LONG-TIME SOLVABILITY OF THE NAVIER-STOKES-BOUSSINESQ EQUATIONS WITH ALMOST PERIODIC INITIAL LARGE DATA

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Abstract. We investigate large time existence of solutions of the Navier-Stokes-Boussinesq equations with spatially almost periodic large data when the density stratification is sufficiently large. In 1996, Kimura and Herring [16] examined numerical simulations to show a stabilizing effect due to the stratification. They observed scattered two-dimensional pancake-shaped vortex patches lying almost in the horizontal plane. Our result is a mathematical justification of the presence of such two-dimensional pancakes. To show the existence of solutions for large times, we use ℓ^1-norm of amplitudes. Existence for large times is then proven using techniques of fast singular oscillating limits and bootstrapping argument from a global-in-time unique solution of the system of limit equations.

1. Introduction

Large-scale fluids such as atmosphere and ocean are parts of geophysical fluids, and the Coriolis force due to the earth rotation plays a significant role in the large scale flows considered in meteorology and geophysics.

Mathematically, it was first investigated by Poincaré [19]. Later on, the problem of strong Coriolis force was extensively studied. Babin, Mahalov and Nicolaenko (BMN) [1, 2] studied the incompressible rotating Navier-Stokes and Euler equations in the periodic case while Chemin, Desjardins, Gallagher and Grenier [8] analyzed the case of decaying data and more recently, the second author [20] considered the almost periodic case. Gallagher in [10] studied a more abstract parabolic system. We also refer to Paicu [18] for anisotropic viscous fluids, Benameur, Ibrahim and Majdoub [5] for rotating Magneto-Hydro-Dynamic system and to Gallagher and Saint-Raymond [11] for inhomogeheous rotating fluid equations.

Moreover on the one hand, the case when fluids are governed by both a strong Coriolis force and vertical stratification effects was investigated by BMN in [3] in the periodic setting and Charve in [6] for decaying data. However, their studies do not cover the case when fluid equations are governed by the only effect of stratification. It is known that a strong Coriolis force has a stabilizing effect (see [1]). However, in BMN [4, Section 9.2] the authors observed that for ideal fluids (i.e., with zero viscosity), the only effect of stratification leads to unbalanced dynamics. Moreover, the case of both strong Coriolis and stratification forces in the almost periodic setting seems to remain open. Finally, note that for the almost periodic case, energy type estimates cannot be used, and instead Fujita-Kato's approach has to be

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used. On the other hand, Kimura and Herring [16] examined numerical simulations to show a stabilizing effect due to the effect of stratification for viscous fluid. They observed scattered two-dimensional pancake-shaped vortex patches lying almost in the horizontal plane. Our result can be seen as a mathematical justification of the presence of such two-dimensional pancakes. More precisely, we study long-time solvability for Navier-Stokes-Boussinesq equation with stratification effects.The Navier-Stokes-Boussinesq equations with stratification effects are governed by the following equations.

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = g\rho e_3 & x\in\mathbb{R}^3, \quad t>0 \\ \partial_t \rho - \kappa\Delta\rho + (u\cdot\nabla)\rho = -\mathcal{N}^2 u_3 & x\in\mathbb{R}^3, \quad t>0 \\ \nabla\cdot u = 0 & x\in\mathbb{R}^3, \quad t>0 \\ u|_{t=0} = u_0\ , \quad \rho|_{t=0} = \rho_0 & \end{cases} \tag{1.1}$$

where the unknown functions $u = u(x,t) = (u_1,u_2,u_3)$, $\rho = \rho(x,t)$ and $p = p(x,t)$ are the fluid velocity, the thermal disturbances and the pressure, respectively. The parameters $\nu > 0$, $\kappa \geq 0$ and $g > 0$ are the viscosity, the thermal diffusivity and the gravity force, respectively. The parameter $\mathcal{N} > 0$ is Brunt-Väisälä frequency (stratification-parameter). Recall that $\Delta := (\partial_1^2 + \partial_2^2 + \partial_3^2)I_3$, $\nabla := (\partial_1, \partial_2, \partial_3)$ and $e_3 := (0,0,1)$.

Our method follows the ideas based on BMN. For the limit equations, we show that it is equivalent to the 2D-Navier-Stokes equations[1], which is known to have, in the almost periodic setting, a unique global solution, see for example [15]. Then, we show that the global existence for the remainder equations in the limit equations. Since we handle not only periodic functions, we have to introduce a new analytic functional setting, which is more suitable for the almost periodic situation as the second author did for the rotating fluid case in [20]. More precisely, a straightforward application of an energy inequality is impossible if the initial data is almost periodic. To overcome this difficulty, we use ℓ^1-norm of amplitudes with sum closed frequency set. We recall the analytic functional setting (see [20]) as follows:

Definition 1.1. *(Countable sum closed frequency set.) A countable set Λ in $\mathbb{R}^3$ is called a sum closed frequency set if it satisfies the following properties:*

$$\Lambda = \{a+b : a, b \in \Lambda\} \quad \textit{and} \quad -\Lambda = \Lambda.$$

Remark 1.2. If $\{e_j\}_{j=1}^3$ is the standard orthogonal basis in $\mathbb{R}^3$, then the sets $\mathbb{Z}^3$, $\{m_1e_1 + \sqrt{2}m_2e_2 + m_3e_3 : m_1, \cdots, m_3 \in \mathbb{Z}\}$ and $\{m_1e_1 + m_2(e_1+e_2\sqrt{2}) + m_3(e_2+e_3\sqrt{3}) : m_1, m_2, m_3 \in \mathbb{Z}\}$ are examples of such countable sum closed frequency sets. Clearly, the case $\mathbb{Z}^3$ corresponds to the periodic. Each of the other two cases are dense in $\mathbb{R}^3$ and therefore they correspond to "purely" almost periodic setting.

Definition 1.3. *(An ℓ^1-type function space) Let BUC be the space of all bounded uniformly continuous functions defined in $\mathbb{R}^3$ equipped with the L^∞-norm. For a countable sum closed frequency set $\Lambda \subset \mathbb{R}^3$, let*

$$X^\Lambda(\mathbb{R}^3) := \{u = \sum_{n\in\Lambda} \hat{u}_n e^{in\cdot x} \in BUC(\mathbb{R}^3) : u_{-n} = u_n^* \quad for \quad n \in \Lambda,\ \|u\| := \sum_{n\in\Lambda} |u_n| < \infty\},$$

where u_n^* is the complex conjugate coefficient of u_n.

The second condition in Definition 1.1 is needed to include real-valued almost periodic functions in X^Λ.

Remark 1.4. Note that functions in ℓ^1 do not necessarily decay as $x \to \infty$. Also, this almost periodic setting is in general, different from the periodic case since the frequency set may have accumulation points. The almost periodic setting is somehow between the periodic and the full non-decaying cases.

[1] in the sense that there is a one to one correspondence between solutions of the two equations

Now, we define anisotropic dilation of the frequency set as follows.

Definition 1.5. *For $\gamma = (\gamma_1, \gamma_2) \in (0, \infty)^2$, let*

$$\Lambda(\gamma) := \{(\gamma_1 n_1, \gamma_2 n_2, n_3) \in \mathbb{R}^3 : (n_1, n_2, n_3) \in \Lambda\}. \tag{1.2}$$

Now, we specify the following Quasi-Geostrophic equation (a part of limiting system), and assume (for the moment) that it has a scalar global solution $\theta = \theta(t) = \theta(t, x_1, x_2, x_3)$,

$$\begin{cases} \partial_t \theta - \Delta_3 \theta + (-\Delta_h)^{-1/2} \left[(w \cdot \nabla) \left((-\Delta_h)^{1/2} \theta \right) \right] = 0, \\ w = \left(-\partial_{x_2} (-\Delta_h)^{-1/2} \theta, \partial_{x_1} (-\Delta_h)^{-1/2} \theta \right) \\ \theta(t)|_{t=0} = \theta_0, \end{cases} \tag{1.3}$$

where $\Delta_h := \partial_{x_1}^2 + \partial_{x_2}^2$ and $\Delta_3 := \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$. We will show that the initial value problem for the QG equation admits a global-in-time unique solution in $C([0, \infty) : X^\Lambda)$ with the initial data $\theta_0 = -\partial_{x_2} (-\Delta_h)^{-1/2} u_{0,1} + \partial_{x_1} (-\Delta_h)^{-1/2} u_{0,2}$.

More precisely, we give an explicit one-to-one correspondence between the QG and a 2D type Navier-Stokes equations (for the existence of the unique global solution to 2D-Navier-Stokes equation with almost periodic initial data, see [15]). Now we state our main result.

Theorem 1.6. *Let Λ be a sum closed frequency set. There exists a set of frequencies dilation factors $\Gamma(\Lambda) \subset (0, \infty)^2$ such that*[2]*:*
for any $\gamma \in \Gamma$, for any zero-mean value and divergence free initial vector field $u_0 \in X^{\Lambda(\gamma)}$, initial thermal disturbance $\rho_0 \in X^{\Lambda(\gamma)}$, $\nu > 0$, $\kappa \geq 0$ and $T > 0$, there exists $N_0 > g$ depending only on ν, κ, u_0, ρ_0 such that if $|N| > N_0$, then there exists a mild solution to the equation (1.1), $u(t) \in C([0, T] : X^{\Lambda(\gamma)})$ with zero-mean value and divergence free, and $\rho(t) \in C([0, T] : X^{\Lambda(\gamma)})$.

