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Solution formula for the vorticity equations in the half plane with application to high vorticity creation at zero viscosity limit

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Abstract

We consider the Navier-Stokes equations for viscous incompressible flows in the half plane under the no-slip boundary condition. In this paper we first establish a solution formula for the vorticity equations through the appropriate vorticity formulation. The formula is then applied to establish the asymptotic expansion of vorticity fields at $t = 0$ that holds at least up to the time $c_1/\nu$, where $\nu$ is the viscosity coefficient and $c_1$ is a constant. As a consequence, we get a natural sufficient condition on the initial data for the vorticity to blow up in the inviscid limit, together with explicit estimates.

Keywords  Navier-Stokes equations; Vorticity equations; No-slip boundary conditions; Inviscid limit

2010 Mathematics Subject Classification  35Q30; 76D05; 76D10

1 Introduction

In this paper we consider the two-dimensional Navier-Stokes equations for viscous incompressible flows under the no-slip boundary conditions:

\[
\begin{cases}
\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p = 0 & t > 0, \quad x \in \Omega, \\
\text{div } u = 0 & t \geq 0, \quad x \in \Omega, \\
u = 0 & t \geq 0, \quad x \in \partial \Omega, \\
\left. u \right|_{t=0} = a & x \in \Omega.
\end{cases}
\]  

(NS)

Here $\Omega$ is a domain with smooth boundary $\partial \Omega$, $u = u(t, x) = (u_1(t, x), u_2(t, x))$ and $p = p(t, x)$ denote the velocity field and the pressure field, and $\nu > 0$ is the viscosity coefficient. We will use the standard notations for derivatives; $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\Delta = \sum_{j=1}^2 \partial_j^2$, $\text{div } u = \sum_{j=1}^2 \partial_j u_j$, and $u \cdot \nabla u = \sum_{j=1}^2 u_j \partial_j u$. In this paper we mostly deal with the case when $\Omega$ is the half plane (Sections 3, 4), but in Section 2 the case of bounded domains is also discussed.

The system (NS) has been studied quite extensively in various settings. In particular, it is well known that (NS) admits a unique smooth solution, for example, in the energy class; see the books [44, 47] and references there in. When $\Omega = \mathbb{R}^2$ the alternative approach using vorticity fields is also useful and has been well developed by now. Here the vorticity $\omega$ of the velocity $u$ is defined by $\omega = \text{Rot } u := \partial_1 u_2 - \partial_2 u_1$, and the equation for $\omega$ is then formally obtained by acting the Rot operator on the first equation of (NS):

\[
\partial_t \omega - \nu \Delta \omega + u \cdot \nabla \omega = 0 \quad t > 0, \quad x \in \Omega.
\]  

When $\Omega = \mathbb{R}^2$ the vorticity equation (1.1) ensures the uniform bound of vorticity fields by the maximum principle, which is essentially used to show the global existence of smooth solutions to (NS) in the infinite energy class [20, 5, 24, 18, 16]; see also [4, 29]. However, in the presence of boundaries, a serious difficulty arises in the study of vorticity fields. Indeed, under the no-slip boundary condition on velocity fields the
The function $\omega$ describing the boundary layer up to the time $c$ is naturally expected from the boundary layer theory. Nevertheless, to the best of the author’s knowledge, this is difficult to gain further insight from such contradiction arguments. If we do not ask for the concrete estimates such as (1.3) or (1.6), but it will be see Corollary 4.2 for details. We note that, when (1.4) holds, the vorticity creation itself may be proved by the following asymptotic expansion holds at $t \to 0$ near the initial time:
\[
\omega(t) \sim \omega_E(t) + \omega_{BL}(t) \quad \text{for} \quad 0 < t \leq c_0 \nu^2.
\]
Here $\omega_E$ is the vorticity field for the solution to the Euler equations with the initial velocity $a$, $\omega_{BL}$ is the function describing the boundary layer up to the time $c_0 \nu^{1/3}$, and $c_0$ is a constant independent of $0 < \nu \ll 1$. The function $\omega_{BL}$ is written rather explicitly in terms of the initial data (see (4.3)), and it is a nontrivial function if and only if
\[
\partial_2(-\Delta_D)^{-1}(a \cdot \nabla \text{Rot} a) \neq 0 \quad \text{on} \quad \partial\mathbb{R}^2_+.
\]
When (1.4) holds $\omega_{BL}$ will be shown to satisfy
\[
ev^{-\frac{1}{2}(1-\frac{1}{p})t^{\frac{1}{2}}(1+\frac{1}{p})} \leq \|\omega_{BL}(t)\|_{L^p(\Omega_{\nu t})} \leq \|\omega_{BL}(t)\|_{L^p} \leq C\nu^{-\frac{1}{2}(1-\frac{1}{p})t^{\frac{1}{2}}(1+\frac{1}{p})}
\]
for all $\nu$, $t > 0$ and $1 \leq p < \infty$, where $\Omega_{\nu t} = \{x \in \mathbb{R}^2_+ : 0 < x_2 \leq (\nu t)^{1/2}\}$ is the region of the boundary layer. In particular, (1.3) and (1.5) imply the high creation of vorticity near the boundary in $L^p$ for $2 < p \leq \infty$ as follows:
\[
\|\omega(0)\|_{L^p(0 \leq x_2 \leq \frac{1}{2}\nu^{1/2})} \geq \nu^{-\frac{1}{2}(1-\frac{1}{p})} \to \infty \quad (\nu \to 0) \quad \text{if} \quad 2 < p \leq \infty,
\]
see Corollary 4.2 for details. We note that, when (1.4) holds, the vorticity creation itself may be proved by contradiction arguments if we do not ask for the concrete estimates such as (1.3) or (1.6). But it will be difficult to gain further insight from such contradiction arguments.

The high creation of vorticity at the zero viscosity limit, which arises due to the nonlinearity of (1.1)-(1.2), is naturally expected from the boundary layer theory. Nevertheless, to the best of the author’s knowledge, this
vorticity creation occurs near the initial time. In such situations it is natural that the boundary layer immediately appears and thus the high vorticity formulation represents the nondegenerate condition for $u$ propagator of the linear vorticity equations which are used in the previous sections. For simplicity, in this section we deal with the case of multi-connected bounded domains and formulate the initial boundary value problems of the vorticity equations by deriving the boundary conditions on vorticity fields. In Section 3 we establish a solution formula to the modified Prandtl equations. Hence, the results of [40, 41] imply the high vorticity creation in $L^p_{loc}$ for any $p > 1$, but under the regularity condition of analyticity on initial data.

In this paper the expansion (1.3) is proved just under the assumptions of some Sobolev regularity on initial data. Furthermore, $\omega_{BL}$ has a simple representation and at least up to the time $c_0 \nu^{1/3}$ we do not need the approximation using the Prandtl-type equations in the boundary layer. This observation of the order $\nu^{1/3}$ is newly obtained by the present paper, though the author does not know if the power $1/3$ can be improved for general initial data in a Sobolev class. Note that we cannot expect the expansion (1.3) up to the time $O(1)$ in general, because the function $\omega_{BL}$ in Theorem 4.1 does not take into account the nonlinear interaction in the boundary layer region.

The condition (1.4) is necessary and sufficient for the vorticity to exhibit an unbounded growth at $T_\nu = c_0 \nu^{1/3}$ as $\nu \to 0$. The meaning of (1.4) is explained as follows. If we recall the Biot-Savart law in $\mathbb{R}^2$ and the vorticity equations associated with the Euler equations, (1.4) asserts nothing but $\partial_t u_{E,1} \big|_{t=0} \neq 0$ on $\partial \mathbb{R}^2_+$. Here $u_E = (u_{E,1}, u_{E,2})$ denotes the solution to the Euler equations with the initial data $u$. Hence (1.4) represents the nondegenerate condition for $u_E$ to be a nonzero velocity field on the boundary right after the initial time. In such situations it is natural that the boundary layer immediately appears and thus the high vorticity creation occurs near the initial time.

In Theorem 4.1 it is also proved that the $L^\infty$ norm of $u$ is uniformly bounded in $0 < \nu \ll 1$ for the time period $(0, c_0 \nu^{1/3})$. In fact, if one does not use the vorticity equations it might be difficult to obtain this uniform bound rigorously for such a “long” time for general initial data. This is one of the advantages of the approach to (NS) from the vorticity formulation.

As a final remark of this section, the boundary conditions on vorticity fields can be derived also for the three-dimensional flows from the similar spirit as in Section 2, and consequently, a solution formula to the three-dimensional vorticity equations is obtained in the case of the half space. However, their representations become more complicated, for the vorticity fields have three components in three-dimensional flows and they interact with each other intricately. The details of this issue will be studied in another paper.

The rest of this paper is organized as follows. In Section 2 we extend the argument in [1] to the case of multi-connected bounded domains and formulate the initial boundary value problems of the vorticity equations by deriving the boundary conditions on vorticity fields. In Section 3 we establish a solution formula for the vorticity equations in the half plane; see Theorem 3.1 and Corollary 3.3. The $L^p - L^q$ estimates for the associated propagator are given in Lemma 3.4. In Section 4 we study the behavior of vorticity fields at the zero viscosity limit and give the asymptotic expansion near the initial time in Theorem 4.1. For the proof we need to introduce a suitable decomposition of the vorticity field and each term has to be estimated carefully. This step requires rather involved calculations. In order to tidy up the arguments and computations we will set out several subsections. Finally in appendices we prove some results on the propagator of the linear vorticity equations which are used in the previous sections.

## 2 Vorticity formulation

In this section we derive an equivalent formulation to (NS) based on vorticity fields. The vorticity formulation to (NS) itself has a long history, and has been studied mostly from the numerical point of view. The key idea for the derivation of the formulation in this section is seen in [1]: the reader is also referred to [39] for another vorticity formulation. For simplicity, in this section we deal with the case of multi-connected bounded domains only. But under suitable spatial decay conditions on velocity fields it is easy to see that
the similar argument works also for \( \Omega = \mathbb{R}^2_+ \) (half plane) or \( \Omega = \mathbb{R}^2 \setminus \Omega_{b dd} \) (exterior domain), where \( \Omega_{b dd} \) is a simply-connected bounded domain.

The assumptions on \( \Omega \) are stated as follows: \( \Omega \) is a bounded domain and \( \partial \Omega \) has connected components \( \Gamma_0, \Gamma_1, \ldots, \Gamma_L \) which are disjoint \( C^\infty \) closed curves, and each \( \Gamma_i, \ 1 \leq i \leq L, \) lies in \( \Omega_0 \), where \( \Omega_0 \) is a simply-connected bounded domain with \( \partial \Omega_0 = \Gamma_0 \).

Before stating the results, let us introduce some function spaces. \( C^\infty_0(\Omega) \) is the set of smooth functions with compact support in \( \Omega \); \( W^{l,p}_0(\Omega), l \in \mathbb{N}, 1 \leq p \leq \infty, \) is the closure of \( C^\infty_0(\Omega) \) with respect to the norm of the Sobolev space \( W^{l,p}(\Omega) \); \( C^\infty_{h \sigma}(\Omega) \) denotes the set of all \( C^\infty \)-vector functions \( \mathbf{u} = (u_1, u_2) \) with compact support in \( \Omega \) such that \( \text{div} \ \mathbf{u} = 0 \); \( L^2_0(\Omega) \) is the closure of \( C^\infty_{0,\sigma}(\Omega) \) with respect to the norm in \((L^p(\Omega))^2\).

### 2.1 Helmholtz-Weyl decomposition

Hereafter \( n \) denotes the outward unit normal to \( \partial \Omega \). The boundary conditions on vorticity fields are closely related with the Helmholtz-Weyl decomposition of vector fields. In particular, we need a decomposition of tangential flows (i.e., vector fields \( \mathbf{u} \) such that \( \text{div} \ \mathbf{u} = 0 \) in \( \Omega \) and \( \mathbf{u} \cdot n = 0 \) on \( \partial \Omega \)) into the irrotational ones and the rotational ones (see [22, 31] and references therein), which is a refinement of the classical decomposition theorem of vector fields [52, 46, 15, 43]. For convenience of reference we follow the notations in [31, Theorem 3.20]. Set

\[
\quad H_{h \sigma}(\Omega) = \{ h \in C^\infty(\Omega) \mid \text{div} \ h = \text{Rot} \ h = 0 \ \text{in} \ \Omega, \ n \cdot h = 0 \ \text{on} \ \partial \Omega \}. \tag{2.1}
\]

Here \( \text{Rot} \ h = \partial_1 h_2 - \partial_2 h_1 \).

**Theorem 2.1** (i) The dimension of \( H_{h \sigma}(\Omega) \) is \( L \) and a basis \( \{ \varphi_1, \ldots, \varphi_L \} \) of \( H_{h \sigma}(\Omega) \) is given by

\[
\varphi_j = \nabla^\perp q_j, \quad 1 \leq j \leq L, \tag{2.2}
\]

where \( \nabla^\perp = (\partial_2, -\partial_1) \) and \( q_j \) is the solution to the Dirichlet boundary value problem

\[
\begin{aligned}
\Delta q_j &= 0, & \text{in} \ \Omega, \\
q_j &= \delta_{ij}, & \text{on} \ \Gamma_i, \quad 0 \leq i \leq L.
\end{aligned} \tag{2.3}
\]

(ii) For any \( u \in L^2(\Omega)^2 \) there exist \( h \in H_{h \sigma}(\Omega), \ \psi \in W^{1,2}_0(\Omega), \ \text{and} \ p \in W^{1,2}(\Omega) \) such that

\[
u = h + \nabla^\perp \psi + \nabla p. \tag{2.4}
\]

This decomposition is unique, and \( h \) and \( \psi \) are given by

\[
h = \sum_{j=1}^L (u, \nabla^\perp \tilde{q}_j)_{L^2} \nabla^\perp \tilde{q}_j, \quad \tilde{q}_j = c_j q_j, \quad \psi = (-\Delta_{D})^{-1} \text{Rot} \ u, \tag{2.5}
\]

where \( g = (-\Delta_{D})^{-1} f \) denotes the solution to the Dirichlet boundary value problem: \( -\Delta g = f \) in \( \Omega, \ g = 0 \) on \( \partial \Omega \). Each \( c_j \) is a positive constant which normalizes the norm of \( \|c_j \nabla q_j\|_{L^2} \).

**Proof.** See, for example, [31, Theorem 3.20].

As a corollary of Theorem 2.1, we get

**Corollary 2.2** Let \( u \in L^2_0(\Omega) \cap W^{1,2}(\Omega)^2 \). Then \( u \in (W^{1,2}_0(\Omega))^2 \) if and only if

\[
\partial_h (-\Delta_{D})^{-1} \text{Rot} \ u + \sum_{j=1}^L (u, \nabla^\perp \tilde{q}_j)_{L^2} \partial_h \tilde{q}_j = 0 \quad \text{on} \ \partial \Omega. \tag{2.6}
\]
In this case we have

\[ u = (-\Delta_D)^{-1}\nabla^2 \omega = \nabla^2 (-\Delta_D)^{-1}\omega + \sum_{j=1}^{L} \left((-\Delta_D)^{-1}\nabla^2 \omega, \nabla^2 q_j \right)_{L^2} \nabla^2 q_j. \]  

(2.7)

Here \( \omega = \text{Rot } u \).

