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ON THE LIOUVILLE THEOREM FOR THE STATIONARY NAVIER-STOKES EQUATIONS IN A CRITICAL SPACE

DONGHO CHAE AND TSUYOSHI YONEDA

Abstract. In this paper we prove Liouville type theorem for the stationary Navier-Stokes equations on $\mathbb{R}^3$. More specifically, if a solution $u \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ to the stationary Navier-Stokes system satisfies additional conditions characterized by the decays near infinity and by the oscillation, then we show that $u = 0$.

AMS Subject Classification(2000):35Q30, 76D05
Key words: Stationary Navier-Stokes equations, the Liouville theorem.

1. Introduction

We consider the stationary problem of the Navier-Stokes equations on $\mathbb{R}^3$:

\begin{align*}
\text{(SNS)} \begin{cases}
(u \cdot \nabla)u = -\nabla P + \Delta u, & x \in \mathbb{R}^3, \\
\text{div } u = 0, & x \in \mathbb{R}^3.
\end{cases}
\end{align*}

Let $\dot{H}^1_\sigma(\mathbb{R}^3) = \{u \in [\dot{H}(\mathbb{R}^3)]^3, \text{div } u = 0\}$. A weak solution of stationary Navier-Stokes equations is $u \in \dot{H}^1_\sigma(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ which satisfy (SNS) in the sense of distribution. More precisely,

\[ \langle (u \cdot \nabla)u, \phi \rangle = -\langle P, \text{div } \phi \rangle - \langle \nabla u, \nabla \phi \rangle \]

for $\phi \in L^3(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, where $\langle f, g \rangle := \int_{\mathbb{R}^3} f(x)g(x)dx$, and $P = \sum_{1 \leq i,j \leq 3} R_i R_j u_i u_j$ with $R_j, j = 1, 2, 3$, the Riesz transform. Using the relation

\[ u = -(-\Delta)^{-1}(\nabla P + (u \cdot \nabla)u) \]

and a bootstrapping argument, we can see $u \in C^\infty(\mathbb{R}^3)$ if $u \in \dot{H}^1_\sigma(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ (see also [1, 3, 4, 5, 9, 10]). Thus, throughout this paper, we may regard $u \in \dot{H}^1_\sigma(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$.

In [4], Galdi showed that if $v \in L^{9/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, then $v \equiv 0$. To prove this Liouville theorem, he used the test function $\psi_R v$, where $\psi_R \in C_0^\infty$, $\psi_R = 0$ for $|x| \geq 2R$, $\psi_R(x) = 1$ for $|x| \leq R$ and satisfying $|\nabla \psi_R| \leq M/R$ for some positive constant $M$ independent of $x \in \mathbb{R}^3$. In this paper we show another type of Liouville theorem, in particular, we treat functions which have slower decay at space infinity than these functions which are in $L^{9/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. We note that there is no mutual implication between our result here and Galdi’s result. Also our theorem is completely different from the recent results by Koch, Nadirashvili, Seregin and Sverak([6]).

To prove our theorem we use the De Giorgi method, which is adapted to the Navier-Stokes equations by Vasseur([11])(see also [2]). To show our Liouville theorem, we introduce function classes defined by “decay-control functions”. We also consider the following
model equation for the time-independent vector field \( u = (u_1, u_2, u_3) \) on \( \mathbb{R}^3 \),

\[
(1) \quad (u \cdot \nabla)u = -\frac{1}{2}u \text{div}u + \Delta u
\]

and the vector Burgers equation

\[
(2) \quad (u \cdot \nabla)u = \Delta u.
\]

Equation (1) is of interest for various reasons (see [8]). It has the same scaling properties and the same energy estimate as (SNS). Moreover (1) has a non-trivial solution \( u \) satisfying

\[
(3) \quad C_1(1 + |x|)^{-\frac{2}{3}} \leq |u(x)| \leq C_2(1 + |x|)^{-\frac{2}{3}} \quad \text{for sufficiently large} \quad x \in \mathbb{R}^3
\]

and \( u \in L^\infty \), where \( 0 < C_1 \leq C_2 \). It means that the non-trivial solution \( u \) is in \( L^{\frac{2}{2}+\epsilon} \) for any \( \epsilon > 0 \). Thus, to consider Liouville type theorem of (1) in \( L^\infty(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) \) is more attractive topic.

