ON THE LIOUVILLE THEOREM FOR THE STATIONARY NAVIER-STOKES EQUATIONS IN A CRITICAL SPACE

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Abstract. In this paper we prove Liouville type theorem for the stationary Navier-Stokes equations on $\mathbb{R}^3$. More specifically, if a solution $u \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ to the stationary Navier-Stokes system satisfies additional conditions characterized by the decays near infinity and by the oscillation, then we show that $u = 0$.

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1. introduction

We consider the stationary problem of the Navier-Stokes equations on $\mathbb{R}^3$:

\begin{equation*}
\begin{aligned}
\langle (u \cdot \nabla)u, \phi \rangle &= -\langle P, \text{div} \phi \rangle - \langle \nabla u, \nabla \phi \rangle \\
\text{for } \phi \in L^3(\mathbb{R}^3) \cap H^1(\mathbb{R}^3), &\text{ where } \langle f, g \rangle := \int_{\mathbb{R}^3} f(x)g(x)dx, \text{ and } P = \sum_{1 \leq i, j \leq 3} R_i R_j u_i u_j \text{ with } R_j, j = 1, 2, 3, \text{ the Riesz transform.}
\end{aligned}
\end{equation*}

and a bootstrapping argument, we can see $u \in C^\infty(\mathbb{R}^3)$ if $u \in \dot{H}^1(\mathbb{R}^3)$ (see also [1, 3, 4, 5, 9, 10]). Thus, throughout this paper, we may regard $u \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$.

In [4], Galdi showed that if $v \in L^{9/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, then $v \equiv 0$. To prove this Liouville theorem, he used the test function $\psi_Rv$, where $\psi_R \in C^\infty_c$, $\psi_R = 0$ for $|x| \geq 2R$, $\psi_R(x) = 1$ for $|x| \leq R$ and satisfying $|\nabla \psi_R| \leq M/R$ for some positive constant $M$ independent of $x \in \mathbb{R}^3$. In this paper we show another type of Liouville theorem, in particular, we treat functions which have slower decay at space infinity than these functions which are in $L^{9/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. We note that there is no mutual implication between our result here and Galdi’s result. Also our theorem is completely different from the recent results by Koch, Nadirashvili, Seregin and Sverak([6]).

To prove our theorem we use the De Giorgi method, which is adapted to the Navier-Stokes equations by Vasseur([11])(see also [2]). To show our Liouville theorem, we introduce function classes defined by “decay-control functions”. We also consider the following
model equation for the time-independent vector field $u = (u_1, u_2, u_3)$ on $\mathbb{R}^3,$

\[(u \cdot \nabla)u = -\frac{1}{2}u \text{div} u + \Delta u\]

and the vector Burgers equation

\[(u \cdot \nabla)u = \Delta u.\]

Equation (1) is of interest for various reasons (see [8]). It has the same scaling properties and the same energy estimate as (SNS). Moreover (1) has a non-trivial solution $u$ satisfying

\[C_1(1 + |x|)^{-\frac{2}{\beta}} \leq |u(x)| \leq C_2(1 + |x|)^{-\frac{2}{\beta}}\]

for sufficiently large $x \in \mathbb{R}^3$ and $u \in L^\infty$, where $0 < C_1 \leq C_2$. It means that the non-trivial solution $u$ is in $L^{\frac{2}{\beta}+\epsilon}$ for any $\epsilon > 0$. Thus, to consider Liouville type theorem of (1) in $L^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is more attractive topic.

Now we define decay-control functions which has a key role in this paper.

**Definition 1.1.** (Decay-control functions.) We say that a monotone decreasing function $\varphi : (0, \infty) \to [0, \infty)$ is a “decay-control function” for $\lambda \leq L$ if the function $\varphi$ satisfies the following property with some $\beta > 0,$

\[\beta \varphi(\lambda) \leq \varphi(4\lambda)\]

for $0 < \lambda < L$.

Note that this condition is related to “doubling condition” (see [7] for example).

