Evolution of regular bent rectangles by the driven crystalline curvature flow in the plane with a non-uniform forcing term

Yoshikazu Giga[1], Przemysław Górka[2], Piotr Rybka[3]
[1] Graduate School of Mathematical Sciences
University of Tokyo
Komaba 3-8-1, Tokyo 153-8914, Japan
e-mail: labgiga@ms.u-tokyo.ac.jp
and
Department of Mathematics, Faculty of Sciences
King Abdulaziz University
P. O. Box 80203, Jeddah 21589, Saudi Arabia

[2] Department of Mathematics and Information Sciences
Warsaw University of Technology
pl. Politechniki 1, 00-661 Warsaw, Poland
e-mail: P.Gorka@mini.pw.edu.pl

[3] Institute of Applied Mathematics and Mechanics, Warsaw University
ul. Banacha 2, 07-097 Warsaw, Poland
fax: +(48 22) 554 4300, e-mail: rybka@mimuw.edu.pl

December 21, 2011

Abstract

We study the motion of regular bent rectangles driven by singular curvature flow with a driving term. The curvature is being interpreted as a solution to a minimization problem. The evolution equation becomes in a local coordinate a system of Hamilton-Jacobi equations with free boundaries, coupled to a system of ODE's with nonlocal nonlinearities. We establish local-in-time existence of variational solutions to the flow and uniqueness is proved under additional regularity assumptions on the data.

Key words: singular energies, bending of facets, driven curvature flow, variational principle, Hamilton-Jacobi equation

2010 Mathematics Subject Classification. Primary: 53C44 Secondary: 35K55

1 Introduction

Our goal is to establish the existence of a properly defined solution to the driven weighted mean curvature (wmc) flow

$$\beta V = \kappa_\gamma + \sigma \quad \text{on}\, \Gamma(t), \tag{1.1}$$

when κ_γ is a singular curvature, $\Gamma(t)$ is from a suitable class of closed curves, corresponding to the anisotropy function γ and β is quite general. We shall succeed, but only for special closed curves, which we call bent rectangles, as initial data. Our motivation is to study facets at the onset of their bending and after that. If the initial curve is a rectangle close to the energy equilibrium shape *i.e.*, the Wulff shape of γ, which is given by (1.2), then the presence of the forcing term σ in (1.1) leads to a 'bent rectangle' with exactly three facets, see Fig. 1.

The first problem here is to give meaning to the curvature term, which is formally defined as

$$\kappa_\gamma = -\mathrm{div}_S \left(\nabla_\zeta \gamma(\zeta)|_{\zeta=\mathbf{n}}\right),$$

where $\mathbf{n}$ is the outer normal to Γ and γ is a surface energy function. This definition makes sense for smooth γ on smooth surfaces. But our goal is to study (1.1), when

$$\gamma(p) = \gamma_\Lambda |p_1| + \gamma_R |p_2|, \tag{1.2}$$

i.e., we deal with crystalline curvature. In Section 2, we recall the definition of κ_γ, which used earlier, see [22].

Our main objective is to study behavior of facets, flat parts of $\Gamma(t)$ with normal vectors defined by the singular directions of γ, *i.e.*, the normal vectors to Wulff shape of γ. The presence of the forcing term will make facets bend, provided that they are long enough. A study of facet stability in a different problem, where (1.1) is coupled to a diffusion equation was performed in [21]. Here, we assume that σ forces bending of an initially flat facet.

Here, we permit a generic driving σ, conforming to the Berg's effect (see [8], [28], [20] and references therein), *i.e.*

$$x_i \frac{\partial \sigma}{\partial x_i}(t, x_1, x_2) > 0 \quad \text{for all}\ x_i \neq 0, \quad i = 1, 2, \quad t \geq 0, \tag{1.3}$$

and the symmetry conditions, for all $t \geq 0$,

$$\sigma(t, x_1, x_2) = \sigma(t, -x_1, x_2), \quad \sigma(t, x_1, x_2) = \sigma(t, x_1, -x_2) \ \text{for all}\ x_i \in \mathbb{R},\ i = 1, 2. \tag{1.4}$$

We studied evolution by (1.1) of graphs with a single facet for rather general kinetic coefficient β, see [16]. In [22], we investigated evolution of bent rectangles by (1.1) for a special choice of β. Here, we are interested in their motion with the same restrictions on β as in [16]. Once we write (1.1) in a local coordinate system, which will be done in Section 2, we will see that the resulting system may be interpreted as a coupled system of two Hamilton-Jacobi equations with free boundaries. We may indicate the main difficulties here using the language suitable for Hamilton-Jacobi equations. These are the rarefaction regions associated to the corners of $\Gamma(t)$ and the free boundaries, the endpoint of the facets, see Fig. 1. This will be explained in Section 3. We do not have tools to study them in the full generality, yet. This is why we do not treat here the corner evolution problem explicitly, but we define them as intersection of facets.