We note that (2.7) gives the Biot-Savart law in the multi-connected bounded domain; the velocity field \( u \) is recovered from the vorticity field \( \omega \) through the formula (2.7), if \( \omega \) satisfies the integral condition

\[ \partial_t (-\Delta_D)^{-1}\omega + \sum_{j=1}^{L} \left((-\Delta_D)^{-1}\nabla^2 \omega, \nabla^2 q_j \right)_{L^2} \partial_n q_j = 0 \quad \text{on } \partial\Omega. \]  

(2.8)

\[ \text{Vorticity equations} \]

Eq. (2.8) gives the integral condition on the vorticity field by which the associated velocity field satisfies the no-slip boundary condition. But it is not so useful for the analysis of vorticity fields even when the topology of the domain is quite simple, for (2.8) is highly non-local. Following [1], we do not deal with (2.8) directly, but instead, we consider the boundary conditions so that (2.8) is preserved under the vorticity equations. Based on this idea we obtain the boundary conditions on vorticity fields which are more local than (2.8).

As is pointed out by [13], these boundary conditions are not necessarily a drastic prescription to overcome difficulties in the numerical analysis. However, as will be seen in Sections 3, 4, they indeed provide a useful information for the mathematical analysis of vorticity fields.

Let \( a \in L^2_t(\Omega) \cap (W^{1,2}_0(\Omega))^2 \). We consider the equation

\[
\begin{cases}
\partial_t \omega - \nu \Delta \omega + u \cdot \nabla \omega = 0 \\
u \left((-\Delta_D)^{-1}\partial_t \omega + \sum_{j=1}^{L} \left((-\Delta_D)^{-1}\nabla^2 \omega, \nabla^2 q_j \right)_{L^2} \partial_n q_j \right) + \sum_{j=1}^{L} \left(\omega u, \nabla^2 q_j \right)_{L^2} \partial_n q_j = 0
\end{cases}
\]

for \( t > 0 \) and \( x \in \partial\Omega \). Here \( q_j \) is the function in Theorem 2.1 and \( \Lambda_{DN} \) is the Dirichlet-Neumann map defined by \( \Lambda_{DN} \omega = \partial_n \omega_{har} \), where \( \omega_{har} \) is the solution to the Dirichlet boundary value problem: \( \Delta \omega_{har} = 0 \) in \( \Omega \), \( \omega_{har} = \omega \) on \( \partial\Omega \). In the next theorem (V)-(BC) is shown to be equivalent with (NS). For simplicity we assume that the solution (and the initial data) is smooth enough to ensure the formal calculations.

**Theorem 2.3** The equation (NS) is equivalent with (V)-(BC) in the sense that if \((u, p)\) is a smooth solution to (NS) then \( \omega = \text{Rot } u \) solves (V)-(BC), and conversely, if \( \omega \) is a smooth solution to (V)-(BC) then \( u \) defined by (V) solves (NS) for some \( p \).

**Proof.** For simplicity of notations we write \( q_j \) for \( \tilde{q}_j \). Let \( \omega \) be a smooth function satisfying (V). Set \( u \times \omega = (\omega u_2, -\omega u_1) \) and

\[ \tilde{u} = \nabla^2 (-\Delta_D)^{-1}\omega + \sum_{j=1}^{L} \left(\alpha - \nu \int_0^t \nabla^2 \omega \, ds + \int_0^t u \times \omega \, ds, \nabla^2 q_j \right)_{L^2} \nabla^2 q_j \]  

(2.9)

Then it is easy to see that \( \tilde{u}|_{t=0} = a \), \( \text{div } \tilde{u} = 0 \) and \( \text{Rot } \tilde{u} = \omega \) for \( t \geq 0 \), \( x \in \Omega \), and \( u \cdot \tilde{u} = 0 \) for \( t \geq 0 \),
\(x \in \partial \Omega\). Let \(\tau = (n_2, -n_1)\) where \(n = (n_1, n_2)\) is the outward unit normal to \(\partial \Omega\). Then we have

\[
\partial_t \tau \cdot \tilde{u} = \partial_n (\Delta_D)^{-1} \partial_\omega + \sum_{j=1}^{L} (-\nu \nabla^\perp \omega + u \times \omega, \nabla^\perp q_j)_{L^2} \partial_n q_j
\]

\[
= \partial_n (\Delta_D)^{-1} \left( \Delta (\omega - \omega_{har}) - u \cdot \nabla \omega \right) + \sum_{j=1}^{L} \left( -\nu \nabla^\perp \omega + u \times \omega, \nabla^\perp q_j \right)_{L^2} \partial_n q_j
\]

\[
= -\partial_n \omega + \partial_n \omega_{har} - \partial_n (\Delta_D)^{-1} (u \cdot \nabla \omega) + \sum_{j=1}^{L} \left( -\nu \nabla^\perp \omega + u \times \omega, \nabla^\perp q_j \right)_{L^2} \partial_n q_j.
\]

Thus \(\omega\) satisfies (BC) if and only if \((\tau \cdot \tilde{u})(t, x) = (\tau \cdot \tilde{u})(0, x) = 0\) for all \(t > 0\) and \(x \in \partial \Omega\).

Assume that \(\omega\) is a smooth solution to (V)-(BC). Then \((\tau \cdot \tilde{u})(t, x) = 0\) for \(t \geq 0, x \in \partial \Omega\) by the above argument, which implies \(\tilde{u}(t) = 0\) on \(\partial \Omega\). In particular, we have from Corollary 2.2 and (V),

\[
\tilde{u} = (\Delta_D)^{-1} \nabla^\perp \omega = \nabla^\perp (\Delta_D)^{-1} \omega + \sum_{j=1}^{L} \left( (\Delta_D)^{-1} \nabla^\perp \omega, \nabla^\perp q_j \right)_{L^2} \nabla^\perp q_j = u.
\]

Hence, \(u\) satisfies \(u(t) = 0\) on \(\partial \Omega\). Next we show that \(u\) solves (NS) with some \(p\). Take any \(v \in C_0^\infty((0, T); C_0^\infty(\Omega))\) and set \(w = \text{Rot} v\). Let us write \(v = \nabla^\perp \psi + \sum_{j=1}^{L} d_j \nabla^\perp q_j\), where \(\psi = (-\Delta_D)^{-1} w\) and \(d_j = (v, \nabla^\perp q_j)_{L^2}\). Then by \(-\Delta u = \nabla^\perp \omega\) and \((\nabla^\perp \omega, \nabla^\perp q_j)_{L^2} = (\nabla^\perp \omega_{har}, \nabla^\perp q_j)_{L^2}\) we have from the integration by parts,

\[
\int_0^T \left( \partial_t u - \nu \Delta u + u \cdot \nabla u, v \right)_{L^2} dt
\]

\[
= \int_0^T \left( \partial_t u - \nu \Delta u + u \cdot \nabla u, \nabla^\perp \psi + \sum_{j=1}^{L} d_j \nabla^\perp q_j \right)_{L^2} dt
\]

\[
= \int_0^T \left( \partial_t \omega - \nu \Delta \omega + u \cdot \nabla \omega, \psi + \sum_{j=1}^{L} d_j \nabla^\perp q_j \right)_{L^2} dt + \int_0^T \int_{\partial \Omega} \nu \partial_n \omega \sum_{j=1}^{L} d_j q_j dS dt
\]

\[
= \nu \sum_{j=1}^{L} \int_0^T \left( \partial_n \omega_{har} - \sum_{j=1}^{L} \left( \nabla^\perp \omega, \nabla^\perp q_j \right)_{L^2} \partial_n q_j \right) \sum_{j=1}^{L} d_j q_j dS dt
\]

\[
- \int_0^T \int_{\partial \Omega} \left( \partial_n (\Delta_D)^{-1} (u \cdot \nabla \omega) - \sum_{j=1}^{L} \left( u \times \omega, \nabla^\perp q_j \right)_{L^2} \partial_n q_j \right) \sum_{j=1}^{L} d_j q_j dS dt
\]

\[
= \nu \sum_{j=1}^{L} \int_0^T \left( \nabla^\perp \omega_{har}, \nabla^\perp q_j \right)_{L^2} dt - \nu \sum_{j=1}^{L} \int_0^T \left( \nabla^\perp \omega_{har}, \nabla^\perp q_j \right)_{L^2} dt
\]

\[
- \int_0^T \int_{\partial \Omega} \left( \partial_n (\Delta_D)^{-1} (u \cdot \nabla \omega) - \sum_{j=1}^{L} \left( u \times \omega, \nabla^\perp q_j \right)_{L^2} \partial_n q_j \right) \sum_{j=1}^{L} d_j q_j dS dt
\]

\[
= -\sum_{j=1}^{L} d_j \int_0^T \left( \nabla (\Delta_D)^{-1} (u \cdot \nabla \omega), \nabla q_j \right)_{L^2} dt
\]

\[
+ \sum_{j=1}^{L} d_j \int_0^T \left( u \cdot \nabla \omega, q_j \right)_{L^2} dt
\]

\[
= -\sum_{j=1}^{L} d_j \int_0^T \left( u \cdot \nabla \omega, q_j \right)_{L^2} dt + \sum_{j=1}^{L} d_j \int_0^T \left( u \times \omega, \nabla^\perp q_j \right)_{L^2} dt = 0.
\]
Since \( v \in C_0^\infty ((0,T);\mathcal{C}_{0,\omega}(\Omega)) \) is arbitrary, \( u \) is a solution to (NS) for some \( p \). Conversely, let \( (u,p) \) be a smooth solution to (NS). Then clearly \( \omega = \text{Rot} \ u \) solves (V) by Corollary 2.2. Furthermore, (NS) implies

\[
\begin{align*}
\omega(t) &= e^{t\Delta_N}b + \int_0^t e^{(t-s)\Delta_N} \left( f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}} \right) ds + \int_0^t \Gamma(t-s) * \left( f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}} \right) ds \\
&\quad - \int_0^t \Gamma(0) * \left( f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}} \right) ds.
\end{align*}
\]

Theorem 3.1 The integral equation for (LV)-(LBC) is given by

\[
\begin{align*}
\omega(t) &= e^{t\Delta_N}b + \Gamma(\nu t) * b - \Gamma(0) * b \\
&\quad + \int_0^t e^{(t-s)\Delta_N} \left( f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}} \right) ds + \int_0^t \Gamma(t-s) * \left( f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}} \right) ds \\
&\quad - \int_0^t \Gamma(0) * \left( f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}} \right) ds.
\end{align*}
\]

Here \( e^{t\Delta_N} \) is the semigroup for the heat equation (with the unit viscosity) in \( \mathbb{R}^2_+ \), subject to the homogeneous Neumann boundary condition, \( \Gamma(0)* := \lim_{t\to0} \Gamma(t)* \), and \( g\mathcal{H}^1_{\{x_2=0\}} \) is a one-dimensional Hausdorff measure with density \( g \) defined by

\[
\langle h, g\mathcal{H}^1_{\{x_2=0\}} \rangle = \int_{\mathbb{R}} h(x_1,0)g(x_1) \, dx_1 \quad \text{for} \quad h \in C_0(\mathbb{R}_+).
\]
The proof of Theorem 3.1 is given in the appendix. We note that $\Gamma(0) \ast h = \Xi E \ast h$ in $R^2_+$. In (3.6) the terms $\Gamma(0) \ast$ seem to cause trouble when solving the vorticity equations, for apparently they could give rise to a derivative loss near the boundary. In fact, these terms do not appear in the vorticity equations, due to the following cancellation property.

**Proposition 3.2** If $g = \partial_2(-\Delta_D)^{-1} f \mid_{x_2=0}$ then

$$
\Xi E \ast (f - g\mathcal{H}^1_{(x_2=0)}) = 0 \quad \text{in } R^2_+.
$$

In particular, we have

$$
\Xi E \ast b = 0 \quad \text{in } R^2_+ \quad \text{if} \quad \partial_2(-\Delta_D)^{-1} b = 0 \quad \text{on } \partial R^2_+.
$$

**Proof.** By the integration by parts we have

$$
\Xi E \ast (f - g\mathcal{H}^1_{(x_2=0)})(x) = \int_{R^2_+} \nabla_y(\Xi E)(x - y^*) \cdot \nabla(-\Delta_D)^{-1} f(y) \, dy
$$

$$
= -2 \int_{R^2_+} (\partial_1 + \partial_2)\Delta E(x - y^*) \partial_1 + (-\partial_1^2)^\frac{1}{2}(-\Delta_D)^{-1} f(y) \, dy
$$

$$
= 0 \quad \text{in } R^2_+.
$$

This completes the proof.

We note that the condition in (3.9) is nothing but (2.8). Thus, reminding also (3.1), we do not have the problematic terms $\Gamma(0) \ast$ in (3.6) for the vorticity equations. It will be useful to rewrite the result of Theorem 3.1 under the conditions in Proposition 3.2.

**Corollary 3.3** Assume that $\partial_2(-\Delta_D)^{-1} b \mid_{x_2=0} = 0$ and $g = \partial_2(-\Delta_D)^{-1} f \mid_{x_2=0}$. Then the integral equation for (LV)-(LBC) is given by

$$
\omega(t) = e^{tB}b + \int_0^t e^{(t-s)B}f(s) - g(s)\mathcal{H}^1_{(x_2=0)} \, ds,
$$

where

$$
e^{tB}h = e^{t\Delta_D}h + \Gamma(t) \ast h.
$$

Corollary 3.3 shows that the integral equation for the vorticity equation is written as

$$
\omega(t) = e^{tB}b - \int_0^t e^{(t-s)B} \left(u \cdot \nabla \omega(s) - \partial_2(-\Delta_D)^{-1}(u \cdot \nabla \omega)(s)\mathcal{H}^1_{(x_2=0)} \right) \, ds, \quad u = \nabla^\perp(-\Delta_D)^{-1} \omega.
$$

It is possible to show the local-in-time solvability of (3.12) in suitable function spaces. Indeed, from $L^p - L^q$ estimates for $e^{tB}$ in Lemma 3.4 below we can construct solutions to (3.12) at least locally in time if $b \in L^p(R^2_+)$ for some $p \in (1, 2)$ by the contraction mapping theorem. Since its proof is rather standard we only state the result in Theorem 3.6 and the details are omitted in this paper.