**Remark 1.2.** $\lambda^{-p} (0 < p < \infty)$ for $\lambda < \infty, -\log(\lambda/2)$ for $\lambda \leq 1/4,$ and $2 - \lambda$ for $\lambda \leq 1/4,$ are decay-control functions.

**Definition 1.3.** (Key function classes.) Let us define the function classes $X$ and $Y$ as follows:

\[(Y := Y_{\varphi,C_1,C_2,M} := \{u \in L^5(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) : C_1 \varphi(\lambda) \leq \|x \in \mathbb{R}^3 : |u(x)| > \lambda\| \leq C_2 \varphi(\lambda)\}\]

with $\lambda \leq \|u\|_\infty/M$}

for $0 < C_1 \leq C_2$ and $M > 1$, and

\[(X := X_{\beta,C_1,C_2,M} := \{u \in L^\infty(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3) : \|\nabla u\|_2 \leq D\|u\|_\infty^{1/2}\}, \quad D = D(M, C_2/(C_1\beta)).\]

Note that the constant $D$ is explicitly expressed (see (7) and (8)).

The definition of $Y$ gives some control on the decay rate of $u$ for sufficiently large $x.$

However this says nothing about $\nabla u$ for the large part $\{x \in \mathbb{R}^3 : \|u\|_\infty/M \leq |u| \leq \|u\|_\infty\}.$

That is why we need $X$ to control large (some kind of highly oscillating) part. It means that we cannot rule out highly oscillating function from our Liouville theorem.

**Remark 1.4.** We can easily find which functions belong to $X$. For all $u \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty$, then $u_{a,b} := au(b) \in X$ if $a << 1$ or $b >> 1$. Actually, $\|\nabla u_{a,b}\|_2/\|u_{a,b}\|_\infty^{1/2} = a^{1/2}b^{-1/2}\|\nabla u\|_2/\|u\|_\infty^{1/2}$.

A direct calculation yields the following lemma (see [11] for example).

**Lemma 1.5.** For any given constants $B$, $\beta > 1$, then there exists some constant $C_0^*$ such that for any sequence $\{a_k\}_{k \geq 0}$ satisfying $0 < a_0 \leq C_0^*$ and $a_k \leq CB^k a_{k-1}^\beta$, for any $k \geq 1$, we have $\lim_{k \to \infty} a_k = 0$. 

2
Proof. Just take \( C_0^* \leq C^{-\sum_{k=1}^{\infty} \frac{1}{p_k}} B^{-\sum_{k=1}^{\infty} \frac{1}{q_k}} \). Then we easily have \( a_n \leq C^{-\frac{n}{2}} B^{-\frac{n+1}{p}} \). \( \blacksquare \)

Now we state the main result.

**Theorem 1.6.** Assume that \( u \in \dot{H}_p^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \) be a weak solution to (SNS), or \( u \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \) be a weak solution to either (1) or (2). If \( u \in X \cap Y \), then \( u = 0 \) in \( \mathbb{R}^3 \).

2. **Proof of the main theorem**

Let \( b = \frac{\|u\|_{L^\infty}}{M} \). For each \( k \geq 0 \) we denote

\[
v_k := (|u| - \frac{b}{2^k}) \chi_{\{|u| > \frac{b}{2^k}\}},
\]

\[
d_k^2 := \frac{b}{|u|} \chi_{\{|u| > \frac{b}{2^k}\}} |\nabla|u||^2 + \frac{v_k}{|u|} |\nabla u|^2,
\]

\[
U_k := \int_{\mathbb{R}^3} d_k^2 dx.
\]

Since \(|\nabla|u|| \leq |\nabla u|, \frac{v_k}{|u|} \leq 1\) and \( \frac{b/2^k}{|u|} \chi_{\{|u| > b/2^k\}} \leq 1 \), we have

\[
U_0 \leq 2\|\nabla u\|^2.
\]