Now we define decay-control functions which has a key role in this paper.

**Definition 1.1.** (Decay-control functions.) We say that a monotone decreasing function \( \varphi : (0, \infty) \to [0, \infty) \) is a "decay-control function" for \( \lambda \leq L \) if the function \( \varphi \) satisfies the following property with some \( \beta > 0 \),

\[
\beta \varphi(\lambda) \leq \varphi(4\lambda) \quad \text{for} \quad 0 < \lambda < L.
\]

Note that this condition is related to "doubling condition" (see [7] for example).

**Remark 1.2.** \( \lambda^{-p} \) (0 < \( p < \infty \)) for \( \lambda < \infty \), \( -\log(\lambda/2) \) for \( \lambda \leq 1/4 \), and \( 2 - \lambda \) for \( \lambda \leq 1/4 \), are decay-control functions.

**Definition 1.3.** (Key function classes.) Let us define the function classes \( X \) and \( Y \) as follows:

\[
(4) \quad Y := Y_{\varphi,C_1,C_2,M} := \{ u \in L^6(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) : C_1 \varphi(\lambda) \leq \{ x \in \mathbb{R}^3 : |u(x)| > \lambda \} \leq C_2 \varphi(\lambda) \quad \text{with} \quad \lambda \leq \|u\|_\infty/M \}
\]

for \( 0 < C_1 \leq C_2 \) and \( M > 1 \), and

\[
(5) \quad X := X_{\beta,C_1,C_2,M} := \{ u \in L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) : \|\nabla u\|_2 \leq D \|u\|^1/2 \} , \quad D = D(M,C_2/(C_1\beta)).
\]

Note that the constant \( D \) is explicitly expressed (see (7) and (8)).

The definition of \( Y \) gives some control on the decay rate of \( u \) for sufficiently large \( x \). However this says nothing about \( \nabla u \) for the large part \( \{ x \in \mathbb{R}^3 : \|u\|_\infty/M \leq |u| \leq \|u\|_\infty \} \).

That is why we need \( X \) to control large (some kind of highly oscillating) part. It means that we cannot rule out highly oscillating function from our Liouville theorem.

**Remark 1.4.** We can easily find which functions belong to \( X \). For all \( u \in H^1(\mathbb{R}^3) \cap L^\infty \), then \( u_{a,b} := au(b) \in X \) if \( a << 1 \) or \( b >> 1 \). Actually, \( \|\nabla u_{a,b}\|_2/\|u_{a,b}\|_\infty^{1/2} = a^{1/2}b^{-1/2}\|\nabla u\|_2/\|u\|_\infty^{1/2} \).

A direct calculation yields the following lemma (see [11] for example).

**Lemma 1.5.** For any given constants \( B, \beta > 1 \), then there exists some constant \( C_0^* \) such that for any sequence \( \{a_k\}_{k \geq 0} \) satisfying \( 0 < a_0 \leq C_0^* \) and \( a_k \leq CB^k a_{k-1}^{\beta} \), for any \( k \geq 1 \), we have \( \lim_{k \to \infty} a_k = 0 \).
Proof. Just take $C_0^* \leq C^{-\sum_{k=1}^{\infty} \frac{1}{\beta k}} B^{-\sum_{k=1}^{\infty} \frac{k}{\beta}}$. Then we easily have $a_n \leq C^{-\frac{n}{\beta}} B^{-\frac{n+1}{\beta}}$.

Now we state the main result.