**Lemma 3.4**

(i) Let $1 \leq q < p \leq \infty$ or $1 < q \leq p < \infty$. Then we have

$$
\|e^{tB}f\|_{L^p} \leq Ct^{-\frac{1}{2} + \frac{q}{p}}\|f\|_{L^q} \quad t > 0.
$$

(ii) Let $1 \leq q \leq p \leq \infty$ and $p > 1$. Then we have

$$
\|e^{tB}(g\mathcal{H}^1_{(x_2=0)})\|_{L^p} \leq Ct^{-\frac{1}{2}(1 + \frac{q}{p} - \frac{p}{2})}\|g\|_{L^q} \quad t > 0.
$$

(iii) Let $1 \leq q \leq p \leq \infty$ and $k \in \mathbb{N}$. Then we have

$$
\|\nabla^k e^{tB}f\|_{L^p} \leq Ct^{-\frac{k}{2} + \frac{q}{p} - \frac{p}{2}}\|f\|_{L^q} \quad t > 0.
$$

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(iv) Let $1 \leq q \leq p \leq \infty$. Assume that $g = \partial q(-\Delta)^{-1} f |_{x_2=0}$. Then we have
\[
\|e^{tB} (f - gH^1_{(x_2=0)})\|_{L^p} \leq Ct^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{q}} \|\nabla^\perp (-\Delta_D)^{-1} f\|_{L^q},
\] (3.16)

Proof. (i) Since it is straightforward to get \(\|e^{tA_N} f\|_{L^p} \leq Ct^{-1/p + 1/q} \|f\|_{L^q}\) for all \(1 \leq q \leq p \leq \infty\), it suffices to show \(\|\Gamma(t) * f\|_{L^p} \leq Ct^{-1/p + 1/q} \|f\|_{L^q}\) if \(1 \leq q \leq p \leq \infty\) or \(1 \leq p \leq q < \infty\). To this end we first write \(\Gamma(t) * f = \Xi(-\Delta_{\mathbb{R}^2})^{-1}(G(t) * f)\) and observe that the symbol \(p(\xi)\) of the operator \(\Xi(-\Delta_{\mathbb{R}^2})^{-1}\) is given by
\[
p(\xi) = 2 - \frac{\xi_1^2 + i |\xi_2|}{|\xi|^2},
\] (3.17)
Thus \(\Xi(-\Delta_{\mathbb{R}^2})^{-1}\) is a singular integral operator in \(\mathbb{R}^2\) (see [12, Theorem 8.14]), so we have for \(1 \leq q < p < \infty\) or \(1 \leq p \leq q < \infty\),
\[
\|\Gamma(t) * f\|_{L^p(\mathbb{R}^2)} \leq C \|G(t) * f\|_{L^p(\mathbb{R}^2)} \leq Ct^{-1/p + 1/q} \|f\|_{L^q}.
\]
Let \(1 \leq q < p = \infty\). Then by the Gagliardo-Nirenberg inequality we have for \(\max\{q, 2\} < \tilde{q} < \infty\) and \(\sigma = 1 - 2/\tilde{q}\),
\[
\|\Gamma(t) * f\|_{L^{\infty}(\mathbb{R}^2)} \leq C \|\nabla \Gamma(t) * f\|_{L^{\tilde{q}}(\mathbb{R}^2)} \|\Gamma(t) * f\|_{L^{\tilde{q}}(\mathbb{R}^2)} \leq C \|\nabla G(t) * f\|_{L^{\tilde{q}}(\mathbb{R}^2)} \|G(t) * f\|_{L^{\tilde{q}}(\mathbb{R}^2)} \leq Ct^{-\frac{1}{2}} \|f\|_{L^q}.
\]
(ii) Again it suffices to show \(\|\Gamma(t) * gH^1_{(x_2=0)}\| \leq Ct^{-1/q - 1/p}/2 \|g\|_{L^q_{\mathbb{R}^1}}\). To prove this we use the pointwise estimate (5.8) and observe that
\[
\|\Gamma(t) * (gH^1_{(x_2=0)}) (x)\| \leq Ct^{-1} \int_{\mathbb{R}} (1 + \frac{|x_1 - y_1|}{\sqrt{t}}) \frac{1}{\log(e + |x_1 - y_1|/\sqrt{t})} + \frac{x_2^2}{t} \right)^{-1} |g(y_1)| \, dy_1.
\] (3.18)
Then the Young inequality implies that
\[
\|\Gamma(t) * (gH^1_{(x_2=0)}) (\cdot, x_2)\|_{L^p_{\mathbb{R}^1}} \leq Ct^{-\frac{1}{2} + \frac{1}{p} + \frac{1}{q} + \frac{1}{p}} \|g\|_{L^q_{\mathbb{R}^1}}.
\]
Hence we get (3.14) since \(p > 1\) and \(p \geq q\).
(iii) It suffices to consider \(\|\nabla \Gamma(t) * f\|_{L^p}\), but the desired estimate follows from the pointwise estimate (5.8) and the Young inequality. The details are omitted here.
(iv) By the definition of \(e^{tB}\) and the integration by parts we have
\[
e^{tB} (f - gH^1_{(x_2=0)}) (x) = \int_{\mathbb{R}^2} \nabla^\perp (G(t, x - y) + G(t, x - y^*) + \Gamma(t, x - y^*)) \cdot \nabla^\perp (-\Delta_D)^{-1} f (y) \, dy.
\]
Then it is easy to get (3.16) from (5.8) and the Young inequality. This completes the proof.

If \(b \in L^p(\mathbb{R}^2_+)\) for some \(p \in [1, 2]\) then it is not difficult to see \(\partial_2(-\Delta_D)^{-1} b |_{x_2=0} \in L^p(\mathbb{R}) + L^\infty(\mathbb{R})\). Moreover, we have

**Proposition 3.5** If \(b \in L^1(\mathbb{R}^2_+)\) and \(\partial_2(-\Delta_D)^{-1} b |_{x_2=0} = 0\) then \(\int_{\mathbb{R}^2_+} b(x) \, dx = 0\).

**Proof.** Since
\[
\partial_2(-\Delta_D)^{-1} b(x_1, 0) = \frac{1}{\pi} \int_{\mathbb{R}^2_+} \frac{y_2}{|x_1 - y_1|^2 + y_2^2} b(y) \, dy,
\] (3.19)
we have
\[
0 = \int_{\mathbb{R}} (\partial_2(-\Delta_D)^{-1} b)(x_1, 0) \, dx_1 = \frac{1}{\pi} \int_{\mathbb{R}^2_+} y_2 b(y) \int_{\mathbb{R}} \frac{1}{|x_1 - y_1|^2 + y_2^2} \, dx_1 \, dy = \int_{\mathbb{R}^2_+} b(y) \, dy.
\]
This completes the proof.

Proposition 3.5 shows that if \( b \in L^1(\mathbb{R}^2_+) \) is the vorticity field of a velocity field satisfying the no-slip boundary condition, then \( b \) must have zero integral mean over \( \mathbb{R}^2_+ \). In other words, there are no nontrivial vorticity fields which are nonnegative and decay fast enough at spatial infinity. This is contrastive if compared with the case \( \Omega = \mathbb{R}^2 \), where there are nontrivial and nonnegative vorticity fields which decay rapidly at \(|x| \to \infty|\).

We conclude this section by stating the local solvability of (3.12).

**Theorem 3.6** Assume that \( b \in L^p(\mathbb{R}^2_+) \) for some \( p \in (1, 2) \). Then there is \( T > 0 \) such that (3.12) has a unique solution \( \omega \in C([0, T); L^p(\mathbb{R}^2_+)) \) satisfying \( \sup_{0 \leq t \leq T} t^{1/p - 1/4} \| \omega(t) \|_{L^1} < \infty \). If \( b \) satisfies the compatibility condition \( \partial_x(-\Delta_D)^{-1} b = 0 \) on \( \partial \mathbb{R}^2_+ \) in addition, then the solution \( \omega(t) \) converges to \( b \) as \( t \to 0 \) in \( L^p(\mathbb{R}^2_+) \).

**Proof.** The solution is constructed in the space \( X_T = \{ f \in C([0, T); L^p(\mathbb{R}^2_+)) \mid \sup_{0 \leq t \leq T} t^{1/p - 1/4} \| f(t) \|_{L^1} < \infty \} \) with the norm \( \| f \|_{X_T} = \sup_{0 \leq t \leq T} \| f(t) \|_{L^p} + \sup_{0 \leq t \leq T} t^{1/p - 1/4} \| f(t) \|_{L^1} \) by the contraction mapping theorem, thanks to Lemma 3.4 and the well-known estimates such as \( \| \nabla^{-1/2} f \|_{L^4} \leq C \| f \|_{L^p} \) for \( 1 < q < \infty \) and \( \| \nabla^{-1} f \|_{L^\infty} \leq C \| f \|_{L^p} \) with \( \sigma = (4 - 2p)/(4 - p) \) for \( 1 < p < 2 \). To show the convergence to the initial data, it suffices to write \( e^{rt\Delta_D} b = e^{rt\Delta_D} b + (\Gamma(t) - \Gamma(0)) \ast b + \Gamma(0) \ast b \) and note that the last term vanishes by Proposition 3.2. Then by the density argument and the estimate \( \sup_{t \geq 0} \| \Gamma(t) \ast b \|_{L^p} \leq C \| b \|_{L^p} \) one can check that \( \| \Gamma(t) - \Gamma(0) \|_{L^p} \) goes to zero as \( t \to 0 \). It is easy to see that \( e^{t\Delta_D} b \) converges to \( b \) in \( L^p \) as \( t \to 0 \). This completes the proof.

**Remark 3.7** Even for \( b \in L^1(\mathbb{R}^2_+) \) we can construct a local unique solution \( \omega \) to (3.12) such that \( t^{1 - 1/r} \| \omega(t) \|_{L^\infty} \leq C \| f \|_{L^p} \) for \( 1 < q < \infty \) and \( \| \nabla^{-1} f \|_{L^\infty} \leq C \| f \|_{L^p} \) with \( \sigma = (4 - 2p)/(4 - p) \) for \( 1 < p < 2 \). To show the convergence to the initial data, it suffices to write \( e^{rt\Delta_D} b = e^{rt\Delta_D} b \) and note that the last term vanishes by Proposition 3.2. Then by the density argument and the estimate \( \sup_{t \geq 0} \| \Gamma(t) \ast b \|_{L^p} \leq C \| b \|_{L^p} \) one can check that \( \| \Gamma(t) - \Gamma(0) \|_{L^p} \) goes to zero as \( t \to 0 \). It is easy to see that \( e^{t\Delta_D} b \) converges to \( b \) in \( L^p \) as \( t \to 0 \). This completes the proof.

**Remark 3.8** By the bootstrap argument using Lemma 3.4 the solution \( \omega \) in Theorem 3.6 is shown to be smooth in positive time. We note that, in order to ensure that \( u = \nabla^{-1}(-\Delta_D)^{-1} \omega \) solves (NS), we need the compatibility condition \( \partial_x(-\Delta_D)^{-1} b = 0 \) on \( \partial \mathbb{R}^2_+ \) for the initial data.

**Remark 3.9** When \( b \in L^p(\mathbb{R}^2_+) \) for some \( p \in (1, 2) \) the related velocity \( a \) belongs to \( L^q_0(\mathbb{R}^2_+) \) with \( 1/q = 1/p - 1/2 \) by the Hardy-Littlewood-Sobolev inequality. Since \( q > 2 \) we already know the solvability of (NS) in this case from the \( L^q \) theory of the Stokes or the Navier-Stokes equations in the half space; for example, see [45, 32, 14, 46, 51, 17, 36, 19, 49, 42, 11] and references therein. On the other hand, if \( b \in L^1(\mathbb{R}^2_+) \) then \( a \) belongs to the weak \( L^2 \) space. The reader is referred to [30] for the analysis of the Navier-Stokes equations in the weak \( L^p \) spaces.

### 4 Application to analysis of vorticity at zero viscosity limit

The inviscid limit behavior of solutions to the Navier-Stokes equations is a classical theme in fluid mechanics. However, if the no-slip boundary conditions are imposed on velocity fields, only partial results are known even in the two-dimensional case; so far we need either the analyticity of initial data or the radial symmetry of the domain and the solutions. More precisely, if the initial data is analytic it is proved in [2, 40, 41] that the inviscid limit is described by the Euler equations and the Prandtl equations. When \( \Omega \) is a disk and the solution possesses a radial symmetry, the inviscid limit is already well studied in various functional settings [37, 6, 34, 35, 27]; see also [38]. On the other hand, [23] gave necessary and sufficient conditions for the convergence of weak solutions of (NS) to that of the Euler equations in the energy class. The analysis in this direction has been developed by [48, 50, 10, 25, 26, 27].
Hence, under the assumption

We note that (4.10) is equivalent to the Euler equations with the boundary condition

Remark 4.3

Here the positive constants \( \omega \) and in particular, it is nontrivial if and only if

Theorem 4.1

Assume that \( b = \text{Rot} a \) with \( a \in L^q_2(\mathbb{R}^2_+) \cap (W_{0}^{1,q}(\mathbb{R}^2_+))^2 \) for some \( 1 < q < \infty \) and \( b \in W^{l,1/3}(\mathbb{R}^2_+) \) for \( l > 1 \). Let \( \omega \) be the solution to \((V)_{BL}(\Omega)\) with \( \Omega = \mathbb{R}^2_+ \). Then there are \( c_0, C > 0 \) such that the following estimates hold for sufficiently small \( \nu > 0 \):

\[
\|u(t)\|_{L^\infty} \leq C \quad \text{for} \quad 0 < t \leq c_0 \nu^\frac{2}{3}, \tag{4.1}
\]

\[
\|\omega(t) - \omega_E(t) - \omega_{BL}(t)\|_{L^p} \leq C \nu^{-\frac{1}{3}}(1+\frac{1}{t})^{\frac{1}{2}(1+\frac{1}{3})} \quad \text{for} \quad 0 < t \leq c_0 \nu^\frac{4}{3}, \quad \frac{4}{3} \leq p \leq \infty. \tag{4.2}
\]

Here \( c_0 \) is independent of \( \nu \), and \( C \) is independent of \( \nu \) and \( t \in [0, c_0 \nu^{1/3}] \). The function \( \omega_E \) is the vorticity field of the solution to the Euler equation with the initial velocity \( a \). The function \( \omega_{BL} \) is defined by

\[
\omega_{BL}(t, x) = 2 \int_0^t (4\pi \nu s)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4\nu s}\right) ds \cdot \partial_2 (-\Delta_D)^{-1} (a \cdot \nabla b)(x_1, 0), \tag{4.3}
\]

and in particular, it is nontrivial if and only if

\[
\partial_2 (-\Delta_D)^{-1} (a \cdot \nabla b) \neq 0 \quad \text{on} \quad \partial \mathbb{R}^2_+. \tag{4.4}
\]

When (4.4) holds \( \omega_{BL} \) satisfies

\[
\|\omega_{BL}(t)\|_{L^p} \leq C \nu^{-\frac{1}{3}}(1+\frac{1}{t})^{\frac{1}{2}(1+\frac{1}{3})} \quad \text{for} \quad t > 0, \quad 1 \leq p \leq \infty, \tag{4.5}
\]

\[
\|\omega_{BL}(t)\|_{L^p((0 \leq x_2 \leq (\nu t)^{\frac{1}{3}}))} \geq c_1 \nu^{-\frac{1}{3}}(1+\frac{1}{t})^{\frac{1}{2}(1+\frac{1}{3})} \quad \text{for} \quad t > 0, \quad 1 < p \leq \infty. \tag{4.6}
\]

Here the positive constants \( c_1 \) and \( C' \) are independent of \( \nu \) and \( t \).