For \( u \in Y \), we have the following property:

\[
\int_{\mathbb{R}^3} \chi_{\{|v_k| > 0\}} = |\{|u| > b/2^k\}| \leq C_2 \varphi(b/2^k) \leq \frac{C_2}{\beta} \varphi(b/2^{k-2})
\]

\[
= \eta C_1 \varphi(b/2^{k-2}) \leq \eta |\{|u| > b/2^{k-2}\}| \leq \eta \int_{\mathbb{R}^3} \chi_{\{|v_k| > b/2^k\}} \quad \text{for} \quad k \geq 1,
\]

where \( \eta = (C_2/(C_1\beta)) \). Note that the above inequality is different from Vasseur’s observation, since we concern Liouville type theorem not regularity criterion. In our case, \( v_k \) grows as \( k \) increases. This explains why the definition of \( Y \) is needed.

The following lemma is minor modification of the lemma described in [2]. However, for the convenience of the readers, we give detailed computations.

**Lemma 2.1.** We have the following inequality:

\[
\begin{cases}
|\nabla v_k| \leq d_k, \\
|\nabla (\frac{v_k}{|u|})| \leq 3d_k.
\end{cases}
\]

**Proof.** First, we show that the inequalities

\[
\chi_{\{|u| > b/2^k\}} |\nabla|u|| \leq d_k \quad \text{and} \quad \frac{v_k}{|u|} |\nabla u| \leq d_k
\]

hold. To justify the second inequality, we derive the definition of \( d_k^2 \). A direct calculation yields

\[
d_k^2 \geq \frac{v_k}{|u|} |\nabla u|^2 \geq \{\frac{v_k}{|u|} |\nabla u|\}^2.
\]
Hence by taking square root, it follows at once that $\frac{v_k}{|u|} |\nabla u| \leq d_k$. To justify the first inequality, we recall that $|\nabla u| \geq |\nabla| u ||$. Hence, it follows from the definition of $d_k^2$ that

$$d_k^2 \geq \frac{b^{2k}}{|u|} \chi_{\{|u|\geq b/2^{2k}\}} |\nabla u|^2 + \frac{1}{|u|} \{ |u| - b/2^k \} \chi_{\{|u|\geq b/2^k\}} |\nabla u|^2 \geq \chi_{\{|u|\geq b/2^k\}} |\nabla u|^2.$$

Since it is obvious to see that $\nabla v_k = \chi_{\{|u|\geq b/2^k\}} |\nabla u|$, we also have the result $|\nabla v_k| \leq d_k$. Next, we want to justify the inequality that $|\nabla(\frac{v_k}{|u|}) u| \leq 3d_k$. So, we notice that, by applying the product rule, we have

$$\nabla(\frac{v_k}{|u|}) u = \nabla(\frac{v_k}{|u|}) \frac{u}{|u|} + \frac{v_k}{|u|} \nabla u - \frac{v_k}{|u|^2} u \nabla |u|.$$

However, since $\frac{v_k}{|u|} |\nabla u| \leq d_k$, and $|\nabla \frac{v_k}{|u|} u| \leq \chi_{\{|u|\geq b/2^k\}} |\nabla u| \leq d_k$, we have the desired estimate.

Thus we have

$$\|\chi_{\{v_k \geq 0\}}\|_{L^q} \leq (\eta C)^{1/q} \left( \frac{2k}{b} \right)^{\frac{6}{q}} U_{k-1}^3.$$
side vanish. Then we have the following estimate.

\[
U_k \leq \left| \int_{\mathbb{R}^3} v_k \frac{u}{|u|} \cdot \nabla P \, dx \right| \leq \frac{3}{2} \int_{\mathbb{R}^3} \chi_{\{|u| > \frac{1}{2} k\}} |P| \, dk \, dx \leq \frac{1}{2} \int_{\mathbb{R}^3} d_k^2 \, dx + C \int_{\mathbb{R}^3} \chi_{\{|v_k| > 0\}} |P|^2 \, dx.
\]

