ON THE LIOUVILLE THEOREM FOR THE STATIONARY NAVIER-STOKES EQUATIONS IN A CRITICAL SPACE

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ABSTRACT. In this paper we prove Liouville type theorem for the stationary Navier-Stokes equations on $\mathbb{R}^3$. More specifically, if a solution $u \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ to the stationary Navier-Stokes system satisfies additional conditions characterized by the decays near infinity and by the oscillation, then we show that $u = 0$.

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1. INTRODUCTION

We consider the stationary problem of the Navier-Stokes equations on $\mathbb{R}^3$:

$$\text{(SNS)} \begin{cases} (u \cdot \nabla)u = -\nabla P + \Delta u, & x \in \mathbb{R}^3, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^3. \end{cases}$$

Let $\dot{H}^1_\sigma(\mathbb{R}^3) = \{u \in [\dot{H}(\mathbb{R}^3)]^3, \operatorname{div} u = 0\}$. A weak solution of stationary Navier-Stokes equations is $u \in \dot{H}^1_\sigma(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ which satisfy (SNS) in the sense of distribution. More precisely,

$$\langle (u \cdot \nabla)u, \phi \rangle = -\langle P, \operatorname{div} \phi \rangle - \langle \nabla u, \nabla \phi \rangle$$

for $\phi \in L^3(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, where $\langle f, g \rangle := \int_{\mathbb{R}^3} f(x)g(x)dx$, and $P = \sum_{1 \le i,j \le 3} R_i R_j u_i u_j$ with R_j, $j = 1, 2, 3$, the Riesz transform. Using the relation

$$u = -(-\Delta)^{-1}(\nabla P + (u \cdot \nabla)u)$$

and a bootstrapping argument, we can see $u \in C^\infty(\mathbb{R}^3)$ if $u \in \dot{H}^1_\sigma(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ (see also [1, 3, 4, 5, 9, 10]). Thus, throughout this paper, we may regard $u \in \dot{H}^1_\sigma(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$.

In [4], Galdi showed that if $v \in L^{9/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, then $v \equiv 0$. To prove this Liouville theorem, he used the test function $\psi_R v$, where $\psi_R \in C_c^\infty$, $\psi_R = 0$ for $|x| \ge 2R$, $\psi_R(x) = 1$ for $|x| \le R$ and satisfying $|\nabla \psi_R| \le M/R$ for some positive constant M independent of $x \in \mathbb{R}^3$. In this paper we show another type of Liouville theorem, in particular, we treat functions which have slower decay at space infinity than these functions which are in $L^{9/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. We note that there is no mutual implication between our result here and Galdi's result. Also our theorem is completely different from the recent results by Koch, Nadirashvili, Seregin and Sverak([6]).

To prove our theorem we use the De Giorgi method, which is adapted to the Navier-Stokes equations by Vasseur([11])(see also [2]). To show our Liouville theorem, we introduce function classes defined by "decay-control functions". We also consider the following model equation for the time-independent vector field $u = (u_1, u_2, u_3)$ on $\mathbb{R}^3$,

$$(u \cdot \nabla)u = -\frac{1}{2} u \,\mathrm{div} u + \Delta u \tag{1}$$

and the vector Burgers equation

$$(u \cdot \nabla)u = \Delta u. \tag{2}$$

Equation (1) is of interest for various reasons (see [8]). It has the same scaling properties and the same energy estimate as (SNS). Moreover (1) has a non-trivial solution u satisfying

$$C_1(1+|x|)^{-\frac{2}{3}} \le |u(x)| \le C_2(1+|x|)^{-\frac{2}{3}} \quad \text{for sufficiently large} \quad x \in \mathbb{R}^3 \tag{3}$$

and $u \in L^\infty$, where $0 < C_1 \le C_2$. It means that the non-trivial solution u is in $L^{\frac{9}{2}+\epsilon}$ for any $\epsilon > 0$. Thus, to consider Lioiville type theorem of (1) in $L^\infty(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$ is more attractive topic.

Now we define decay-control functions which has a key role in this paper.

Definition 1.1. (Decay-control functions.) We say that a monotone decreasing function $\varphi : (0, \infty) \to [0, \infty)$ is a "decay-control function" for $\lambda \le L$ if the function φ satisfies the following property with some $\beta > 0$,

$$\beta\varphi(\lambda) \le \varphi(4\lambda) \quad \text{for} \quad 0 < \lambda < L.$$

Note that this condition is related to "doubling condition" (see [7] for example).

Remark 1.2. λ^{-p} $(0 < p < \infty)$ for $\lambda < \infty$, $-\log(\lambda/2)$ for $\lambda \le 1/4$, and $2-\lambda$ for $\lambda \le 1/4$, are decay-control functions.

