



Title	Eikonal equations in metric spaces
Author(s)	Giga, Yoshikazu; Hamamuki, Nao; Nakayasu, Atsushi
Citation	Hokkaido University Preprint Series in Mathematics, 991, 1-18
Issue Date	2011-12-25
DOI	10.14943/84138
Doc URL	<a href="http://hdl.handle.net/2115/69797">http://hdl.handle.net/2115/69797</a>
Type	bulletin (article)
File Information	pre991.pdf



[Instructions for use](#)

# Eikonal equations in metric spaces

Yoshikazu Giga<sup>a</sup>      Nao Hamamuki<sup>b</sup>      Atsushi Nakayasu<sup>c</sup>

<sup>a</sup> *Graduate School of Mathematical Sciences, University of Tokyo*  
*3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914 Japan and*  
*Department of Mathematics, Faculty of Sciences, King Abdulaziz University*  
*P. O. Box 80203, Jeddah 21589, Saudi Arabia, labgiga@ms.u-tokyo.ac.jp*  
<sup>b</sup> *Graduate School of Mathematical Sciences, University of Tokyo*  
*3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914 Japan, hnao@ms.u-tokyo.ac.jp*  
<sup>c</sup> *Graduate School of Mathematical Sciences, University of Tokyo*  
*3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914 Japan, ankys@ms.u-tokyo.ac.jp*

## Abstract

A new notion of a viscosity solution for Eikonal equations in a general metric space is introduced. A comparison principle is established. The existence of a unique solution is shown by constructing a value function of the corresponding optimal control theory. The theory applies to infinite dimensional setting as well as topological networks, surfaces with singularities.

**Key words:** Eikonal equation, metric space, metric viscosity solutions  
**Mathematics Subject Classification:** 35D40, 35F30, 49L25

## 1 Introduction

The goal of this paper is to establish a notion of viscosity solutions for Eikonal equations in an open set of a general metric space which is consistent with usual notion when the metric space is Euclidean. Let  $\mathcal{X}$  be a metric space and let  $\Omega$  be an open set in  $\mathcal{X}$ . We would like to consider an Eikonal equation for a function  $u$  defined on  $\Omega$  of the form

$$|Du| = f(x) \quad \text{in } \Omega, \tag{1.1}$$

where  $f$  is a given function on  $\Omega$ . The symbol  $Du$  formally denotes the gradient of  $u$  but it is not well-defined in a general metric space. However, as we see later its modulus  $|Du|$  is able to be characterized.

Eikonal equations are fundamental to describe propagation of a wave front or interfaces in various disciplines of sciences and technology. The theory of viscosity solutions for Eikonal equations or more general Hamilton-Jacobi equations is well-developed when  $\mathcal{X}$  is a Euclidean space; see, e.g., [9], [15] or more generally a Banach space [10]. In these days it is also desirable to extend the

notion for more general metric spaces. For example, the theory of viscosity solutions is extended to the spaces where  $Du$  is well-defined such as Wasserstein metric spaces [12] and Riemannian manifolds [11]. A Hamilton-Jacobi equation on a topological network is also considered in [1], [8], and [14], which seems to be important in handling a social network problem. There are several works on elliptic and parabolic equations in a singular manifold. The reader is referred to [2] and references therein. The theory for a gradient flow in a general complete metric space is discussed in [4]. However, there seem to be no theories for the first-order nonlinear partial differential equations. Thus it is very natural to extend the notion of solutions for (1.1) to general metric spaces.

Let us describe our idea to define the notion of viscosity solutions for (1.1). For a given curve  $\xi = \xi(t)$  in  $\mathcal{X}$ , one is able to define its metric derivative  $|\xi'(t)|$  although  $\xi'(t)$  may not be well-defined; see (2.4) and [4] for definition. In a Euclidean space we have  $|Du(x)| = \sup_{\xi} |(u \circ \xi)'(0)|/|\xi'(0)|$ , when  $\xi$  is a smooth curve passing  $x$  at  $t = 0$ , i.e.,  $\xi(0) = x$ . Reflecting this property, we say that  $u$  is a metric viscosity subsolution of (1.1) if  $|(u \circ \xi)'(0)| \leq f(x)$  (in the viscosity sense) for each  $x \in \Omega$  and curve  $\xi$  satisfying  $|\xi'| \leq 1$  and  $\xi(0) = x$ . The definition of a supersolution is more involved. Roughly speaking, we say that  $u$  is a metric viscosity supersolution if for each  $x \in \Omega$  there is a curve  $\xi$  with  $|\xi'| \leq 1$  and  $\xi(0) = x$  such that  $|w'(t)| \gtrsim f(\xi(t))$  for all  $t$  until  $\xi$  hits the boundary  $\partial\Omega$  where  $w$  is an upper approximation of  $u \circ \xi$  with  $w(x) = (u \circ \xi)(0)$ . More rigorous definition is found in Section 2. The point is that we reduce the notion to one-dimension. Fortunately, we are able to establish a standard comparison principle by reflecting the classical idea of Ishii [15] even when  $\Omega$  is unbounded under the assumption  $\inf_{\Omega} f > 0$ . (If  $f$  is allowed to be zero, we know that the comparison principle fails because the set  $\{f = 0\}$  is an Aubry set.)

The existence of a metric viscosity solution for (1.1) with a given boundary condition is proved by constructing the value function of the corresponding optimal control problem. Since there may be no optimal curves, we need an approximation  $w$  in the definition of supersolutions. By the comparison principle and the verification that the value function is a solution, we are able to establish a unique existence result for a boundary value problem.

Our solution also enjoys a stability property. For a subsolution it is similar to the Euclidean case. However, for a supersolution it is valid in a restrictive setting. Our argument requires uniform convergence of supersolutions.

The notion of a metric viscosity subsolution is consistent with the classical one when  $\mathcal{X}$  is Euclidean. However, the notion of a metric viscosity supersolution is stronger than the Euclidean one since our notion is not a local notion. Fortunately, for (1.1) it turns out that Euclidean viscosity solution is a metric one when a suitable comparison principle holds. We establish this property by representing a solution as a value function of the corresponding optimal control problem.

This paper is organized as follows. In Section 2 we give our definition of metric viscosity solutions. In Section 3 we establish a comparison principle while in Section 4 we verify that the value function of the corresponding control problem

is indeed a metric viscosity solution. We also discuss its stability in Section 5. In Section 6 we discuss the consistency of our metric viscosity solutions when the metric space is Euclidean.

## 2 Definition

Let  $(\mathcal{X}, d)$  be a metric space. Let  $f$  be a nonnegative continuous function on an open set  $\Omega$  in  $\mathcal{X}$ . We consider the Eikonal equation of the form

$$|Du| = f(x) \quad \text{in } \Omega. \quad (2.1)$$

To motivate the problem we begin with the case that  $(\mathcal{X}, d) = (\mathbb{R}^N, d_E)$ , where  $d_E$  is the standard Euclidean metric, i.e.,  $d_E(a, b) = |a - b|$ . Here  $|a|$  is the standard Euclidean norm of  $a$  defined by  $|a|^2 = a \cdot a$ . Let  $u$  be a  $C^1$  function satisfying  $|Du(x)| = f(x)$  for all  $x \in \Omega$ . Fix  $x \in \Omega$  and let  $\xi$  be a  $C^1$  curve on  $\mathcal{X} = \mathbb{R}^N$  satisfying  $\xi(0) = x$ . Differentiating a composite function yields

$$|(u \circ \xi)'(0)| = |Du(x) \cdot \xi'(0)| \leq |Du(x)| |\xi'(0)| \quad (2.2)$$

by the Schwarz inequality. Moreover, there exists a curve  $\xi$  such that the equality of (2.2) holds. Hence we have

$$|Du(x)| = \sup_{\xi} \frac{|(u \circ \xi)'(0)|}{|\xi'(0)|}. \quad (2.3)$$

We define a new notion of solutions to (2.1) based on this fact.