We note that since \( u \in L^\infty(\mathbb{R}^3) \cap \dot{H}^1_0(\mathbb{R}^3) \), and \( \dot{H}^1(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3) \) we have \( u \in L^p(\mathbb{R}^3) \) for all \( p \in (6, \infty) \). Hence,

\[
\frac{U_k}{2} \leq C \int_{\mathbb{R}^3} \chi_{\{|v_k| \geq 0\}} |P|^2 \, dx \leq C \|P\|_{L^2}^2 \|\chi_{\{|v_k| \geq 0\}}\|_{L^{\frac{p}{2}}},
\]

\[
\leq \eta C \left( \frac{2k}{b} \right)^{\frac{6(p-4)}{p}} \|u\|_{L^p}^{\frac{24}{p}} U_{k-1}^{\frac{3(p-4)}{2}}
\]

\[
\leq \eta C(2k M)^{\frac{6(p-4)}{p}} \|\nabla u\|_{L^2} \|u\|_{L^\infty}^{\frac{24}{p}} U_{k-1}^{\frac{3(p-4)}{2}}
\]

for \( k \geq 1 \). We can also obtain the same estimate for the cases of (1) and (2). Note that

\[
\left| \int_{\mathbb{R}^3} \frac{v_k^2}{2} \text{div} u \right| = \left| \int_{\mathbb{R}^3} \frac{\nabla v_k^2 \cdot u}{2} \right| = \left| \int_{\mathbb{R}^3} v_k (\nabla v_k \cdot u) \right| \leq \int_{\mathbb{R}^3} \chi_{\{|v_k| \geq 0\}} |u|^2 \, dk \, dx
\]

and

\[
\left| \int_{\mathbb{R}^3} \frac{u \text{div} u}{2} \cdot \left( \frac{v_k}{|u|} \right) \right| = \left| \frac{1}{2} \int_{\mathbb{R}^3} |u| v_k \text{div} u \right| \leq \frac{1}{2} \int_{\mathbb{R}^3} (\nabla |u| \cdot u) v_k + \frac{1}{2} \int_{\mathbb{R}^3} (\nabla v_k \cdot u) |u| \leq \int_{\mathbb{R}^3} \chi_{\{|v_k| \geq 0\}} |u|^2 \, dk \, dx,
\]

and we can estimate these terms as a pressure term in (SNS).

Since \( \frac{3(p-4)}{p} > 1 \) for \( p > 6 \), we obtain \( U_k \rightarrow 0 \) as \( k \rightarrow \infty \) by Lemma 1.5, if \( \|\nabla u\|_2 \) and \( \|u\|_\infty \) satisfy the following inequality:

\[
\|\nabla u\|_2^2 \leq \left( \eta C(2M)^{\kappa_2(p)} \|\nabla u\|_2^{\frac{24}{p}} \|u\|_{L^\infty}^{-2} \right)^{-\kappa_4(p)} 2^{-\kappa_2(p)} \quad \text{for} \quad p > 6, \quad M > 1,
\]

where

\[
\begin{aligned}
\kappa_1(p) := & \sum_{k=1}^\infty (\frac{3(p-4)}{p})^{-k} > 0, \\
\kappa_2(p) := & \frac{6(p-4)}{p} \sum_{k=2}^\infty k (\frac{3(p-4)}{p})^{-k} > 0, \\
\kappa_3(p) := & \frac{6(p-4)}{p} > 0.
\end{aligned}
\]

**Remark 2.2.** The following inequality is the simplification of (7):

\[
\|\nabla u\|_2 \leq C_{M,p,\eta} \|u\|_\infty^{\alpha(p)} \quad \text{for} \quad p > 6,
\]

where \( \alpha(p) := \frac{1}{(\frac{1}{\kappa_1(p)} + \frac{12}{p})} \). Since \( \kappa_1(p) \) is a monotone decreasing function and \( \kappa_1(12) = 1 \), we have that \( \alpha(p) < 1 \) for \( p > 6 \). We easily see that \( \alpha(12) = 1/2 \). Thus we take \( p = 12 \) in our main theorem. However we can easily generalize the index \( p \).
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