Definition 1.3. (Key function classes.) Let us define the function classes X and Y as follows:

$$Y := Y_{\varphi,C_1,C_2,M} := \{u \in L^6(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) : C_1\varphi(\lambda) \le |\{x \in \mathbb{R}^3 : |u(x)| > \lambda\}| \le C_2\varphi(\lambda) \quad \text{with} \quad \lambda \le \|u\|_\infty/M\} \tag{4}$$

for $0 < C_1 \le C_2$ and $M > 1$, and

$$X := X_{\beta,C_1,C_2,M} := \{u \in L^\infty(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3) : \|\nabla u\|_2 \le D\|u\|_\infty^{1/2}\}, \quad D = D(M, C_2/(C_1\beta)). \tag{5}$$

Note that the constant D is explicitly expressed (see (7) and (8)).

The definition of Y gives some control on the decay rate of u for sufficiently large x. However this says nothing about ∇u for the large part $\{x \in \mathbb{R}^3 : \|u\|_\infty/M \le |u| \le \|u\|_\infty\}$. That is why we need X to control large (some kind of highly oscillating) part. It means that we cannot rule out highly oscillating function from our Liouville theorem.

Remark 1.4. We can easily find which functions belong to X. For all $u \in \dot{H}^1_\sigma(\mathbb{R}^3) \cap L^\infty$, then $u_{a,b} := au(b\cdot) \in X$ if $a << 1$ or $b >> 1$. Actually, $\|\nabla u_{a,b}\|_2/\|u_{a,b}\|_\infty^{1/2} = a^{1/2}b^{-1/2}\|\nabla u\|_2/\|u\|_\infty^{1/2}$.

A direct calculation yields the following lemma (see [11] for example).

Lemma 1.5. *For any given constants B, $\beta > 1$, then there exists some constant C_0^* such that for any sequence $\{a_k\}_{k\ge0}$ satisfying $0 < a_0 \le C_0^*$ and $a_k \le CB^k a_{k-1}^\beta$, for any $k \ge 1$, we have $\lim_{k\to\infty} a_k = 0$.*

Proof. Just take $C_0^* \le C^{-\sum_{k=1}^{\infty}\frac{1}{\beta^k}} B^{-\sum_{k=1}^{\infty}\frac{k}{\beta^k}}$. Then we easily have $a_n \le C^{-\frac{1}{\beta}} B^{-\frac{n+1}{\beta}}$. ∎

Now we state the main result.

Theorem 1.6. *Assume that $u \in \dot{H}^1_\sigma(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ be a weak solution to (SNS), or $u \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ be a weak solution to either (1) or (2). If $u \in X \cap Y$, then $u = 0$ in $\mathbb{R}^3$.*

2. Proof of the main theorem

Let $b = \frac{\|u\|_\infty}{M}$. For each $k \ge 0$ we denote

$$\begin{aligned} v_k &:= (|u| - \frac{b}{2^k})\chi_{\{|u|>\frac{b}{2^k}\}},\\ d_k^2 &:= \frac{b/2^k}{|u|}\chi_{\{|u|>b/2^k\}}|\nabla|u||^2 + \frac{v_k}{|u|}|\nabla u|^2,\\ U_k &:= \int_{\mathbb{R}^3} d_k^2 dx. \end{aligned}$$

Since $|\nabla|u|| \le |\nabla u|$, $|\frac{v_k}{|u|}| \le 1$ and $|\frac{b/2^k}{|u|}\chi_{\{|u|>b/2^k\}}| \le 1$, we have

$$U_0 \le 2\|\nabla u\|_2^2.$$

For $u \in Y$, we have the following property:

$$\begin{aligned} (6) \quad \int_{\mathbb{R}^3} \chi_{\{v_k>0\}} &= |\{|u| > b/2^k\}| \le C_2\varphi(b/2^k) \le \frac{C_2}{\beta}\varphi(b/2^{k-2})\\ &= \eta C_1 \varphi(b/2^{k-2}) \le \eta|\{|u| > b/2^{k-2}\}| \le \eta\int_{\mathbb{R}^3}\chi_{\{v_{k-1}>b/2^k\}} \quad \text{for} \quad k \ge 1, \end{aligned}$$

where $\eta = (C_2/(C_1\beta))$. Note that the above inequality is different from Vasseur's observation, since we concern Liouville type theorem not regularity criterion. In our case, v_k grows as k increases. This explains why the definition of Y is needed.

The following lemma is minor modification of the lemma described in [2]. However, for the convenience of the readers, we give detailed computations.

