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ON THE LIOUVILLE THEOREM FOR THE STATIONARY NAVIER-STOKES EQUATIONS IN A CRITICAL SPACE

DONGHO CHAE AND TSUYOSHI YONEDA

Abstract. In this paper we prove Liouville type theorem for the stationary Navier-Stokes equations on $\mathbb{R}^3$. More specifically, if a solution $u \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ to the stationary Navier-Stokes system satisfies additional conditions characterized by the decays near infinity and by the oscillation, then we show that $u = 0$.

AMS Subject Classification(2000):35Q30, 76D05
Key words: Stationary Navier-Stokes equations, the Liouville theorem.

1. Introduction

We consider the stationary problem of the Navier-Stokes equations on $\mathbb{R}^3$:

\[(\text{SNS}) \begin{cases} (u \cdot \nabla)u = -\nabla P + \Delta u, & x \in \mathbb{R}^3, \\ \text{div} u = 0, & x \in \mathbb{R}^3. \end{cases} \]

Let $\dot{H}^1_2(\mathbb{R}^3) = \{ u \in [\dot{H}(\mathbb{R}^3)]^3, \text{div} u = 0 \}$. A weak solution of stationary Navier-Stokes equations is $u \in \dot{H}^1_2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ which satisfy (SNS) in the sense of distribution. More precisely,

\[\langle (u \cdot \nabla)u, \phi \rangle = -\langle P, \text{div} \phi \rangle - \langle \nabla u, \nabla \phi \rangle\]

for $\phi \in L^3(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, where $\langle f, g \rangle := \int_{\mathbb{R}^3} f(x)g(x)dx$, and $P = \sum_{1 \leq i,j \leq 3} R_i R_j u_i u_j$ with $R_j, j = 1, 2, 3$, the Riesz transform. Using the relation

\[u = -(-\Delta)^{-1}(\nabla P + (u \cdot \nabla)u)\]

and a bootstrapping argument, we can see $u \in C^\infty(\mathbb{R}^3)$ if $u \in \dot{H}^1_2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ (see also [1, 3, 4, 5, 9, 10]). Thus, throughout this paper, we may regard $u \in \dot{H}^1_2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$.

In [4], Galdi showed that if $v \in L^{9/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, then $v \equiv 0$. To prove this Liouville theorem, he used the test function $\psi_R v$, where $\psi_R \in C^\infty_c$, $\psi_R = 0$ for $|x| \geq 2R$, $\psi_R(x) = 1$ for $|x| \leq R$ and satisfying $|\nabla \psi_R| \leq M/R$ for some positive constant $M$ independent of $x \in \mathbb{R}^3$. In this paper we show another type of Liouville theorem, in particular, we treat functions which have slower decay at space infinity than these functions which are in $L^{9/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. We note that there is no mutual implication between our result here and Galdi’s result. Also our theorem is completely different from the recent results by Koch, Nadirashvili, Seregin and Sverak([6]).

To prove our theorem we use the De Giorgi method, which is adapted to the Navier-Stokes equations by Vasseur([11])(see also [2]). To show our Liouville theorem, we introduce function classes defined by “decay-control functions”. We also consider the following
model equation for the time-independent vector field \( u = (u_1, u_2, u_3) \) on \( \mathbb{R}^3 \),

\[
(1) \quad (u \cdot \nabla)u = -\frac{1}{2}u \text{div} u + \Delta u
\]

and the vector Burgers equation

\[
(2) \quad (u \cdot \nabla)u = \Delta u.
\]

Equation (1) is of interest for various reasons (see [8]). It has the same scaling properties and the same energy estimate as (SNS). Moreover (1) has a non-trivial solution \( u \) satisfying

\[
(3) \quad C_1(1 + |x|)^{-\frac{1}{2}} \leq |u(x)| \leq C_2(1 + |x|)^{-\frac{3}{2}} \quad \text{for sufficiently large} \quad x \in \mathbb{R}^3
\]

and \( u \in L^\infty \), where \( 0 < C_1 \leq C_2 \). It means that the non-trivial solution \( u \) is in \( L^{2+\epsilon} \) for any \( \epsilon > 0 \). Thus, to consider Liouville type theorem of (1) in \( L^{\infty}(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \) is more attractive topic.

