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ON THE LIOUVILLE THEOREM FOR THE STATIONARY NAVIER-STOKES EQUATIONS IN A CRITICAL SPACE

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Abstract. In this paper we prove Liouville type theorem for the stationary Navier-Stokes equations on $\mathbb{R}^3$. More specifically, if a solution $u \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ to the stationary Navier-Stokes system satisfies additional conditions characterized by the decays near infinity and by the oscillation, then we show that $u = 0$.

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Key words: Stationary Navier-Stokes equations, the Liouville theorem.

1. INTRODUCTION

We consider the stationary problem of the Navier-Stokes equations on $\mathbb{R}^3$:

\[
(SNS) \quad \begin{cases}
(u \cdot \nabla) u = -\nabla P + \Delta u, & x \in \mathbb{R}^3, \\
\text{div} \ u = 0, & x \in \mathbb{R}^3.
\end{cases}
\]

Let $\dot{H}^1_0(\mathbb{R}^3) = \{u \in \dot{H}(\mathbb{R}^3)^3, \text{div} \ u = 0\}$. A weak solution of stationary Navier-Stokes equations is $u \in \dot{H}^1_0(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ which satisfy (SNS) in the sense of distribution. More precisely,

\[
\langle (u \cdot \nabla) u, \phi \rangle = -\langle P, \text{div} \ \phi \rangle - \langle \nabla u, \nabla \phi \rangle
\]

for $\phi \in L^3(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, where $\langle f, g \rangle := \int_{\mathbb{R}^3} f(x)g(x)dx$, and $P = \sum_{i,j=1,2,3} R_i R_j u_i u_j$ with $R_j, j = 1, 2, 3$, the Riesz transform. Using the relation

\[
u = -(-\Delta)^{-1}(\nabla P + (u \cdot \nabla) u)
\]

and a bootstrapping argument, we can see $u \in C^\infty(\mathbb{R}^3)$ if $u \in \dot{H}^1_0(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ (see also [1, 3, 4, 5, 9, 10]). Thus, throughout this paper, we may regard $u \in \dot{H}^1_0(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$.

In [4], Galdi showed that if $v \in L^{9/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, then $v \equiv 0$. To prove this Liouville theorem, he used the test function $\psi_R v$, where $\psi_R \in C^\infty_c$, $\psi_R = 0$ for $|x| \geq 2R$, $\psi_R(x) = 1$ for $|x| \leq R$ and satisfying $|\nabla \psi_R| \leq M/R$ for some positive constant $M$ independent of $x \in \mathbb{R}^3$. In this paper we show another type of Liouville theorem, in particular, we treat functions which have slower decay at space infinity than these functions which are in $L^{9/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. We note that there is no mutual implication between our result here and Galdi’s result. Also our theorem is completely different from the recent results by Koch, Nadirashvili, Seregin and Sverak([6]).

To prove our theorem we use the De Giorgi method, which is adapted to the Navier-Stokes equations by Vasseur([11]) (see also [2]). To show our Liouville theorem, we introduce function classes defined by “decay-control functions”. We also consider the following
Definition 1.1. (Decay-control functions.) We say that a monotone decreasing function \( \varphi : (0, \infty) \to [0, \infty) \) is a “decay-control function” for \( \lambda \leq L \) if the function \( \varphi \) satisfies the following property with some \( \beta > 0 \),
\[
\beta \varphi(\lambda) \leq \varphi(4\lambda) \quad \text{for} \quad 0 < \lambda < L.
\]

Note that this condition is related to “doubling condition” (see [7] for example).

Remark 1.2. \( \lambda^{-p} \) (\( 0 < p < \infty \)) for \( \lambda < \infty, -\log(\lambda/2) \) for \( \lambda \leq 1/4 \), and \( 2 - \lambda \) for \( \lambda \leq 1/4 \), are decay-control functions.

Definition 1.3. (Key function classes.) Let us define the function classes \( X \) and \( Y \) as follows:
\[
Y := Y_{\varphi,C_1,C_2,M} := \{ u \in L^5(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) : C_1 \varphi(\lambda) \leq |\{ x \in \mathbb{R}^3 : |u(x)| > \lambda \}| \leq C_2 \varphi(\lambda) \}
\]
with \( \lambda \leq \|u\|_\infty/M \)

for \( 0 < C_1 \leq C_2 \) and \( M > 1 \), and
\[
X := X_{\beta,C_1,C_2,M} := \{ u \in L^\infty(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3) : \|\nabla u\|_2 \leq D\|u\|^{1/2}_\infty \}, \quad D = D(M,C_2/(C_1\beta)).
\]

The definition of \( Y \) gives some control on the decay rate of \( u \) for sufficiently large \( x \). However this says nothing about \( \nabla u \) for the large part \( \{ x \in \mathbb{R}^3 : \|u\|_\infty/M \leq |u| \leq \|u\|_\infty \} \).