Corollary 4.2

Under the assumptions of Theorem 4.1, if (4.4) holds in addition, then there is \( c_2 > 0 \) such that

\[
\|\omega(t) - \omega_E(t)\|_{L^p((0 \leq x_2 \leq (\nu t)^{\frac{1}{3}}))} \geq c_2 \nu^{-\frac{1}{3}}(1+\frac{1}{t})^{\frac{1}{2}(1+\frac{1}{3})} \quad \text{for} \quad 0 \leq t \leq c_0 \nu^{\frac{4}{3}}, \quad \frac{4}{3} \leq p \leq \infty. \tag{4.7}
\]

Here \( c_0 \) is the constant in Theorem 4.1 and \( c_2 \) is independent of \( \nu \) and \( t \in [0, c_0 \nu^{1/3}] \). In particular, the high creation of vorticity near the boundary in \( L^p \) occurs in the following sense.

\[
\|\omega(c_0 \nu^{\frac{4}{3}})\|_{L^p((0 \leq x_2 \leq c_0 \nu^{\frac{4}{3}}))} \geq c_1 \nu^{-\frac{1}{3}}(1+\frac{1}{t})^{\frac{1}{2}(1+\frac{1}{3})} \rightarrow \infty (\nu \rightarrow 0) \quad \text{if} \quad 2 < p \leq \infty. \tag{4.8}
\]

Remark 4.3

In Theorem 4.1 the assumption on \( b \) in \( L^{4/3}(\mathbb{R}^2_+) \) is just for technical reasons, and we can also handle with \( b \) in \( W^{l,p}(\mathbb{R}^2_+) \) if \( p \in [1, 2] \) and \( l > 1 \).

The proof of Theorem 4.1 requires rather lengthy calculations, and we divide it into several steps. Let \( J(f) \) be the velocity field recovered from \( f \) via the Biot-Savart law, i.e.,

\[
J(f) = (J_1(f), J_2(f)) = \nabla^\perp (-\Delta_D)^{-1} f, \quad \nabla^\perp = (\partial_2, -\partial_1). \tag{4.9}
\]

Note that \( J(f) \) satisfies \( \nabla \cdot J(f) = 0 \) in \( \mathbb{R}^2_+ \) and \( J_2(f) = 0 \) on \( \partial \mathbb{R}^2_+ \). The function \( \omega_E \) satisfies the equation

\[
\begin{cases}
\partial_t \omega_E + u_E \cdot \nabla \omega_E = 0 & t > 0, \quad x \in \mathbb{R}^2_+, \\
u E = J(\omega_E) & t > 0, \quad x \in \mathbb{R}^2_+, \\
\omega_E|_{t=0} = b, & x \in \mathbb{R}^2_.
\end{cases} \tag{4.10}
\]

We note that (4.10) is equivalent to the Euler equations with the boundary condition \( u_{E,2} = 0 \) on \( \partial \mathbb{R}^2_+ \). Hence, under the assumption \( b \in W^{l,4/3}(\mathbb{R}^2_+) \) with \( l > 1 \) the existence and the uniqueness of solutions to
(4.10) follow from the methods developed in the literature [53, 54, 22, 3, 28, 7, 9, 8]. In particular, we can show that $\omega_E \in C^1([0,T); W^{l,4/3}(\mathbb{R}^2_+))$ with $l' \gg 1$ for all $T > 0$, and this fact will be freely used in the rest of the paper. Next we consider the second and third expansions of $\omega$ which are directly related with $\omega_E$:

$$
\left\{ \begin{array}{l}
\partial_t w_{E,1} - \nu \Delta w_{E,1} = 0 \quad \quad t > 0, \ x \in \mathbb{R}^2_+,
\nu(\partial_t w_{E,1} + (-\partial_t^2)^{1/2} w_{E,1}) = -J_1(w_E \cdot \nabla \omega_E) \quad \quad t > 0, \ x \in \partial \mathbb{R}^2_+,
\end{array} \right. \quad \quad (4.11)
$$

$$
\left\{ \begin{array}{l}
\partial_t w_{E,2} - \nu \Delta w_{E,2} = \nu \Delta \omega_E \quad \quad t > 0, \ x \in \mathbb{R}^2_+,
\nu(\partial_t w_{E,2} + (-\partial_t^2)^{1/2} w_{E,2}) = -\nu J_1(\Delta \omega_E) \quad \quad t > 0, \ x \in \partial \mathbb{R}^2_+.
\end{array} \right. \quad \quad (4.12)
$$

The function $w_{E,1}$ is responsible for the creation of vorticity near the boundary. Set

$$
w_E = w_{E,1} + w_{E,2}, \quad \quad F = J(\omega_E + w_E) \cdot \nabla w_E + J(w_E) \cdot \nabla \omega_E. \quad \quad (4.13)
$$

Then $w = \omega - \omega_E - w_E$ satisfies $w|_{t=0} = 0$ and

$$
\left\{ \begin{array}{l}
\partial_t w - \nu \Delta w = -L(\omega_E + w_E)w - N(w,w) - F \quad \quad t > 0, \ x \in \mathbb{R}^2_+,
\nu(\partial_t w + (-\partial_t^2)^{1/2} w) = -J_1(L(\omega_E + w_E)w + N(w,w) + F) \quad \quad t > 0, \ x \in \partial \mathbb{R}^2_+.
\end{array} \right. \quad \quad (4.14)
$$

Here

$$
L(f)w = J(f) \cdot \nabla w + J(w) \cdot \nabla f, \quad \quad N(f,g) = J(f) \cdot \nabla g. \quad \quad (4.15)
$$

By the above definitions we can check that each of $J(\omega_E + w_{E,1})$, $J(w_{E,2})$, and $J(w)$, satisfies the no-slip boundary condition (see the proof of Theorem 2.3), and this property will be essentially used in the proof of Theorem 4.1. We note that the above decomposition of $\omega$ should be effective only near the initial time $0 < t \leq \nu^3$ for some $\beta > 0$. For a longer time period we need to take into account the vorticity counterpart of the Prandtl equations, where the verification of such expansion is widely open except for the analytic initial data. Finally we set for $\delta > 0$,

$$
\Omega_\delta = \{ x \in \mathbb{R}^2_+ \mid 0 \leq x_2 \leq \delta^{1/2} \}, \quad \quad \Omega_\delta^\circ = \mathbb{R}^2_+ \backslash \Omega_\delta = \{ x \in \mathbb{R}^2_+ \mid x_2 \geq \delta^{1/2} \}. \quad \quad (4.16)
$$

In the sequel we will focus on the a priori estimates of $\omega$ (especially, of $w$) based on the above decompositions. The basic strategy is as follows: we will use the integral equations (3.6) or (3.10) for the estimates of $w_{E,1}$ and $w_{E,2}$, and also of $w$ near the boundary. The estimates of $w$ away from the boundary will be obtained by the energy argument. Theorem 4.1 then follows from these a priori estimates.

### 4.1 Preliminary estimates

In this section we prepare several linear estimates which will be used to study $w_E$ and $w$.

#### 4.1.1 Estimate for layer potential

We set

$$
G_i(t,x) = \frac{1}{(4\pi t)^{3/2}} \exp(-\frac{|x_i|^2}{4t}), \quad \quad (4.17)
$$

$$
K_{0,\nu}(g)(t,x) = 2 \int_0^t G_2(\nu(t-s),x) \, ds \, g(0,x_1), \quad \quad (4.18)
$$

$$
K_{1,\nu}(g)(t) = 2 \int_0^t G(\nu(t-s)) \ast g(s) \mathcal{H}^1_{|x_2=0} \, ds. \quad \quad (4.19)
$$

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Lemma 4.4 Let $1 \leq p \leq \infty$, $k \in \mathbb{N} \cup \{0\}$, $l = 0, 1, 2$. Then we have

$$
\| \partial_t^k K_{0,\nu}(g)(t) \|_{L^p(\Omega_t)} \geq C \nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})} \| \partial_t^k g(0) \|_{L^p_t}, \quad (4.20)
$$

$$
\| \partial_t^l \partial_x^k K_{1,\nu}(g)(t) \|_{L^p} \leq C \nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})} (\| \partial_t^k g \|_{L^p_t L^p_x} + t \| \partial_t \partial_x^k g \|_{L^p_t L^p_x} + \nu t \| \partial_t^{l+k} g \|_{L^p_t L^p_x}), \quad (4.21)
$$

$$
\| \partial_t^k K_{1,\nu}(g)(t) \|_{L^1_t L^\infty_x} \leq C t \| \partial_t^k g \|_{L^p_t L^p_x}, \quad (4.22)
$$

$$
\| \partial_t^k (K_{1,\nu} - K_{0,\nu})(g)(t) \|_{L^p} \leq C \nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})} (t^{\frac{1}{2}} \| \partial_t \partial_x^k g \|_{L^p_t L^p_x} + \nu^{\frac{1}{2}} \| \partial_t^{l+k} g \|_{L^p_t L^p_x}), \quad (4.23)
$$

Proof. We may assume that $k = 0$. Since

$$
|K_0(g)(t, x)| = 2|g(0, x_1)| \int_0^t G_2(\nu(t - s), x) \, ds \geq 2|g(0, x_1)| \int_0^t G_2(\nu(t - s), x) \, ds \geq |g(0, x_1)| \frac{t^\frac{1}{2}}{2(\pi \nu)^\frac{1}{2}} \exp(-\frac{x^2}{2 \nu t}),
$$

we have

$$
\|K_0(g)(t)\|_{L^p(\Omega_t)} \geq \frac{t^\frac{1}{2}}{(2\pi \nu)^\frac{1}{2}} \left( \int_0^\infty \exp(-\frac{px^2}{2\nu t}) \, dx \right)^\frac{1}{2} \|g(0)\|_{L^p_t} = C \nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})} \| \partial_t^k g(0) \|_{L^p_t}. \quad (4.24)
$$

This proves (4.20). When $l = 0$, (4.21) is a direct consequence of the Young inequality. When $l = 1$ we rewrite $\partial_t K_{1,\nu}(g)$ as

$$
\partial_t K_{1,\nu}(g)(t, x) = 2 \int_0^t \int_\mathbb{R} \partial_x G((\nu(t - s), x_1 - y_1, y_2) g(s, y_1) \, dy_1 \, ds \quad (4.24)
$$

$$
= 2 \int_0^t \int_\mathbb{R} \partial_x^2 G((\nu(t - s), x_1 - y_1, y_2) g(s, y_1) \, dy_1 \, ds \quad (4.25)
$$

Thus the Young inequality implies

$$
\| \partial_t^2 K_{1,\nu}(g)(t) \|_{L^1_t L^\infty_x} \leq C \nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})} \| \partial_t g \|_{L^p_t L^p_x} + \nu t \| \partial_t^2 g \|_{L^p_t L^p_x}, \quad (4.26)
$$

which leads to

$$
\| \partial_x^2 K_{1,\nu}(g)(t) \|_{L^p(\{x_2 \leq R\})} \leq C R^{\frac{1}{2} \nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2} \frac{1}{p} + \frac{1}{2} \frac{1}{p}} t^{\frac{1}{2}(1+\frac{1}{p})}} \|g\|_{L^p_t L^p_x}. \quad (4.27)
$$

On the other hand, we have from (4.24) and $|\partial_x G(t, x)| \leq C x_2^{-1} |G(2t, x)|$,

$$
\| \partial_t K_{1,\nu}(g)(t) \|_{L^p(\{x_2 \geq R\})} \leq C R^{-1} \nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})} \|g\|_{L^p_t L^p_x}. \quad (4.28)
$$

Taking $R = (\nu t)^{1/2}$ we get (4.21) with $l = 1$. The case $l = 2$ for (4.21) is obtained by the equality

$$
\partial_t^2 K_{1,\nu}(g)(t, x) = 2 t^{-1} \int_\mathbb{R} G(\nu(t - s), x_1 - y_1, y_2) g(t, y_1) \, dy_1 \quad (4.29)
$$

$$
+ 2 \int_0^t \int_\mathbb{R} G((\nu(t - s), x_1 - y_1, y_2)(\nu^{-1} \partial_x g(s, y_1) - \partial_t^2 g(s, y_1)) \, dy_1 \, ds, \quad (4.29)
$$

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which is derived from (4.25). The details are omitted here. Est.(4.22) is easily checked from the definition of $K_{1,\nu}(g)$. To prove (4.23) we observe that

$$
K_{1,\nu}(g)(t, x) - K_{0,\nu}(g)(t, x) = 2 \int_0^t \int_{\mathbb{R}} G(\nu(t - s), x_1 - y_1, x_2)(g(s, y_1) - g(0, x_1)) \, dy_1 \, ds
$$

$$
= 2 \int_0^t \int_{\mathbb{R}} G(\nu(t - s), x_1 - y_1, x_2)(g(s, y_1) - g(0, y_1)) \, dy_1 \, ds
$$

$$
+ 2 \int_0^t \int_{\mathbb{R}} G(\nu(t - s), x_1 - y_1, x_2)(g(0, y_1) - g(0, x_1)) \, dy_1 \, ds
$$

$$
= I_1(t, x) + I_2(t, x).
$$

Then it is easy to see $\|I_1(t)\|_{L^p} \leq C\nu^{-(1-1/p)/2+l(3+1/p)/2} \|\partial_t g\|_{L^p_x L^p_t}$. As for $I_2$, we have

$$
\|I_2(t)\|_{L^p} \leq C \int_0^t \|x_1 G_1(\nu(t - s))\|_{L^p_x L^p_t} \|G_2(\nu(t - s))\|_{L^p_x L^\infty_t} \, ds \|\partial_t g\|_{L^p_x L^p_t} \leq C\nu^{1/2} t^{1/2} \|\partial_t g\|_{L^p_x L^p_t}.
$$

This completes the proof.