Remark 1.7. For the periodic case, we do not need to restrict the frequency set to Γ i.e. we can take $\Gamma(\Lambda) = (0, \infty)^2$. However, the computation in this case is more complicated and needs a"restricted convolution" type result in the spirit of [2].

2. Preliminaries

Before going any further, we first recall the following few facts about the space X^Λ:

- $(X^\Lambda, \| \cdot \|)$ is a Banach space, and any almost periodic function $u \in X^\Lambda$ can be decomposed $u(x) = \Sigma_{n \in \Lambda} \hat{u}_n e^{inx}$, where each "Fourier coefficient" $\hat{u}_n$ is uniquely determined by
$$\hat{u}_n = \lim_{|B| \to \infty} \frac{1}{|B|} \int_B u(x) e^{ix \cdot n} \, dx,$$
and B stands for a ball in $\mathbb{R}^3$ (see for example [7]).
- X^Λ is closed subspace of FM, the Fourier preimage of the space of all finite Radon measures proposed by Giga, Inui, Mahalov and Matsui in 2005 (see [12, 13, 14]).
- Leray projection on almost periodic functions $\bar{P} = \{\bar{P}_{jk}\}_{j,k=1,2,3}$ is defined as
$$\bar{P}_{jk} := \delta_{jk} + R_j R_k \quad (1 \leq j, k \leq 3)$$
with δ_{jk} is Kronecker's delta and R_j is the Riesz transform defined by
$$R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-\frac{1}{2}} \; for \; j = 1, 2, 3.$$

[2]the complement set Γ^c is at most countable.

The symbol $\sigma(R_j)$ of R_j is $in_j/|n|$, where $i=\sqrt{-1}$ (see [7]). Let P be the extended Leray projection with Fourier-multiplier $P_n = \{P_{n,ij}\}_{i,j=1,2,3,4}$ given by

$$P_{n,ij} := \begin{cases} \delta_{ij} - \frac{n_i n_j}{|n|^2} & (1\leq i,j\leq 3),\\ \delta_{ij} & (otherwise). \end{cases}$$

- Helmotz-Leray decomposition is defined on almost periodic functions in the same way as in the periodic case. Namely, u is uniquely decomposed as

$$u = w + \nabla\pi,$$

where $\pi = -(-\Delta)^{-1}\mathrm{div}\ u$ and $w = \bar{P}u$.

Now we rewrite the system (1.1) in a more abstract way. Set $N := \mathcal{N}\sqrt{g}$ and $v \equiv (v_1, v_2, v_3, v_4) := (u_1, u_2, u_3, \frac{\sqrt{g}}{\mathcal{N}}\rho)$. Then v solves

$$\begin{cases} \partial_t v - \tilde{\nu}\Delta v + NJv + \nabla_3 p = -(v\cdot\nabla_3)v\\ v|_{t=0} = v_0\\ \nabla_3\cdot v = 0, \end{cases} \tag{2.1}$$

with $\tilde{\nu} = \mathrm{diag}(\nu,\nu,\nu,\kappa)$, the initial data $v_0 = (u_{0,1}, u_{0,2}, u_{0,3}, \frac{\sqrt{g}}{\mathcal{N}}\rho_0)$, $\nabla_3 := (\partial_1, \partial_2, \partial_3, 0)$,

$$J := \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and $(v\cdot\nabla_3) = (v_1\partial_1 + v_2\partial_2 + v_3\partial_3)$.

Observe that under the condition $N > g$ we have $\mathcal{N} > \sqrt{g}$ and therefore $\|v_{0,4}\| = \|\frac{\sqrt{g}}{\mathcal{N}}\rho_0\| < \|\rho_0\|$. We will assume this condition throughout the paper.

Applying the extended Leray projection P to (2.1), we obtain

$$\begin{cases} dv/dt + (-\tilde{\nu}\Delta + NS)v = -P(v\cdot\nabla_3)v,\\ v|_{t=0} = Pv_0 = v_0, \end{cases} \tag{2.2}$$

with $S := PJP$. Recall that for $|n|_h \neq 0$, the matrix $S_n := P_n J P_n$ has the following Craya-Herring orthonormal eigen basis $\{q_n^1, q_n^{-1}, q_n^0, q_n^{div}\}$ (see [3, 9]) associated to the eigenvalues $\{i\omega_n, -i\omega_n, 0, 0\}$ with

$$\omega_n = \frac{|n|_h}{|n|},\ |n|_h = \sqrt{n_1^2 + n_2^2}$$

and

$$\begin{aligned} q_n^1 &:= (q_{1,n}^1, q_{2,n}^1, q_{3,n}^1, q_{4,n}^1) := \frac{1}{\sqrt{2}|n|_h^2}(i\omega_n n_1 n_3, i\omega_n n_2 n_3, -i|n|_h^2\omega_n, |n|_h^2) = q_n^{-1*}\\ q_n^{-1} &:= (q_{1,n}^{-1}, q_{2,n}^{-1}, q_{3,n}^{-1}, q_{4,n}^{-1}) := \frac{1}{\sqrt{2}|n|_h^2}(-i\omega_n n_1 n_3, -i\omega_n n_2 n_3, i|n|_h^2\omega_n, |n|_h^2) = q_n^{1*}\\ q_n^0 &:= (q_{1,n}^0, q_{2,n}^0, q_{3,n}^0, q_{4,n}^0) := \frac{1}{|n|_h}(-n_2, n_1, 0, 0) = q_n^{0*}\\ q_n^{div} &:= (q_{1,n}^{div}, q_{2,n}^{div}, q_{3,n}^{div}, q_{4,n}^{div}) := \frac{1}{|n|}(n_1, n_2, n_3, 0), \end{aligned}$$

where $q_n^{1*} = (q_n^1)^*$ is the conjugate of q_n^1. The case when $|n|_h = 0$ and $n_3 \neq 0$, we define

$$\begin{aligned} q_n^1 :=& (1/2, 1/2, 0, 1/\sqrt{2}) \\ q_n^{-1} :=& (-1/2, -1/2, 0, 1/\sqrt{2}) \\ q_n^0 :=& (-1/\sqrt{2}, 1/\sqrt{2}, 0, 0) \\ q_n^{div} :=& (0, 0, 1, 0). \end{aligned}$$

In fact, for $|n|_h = 0$ and $n_3 \neq 0$, we have $S_n = P_n J P_n = 0$. However, the above choice of the basis is uniquely determined by the conditions $(\tilde{\nu} q_n^1 \cdot q_n^{1*}) = \left(\frac{\nu+\kappa}{2}\right)$, $(\tilde{\nu} q_n^{-1} \cdot q_n^{-1*}) = \left(\frac{\nu+\kappa}{2}\right)$ and $(\tilde{\nu} q_n^0 \cdot q_n^{0*}) = \nu$ for (2.6). Moreover, the divergence-free condition requires that $(\hat{v}_n(t) \cdot q_n^{div}) = 0$, giving $q_n^{div} := (0, 0, 1, 0)$.

Using Craya-Herring basis, one obtains an explicit representation of the solution to the linear version of (2.2). For $n \in \Lambda$ and $\hat{v}_n := (\hat{v}_{n,1}, \hat{v}_{n,2}, \hat{v}_{n,3}, \hat{v}_{n,4})$ such that $\hat{v}_n \cdot \vec{n} = 0$, we have

$$e^{tNS_n} \hat{v}_n = \sum_{\sigma_0 \in \{-1,0,1\}} a_n^{\sigma_0} e^{tNS_n} q_n^{\sigma_0} = \sum_{\sigma_0 \in \{-1,0,1\}} a_n^{\sigma_0} e^{i\sigma_0 \omega_n N t} q_n^{\sigma_0},$$

with $\vec{n} := (n, 0) = (n_1, n_2, n_3, 0)$,

$$\hat{v}_n = \sum_{\sigma_0 \in \{-1,0,1\}} a_n^{\sigma_0} q_n^{\sigma_0} \quad \text{and} \quad a_n^{\sigma_0} := (\hat{v}_n \cdot q_n^{\sigma_0 *}).$$

Similarly, write a solution v of (2.2) as

$$v(t, x) = \sum_{n \in \Lambda} \hat{v}_n(t) e^{in\cdot x}.$$

From (2.2), we derive for $n \in \Lambda$,

$$\partial_t \hat{v}_n(t) = -\tilde{\nu}|n|^2 \hat{v}_n(t) - S_n \hat{v}_n(t) - iP_n \sum_{n=k+m} (\hat{v}_k(t) \cdot \vec{m}) \hat{v}_m(t) \quad \text{with} \quad (\vec{n} \cdot \hat{v}_n(t)) = 0. \tag{2.3}$$