We return to the general case. We should recall a modulus of tangent vectors for a curve on a metric space. Let  $\xi$  be a curve in  $\mathcal{X}$ . In other words,  $\xi$  is a mapping from an interval  $I$  of  $\mathbb{R}$  to  $\mathcal{X}$ . For each  $t \in I$  we define  $|\xi'| (t)$  as a *metric derivative*

$$|\xi'| (t) := \lim_{s \rightarrow t} \frac{d(\xi(s), \xi(t))}{|s - t|} \quad (2.4)$$

although  $\xi'(t)$  itself is not well-defined. We say that  $\xi$  is an *absolutely continuous* curve if the limit of (2.4) exists for a.e.  $t \in I$  and  $|\xi'|$  belongs to  $L^1_{loc}(I)$  and satisfies

$$d(\xi(s), \xi(t)) \leq \int_s^t |\xi'| (r) dr$$

for all  $s, t \in I, s \leq t$ . For an equivalent definition of absolute continuity and its properties, the reader is referred to a book of L. Ambrosio et al. [4], where the metric space is assumed to be complete. However, to establish a notion of absolute continuity the completeness is unnecessary [3].

We hereafter only consider a curve whose speed does not exceed one, i.e.,

$$|\xi'| \leq 1 \quad \text{a.e. in } I, \quad (2.5)$$

and we say that an absolutely continuous curve  $\xi$  is *admissible* if  $\xi$  satisfies (2.5). The set of all admissible curves defined on an interval  $I$  is denoted by  $\mathcal{A}(I, \mathcal{X})$ .

In addition, for a fixed point  $x \in \mathcal{X}$  we say that an admissible curve  $\xi$  belongs to  $\mathcal{A}_x(I, \mathcal{X})$  with  $0 \in I$  if  $\xi$  satisfies  $\xi(0) = x$ . For  $\Omega \subset \mathcal{X}$  and  $\xi \in \mathcal{A}_x(I, \mathcal{X})$  with  $x \in \Omega$ , define the *exit time* and *entrance time* as below respectively:

$$\begin{aligned} T_{\Omega}^+[\xi] &:= \inf \{t \in I \cap [0, \infty) \mid \xi(t) \notin \Omega\} \in [0, \infty], \\ T_{\Omega}^-[\xi] &:= \sup \{t \in I \cap (-\infty, 0] \mid \xi(t) \notin \Omega\} \in [-\infty, 0]. \end{aligned}$$

To introduce our notion we recall super- and subdifferentials. For a general function  $f$  defined on an open set  $W$  in  $\mathbb{R}^N$ , let  $D^+f(x)$  be the superdifferential of  $f$  at  $x \in W$  and let  $D^-f(x)$  be the subdifferential of  $f$  at  $x$ . Namely, let

$$\begin{aligned} D^+f(x) &:= \{D\varphi(x) \mid \varphi \in C^1(W), f - \varphi \text{ has a local maximum at } x\}, \\ D^-f(x) &:= \{D\varphi(x) \mid \varphi \in C^1(W), f - \varphi \text{ has a local minimum at } x\}. \end{aligned}$$

For a subset  $\Omega$  of  $\mathcal{X}$  the set of all upper (resp. lower) semicontinuous functions on  $\Omega$  is denoted by  $USC(\Omega)$  (resp.  $LSC(\Omega)$ ). We introduce a weaker notion of continuity for our solutions. We say that a function  $u$  defined on  $\Omega$  is *arcwise upper* (resp. *lower*) *semicontinuous* if for each admissible curve  $\xi \in \mathcal{A}(I, \Omega)$  with an interval  $I$  the composite function  $u \circ \xi$  is upper (resp. lower) semicontinuous on  $I$ . The set of all arcwise upper (resp. lower) semicontinuous functions on  $\Omega$  is represented by  $USC_a(\Omega)$  (resp.  $LSC_a(\Omega)$ ). We say that a function defined on  $\Omega$  is *arcwise continuous* if it is both arcwise upper and lower semicontinuous. Let  $C_a(\Omega)$  be the set of all arcwise continuous functions.

**Definition 2.1** (Metric viscosity solution). We say that  $u \in USC_a(\Omega)$  is a *metric viscosity subsolution* of (2.1) if for each  $x \in \Omega$  and  $\xi \in \mathcal{A}_x(\mathbb{R}, \Omega)$  the inequality

$$|p| \leq f(x)$$

holds for all  $p \in D^+w(0)$  with  $w = u \circ \xi$ .

We say that  $u \in LSC_a(\Omega)$  is a *metric viscosity supersolution* of (2.1) if for each  $x \in \Omega$  and  $\varepsilon > 0$  there exist  $\xi \in \mathcal{A}_x(\mathbb{R}, \mathcal{X})$  and  $w \in LSC(T^-, T^+)$  such that

$$\text{both } T^{\pm} := T_{\Omega}^{\pm}[\xi] \text{ are finite,} \quad (2.6)$$

$$u(\xi(t)) - \varepsilon \leq w(t), \quad w(0) = u(x), \quad (2.7)$$

and the inequality

$$|p| \geq f(\xi(t)) - \varepsilon \quad (2.8)$$

holds for all  $t \in (T^-, T^+)$  and  $p \in D^-w(t)$ . We call this pair  $(\xi, w)$  an  $\varepsilon$ -pair at  $x$  for  $u$  and  $f$ . Since  $x \in \Omega$ , i.e.,  $x$  is not on the boundary,  $T^{\pm} \neq 0$ . The existence of  $\xi$  satisfying (2.6) implicitly assumes that  $\partial\Omega$  is nonempty.

Finally, we say that  $u \in C_a(\Omega)$  is a *metric viscosity solution* if it is both a metric viscosity subsolution and a metric viscosity supersolution.

We hereafter suppress the words ‘‘metric viscosity’’ unless confusion occurs. For example, we simply say ‘‘a subsolution’’ instead of ‘‘a metric viscosity subsolution’’.

*Remark 2.2.* (i) The formula (2.3) means that we measure  $Du$  in (2.1) with the dual norm when  $\mathcal{X}$  is a Banach space.

(ii) We actually do not invoke the symmetry property of the metric  $d$ , i.e.,  $d(x, y) = d(y, x)$  throughout this paper. Thus our theory applies to a general quasi-metric space.

*Remark 2.3.* The notion of our subsolutions is local in the sense that  $u$  is a subsolution in  $\Omega$  if and only if  $u$  is a subsolution in some open neighborhood of each point in  $\Omega$ . However, for a supersolution this type of locality does not hold in general although a weaker version is valid (Lemma 6.4). In fact, as shown in the next example, the notion of our supersolutions is a concept stronger than that of conventional viscosity supersolutions.

**Example 2.4.** Let  $(\mathcal{X}, d) = (\mathbb{R}, d_E)$ . We consider the boundary value problem

$$\begin{cases} |u'(x)| = 1 & \text{in } (0, \infty), \\ u(0) = 0. \end{cases} \quad (2.9)$$

A function  $u(x) := x$  is a unique solution by Theorem 4.5. In particular, another function  $v(x) := -x$  is not a solution although  $v$  satisfies (2.9) in the classical sense. In fact, we are able to confirm that  $v$  is not a supersolution directly from the definition. Fix  $x > 0$  and choose  $\varepsilon < \min\{1, x\}$ . For each  $\xi \in \mathcal{A}_x(\mathbb{R}, \mathcal{X})$  and  $w \in LSC(T^-, T^+)$  satisfying (2.6) and (2.7), we have  $\liminf_{t \rightarrow T^\pm} w(t) \geq -\varepsilon > -x = w(0)$ . This means that  $w$  attains its minimum at some  $t \in (T^-, T^+)$ . Hence  $0 \in D^-w(t)$  but (2.8) does not hold since  $0 < 1 - \varepsilon$ .

*Remark 2.5.* Such a kind of asymmetry between the definition of a subsolution and a supersolution also occurs in the theory of viscosity solutions on topological networks [8]. That is to say, their notion and ours are similar in spirit: For a subsolution we test all curves passing through a given point while for a supersolution we need find an appropriate curve satisfying a desired inequality of subdifferentials. Indeed, we see that the notion of our metric subsolutions consists with their notion of viscosity subsolutions on a topological network. We also expect that our metric supersolution should consist with the network supersolution although it is not verified.

In [1] the authors construct a unique state constraint solution for a Bellman equation, which excludes our eikonal equation, by considering the infinite horizon problem. In [8] a class of equations including Eikonal equations is treated. Moreover, the network is allowed to have finitely many junctions. The paper [14] studies a class of Hamiltonians of evolution type and it allows a discontinuity with respect to the gradient variable. The network they study allows only one junction. However, the authors of [14] deduce their definition from the one of weak solution of a conservation law on a network, which is quite natural. In any case it is nontrivial to compare definitions of solutions in these three papers with ours for Eikonal equations on a topological network. We also point out that a numerical scheme for Eikonal equations on a topological network is proposed in a recent preprint [7] and it is actually used to compute numerically the distance to a target and the corresponding shortest path in some settings.