4.1.2 Estimate for $(\Gamma(t) - \Gamma(0))$

**Proposition 4.5** Let $1 \leq p \leq \infty$, $k \in \mathbb{N} \cup \{0\}$, $l = 0, 1$, and $m > 0$. Then

$$
\|\partial_t^k \partial_x^l (\Gamma(t) - \Gamma(0)) \ast f\|_{L^p} \leq Ct^{k/2} \|\partial_t^{1+k+l} f\|_{L^p} + \|\partial_t^{1+k+l} (-\partial_t^l)^{3/2} f\|_{L^p} + C\|\partial_t^k (-\partial_x^l)^{3/2} f\|_{L^p}.
$$

(4.30)

$$
\|\partial_t^k (\Gamma(t) - \Gamma(0)) \ast (gH^1_{\{x_2 = 0\}})\|_{L^p} + t^{-m/2} \|x_2^m \partial_t^k \partial_x^l (\Gamma(t) - \Gamma(0)) \ast (gH^1_{\{x_2 = 0\}})\|_{L^p}
$$

$$
\leq Ct^{k/2} \left( \|\partial_t^{1+k} g\|_{L^p_t} + t^{3/2} \|\partial_t^{2+k} g\|_{L^p_t} + t \|\partial_t^{1+k} (-\partial_x^l)^{3/2} g\|_{L^p_t} \right).
$$

(4.31)

**Proof.** We may assume that $k = 0$. In view of (5.5) and (5.6) we have

$$
(\Gamma(t) - \Gamma(0)) \ast f = - \int_0^t \Xi G(\tau) \ast f \, d\tau = -2 \int_0^t \partial_t G(\tau) \ast \partial_t f \, d\tau - 2 \int_0^t \partial_x G(\tau) \ast (-\partial_x^l)^{3/2} f \, d\tau.
$$

(4.32)

Hence (4.30) with $l = 0$ follows from the Young inequality. When $l = 1$ by the equality $\partial^2_x G(\tau) = \partial_t G(\tau) - \partial^2_t G(\tau)$ we observe that in $\mathbb{R}^2_+$,

$$
\partial_2 (\Gamma(t) - \Gamma(0)) \ast f = -2 \int_0^t \partial_2 G(\tau) \ast \partial_2^l f \, d\tau - 2 G(t) \ast (-\partial_x^l)^{3/2} f - 2 \int_0^t \partial_1 G(\tau) \ast \partial_1 (-\partial_x^l)^{3/2} f \, d\tau.
$$

Hence it is easy to get (4.30) also for $l = 1$ by the Young inequality. As for (4.31), we have the equality (4.32) with $f$ replaced by $gH^1_{\{x_2 = 0\}}$, and thus,

$$
(\Gamma(t) - \Gamma(0)) \ast (gH^1_{\{x_2 = 0\}}) = -K_{1,1}(\partial^2_t g)(t) - \partial_2 K_{1,1}((-\partial_x^l)^{3/2} g)(t).
$$

(4.33)

Here $K_{1,1}$ is defined by (4.19) and we have used the fact that $g$ is time-independent in this case. Then by using (4.21) and by noting that $\partial_2 g = 0$, we conclude that $\|\Gamma(t) - \Gamma(0)\ast (gH^1_{\{x_2 = 0\}})\|_{L^p}$ is bounded by the right-hand side of (4.31). As for $\|x_2^m \partial_2^l (\Gamma(t) - \Gamma(0)) \ast (gH^1_{\{x_2 = 0\}})\|_{L^p}$, from the inequality $|x_2^m \partial_2^l G(\tau, x)| \leq C\tau^{(m-\gamma)/2} G(2\tau, x)$ it is not difficult to see that $t^{-(m-\gamma)/2} \|x_2^m \partial_2^l (\Gamma(t) - \Gamma(0)) \ast (gH^1_{\{x_2 = 0\}})\|_{L^p}$ is estimated just as same as $\|\Gamma(t) - \Gamma(0)\ast (gH^1_{\{x_2 = 0\}})\|_{L^p}$. We omit the details here. This completes the proof.
4.1.3 Estimate for velocity field

**Proposition 4.6** Let \( J(f) \) be the vector field defined by (4.9). Then it follows that

\[
\|J(f)\|_{L^p} \leq C \|f\|_{L^2}^{1/2} \|f\|_{L^1}^{1/2}, \quad 4 \leq p \leq \infty, \tag{4.34}
\]

\[
\|J(f)\|_{L^\infty} \leq C \|f\|_{L^1} + C_m \|x_2^n f\|_{L^2} \tag{4.35}
\]

\[
\|J_1(f)\|_{L^2 L^2} \leq C \|f\|_{L^2}, \tag{4.36}
\]

\[
\|x_2^{-1} J_2(f)\|_{L^2 L^2} \leq C \|f\|_{L^2}, \tag{4.37}
\]

\[
\|\nabla J(f)\|_{L^p} + \|J(\nabla f)\|_{L^p} \leq C \|f\|_{L^p} \tag{4.38}
\]

**Proof.** We first note that (4.38) follows from the Calderón-Zygmund inequality. As for (4.34), the case \( p = 4 \) is derived from the Hardy-Littlewood-Sobolev inequality, and the case \( p = \infty \) follows from the Gagliardo-Nirenberg inequality \( \|J(f)\|_{L^\infty(\mathbb{R}^2)} \leq C \|\nabla J(f)\|_{L^2}^{1/2} \|J(f)\|_{L^2}^{1/2} \) and by applying (4.38) and (4.34) with \( p = 4 \). The case \( 4 < p < \infty \) is then obtained by the interpolation. Est.(4.35) is derived from the inequality

\[
\|J(f)(x)\| \leq C \int_{y_2 \leq \frac{y_1}{2}} |f(y)| \, dy + C_m \int_{y_2 \geq \frac{y_1}{2}} \frac{1}{|x - y|} |y_2^n f(y)| \, dy \quad \text{for } x_2 \geq 1,
\]

and then by applying the same argument as in the proof of (4.34) with \( p = \infty \) to the second term of the right-hand side of the above inequality. To prove (4.36) we note that

\[
\|J_1(f)(x)\| \leq C \int_{\mathbb{R}^2} \left( \frac{|x_2 - y_2|}{|x_1|^2} + \frac{|x_2 + y_2|}{|x_1|^2} \right) |f(y)| \, dy,
\]

then the desired estimate follows from the Young inequality

\[
\|J_1(f)(x)\|_{L^2 L^2} \leq C \int_{\mathbb{R}^2} \left( \frac{|x_2 - y_2|}{|x_1|^2} + \frac{|x_2 + y_2|}{|x_1|^2} \right) |f(x, y)| \, dy \leq C \|f\|_{L^2 L^2}.
\]

Finally, we have for \( J_2(f) \),

\[
J_2(f)(x) = \int_0^2 \partial_2 J_1(f)(x_1, y_2) \, dy_2 - \int_0^2 J_1(\partial_1 f)(x_1, y_2) \, dy_2.
\]

Thus (4.37) holds by (4.36). This completes the proof.

4.2 Estimate for \( w_{E,1} \)

Set \( g = -J_1(u_E \cdot \nabla \omega_E) \big|_{x_2=0} \). Then Theorem 3.1 implies

\[
w_{E,1}(t) = - \int_0^t e^{(t-s) \Delta_E} (g(s) \mathcal{H}^1_{x_2=0}) \, ds - \int_0^t \left( \Gamma(\nu(t-s)) - \Gamma(0) \right) * (g(s) \mathcal{H}^1_{x_2=0}) \, ds \tag{4.39}
\]

We also recall that \( \omega_{BL} \) is defined by (4.3).

**Proposition 4.7** Let \( 1 \leq p \leq \infty, \ 0 \leq k \leq 4, \ l = 0, 1, \) and \( m > 0 \). Let \( 0 < t \leq 1 \). Then we have

\[
\|\partial_t^k \omega_{BL}(t)\|_{L^p} \geq C_1 \nu^{(1-k)/2} t^{(1-k)/2} \|\partial_t^k g(0)\|_{L^2}, \tag{4.40}
\]

\[
\|\partial_t^k \omega_{BL}(t)\|_{L^p} \leq C \nu^{(1-k)/2} t^{(1-k)/2}, \tag{4.41}
\]

\[
\|\partial_t^k w_{E,1,1}(t) - \partial_t^k \omega_{BL}(t)\|_{L^p} \leq C \nu^{(1-k)/2} t^{(1-k)/2} (t^{1+k} + \nu^2), \tag{4.42}
\]

\[
\|\partial_t^k w_{E,1,1}(t) + (\nu t)^{-m/2} |x_2|^m \partial_t^k \omega_{E,1,1}(t)\|_{L^p} \leq C \nu^{(1-k)/2} t^{(1-k)/2} (t^{1+k} + \nu^2), \tag{4.43}
\]

\[
\|\partial_t^k w_{E,1,2}(t) + (\nu t)^{-m/2} |x_2|^m \partial_t^k \omega_{E,1,2}(t)\|_{L^p} \leq C \nu^{(1-k)/2} t^{(1-k)/2}, \tag{4.44}
\]

\[
\|\partial_t^k w_{E,1}(t)\|_{L^2 L^2} \leq C t. \tag{4.45}
\]
Proposition 4.9

Proposition 4.8

4.4 Estimate for $w_{E,2}$

By Theorem 3.1 the function $w_{E,2}$ is written as

$$w_{E,2}(t) = \nu \int_0^t (\nu(t-s)^B - \Gamma(0) \star \nu t^\Delta \omega_E) \, ds - \nu \int_0^t (\nu(t-s)^B - \Gamma(0) \star J_1(\Delta \omega_E) \mathcal{H}^1_{\{(x_2=0)\}}) \, ds =: w_{E,2,1}(t) + w_{E,2,2}(t). \tag{4.46}$$

Proposition 4.8 Let $4/3 \leq p \leq \infty$, $0 \leq k \leq 4$, $l = 0, 1$, and $m > 0$. Then we have

$$\| \partial_1^k \partial_2^l w_{E,2,1}(t) \|_{L^p} \leq C \nu t, \tag{4.47}$$

$$\| \partial_1^k \partial_2^l w_{E,2,2}(t) \|_{L^p} + (\nu t)^{-\frac{m-1}{2}} \| \partial_1^k \partial_2^l w_{E,2,2}(t) \|_{L^p} \leq C(\nu t)^{\frac{1}{2} + \frac{l}{2}}, \tag{4.48}$$

$$\| \partial_1^k \partial_2^l w_{E,2,2}(t) \|_{L^1_{x_1} L^\infty_{x_2}} \leq C \nu t. \tag{4.49}$$

Proof. The proof of (4.48) and (4.49) is the same as in (4.43)-(4.45). Indeed, in this case it suffices to take $g$ as $-\nu J_1(\Delta \omega_E) |_{x_2=0}$. So we omit the details. To estimate $w_{E,2,1}$ we decompose it as

$$w_{E,2,1}(t) = \nu \int_0^t (\nu(t-s)^\Delta \omega_E) \, ds + \nu \int_0^t (\Gamma(\nu(t-s) - \Gamma(0)) \star \Delta \omega_E) \, ds =: w_{E,2,1,1}(t) + w_{E,2,1,2}(t).$$

From $\partial_2 \nu^\Delta \omega f = \nu^\Delta \partial_2 f$ and the Young inequality we have

$$\| \partial_1^k \partial_2^l w_{E,2,1,1}(t) \|_{L^p} \leq C \nu \int_0^t \| \partial_1^k \partial_2^l \Delta \omega_E \|_{L^p} \, ds \leq C \nu t.$$

By using (4.30) the function $w_{E,2,1,2}$ is estimated as $\| \partial_1^k \partial_2^l w_{E,2,1,2}(t) \|_{L^p} \leq C(\nu t)^{\frac{3}{2} - \frac{m}{2}}$. This completes the proof.

4.4 Estimate for $F$

Let $F$ be the function defined by (4.13), which is decomposed as $F = \sum_{i=1}^3 F_i$ with

$$F_1 = J(\omega_E + w_E) \cdot \nabla w_{E,1}, \quad F_2 = J(\omega_E + w_E) \cdot \nabla w_{E,2}, \quad F_3 = J(w_E) \cdot \nabla \omega_E. \tag{4.50}$$

Proposition 4.9 Let $1 \leq p \leq \infty$, $4/3 \leq q \leq \infty$, $k = 0, 1$, and $m \geq 0$. Let $0 < t \leq 1$. Then

$$\| \partial_1^k \partial_2^l F_1(t) \|_{L^p} \leq C \nu^{-1} (\nu t^{-\frac{1}{2}} + 1)(\nu t)^{\frac{1}{2} + \frac{m}{2}} + 1), \tag{4.51}$$

$$\| \partial_1^k \partial_2^l F_2(t) \|_{L^p} \leq C \nu t + C \nu t^{\frac{1}{2} + \frac{m}{2}} + C(\nu t)^{\frac{3}{2}}, \tag{4.52}$$

$$\| \partial_1^k \partial_2^l F_3(t) \|_{L^p} \leq C \nu t^{\frac{1}{2} + \frac{m}{2}} + C(\nu t)^{\frac{3}{2}}, \tag{4.53}$$

$$\| \partial_1^k \partial_2^l F_3(t) \|_{L^p} \leq C \nu t^{\frac{1}{2} + \frac{m}{2}} + C(\nu t)^{\frac{3}{2}}. \tag{4.54}$$
In particular, we have
\[
\| \partial^k F(t) \|_{L^p} \leq C \nu^{-2}(\nu t)^{\frac{3}{4} + \frac{1}{p}} + C \nu^{-1}(\nu t)^{1 + \frac{1}{p}} + C(\nu t)^{\frac{2}{3}} \quad 1 \leq p < \frac{4}{3},
\]
(4.55)
\[
\| \partial^k F(t) \|_{L^p} \leq C \nu^{-2}(\nu t)^{\frac{3}{4} + \frac{1}{p}} + C \nu^{-1}(\nu t)^{1 + \frac{1}{p}} + C(\nu t)^{\frac{4}{3}} \quad \frac{4}{3} \leq p \leq \infty.
\]
(4.56)

**Remark 4.10** Although it is possible to derive slightly better estimates for \(F_2\) and \(F_3\), (4.52) and (4.53) are enough for the proof of Theorem 4.1. If \(b \in W^{1,1}(\mathbb{R}^4_+\)) in addition, then (4.54) and (4.56) hold also for \(1 \leq q \leq (p) \leq 4/3\).

**Proof.** It suffices to consider the case \(k = 0\). Since \(J(\omega + w)\) satisfies the no-slip boundary condition, we have
\[
J_1(\omega + w)(x) = -\int_0^{y_2} (\omega + w)(x_1, y_2) \, dy_2 - \int_0^{y_2} \int_0^{y_2} J_1(\partial^2_1(\omega + w))(x_1, z_2) \, dz_2 \, dy_2,
\]
and
\[
J_2(\omega + w)(x) = \int_0^{y_2} \int_0^{y_2} \partial_1(\omega + w)(x_1, z_2) \, dz_2 \, dy_2 - \int_0^{y_2} \int_0^{y_2} J_2(\partial^2_1(\omega + w))(x_1, z_2) \, dz_2 \, dy_2.
\]
The estimates (4.34)-(4.37) with \(m \gg 1\) and Propositions 4.7-4.8 imply
\[
\|J(\partial^2_1(\omega + w))\|_{L^\infty} \leq C \quad \text{for} \quad 0 \leq t \leq 1.
\]
Thus we have from (4.45), (4.47), and (4.49),
\[
|J_1(\omega + w)(x)| \leq C(x_2 + x_2^2 + t), \quad |J_2(\omega + w)(x)| \leq Cx_2(x_2 + t).
\]
(4.57)
This yields
\[
|F_1(t, x)| \leq C(x_2 + x_2^2 + t)|\partial_1 w_{E,1}(t, x)| + Cx_2(x_2 + t)|\partial_2 w_{E,2}(t, x)|,
\]
and hence, (4.51) holds by (4.43) and (4.44). Next we consider (4.52). We decompose \(w_{E,2}\) as in (4.46), and then (4.47) yields
\[
\|J(\omega + w) \cdot \nabla w_{E,2} \|_{L^p} \leq C \|J(\omega + w)\|_{L^\infty} \|\nabla w_{E,2}\|_{L^p} \leq C\nu t.
\]
On the other hand, the term \(J(\omega + w) \cdot \nabla w_{E,2}\) is estimated as in the proof of (4.51) by using (4.48) and we get
\[
\|J(\omega + w) \cdot \nabla w_{E,2} \|_{L^p} \leq C(t + (\nu t)^{\frac{1}{4}}) (\nu t)^{\frac{3}{2} + \frac{1}{p}}.
\]
This shows (4.52). As for (4.53), we write \(F_3 = F_{3,1} + F_{3,2}\) with \(F_{3,1} = J(\omega_{E,1}) \cdot \nabla \omega_{E}\). In order to estimate \(F_{3,1}\) we observe that if \(x_2 \geq (\nu t)^{1/4}\) then
\[
|J(w_{E,1})(t, x)| \leq C \int_{\mathbb{R}^2} \frac{y_2}{|x - y| |x - y|^2} |w_{E,1}(t, y)| \, dy \leq C(\nu t)^{-\frac{1}{4}} \int_{\mathbb{R}^2} \frac{y_2}{|x - y|} |w_{E,1}(t, y)| \, dy.
\]
(4.58)
Here, in the first inequality of the above estimate we have used
\[
|x_1 - y_1| \frac{1}{|x - y|^2} - \frac{1}{|x - y|^2} + \frac{y_2}{|x - y|^2} - \frac{y_2}{|x - y|^2} \leq \frac{C y_2}{|x - y||x - y|^2}.
\]
Hence by the Hölder inequality and the estimates for the Riesz potential we have
\[
\|J(w_{E,1}) \cdot \nabla \omega_{E}\|_{L^1((x_2 \geq (\nu t)^{1/4}))} \leq C(\nu t)^{-\frac{1}{4}} \|w_{E,1}\|_{L^4} = C(\nu t)^{-\frac{1}{4}} \|w_{E,1}\|_{L^4} \leq C \
\|J(w_{E,1}) \cdot \nabla \omega_{E}\|_{L^\infty((x_2 \geq (\nu t)^{1/4}))} \leq C(\nu t)^{-\frac{1}{4}} \|w_{E,1}\|_{L^4} \|w_{E,1}\|_{L^4} \leq C t.
\]
where we have used (4.43) and (4.44). Then by the interpolation we have $\|J(w, t) \cdot \nabla w\|_{L^p((x_2 \geq (\nu t)^{3/4}))} \leq C\nu^{1/(8p) + 1/(8p)}$. On the other hand, we have from Proposition 4.6,