In the sequel, we do not distinguish between $\vec{n}$ and n unless a confusion occurs. For $n \in \Lambda$ we have

$$e^{tNS_n} \hat{v}_n(t) = \sum_{\sigma_0 \in \{-1,0,1\}} a_n^{\sigma_0}(t) e^{tNS_n} q_n^{\sigma_0} = \sum_{\sigma_0 \in \{-1,0,1\}} a_n^{\sigma_0}(t) e^{i\sigma_0 \omega_n N t} q_n^{\sigma_0},$$

where $a_n^{\sigma_0}(t) := (\hat{v}_n(t) \cdot q_n^{\sigma_0 *})$. From equation (2.3), we get for $\sigma_0 = -1, 0, 1$,

$$\begin{aligned} \partial_t a_n^{\sigma_0}(t) &= -a_n^{\sigma_0}(t)((|n|^2 \tilde{\nu} + NS_n) q_n^{\sigma_0} \cdot q_n^{\sigma_0 *}) \\ &- i \sum_{n=k+m,\ \sigma_1, \sigma_2 \in \{-1,0,1\}} c_k^{\sigma_1} c_m^{\sigma_2} (q_k^{\sigma_1} \cdot m)(P_n q_m^{\sigma_2} \cdot q_n^{\sigma_0 *}). \end{aligned}$$

Note that P_n is self adjoint and $P_n q_n^{\sigma_0 *} = q_n^{\sigma_0 *}$. Setting $c_n^{\sigma_0}(t) := e^{-itN\sigma_0\omega_n} a_n^{\sigma_0}(t)$ leads to the following equation

$$\begin{aligned} \partial_t c_n^{\sigma_0}(t) &= -c_n^{\sigma_0}(t) |n|^2 (\tilde{\nu} q_n^{\sigma_0} \cdot q_n^{\sigma_0 *}) \\ &- i \sum_{n=k+m,\ \sigma_1, \sigma_2 \in \{-1,0,1\}} e^{iNt\omega_{nkm}^{\sigma}} c_k^{\sigma_1} c_m^{\sigma_2} (q_k^{\sigma_1} \cdot m)(q_m^{\sigma_2} \cdot q_n^{\sigma_0 *}). \end{aligned}$$

where, $\omega_{nkm}^{\sigma} := (-\sigma_0 \omega_n + \sigma_1 \omega_k + \sigma_2 \omega_m)$. Now we split the nonlinear part into the "resonant" (independent of N) and non "resonant" two parts defined by

$$\bar{B}_n^{\sigma_0}(g^{\sigma_1}, h^{\sigma_2}) := -i \sum_{n=k+m,\ \omega_{nkm}^{\sigma}=0} (q_k^{\sigma_1} \cdot m)(q_m^{\sigma_2} \cdot q_n^{\sigma_0 *}) g_k^{\sigma_1} h_m^{\sigma_2}$$

and

$$\tilde{B}_n^{\sigma_0}(Nt, g^{\sigma_1}, h^{\sigma_2}) := -i \sum_{n=k+m,\ \omega_{nkm}^{\sigma} \neq 0} (q_k^{\sigma_1} \cdot m)(q_m^{\sigma_2} \cdot q_n^{\sigma_0 *}) g_k^{\sigma_1} h_m^{\sigma_2} \exp(i\omega_{nkm}^{\sigma} N t),$$

respectively. In addition, observe that we have the following estimates:

$$\begin{cases} \|e^{-\nu|n|^2t}\tilde{B}_n^{\sigma_0}(Nt,g^{\sigma_1},h^{\sigma_2})\| \le \frac{C_\nu}{t^{1/2}}\|g^{\sigma_1}\|\|h^{\sigma_2}\| \\ \|e^{-\nu|n|^2t}\bar{B}_n^{\sigma_0}(g^{\sigma_1},h^{\sigma_2})\| \le \frac{C_\nu}{t^{1/2}}\|g^{\sigma_1}\|\|h^{\sigma_2}\| \end{cases} \tag{2.4}$$

(for $\sigma_0 = -1, 0, 1$) obtained by estimating the first derivative of the heat kernel as follows

$$\sup_{n\in\Lambda}\left||n|e^{-\nu|n|^2t}\right| \le \frac{C_\nu}{t^{1/2}}.$$

The constant $C_\nu > 0$ is independent of N.

Then we have the following equations:

$$\begin{cases} \partial_t c_n^0(t) = -\nu|n|^2 c_n^0(t) + \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2}\left(\bar{B}_n^0(c^{\sigma_1},c^{\sigma_2}) + \tilde{B}_n^0(Nt,c^{\sigma_1},c^{\sigma_2})\right) \\ \\ \partial_t c_n^{\sigma_0}(t) = -\left(\frac{\nu+\kappa}{2}\right)|n|^2 c_n^{\sigma_0}(t) + \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2}\left(\bar{B}_n^{\sigma_0}(c^{\sigma_1},c^{\sigma_2}) + \tilde{B}_n^{\sigma_0}(Nt,c^{\sigma_1},c^{\sigma_2})\right), \end{cases} \tag{2.5}$$

for $\sigma_0 = \pm 1$. From the condition $\omega_{nkm}^{\sigma} = 0$, we easily see that the terms $\bar{B}_n^0(c^1,c^1)$, $\bar{B}_n^0(c^{-1},c^{-1})$, $\bar{B}_n^0(c^0,c^{\pm1})$, $\bar{B}_n^0(c^{\pm1},c^0)$, $\bar{B}_n^{\pm1}(c^{\mp1},c^0)$, $\bar{B}_n^{\pm1}(c^0,c^{\mp1})$ and $\bar{B}_n^{\pm1}(c^0,c^0)$ disappear. Now, we define the "limit equations" by

$$\begin{cases} \partial_t c_n^0(t) = -\nu|n|^2 c_n^0(t) + \bar{B}_n^0(c^0,c^0) + \bar{B}_n^0(c^1,c^{-1}) + \bar{B}_n^0(c^{-1},c^1), \\ \\ \partial_t c_n^{\sigma_0}(t) = -\left(\frac{\nu+\kappa}{2}\right)|n|^2 c_n^{\sigma_0}(t) + \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2\setminus D}\bar{B}_n^{\sigma_0}(c^{\sigma_1},c^{\sigma_2}), \quad \sigma_0 = \pm1, \end{cases} \tag{2.6}$$

where $D := \{(0,0),(-1,0),(0,-1)\}$ for $\sigma_0 = 1$ and $D := \{(0,0),(1,0),(0,1)\}$ for $\sigma_0 = -1$. Formally, we can get (2.6) from (2.5) when $N \to \infty$. We will justify this convergence in Lemma 3.2. Now we show that there is more non trivial cancellation in the limit equations. More precisely,

Lemma 2.1. *We have*

$$\bar{B}_n^0(c^1,c^{-1}) + \bar{B}_n^0(c^{-1},c^1) = 0.$$

Proof. To prove the lemma, it suffices to show

$$(q_k^1\cdot m)(q_m^{-1}\cdot q_n^{0*}) + (q_m^{-1}\cdot k)(q_k^1\cdot q_n^{0*}) = 0 \quad \text{for any} \quad n = k+m \quad \text{with} \quad \omega_k = \omega_m. \tag{2.7}$$

First we show that $\omega_k = \omega_m$ if and only if

$$k, m \in \{n \in \mathbb{Z}^3 : |n|_h^2 = \lambda n_3^2\} \quad \text{for some} \quad \lambda > 0.$$

($\Leftarrow$): This direction is clear. Thus we omit it.

($\Rightarrow$): Rewrite the identity $\omega_k = \omega_m$ as $F(X) = F(Y)$, where $X := |k|_h^2/k_3^2$, $Y := |m|_h^2/m_2^3$ and $F(X) := X/(X+1)$. Since the function F is monotone increasing, we see $X = Y$. This means that

$$k_3 = \pm\frac{|k|_h}{\sqrt{\lambda}} \quad \text{and} \quad m_3 = \pm\frac{|m|_h}{\sqrt{\lambda}}.$$

We only consider the case $k_3 = \frac{|k|_h}{\sqrt{\lambda}}$ and $m_3 = \frac{|m|_h}{\sqrt{\lambda}}$, since the other cases are similar. A direct calculation shows that

$$\begin{aligned} (q_k^1\cdot m)(q_m^{-1}\cdot q_n^{0*}) &= \frac{1}{\sqrt{2}\lambda|m||k||n|_h}\left(\frac{k_h\cdot m_h}{\lambda} - \frac{m_3|k|}{\sqrt{1+\lambda^2}}\right)(-m_2k_1 + m_1k_2), \\ (q_m^{-1}\cdot k)(q_k^1\cdot q_n^{0*}) &= \frac{1}{\sqrt{2}\lambda|m||k||n|_h}\left(\frac{k_h\cdot m_h}{\lambda} - \frac{k_3|m|}{\sqrt{1+\lambda^2}}\right)(-k_2m_1 + k_1m_2). \end{aligned}$$

By $k_3 = \frac{|k|_h}{\sqrt{\lambda}}$ and $m_3 = \frac{|m|_h}{\sqrt{\lambda}}$, we have (2.7). $\square$

Now we show that the function c^0 in the limit equations satisfies a quasi geostrophic (QG) equation type and that this QG equation is equivalent to the 2D type Navier-Stokes equation. By the following lemma, we can see that the function c^0 satisfies the QG equation (1.3).