We establish a necessary and sufficient condition for subsolutions.

**Proposition 2.6.** *Assume*

$$f \in C_a(\Omega) \text{ and } f \geq 0 \text{ in } \Omega. \quad (2.10)$$

Let  $u \in USC_a(\Omega)$ . Then the following statements are equivalent:

(i)  $u$  is a subsolution of (2.1).

(ii) The inequality

$$u(\xi(s)) \leq \int_s^t f(\xi(r)) dr + u(\xi(t)) \quad (2.11)$$

holds for all  $\xi \in \mathcal{A}(\mathbb{R}, \Omega)$  and  $s, t \in \mathbb{R}$ ,  $s < t$ .

(iii) The inequality

$$|u(\xi(s)) - u(\xi(t))| \leq \int_s^t f(\xi(r)) dr \quad (2.12)$$

holds for all  $\xi \in \mathcal{A}(\mathbb{R}, \Omega)$  and  $s, t \in \mathbb{R}$ ,  $s < t$ .

*Proof.* (ii)  $\Rightarrow$  (iii): This is clear by taking a curve  $r \mapsto \xi(-r)$ .

(i)  $\Rightarrow$  (ii): By Lemma A.1 a function  $t \mapsto u(\xi(t)) - \int_0^t f(\xi(s)) ds$  is nonincreasing. Therefore, we have (2.11) for all  $s < t$ .

(iii)  $\Rightarrow$  (i): Fix  $x \in \Omega$  and  $\xi \in \mathcal{A}_x(\mathbb{R}, \Omega)$ . Suppose that  $u \circ \xi - \varphi$  has a local maximum at 0 for  $\varphi \in C^1(\mathbb{R})$ . Then we observe by (2.12) that

$$\varphi(0) - \varphi(h) \leq u(x) - u(\xi(h)) \leq \begin{cases} \int_0^h f(\xi(s)) ds & \text{for } h > 0, \\ \int_h^0 f(\xi(s)) ds & \text{for } h < 0. \end{cases}$$

Therefore, we have  $|\varphi'(0)| \leq f(\xi(0)) = f(x)$ .  $\square$

Similarly, we show a sufficient condition for supersolutions.

**Proposition 2.7.** *Assume (2.10). Let  $u \in LSC_a(\Omega)$ . Then  $u$  is a supersolution of (2.1) if for every  $x \in \Omega$  and  $\varepsilon > 0$  there exists  $\xi \in \mathcal{A}_x([0, \infty), \mathcal{X})$  satisfying  $T := T_\Omega^+[\xi] < \infty$  and*

$$u(x) + \varepsilon \geq \int_0^t f(\xi(s)) ds + u(\xi(t)) \quad (2.13)$$

for all  $0 \leq t < T$ .

*Proof.* Define  $\tilde{\xi} \in \mathcal{A}_x(\mathbb{R}, \mathcal{X})$  as  $\tilde{\xi}(t) = \xi(|t|)$  and let

$$w(t) := u(x) - \int_0^{|t|} f(\xi(s)) ds$$

for  $t \in (-T, T)$ . We have  $w(0) = u(x)$  and  $w(t) \geq u(\tilde{\xi}(t)) - \varepsilon$  by (2.13). We also observe that

$$w'(t) = \begin{cases} -f(\tilde{\xi}(t)) & \text{for } t \in (0, T), \\ f(\tilde{\xi}(t)) & \text{for } t \in (-T, 0), \end{cases}$$

and  $D^-w(0) = \emptyset$  if  $f(x) > 0$ . We hence have  $|p| \geq f(\tilde{\xi}(t))$  for all  $t \in (-T, T)$  and  $p \in D^-w(0)$ . Therefore,  $(\tilde{\xi}, w)$  is an  $\varepsilon$ -pair at  $x$  for  $u$ .  $\square$

### 3 Comparison principle

**Theorem 3.1** (Comparison). *Assume  $f \in C_a(\Omega)$  and*

$$\sigma := \inf_{x \in \Omega} f(x) > 0. \quad (3.1)$$

*Let  $u \in USC_a(\bar{\Omega})$  be a subsolution of (2.1) and let  $v \in LSC_a(\bar{\Omega})$  be a supersolution of (2.1). If  $u \leq v$  on  $\partial\Omega$  and  $c \leq v$  on  $\partial\Omega$  for some constant  $c \in \mathbb{R}$ , then  $u \leq v$  in  $\Omega$ .*

Before proving this theorem we recall a typical comparison principle ([15]) for the Eikonal equation (2.1) when  $(\mathcal{X}, d) = (\mathbb{R}^N, d_E)$ . In [15] it is shown in more general setting that under the assumption that

$$\Omega \text{ is bounded, } f \in C(\bar{\Omega}), \text{ and } f > 0 \text{ in } \Omega \quad (3.2)$$

we have  $u \leq v$  in  $\Omega$  for a conventional viscosity subsolution  $u \in USC(\bar{\Omega})$  and supersolution  $v \in LSC(\bar{\Omega})$  of (2.1) if  $u \leq v$  on  $\partial\Omega$ . The reason we do not need the boundedness of  $\Omega$  in Theorem 3.1 is that we compare the sub- and supersolution not in the whole of  $\Omega$  but in the bounded interval  $(T^-, T^+)$  which appears in the definition of a metric supersolution.

*Proof.* Suppose that  $m := (u - v)(x) > 0$  for some  $x \in \Omega$ . Take  $\lambda \in (0, 1)$  satisfying  $(1 - \lambda)u(x) < m/2$  and  $(\lambda - 1)c < m/4$ . By this choice we have

$$(\lambda u - v)(x) = (\lambda - 1)u(x) + (u - v)(x) > m/2 \quad (3.3)$$

and

$$(\lambda u - v)(z) = (\lambda - 1)v(z) + \lambda(u - v)(z) < m/4 \quad (3.4)$$

for all  $z \in \partial\Omega$  because of the assumptions  $u \leq v$  on  $\partial\Omega$  and  $c \leq v$  on  $\partial\Omega$ . For each  $\varepsilon > 0$  take an  $\varepsilon$ -pair  $(\xi, w)$  at  $x$  for the supersolution  $v$  and set  $T^\pm := T_\Omega^\pm[\xi]$ . By extending  $w$  as  $w(T^\pm) := v(\xi(T^\pm)) - \varepsilon$  we have  $w \in LSC([T^-, T^+])$  and  $v(\xi(t)) - \varepsilon \leq w(t)$  for all  $t \in [T^-, T^+]$ .

Define  $\Phi_\alpha \in USC([T^-, T^+]^2)$  as

$$\Phi_\alpha(t, s) := \lambda u(\xi(t)) - w(s) - \alpha(t - s)^2$$



for  $\alpha > 0$  and choose a maximum point  $(t_\alpha, s_\alpha) \in [T^-, T^+]^2$  of  $\Phi_\alpha$ . We then have

$$\Phi_\alpha(t_\alpha, s_\alpha) \geq \Phi_\alpha(0, 0) = (\lambda u - v)(x) > m/2 \quad (3.5)$$

by (3.3). This implies

$$\alpha(t_\alpha - s_\alpha)^2 \leq \lambda \max_{[T^-, T^+]} (u \circ \xi) - \min_{[T^-, T^+]} w - m/2 < \infty.$$

Thus, we may assume that  $\lim_{\alpha \rightarrow \infty} (t_\alpha, s_\alpha) = (\hat{t}, \hat{t})$  for some  $\hat{t} \in [T^-, T^+]$ .

We claim  $\hat{t} \in (T^-, T^+)$ . Taking  $\limsup_{\alpha \rightarrow \infty}$  in (3.5), we have

$$m/2 \leq \lambda u(\xi(\hat{t})) - w(\hat{t}) \leq (\lambda u - v)(\xi(\hat{t})) + \varepsilon.$$

If  $\xi(\hat{t}) \in \partial\Omega$ , we see that  $m/2 \leq m/4 + \varepsilon$  by (3.4). This is impossible since one may choose  $\varepsilon < m/4$ . Therefore, we have  $\hat{t} \in (T^-, T^+)$  and so  $t_\alpha, s_\alpha \in (T^-, T^+)$  for sufficiently large  $\alpha$ .