Remark 1.4. We can easily find which functions belong to \( X \). For all \( u \in \dot{H}_s^1(\mathbb{R}^3) \cap L^\infty \), then \( u_{a,b} := au(b) \in X \) if \( a \ll 1 \) or \( b >> 1 \). Actually, \( \|\nabla u_{a,b}\|_2/\|u_{a,b}\|^{1/2}_\infty = a^{1/2}b^{-1/2}\|\nabla u\|_2/\|u\|^{1/2}_\infty \).

A direct calculation yields the following lemma (see [11] for example).

Lemma 1.5. For any given constants \( B, \beta > 1 \), then there exists some constant \( C_0^* \) such that for any sequence \( \{a_k\}_{k \geq 0} \) satisfying \( 0 < a_0 \leq C_0^* \) and \( a_k \leq CB^k a_{k-1}^\beta \), for any \( k \geq 1 \), we have \( \lim_{k \to \infty} a_k = 0 \).
Proof. Just take \( C_0^* \leq C^{-\sum_{k=1}^{\infty} \frac{1}{\beta_k}} B^{-\sum_{k=1}^{\infty} \frac{k}{\beta_k}} \). Then we easily have \( a_n \leq C^{-\frac{1}{\beta}} B^{-\frac{n+1}{\beta}} \). 

Now we state the main result.

**Theorem 1.6.** Assume that \( u \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \) be a weak solution to (SNS), or \( u \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \) be a weak solution to either (1) or (2). If \( u \in X \cap Y \), then \( u = 0 \) in \( \mathbb{R}^3 \).

2. PROOF OF THE MAIN THEOREM

Let \( b = \frac{\|u\|_\infty}{M} \). For each \( k \geq 0 \) we denote

\[
v_k := (|u| - \frac{b}{2^k})\chi_{\{|u| > \frac{b}{2^k}\}},
\]

\[
d_k^2 := \frac{b/2^k}{|u|} \chi_{\{|u| > b/2^k\}} |\nabla|u||^2 + \frac{v_k}{|u|} |\nabla u||^2,
\]

\[
U_k := \int_{\mathbb{R}^3} d_k^2 dx.
\]

Since \( |\nabla|u| \leq |\nabla u| \), \( \left|\frac{v_k}{|u|}\right| \leq 1 \) and \( \left|\frac{b/2^k}{|u|} \chi_{\{|u| > b/2^k\}}\right| \leq 1 \), we have

\[
U_0 \leq 2\|\nabla u\|_2^2.
\]

For \( u \in Y \), we have the following property:

\[
(6) \quad \int_{\mathbb{R}^3} \chi_{\{v_k > 0\}} = |\{|u| > b/2^k\}| \leq C_2 \varphi(b/2^k) \leq \frac{C_2}{\beta} \varphi(b/2^{k-2}) \leq \eta \int_{\mathbb{R}^3} \chi_{\{v_{k-1} > b/2^k\}} \quad \text{for } k \geq 1,
\]

where \( \eta = (C_2/(C_1\beta)) \). Note that the above inequality is different from Vasseur’s observation, since we concern Liouville type theorem not regularity criterion. In our case, \( v_k \) grows as \( k \) increases. This explains why the definition of \( Y \) is needed.

The following lemma is minor modification of the lemma described in [2]. However, for the convenience of the readers, we give detailed computations.