Lemma 2.2. *Let $|n|_h = \sqrt{n_1^2+n_2^2}$. The resonant part $\bar{B}_n^0(c^0,c^0)$ can be expressed as follows:*

$$\bar{B}_n^0(c^0,c^0) = -\sum_{n=k+m} \frac{i(k\times m)|m|_h}{|k|_h|n|_h} c_k^0 c_m^0.$$

Proof. Since $q_n^0 := \frac{1}{|n|_h}(-n_2,n_1,0,0)$ and $q_n^0 = \frac{1}{|n|_h}(|k|_h q_k^0 + |m|_h q_m^0)$ for $n=k+m$, we have

$$\begin{aligned}
\bar{B}_n^0(c^0,c^0) &= -\sum_{n=k+m} c_k^0 c_m^0 (q_k^0\cdot im)(q_m^0\cdot q_n^{0*})\\
&= -\sum_{n=k+m} \frac{i}{|k|_h|n|_h}(k_2m_1-k_1m_2)\left(q_m^0\cdot(|k|_h q_k^{0*}+|m|_h q_m^{0*})\right) c_k^0 c_m^0\\
&= \sum_{n=k+m} -\frac{i|m|_h}{|k|_h|n|_h}(k_2m_1-k_1m_2)c_k^0c_m^0\\
&\quad -\sum_{n=k+m}\frac{i}{|n|_h}(k_2m_1-k_1m_2)(q_m^0\cdot q_k^{0*})c_k^0c_m^0.
\end{aligned}$$

Since $k\times m = -(m\times k)$, we see that

$$\sum_{n=k+m}\frac{i}{|n|_h}(k_2m_1-k_1m_2)(q_m^0\cdot q_k^{0*})c_k^0c_m^0 = 0,$$

which leads to the desired formula. □

Now we show that there is a one-to-one correspondence between the QG and a 2D type Navier-Stokes equations.

Lemma 2.3. *Let $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ and*

$$w := (w_1(x_1,x_2,x_3,t), w_2(x_1,x_2,x_3,t)) := \left(\sum_{n\in\Lambda}\hat{w}_{1,n}(t)e^{in\cdot x}, \sum_{n\in\Lambda}\hat{w}_{2,n}(t)e^{in\cdot x}\right)$$

and define $\theta = \theta(t,x_1,x_2,x_3) := (-\Delta_h)^{-1/2}rot_2 w$, with rot_2 is the 2 dimensional curl given by

$$rot_2 w = \partial_2 w_1 - \partial_1 w_2.$$

Then, w solves the following 2D type Navier-Stokes equation

$$\begin{cases}\partial_t w - \Delta w + (w\cdot\nabla_2)w + \nabla_2 p = 0,\\ \nabla_2\cdot w = 0,\ \ w|_{t=0} = w_0\end{cases} \tag{2.8}$$

if and only if θ solves (1.3).

Proof. Recall that $-\Delta_h = -(\partial_{x_1}^2+\partial_{x_2}^2)$. First observe that for $\theta = (-\Delta_h)^{-1/2}\mathrm{rot}_2 w = \sum_{n\in\Lambda}\hat{\theta}_n(t)e^{in\cdot x}$, we have by $\nabla_2\cdot w = 0$,

$$w = (i\sum_{n\in\Lambda}\frac{n_2}{|n|_h}\hat{\theta}_n(t)e^{in\cdot x}, -i\sum_{n\in\Lambda}\frac{n_1}{|n|_h}\hat{\theta}_n(t)e^{in\cdot x}) = (\partial_2(-\Delta_h)\theta, -\partial_1(-\Delta_h)\theta).$$

Then, apply rot_2 to (2.8), we get

$$\partial_t \mathrm{rot}_2 w - \Delta \mathrm{rot}_2 w + (w\cdot\nabla_2)\mathrm{rot}_2 w = 0, \tag{2.9}$$

here, we used the fact that

$$(w\cdot\nabla_2)\mathrm{rot}_2 w=\mathrm{rot}_2[(w\cdot\nabla_2)w]. \tag{2.10}$$

Finally, apply $(-\Delta_h)^{-1/2}$ to both sides of (2.9), we see that $\theta=(-\Delta_h)^{-1/2}\mathrm{rot}w$ satisfies the desired QG equation (1.3). Conversely, applying $L:=(-(-\Delta_h)^{-1}\partial_{x_2},(-\Delta_h)^{-1}\partial_{x_1})$ (which commutes with Δ) to (2.9), and by (2.10) we can see that $L\mathrm{rot}_2$ is nothing but the two dimensional Leray projection. Therefore, this implies (2.8) as desired. $\square$

Remark 2.4. We refer to [15] for the existence of the unique global solution to 2D type Navier-Stokes equation (2.8) with almost periodic initial data.

In what follows and in order to show the main theorem, we need the following lemma (which is needed only for the almost periodic case) on the dilation of the frequency set (1.2). This kind of restrictions is technical. However we do not know whether or not such constraints are removable. This means that the general almost periodic setting seems to remain open.

Lemma 2.5. *For $n,k,m\in\Lambda$ and $\gamma=(\gamma_1,\gamma_2)\in(0,\infty)^2$, define $\tilde n=(\gamma_1 n_1,\gamma_2 n_2,n_3)$, $\tilde k=(\gamma_1 k_1,\gamma_2 k_2,k_3)$ and $\tilde m=(\gamma_1 m_1,\gamma_2 m_2,m_3)$. Let*

$$P_{nkm}(\gamma):=|\tilde n|^8|\tilde k|^8|\tilde m|^8\prod_{\sigma\in\{-1,1\}^3}\omega^{\sigma}_{\tilde n\tilde k\tilde m}.$$

Given a frequency set Λ, there is $\Gamma:=\Gamma(\Lambda)\subset(0,\infty)^2$ s.t. for any $\gamma\in\Gamma$, $P_{nkm}(\gamma)\neq 0$ for any n, k, $m\in\Lambda$ such that $(n_h,k_h,m_h)\neq(0,0,0)$.

Remark 2.6. If n_h, k_h, $m_h=0$, then $\bar B_n^{\pm1}(c^{\pm1},c^{\pm1})=0$.

Proof. Define Γ by

$$\Gamma:=\{\gamma\in(0,\infty)^2:P_{nkm}(\gamma)\neq 0\quad for\quad all\quad n,k,m\in\Lambda\quad with\quad |n|_h,|k|_h,|m|_h\neq 0\}.$$

Nothe that if $\gamma\in\Gamma$, then $\omega^{\sigma}_{\tilde n\tilde k\tilde m}\neq 0$. We show that Γ cannot be empty. By a direct calculation, we have

$$\begin{aligned}
P_{nkm}(\gamma)&=|\tilde n|^8|\tilde k|^8|\tilde m|^8\\
&\quad\left((\omega_{\tilde n}+\omega_{\tilde k}+\omega_{\tilde m})(-\omega_{\tilde n}+\omega_{\tilde k}+\omega_{\tilde m})(\omega_{\tilde n}-\omega_{\tilde k}+\omega_{\tilde m})(\omega_{\tilde n}+\omega_{\tilde k}-\omega_{\tilde m})\right)^2\\
&=|\tilde n|^8|\tilde k|^8|\tilde m|^8\left(\omega_{\tilde n}^2-(\omega_{\tilde k}+\omega_{\tilde m})^2\right)^2\left((\omega_{\tilde n}^2-(\omega_{\tilde k}-\omega_{\tilde m})^2\right)^2\\
&=|\tilde n|^8|\tilde k|^8|\tilde m|^8\left((\omega_{\tilde n}^2-\omega_{\tilde k}^2-\omega_{\tilde m}^2)^2-4\omega_{\tilde k}^2\omega_{\tilde m}^2\right)^2\\
&=|\tilde n|^8|\tilde k|^8|\tilde m|^8\left(\omega_{\tilde n}^4+\omega_{\tilde k}^4+\omega_{\tilde m}^4-2\omega_{\tilde k}^2\omega_{\tilde m}^2-2\omega_{\tilde m}^2\omega_{\tilde n}^2-2\omega_{\tilde n}^2\omega_{\tilde k}^2\right)^2\\
&=|\tilde n|_h^4|\tilde k|^4|\tilde m|^4+|\tilde n|^4|\tilde k|_h^4|\tilde m|^4+|\tilde n|^4|\tilde k|^4|\tilde m|_h^4\\
&\quad-2|\tilde n|^2|\tilde n|_h^2|\tilde k|^2|\tilde k|_h^2|\tilde m|^4-2|\tilde n|^2|\tilde n|_h^2|\tilde k|^4|\tilde m|^2|\tilde m|_h^2-2|\tilde n|^4|\tilde k|^2|\tilde k|_h^2|\tilde m|^2|\tilde m|_h^2\\
&=-3n_1^2k_1^2m_1^2\gamma_1^6-3n_2^2k_2^2m_2^2\gamma_2^6\\
&\quad-3n_1^2k_1^2m_2^2\gamma_1^4\gamma_2^2-3n_1^2k_2^2m_1^2\gamma_1^4\gamma_2^2-3n_2^2k_1^2m_1^2\gamma_1^4\gamma_2^2\\
&\quad-3n_2^2k_2^2m_1^2\gamma_1^2\gamma_2^4-3n_2^2k_1^2m_2^2\gamma_1^2\gamma_2^4-3n_1^2k_2^2m_2^2\gamma_1^2\gamma_2^4+\cdots.
\end{aligned}$$

Since $|n|_h,|k|_h,|m|_h\neq 0$, then the highest order terms never disappear. This means that

$$|\{\gamma:\cup_{n,k,m}P_{nkm}(\gamma)=0\}|=0.$$

Thus the complement set of Γ is countable (which means that Γ is a non-empty set). $\square$

Now we show that the limit equations have a global solution. In the almost periodic case, the non-resonant part $\bar B^{\pm1}(c^{\pm1},c^{\pm1})$ disappears just by restricting the frequencies set to $\Lambda(\gamma)$. However, the periodic case is more subtle as we need a lemma on restricted convolution (see [2]).