$$
\|J(w, t) \cdot \nabla w\|_{L^p((x_2 \geq (\nu t)^{3/4}))} \leq C(\nu t)^{\frac{3}{4}} \|J(w, t) \cdot \nabla w\|_{L^2_{x_2} L^p_{t_1}} + C(\nu t)^{1 + \frac{3}{4}} \|\partial_{x_2}^{-1} J_2(w, t) \cdot \nabla w\|_{L^2_{x_2} L^p_{t_1}}
$$

$$
\leq C(\nu t)^{\frac{3}{4}} \|w_{E, 1}\|_{L^{3/5}_{x_1}} + C(\nu t)^{1 + \frac{3}{4}} \|\partial_{x_1} w_{E, 1}\|_{L^{3/5}_{x_1}}
$$

$$
\leq C\nu^{1 + \frac{3}{4}}.
$$

In the last line we have used $\|f\|_{L^p_{x_2} L^r_{t_1}} \leq C\|f\|^{rac{1}{p}}_{L^1_{x_2} L^r_{t_1}}$ and the estimates (4.43)-(4.44). Next we have from (4.47) and (4.48) that $\|J(w, t) \cdot \nabla w\|_{L^1} \leq C\|w\|_{L^{4/3}} \leq C(\nu t)^{7/8}$ and $\|J(w, t) \cdot \nabla w\|_{L^\infty} \leq C\|w, t\|_{L^{4/3}} \|w, t\|_{L^4} \leq C(\nu t)^{3/4}$. This yields $\|J(w, t) \cdot \nabla w\|_{L^p} \leq C(\nu t)^{3/4}$, and (4.53) is proved. For (4.54) it suffices to consider the case $q \leq \infty$. Instead of (4.58), we use

$$
|J(w, t)(t, x)| \leq C \int_{\mathbb{R}^2} \frac{y_2}{|x - y||x - y|^2} |w(t, y)| \, dy \leq C \int_0^\infty \frac{y_2}{|x_2 - y_2|^2} |w(t, y)| \, dy.
$$

Then, if $x_2 \geq (\nu t)^{3/4}$ and $0 < \alpha < 1$ we have

$$
\|J(w, t)(t, x, \nu t)^{3/4}\|_{L^1_{x_1}} \leq C(\nu t)^{-\frac{3}{4}} \int_0^\infty \frac{1}{|x_2 - y_2|^{1 + \frac{3}{4}}} \|y_2 w(t, y, y_2)\|_{L^1_{x_1}} \, dy.
$$

Set $b(t, x_2) = \|J(w, t)(t, x_2)\|_{L^1_{x_1}} \|\nabla w(t)\|_{L^1_{x_2}}$. Then the Hardy-Littlewood-Sobolev inequality leads to

$$
\|J(w, t) \cdot \nabla w\|_{L^q((x_2 \geq (\nu t)^{3/4}))} \leq C(\nu t)^{\frac{3}{4}} \|\nabla w_{E, 1}(t)\|_{L^{3/5}_{x_1}}
$$

$$
\leq C(\nu t)^{\frac{3}{4}} \|\nabla w_{E, 1}(t)\|_{L^{3/5}_{x_1}} \|\nabla w_{E, 1}(t)\|_{L^1_{t_1}}^{1 - \frac{3}{4}}
$$

where $1/q = 1/r - \alpha/2$. Hence (4.43)-(4.44) imply that $\|J(w, t) \cdot \nabla w\|_{L^q((x_2 \geq (\nu t)^{3/4}))} \leq C\nu^{-1}(\nu t)^{1 + 1/(2q)}$.

By the same argument as in the proof of (4.53) we also have $\|J(w, t) \cdot \nabla w\|_{L^q((x_2 \leq (\nu t)^{3/4}))} \leq C\nu^{-1}(\nu t)^{1 + 1/(2q)}$, and thus, (4.54) holds if $4/3 \leq q \leq \infty$. This completes the proof.

### 4.5 Estimate for $w$

By (4.14) and Corollary 3.3 the function $w$ is expressed as

$$
w(t) = -\frac{3}{5} \int_0^t e^{(t-s)B} (s) f_{1}(s) H_{t_0}(s) \, ds =: -\frac{3}{5} \int_0^t W_i(t).
$$

Here $g_i(t) = J(f_i(t))(x_1, 0)$ and

$$
f_0 = F, \quad f_1 = J(w, t) \cdot \nabla w,
$$

$$
f_2 = J(w) \cdot \nabla (w + w), \quad f_3 = N(w, w).
$$

**Proposition 4.11** Let $0 < \nu \ll 1$. Then there are $c_0, C > 0$ such that

$$
\|w(t)\|_{L^p} \leq C\nu^{\frac{3}{4}} t \quad \text{for} \quad 0 < t \leq c_0 \nu^{\frac{3}{4}}, \quad \frac{4}{3} \leq p \leq 4,
$$

$$
\|w(t)\|_{L^p} \leq C\nu^{\frac{3}{4}} t \quad \text{for} \quad 0 < t \leq c_0 \nu^{\frac{3}{4}}, \quad 4 < p \leq \infty.
$$

Here $c_0$ is independent of $\nu$, and $C$ is independent of $\nu$ and $t \in [0, c_0 \nu^{1/3}]$.

**Remark 4.12** In order for the arrangement of the proof we will establish the $L^p$ estimates of $w$ based on the estimates in $L^{4/3}$ and $L^1$. As a result, the $L^p$ estimates in Proposition 4.11 become slightly rough when $p > 4$. In fact, from the argument below it is possible to show that $\|w(t)\|_{L^p} \leq C\nu^{2/(3p)} t$ also for $4 < p < \infty$, but if $0 < t \leq c_p \nu^{1/3}$ with $c_p$ depending on $p$. 

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4.5.1 Estimate away from the boundary

In the region away from the boundary the energy argument is useful to estimate \( w \). The divergence free condition of \( u \) plays an essential role. We recall that \( \Omega_s \) and \( \Omega_s^\pm \) are defined by (4.16).

**Proposition 4.13** Let \( 4 \leq p < \infty \). Assume that \( \delta = \nu^{2/3} \) and \( 0 < t \leq \nu^{1/3} \). Then

\[
\|w(t)\|_{L^4(\Omega_s^\pm)} \leq C \int_0^t \|w\|_{L^4} \, ds + C\nu^{-\frac{1}{3}} \int_0^t \|w\|_{L^4} \|w\|_{L^4} \, ds + C\nu^{\frac{2}{3}} t^{\frac{7}{3}} + C\nu^{\frac{2}{3}} t^{\frac{7}{3}}, \tag{4.64}
\]

\[
\|w(t)\|_{L^p(\Omega_s^\pm)} \leq C\nu^{-\frac{1}{2}} \int_0^t \|w\|_{L^4}^\frac{p+2}{2} \|w\|_{L^4}^\frac{p-2}{2} \, ds + C\nu^{\frac{p-2}{2}} \int_0^t \|w\|_{L^p} \, ds
\]

\[
+ C\nu^{-\frac{1}{2}} \int_0^t \|w\|_{L^p} \|w\|_{L^4} \|w\|_{L^4} \, ds + C\nu^{\frac{p-2}{2}} t^{\frac{2}{3}} + C\nu^{\frac{p-2}{2}} t^{\frac{7}{3}}. \tag{4.65}
\]

Here the constant \( C \) is taken independently also of \( p \).

**Proof.** Let \( \epsilon > 0 \) and let \( \chi_\epsilon(x_o) \) be a cutoff function such that \( \chi_\epsilon = 1 \) if \( x_2 \geq 2\epsilon \) and \( \chi = 0 \) if \( x_2 \leq \epsilon \), and \( |\partial_2 \chi_\epsilon| \leq C\epsilon^{-1} \). Note that we can take \( \chi_\epsilon \) of the form \( (\chi_\epsilon)^2 \) for a suitable \( \overline{\chi}_\epsilon \). Set \( w_\epsilon = w\chi_\epsilon \). Then \( w_\epsilon \) satisfies

\[
\partial_t w_\epsilon - \nu \Delta w_\epsilon + u \cdot \nabla w_\epsilon = -\chi_\epsilon J(w) \cdot \nabla (\omega_E + w_E) - F\chi_\epsilon + w_\epsilon u_2 \partial_2 \chi_\epsilon - \nu (\partial_2 \chi_\epsilon \partial_2 w + w \partial_2^2 \chi_\epsilon), \tag{4.66}
\]

which is now considered as the equation in \( \mathbb{R}^2 \). For \( \eta > 0 \) we set

\[
\Psi_\eta(z) = (z^2 + \eta^2)^{\frac{1}{2}} - \eta, \tag{4.67}
\]

which satisfies

\[
0 \leq \Psi_\eta(z) \leq |z|, \quad |\Psi_\eta'(z)| \leq 1, \quad \Psi_\eta''(z) > 0, \quad |z|^q \Psi_\eta''(z) \leq \eta^{q-1} \quad \text{for } q \geq 1. \tag{4.68}
\]

Let \( 1 < p < \infty \). Then by the integration by parts we have

\[
\frac{d}{dt} \|\Psi_\eta(w_\epsilon)\|_{L^p}^p = -\nu p \int \Psi_\eta'(w_\epsilon)|\nabla w_\epsilon|^2 \Psi_\eta^{-1}(w_\epsilon) - \nu(p - 1) \int |\Psi_\eta'(w_\epsilon)|^2 |\nabla w_\epsilon|^2 \Psi_\eta^{-2}(w_\epsilon) \tag{4.69}
\]

\[
- p \int \Psi_\eta'(w_\epsilon) \Psi_\eta^{-1}(w_\epsilon) \chi_\epsilon J(w) \cdot \nabla (\omega_E + w_E) - p \int \Psi_\eta'(w_\epsilon) \Psi_\eta^{-1}(w_\epsilon) F\chi_\epsilon
\]

\[
+ 2\nu \int w_\epsilon \partial_2 \chi_\epsilon \partial_2 w_\epsilon \left\{ \Psi_\eta''(w_\epsilon) \Psi_\eta^{-1}(w_\epsilon) + (p - 1) \Psi_\eta'(w_\epsilon) \Psi_\eta^{-2}(w_\epsilon) \right\} \tag{4.70}
\]

\[
+ \nu \int w_\epsilon \partial_2^2 \chi_\epsilon \Psi_\eta'(w_\epsilon) \Psi_\eta^{-1}(w_\epsilon) + p \int \Psi_\eta'(w_\epsilon) \Psi_\eta^{-1}(w_\epsilon) w_\epsilon u_2 \partial_2 \chi_\epsilon \tag{4.71}
\]

\[
\leq -\frac{\nu p}{2} \int \Psi_\eta''(w_\epsilon)|\nabla w_\epsilon|^2 \Psi_\eta^{-1}(w_\epsilon) - \frac{\nu(p - 1)}{2} \int |\Psi_\eta'(w_\epsilon)|^2 |\nabla w_\epsilon|^2 \Psi_\eta^{-2}(w_\epsilon) \tag{4.72}
\]

\[
+ p \int |w_\epsilon|^{p-1} \chi_\epsilon |J(w) \cdot \nabla (\omega_E + w_E)| + p \int |w_\epsilon|^{p-1} |F| \chi_\epsilon
\]

\[
+ C\nu \int w_\epsilon^2 |\partial_2 \chi_\epsilon|^2 \Psi_\eta''(w_\epsilon)|w_\epsilon|^{p-1} + C\nu^2 \int |w_\epsilon|^{p-1} \tag{4.73}
\]

\[
+ C\nu \int \int_{|x_2| < 2\epsilon} |w_\epsilon|^{p-1} |w_\epsilon|, \tag{4.74}
\]

From (4.68) and \( \chi_\epsilon = (\overline{\chi}_\epsilon)^2 \) it is easy to check that

\[
\int w_\epsilon^2 |\partial_2 \chi_\epsilon|^2 \Psi_\eta''(w_\epsilon)|w_\epsilon|^{p-1} \to 0 \quad \text{as } \eta \to 0
\]

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by the Lebesgue convergence theorem. Hence by integrating over 0 to \( t \) and letting \( \eta \to 0 \) we get

\[
\| w(t) \|_{L^p} \leq p \int_0^t \| w(s) \|_{L^p}^p (\| \chi_\eta J(w) \cdot \nabla (\omega_E + w_E) \| + \| \chi_\eta F \|_{L^p}) \, ds \\
+ C p^2 \nu^{-2} \int_0^t \| w \|_{L^p} \, ds + C p \nu^{-1} \int_0^t \| w \|_{L^p} \| w \|_{L^p} \, ds + C p \nu^{-1} \int_0^t \| \chi_\eta J(w) \cdot \nabla (\omega_E + w_E) \|_{L^p} \, ds 
\]

Let \( \epsilon = \rho \delta^{1/2} \geq \delta^{1/2} \). Then this inequality implies

\[
\sup_{0 \leq s \leq t} \| w(s) \|_{L^p(\Omega_T)} \leq p \int_0^t \| J(w) \cdot \nabla (\omega_E + w_E) \|_{L^p(\Omega_T)} + \| F \|_{L^p(\Omega_T)} \, ds \\
+ C \int_0^t (\nu \delta^{-1} \| w \|_{L^p} + \delta^{-\frac{3}{2}} \| w \|_{L^p} \| w \|_{L^p} ) \, ds. 
\]

From (4.34) and (4.57) we have

\[
\| w_{t} \|_{L^p((\tau < \tau_{2} \rho^{\frac{1}{2}}))} \leq C \| w \|_{L^p} (\| j_{2}(w) \|_{L^\infty} + \| j_{2}(\omega_E + w_E) \|_{L^\infty((\tau < \tau_{2} \rho^{\frac{1}{2}}))}) \\
= C \| w \|_{L^p} (\| w \|_{L^{q}(\Omega_T)}^{\frac{1}{2}} \| w \|_{L^{q}(\Omega_T)}^{\frac{1}{2}} + C \rho \delta^{\frac{1}{2}} (p \delta^{\frac{1}{2}} + s)). 
\]