Let us stress that our purpose is to construct solutions to (1.1) in relatively simple cases. We mean by this *regular bent rectangles*, (see Definition 2.1 at the end of Subsection 2.1), that is we impose some smoothness assumptions on the curved parts of $\Gamma(t)$ as well as on σ so that we may use the method of characteristics to solve the Hamilton-Jacobi equations. Our main result is Theorem 3.5 in subsection 3.3 stating existence of variational solutions to (1.1) for a number of possible types of behavior of the facet endpoints. The basic, generic case corresponds to such data that $\Gamma(0)$ has kinks at the facet endpoints. Such a facet may shrink or expand. On the other hand, if $\Gamma(0)$ is smooth in a neighborhood of the endpoints of a facet, and this facet shrinks, then smoothness is preserved. Separately we consider the case when the interfaces coalesce. This is treated in Theorem 3.6.

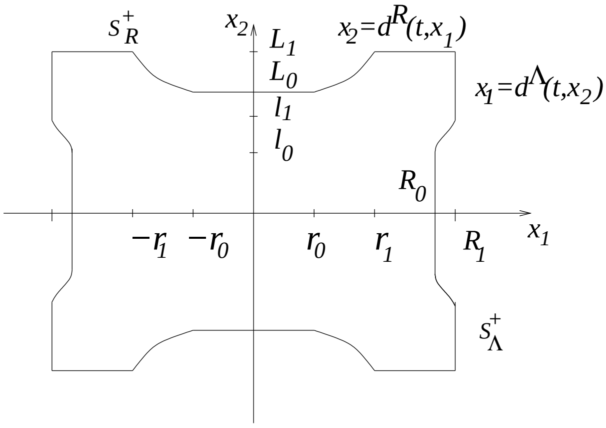

Fig. 1

A side effect of our work is to create supply of super- and subsolutions necessary for the development of viscosity theory for equations like (1.1), see [15].

In [16], we discussed in detail the construction of the inner interfaces, called l_0, r_0, also when there is no corner at the junctions of facets with the curved parts. The curve was constructed as a shock wave, *i.e.*, the result of crashing of the characteristics of a Hamilton-Jacobi equation on the central facet. This approach does not work for the outer interfaces, called l_1, r_1, because, loosely speaking, they are trying to escape from the characteristics of a Hamilton-Jacobi equation governing the curved part. Thus, geometrically the behavior of the interfaces is different.

We must stress that we assume that the corners of the bent rectangles evolve as intersections of outer facets. This may not be suitable for general kinetic coefficients. A corner may round off. Even if σ is constant we need a condition for β such that the corner never rounds off as pointed out in [13] if the flow is to enjoy the comparison principle. In our current setting it seems that a sufficient condition for the corner never to round off is that the Wulff shape $W_{1/\beta}$ is a rectangle, although we will not discuss this problem in this paper. The construction of solutions is conducted here in a such way, that accommodating such a rounding off will be easy.

We are able to prove uniqueness of our solutions only in special cases. They are related to the behavior of the interfaces. If all the facets shrink and $\Gamma(t)$ is smooth at the interfaces, then we can guarantee uniqueness. If at least one facets expands, so that $\Gamma(t)$ is no longer smooth at the interfaces, then our proof fails. The problem is, compared to [23, Theorem 4.1], that in the present case the general β destroys the gradient flow structure exploited there. Here, we use tools typical for transport problems, see [1], [9].

We use methods based on the method of characteristics, because the viscosity theory, does not seem developed sufficiently to handle the case of closed curves for general β. This approach was quite successful for graphs, see [16], [12], [15]. We expect that we could approximate the left out cases by regular bent rectangles, but we postpone it for further studies, as this seems to require the viscosity methods which are being developed, see [15]. Another difficulty associated to the application of the general viscosity theory for the first order Hamilton-Jacobi equations is that the notion of the variational solution no longer makes sense, a new notion has to be invented.

2 Setting up the problem

2.1 Definitions

Here, we recall the notions we used in [22]–[23]. We restrict our attention to evolution of a subclass of bent rectangles. We shall call a Lipschitz closed curve Γ a *bent rectangle* (see [23, §2]) if the following conditions are satisfied:

> There exist even, Lipschitz continuous functions $d^R,\ d^\Lambda : \mathbb{R} \to \mathbb{R}_+$, which are non-decreasing for positive arguments and there are positive numbers L_1, R_1 such that
> $$d^\Lambda(L_1) = R_1, \qquad d^R(R_1) = L_1.$$
> In addition d^Λ is constant in a neighborhood of zero and L_1 (respectively, d^R is constant in a neighborhood of zero and R_1), furthermore

$$(BR) \qquad \Gamma = \partial\{(x_1, x_2) :\ |x_1| \le d^\Lambda(x_2)\ , |x_2| \le d^R(x_1)\}.$$

We shall call $d^R,\ d^\Lambda$ *a pair of admissible functions*.