Since  $t \mapsto \Phi_\alpha(t, s_\alpha)/\lambda$  attains its maximum at  $t_\alpha \in (T^-, T^+)$  and  $u$  is a subsolution, we have

$$|2\alpha(t_\alpha - s_\alpha)| \leq \lambda f(\xi(t_\alpha)).$$

Similarly, since  $s \mapsto -\Phi_\alpha(t_\alpha, s)$  attains its minimum at  $s_\alpha \in (T^-, T^+)$  and  $v$  is a supersolution, we have

$$|2\alpha(t_\alpha - s_\alpha)| \geq f(\xi(s_\alpha)) - \varepsilon.$$

These two inequalities yield

$$0 \geq f(\xi(s_\alpha)) - \lambda f(\xi(t_\alpha)) - \varepsilon.$$

Sending  $\alpha \rightarrow \infty$ , we obtain

$$0 \geq (1 - \lambda)f(\xi(\hat{t})) - \varepsilon \geq (1 - \lambda)\sigma - \varepsilon,$$

which is a contradiction if we choose  $\varepsilon < (1 - \lambda)\sigma$ .  $\square$

It is impossible to remove the assumption that  $c \leq v$  on  $\partial\Omega$  in general.

**Example 3.2.** Let  $(\mathcal{X}, d) = (\mathbb{R}^2, d_E)$ . We consider the boundary value problem

$$\begin{cases} |Du(x, y)| = 1 & \text{in } \Omega := \mathbb{R} \times (0, \infty), \\ u(x, 0) = x & \text{on } \partial\Omega = \mathbb{R} \times \{0\}. \end{cases}$$

Then  $u(x, y) = x$  is a subsolution while  $v(x, y) = x + ky$  is a supersolution for each  $k \in \mathbb{R}$ . Evidently, the comparison principle is violated because  $u > v$  in  $\mathbb{R} \times (0, \infty)$  when  $k < 0$ . (Note that  $v$  is not bounded from below on  $\partial\Omega$ .) Let us check that  $v$  is indeed a supersolution. Fix  $P = (a, b) \in \Omega$  and let  $Q_z = (z, 0) \in \partial\Omega$ . Define  $\xi_z \in \mathcal{A}_P(\mathbb{R}, \mathcal{X})$  as  $\xi_z(t) = (1 - |t|/l_z)P + (|t|/l_z)Q_z$ , where  $l_z$  is the length of the line segment joining  $P$  and  $Q_z$ , i.e.,  $l_z = \sqrt{(z - a)^2 + b^2}$ . A direct calculation yields  $(v \circ \xi)(t) = (z - a - kb)|t|/l_z + a + kb$  and  $(z - a - kb)/l_z \rightarrow -1$  as  $z \rightarrow -\infty$ . This implies that  $v$  is a supersolution.

## 4 Solutions by optimal control theory

We next construct a unique solution of (2.1) with a boundary condition

$$u = g \quad \text{on } \partial\Omega \quad (4.1)$$

by applying the optimal control theory. Here  $g$  is a given function on  $\partial\Omega$ . We say that  $u \in C_a(\bar{\Omega})$  is a solution of the boundary value problem (2.1) and (4.1) if  $u$  is a solution of (2.1) and satisfies (4.1).

For  $x \in \bar{\Omega}$  and a curve  $\xi \in \mathcal{C}_x := \{\xi \in \mathcal{A}_x([0, \infty), \mathcal{X}) \mid T_\Omega^+[\xi] \in [0, \infty)\}$ , we consider the cost functional

$$C[\xi] := \int_0^{T_\Omega^+[\xi]} f(\xi(s)) ds + g(\xi(T_\Omega^+[\xi])).$$

We define the value function  $u$  as the infimum of the cost, i.e.,

$$u(x) := \inf_{\xi \in \mathcal{C}_x} C[\xi].$$

The goal of this section is to show that the value function  $u$  is a unique solution of the boundary value problem (2.1) and (4.1). It is clear that  $u$  satisfies (4.1).

Our basic assumptions are the following:

$$\mathcal{C}_x \text{ is nonempty for each } x \in \bar{\Omega}. \quad (4.2)$$

$$f \in C_a(\bar{\Omega}) \text{ and } f \geq 0 \text{ in } \bar{\Omega}. \quad (4.3)$$

$$g \text{ is bounded from below on } \partial\Omega. \quad (4.4)$$

These assumptions imply that  $u$  is well-defined as a real-valued function.

**Lemma 4.1** (Dynamic programming principle). *Assume (4.2)–(4.4). Let  $u(x) = \inf_{\xi \in \mathcal{C}_x} C[\xi]$ . Then we have*

$$u(x) = \inf_{\xi \in \mathcal{C}_x} \left\{ \int_0^T f(\xi(s)) ds + u(\xi(T)) \right\}$$

with  $T = \min\{T_0, T_\Omega^+[\xi]\}$  for all  $x \in \bar{\Omega}$  and  $T_0 \geq 0$ .

This lemma is proved by an argument similar to that in the conventional theory; see, e.g., [5, Proposition IV.2.1].

We show that the value function  $u$  is a solution of (2.1).

**Theorem 4.2.** *Assume (4.2)–(4.4). Then  $u(x) = \inf_{\xi \in \mathcal{C}_x} C[\xi]$  is a solution of (2.1).*

*Proof.* By Lemma 4.1 we see that  $u$  satisfies (ii) in Proposition 2.6. It now follows that  $u \in C_a(\Omega)$  and  $u$  is a subsolution.

Let  $x \in \Omega$  and  $\varepsilon > 0$ . Then take  $\xi \in \mathcal{C}_x$  satisfying  $u(x) + \varepsilon \geq \int_0^T f(\xi(s)) ds + g(\xi(T))$  with  $T = T_\Omega^+[\xi]$ . By the definition of  $u(\xi(t))$  we also have  $u(\xi(t)) \leq \int_t^T f(\xi(s)) ds + g(\xi(T))$  for each  $t \in [0, T]$ . Combining these two inequalities, we obtain (2.13).  $\square$

We next show that the value function  $u$  is a unique solution of the boundary value problem (2.1) and (4.1). It remains to establish  $u \in C_a(\bar{\Omega})$ , which is required to apply Theorem 3.1 for uniqueness. We have already shown  $u \in C_a(\Omega)$  but the arcwise continuity on the boundary turns out to be an issue.

**Example 4.3.** Let  $(\mathcal{X}, d) = (\mathbb{R}, d_E)$ . We consider the boundary value problem

$$\begin{cases} |u'(x)| = 1 & \text{in } (0, 1), \\ u(0) = 0, u(1) = a \end{cases}$$

with  $a \geq 0$ . Then the value function is of the form  $u(x) = \min\{x, -x + a + 1\}$  for  $x \in [0, 1)$  and  $u(1) = a$ . When  $a > 1$ , this is not arcwise continuous at  $x = 1$ .

This example suggests that we have to impose a certain growth condition on  $g$  in order to guarantee the continuity of  $u$ . We use the following condition:

$$\begin{aligned} \text{The inequality } g(x) \leq \int_0^T f(\xi(s))ds + g(\xi(T)) \text{ holds} & \quad (4.5) \\ \text{for all } x \in \partial\Omega \text{ and } \xi \in \mathcal{A}_x([0, \infty), \bar{\Omega}) \text{ satisfying } \xi(T) \in \partial\Omega. \end{aligned}$$

**Lemma 4.4.** Assume (4.2)–(4.4). Let  $u(x) = \inf_{\xi \in \mathcal{C}_x} C[\xi]$ . Then  $u \in C_a(\bar{\Omega})$  if and only if (4.5) holds.

*Proof.* Assume (4.5). We only have to show  $\lim_{t \rightarrow T} u(\xi(t)) = g(\xi(T))$  for all  $\xi \in \mathcal{A}([0, \infty), \bar{\Omega})$  such that  $\xi(T) \in \partial\Omega$  with  $T > 0$ . By the definition of  $u(\xi(t))$  and (4.5), we observe that

$$u(\xi(t)) \leq \int_t^{T_t} f(\xi(s))ds + g(\xi(T_t)) \leq \int_t^T f(\xi(s))ds + g(\xi(T))$$

for all  $t \in [0, T]$ , where  $T_t = \inf\{s \in [t, \infty) \mid \xi(s) \notin \Omega\} \in [t, T]$ . We thus have  $\limsup_{t \rightarrow T} u(\xi(t)) \leq g(\xi(T))$ . We next observe by (4.5) that

$$g(\xi(T)) \leq \int_t^T f(\xi(s))ds + \int_0^{\tilde{T}} f(\tilde{\xi}(s))ds + g(\tilde{\xi}(\tilde{T}))$$

for all  $t \in [0, T]$  and  $\tilde{\xi} \in \mathcal{C}_{\xi(t)}$ , where  $\tilde{T} = T_{\Omega}^+[\tilde{\xi}]$ . Thus we have

$$u(\xi(t)) = \inf_{\tilde{\xi} \in \mathcal{C}_{\xi(t)}} C[\tilde{\xi}] \geq - \int_t^T f(\xi(s))ds + g(\xi(T)),$$

and so  $\liminf_{t \rightarrow T} u(\xi(t)) \geq g(\xi(T))$ .