Let us consider \( \| J(w) \cdot \nabla (\omega_E + w_E) \|_{L^p(\Omega_T)} \). For \( p = 4/3 \) we have from Propositions 4.6, 4.7, and 4.8,

\[
\| J(w) \cdot \nabla (\omega_E + w_E) \|_{L^p(\Omega_T)} \leq \| J(w) \|_{L^1} (\| \nabla \omega_E \|_{L^2} + \| \nabla w_{E,2.1} \|_{L^2} + \delta^{-1} \| x_2 \nabla (w_{E,1} + w_{E,2.2}) \|_{L^2}) \\
\leq C \| w \|_{L^\frac{3}{2}} (1 + \nu s + \delta^{-1} \nu + \delta^{-1} s). 
\]

Similarly, for \( p \in [4, \infty) \) we have from (4.34) and Propositions 4.7 and 4.8,

\[
\| J(w) \cdot \nabla (\omega_E + w_E) \|_{L^p(\Omega_T)} \leq \| J(w) \|_{L^1} (\| \nabla \omega_E \|_{L^\infty} + \| \nabla w_{E,2.1} \|_{L^\infty} + \delta^{-1} \| x_2 \nabla (w_{E,1} + w_{E,2.2}) \|_{L^\infty}) \\
\leq C \| w \|_{L^\frac{3}{2}} (1 + \nu s + \delta^{-1} s). 
\]

As for \( F \), we have from Proposition 4.9,

\[
\| F \|_{L^p(\Omega_T)} \leq \| F_{1} \|_{L^p(\Omega_T)} + \| F_{2} \|_{L^p} + \| F_{3} \|_{L^p} \\
\leq \delta^{-\frac{3}{2}} \| x_2 \|_{L^p}^{\frac{3}{2}} \| F_{1} \|_{L^p} + C(\nu t^{\frac{1}{2}}) + C \nu^{\frac{1}{2}} t^{\frac{1}{2}} \\
\leq C \delta^{-\frac{3}{2}} \nu^{-2} (\nu t^{\frac{1}{2}}) + C(\nu t^{\frac{1}{2}}) + C \nu^{\frac{1}{2}} t^{\frac{1}{2}} 
\]

for \( 4/3 \leq p \leq \infty \). Let us take \( m > 0 \) large enough. Then, collecting these above, we have for \( 0 < t < 1 \),

\[
\| w(t) \|_{L^\frac{3}{2}(\Omega_T)} \leq C (1 + \nu^{-\frac{3}{2}} t^{\frac{1}{2}}) \int_0^t \| w \|_{L^\frac{3}{2}} \, ds + C \nu^{-\frac{1}{2}} \int_0^t \| w \|_{L^\frac{1}{2}} \, ds + C \nu^{\frac{1}{2}} t^{\frac{1}{2}} + C \nu^{\frac{1}{2}} t^{\frac{1}{2}}, 
\]

and for \( p \in [4, \infty) \),

\[
\| w(t) \|_{L^p(\Omega_T)} \leq C p (1 + \nu^{-\frac{3}{2}} t) \int_0^t \| w \|_{L^\frac{3}{2}} \, ds + C (p^2 \nu^{\frac{1}{2}} + pt) \int_0^t \| w \|_{L^p} \, ds \\
+ C \nu^{-\frac{1}{2}} \int_0^t \| w \|_{L^p} \, ds + C \nu^{\frac{1}{2}} t^{\frac{1}{2}} + C \nu^{\frac{1}{2}} t^{\frac{1}{2}}. 
\]

Hence Proposition 4.13 follows from \( 0 < t \leq \nu^{1/3} \). This completes the proof.
4.5.2 Estimate for $W_0$

**Proposition 4.14** Let $1 < p < \infty$ and $0 < \kappa < 1$. Assume that $0 < t \leq \nu^{1/3}$. Then

$$
\|W_0(t)\|_{L^p} \leq C\nu^{-2+\frac{1}{p}}(\nu t)^{\frac{1}{2}}, \\
\|W_0(t)\|_{L^\infty} \leq C\nu^{-2-\kappa}(\nu t)^{\frac{1}{2}}.
$$

**Proof.** If $1 < p < \infty$ then we have from (3.13) and (3.14),

$$
\|W_0(t)\|_{L^p} \leq C \int_0^t \|F(s)\|_{L^p} ds + C \int_0^t (\nu(t-s))^{-\frac{1}{2}}|g_0(s)|_{L^p_1} ds.
$$

By (4.36) we have $\|g_0(s)\|_{L^p_1} \leq C\|F(s)\|_{L^p_2} \leq C\|F(s)\|_{L^p_1}^{1/p}\|\partial_t F(s)\|_{L^p_1}^{1-1/p}$. Thus (4.55) yields

$$
\|W_0(t)\|_{L^p} \leq C\nu^{-3}(\nu t)^{-\frac{3}{4}+\frac{1}{p}} + C\nu^{-2}(\nu t)^{2+\frac{1}{p}} + C\nu^{-1}(\nu t)^{\frac{1}{2}} \\
+ C\nu^{-1}(\nu t)^{-\frac{1}{2}+\frac{1}{p}}(\nu^{-3}(\nu t)^{\frac{1}{2}} + \nu^{-2}(\nu t)^{\frac{1}{2}} + \nu^{-1}(\nu t)^{\frac{1}{2}}).
$$

if $0 < t \leq \nu^{1/3}$. This gives (4.76). If $p = \infty$ then we have, instead of (4.78),

$$
\|W_0(t)\|_{L^\infty} \leq C \int_0^t ((\nu(t-s))^{-\frac{1}{2}}|F(s)|_{L^\infty} ds + C \int_0^t (\nu(t-s))^{-\frac{1}{2}}|g_0(s)|_{L^\infty_1} ds
$$

for all $2 < q < \infty$. This implies $\|W_0(t)\|_{L^\infty} \leq C\nu^{-2-2/(3\delta)}(\nu t)^{\frac{1}{2}}$ if $0 < t \leq \nu^{1/3}$, which completes the proof.

4.5.3 Estimate for $W_1$

**Proposition 4.15** Let $4 \leq p \leq \infty$. Assume that $0 < t \leq \nu^{1/3}$. Then

$$
\|W_1(t)\|_{L^\frac{4}{3}} \leq C\nu^{-\frac{1}{3}}\int_0^t (t-s)^{-\frac{1}{2}}\|w\|_{L^\frac{4}{3}} ds \\
+ C\nu^{-\frac{1}{3}}\int_0^t \|w\|_{L^\frac{4}{3}} ds + C\nu^{-\frac{1}{3}}\int_0^t \|w\|_{L^\frac{4}{3}}^\frac{1}{2} \|w\|_{L^4}^\frac{1}{2} ds + C\nu^{-\frac{1}{3}}\|w\|_{L^4},
$$

$$
\|W_1(t)\|_{L^p} \leq C\nu^{-\frac{1}{3}+\frac{1}{p}}\int_0^t (t-s)^{-\frac{1}{2}+\frac{1}{p}} \|w\|_{L^p_1} ds + C\nu^{-1+\frac{1}{p}}\int_0^t \|w\|_{L^p_2} ds + C\nu^{-\frac{1}{3}+\frac{1}{p}}\int_0^t \|w\|_{L^p_1} ds \\
+ C\nu^{-1+\frac{1}{p}}\int_0^t \|w\|_{L^p_2} ds + C\nu^{-2}(\nu t)^{\frac{1}{2}} + C\nu^{-3}(\nu t)^{\frac{1}{2}} + C\nu^{-3}(\nu t)^{\frac{1}{2}}.
$$

**Proof.** We give the proof only for (4.81) since (4.80) is obtained in the same manner. From (3.16) and (4.38) we have

$$
\|W_1(t)\|_{L^p} \leq C \int_0^t (\nu(t-s))^{-\frac{1}{2}+\frac{1}{p}}\|\nabla^\perp(-\Delta_D)^{-1}(J(\omega_E + w_E) \cdot \nabla w)\|_{L^1} ds \\
\leq C \int_0^t (\nu(t-s))^{-\frac{1}{2}+\frac{1}{p}}\|J(\omega_E + w_E)w\|_{L^1} ds.
$$

Set $\delta = \nu^{2/3}$. Then by (4.57) we have for $0 < s \leq t \leq \nu^{1/3}$,

$$
\|J(\omega_E + w_E)w\|_{L^1(\Omega^c)} \leq C(\delta^{\frac{1}{2}} + s)\|w\|_{L^1} \leq C\nu^{\frac{1}{2}}\|w\|_{L^1}.
$$

On the other hand, from (4.34) for $J(\omega_E)$, (4.36) for $J_1(w_E)$, and (4.35)-(4.37) with $\mu \gg 1$ for $J_2(w_E)$, we have

$$
\|J(\omega_E + w_E)w\|_{L^1(\Omega^c)} \leq C(\|J(\omega_E)\|_{L^\infty_1} + \|J_1(w_E)\|_{L^\infty_1} + \|J_2(w_E)\|_{L^\infty_1})\|w\|_{L^1(\Omega^c)} \leq C\|w\|_{L^1(\Omega^c)}.
$$

Here we have also used Propositions 4.7, 4.8, and $0 < t \leq 1$. Then (4.81) follows from (4.65). This completes the proof.
4.5.4 Estimate for $W_2$

Set $f_{2,1}(t) = J(w) \cdot \nabla \omega_E$, $f_{2,2}(t) = J(w) \cdot \nabla w_E$, $g_{2,j}(t,x) = J_t(f_{2,j}(t))(x,0)$, and

$$W_{2,j}(t) = \int_0^t e^{\nu(t-s)B} (f_{2,j}(s) - g_{2,j}(s)H_{1\nu}(t_2=0)) \, ds.$$  \hfill (4.82)

**Proposition 4.16** Let $4 \leq p \leq \infty$. Assume that $0 < t \leq \nu^{1/3}$. Then

$$\|W_{2,1}(t)\|_{L^4} \leq C\nu^{-\frac{1}{4}} \int_0^t (t-s)^{-\frac{1}{4}} \|w\|_{L^4} \, ds,$$  \hfill (4.83)

$$\|W_{2,1}(t)\|_{L^p} \leq C\nu^{-\frac{1}{4}} \int_0^t (t-s)^{-\frac{1}{4}} \|w\|_{L^4} \, ds,$$  \hfill (4.84)

$$\|W_{2,2}(t)\|_{L^4} \leq C\nu^{-\frac{1}{4}} \int_0^t (t-s)^{-\frac{1}{4}} \|w\|_{L^4} \, ds,$$  \hfill (4.85)

$$\|W_{2,2}(t)\|_{L^p} \leq C\nu^{-\frac{1}{4}} \int_0^t (t-s)^{-\frac{1}{4}} \left(\|w\|_{L^4} + \nu^{\frac{1}{2}} \|w\|_{L^8}^2\right) \, ds.$$  \hfill (4.86)

The other estimates are proved by the similar arguments. By the definition of $W_{2,1}$, (3.13) and (3.14) yield

$$\|W_{2,1}(t)\|_{L^p} \leq C\int_0^t \left(\nu(t-s)^{-\frac{1}{4}} \|f_{2,1}\|_{L^4} + \nu(t-s)^{-\frac{1}{4}} \|w\|_{L^4} \right) \, ds$$

Then (4.84) follows from

$$\|f_{2,1}(s)\|_{L^4} \leq C\|J(w)\|_{L^4} \leq C\|w\|_{L^4}^2,$$

$$\|g_{2,1}(s)\|_{L^4} \leq C\|J(w)\cdot \nabla \omega_E\|_{L^2_t L^4_x} \leq C\|J(w)\|_{L^4} \|\nabla \omega_E\|_{L^4_t L^4_x} \leq C\|w\|_{L^4}.$$  \hfill (4.87)

Here we have used Proposition 4.6. Next we consider $W_{2,2}$. By (3.16) we have

$$\|W_{2,2}(t)\|_{L^p} \leq C\int_0^t (\nu(t-s))^{-\frac{1}{4} + \frac{1}{p}} \|\nabla (-D)^{-1} (J(w) \cdot \nabla w_E)\|_{L^4} \, ds$$

$$\leq C\int_0^t (\nu(t-s))^{-\frac{1}{4} + \frac{1}{p}} \left(\|J(w)w_{E,1}\|_{L^4} + \|J(w)w_{E,2}\|_{L^4}\right) \, ds.$$  \hfill (4.88)

Since $J(w)$ satisfies the no-slip boundary condition we have from (4.33)-(4.44),

$$\|J(w)w_{E,1}\|_{L^4} = \|\int_0^t \partial_2 J(w) \, dy w_{E,1}\|_{L^4} \leq C\|\partial_2 J(w)\|_{L^4} \|w_{E,1}\|_{L^2_t L^\infty_x}$$

$$\leq C\|w\|_{L^4} \leq C\nu^\frac{1}{2} \|w\|_{L^4}.$$  \hfill (4.89)

By using Proposition 4.8 the term $J(w)w_{E,2}$ is estimated as

$$\|J(w)w_{E,2}\|_{L^4} \leq \|J(w)\|_{L^4} \|w_{E,2}\|_{L^\infty} \leq C(\nu t)\|w\|_{L^4} \leq C\nu^\frac{1}{2} \|w\|_{L^4}$$

if $0 < t \leq \nu^{1/3}$. This shows (4.86). The proof of Proposition 4.16 is completed.

4.5.5 Estimate for $W_3$

**Proposition 4.17** Let $4 \leq p \leq \infty$. Then

$$\|W_3(t)\|_{L^4} \leq C\int_0^t (\nu(t-s))^{-\frac{1}{4}} \|w(s)\|_{L^4}^\frac{3}{2} \|w(s)\|_{L^4}^\frac{1}{2} \, ds,$$  \hfill (4.88)

$$\|W_3(t)\|_{L^p} \leq C\int_0^t (\nu(t-s))^{-\frac{1}{4} + \frac{1}{p}} \|w(s)\|_{L^4}^\frac{3}{2} \|w(s)\|_{L^4}^\frac{1}{2} \, ds.$$  \hfill (4.89)
4.7. This completes the proof of Theorem 4.1.

By Propositions 4.6, 4.7, 4.8, and 4.11, one can check that

Thus we get

which implies

Collecting the estimates in Propositions 4.14 - 4.17, we get for $0 \leq t \leq c_0 t^{1/3}$.

\[ \| \nabla^\perp (J(w) \cdot \nabla w) \|_{L^4} \leq C \| J(w) w \|_{L^4} \leq C \| J(w) \|_{L^\infty} \| w \|_{L^4} \leq C \| w \|_{L^4}^{1/4} \| w \|_{L^4}^{3/4} \] .

Here we have used Proposition 4.6. This completes the proof.

4.5.6 Proof of Proposition 4.11

Let $0 < t \leq c_0 t^{1/3}$, where $0 < c_0 < 1$ will be taken small enough. Set $\| w \|_{X_0} = \sup_{0 \leq t \leq c_0 t^{1/3}} \| w(t) \|_{L^p}$.

