Abstract. In this paper we prove Liouville type theorem for the stationary Navier-Stokes equations on $\mathbb{R}^3$. More specifically, if a solution $u \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ to the stationary Navier-Stokes system satisfies additional conditions characterized by the decays near infinity and by the oscillation, then we show that $u = 0$.

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1. INTRODUCTION

We consider the stationary problem of the Navier-Stokes equations on $\mathbb{R}^3$:

\[
\begin{aligned}
(u \cdot \nabla)u &= -\nabla P + \Delta u, \\
\text{div } u &= 0,
\end{aligned}
\]

Let $\dot{H}^1(\mathbb{R}^3) = \{u \in [\dot{H}(\mathbb{R}^3)]^3, \text{div } u = 0\}$. A weak solution of stationary Navier-Stokes equations is $u \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ which satisfy (SNS) in the sense of distribution. More precisely,

\[
\langle (u \cdot \nabla)u, \phi \rangle = -\langle P, \text{div } \phi \rangle - \langle \nabla u, \nabla \phi \rangle
\]

for $\phi \in L^3(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, where $\langle f, g \rangle := \int_{\mathbb{R}^3} f(x)g(x)dx$, and $P = \sum_{1 \leq i, j \leq 3} R_i R_j u_i u_j$ with $R_j, j = 1, 2, 3$, the Riesz transform. Using the relation

\[
u = -(-\Delta)^{-1}(\nabla P + (u \cdot \nabla)u)
\]

and a bootstrapping argument, we can see $u \in C^\infty(\mathbb{R}^3)$ if $u \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ (see also [1, 3, 4, 5, 9, 10]). Thus, throughout this paper, we may regard $u \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$.

In [4], Galdi showed that if $v \in L^{9/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, then $v \equiv 0$. To prove this Liouville theorem, he used the test function $\psi_R v$, where $\psi_R \in C^\infty$, $\psi_R = 0$ for $|x| \geq 2R$, $\psi_R(x) = 1$ for $|x| \leq R$ and satisfying $|\nabla \psi_R| \leq M/R$ for some positive constant $M$ independent of $x \in \mathbb{R}^3$. In this paper we show another type of Liouville theorem, in particular, we treat functions which have slower decay at space infinity than these functions which are in $L^{9/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. We note that there is no mutual implication between our result here and Galdi’s result. Also our theorem is completely different from the recent results by Koch, Nadirashvili, Seregin and Sverak([6]).

To prove our theorem we use the De Giorgi method, which is adapted to the Navier-Stokes equations by Vasseur([11])(see also [2]). To show our Liouville theorem, we introduce function classes defined by “decay-control functions”. We also consider the following
model equation for the time-independent vector field $u = (u_1, u_2, u_3)$ on $\mathbb{R}^3$,
\[ (u \cdot \nabla)u = -\frac{1}{2} u \text{div} u + \Delta u \]  
and the vector Burgers equation
\[ (u \cdot \nabla)u = \Delta u. \]
Equation (1) is of interest for various reasons (see [8]). It has the same scaling properties and the same energy estimate as (SNS). Moreover (1) has a non-trivial solution $u$ satisfying
\[ C_1(1 + |x|)^{-\frac{3}{2}} \leq |u(x)| \leq C_2(1 + |x|)^{-\frac{3}{2}} \]
for sufficiently large $x \in \mathbb{R}^3$ and $u \in L^\infty$, where $0 < C_1 \leq C_2$. It means that the non-trivial solution $u$ is in $L^{\frac{2}{5} + \epsilon}$ for any $\epsilon > 0$. Thus, to consider Liouville type theorem of (1) in $L^\infty(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$ is more attractive.

Now we define decay-control functions which has a key role in this paper.

**Definition 1.1.** (Decay-control functions.) We say that a monotone decreasing function $\varphi : (0, \infty) \to [0, \infty)$ is a “decay-control function” for $\lambda \leq L$ if the function $\varphi$ satisfies the following property with some $\beta > 0$,
\[ \beta \varphi(\lambda) \leq \varphi(4\lambda) \quad \text{for} \quad 0 < \lambda < L. \]
Note that this condition is related to “doubling condition” (see [7] for example).

**Remark 1.2.** $\lambda^{-p}$ $(0 < p < \infty)$ for $\lambda < \infty$, $-\log(\lambda/2)$ for $\lambda \leq 1/4$, and $2 - \lambda$ for $\lambda \leq 1/4$, are decay-control functions.