Now we define decay-control functions which has a key role in this paper.

**Definition 1.1.** (Decay-control functions.) We say that a monotone decreasing function \( \varphi : (0, \infty) \to [0, \infty) \) is a “decay-control function” for \( \lambda \leq L \) if the function \( \varphi \) satisfies the following property with some \( \beta > 0 \),

\[
\beta \varphi(\lambda) \leq \varphi(4\lambda) \quad \text{for} \quad 0 < \lambda < L.
\]

Note that this condition is related to “doubling condition” (see [7] for example).

**Remark 1.2.** \( \lambda^{-p} (0 < p < \infty) \) for \( \lambda < \infty, -\log(\lambda/2) \) for \( \lambda \leq 1/4 \), and \( 2 - \lambda \) for \( \lambda \leq 1/4 \), are decay-control functions.

**Definition 1.3.** (Key function classes.) Let us define the function classes \( X \) and \( Y \) as follows:

\[
(4) \quad Y := Y_{\varphi,C_1,C_2,M} := \{ u \in L^6(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) : C_1 \varphi(\lambda) \leq |\{ x \in \mathbb{R}^3 : |u(x)| > \lambda \}| \leq C_2 \varphi(\lambda) \quad \text{with} \quad \lambda \leq \| u \|_{\infty}/M \}
\]

for \( 0 < C_1 \leq C_2 \) and \( M > 1 \), and

\[
(5) \quad X := X_{\beta,C_1,C_2,M} := \{ u \in L^\infty(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3) : \| \nabla u \|_2 \leq D \| u \|_{\infty}^{1/2} \}, \quad D = D(M,C_2/(C_1 \beta)).
\]

Note that the constant \( D \) is explicitly expressed (see (7) and (8)).

The definition of \( Y \) gives some control on the decay rate of \( u \) for sufficiently large \( x \). However this says nothing about \( \nabla u \) for the large part \( \{ x \in \mathbb{R}^3 : \| u \|_{\infty}/M \leq |u| \leq \| u \|_{\infty} \} \). That is why we need \( X \) to control large (some kind of highly oscillating) part. It means that we cannot rule out highly oscillating function from our Liouville theorem.

**Remark 1.4.** We can easily find which functions belong to \( X \). For all \( u \in \dot{H}^1_{\sigma}(\mathbb{R}^3) \cap L^\infty \), then \( u_{a,b} := au(b \cdot) \in X \) if \( a \ll 1 \) or \( b >> 1 \). Actually, \( \| \nabla u_{a,b} \|_2/\| u_{a,b} \|_{\infty}^{1/2} = a^{1/2}b^{-1/2}\| \nabla u \|_2/\| u \|_{\infty}^{1/2} \).

A direct calculation yields the following lemma (see [11] for example).

**Lemma 1.5.** For any given constants \( B, \beta > 1 \), then there exists some constant \( C_0^* \) such that for any sequence \( \{ a_k \}_{k \geq 0} \) satisfying \( 0 < a_0 \leq C_0^* \) and \( a_k \leq CB^k a_{k-1}^{\beta} \), for any \( k \geq 1 \), we have \( \lim_{k \to \infty} a_k = 0 \).
Proof. Just take $C_0^* \leq C^{-\sum_{k=1}^{\infty} \frac{1}{2^k}} B^{-\sum_{k=1}^{\infty} \frac{k}{2^k}}$. Then we easily have $a_n \leq C^{-\frac{1}{2}} B^{-\frac{n+1}{2}}$.

Now we state the main result.