**Lemma 2.1.** We have the following inequality:

\[
\begin{cases}
|\nabla v_k| \leq d_k, \\
|\nabla (\frac{v_k}{|u|})| \leq 3d_k.
\end{cases}
\]

Proof. First, we show that the inequalities

\[
\chi_{\{|u| > b/2^k\}} |\nabla|u| | \leq d_k \quad \text{and} \quad \frac{v_k}{|u|} |\nabla u| \leq d_k
\]

hold. To justify the second inequality, we derive the definition of \( d_k^2 \). A direct calculation yields

\[
d_k^2 \geq \frac{v_k}{|u|} |\nabla u|^2 \geq \left\{\frac{v_k}{|u|} |\nabla u|\right\}^2.
\]
Hence by taking square root, it follows at once that \( \frac{v_k}{|u|} |\nabla u| \leq d_k \). To justify the first inequality, we recall that \( |\nabla u| \geq |\nabla|u|| \). Hence, it follows from the definition of \( d_k^2 \) that
\[
d_k^2 \geq \frac{b^2}{2k} \chi_{\{|u| \geq b/2^k\}} |\nabla u|^2 + \frac{1}{|u|} \{|u| - b/2^k\} \chi_{\{|u| \geq b/2^k\}} |\nabla u|^2 \geq \chi_{\{|u| \geq b/2^k\}} |\nabla u|^2.
\]

Since it is obvious to see that \( \nabla v_k = \chi_{\{|u| \geq b/2^k\}} |\nabla u| \), we also have the result \( |\nabla v_k| \leq d_k \).

Next, we want to justify the inequality that \( |\nabla (\frac{v_k}{|u|} u)| \leq 3d_k \). So, we notice that, by applying the product rule, we have
\[
\nabla (\frac{v_k}{|u|} u) = \nabla (\frac{v_k}{|u|}) u + \frac{v_k}{|u|} u - \frac{v_k}{|u|^2} u |\nabla u|.
\]

However, since \( \frac{v_k}{|u|} |\nabla u| \leq d_k \), and \( \frac{v_k}{|u|^2} u |\nabla u| \leq \chi_{\{|u| \geq b/2^k\}} |\nabla u| \leq d_k \), we have the desired estimate.

Now we return to prove the main theorem. By (6), we see that
\[
\int_{\mathbb{R}^3} \chi_{\{v_k > 0\}} \, dx \leq \eta \int_{\mathbb{R}^3} \chi_{\{v_{k-1} > \frac{2^k}{b}\}} \, dx
\]
\[
\leq \eta \left( \frac{2^k}{b} \right)^6 \|v^{k-1}\|_{L^6}^6 \leq \eta C \left( \frac{2^k}{b} \right)^6 \|\nabla v_{k-1}\|_{L^6}^6
\]
\[
\leq \eta C \left( \frac{2^k}{b} \right)^6 \|d_{k-1}\|_{L^2}^6 \leq \eta C \left( \frac{2^k}{b} \right)^6 U_{k-1}^3.
\]

Thus we have
\[
\|\chi_{\{v_k \geq 0\}}\|_{L^q} \leq (\eta C)^{1/q} \left( \frac{2^k}{b} \right)^{\frac{6}{q}} U_{k-1}^3.
\]

Now let us construct a level set energy equality of (SNS). (The case of (1) and (2) are similar. Thus we omit these cases.)

Multiplying \( u \) to (SNS), we obtain
\[
\text{div}(u \frac{|u|^2}{2}) + \text{div}(uP) + |\nabla u|^2 - \Delta |u|^2 = 0.
\]

Also, multiplying \( u \frac{v_k^2}{|u|^2} - 1 \) to (SNS), we have
\[
\text{div}(u \frac{v_k^2}{|u|^2} - \frac{|u|^2}{2}) + u \frac{v_k}{|u|} - 1) \nabla P - u \frac{v_k}{|u|} - 1 \Delta u = 0.
\]

Since
\[
-u \frac{v_k}{|u|} - 1 \Delta u = -\Delta \frac{v_k^2}{2} + d_k^2 - |\nabla u|^2,
\]
we find that
\[
d_k^2 - \Delta \frac{v_k^2}{2} + \text{div}(\frac{v_k^2}{2} u) + \frac{v_k}{|u|} u \cdot \nabla P = 0.
\]

(See [11, Lemma 5].) The equality is valid for \( x \in \mathbb{R}^3 \), since we are always treating smooth functions. Taking integral over \( \mathbb{R}^3 \) the second and the third terms of left hand
side vanish. Then we have the following estimate.

\[ U_k \leq \left| \int_{\mathbb{R}^3} v_k \frac{u}{|u|} \cdot \nabla P \, dx \right| = \left| \int_{\mathbb{R}^3} P \text{div} \left( v_k \frac{u}{|u|} \right) \, dx \right| \]

\[ \leq 3 \int_{\mathbb{R}^3} \chi_{\{|u| > \frac{1}{2k}\}} |P| \, dk \, dx \leq \frac{1}{2} \int_{\mathbb{R}^3} d_k^2 \, dx + C \int_{\mathbb{R}^3} \chi_{\{|v_k| > 0\}} |P|^2 \, dx. \]