**Definition 1.3.** (Key function classes.) Let us define the function classes $X$ and $Y$ as follows:
\[ Y := Y_{\varphi,C_1,C_2,M} := \{ u \in L^6(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) : C_1 \varphi(\lambda) \leq \{ x \in \mathbb{R}^3 : |u(x)| > \lambda \} \leq C_2 \varphi(\lambda) \quad \text{with} \quad \lambda \leq \|u\|_\infty/M \} \]
for $0 < C_1 \leq C_2$ and $M > 1$, and
\[ X := X_{\beta,C_1,C_2,M} := \{ u \in L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) : \| \nabla u \|_2 \leq D \|u\|_\infty^{1/2} \}, \quad D = D(M, C_2/(C_1 \beta)). \]
Note that the constant $D$ is explicitly expressed (see (7) and (8)).

The definition of $Y$ gives some control on the decay rate of $u$ for sufficiently large $x$. However this says nothing about $\nabla u$ for the large part $\{ x \in \mathbb{R}^3 : \|u\|_\infty/M \leq |u| \leq \|u\|_\infty \}$. That is why we need $X$ to control large (some kind of highly oscillating) part. It means that we cannot rule out highly oscillating function from our Liouville theorem.

**Remark 1.4.** We can easily find which functions belong to $X$. For all $u \in H^1_\sigma(\mathbb{R}^3) \cap L^\infty$, then $u_{a,b} := au(b) \in X$ if $a << 1$ or $b >> 1$. Actually, $\| \nabla u_{a,b} \|_2 / \|u_{a,b}\|_\infty^{1/2} = a^{1/2}b^{-1/2} \| \nabla u \|_2 / \|u\|_\infty^{1/2}$. A direct calculation yields the following lemma (see [11] for example).

**Lemma 1.5.** For any given constants $B, \beta > 1$, then there exists some constant $C_0^*$ such that for any sequence $\{ a_k \}_{k \geq 0}$ satisfying $0 < a_0 \leq C_0^*$ and $a_k \leq CB^k a_{k-1}^\beta$, for any $k \geq 1$, we have $\lim_{k \to \infty} a_k = 0$. 

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Proof. Just take $C_0^* \leq C^{-\sum_{k=1}^{\infty} \frac{1}{2^k}} B^{-\sum_{k=1}^{\infty} \frac{b}{2^k}}$. Then we easily have $a_n \leq C^{-\frac{1}{2}} B^{-\frac{n+1}{2}}$. \[ \]

Now we state the main result.

**Theorem 1.6.** Assume that $u \in \dot{H}^1_s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ be a weak solution to (SNS), or $u \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ be a weak solution to either (1) or (2). If $u \in X \cap Y$, then $u = 0$ in $\mathbb{R}^3$.

2. Proof of the main theorem

Let $b = \|u\|_\infty$. For each $k \geq 0$ we denote

\[
\begin{align*}
 v_k & := \left( |u| - \frac{b}{2^k} \right) \chi_{\{|u| > \frac{b}{2^k}\}}, \\
 d_k^2 & := \frac{b/2^k}{|u|} \chi_{\{|u| > b/2^k\}} |\nabla|u||^2 + \frac{v_k}{|u|} |\nabla u||^2, \\
 U_k & := \int_{\mathbb{R}^3} d_k^2 dx.
\end{align*}
\]

Since $|\nabla|u|| \leq |\nabla u|$, $|\frac{v_k}{|u|}| \leq 1$ and $|\frac{b/2^k}{|u|} \chi_{\{|u| > b/2^k\}}| \leq 1$, we have $U_0 \leq 2 \|\nabla u\|_2^2$.

For $u \in Y$, we have the following property:

\[
\int_{\mathbb{R}^3} \chi_{\{|v_k| > 0\}} = \{ |u| > b/2^k \} \leq C_2 \varphi(b/2^k) \leq \frac{C_2}{\beta} \varphi(b/2^{k-2})
\]

\[
= \eta C_1 \varphi(b/2^{k-2}) \leq \eta |\{|u| > b/2^{k-2}\}| \leq \eta \int_{\mathbb{R}^3} \chi_{\{v_{k-1} > b/2^k\}} \text{ for } k \geq 1,
\]

where $\eta = (C_2/(C_1\beta))$. Note that the above inequality is different from Vasseur’s observation, since we concern Liouville type theorem not regularity criterion. In our case, $v_k$ grows as $k$ increases. This explains why the definition of $Y$ is needed.

The following lemma is minor modification of the lemma described in [2]. However, for the convenience of the readers, we give detailed computations.