Lemma 2.7. *Let Λ be a sum closed frequency set. If $\Lambda = \mathbb{Z}^3$, take $\gamma \in (0,\infty)^2$, otherwise we restrict it to $\gamma \in \Gamma(\Lambda)$. Then for $\sigma_0 = \pm 1$ there exists a global-in-time unique solution $c^{\sigma_0}(t)$ to equations (2.6) such that $c^{\sigma_0}(t) \in C([0,\infty) : \ell^1(\Lambda(\gamma)))$ with $(c_n^{\sigma_0}(t) \cdot n) = 0$ for all $n \in \Lambda(\gamma)$ and $c_0^{\sigma_0}(t) = 0$.*

Proof. Recall that $\tilde{n} = (\gamma_1 n_1, \gamma_2 n_2, n_3)$, $\tilde{k} = (\gamma_1 k_1, \gamma_2 k_2, k_3)$ and $\tilde{m} = (\gamma_1 m_1, \gamma_2 m_2, m_3)$. First we consider the almost periodic case. By restricting $\gamma \in \Gamma$, we can eliminate the worst non-linear term using Lemma 2.5. More precisely, for all $n \in \Lambda$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \{-1,1\}^3$, the term

$$\bar{B}_{\tilde{n}}^{\sigma_0}(c^{\sigma_1}, c^{\sigma_2}) = \sum_{\substack{\tilde{n}=\tilde{k}+\tilde{m} \\ \omega^{\sigma}_{\tilde{n}\tilde{k}\tilde{m}}=0}} (q_{\tilde{k}}^{\sigma_1} \cdot i\tilde{m})(q_{\tilde{m}}^{\sigma_2} \cdot q_{\tilde{n}}^{\sigma_0 *}) c_{\tilde{k}}^{\sigma_1} c_{\tilde{m}}^{\sigma_2} \quad \text{disappears.}$$

Then we have two coupled linear equations for $\{c_n^{-1}\}_n$ and $\{c_n^1\}_n$. In this case, the global existence will immediately follow from estimates (2.4).

However, the periodic case requires more details. For $\alpha > 0$ and $p \geq 1$, define the weighted ℓ_p^α norm as

$$\|c\|_{\ell_p^\alpha} := \left(\sum_{n \in \Lambda} |n|^{p\alpha} |c_n|^p \right)^{1/p}.$$

The main step is to show an *à-priori* bound on $c^{\pm 1}$ in ℓ_2^1. Since the 3D type Navier-Stokes equation (2.6) is subcritical in the space ℓ_2^s with $s > 1/2$, then a bootstrap argument (using the dissipation, see [17, Proposition 15.1] for example) enables us to conclude that

$$c^{\pm 1}(t) \in L^\infty_{loc}([0,\infty) : \ell_2^1) \cap L^\infty_{loc}((0,\infty) : \ell_2^s) \quad \text{for} \quad 1 < s < 2,$$

whenever the initial data $c^{\pm 1}(0) \in \ell_2^1$. More precisely, by (2.6) we write the following mild formulation:

$$c_n^{\sigma_0}(t) = e^{\frac{\nu+\kappa}{2}|n|^2 t} c_n^{\sigma_0}(t_0) - \sum_{(\sigma_1,\sigma_2) \in \{-1,0,1\}^2 \setminus D} \int_{t_0}^t e^{-\frac{\nu+\kappa}{2}(t-\tau+t_0)|n|^2} \bar{B}_n^{\sigma_0}(c^{\sigma_1}, c^{\sigma_2}) d\tau \tag{2.11}$$

for $0 < t_0 < t$. We have a good estimate for the heat kernel with fractional Laplacian,

$$|n|^s e^{-|n|^2 t} = t^{-s/2} |t^{1/2} n|^s e^{-|t^{1/2} n|^2} \leq t^{-s/2} \frac{C_\epsilon}{1 + |t^{1/2} n|^{1/\epsilon}} \leq t^{-s/2-\epsilon/2} \frac{C_\epsilon}{1 + |n|^{1/\epsilon}}$$

for $0 < t \leq 1$ and $\epsilon > 0$. Then by Hölder's and Young's inequality for the discrete case, we have the following estimate from (2.11):

$$\begin{aligned} \|c^{\sigma_0}(t)\|_{\ell_2^s} \leq \; & C_{\kappa,\nu,s} \Bigg[\|c^{\sigma_0}(t_0)\|_{\ell_2^1} + \int_{t_0}^t (t-\tau+t_0)^{-s/2-\epsilon/2} \times \\ & \sum_{(\sigma_1,\sigma_2) \in \{-1,0,1\}^2} \left(\|c^{\sigma_1}(\tau)\|_{\ell_2^1} \|c^{\sigma_2}(\tau)\|_{\ell_2^0} + \|c^{\sigma_1}(\tau)\|_{\ell_2^0} \|c^{\sigma_2}(\tau)\|_{\ell_2^1} \right) d\tau \Bigg]. \end{aligned} \tag{2.12}$$

If $1 < s < 2$, then $\|c^{\sigma_0}(t)\|_{\ell_2^s}$ is finite since the right hand side of (2.12) is finite. This means that $c^{\sigma_0} \in L^\infty_{loc}((0,\infty) : \ell_2^s)$ for $1 < s < 2$. Then, thanks to Bernstein's lemma (which can be applied only for the periodic case), $\sum_n |c_n| \leq \|c\|_{\ell_2^\alpha}$ for $\alpha > 3/2$, we get an *à-priori* bound of the ℓ^1-norm. Now we show an *à-priori* bound of $c^{\pm 1}$ in ℓ_2^1. Multiply equation (2.6) by $|n|^2 (c_n^{\sigma_0})^*$ and summing, we obtain

$$\begin{aligned} \sum_{\sigma_0 \in \{-1,1\}} \sum_{n \in \mathbb{Z}^3} & \left[\frac{1}{2} \partial_t (|n| c_n^{\sigma_0}(t))^2 + \left(\frac{\nu+\kappa}{2} \right) (|n|^2 c_n^{\sigma_0}(t))^2 \right] \\ & = \sum_{n \in \mathbb{Z}^3} \sum_{\sigma_0 \in \{-1,1\}} \sum_{(\sigma_1,\sigma_2) \in \{-1.0,1\}^2 \setminus D} \bar{B}_n^{\sigma_0}(c^{\sigma_1}, c^{\sigma_2}) |n|^2 c_n^{\sigma_0 *}(t). \end{aligned}$$

Notice that

$$\begin{aligned}(c_n^{\sigma_0})^* &= \left[(\hat{v}_n \cdot q_n^{\sigma_0 *})e^{-itN\sigma_0\omega_n}\right]^* = (\hat{v}_n^* \cdot q_n^{\sigma_0})e^{itN\sigma_0\omega_n}\\ &= (\hat{v}_{-n} \cdot (q_{-n}^{-\sigma_0})^*)e^{-itN(-\sigma_0)\omega_{-n}} = c_{-n}^{-\sigma_0}.\end{aligned}$$

Moreover, from one hand, a direct calculation using Hölder's then young's inequalities shows that

$$\left|\sum_n \bar{B}_n^{\pm 1}(c^0, c^{\pm 1})|n|^2 c_{-n}^{\mp 1}\right| + \left|\sum_n \bar{B}_n^{\pm 1}(c^{\pm 1}, c^0)|n|^2 c_{-n}^{\mp 1}\right| \tag{2.13}$$

$$\leq C\|c^{\mp 1}\|_{\ell_2^2}\left(\|c^{\pm 1}\|_{\ell_2^0}\|c^0\|_{\ell_1^1} + \|c^{\pm 1}\|_{\ell_2^1}\|c^0\|_{\ell_1^0}\right) \leq \epsilon\|c^{\mp 1}\|_{\ell_2^2}^2 + C\left(\|c^{\pm 1}\|_{\ell_2^0}^2\|c^0\|_{\ell_1^1}^2 + \|c^{\pm 1}\|_{\ell_2^1}^2\|c^0\|_{\ell_1^0}^2\right)$$

for sufficiently small $\epsilon > 0$. On the other hand, we show that

$$\left|\sum_n \bar{B}_n^{\pm 1}(c^{\pm 1}, c^{\pm 1})|n|^2 c_{-n}^{\mp 1}\right| \leq \sum_n \sum_{-n=k+m} |m||c_m^{\pm 1}||k||c_k^{\pm 1}||n||c_n^{\mp 1}|. \tag{2.14}$$