**Theorem 1.6.** Assume that $u \in \dot{H}^1 (\mathbb{R}^3) \cap L^\infty (\mathbb{R}^3)$ be a weak solution to (SNS), or $u \in H^1 (\mathbb{R}^3) \cap L^\infty (\mathbb{R}^3)$ be a weak solution to either (1) or (2). If $u \in X \cap Y$, then $u = 0$ in $\mathbb{R}^3$.

### 2. Proof of the main theorem

Let $b = \|u\|_\infty$. For each $k \geq 0$ we denote

\[ v_k := (|u| - \frac{b}{2^k})\chi_{\{|u| > \frac{b}{2^k}\}}, \]

\[ d_k^2 := \frac{b/2^k}{|u|} \chi_{\{|u| > \frac{b}{2^k}\}} |\nabla|u||^2 + \frac{v_k}{|u|} |\nabla u||^2, \]

\[ U_k := \int_{\mathbb{R}^3} d_k^2 \, dx. \]

Since $|\nabla u| \leq |\nabla u|$, $|\frac{v_k}{|u|}| \leq 1$ and $|\frac{b/2^k}{|u|} \chi_{\{|u| > \frac{b}{2^k}\}}| \leq 1$, we have

\[ U_0 \leq 2\|\nabla u\|_2^2. \]

For $u \in Y$, we have the following property:

\[ \int_{\mathbb{R}^3} \chi_{\{v_k > 0\}} = |\{|u| > \frac{b}{2^k}\}| \leq C_2 \varphi (b/2^k) \leq \frac{C_2}{\beta} \varphi (b/2^{k-2}) \]

\[ \leq \eta C_1 \varphi (b/2^{k-2}) \leq \eta \left[ \left| \left| u \right| > \frac{b}{2^{k-2}} \right| \right] \leq \eta \int_{\mathbb{R}^3} \chi_{\{v_{k-1} > \frac{b}{2^k}\}} \quad \text{for} \quad k \geq 1, \]

where $\eta = (C_2/(C_1 \beta))$. Note that the above inequality is different from Vasseur’s observation, since we concern Liouville type theorem not regularity criterion. In our case, $v_k$ grows as $k$ increases. This explains why the definition of $Y$ is needed.

The following lemma is minor modification of the lemma described in [2]. However, for the convenience of the readers, we give detailed computations.

**Lemma 2.1.** We have the following inequality:

\[
\begin{cases}
|\nabla v_k| \leq d_k, \\
|\nabla (\frac{v_k}{|u|})| \leq 3d_k.
\end{cases}
\]

**Proof.** First, we show that the inequalities

\[ \chi_{\{|u| > \frac{b}{2^k}\}} |\nabla|u|| \leq d_k \quad \text{and} \quad \frac{v_k}{|u|} |\nabla u|| \leq d_k \]

hold. To justify the second inequality, we derive the definition of $d_k^2$. A direct calculation yields

\[ d_k^2 \geq \frac{v_k}{|u|} |\nabla u||^2 \geq \left( \frac{v_k}{|u|} |\nabla u|| \right)^2. \]
Hence by taking square root, it follows at once that \( \frac{v_k}{|u|} \) \( \leq d_k \). To justify the first inequality, we recall that \( |\nabla u| \geq |\nabla v_k| \). Hence, it follows from the definition of \( d_k \) that
\[
d^2_k \geq \frac{b/2}{|u|} \chi_{\{|u| \geq b/2^k\}} |\nabla u|^2 + \frac{1}{|u|} \{|u| - b/2^k\} \chi_{\{|u| \geq b/2^k\}} |\nabla u|^2 \geq \chi_{\{|u| \geq b/2^k\}} |\nabla u|^2.
\]
Since it is obvious to see that \( \nabla v_k = \chi_{\{|u| \geq b/2^k\}} \nabla |u| \), we also have the result \( |\nabla v_k| \leq d_k \). Next, we want to justify the inequality that \( |\nabla (v_k |u|) - u| \leq 3d_k \). So, we notice that, by applying the product rule, we have
\[
\nabla (v_k |u|) = \nabla (v_k) \frac{|u|}{|u|} \nabla u + \frac{v_k}{|u|^2} u \nabla |u|.
\]
However, since \( \frac{v_k}{|u|} \leq \frac{d_k}{2} \) and \( |u|^2 u \nabla |u| \leq \chi_{\{|u| \geq b/2^k\}} |\nabla |u|| \leq d_k \), we have the desired estimate.}