Suppose that (4.5) were false, i.e., there would exist  $x \in \partial\Omega$  and  $\xi \in \mathcal{A}_x([0, \infty), \bar{\Omega})$  satisfying  $\xi(T) \in \partial\Omega$  and  $g(x) > \int_0^T f(\xi(s))ds + g(\xi(T))$  for some  $T > 0$ . Since  $u(\xi(t)) \leq \int_t^T f(\xi(s))ds + g(\xi(T))$  for all  $t > 0$ , we would have

$$\limsup_{t \rightarrow 0} u(\xi(t)) \leq \int_0^T f(\xi(s))ds + g(\xi(T)) < g(x).$$

Hence,  $u$  would not be arcwise continuous at  $x$ . The proof is now complete.  $\square$

Combining Theorem 3.1, 4.2, and Lemma 4.4, we have

**Theorem 4.5.** *Assume (3.1), (4.2)–(4.4), and (4.5). Then  $u(x) = \inf_{\xi \in \mathcal{C}_x} C[\xi]$  is a unique solution of (2.1) and (4.1).*

Let  $d_g(x, y)$  be a geodesic distance

$$d_g(x, y) = \inf \left\{ T_{\mathcal{X} \setminus \{x\}}^+[\xi] \mid \xi \in \mathcal{A}_y([0, \infty), \mathcal{X}) \right\} \in [0, \infty]$$

for each  $x, y \in \mathcal{X}$ . When  $\mathcal{X}$  is a Banach space equipped with a norm  $\|\cdot\|$ , this metric  $d_g$  is nothing but  $d$  defined by  $d(x, y) = \|x - y\|$ .

**Example 4.6.** We consider the boundary value problem

$$\begin{cases} |Du| = 1 & \text{in } \Omega := \mathcal{X} \setminus \{a\}, \\ u(a) = 0 \end{cases} \quad (4.6)$$

with  $a \in \mathcal{X}$ . Then the value function is  $u(x) = d_g(a, x)$  and  $u$  is a unique solution of (4.6) provided that  $\mathcal{C}_x$  is nonempty for all  $x \in \mathcal{X}$ .

*Remark 4.7.* One of sufficient conditions for (4.5) is that  $g$  is a Lipschitz continuous function on  $\partial\Omega$  with the Lipschitz constant less than or equal to the infimum of  $f$ , i.e.,  $|g(x) - g(y)| \leq (\inf_{\bar{\Omega}} f)d(x, y)$  for all  $x, y \in \partial\Omega$ .

*Remark 4.8.* The value function  $u$  is arcwise continuous in  $\Omega$ . However, it may not be continuous in general (Example 4.9). The following condition is sufficient to guarantee that  $u$  is continuous at  $a \in \Omega$ :

$$d_g(x, a) \rightarrow 0 \text{ as } x \rightarrow a \text{ in the sense that } d(x, a) \rightarrow 0. \quad (4.7)$$

$$f \text{ is bounded from above on } \{x \in \Omega \mid d_g(x, a) \leq r\} \text{ for some } r > 0. \quad (4.8)$$

Indeed, for each  $\varepsilon \in (0, r]$  we take  $\delta > 0$  such that  $d_g(x, a) < \varepsilon$  whenever  $d(x, a) < \delta$ . Since there exists  $\xi \in \mathcal{A}_a([0, \infty), \mathcal{X})$  satisfying  $\xi(\varepsilon) = x$ , we have  $|u(a) - u(x)| \leq \int_0^\varepsilon f(\xi(s))ds \leq M\varepsilon$  for some  $M < \infty$ .

**Example 4.9.** Let  $\mathcal{X} = ([0, 2] \times \{0\}) \cup (\{0\} \times [0, 1]) \cup (\bigcup_{n=1}^\infty [0, 1] \times \{1/n\}) \subset \mathbb{R}^2$  and  $d = d_E$ . We consider the boundary value problem (4.6) with  $a = (2, 0)$ . Then the value function is

$$u(x, y) = \begin{cases} 2 - x & \text{if } y = 0, \\ 2 + y & \text{if } x = 0, \\ 2 + 1/n + x & \text{if } y = 1/n. \end{cases}$$

However, this is not continuous at  $(1, 0)$ , where (4.7) does not hold.

## 5 Stability

By applying Proposition 2.6 we easily obtain stability results for subsolutions.

**Proposition 5.1.** *Let  $\Lambda$  be a nonempty index set. Assume  $f_\lambda, f \in C_a(\Omega)$  and let  $u_\lambda \in USC_a(\Omega)$  be a subsolution of (2.1) with  $f = f_\lambda$  for each  $\lambda \in \Lambda$ .*

- (1) *If  $\sup_{\lambda \in \Lambda} f_\lambda(x) \leq f(x)$  and  $\bar{u}_1(x) := \sup_{\lambda \in \Lambda} u_\lambda(x) < \infty$  for all  $x \in \Omega$ , then  $\bar{u}_1$  is a subsolution of (2.1).*
- (2) *Let  $\Lambda = \mathbb{N}$ . Assume that there exists  $g \in C_a(\Omega)$  such that  $\sup_{n \in \mathbb{N}} f_n \leq g$ . If  $\limsup_{n \rightarrow \infty} f_n(x) \leq f(x)$  and  $\bar{u}_2(x) := \limsup_{n \rightarrow \infty} u_n(x) < \infty$  for all  $x \in \Omega$ , then  $\bar{u}_2$  is a subsolution of (2.1).*

*Proof.* We only prove (2) because (1) is verified by a similar argument. Fix  $\xi \in \mathcal{A}(\mathbb{R}, \Omega)$  and  $s < t$ . By Proposition 2.6 we have

$$u_n(\xi(s)) \leq \int_s^t f_n(\xi(r)) dr + u_n(\xi(t))$$

for all  $n \in \mathbb{N}$ . Since  $\limsup_{n \rightarrow \infty} \int_s^t f_n(\xi(r)) dr \leq \int_s^t f(\xi(r)) dr$  by Fatou's lemma, we see that  $\bar{u}_2$  satisfies (2.11).  $\square$

*Remark 5.2.* In the literature ([6, Theorem A.2], [16, Proposition 1.2] see also [13, Chapter 2]) the stability is often shown in the sense of a *relaxed limit*, i.e.,

$$\bar{u}_3(x) := \lim_{n \rightarrow \infty} \sup \{u_k(y) \mid k \geq n, d_g(y, x) < 1/n\}. \quad (5.1)$$

In our situation, however, this relaxed limit  $\bar{u}_3$  is nothing but  $\bar{u}_2$  if  $\sup_{n \in \mathbb{N}} f_n$  satisfies (4.8) for all  $a \in \Omega$ . Let us check this fact. For fixed  $x \in \Omega$  and  $n \in \mathbb{N}$  we let  $k \geq n$  and  $d_g(y, x) < 1/n$ . Then we have  $u_k(y) \leq \int_0^{1/n} f_k(\xi(s)) ds + u_k(x)$  for some  $\xi \in \mathcal{A}_x([0, \infty), \Omega)$  such that  $\xi(1/n) = y$ . Since  $\int_0^{1/n} f_k(\xi(s)) ds \rightarrow 0$  as  $n \rightarrow \infty$ , we obtain  $\bar{u}_3(x) \leq \bar{u}_2(x)$ . It is clear that  $\bar{u}_3(x) \geq \bar{u}_2(x)$  by the definitions.