**Lemma 2.1.** We have the following inequality:

\[
\begin{cases}
|\nabla v_k| \leq d_k, \\
|\nabla (\frac{v_k}{|u|})| \leq 3d_k.
\end{cases}
\]

**Proof.** First, we show that the inequalities

\[
\chi_{\{|u| > b/2^k\}} |\nabla|u|| \leq d_k \quad \text{and} \quad \frac{v_k}{|u|} |\nabla u|| \leq d_k
\]

hold. To justify the second inequality, we derive the definition of $d_k^2$. A direct calculation yields

\[
d_k^2 \geq \frac{v_k}{|u|} |\nabla u|^2 \geq \left\{ \frac{v_k}{|u|} |\nabla u| \right\}^2.
\]
Hence by taking square root, it follows at once that \( \frac{v_k}{|u|} |\nabla u| \leq d_k \). To justify the first inequality, we recall that \( |\nabla u| \geq |\nabla|u||. \) Hence, it follows from the definition of \( d_k^2 \) that

\[
d_k^2 \geq b/2^k |\nabla|_{|u| \geq b/2^k}|\nabla|u|^2 + \frac{1}{|u|} \{ |u| - b/2^k \} \chi_{|u| \geq b/2^k} |\nabla|u|^2 \geq \chi_{|u| \geq b/2^k} |\nabla|u|^2.
\]

Since it is obvious to see that \( \nabla v_k = \chi_{|u| \geq b/2^k} |\nabla|u| \), we also have the result \( |\nabla v_k| \leq d_k \). Next, we want to justify the inequality that \( |\nabla(v_k/|u|)| \leq 3d_k \). So, we notice that, by applying the product rule, we have

\[
\nabla(\frac{v_k}{|u|}) = \nabla(v_k) \frac{u}{|u|} + \frac{v_k}{|u|^2} \nabla u - \frac{v_k}{|u|^2} \nabla |u|.
\]

However, since \( \frac{v_k}{|u|}|\nabla u| \leq d_k \), and \( \frac{v_k}{|u|^2} u \nabla u || \leq \chi_{|u| \geq b/2^k} |\nabla|u|| \leq d_k \), we have the desired estimate.

Now we return to prove the main theorem. By (6), we see that

\[
\int_{\mathbb{R}^3} \chi_{\{v_k > 0\}} \, dx \leq \eta \int_{\mathbb{R}^3} \chi_{\{v_{k-1} > \frac{2^k}{b}\}} \, dx
\]

\[
\leq \eta \left( \frac{2^k}{b} \right)^6 \|v^{k-1}\|_{L^6}^6 \leq \eta C \left( \frac{2^k}{b} \right)^6 \|\nabla v_{k-1}\|_{L^2}^6
\]

\[
\leq \eta C \left( \frac{2^k}{b} \right)^6 \|d_{k-1}\|_{L^2}^6 \leq \eta C \left( \frac{2^k}{b} \right)^6 U_{k-1}^3.
\]

Thus we have

\[
\|\chi_{\{v_k \geq 0\}}\|_{L^q} \leq (\eta C)^{1/q} \left( \frac{2^k}{b} \right)^{\frac{6}{q}} U_{k-1}^\frac{3}{q}.
\]

Now let us construct a level set energy equality of (SNS). (The case of (1) and (2) are similar. Thus we omit these cases.)

Multiplying \( u \) to (SNS), we obtain

\[
\text{div}(u \frac{|u|^2}{2}) + \text{div}(uP) + |\nabla u|^2 - \Delta \frac{|u|^2}{2} = 0.
\]

Also, multiplying \( u(u^2/|u|^2 - 1) \) to (SNS), we have

\[
\text{div}(u \frac{v_k^2 - |u|^2}{2}) + u(v_k/|u| - 1) \nabla P - (v_k/|u| - 1) \Delta u = 0.
\]

Since

\[
-v_k/|u| - 1) \Delta u = - \Delta \frac{v_k^2 - |u|^2}{2} + d_k^2 - |\nabla u|^2;
\]

we find that

\[
d_k^2 - \Delta \left( \frac{1}{2} v_k^2 \right) + \text{div}(\frac{v_k^2}{2} u) + \frac{v_k}{|u|} u \cdot \nabla P = 0.
\]

(See [11, Lemma 5].) The equality is valid for \( x \in \mathbb{R}^3 \), since we are always treating smooth functions. Taking integral over \( \mathbb{R}^3 \) the second and the third terms of left hand
side vanish. Then we have the following estimate.