**Theorem 1.6.** Assume that $u \in \dot{H}^1_s (\mathbb{R}^3) \cap L^\infty (\mathbb{R}^3)$ be a weak solution to (SNS), or $u \in \dot{H}^1 (\mathbb{R}^3) \cap L^\infty (\mathbb{R}^3)$ be a weak solution to either (1) or (2). If $u \in X \cap Y$, then $u = 0$ in $\mathbb{R}^3$.

2. Proof of the main theorem

Let $b = \|u\|_\infty$. For each $k \geq 0$ we denote

$$v_k := (|u| - \frac{b}{2^k}) \chi_{\{|u| > \frac{b}{2^k}\}},$$

$$d_k^2 := \frac{b/2^k}{|u|} \chi_{\{|u| > b/2^k\}} |\nabla|u||^2 + \frac{v_k}{|u|} |\nabla u||^2,$$

$$U_k := \int_{\mathbb{R}^3} d_k^2 dx.$$

Since $|\nabla|u|| \leq |\nabla u|, \|\frac{v_k}{|u|}\| \leq 1$ and $|\frac{b/2^k}{|u|} \chi_{\{|u| > b/2^k\}}| \leq 1$, we have

$$U_0 \leq 2|\nabla u||^2.$$

For $u \in Y$, we have the following property:

$$\int_{\mathbb{R}^3} \chi_{\{v_k > 0\}} = |\{|u| > b/2^k\}| \leq C_2 \phi \left(\frac{b}{2^k}\right) \leq C_2 \beta \phi \left(\frac{b}{2^k}\right)$$

$$\quad = \eta C_1 \phi \left(\frac{b}{2^{k-2}}\right) \leq \eta |\{|u| > b/2^{k-2}\}| \leq \eta \int_{\mathbb{R}^3} \chi_{\{|v_k| > b/2^k\}} \text{ for } k \geq 1,$$

where $\eta = (C_2/(C_1 \beta))$. Note that the above inequality is different from Vasseur’s observation, since we concern Liouville type theorem not regularity criterion. In our case, $v_k$ grows as $k$ increases. This explains why the definition of $Y$ is needed.

The following lemma is minor modification of the lemma described in [2]. However, for the convenience of the readers, we give detailed computations.

**Lemma 2.1.** We have the following inequality:

$$\begin{cases} \frac{|\nabla v_k|}{|u|} \leq d_k, \\ |\nabla (\frac{v_k}{|u|})| \leq 3d_k. \end{cases}$$

Proof. First, we show that the inequalities

$$\chi_{\{|u| > b/2^k\}} |\nabla|u|| \leq d_k \quad \text{and} \quad \frac{v_k}{|u|} |\nabla u| \leq d_k$$

hold. To justify the second inequality, we derive the definition of $d_k^2$. A direct calculation yields

$$d_k^2 \geq \frac{v_k}{|u|} |\nabla u|^2 \geq \left\{\frac{v_k}{|u|} |\nabla u|\right\}^2.$$
Hence by taking square root, it follows at once that \( \frac{v}{|u|} |\nabla u| \leq d_k \). To justify the first inequality, we recall that \( |\nabla u| \geq |\nabla|u|| \). Hence, it follows from the definition of \( d_k^2 \) that
\[
 d_k^2 \geq b/2^k |\nabla|u| \geq \frac{1}{|u|} \{ |u| - b/2^k \} \chi_{\{|u| \geq b/2^k\}} |\nabla|u||^2 \geq \chi_{\{|u| \geq b/2^k\}} |\nabla|u||^2.
\]
Since it is obvious to see that \( \nabla v_k = \chi_{\{|u| \geq b/2^k\}} \nabla|u| \), we also have the result \( |\nabla v_k| \leq d_k \). Next, we want to justify the inequality that \( |\nabla(v_k |u| u)| \leq 3d_k \). So, we notice that, by applying the product rule, we have
\[
 \nabla(v_k |u| u) = \nabla(v_k) |u| u + v_k |u| \nabla|u| - v_k |u|^2 u \nabla|u|.
\]
However, since \( \frac{v_k}{|u|} |\nabla u| \leq d_k \), and \( \frac{v_k}{|u|^2} u |\nabla|u|| \leq \chi_{\{|u| \geq b/2^k\}} |\nabla|u|| \leq d_k \), we have the desired estimate.