Now we return to prove the main theorem. By (6), we see that
\[
\int_{\mathbb{R}^3} \chi_{\{v_k > 0\}} \, dx \leq \eta \int_{\mathbb{R}^3} \chi_{\{v_{k-1} > \frac{b}{2^k}\}} \, dx
\]
\[
\leq \eta \left( \frac{2^k}{b} \right)^6 \|v^{k-1}\|_{L^6}^6 \leq \eta C \left( \frac{2^k}{b} \right)^6 \|\nabla v_{k-1}\|_{L^2}^6
\]
\[
\leq \eta C \left( \frac{2^k}{b} \right)^6 \|d_{k-1}\|_{L^2}^6 \leq \eta C \left( \frac{2^k}{b} \right)^6 U_{k-1}^3.
\]
Thus we have
\[
\|\chi_{\{v_k > 0\}}\|_{L^q} \leq (\eta C)^{1/q} \left( \frac{2^k}{b} \right)^{6/q} U_{k-1}^{3/q}.
\]
Now let us construct a level set energy equality of (SNS). (The case of (1) and (2) are similar. Thus we omit these cases.)

Multiplying \( u \) to (SNS), we obtain
\[
\text{div}\left(u \frac{|u|^2}{2}\right) + \text{div}(uP) + |\nabla u|^2 - \Delta \frac{|u|^2}{2} = 0.
\]
Also, multiplying \( u (\frac{v_k}{|u|} - 1) \) to (SNS), we have
\[
\text{div}\left(u \frac{v_k^2}{2} - \frac{|u|^2}{2}\right) + u (\frac{v_k}{|u|} - 1) \nabla P - u (\frac{v_k}{|u|} - 1) \Delta u = 0.
\]
Since
\[
-u (\frac{v_k}{|u|} - 1) \Delta u = -\Delta \frac{v_k^2 - |u|^2}{2} + d_k^2 - |\nabla u|^2,
\]
we find that
\[
d_k^2 - \Delta \left( \frac{1}{2} v_k^2 \right) + \text{div}\left( \frac{v_k^2}{2} u \right) + \frac{v_k}{|u|} u \cdot \nabla P = 0.
\]
(See [11, Lemma 5].) The equality is valid for \( x \in \mathbb{R}^3 \), since we are always treating smooth functions. Taking integral over \( \mathbb{R}^3 \) the second and the third terms of left hand
side vanish. Then we have the following estimate.
\[ U_k \leq \left| \int_{\mathbb{R}^3} \frac{v_k u}{|u|} \cdot \nabla P \, dx \right| = \left| \int_{\mathbb{R}^3} P \div \left( \frac{v_k u}{|u|} \right) \, dx \right| \]
\[ \leq 3 \int_{\mathbb{R}^3} \chi_{\{|u|>\frac{1}{3}\}} |P| \, dk \, dx \leq \frac{1}{2} \int_{\mathbb{R}^3} d_k^2 \, dx + C \int_{\mathbb{R}^3} \chi_{\{|u|>0\}} |P|^2 \, dx. \]