Lemma 2.1. *We have the following inequality:*

$$\begin{cases} |\nabla v_k| \le d_k,\\ |\nabla(\frac{v_k}{|u|}u)| \le 3d_k. \end{cases}$$

Proof. First, we show that the inequalities

$$\chi_{\{|u|>b/2^k\}}|\nabla|u|| \le d_k \quad \text{and} \quad \frac{v_k}{|u|}|\nabla u| \le d_k$$

hold. To justify the second inequality, we derive the definition of d_k^2. A direct calculation yields

$$d_k^2 \ge \frac{v_k}{|u|}|\nabla u|^2 \ge \{\frac{v_k}{|u|}|\nabla u|\}^2.$$

Hence by taking square root, it follows at once that $\frac{v_k}{|u|}|\nabla u| \leq d_k$. To justify the first inequality, we recall that $|\nabla u| \geq |\nabla |u||$. Hence, it follows from the definition of d_k^2 that

$$d_k^2 \geq \frac{b/2^k}{|u|}\chi_{\{|u|\geq b/2^k\}}|\nabla |u||^2 + \frac{1}{|u|}\{|u| - b/2^k\}\chi_{\{|u|\geq b/2^k\}}|\nabla |u||^2 \geq \chi_{\{|u|\geq b/2^k\}}|\nabla |u||^2.$$

Since it is obvious to see that $\nabla v_k = \chi_{\{|u|\geq b/2^k\}}\nabla |u|$, we also have the result $|\nabla v_k| \leq d_k$. Next, we want to justify the inequality that $|\nabla(\frac{v_k}{|u|}u)| \leq 3d_k$. So, we notice that, by applying the product rule, we have

$$\nabla(\frac{v_k}{|u|}u) = \nabla(v_k)\frac{u}{|u|} + \frac{v_k}{|u|}\nabla u - \frac{v_k}{|u|^2}u\nabla |u|.$$

However, since $\frac{v_k}{|u|}|\nabla u| \leq d_k$, and $|\frac{v_k}{|u|^2}u\nabla |u|| \leq \chi_{\{|u|\geq b/2^k\}}|\nabla |u|| \leq d_k$, we have the desired estimate. ∎

Now we return to prove the main theorem. By (6), we see that

$$\begin{aligned}
\int_{\mathbb{R}^3} \chi_{\{v_k>0\}}\, dx &\leq \eta \int_{\mathbb{R}^3} \chi_{\{v_{k-1}>\frac{b}{2^k}\}}\, dx \\
&\leq \eta \left(\frac{2^k}{b}\right)^6 \|v^{k-1}\|_{L^6}^6 \leq \eta C \left(\frac{2^k}{b}\right)^6 \|\nabla v_{k-1}\|_{L^2}^6 \\
&\leq \eta C \left(\frac{2^k}{b}\right)^6 \|d_{k-1}\|_{L^2}^6 \leq \eta C \left(\frac{2^k}{b}\right)^6 U_{k-1}^3.
\end{aligned}$$

Thus we have

$$\|\chi_{\{v_k\geq 0\}}\|_{L^q} \leq (\eta C)^{1/q} \left(\frac{2^k}{b}\right)^{\frac{6}{q}} U_{k-1}^{\frac{3}{q}}.$$

Now let us construct a level set energy equality of (SNS). (The case of (1) and (2) are similar. Thus we omit these cases.)

Multiplying u to (SNS), we obtain

$$\mathrm{div}(u\frac{|u|^2}{2}) + \mathrm{div}(uP) + |\nabla u|^2 - \Delta\frac{|u|^2}{2} = 0.$$

Also, multiplying $u(\frac{v_k}{|u|} - 1)$ to (SNS), we have

$$\mathrm{div}(u\frac{v_k^2 - |u|^2}{2}) + u(\frac{v_k}{|u|} - 1)\nabla P - u(\frac{v_k}{|u|} - 1)\Delta u = 0.$$

Since

$$-u(\frac{v_k}{|u|} - 1)\Delta u = -\Delta\frac{v_k^2 - |u|^2}{2} + d_k^2 - |\nabla u|^2,$$

we find that

$$d_k^2 - \Delta(\frac{1}{2}v_k^2) + \mathrm{div}(\frac{v_k^2}{2}u) + \frac{v_k}{|u|}u \cdot \nabla P = 0.$$

(See [11, Lemma 5].) The equality is valid for $x \in \mathbb{R}^3$, since we are always treating smooth functions. Taking integral over $\mathbb{R}^3$ the second and the third terms of left hand side vanish. Then we have the following estimate.

$$
\begin{aligned}
U_k &\le \left|\int_{\mathbb{R}^3} v_k \frac{u}{|u|} \cdot \nabla P \, dx\right| = \left|\int_{\mathbb{R}^3} P \mathrm{div} \left(v_k \frac{u}{|u|}\right) dx\right| \\
&\le 3 \int_{\mathbb{R}^3} \chi_{\{|u| > \frac{b}{2^k}\}} |P| \, d_k dx \le \frac{1}{2} \int_{\mathbb{R}^3} d_k^2 \, dx + C \int_{\mathbb{R}^3} \chi_{\{v_k > 0\}} |P|^2 \, dx.
\end{aligned}
$$