**Example 5.3.** If one replaces  $d_g$  by  $d$  in (5.1), i.e.,

$$\bar{u}_4(x) := \lim_{n \rightarrow \infty} \sup \{u_k(y) \mid k \geq n, d(y, x) < 1/n\}, \quad (5.2)$$

then it may happen that  $\bar{u}_4$  is not a subsolution of (2.1). For example, we consider the same setting as in Example 4.9. Set  $u_n = u$  in (5.2) and then we have  $\bar{u}_4(x, 0) = 2 + x$  for  $0 \leq x \leq 1$  and  $\bar{u}_4(x, 0) = 2 - x$  for  $1 < x \leq 2$ . Discontinuity at  $(1, 0)$  implies that  $\bar{u}_4$  is not a subsolution.

We establish a stability result for supersolutions.

**Proposition 5.4.** *Assume  $f_n, f \in C_a(\Omega)$  and let  $u_n \in LSC_a(\Omega)$  be a supersolution of (2.1) with  $f = f_n$  for each  $n \in \mathbb{N}$ . If  $\liminf_{n \rightarrow \infty} \sup_\Omega (f - f_n) \leq 0$  and  $u_n$  converges to  $u \in LSC_a(\Omega)$  uniformly in  $\Omega$ , then  $u$  is a supersolution of (2.1).*

*Proof.* Fix  $\varepsilon > 0$ . Then we have  $\sup_{\Omega} |u_N - u| < \varepsilon$  and  $\sup_{\Omega} (f - f_N) < \varepsilon$  for some  $N \in \mathbb{N}$ . For each  $x \in \Omega$  we take an  $\varepsilon$ -pair  $(\xi_N, w_N)$  for  $u_N$  and  $f_N$ .

Let  $\xi$  be  $\xi_N$  and define  $w$  as  $w(t) = w_N(t) + u(x) - u_N(x)$ . We claim that  $(\xi, w)$  is a  $3\varepsilon$ -pair at  $x$  for  $u$  and  $f$ . Indeed, for all  $t \in (T_{\Omega}^{-}[\xi], T_{\Omega}^{+}[\xi])$  we have  $w(0) = u(x)$  and  $u(\xi(t)) - 2\varepsilon \leq u_N(\xi(t)) - \varepsilon \leq w_N(t) \leq w(t) + \varepsilon$ . We also observe that  $|p| \geq f_N(\xi(t)) - \varepsilon \geq f(\xi(t)) - 2\varepsilon$  for all  $p \in D^{-}w(t) = D^{-}w_N(t)$ . Therefore,  $u$  is a supersolution.  $\square$

## 6 Consistency with Euclidean viscosity solution

In this section we investigate the consistency of our metric solutions with Euclidean solutions. Let  $\Omega$  be an open set in  $\mathcal{X} = \mathbb{R}^N$ . In this situation our absolute continuity is equivalent to conventional absolute continuity. In addition, we have  $USC_a(\Omega) = USC(\Omega)$  and  $LSC_a(\Omega) = LSC(\Omega)$ . Indeed, for each sequence of points  $x_n \in \Omega$  converging to  $x \in \Omega$ , there exists a zigzag line  $\xi$  consisting of segments  $[x_n, x_{n+1}]$  and going to  $x$ . Since we may assume that  $\sum_n |x_n - x_{n+1}| < \infty$ , the curve  $\xi$  is admissible so that the inclusion  $USC_a(\Omega) \subset USC(\Omega)$  follows. The other direction is easier.

We recall the definition of conventional viscosity solutions. We say that  $u \in USC(\Omega)$  (resp.  $u \in LSC(\Omega)$ ) is a *Euclidean viscosity subsolution* (resp. *Euclidean viscosity supersolution*) of (2.1) if the inequality  $|p| \leq f(x)$  (resp.  $|p| \geq f(x)$ ) holds for all  $x \in \Omega$  and  $p \in D^{+}u(x)$  (resp.  $p \in D^{-}u(x)$ ). We say that  $u \in C(\Omega)$  is a *Euclidean viscosity solution* if it is both a Euclidean viscosity subsolution and a Euclidean viscosity supersolution.

We first assert equivalence of a metric subsolution and a Euclidean subsolution.

**Proposition 6.1.** *Assume (2.10), i.e.,*

$$f \in C(\Omega) \text{ and } f \geq 0 \text{ in } \Omega. \quad (6.1)$$

*Let  $u \in USC(\Omega)$ . Then  $u$  is a metric viscosity subsolution of (2.1) if and only if  $u$  is a Euclidean viscosity subsolution of (2.1).*

*Proof.* Let  $u$  be a metric viscosity subsolution. Fix  $x \in \Omega$  and suppose that  $u - \varphi$  has a local maximum at  $x$  for  $\varphi \in C^1(\Omega)$ . For each unit vector  $v$  we set  $\xi(t) = x + vt$ . Since  $u \circ \xi - \varphi \circ \xi$  has a local maximum at 0, we have  $|(\varphi \circ \xi)'(0)| = |D\varphi(x) \cdot v| \leq f(x)$ . We thus obtain  $|D\varphi(x)| \leq f(x)$  by taking  $v = D\varphi(x)/|D\varphi(x)|$  if  $|D\varphi(x)| \neq 0$ . The case when  $|D\varphi(x)| = 0$  is trivial.

Let  $u$  be a Euclidean viscosity subsolution. Fix  $x \in \Omega$  and  $\xi \in \mathcal{A}_x(\mathbb{R}, \Omega)$ . Suppose that  $u \circ \xi - \varphi$  has a local maximum at 0 for  $\varphi \in C^1(\mathbb{R})$ . Then we observe that

$$\begin{aligned} -(\varphi(h) - \varphi(0))/|h| &\leq -(u(\xi(h)) - u(x))/|h| \leq |u(\xi(h)) - u(x)|/|h| \\ &\leq \begin{cases} |u(\xi(h)) - u(x)|/|\xi(h) - x| & \text{if } \xi(h) \neq x, \\ 0 & \text{if } \xi(h) = x \end{cases} \end{aligned}$$

for  $h \in \mathbb{R} \setminus \{0\}$ . Taking  $\limsup_{h \downarrow 0}$  and  $\limsup_{h \uparrow 0}$ , we conclude that  $|\varphi'(0)| \leq f(x)$  by Lemma A.2.  $\square$

We also show that a metric supersolution is a Euclidean supersolution.

**Proposition 6.2.** *Assume (6.1). If  $u \in LSC(\Omega)$  is a metric viscosity supersolution of (2.1), then  $u$  is a Euclidean viscosity supersolution of (2.1).*

*Proof.* Suppose that  $u - \varphi$  attains its local minimum at  $\hat{x}$  for  $\varphi \in C^1(\Omega)$ . We may assume that  $(u - \varphi)(\hat{x}) = 0$  and  $u - \varphi \geq |x - \hat{x}|^2$  in  $B := \{x \in \mathbb{R}^N \mid |x - \hat{x}| \leq R\} \subset \Omega$  with  $R > 0$ . For each  $\varepsilon > 0$  we take an  $\varepsilon$ -pair  $(\xi_\varepsilon, w_\varepsilon)$  at  $\hat{x}$ . Set  $T_\varepsilon^\pm = T_{B_2}^\pm[\xi_\varepsilon]$  with  $B_2 := \{x \in \mathbb{R}^N \mid |x - \hat{x}| < R/2\}$ . We mollify each component of  $\xi_\varepsilon$  by convolving the Friedrichs' mollifier. For each index  $\nu \in \mathbb{N}$  let  $\xi_\varepsilon^\nu \in C^\infty(\mathbb{R}, \mathbb{R}^n)$  be the mollification, which converges to  $\xi_\varepsilon$  uniformly on  $[T_\varepsilon^-, T_\varepsilon^+]$  as  $\nu \rightarrow \infty$ . We thus have  $|\xi_\varepsilon^\nu(t) - \xi_\varepsilon(t)| \leq R/2$  and  $|\varphi(\xi_\varepsilon^\nu(t)) - \varphi(\xi_\varepsilon(t))| \leq \varepsilon$  by uniform continuity of  $\varphi$  on  $B$  for all  $t \in [T_\varepsilon^-, T_\varepsilon^+]$  and sufficiently large  $\nu$ . We also remark that  $|(\xi_\varepsilon^\nu)'| \leq 1$  since  $|\xi_\varepsilon'| \leq 1$  a.e. and  $(\xi_\varepsilon^\nu)'$  is nothing but the mollification of  $\xi_\varepsilon'$ .