Now we return to prove the main theorem. By (6), we see that
\[
 \int_{\mathbb{R}^3} \chi_{\{v_k > 0\}} \, dx \leq \eta \int_{\mathbb{R}^3} \chi_{\{v_{k-1} > \frac{b}{2^k}\}} \, dx
\leq \eta \left( \frac{2^k}{b} \right)^6 \|v^{k-1}\|_{L^6}^6 \leq \eta C \left( \frac{2^k}{b} \right)^6 \|\nabla v_{k-1}\|_{L^2}^6
\leq \eta C \left( \frac{2^k}{b} \right)^6 \|d_{k-1}\|_{L^2}^6 \leq \eta C \left( \frac{2^k}{b} \right)^6 U_{k-1}^3.
\]
Thus we have
\[
\|\chi_{\{v_k \geq 0\}}\|_{L^q} \leq (\eta C)^{1/q} \left( \frac{2^k}{b} \right)^{\frac{6}{q}} U_{k-1}^\frac{3}{q}.
\]
Now let us construct a level set energy equality of (SNS). (The case of (1) and (2) are similar. Thus we omit these cases.)

Multiplying \( u \) to (SNS), we obtain
\[
 \text{div}(u |u|^2) + \text{div}(u P) + |\nabla u|^2 - \Delta |u|^2 = 0.
\]
Also, multiplying \( u(v_k - 1) \) to (SNS), we have
\[
 \text{div}(u(v_k^2 - |u|^2) + u(v_k - 1) \nabla P - u(v_k - 1) \Delta u = 0.
\]
Since
\[
 -u(v_k - 1) \Delta u = -\Delta \frac{v_k^2 - |u|^2}{2} + d_k^2 - |\nabla u|^2,
\]
we find that
\[
 d_k^2 - \Delta \left( \frac{1}{2} v_k^2 \right) + \text{div} \left( \frac{v_k^2}{2} u \right) + \frac{v_k}{|u|} u \cdot \nabla P = 0.
\]
(See [11, Lemma 5].) The equality is valid for \( x \in \mathbb{R}^3 \), since we are always treating smooth functions. Taking integral over \( \mathbb{R}^3 \) the second and the third terms of left hand
side vanish. Then we have the following estimate.

\[
U_k \leq \left| \int_{\mathbb{R}^3} v_k \frac{u}{|u|} \cdot \nabla P \, dx \right| = \left| \int_{\mathbb{R}^3} P \text{div} \left( v_k \frac{u}{|u|} \right) \, dx \right|
\leq 3 \int_{\mathbb{R}^3} \chi_{\{|u| > \frac{1}{2k}\}} |P| \, dk \, dx \leq \frac{1}{2} \int_{\mathbb{R}^3} d_k^2 \, dx + C \int_{\mathbb{R}^3} \chi_{\{|v_k| > 0\}} |P|^2 \, dx.
\]

We note that since \( u \in L^\infty(\mathbb{R}^3) \cap \dot{H}^1_0(\mathbb{R}^3) \), and \( \dot{H}^1_0(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \) we have \( u \in L^p(\mathbb{R}^3) \) for all \( p \in (6, \infty) \). Hence,