We note that since $u \in L^\infty(\mathbb{R}^3) \cap \dot{H}^1_\sigma(\mathbb{R}^3)$, and $\dot{H}^1_\sigma(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ we have $u \in L^p(\mathbb{R}^3)$ for all $p \in (6, \infty)$. Hence,

$$
\begin{aligned}
\frac{U_k}{2} &\le C \int_{\mathbb{R}^3} \chi_{\{v_k \ge 0\}} |P|^2 \, dx \le C \|P\|^2_{L^{\frac{p}{2}}} \|\chi_{\{v_k \ge 0\}}\|_{L^{\frac{p}{p-4}}} \\
&\le \eta C \left(\frac{2^k}{b}\right)^{\frac{6(p-4)}{p}} \|u\|^4_{L^p} U_{k-1}^{\frac{3(p-4)}{p}} \\
&\le \eta C \left(\frac{2^k}{b}\right)^{\frac{6(p-4)}{p}} \|\nabla u\|^{\frac{24}{p}}_{L^2} \|u\|^{\frac{4(p-6)}{p}}_{L^\infty} U_{k-1}^{\frac{3(p-4)}{p}} \\
&\le \eta C (2^k M)^{\frac{6(p-4)}{p}} \|\nabla u\|^{\frac{24}{p}}_{L^2} \|u\|^{-2}_{L^\infty} U_{k-1}^{\frac{3(p-4)}{p}}
\end{aligned}
$$

for $k \ge 1$. We can also obtain the same estimate for the cases of (1) and (2). Note that

$$
\left|\int_{\mathbb{R}^3} \frac{v_k^2}{2} \mathrm{div} u\right| = \left|\int_{\mathbb{R}^3} \frac{(\nabla v_k^2 \cdot u)}{2}\right| = \left|\int_{\mathbb{R}^3} v_k (\nabla v_k \cdot u)\right| \le \int_{\mathbb{R}^3} \chi_{\{v_k \ge 0\}} |u|^2 d_k \, dx
$$

and

$$
\begin{aligned}
\left|\int_{\mathbb{R}^3} \frac{u \mathrm{div} u}{2} \cdot \left(v_k \frac{u}{|u|}\right)\right| &= \left|\frac{1}{2} \int_{\mathbb{R}^3} |u| v_k \mathrm{div} u\right| \\
&\le \left|\frac{1}{2} \int_{\mathbb{R}^3} (\nabla |u| \cdot u) v_k\right| + \left|\frac{1}{2} \int_{\mathbb{R}^3} (\nabla v_k \cdot u) |u|\right| \le \int_{\mathbb{R}^3} \chi_{\{v_k \ge 0\}} |u|^2 d_k \, dx,
\end{aligned}
$$

and we can estimate these terms as a pressure term in (SNS).

Since $\frac{3(p-4)}{p} > 1$ for $p > 6$, we obtain $U_k \to 0$ as $k \to \infty$ by Lemma 1.5, if $\|\nabla u\|_2$ and $\|u\|_\infty$ satisfy the following inequality:

$$
\|\nabla u\|_2^2 \le \left(\eta C (2M)^{\kappa_3(p)} \|\nabla u\|_2^{\frac{24}{p}} \|u\|_\infty^{-2}\right)^{-\kappa_1(p)} 2^{-\kappa_2(p)} \quad \text{for} \quad p > 6, \quad M > 1, \tag{7}
$$

where

$$
\begin{cases}
\kappa_1(p) := \sum_{k=1}^\infty \left(\frac{3(p-4)}{p}\right)^{-k} > 0, \\
\kappa_2(p) := 2^{\frac{6(p-4)}{p}} \sum_{k=2}^\infty k \left(\frac{3(p-4)}{p}\right)^{-k} > 0, \\
\kappa_3(p) := \frac{6(p-4)}{p} > 0.
\end{cases}
$$

Remark 2.2. The following inequality is the simplification of (7):

$$
\|\nabla u\|_2 \le C_{M,p,\eta} \|u\|_\infty^{\alpha(p)} \quad \text{for} \quad p > 6, \tag{8}
$$

where $\alpha(p) := 1/(\frac{1}{\kappa_1(p)} + \frac{12}{p})$. Since $\kappa_1(p)$ is a monotone decreasing function and $\kappa_1(12) = 1$, we have that $\alpha(p) < 1$ for $p > 6$. We easily see that $\alpha(12) = 1/2$. Thus we take $p = 12$ in our main theorem. However we can easily generalize the index p.

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