Using identities $\alpha \times (\alpha \times \beta) = -|\alpha|^2\beta$ and $(\alpha \cdot (\beta \times \gamma)) = ((\alpha \times \beta) \cdot \gamma)$, we see that

$$\begin{aligned}\sum_n \bar{B}_n^{\pm 1}(c^{\pm 1}, c^{\pm 1})|n|^2 c_{-n}^{\mp 1} &= \sum_n \sum_{-n=k+m} (im \cdot q_k^{\pm 1})\left(q_m^{\pm 1} \cdot (in \times (in \times q_n^{\mp 1}))\right) c_k^{\pm 1} c_m^{\pm 1} c_n^{\mp 1}\\ &= \sum_n \sum_{-n=k+m} (im \cdot q_k^{\pm 1})\left((q_m^{\pm 1} \times in) \cdot (in \times q_n^{\mp 1})\right) c_k^{\pm 1} c_m^{\pm 1} c_n^{\mp 1}\\ &= -\sum_n \sum_{-n=k+m} (im \cdot q_k^{\pm 1})(q_m^{\pm 1} \times ik)(in \times q_n^{\pm 1}) c_k^{\mp 1} c_m^{\pm 1} c_n^{\mp 1}\\ &\quad -\sum_n \sum_{-n=k+m} (im \cdot q_k^{\pm 1})(q_m^{\pm 1} \times im)(in \times q_n^{\pm 1}) c_k^{\mp 1} c_m^{\pm 1} c_n^{\mp 1}.\end{aligned}$$

By the skew-symmetry, namely,

$$\begin{aligned}&\sum_n \sum_{-n=k+m} (im \cdot q_k^{\pm 1})(q_m^{\pm 1} \times im)(in \times q_n^{\mp 1}) c_k^{\pm 1} c_m^{\pm 1} c_n^{\mp 1} =\\ -&\sum_n \sum_{-n=k+m} (in \cdot q_k^{\pm 1})(q_m^{\pm 1} \times im)(in \times q_n^{\mp 1}) c_k^{\pm 1} c_m^{\pm 1} c_n^{\mp 1} =\\ -&\sum_m \sum_{-m=k+n} (in \cdot q_k^{\pm 1})(q_m^{\pm 1} \times im)(in \times q_n^{\mp 1}) c_k^{\pm 1} c_m^{\pm 1} c_n^{\mp 1},\end{aligned}$$

the second term disappears which obviously leads to (2.14). Now to have the *à priori* bound of $c^{\pm 1}$ in ℓ_2^1, we apply a smoothing via time averaging effect due to [2].

Proposition 2.8. *Restricted convolution*[2, Theorem 3.1 and Lemma 3.1] *Assume that the following holds for* $|n|_h \neq 0$

$$\sup_n \sum_{k:k+m+n=0, k\in\Sigma_i} \chi(n,k,m)|k|^{-1} \leq C2^i \tag{2.15}$$

for every $i = 1, 2, \cdots,$ *where*

$$\Sigma_i := \{k : 2^i \leq |k| \leq 2^{i+1}\}, \quad \chi(n,k,m) = \begin{cases} 1 & \text{if} \quad P_{nkm}(1) = 0\\ 0 & \text{if} \quad P_{nkm}(1) \neq 0.\end{cases}$$

Then we have

$$\left|\sum_n \bar{B}_n^{\pm 1}(c^{\pm 1}, c^{\pm 1})|n|^2 c_{-n}^{\mp 1}\right| \le C\|c^{\pm 1}\|_{\ell_2^2}\|c^{\pm 1}\|^2_{\ell_2^1} \le C\|c^{\pm 1}\|^4_{\ell_2^1} + \epsilon\|c^{\pm 1}\|^2_{\ell_2^2}. \tag{2.16}$$

Now, combining (2.13) and (2.16), we obtain the following estimate on $c^{\pm 1}$ in ℓ_2^1, namely,

$$\|c^{\pm 1}(t)\|^2_{\ell_2^1} \le \|c^{\pm 1}(0)\|^2_{\ell_2^1} + C\int_0^t \left(\|c^{\pm 1}(s)\|^4_{\ell_2^1} + \|c^{\pm 1}(s)\|^2_{\ell_2^1}\|c^0(s)\|^2_{\ell_1^1}\right) ds. \tag{2.17}$$

Moreover we have the following energy inequality:

$$\begin{aligned} \|c^{\pm 1}(t)||_{\ell_2^0} &+ \left(\frac{\nu+\kappa}{2}\right)\int_0^t \|c^{\pm 1}(s)\|_{\ell_2^1} ds \\ &\le \|c^{\pm 1}(t)||_{\ell_2^0} + \|c^0(t)\|_{\ell_2^0} + \int_0^t \left(\left(\frac{\nu+\kappa}{2}\right)\|c^{\pm 1}(s)\|_{\ell_2^1} + \nu\|c^0(s)\|_{\ell_2^1}\right) ds \\ &\le \|c^{\pm 1}(0)\|^2_{\ell_2^0} + \|c^0(0)\|^2_{\ell_2^1}. \end{aligned} \tag{2.18}$$

In fact, multiply the first equation of (2.6) by $c^{0*}(t)$ and the second one by $c^{\pm 1*}(t)$, we have (2.18). Since all convection terms disappear due to the skew-symmetry, for example,

$$\begin{aligned} \sum_n \left(\bar{B}_n^1(c^1, c^1)\cdot c^{1*}\right) &= -i\sum_n \sum_{-n=k+m,\ -\omega_n^1+\omega_k^1+\omega_m^1=0} (q_k^1\cdot m)(q_m^1\cdot q_n^{-1})c_k^1 c_m^1 c_n^{-1} \\ &= i\sum_n \sum_{-n=k+m,\ -\omega_n^1+\omega_k^1+\omega_m^1=0} (q_k^1\cdot n)(q_{-m}^{-1*}\cdot q_n^{-1})c_k^1 c_{-m}^{-1*} c_n^{-1} \\ &= i\sum_m \sum_{m=k+n,\ \omega_n^{-1}+\omega_k^1-\omega_m^{-1}=0} (q_k^1\cdot n)(q_m^{-1*}\cdot q_n^{-1})c_k^1 c_m^{-1*} c_n^{-1} \\ &= -\sum_m \left(\bar{B}_m^{-1}(c^1, c^{-1})\cdot c^{-1*}\right). \end{aligned}$$

To apply Gronwall's inequality, we need the following definition. Let C be the positive constant appearing in (2.17). Then define h as

$$h := \inf\{h' \in [0,\infty) : \int_\tau^{\tau+h'} \|c^{\pm 1}(s)\|^2_{\ell_2^1} ds \le \frac{1}{2C} \quad \text{for any} \quad \tau > 0\}.$$

Note that h is independent of N and can be chosen positive thaks to (2.18). From (2.17) and an absorbing argument, we see that

$$\sup_{0<s\le t}\|c^{\pm 1}(s)\|^2_{\ell_2^1} \le 2\|c^{\pm 1}(0)\|^2_{\ell_2^1} + 2C\int_0^t \sup_{0<s''\le s'}\|c^{\pm 1}(s'')\|^2_{\ell_2^1}\|c^0(s')\|^2_{\ell_1^1} ds' \quad \text{for} \quad t < h.$$

Then by Gronwall's inequality,

$$\sup_{0<s\le t}\|c^{\pm 1}(s)\|^2_{\ell_2^1} \le 2\|c^{\pm 1}(0)\|^2_{\ell_2^1}\exp\left(2C\int_0^t \|c^0(s)\|^2_{\ell_1^1} ds\right) \quad \text{for} \quad t < h.$$

Iterating the same argument, one more time, we obtain

$$\sup_{0<s\le t}\|c^{\pm 1}(s)\|^2_{\ell_2^1} \le 2\|c^{\pm 1}(h)\|^2_{\ell_2^1}\exp\left(2C\int_h^t \|c^0(s)\|^2_{\ell_1^1} ds\right) \quad \text{for} \quad h \le t < 2h.$$

Note that $\int_0^t \|c^0(s)\|^2_{\ell_1^1} ds$ is always finite for any fixed $t > 0$, since there is a global solution to the 2D Navier-Stokes equations in ℓ_1^s-type function spaces. Fixing $T > 0$ and repeating this argument finitely many times, we have an *à priori* bound of $c^{\pm 1}$ in ℓ_2^1 over $t \in [0, T]$.

