T^D(s) = D \cup S^D(s) \cup T^D(s) \). Piecewise smoothness of \( \partial T^D(s) \) then follows from the piecewise smoothness of \( \partial D \) and smoothness of solutions to the vector field. Let \( T = \{ x \in T^D(s) \mid r < |x| < \alpha \} \) and let \( V = \{ x \in T^D(s) \mid r < |x| < \alpha \} \), and let \( D^* \) be as defined above. Note that \( T \) has piecewise smooth boundary since it is the intersection of two sets with piecewise smooth boundary. Write \( \partial T = D \cup D^* \cup V \). If \( x \in V \) then a part of the streamline through \( x \) lies in \( V \), therefore \( u(x) \) is in the tangent space of \( V \) at \( x \). Then, applying the divergence theorem and using div \( u \equiv 0 \) gives the stated result.

Observe that the implication \( D_1 \cap D_2 = \emptyset \Rightarrow D_1^* \cap D_2^* = \emptyset \) follows from the uniqueness of solutions in the same way as above. \( \square \)

**Claim 2.8.** \( A_r \) is open. Moreover, for each \( \eta \in A_r \), there is a \( \delta > 0 \) such that \( D \equiv \{ \xi \in \partial B_\alpha : |\xi - \eta| < \delta \} \) satisfies the assumptions of the above lemma.

**Proof.** Let \( \eta \in A_r \) and \( s > 0 \) be as in the definition of \( A^\eta_r \). Then \( (u \cdot \nu)(\eta) < 0 \). By continuity there exists \( \delta > 0 \) so that \( E \equiv \{ \xi \in \partial B_\alpha : |\xi - \eta| \leq \delta \} \) has \( (u \cdot \nu)(\lambda) < 0 \) for \( \lambda \in E \). \( E \) is compact, and by a property of compact sets, there exists \( \alpha > 0 \) so that dist\((\xi, E) < \alpha \) implies \( (u \cdot \nu)(\eta) < 0 \). Let \( t = \inf\{ s' > 0 : |\gamma_\eta(s') - \eta| > \alpha/2 \} \) and let \( \beta(s) = \inf\{ |\gamma_\eta(s') - \partial B_\alpha| : t \leq s' \leq s \} \). Observe that \( \beta > 0 \) since the sets \( \{ \gamma_\eta(s') : t \leq s' \leq s \} \) and \( \partial B_\alpha \) are compact and disjoint.

Let \( \beta' > 0 \) be such that \( |\xi - \gamma_\eta(s)| < \beta' \) implies \( \xi \in \partial B_r \). Let \( \alpha' = \min\{ \alpha/2, \beta, \beta' \} \). By continuous dependence on initial data, there is a \( \delta' > 0 \), \( \delta' \leq \delta \) so that \( |\xi - \eta| < \delta' \) implies \( |\xi(s') - \gamma_\eta(s)| \leq \alpha' \) for \( 0 \leq s' \leq s \). For these \( \xi \), \( |\xi(s') - E| < \alpha \) for \( 0 \leq s' \leq s \) and so \( (u \cdot \nu)(\xi(s')) < 0 \) for \( 0 \leq s' \leq t \), from which it follows that \( \gamma_\xi(s') \in B_\alpha \) for \( 0 < s' \leq t \). Then, \( |\gamma_\xi(s') - \gamma_\eta(s)| < \beta \) implies \( \gamma_\xi(s') \in B_r \), and \( |\gamma_\xi(s') - \gamma_\eta(s)| < \beta \) implies \( \gamma_\xi(s') \in B_\alpha \), for \( t \leq s' \leq s \). Therefore \( \delta' \) gives \( D \) that satisfies the claim. \( \square \)

*End of the proof of Theorem 2.5.* Since \( A_r \) is open it is Lebesgue measurable. It follows that for each \( \epsilon > 0 \), by a theorem for measurable sets there exists \( K \) closed, \( K \subset A_r \), such that \( m(A_r \setminus K) < \epsilon \), where \( m \) denotes Lebesgue measure. For each \( \eta \in A_r \) let \( D_\eta \) be as in the above claim, then \( \{ D_\eta \}_{\eta \in K} \) is an open cover of \( K \). Since \( K \) is a closed and bounded subset of \( \mathbb{R}^3 \), it is compact and therefore from the above cover one can take a finite subcover \( \{ D_{\eta_i} \}_{1 \leq i \leq k} \). Let \( E_1 = D_m \) and for \( 2 \leq i \leq k \) let \( E_i = D_{\eta_i} \setminus E_{i-1} \); then the \( E_i \) are pairwise disjoint and have piecewise smooth boundary, and \( \bigcup_{i=1}^k E_i \) covers \( K \). For each \( i \) let \( E_i^* = T^E(s) \cap \partial D_r \). Then

\[
\int_{\bigcup_{i=1}^k E_i} u \cdot \mathbf{n} d\sigma = \int_{\bigcup_{i=1}^k E_i^*} u \cdot \mathbf{n} d\sigma
\]

using \( \int_{E_i} u \cdot \mathbf{n} d\sigma = \int_{E_i^*} u \cdot \mathbf{n} d\sigma \) (from Lemma 2.7) for each \( i \) and \( E_i \cap E_j = \emptyset \) implies that \( E_i^* \cap E_j^* = \emptyset \). Since \( \bigcup_{i=1}^k E_i^* \subset \partial D_r \) and \( m(A_r \setminus \bigcup_{i=1}^k E_i) \leq m(A_r \setminus K) < \epsilon \) it follows that

\[
\int_{\partial D_r} |u \cdot \mathbf{n} d\sigma| \geq \int_{A_r} |u \cdot \mathbf{n} d\sigma| - \epsilon \|u\|_{L^\infty(\partial B_\alpha)}
\]

Since \( u \in C^1(D \setminus \{0\}, \mathbb{R}^3) \) by assumption then \( \|u\|_{L^\infty(\partial B_\alpha)} < \infty \). Moreover, since \( \epsilon > 0 \) is arbitrary we have

\[
\int_{\partial D_r} |u \cdot \mathbf{n} d\sigma| \geq \int_{A_r} |u \cdot \mathbf{n} d\sigma| = \int_{A^*_r} |u \cdot \mathbf{n} d\sigma|
\]

as claimed.
Remark 2.9.  
1. Note that condition $|\int_{A_r^\alpha} u \cdot \check{n} d\sigma| \geq C_r^{1/2}$ in the theorem implicitly requires that the Lebesgue measure of the set $A_r^\alpha$ is non zero for some $\alpha > 0$ and any $0 < r < \alpha$. The example of a rotating vector field $u(x) = \frac{(x_3, -x_1, 0)}{|x|^3}$ shows that for any $\alpha > 0$, and for any $r < \alpha$ the set $A_r^\alpha$ is empty. Moreover, this example shows that the vector field $u$ can be in $L^2$ as well as not in $L^2$ depending whether or not $\gamma < 4$ or $\gamma > 4$.

2. We can easily generalize the main theorem (Theorem 2.5) to $L^p$ spaces ($1 \leq p \leq \infty$). In fact, we just use Hölder inequality instead of Jensen’s inequality which is used in (2.2) and (2.3). More precisely we have the following statement: If for some $\alpha > 0$ and for some $C > 0$ independent of $r$, $|\int_{A_r^\alpha} u \cdot \check{n} d\sigma| \geq C_r r^{2(1-1/p)}$ as $r \to 0$, then $u \notin L^p(\mathbb{R}^3)$.

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References


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