We note that since \( u \in L^\infty(\mathbb{R}^3) \cap H_0^1(\mathbb{R}^3) \), and \( H_0^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3) \) we have \( u \in L^p(\mathbb{R}^3) \) for all \( p \in (6, \infty) \). Hence,
\[ \frac{U_k}{2} \leq C \int_{\mathbb{R}^3} \chi_{\{|u|>0\}} |P|^2 \, dx \leq C \| P \|_{L^p}^2 \| \chi_{\{|u|>0\}} \|_{L^\frac{p}{p-1}} \]
\[ \leq \eta C \left( \frac{2^k}{b} \right) - \frac{6(p-4)}{p} \| u \|_{L^p}^4 \| \chi_{\{|u|>0\}} \|_{L^\frac{p}{p-1}} \]
\[ \leq \eta C \left( \frac{2^k}{b} \right) - \frac{6(p-4)}{p} \| \nabla u \|_{L^2}^2 \| u \|_{L^\infty} 2 \| \chi_{\{|u|>0\}} \|_{L^\frac{p}{p-1}} \]
\[ \leq \eta C \left( 2^k M \right) - \frac{6(p-4)}{p} \| \nabla u \|_{L^2}^2 \| u \|_{L^\infty} 2 \| \chi_{\{|u|>0\}} \|_{L^\frac{p}{p-1}} \]

for \( k \geq 1 \). We can also obtain the same estimate for the cases of (1) and (2). Note that
\[ \left| \int_{\mathbb{R}^3} \frac{v_k^2}{2} \div u \right| = \left| \int_{\mathbb{R}^3} \nabla v_k \cdot u \right| = \int_{\mathbb{R}^3} v_k (\nabla v_k \cdot u) \leq \int_{\mathbb{R}^3} v_k (\nabla v_k \cdot u) \right| \]
and
\[ \left| \int_{\mathbb{R}^3} \frac{u \div v}{2} \cdot \left( \frac{v_k u}{|u|} \right) \right| = \left| \frac{1}{2} \int_{\mathbb{R}^3} |u| v_k \div u \right| \]
\[ \leq \frac{1}{2} \int_{\mathbb{R}^3} (\nabla |u| \cdot u) v_k + \frac{1}{2} \int_{\mathbb{R}^3} (\nabla v_k \cdot u) |u| \leq \int_{\mathbb{R}^3} \chi_{\{|u|>0\}} |u| \right|^2 \, dk \, dx, \]
and we can estimate these terms as a pressure term in (SNS).

Since \( \frac{3(p-4)}{p} > 1 \) for \( p > 6 \), we obtain \( U_k \to 0 \) as \( k \to \infty \) by Lemma 1.5, if \( \| \nabla u \|_2 \) and \( \| u \|_\infty \) satisfy the following inequality:
\[ \| \nabla u \|_2^2 \leq \left( \eta C(2M)^\kappa_1(p) \| \nabla u \|_{L^2}^{24} \| u \|_{L^\infty}^{-2} \right)^{-\kappa_1(p)} 2^{-\kappa_2(p)} \] for \( p > 6, \ M > 1 \),
where
\[ \left\{ \begin{array}{l}
\kappa_1(p) := \sum_{k=1}^\infty \frac{3(p-4)}{p} - k > 0,
\kappa_2(p) := 2 \frac{6(p-4)}{p} \sum_{k=2}^\infty k \left( \frac{3(p-4)}{p} \right) - k > 0,
\kappa_3(p) := \frac{6(p-4)}{p} > 0.
\end{array} \right. \]

Remark 2.2. The following inequality is the simplification of (7):
\[ \| \nabla u \|_2 \leq C_{M,p,\eta} \| u \|^{\alpha(p)}_\infty \] for \( p > 6 \),
where \( \alpha(p) := 1/(\frac{1}{\kappa_1(p)} + \frac{12}{p}) \). Since \( \kappa_1(p) \) is a monotone decreasing function and \( \kappa_1(12) = 1 \),
we have that \( \alpha(p) < 1 \) for \( p > 6 \). We easily see that \( \alpha(12) = 1/2 \). Thus we take \( p = 12 \) in our main theorem. However we can easily generalize the index \( p \).
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