Finally, to use Proposition 2.8 on the restricted convolution, we need to verify that (2.15) holds. Observe that

$$\begin{aligned} P_{n,k,-n-k}(1) &= |n|_h^4|k|^4|n+k|^4+|n|^4|k|_h^4|n+k|^4+|n|^4|k|^4|n+k|_h^4 \\ &\quad -2|n|^2|n|_h^2|k|^2|k|_h^2|n+k|^4-2|n|^2|n|_h^2|k|^4|n+k|^2|n+k|_h^2 \\ &\quad -2|n|^4|k|^2|k|_h^2|n+k|^2|n+k|_h^2 \\ &= |n|_h^4k_3^8+l.o.t. \end{aligned}$$

where $l.o.t.$ stands for lower order terms. Thus, it follows that $P_{n,k,-n-k}(1)$ is a polynomial of degree eight in k_3 with a nonzero leading coefficient whenever $|n_1|+|n_2|\neq 0$. Then for fixed k_1, k_2 and n, there are at most eight k_3 satisfying $\chi(n,k,-n-k)=1$. Thus,

$$\begin{aligned} \sum_{2^i\le|k|\le 2^{i+1}}|k|^{-1}\chi(n,k,-n-k) &\le \sum_{0\le|k|_h\le 2^{i+1},k_3\in\mathbb{R}}|k|_h^{-1}\chi(n,k,-n-k) \\ &\le 8\sum_{j=1}^{i}\sum_{2^j\le|k|_h\le 2^{j+1}}|k|_h^{-1}\le 8\sum_{j=1}^{i}2^{2(j+1)}2^{-j}\le C2^i. \end{aligned}$$

□

3. Proof of the main theorem

Before proving the main theorem, we first mention the local existence result. Using estimate (2.4), we obtain a local-in-time unique solution to (2.5) in $C([0,T]:\ell^1(\Lambda))$ as stated in the following lemma.

Lemma 3.1. *Assume that* $c(0):=\{c_n^{\sigma_0}(0)\}_{n\in\Lambda,\sigma_0\in\{-1,0,1\}}\in\ell^1(\Lambda)$ *and* $c_0^{\sigma_0}(0)=0$ *for* $\sigma_0\in\{-1,0,1\}$. *Then there is a local-in-time unique solution* $c(t)\in C([0,T_L]:\ell^1(\Lambda))$ *and* $c_0^{\sigma_0}(t)=0$ *for* $\sigma_0\in\{-1,0,1\}$ *satisfying*

$$T_L\ge\frac{C}{\|c(0)\|^2},\quad \sup_{0<t<T_L}\|c(t)\|\le 10\|c(0)\|, \tag{3.1}$$

where C *is a positive constant independent of* N.

Proof. First we recall the mild formulation of (2.5):

$$\begin{aligned} c_n^0(t) &= e^{-\nu|n|^2t}c_n^0(0) \\ &\quad +\sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2}\int_0^t e^{-\nu(t-s)|n|^2}\left(\bar{B}_n^0(c^{\sigma_1},c^{\sigma_2})+\tilde{B}_n^0(Nt,c^{\sigma_1},c^{\sigma_2})\right)ds \end{aligned}$$

and

$$\begin{aligned} c_n^{\sigma_0}(t) &= e^{-\frac{\nu+\kappa}{2}|n|^2t}c_n^{\sigma_0}(0) \\ &\quad +\sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2}\int_0^t e^{-\frac{\nu+\kappa}{2}(t-s)|n|^2}\left(\bar{B}_n^{\sigma_0}(c^{\sigma_1},c^{\sigma_2})+\tilde{B}_n^{\sigma_0}(Nt,c^{\sigma_1},c^{\sigma_2})\right)ds. \end{aligned}$$

By (2.4), we have the estimates

$$\|c_n^0(t)\|\le\|c_n^0(0)\|+C_\nu t^{1/2}\sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2}\left(\sup_{0\le s<t}\|c^{\sigma_1}(s)\|\sup_{0\le s<t}\|c^{\sigma_2}(s)\|\right)$$

and

$$\begin{aligned} \|c_n^{\sigma_0}(t)\| &\le \|c_n^{\sigma_0}(0)\| \\ &\quad +C_{\left(\frac{\nu+\kappa}{2}\right)}t^{1/2}\sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2}\left(\sup_{0\le s<t}\|c^{\sigma_1}(s)\|\sup_{0\le s<t}\|c^{\sigma_2}(s)\|\right). \end{aligned}$$

These *à-priori* estimates of $\sup_t \|c^0(t)\|$ and $\sup_t \|c^{\sigma_0}(t)\|$ give us through a standard fixed point argument the existence of a local-in-time unique solution (for the detailed computation, see [12] for example). $\square$

Let $b^{\sigma_0}(t)$ be the solution to the limit equations (2.6) and $c^{\sigma_0}(t)$ be the solution to the original equation (2.5). The point is to control, in the ℓ^1-norm, the remainder term $r_n^{\sigma_0}(t) := c_n^{\sigma_0}(t) - b_n^{\sigma_0}(t)$ $(\sigma_0 = -1, 0, 1)$ by the large parameter N. More precisely, r^0 and r^{σ_0} satisfy

$$\partial_t r_n^0(t) = -\nu|n|^2 r_n^0(t) + \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2} \left(\bar{B}_n^0(r^{\sigma_1}, c^{\sigma_2}) + \bar{B}_n^0(b^{\sigma_1}, r^{\sigma_2}) + \tilde{B}_n^0(Nt, c^{\sigma_1}, c^{\sigma_2})\right)$$

and

$$\begin{aligned}\partial_t r_n^{\sigma_0}(t) &= -\left(\frac{\nu+\kappa}{2}\right)|n|^2 r_n^{\sigma_0}(t)\\ &\quad + \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2} \left(\bar{B}_n^{\sigma_0}(r^{\sigma_1}, c^{\sigma_2}) + \bar{B}_n^{\sigma_0}(b^{\sigma_1}, r^{\sigma_2}) + \tilde{B}_n^{\sigma_0}(Nt, c^{\sigma_1}, c^{\sigma_2})\right),\end{aligned}$$

respectively. Once we control the remainder term in the ℓ^1-norm, we easily have the main result by a usual bootstrapping argument (see [20] for example). Now we show the following lemma concerning the smallness of the reminder term. Let $b(t) := \{b_n^{\sigma_0}(t)\}_{n\in\Lambda,\sigma_0\in\{-1,0,1\}}$ and $r(t) := \{r_n^{\sigma_0}(t)\}_{n\in\Lambda,\sigma_0\in\{-1,0,1\}}$.

Lemma 3.2. *For all $\epsilon > 0$, there is $N_0 > 0$ such that $\|r_n(t)\| \le \epsilon$ for $0 < t < T_L$ and $|N| > N_0$, where T_L is the local existence time (see Lemma 3.1).*

Proof. To simplify the remainder equation, we introduce the following notation. Let

$$\begin{aligned}\bar{R}_n^{\sigma_0}(r,c,b) :&= \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2} \left(\bar{B}_n^{\sigma_0}(r^{\sigma_1}, c^{\sigma_2}) + \bar{B}_n^{\sigma_0}(b^{\sigma_1}, r^{\sigma_2})\right),\\ \tilde{R}_n^{\sigma_0}(Nt,c) :&= \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2} \tilde{B}_n^{\sigma_0}(Nt, c^{\sigma_1}, c^{\sigma_2}).\end{aligned}$$

We rewrite the remainder equations as follows:

$$\begin{cases}\partial_t r_n^0(t) = -\nu|n|^2 r_n^0(t) + \bar{R}_n^0(r,c,b) + \tilde{R}_n^0(Nt,c)\\ \partial_t r_n^{\sigma_0}(t) = -\left(\frac{\nu+\kappa}{2}\right)|n|^2 r_n^{\sigma_0}(t) + \bar{R}_n^{\sigma_0}(r,c,b) + \tilde{R}_n^{\sigma_0}(Nt,c) \quad \text{for} \quad \sigma_0 = -1, 1.\end{cases} \tag{3.2}$$

To control r, the key is to control $\tilde{R}_n^0(Nt,c)$ and $\tilde{R}_n^{\sigma_0}(Nt,c)$ in (3.2). To do so, we need to analyze the following oscillatory integral of the non-resonant part as follows:

$$\tilde{\mathcal{B}}_n^{\sigma_0}(Nt, g^{\sigma_1}, h^{\sigma_2}) := \sum_{n=k+m,\omega_{nkm}^{\sigma}\neq 0} \frac{1}{iN\omega_{nkm}^{\sigma}} e^{iNt\omega_{nkm}^{\sigma}} (q_k^{\sigma_1}\cdot im)(q_m^{\sigma_2}\cdot q_n^{\sigma_0 *}) g_k^{\sigma_1} h_m^{\sigma_2}$$

and

$$\tilde{\mathcal{R}}_n^{\sigma_0}(Nt,c) := \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2} \tilde{\mathcal{B}}^{\sigma_0}(c^{\sigma_1}, c^{\sigma_2}). \tag{3.3}$$