\[ U_k \leq \left| \int_{\mathbb{R}^3} v_k u |u|^{-1} \cdot \nabla P \, dx \right| = \left| \int_{\mathbb{R}^3} P \text{div}\left( v_k |u|^{-1} \right) \, dx \right| \]
\[ \leq 3 \int_{\mathbb{R}^3} x(|u| > \frac{3}{P}) |P| P \, dx \leq \frac{1}{2} \int_{\mathbb{R}^3} d_k^2 \, dx + C \int_{\mathbb{R}^3} x(v_k > 0) |P|^2 \, dx. \]

We note that since \( u \in L^\infty(\mathbb{R}^3) \cap H^1_0(\mathbb{R}^3) \), and \( \dot{H}^1_0(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \) we have \( u \in L^p(\mathbb{R}^3) \) for all \( p \in (6, \infty) \). Hence,
\[
\frac{U_k}{2} \leq C \int_{\mathbb{R}^3} x(v_k > 0) |P|^2 \, dx \leq C \| P \|_{L^1}^2 \| x(v_k > 0) \|_{L^p(\mathbb{R}^3)}
\leq \eta C \left( \frac{2k}{b} \right)^{6(p-4)} \frac{p}{p-4} \| u \|_{L^p(\mathbb{R}^3)} \frac{3(p-4)}{p-4} \]
\leq \eta C \left( \frac{2k}{b} \right)^{6(p-4)} \frac{p}{p-4} \| \nabla u \|_{L^p(\mathbb{R}^3)} \frac{2(4-6)}{2} \frac{3(p-4)}{p-4} \]
\leq \eta C (2k M)^{6(p-4)} \frac{p}{p-4} \| \nabla u \|_{L^p(\mathbb{R}^3)} \frac{24}{2} \frac{3(p-4)}{p-4} \]
\]
for \( k \geq 1 \). We can also obtain the same estimate for the cases of (1) and (2). Note that
\[
\left| \int_{\mathbb{R}^3} \frac{v_k^2}{2} \text{div} u \right| = \left| \int_{\mathbb{R}^3} \nabla v_k \cdot u \right| = \left| \int_{\mathbb{R}^3} v_k (\nabla v_k \cdot u) \right| \leq \int_{\mathbb{R}^3} x(v_k > 0) |u|^2 d_k \, dx
\]
and
\[
\left| \int_{\mathbb{R}^3} \frac{u \text{div} u}{2} \left( v_k \frac{u}{|u|} \right) \right| = \left| \frac{1}{2} \int_{\mathbb{R}^3} |u| v_k \text{div} u \right|
\leq \left| \frac{1}{2} \int_{\mathbb{R}^3} (\nabla |u| \cdot u) v_k \right| + \frac{1}{2} \int_{\mathbb{R}^3} (\nabla v_k \cdot u) |u| \right| \leq \int_{\mathbb{R}^3} x(v_k > 0) |u|^2 d_k \, dx,
\]
and we can estimate these terms as a pressure term in (SNS).

Since \( \frac{3(p-4)}{p} > 1 \) for \( p > 6 \), we obtain \( U_k \to 0 \) as \( k \to \infty \) by Lemma 1.5, if \( \| \nabla u \|_2 \) and \( \| u \|_\infty \) satisfy the following inequality:

\[ \| \nabla u \|_2 \leq \left( \eta C (2M)^{\kappa_3(p)} \| u \|_2 \right)^{-\kappa_1(p)} \| u \|_\infty^{-2\kappa_2(p)} \quad \text{for} \quad p > 6, \quad M > 1, \]

where
\[
\begin{align*}
\kappa_1(p) &:= \sum_{k=1}^{\infty} \left( \frac{3(p-4)}{p} \right)^{-k} > 0, \\
\kappa_2(p) &:= 2 \frac{6(p-4)}{p} \sum_{k=2}^{\infty} k \left( \frac{3(p-4)}{p} \right)^{-k} > 0, \\
\kappa_3(p) &:= \frac{6(p-4)}{p} > 0.
\end{align*}
\]

**Remark 2.2.** The following inequality is the simplification of (7):

\[ \| \nabla u \|_2 \leq C_{M,p,q} \| u \|_{L^p(\mathbb{R}^3)}^{\alpha(p)} \quad \text{for} \quad p > 6, \]

where \( \alpha(p) := 1/(\kappa_1(p) + \frac{12}{p}) \). Since \( \kappa_1(p) \) is a monotone decreasing function and \( \kappa_1(12) = 1 \), we have that \( \alpha(p) < 1 \) for \( p > 6 \). We easily see that \( \alpha(12) = 1/2 \). Thus we take \( p = 12 \) in our main theorem. However we can easily generalize the index \( p \).
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