Take a minimum point  $t_\varepsilon^\nu$  of  $w_\varepsilon - \varphi \circ \xi_\varepsilon^\nu$  on  $[T_\varepsilon^-, T_\varepsilon^+]$ . Then we have

$$\begin{aligned} \varepsilon &= (u - \varphi)(\hat{x}) + \varepsilon \geq (w_\varepsilon - \varphi \circ \xi_\varepsilon^\nu)(0) \geq (w_\varepsilon - \varphi \circ \xi_\varepsilon^\nu)(t_\varepsilon^\nu) \\ &\geq (u - \varphi)(\xi_\varepsilon(t_\varepsilon^\nu)) - 2\varepsilon, \end{aligned}$$

which implies  $|\xi_\varepsilon(t_\varepsilon^\nu) - \hat{x}|^2 \leq 3\varepsilon$ . Noting that  $|\xi_\varepsilon(T_\varepsilon^\pm) - \hat{x}| = R/2$ , we see that  $t_\varepsilon^\nu \in (T_\varepsilon^-, T_\varepsilon^+)$  for sufficiently small  $\varepsilon$ . Since  $(\xi_\varepsilon, w_\varepsilon)$  is an  $\varepsilon$ -pair, we have

$$f(\xi_\varepsilon(t_\varepsilon^\nu)) - \varepsilon \leq |(\varphi \circ \xi_\varepsilon^\nu)'(t_\varepsilon^\nu)| \leq |D\varphi(\xi_\varepsilon^\nu(t_\varepsilon^\nu))|.$$

We may assume that  $t_\varepsilon^\nu \rightarrow t_\varepsilon \in [T_\varepsilon^-, T_\varepsilon^+]$  as  $\nu \rightarrow \infty$  by choosing a subsequence. Then we obtain

$$|\xi_\varepsilon(t_\varepsilon) - \hat{x}|^2 \leq 3\varepsilon, \quad |D\varphi(\xi_\varepsilon(t_\varepsilon))| \geq f(\xi_\varepsilon(t_\varepsilon)) - \varepsilon.$$

We thus have  $\xi_\varepsilon(t_\varepsilon) \rightarrow \hat{x}$  as  $\varepsilon \rightarrow 0$  and hence  $|D\varphi(\hat{x})| \geq f(\hat{x})$ .  $\square$

As we observed in Example 2.4, a Euclidean supersolution is not necessarily a metric supersolution. We give a sufficient condition that a Euclidean solution is indeed a metric solution.

**Proposition 6.3.** *Assume (4.2) and (4.4). Assume that  $f \in C(\overline{\Omega})$  and  $f \geq 0$  in  $\overline{\Omega}$ . If  $u \in C(\overline{\Omega})$  is a unique Euclidean viscosity solution of (2.1) and (4.1), then  $u$  is a metric viscosity solution of (2.1) and (4.1).*

*Proof.* Let  $\tilde{u}$  be the value function, i.e.,  $\tilde{u}(x) := \inf_{\xi \in \mathcal{C}_x} C[\xi]$  for  $x \in \overline{\Omega}$ . Theorem 4.2 implies that  $\tilde{u} \in C_a(\Omega)$  and this is a metric solution of (2.1). Thus  $\tilde{u}$  is also a Euclidean solution of (2.1) by Proposition 6.1 and 6.2. If we prove  $\tilde{u} \in C(\overline{\Omega})$ , the conclusion follows since we are able to conclude that  $u = \tilde{u}$  by uniqueness. Since  $\tilde{u} \in C_a(\Omega) = C(\Omega)$ , it is sufficient to show  $\lim_{x \rightarrow z, x \in \overline{\Omega}} \tilde{u}(x) = g(z)$  for all  $z \in \partial\Omega$ .

We first show  $u(x) \leq \tilde{u}(x)$  for all  $x \in \Omega$ . Since  $u$  is a metric subsolution by Proposition 6.1, we observe that  $u$  satisfies (2.11) for all  $\xi \in \mathcal{C}_x$  and  $s, t \in [0, T_\Omega^+(\xi)]$ ,  $s < t$ . By the continuity of  $u$  up to the boundary we have  $u(x) \leq C[\xi]$  so that  $u(x) \leq \tilde{u}(x)$ . Hence we obtain  $\liminf_{x \rightarrow z} \tilde{u}(x) \geq g(z)$ .

We next let  $\xi_x(t) = x + (z - x)t/|z - x|$  for  $x \in \bar{\Omega} \setminus \{z\}$ . Then we see that

$$\tilde{u}(x) \leq \int_0^{T_x} f(\xi_x(s)) ds + g(\xi_x(T_x)),$$

where  $T_x = T_\Omega^+(\xi_x)$ . Since  $T_x \rightarrow 0$  and  $\xi_x(T_x) \rightarrow z$  as  $x \rightarrow z$ , we obtain  $\limsup_{x \rightarrow z} \tilde{u}(x) \leq g(z)$ .  $\square$

We prepare Lemma 6.4 to remove the assumption of the continuity of  $u$  on the boundary in Proposition 6.3.

**Lemma 6.4.** *Let  $\Omega$  be an open set in a metric space  $\mathcal{X}$ . Let  $\Omega_n$  be an open subset of  $\Omega$  for each  $n \in \mathbb{N}$  such that  $\sup_{z \in \partial\Omega_n} d_g(\partial\Omega, z) \rightarrow 0$  as  $n \rightarrow \infty$ , where  $d_g(\partial\Omega, z) = \inf_{y \in \partial\Omega} d_g(y, z)$ . Assume (2.10) and that  $f$  is bounded. If an arcwise uniformly continuous function  $u$  defined on  $\Omega$  is a metric viscosity supersolution of  $|Du| = f(x)$  in  $\Omega_n$  for all  $n \in \mathbb{N}$ , then  $u$  is a metric viscosity supersolution in  $\Omega$ .*

Here we say that a function  $u$  defined on  $\Omega$  is *arcwise uniformly continuous* if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$|u(\xi(s)) - u(\xi(t))| < \varepsilon \text{ for all } \xi \in \mathcal{A}(I, \Omega) \text{ and } s, t \in I, |s - t| < \delta. \quad (6.2)$$

*Proof.* Fix  $x \in \Omega$  and  $\varepsilon > 0$ . Take  $\delta > 0$  satisfying (6.2) and  $\delta \sup f < \varepsilon$ . We have  $x \in \Omega_n$  and  $d_g(\partial\Omega, z) < \delta$  for all  $z \in \partial\Omega_n$  by choosing  $n \in \mathbb{N}$  large enough. Take an  $\varepsilon$ -pair  $(\xi, w)$  at  $x$  for the supersolution  $u$  in  $\Omega_n$ . Let  $T^\pm = T_{\Omega_n}^\pm[\xi]$ . Since  $d_g(\partial\Omega, \xi(T^\pm)) < \delta$ , it is possible to construct  $\tilde{\xi} \in \mathcal{A}(\mathbb{R}, \mathcal{X})$  satisfying

$$\tilde{\xi} = \xi \text{ on } [T^-, T^+] \text{ and } [\tilde{T}^-, \tilde{T}^+] \subset [T^- - \delta, T^+ + \delta]$$

with  $\tilde{T}^\pm := T_\Omega^\pm[\tilde{\xi}]$ . Define  $\tilde{w} \in LSC(\tilde{T}^-, \tilde{T}^+)$  as

$$\tilde{w}(t) = \begin{cases} w(t) & \text{for } T^- < t < T^+, \\ u(\xi(T^+)) - \varepsilon - \int_{T^+}^t f(\tilde{\xi}(s)) ds & \text{for } T^+ \leq t < \tilde{T}^+, \\ u(\xi(T^-)) - \varepsilon - \int_t^{T^-} f(\tilde{\xi}(s)) ds & \text{for } \tilde{T}^- < t \leq T^-. \end{cases}$$

Then we have

$$u(\tilde{\xi}(t)) - 3\varepsilon \leq u(\tilde{\xi}(T^+)) - 2\varepsilon = \tilde{w}(t) + \int_{T^+}^t f(\tilde{\xi}(s)) ds - \varepsilon \leq \tilde{w}(t)$$

for all  $T^+ \leq t < \tilde{T}^+$ . We also observe that  $\tilde{w}'(t) = -f(\tilde{\xi}(t))$  for all  $T^+ < t < \tilde{T}^+$  and that  $p \leq -f(\tilde{\xi}(T^+))$  if  $p \in D^- \tilde{w}(T^+)$ . We have a similar result for  $\tilde{T}^- < t \leq T^-$ . Therefore,  $(\tilde{\xi}, \tilde{w})$  is a  $3\varepsilon$ -pair for  $u$  in  $\Omega$ .  $\square$



To simplify assumptions for uniqueness we restrict ourselves to the case when (3.2) holds so that Ishii's comparison result [15] applies to  $|Du| = f(x)$  in  $U$  for every open subset  $U$  of  $\Omega$ .