\[
\frac{U_k}{2} \leq C \int_{\mathbb{R}^3} \chi_{\{|v_k| \geq 0\}} |P|^2 \, dx \leq C \|P\|_{L^\frac{6}{5}}^2 \|\chi_{\{|v_k| \geq 0\}}\|_{L^\frac{p}{p+4}}
\leq \eta C\left( \frac{2^k}{b} \right)^{\frac{6(p-4)}{p}} \|u\|_{L^p}^4 \|\chi_{\{|v_k| \geq 0\}}\|_{L^{\frac{p}{p+4}}}
\leq \eta C\left( \frac{2^k}{b} \right)^{\frac{6(p-4)}{p}} \|\nabla u\|_{L^p}^{\frac{24}{p}} \|u\|_{L^\infty}^{\frac{4(p-6)}{p}} \|\chi_{\{|v_k| \geq 0\}}\|_{L^{\frac{p}{p+4}}} \leq \eta C(2^k M)^{\frac{6(p-4)}{p}} \|\nabla u\|_{L^p}^{\frac{24}{p}} \|u\|_{L^\infty}^{\frac{2}{p} \|\chi_{\{|v_k| \geq 0\}}\|_{L^{\frac{p}{p+4}}}}
\]

for \( k \geq 1 \). We can also obtain the same estimate for the cases of (1) and (2). Note that

\[
\left| \int_{\mathbb{R}^3} \frac{v_k^2}{2} \text{div} u \right| = \left| \int_{\mathbb{R}^3} \left( \nabla v_k^2 \cdot u \right) \right| = \left| \int_{\mathbb{R}^3} v_k (\nabla v_k \cdot u) \right| \leq \int_{\mathbb{R}^3} \chi_{\{|v_k| \geq 0\}} |u| \, d_k \, dx
\]

and

\[
\left| \int_{\mathbb{R}^3} \frac{u \text{div} v_k}{2} \cdot \left( v_k \frac{u}{|u|} \right) \right| = \left| \frac{1}{2} \int_{\mathbb{R}^3} |u| v_k \text{div} u \right| 
\leq \frac{1}{2} \int_{\mathbb{R}^3} (\nabla |u| \cdot u) v_k \right| + \frac{1}{2} \int_{\mathbb{R}^3} (\nabla v_k \cdot u) |u| \right| \leq \int_{\mathbb{R}^3} \chi_{\{|v_k| \geq 0\}} |u|^2 \, d_k \, dx,
\]

and we can estimate these terms as a pressure term in (SNS).

Since \( \frac{3(p-4)}{p} > 1 \) for \( p > 6 \), we obtain \( U_k \to 0 \) as \( k \to \infty \) by Lemma 1.5, if \( \|\nabla u\|_2 \) and \( \|u\|_\infty \) satisfy the following inequality:

\[
\|\nabla u\|_2^2 \leq \left( \eta C(2M)^{\kappa_1(p)} \|\nabla u\|_2^{\kappa_2(p)} \|u\|_\infty^{-\kappa_3(p)} \right)^{2-(\kappa_1(p))} \quad \text{for} \quad p > 6, \quad M > 1,
\]

where

\[
\kappa_1(p) := \sum_{k=1}^{\infty} \left( \frac{3(p-4)}{p} \right)^{-k} > 0,
\kappa_2(p) := 2 \left( \frac{3(p-4)}{p} \right)^{-1} \sum_{k=2}^{\infty} k \left( \frac{3(p-4)}{p} \right)^{-k} > 0,
\kappa_3(p) := \frac{6(p-4)}{p} > 0.
\]

**Remark 2.2.** The following inequality is the simplification of (7):

\[
\|\nabla u\|_2 \leq C_{M,p,q} \|u\|_{\infty}^{\alpha(p)} \quad \text{for} \quad p > 6,
\]

where \( \alpha(p) := 1/(\kappa_1(p) + \kappa_2(p)) \). Since \( \kappa_1(p) \) is a monotone decreasing function and \( \kappa_1(12) = 1 \), we have that \( \alpha(p) < 1 \) for \( p > 6 \). We easily see that \( \alpha(12) = 1/2 \). Thus we take \( p = 12 \) in our main theorem. However we can easily generalize the index \( p \).
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References


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