Collecting the estimates in Propositions 4.14 - 4.17, we get for $0 < t \leq c_0 t^{1/3}$,

\[ \| w(t) \|_{L^4} \leq \sum_{i=0}^{3} \| W_i(t) \|_{L^4} \leq C \| X_0 \| \| w \|_{X_4} + C \nu^{-\frac{1}{4}} \| w \|_{X_4}^\frac{3}{2} \| w \|_{X_4}^\frac{1}{2} + C \nu^\frac{3}{2} , \] (4.90)

and

\[ \| w(t) \|_{L^4} \leq \sum_{i=0}^{3} \| W_i(t) \|_{L^4} \leq C \| X_0 \| \| w \|_{X_4} + C \nu^{-\frac{1}{4}} \| w \|_{X_4}^\frac{3}{2} \| w \|_{X_4}^\frac{1}{2} + C \nu^\frac{3}{2} , \] (4.91)

Then it is easy to see that $\| w \|_{X_{1/3}} \leq C \nu^{\frac{3}{4}}$ and $\| w \|_{X_4} \leq C \nu^{\frac{3}{2}}$ for some $C > 0$ if $c_0$ and $\nu$ are sufficiently small. Note that $c_0$ and $C$ are taken independent of $\nu$ if $0 < \nu \ll 1$. Then Propositions 4.14 - 4.17, $\| w \|_{X_{1/3}} \leq C \nu^{\frac{3}{4}}$, and $\| w \|_{X_4} \leq C \nu^{\frac{3}{2}}$, yield

\[ \| W_i(t) \|_{L^4} \leq C \nu^{\frac{3}{4}} t^{\frac{1}{2}} , \quad \| W_i(t) \|_{L^4} \leq C \nu^{\frac{3}{2}} t^{\frac{1}{2}} \quad i = 0, 1, 2, 3 , \]

which implies $\| w(t) \|_{L^4} \leq C \nu^{3/2} t^{1/2}$ and $\| w(t) \|_{L^4} \leq C \nu^{1/3} t^{1/2}$. Repeating this argument again, we get

\[ \| W_i(t) \|_{L^4} \leq C \nu^\frac{3}{4} t , \quad \| W_i(t) \|_{L^4} \leq C \nu^\frac{3}{2} t \quad i = 0, 1, 2, 3 . \]

Thus we get $\| w(t) \|_{L^4} \leq C \nu^{1/2} t$ and $\| w(t) \|_{L^4} \leq C \nu^{1/6} t$. By the interpolation we have $\| w(t) \|_{L^p} \leq C \nu^{2/(3p)} t$ if $4/3 \leq p \leq 4$. Then Propositions 4.14 - 4.17 yield $\| w(t) \|_{L^p} \leq C \nu^{-1/6 + 4/(3p)} t$ for all $4 < p \leq \infty$. This completes the proof of Proposition 4.11.

4.6 Proof of Theorem 4.1

By Propositions 4.6, 4.7, 4.8, and 4.11, one can check that $u = J(\omega) = J(\omega_E) + J(w_{E,1}) + J(w_{E,2}) + J(w)$ is uniformly bounded in $(L^\infty(\mathbb{R}^2))^2$ with respect to $0 < \nu \ll 1$ and $0 < t \leq c_0 \nu^{1/3}$. Let $\omega_{BL}$ be the function defined by (4.3). Then from (4.42), (4.47), (4.48), (4.63), and (4.63), we have for $0 < t \leq c_0 \nu^{1/3}$,

\[ \| \omega(t) - \omega_{E}(t) - \omega_{BL}(t) \|_{L^p} \leq C \nu^{-\frac{1}{4}} \| t \|^{\frac{1}{2}} + C \nu^{\frac{3}{4}} t + C t + C(\nu t)^{\frac{1}{2} (1 + \frac{\nu}{2})} + C \nu^{-\frac{1}{4}} \| t \|^{\frac{1}{2} (1 + \frac{\nu}{2})} . \]

This proves (4.2). The other statements in the theorem follow from the definition of $\omega_{BL}$ and Proposition 4.7. This completes the proof of Theorem 4.1.
5 Appendix

5.1 Proof of solution formula

For simplicity we consider the case $b = 0$ in (LV)-(LBC). It is easy to recover the case $b \neq 0$ from this case. Let $\tilde{\omega}$ be the Fourier-Laplace transform of $\omega$ defined by

$$\tilde{\omega}(s, \xi_1, x_2) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_0^\infty \int_\mathbb{R} \omega(t, x_1, x_2) e^{-ix_1\xi_1-t s} dx_1 dt. \tag{5.1}$$

Then (LV)-(LBC) is converted to

$$\partial_t^2 \tilde{\omega} - \left(\frac{s}{\nu} + \xi_1^2\right) \tilde{\omega} = -\frac{\tilde{f}}{\nu}, \quad x_2 > 0, \tag{5.2}$$

$$\partial_t \tilde{\omega} + |\xi_1| \tilde{\omega} = \frac{\tilde{g}}{\nu}, \quad x_2 = 0. \tag{5.3}$$

Set $\alpha = s/\nu + \xi_1^2$. Solving this ODE under a decay condition at spatial infinity, we get

$$\tilde{\omega}(s, \xi_1, x_2) = \left(\frac{1}{s} \left(\frac{\xi_1^2}{\sqrt{\alpha}} + |\xi_1|\right) + \frac{1}{2\nu \sqrt{\alpha}}\right) \int_0^\infty e^{-\sqrt{\alpha}(x_2+y_2)} \tilde{f}(s, \xi_1, y_2) dy_2$$

$$+ \frac{1}{2\nu \sqrt{\alpha}} \left(\int_0^{x_2} e^{-\sqrt{\alpha}(x_2-y_2)} \tilde{f}(s, \xi_1, y_2) dy_2 + \int_{x_2}^\infty e^{-\sqrt{\alpha}(y_2-x_2)} \tilde{f}(s, \xi_1, y_2) dy_2 \right)$$

$$- \left(\frac{1}{s} \left(\frac{\xi_1^2}{\sqrt{\alpha}} + |\xi_1|\right) + \frac{1}{2\nu \sqrt{\alpha}}\right) e^{-\sqrt{\alpha}x_2} \tilde{g}(s, \xi_1). \tag{5.4}$$

Inverting the Fourier-Laplace transform, we have

$$\omega(t, x_1, x_2)$$

$$= \int_0^t \int_{\mathbb{R}^2} \left(G(\nu(t-s), x-y) + G(\nu(t-s), x-y^*)\right) f(s, y) dy ds$$

$$- 2 \int_0^t \int_{\mathbb{R}} G(\nu(t-s), x_1-y_1, x_2) g(s, y_1) dy_1 ds$$

$$- 2\nu \int_0^t \int_{\mathbb{R}^2} \left(\partial_1^2 + (-\partial_1^2)^{\frac{1}{2}} \partial_2\right) G(\nu(s-\tau), x-y^*) f(\tau, y) dy d\tau ds$$

$$+ 2\nu \int_0^t \int_{\mathbb{R}^2} \left(\partial_1^2 + (-\partial_1^2)^{\frac{1}{2}} \partial_2\right) G(\nu(s-\tau), x_1-y_1, x_2) g(\tau, y_1) dy_1 d\tau ds$$

$$= \int_0^t e^{\nu(t-s)\Delta} \left(f(s) - g(s) \mathcal{H}_{x_2=0}^1\right) ds - \nu \int_0^t \int_0^\infty \left(\mathcal{F}G(\nu(s-\tau))\right) * \left(f(\tau) - g(\tau) \mathcal{H}_{x_2=0}^1\right) d\tau ds \tag{5.5}$$

From the equality $G(t) = -\partial_t(-\Delta_{\mathbb{R}^2})^{-1} G(t)$ the second term of the right-hand side of (5.5) is written in $\mathbb{R}^2_+$.
as

\[-\nu \int_0^t \int_0^s (\Xi G(\nu(s - \tau))) \ast (f(\tau) - g(\tau) H^1_{\{x_2 = 0\}}) \, d\tau \, ds \]

\[= \nu \int_0^t \int_0^s (\Xi(\partial_k(-\Delta R^{-1}G(\nu(s - \tau)))) \ast (f(\tau) - g(\tau) H^1_{\{x_2 = 0\}}) \, d\tau \, ds \]

\[= \int_0^t \int_0^s \partial_s (\Xi(-\Delta R^{-1}G(\nu(s - \tau))) \ast (f(\tau) - g(\tau) H^1_{\{x_2 = 0\}}) \, d\tau \, ds \]

\[= \int_0^t \int_0^s \partial_s (\Xi(-\Delta R^{-1}G(\nu(s - \tau))) \ast (f(\tau) - g(\tau) H^1_{\{x_2 = 0\}}) \, d\tau \, ds \]

\[= \nu \int(t - s) \ast (f(s) - g(s) H^1_{\{x_2 = 0\}}) \, ds - \int_0^t \Gamma(0) \ast (f(s) - g(s) H^1_{\{x_2 = 0\}}) \, ds. \tag{5.6} \]

This completes the proof.

### 5.2 Pointwise estimate of $\Gamma$

**Proposition 5.1** Let $k, l \in \mathbb{N} \cup \{0\}$. Then there is $C$ such that

\[|\partial^k_1 \partial^l_2 \Gamma(1, x)| \leq C \left(1 + \frac{|x_1|^{2+k}}{\log(e + |x_1|)^{\delta_n}} + |x_2|^{2+k+l}\right)^{-1}. \tag{5.7} \]

Here $\delta_n$ is Kronecker's delta. In particular, it follows that

\[|\partial^k_1 \partial^l_2 \Gamma(t, x)| \leq Ct^{-\frac{k+l+2}{2}} \left(1 + \frac{|x_1|^{2+k}}{\log(e + |x_1|^{\frac{1}{2}})} + |x_2|^{2+k+l}\right)^{-1}. \tag{5.8} \]

**Proof.** Let $0 < R < 1$ and let $\chi_R$ be a cutoff function on $\mathbb{R}$ such that $\chi_R(r) = 1$ if $|r| \leq R$ and $\chi_R(r) = 0$ if $|r| \geq 2R$, and $|\partial^k_1 \chi_R(r)| \leq CR^{-k}$. Set $\tilde{\chi}_R = 1 - \chi_R$. With the definition of $p(\xi)$ in (3.17) we observe that

\[\partial^k_1 \partial^l_2 \Gamma(1, x) = \frac{i^{k+l}}{2\pi} \int_{\mathbb{R}^2} \xi^l_1 \xi^l_2 p(\xi) e^{-|\xi|^2+i\xi \cdot x} \, d\xi \]

\[= \frac{i^{k+l}}{2\pi} \int_{\mathbb{R}^2} \left(\chi_R(\xi_1)\chi_R(\xi_2) + \chi_R(\xi_1)\chi_R(\xi_2) + \chi_R(\xi_1)\chi_R(\xi_2) + \chi_R(\xi_1)\chi_R(\xi_2)\right) \]

\[\times \xi^l_1 \xi^l_2 p(\xi) e^{-|\xi|^2+i\xi \cdot x} \, d\xi \]

\[= \sum_{j=1}^4 I_j(x). \tag{5.9} \]

Then we have $|I_1(x)| \leq CR^{k+l+2}$ and

\[|I_2(x)| \leq CR^k \int_{|\xi_1| \leq 2R, |\xi_2| \geq R} \frac{|\xi_1||\xi_2|^l(|\xi_1| + |\xi_2|)}{|\xi|^2} e^{-|\xi|^2} \, d\xi \leq CR^{k+2} \int_R^\infty |\xi_2|^l e^{-\xi_2^2} \, d\xi_2 \]

\[\leq C(1 + |\log R(\delta_n)|)R^{k+2}. \tag{5.10} \]

For $I_3$ we use the equality

\[x^m_1 I_3(x) = (-1)^m i^{k+l-m} \int_{\mathbb{R}^2} \chi_R(\xi_2) \xi^l_2 e^{i\xi \cdot x} \partial^m_1 \left(\chi_R(\xi_1)\xi^l_1 p(\xi) e^{-|\xi|^2}\right) \, d\xi. \tag{5.11} \]
If \( m \geq k + 2 \) we have

\[
|\partial_1^m (\chi_R(\xi_1)\xi_1^k p(\xi)e^{-|\xi|^2})| \leq C(1 + \frac{|\xi_1|^{k-m+1}}{|\xi|}) e^{-\frac{\xi_1^2}{2}} \xi_1 \chi_2(\xi_1).
\]  

(5.12)

Hence

\[
|x_1^m I_3(x)| \leq C R^{k+1}(1 + R^{k-m+1}) \leq C R^{k+l+2-m}.
\]  

(5.13)

The term \( I_4 \) is estimated similarly. Indeed, we have from (5.12),

\[
|x_1^m I_4(x)| \leq C \int_{|\xi_1| \geq R, |\xi_2| \geq R} |\xi_1|^{k-m+1} e^{-\frac{\xi_1^2}{2}} d\xi_1 d\xi_2 \leq C \int_R^\infty |\xi_1|^{k-m+1} e^{-\frac{\xi_1^2}{2}} d\xi_1 d\xi_2.
\]

Then by taking \( R = |x_1|^{-1} \) for \( |x_1| > 2 \) we get \( |\partial_1^k \partial_2^l \Gamma(1, x)| \leq C|x_1|^{-k+2}\{\log(e + |x_1|)|^{\delta_0} \), which implies

\[
|\partial_1^k \partial_2^l \Gamma(1, x)| \leq C(1 + \frac{|x_1|^{k+2}}{(\log(e + |x_1|)|^{\delta_0})}^{-1}
\]  

(5.15)

for all \( x \in \mathbb{R}^2 \). To show the spatial decay in \( x_2 \) direction, instead of (5.9), we write

\[
\partial_1^k \partial_2^l \Gamma(1, x) = \frac{e^{k+l}}{2\pi} \int_{\mathbb{R}^2} (\chi_R(\xi) + \chi_R(\xi)) \xi_1^k \xi_2^l p(\xi)e^{-|\xi|^2 + ix_2 \xi} d\xi.
\]  

(5.16)

It is clear that \(|\Gamma_1(x)| \leq CR^{k+l+2}\), while we have

\[
x_2^m \Gamma_2(x) = (-1)^m e^{k+l-m} \int_{\mathbb{R}^2} \xi_1^k \xi_2^l e^{-|\xi|^2} \xi_1^m (\chi_R(\xi)) \xi_2^l p(\xi)e^{-|\xi|^2} d\xi.
\]  

(5.17)

From

\[
|\xi_1^k \partial_2^l (\chi_R(\xi)) \xi_2^l p(\xi)e^{-|\xi|^2})| \leq C|\xi_1|^{k+l-m} e^{-\frac{|\xi|^2}{2}} \chi_2(|\xi|),
\]  

(5.18)

for \( m \gg 1 \), we get

\[
x_2^m \Gamma_2(x) \leq C R^{k+l-m+2} m \gg 1.
\]  

(5.19)

By taking \( R = |x_2|^{-1} \) for \( |x_2| > 2 \) we see \( |\partial_1^k \partial_2^l \Gamma(1, x)| \leq C|x_2|^{-k-l-2} \) for \( |x_2| > 2 \), which implies

\[
|\partial_1^k \partial_2^l \Gamma(1, x)| \leq C(1 + |x_2|^{k+l+2})^{-1}
\]  

(5.20)

for all \( x \in \mathbb{R}^2 \). Then (5.15) and (5.20) yield (5.7). Est.(5.8) is then obtained by the relation \( \Gamma(t, x) = t^{-1} \Gamma(1, x/\sqrt{t}) \). This completes the proof.

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**References**