**Proposition 6.5.** *Assume (3.2). If  $u \in C(\Omega)$  is a Euclidean viscosity solution of (2.1), then  $u$  is a metric viscosity solution.*

*Proof.* Let  $\Omega_n = \{x \in \Omega \mid \inf_{y \in \partial\Omega} |y - x| > 1/n\}$  for each  $n \in \mathbb{N}$ . Since  $u \in C(\overline{\Omega_n})$ , Proposition 6.3 yields that  $u$  is a metric solution in  $\Omega_n$ . In addition, since  $u$  is a metric subsolution in  $\Omega$  by Proposition 6.1, we see that  $u$  is arcwise uniformly continuous by (2.12) and the boundedness of  $f$ . We now apply Lemma 6.4 to conclude that  $u$  is a metric solution of (2.1).  $\square$

## A Results on Euclidean viscosity solutions

In this section we gather some results for Euclidean viscosity solutions used in this paper.

**Lemma A.1.** *Let  $I$  be an open interval of  $\mathbb{R}$  and let  $w \in USC(I)$ . Then  $w$  is a Euclidean viscosity subsolution of  $w'(t) = 0$  in  $I$ , i.e.,  $p \leq 0$  for all  $t \in I$  and  $p \in D^+w(t)$  if and only if  $w$  is nonincreasing in  $I$ .*

This lemma is more or less known with extra assumptions; see, e.g., [5, Lemma II.5.15]. Since their argument requires  $w \in C(I)$ , we give a different proof.

*Proof.* It is easy to show that  $w$  is a subsolution if  $w$  is nonincreasing. Assume that  $w$  is a subsolution. We suppose that  $w(a) < w(b)$  for some  $a, b \in I$ ,  $a < b$ . Take  $c \in I$  satisfying  $b < c$ . Define  $\varphi \in C^1(I)$  as

$$\varphi(t) = \begin{cases} C(t - a) + w(a) & \text{if } t \leq b, \\ C(t - a) + w(a) + k(t - b)^2 & \text{if } t \geq b \end{cases}$$

with  $C = (w(b) - w(a))/(b - a) > 0$  and  $k > 0$ . Obviously,  $w(a) = \varphi(a)$  and  $w(b) = \varphi(b)$ . We also have  $w(c) \leq \varphi(c)$  by taking  $k$  large enough. Hence  $w - \varphi \in USC(I)$  attains its maximum on  $[a, c]$  at some  $\hat{t} \in (a, c)$ . We have  $\varphi'(\hat{t}) \leq 0$  since  $w$  is a subsolution while  $\varphi'(\hat{t}) \geq C > 0$  by the definition of  $\varphi$ . This is a contradiction.  $\square$

**Lemma A.2.** *Let  $\Omega$  be an open set in  $\mathbb{R}^N$  and assume (6.1). If  $u \in USC(\Omega)$  is a Euclidean viscosity subsolution of (2.1), then the inequality*

$$\limsup_{y \rightarrow x} \frac{|u(y) - u(x)|}{|y - x|} \leq f(x) \tag{A.1}$$

*holds for all  $x \in \Omega$ .*

*Proof.* Let  $B_r = \{y \in \mathbb{R}^N \mid |y - x| < r\} \subset \Omega$  and  $M_r = \max_{\overline{B_r}} f$  for  $r > 0$ . Since  $u$  is a subsolution of (2.1), it is also a subsolution of  $|Du| = M_r$  in  $B_r$ .

We show

$$u(b) - u(a) \leq M_r |b - a| \tag{A.2}$$

for all  $a, b \in B_{r/4}$ . Fix  $a, b \in B_{r/4}$  such that  $u(b) > u(a)$ . Define  $\varphi \in C(\mathbb{R}^N)$  as

$$\varphi(y) = \begin{cases} C|y - a| + u(a) & \text{if } |y - a| \leq r/2, \\ C|y - a| + u(a) + k(|y - a| - r/2)^2 & \text{if } |y - a| \geq r/2 \end{cases}$$

with  $C = (u(b) - u(a))/|b - a| > 0$  and  $k > 0$ . Note that  $u(a) = \varphi(a)$ ,  $u(b) = \varphi(b)$ , and  $|D\varphi| \geq C = (u(b) - u(a))/|b - a|$  in  $\mathbb{R}^N \setminus \{a\}$ . We also have  $\max_{\partial B_r} (u - \varphi) \leq 0$  for sufficiently large  $k$ . Hence  $u - \varphi \in USC(\overline{B_r})$  attains its maximum at some  $\hat{x} \in B_r \setminus \{a\}$ . We then have  $|D\varphi(\hat{x})| \leq M_r$  since  $u$  is a subsolution. Thus (A.2) holds.

We now have  $|u(y) - u(x)|/|y - x| \leq M_r$  for all  $y \in B_{r/4} \setminus \{x\}$ . Taking  $\limsup_{y \rightarrow x}$  and  $r \rightarrow 0$ , we obtain (A.1).  $\square$

## Acknowledgments

The work of the first author was partly supported by a Grant-in-Aid for Scientific Research (S) No.21224001, Japan Society for the Promotion of Science (JSPS). The work of the second author was supported by a Grant-in-Aid for JSPS Fellows No. 23-4365.

## References

- [1] Y. Achdou, F. Camilli, A. Cutri, and N. Tchou, *Hamilton-Jacobi equations on networks*, preprint, available at <http://hal.archives-ouvertes.fr/hal-00503910/>.
- [2] H. Amann, *Function Spaces on Singular Manifolds*, Math. Nachr., to appear, available at <http://arxiv.org/abs/1106.2033>.
- [3] L. Ambrosio, *Minimizing movements*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) **19** (1995), 191–246. MR 1387558 (97c:49044)
- [4] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, second ed., Birkhäuser Verlag, Basel, 2008. MR 2401600 (2009h:49002)
- [5] M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser Boston Inc., Boston, MA, 1997. MR 1484411 (99e:49001)
- [6] G. Barles and B. Perthame, *Discontinuous solutions of deterministic optimal stopping time problems*, RAIRO Modél. Math. Anal. Numér. **21** (1987), no. 4, 557–579. MR 921827 (88k:49032)

- [7] F. Camilli, A. Festa, and D. Schieborn, *Shortest paths and Eikonal equations on a graph*, preprint, available at <http://arxiv.org/abs/1105.5725>.
- [8] F. Camilli and D. Schieborn, *Viscosity solutions of Eikonal equations on topological networks*, *Calc. Var. Partial Differential Equations*, to appear, available at <http://arxiv.org/abs/1103.4041>.
- [9] M. G. Crandall and P.-L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, *Trans. Amer. Math. Soc.* **277** (1983), no. 1, 1–42. MR 690039 (85g:35029)
- [10] ———, *Hamilton-Jacobi equations in infinite dimensions. I. Uniqueness of viscosity solutions*, *J. Funct. Anal.* **62** (1985), no. 3, 379–396. MR 794776 (86j:35154)
- [11] J. Feng and M. Katsoulakis, *A comparison principle for Hamilton-Jacobi equations related to controlled gradient flows in infinite dimensions*, *Arch. Ration. Mech. Anal.* **192** (2009), no. 2, 275–310. MR 2486597 (2010e:35048)
- [12] W. Gangbo, T. Nguyen, and A. Tudorascu, *Hamilton-Jacobi equations in the Wasserstein space*, *Methods Appl. Anal.* **15** (2008), no. 2, 155–183. MR 2481677 (2010f:49061)
- [13] Y. Giga, *Surface evolution equations*, Birkhäuser Verlag, Basel, 2006. MR 2238463 (2007j:53071)
- [14] C. Imbert, R. Monneau, and H. Zidani, *A Hamilton-Jacobi approach to junction problems and application to traffic flows*, preprint, available at <http://arxiv.org/abs/1107.3250>.
- [15] H. Ishii, *A simple, direct proof of uniqueness for solutions of the Hamilton-Jacobi equations of eikonal type*, *Proc. Amer. Math. Soc.* **100** (1987), no. 2, 247–251. MR 884461 (88d:35040)
- [16] ———, *A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations*, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **16** (1989), no. 1, 105–135. MR 1056130 (91f:35071)