We note that since \( u \in L^\infty(\mathbb{R}^3) \cap H^1_0(\mathbb{R}^3) \), and \( \dot{H}^1(\mathbb{R}^3) \to L^p(\mathbb{R}^3) \) we have \( u \in L^p(\mathbb{R}^3) \) for all \( p \in (6, \infty) \). Hence,

\[ \frac{U_k}{2} \leq C \int_{\mathbb{R}^3} \chi_{\{|v_k| \geq 0\}} |P|^2 \, dx \leq C \|P\|_{L^p}^2 \|\chi_{\{|v_k| \geq 0\}}\|_{L^{\frac{p}{p-1}}} \]

\[ \leq \eta C \left( \frac{2k}{b} \right)^{\frac{6(p-4)}{p}} \|u\|_{L^p}^{\frac{3(p-4)}{2}} \]

\[ \leq \eta C \left( \frac{2k}{b} \right)^{\frac{6(p-4)}{p}} \|\nabla u\|_{L^2} \|u\|_{L^\infty} \cdot \frac{24}{L_k} U_k^{-1} \]

\[ \leq \eta C(2k M) \left( \frac{6(p-4)}{p} \right) \|\nabla u\|_{L^2} \|u\|_{L^\infty} \cdot \frac{24}{L_k} U_k^{-1} \]

for \( k \geq 1 \). We can also obtain the same estimate for the cases of (1) and (2). Note that

\[ \left| \int_{\mathbb{R}^3} \frac{v_k^2}{2} \text{div} u \right| = \left| \int_{\mathbb{R}^3} \left( \nabla v_k^2 \cdot u \right) \right| \leq \int_{\mathbb{R}^3} \chi_{\{|v_k| \geq 0\}|u|^2} \, dk \, dx \]

and

\[ \left| \int_{\mathbb{R}^3} \frac{u \text{div} v_k}{2} \cdot \left( v_k \frac{u}{|u|} \right) \right| = \left| \frac{1}{2} \int_{\mathbb{R}^3} |u| v_k \text{div} u \right| \]

\[ \leq \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u| \cdot u) v_k \right| + \left| \frac{1}{2} \int_{\mathbb{R}^3} (\nabla v_k \cdot u)|u| \right| \leq \int_{\mathbb{R}^3} \chi_{\{|v_k| \geq 0\}|u|^2} \, dk \, dx, \]

and we can estimate these terms as a pressure term in (SNS).

Since \( \frac{3(p-4)}{p} > 1 \) for \( p > 6 \), we obtain \( U_k \to 0 \) as \( k \to \infty \) by Lemma 1.5, if \( \|\nabla u\|_2 \) and \( \|u\|_\infty \) satisfy the following inequality:

\[ \|\nabla u\|_2^2 \leq \left( \eta C(2M)^{\kappa_2(p)} \|\nabla u\|_2^{24} \|u\|_{L^\infty}^{-2} \right)^{-\kappa_1(p)} \]

\[ 2^{-\kappa_2(p)} \quad \text{for} \quad p > 6, \quad M > 1, \]

where

\[ \kappa_1(p) := \sum_{k=1}^{\infty} \left( \frac{3(p-4)}{p} \right)^{-k} > 0, \]

\[ \kappa_2(p) := \frac{6(p-4)}{p} \sum_{k=2}^{\infty} k \left( \frac{3(p-4)}{p} \right)^{-k} > 0, \]

\[ \kappa_3(p) := \frac{6(p-4)}{p} > 0. \]

**Remark 2.2.** The following inequality is the simplification of (7):

\[ \|\nabla u\|_2 \leq C_{M,p,\eta} \|u\|_{L^\infty}^{\alpha(p)} \quad \text{for} \quad p > 6, \]

where \( \alpha(p) := \frac{1}{\kappa_1(p)} + \frac{12}{p} \). Since \( \kappa_1(p) \) is a monotone decreasing function and \( \kappa_1(12) = 1 \), we have that \( \alpha(p) < 1 \) for \( p > 6 \). We easily see that \( \alpha(12) = 1/2 \). Thus we take \( p = 12 \) in our main theorem. However we can easily generalize the index \( p \).
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