Note that we have the following relation between $\tilde{B}$ and $\tilde{\mathcal{B}}$ ($\tilde{R}$ and $\tilde{\mathcal{R}}$):

$$\begin{aligned}\partial_t\left(\tilde{\mathcal{B}}_n^{\sigma_0}(Nt, g^{\sigma_1}, h^{\sigma_2})\right) &= \tilde{B}_n^{\sigma_0}(Nt, g^{\sigma_1}, h^{\sigma_2}) + \tilde{\mathcal{B}}_n^{\sigma_0}(Nt, \partial_t g^{\sigma_1}, h^{\sigma_2})\\ &\quad + \tilde{\mathcal{B}}_n^{\sigma_0}(Nt, g^{\sigma_1}, \partial_t h^{\sigma_2})\end{aligned}$$

and

$$\partial_t\left(\tilde{\mathcal{R}}_n^{\sigma_0}(Nt,c)\right) = \tilde{R}_n^{\sigma_0}(Nt,c) + \tilde{\mathcal{R}}_n^{\sigma_0}(Nt,\partial_t c).$$

To control r, we split (3.2) into two parts: finitely many terms and small (in $\ell^1(\Lambda)$) remainder terms, respectively (cf. [1, Theorem 6.3]). For $\eta = 1, 2, \cdots$, we choose $\{s_j\}_{j=1}^\infty \subset \mathbb{N}$ ($s_1 < s_2 < \cdots$) in order to satisfy $\|(I - \mathcal{P}_\eta)r\| \to 0 \quad (\eta \to \infty)$, where

$$\begin{aligned} \mathcal{P}_\eta r \quad := \quad & \Big\{ r_{n_1}, r_{n_2}, \cdots, r_{n_{s_\eta}} : \\ & n_1, \cdots, n_{s_\eta} \in \Lambda : n_k \neq n_\ell \ (k \neq \ell), |n_j| \leq \eta \quad \text{for all} \quad j = 1, \cdots, s_\eta \Big\}. \end{aligned}$$

The choice of $n_1 \cdots n_{s_\eta}$ is not uniquely determined, however this does not matter. Then we can divide r into two parts: finitely many terms $r_{n_1}, \cdots, r_{n_{s_\eta}}$ and small remainder terms $\{(I - \mathcal{P}_\eta)r_n\}_{n \in \Lambda}$.

Remark 3.3. We have the following estimates:

$$\|\mathcal{P}_\eta \tilde{\mathcal{B}}_n^{\sigma_0}(\mathcal{P}_\eta c, \mathcal{P}_\eta c)\|_0 \leq \frac{\beta(\eta)}{N}(1 + \eta^2)^{1/2} \|\mathcal{P}_\eta c\|_0^2,$$

$$\|\mathcal{P}_\eta \bar{R}_n^{\sigma_0}(\mathcal{P}_\eta y, c, b)\| \leq (1 + \eta^2)^{1/2} \|\mathcal{P}_\eta y\| (\|c\| + \|b\|)$$

and

$$\||n|^2 \mathcal{P}_\eta y\| \leq (1 + \eta^2) \|\mathcal{P}_\eta y\|_0$$

for $0 < t < T_L$ (T_L is a local existence time, see (3.1)), where

$$\beta(\eta) := \max\{|\omega_{nkm}^\sigma|^{-1} : k = k_1, \cdots k_{s_\eta}, \ n = n_1, \cdots, n_{s_\eta}, \ m = n - k\}.$$

Note that $\beta(\eta)$ is always finite, since it only have finite combinations for the choice of n, k and m. We can also have the same type estimate for $\|\partial_t \mathcal{P}_\eta c\|$ using (2.6).

We use a change of variables to control $\tilde{R}^0$ and $\tilde{R}^{\sigma_0}$. Let us set y as

$$y_n^0(t) := r_n^0(t) - \tilde{\mathcal{R}}_n^0(Nt, \mathcal{P}_\eta c) \quad \text{and} \quad y_n^{\sigma_0}(t) := r_n^{\sigma_0}(t) - \tilde{\mathcal{R}}_n^{\sigma_0}(Nt, \mathcal{P}_\eta c).$$

From (3.2), we see that

$$\begin{aligned} \partial_t \left(y_n^0 + \tilde{\mathcal{R}}_n^0 \right) &= -\nu |n|^2 (y_n^0 + \tilde{\mathcal{R}}_n^0) + \bar{R}_n^0(y^0 + \tilde{\mathcal{R}}^0, c, b) \\ &\quad + \tilde{R}_n^0(Nt, \mathcal{P}_\eta c) + \tilde{R}_n^0(Nt, (I - \mathcal{P}_\eta)c), \\ \partial_t \left(y_n^{\sigma_0} + \tilde{\mathcal{R}}_n^{\sigma_0} \right) &= -\left(\frac{\nu + \kappa}{2} \right) |n|^2 (y_n^{\sigma_0} + \tilde{\mathcal{R}}_n^{\sigma_0}) + \bar{R}_n^{\sigma_0}(y^{\sigma_0} + \tilde{\mathcal{R}}^{\sigma_0}, c, b) \\ &\quad + \tilde{R}_n^{\sigma_0}(Nt, \mathcal{P}_\eta c) + \tilde{R}_n^{\sigma_0}(Nt, (I - \mathcal{P}_\eta)c). \end{aligned}$$

Now we control $\mathcal{P}_\eta y^0$ and $\mathcal{P}_\eta y^{\sigma_0}$ for fixed η. By (3.3),

$$\begin{aligned} \partial_t \mathcal{P}_\eta y_n^0(t) &= -\nu |n|^2 \mathcal{P}_\eta y_n^0 + \mathcal{P}_\eta \bar{R}_n^0(\mathcal{P}_\eta y, c, b) + E_n^0, \\ \partial_t \mathcal{P}_\eta y_n^{\sigma_0}(t) &= -\left(\frac{\nu + \kappa}{2} \right) |n|^2 \mathcal{P}_\eta y_n^{\sigma_0} + \mathcal{P}_\eta \bar{R}_n^{\sigma_0}(\mathcal{P}_\eta y, c, b) + E_n^{\sigma_0}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} E_n^0 : &= -\mathcal{P}_\eta \tilde{\mathcal{R}}_n^0(Nt, \mathcal{P}_\eta \partial_t c) + \mathcal{P}_\eta \bar{R}_n^0(\mathcal{P}_\eta \tilde{\mathcal{R}}_n^0(Nt, \mathcal{P}_\eta c), c, b) \\ &\quad -\nu |n|^2 \mathcal{P}_\eta \tilde{\mathcal{R}}_n^0(Nt, \mathcal{P}_\eta c) + \mathcal{P}_\eta \tilde{R}_n^0(Nt, (I - \mathcal{P}_\eta)c) \end{aligned}$$

and

$$\begin{aligned} E_n^{\sigma_0} : &= -\mathcal{P}_\eta \tilde{\mathcal{R}}_n^{\sigma_0}(Nt, \mathcal{P}_\eta \partial_t c) + \mathcal{P}_\eta \bar{R}_n^{\sigma_0}(\mathcal{P}_\eta \tilde{\mathcal{R}}_n^{\sigma_0}(Nt, \mathcal{P}_\eta c), c, b) \\ &\quad - \left(\frac{\nu + \kappa}{2} \right) |n|^2 \mathcal{P}_\eta \tilde{\mathcal{R}}_n^{\sigma_0}(Nt, \mathcal{P}_\eta c) + \mathcal{P}_\eta \tilde{R}_n^{\sigma_0}(Nt, (I - \mathcal{P}_\eta)c). \end{aligned}$$

Note that (3.4) are linear heat type equations with external force E^0 and E^{σ_0}. Thus the point is to control E^0 and E^{σ_0}. By Remark 3.3, we can see that for any $\epsilon > 0$, there is η_0 and N_0 (depending on η_0) such that if $N > N_0$ and $\eta > \eta_0$, then $\|E^{\sigma_0}\| < \epsilon$ and $\|E^{\sigma_0}\| < \epsilon$. Thus we have from (3.4),

$$\begin{aligned}\|\mathcal{P}_\eta y_n^0(t)\| \leq & \int_0^t \Big(C_\nu(1+\eta^2)\|\mathcal{P}_\eta y_n^0(s)\| \\ & +(1+\eta^2)^{1/2}\|\mathcal{P}_\eta y^0(s)\|(\|c(s)\|+\|b(s)\|)+\epsilon\Big) ds\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{P}_\eta y_n^{\sigma_0}(t)\| \leq & \int_0^t \Big(C_{\nu,\kappa}(1+\eta^2)\|\mathcal{P}_\eta y_n^{\sigma_0}(s)\| \\ & +(1+\eta^2)^{1/2}\|\mathcal{P}_\eta y^{\sigma_0}(s)\|(\|c(s)\|+\|b(s)\|)+\epsilon\Big) ds.\end{aligned}$$

By Gronwall's inequality, we have that for any $\epsilon > 0$, there is η_0 and N_0 (depending on η_0) such that if $\eta > \eta_0$ and $N > N_0$, then $\|\mathcal{P}_\eta y^0\| < \epsilon$ and $\|\mathcal{P}_\eta y^{\sigma_0}\| < \epsilon$ for $0 < t < T_L$. Clearly, we can also control $(I - \mathcal{P}_\eta)y$ with sufficiently large η (independent of N), and $\mathcal{P}_\eta \tilde{\mathcal{R}}_n^{\sigma_0}(Nt, \mathcal{P}_\eta c)$ with sufficiently large N for fixed η. Thus we can control r for sufficiently large η and N.

□

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