We shall call the points $(\pm R_1, \pm L_1)$ vertexes of Γ. Thus, after we set

$$S^\pm_\Lambda = \{(x_1, x_2) \in \Gamma :\ x_1 = \pm d^\Lambda(x_2),\ x_2 \in [-L_1, L_1]\},$$
$$S^\pm_R = \{(x_1, x_2) \in \Gamma :\ x_2 = \pm d^R(x_1),\ x_1 \in [-R_1, R_1]\},$$

we notice that the graphs of $\pm\, d^\Lambda\big|_{[-L_1,L_1]}$, $\pm d^R|_{[-R_1,R_1]}$ make up the whole $\Gamma(t)$, *i.e.*

$$\Gamma = S^-_R \cup S^+_R \cup S^-_\Lambda \cup S^+_\Lambda.$$

We will collectively write S_R for $S^\pm_R$ and S_Λ for $S^\pm_\Lambda$. We will call them *sides* of $\Gamma(t)$. Vertexes of Γ are the intersections $S^\pm_R \cap S^\pm_\Lambda$. Moreover, the sides meet at vertexes at the right angle. However, a general kinetic coefficient may lead to rounding off the corner. This happens when the Wulff shape $W_{1/\beta}$ has a round corner or not. Such phenomenon is indicated by physics literature, see *e.g.*, [10], [18] and studied in [13] when σ is spatially constant. We do not consider this here, but state the major theorems in such a way, that they can be easily adapted if the corresponding dynamics will become known.

We will denote by $\mathbf{n}$ the outer normal to Γ and in particular,

$$\mathbf{n}_\Lambda = (1, 0), \qquad (\text{resp.} \quad \mathbf{n}_R = (0, 1))$$

are normals to the faceted regions of S_Λ, (resp. S_R). A rigorous definition of the notion of faceted regions is given later just before formula (2.7) in this section.

The curvature, κ_γ, appearing in (1.1) is defined by

$$\kappa_\gamma = -\mathrm{div}_S \left(\nabla_\zeta \gamma(\zeta)|_{\zeta=\mathbf{n}}\right),$$

where $\mathbf{n}$ is the outer normal to Γ and γ is a surface energy function. In our case vector $\mathbf{n}$ is defined only $\mathcal{H}^1$-a.e. The physical examples, we have in mind, see [24], [19], give us the motivation to consider

$$\gamma(p_1, p_2) = \gamma_\Lambda |p_1| + \gamma_R |p_2|. \tag{2.1}$$

We notice that the flat parts with normals belonging to the set of normals of the Wulff shape W_γ are energetically preferred. We refer the reader to [24, §7.5] (see also [23, §2]) for the definition of the Wulff shape. In our case it is a rectangle.

Now, the fundamental problem is apparent: $\nabla\gamma(\mathbf{n})$ is not defined on bent rectangles on sets of positive $\mathcal{H}^1$-measure. We resolved this issue by replacing $\nabla\gamma$ by $\partial\gamma$, which is always well-defined, because of convexity of γ. The subdifferential coincides with $\{\nabla\gamma(x_0)\}$ when γ is differentiable at x_0.

Since in general $\partial\gamma$ is not a singleton, this leaves us with a necessity to select the proper Cahn-Hoffman vector field $\xi(x) \in \partial\gamma(\mathbf{n}(x))$. We note that this task is obvious on curved parts of S^Λ, S^R, because $\partial\gamma$ is a singleton there, while it is not trivial on flat facets. Thus, we have to find a proper selection of $\nabla\gamma$. For this purpose, we use a variational principle as in [22], [23]. A similar approach was first introduced by [11] for graph-like solutions and was developed in several ways by the authors of [2] – [7].

We impose quite natural constraints on ξ, see [22],

$$\mathrm{div}_S \xi \in L^2(S_i), \qquad i = R, \Lambda.$$

This implies that $\xi \cdot \nu$ has a trace, where $\nu \in T_x S_i$ is a normal vector to S_i, $i = R$, Λ. If we combine it with

$$\partial\gamma(\mathbf{n}_R) \cap \partial\gamma(\mathbf{n}_\Lambda) = \{p\},$$

then we see that ξ satisfies a boundary condition

$$\xi|_{vertex} = p.$$

The necessity of selecting ξ implies that in order to define a solution to (1.1), we need to specify not only a curve $\Gamma(t)$ but also $\xi(t, \cdot)$. After [22], we recall the notion of solution. Namely, by *a solution to* (1.1) we call a family of couples $(\Gamma(t), \xi(t))$, $t \in [0, T)$, such that for some $T > 0$, the following conditions are satisfied:

(a) For each $t \in [0, T)$ the curve $\Gamma(t)$ is a bent rectangle and d^Λ, d^R are continuous functions of its arguments, for each x, $d^j(\cdot, x)$, $j = \Lambda, R$ are Lipschitz continuous and for each $t \in [0, T)$, the functions $d^j(t, \cdot)$, $j = \Lambda, R$ are admissible;

(b) $\xi : \bigcup_{t\in[0,T)} \{t\} \times \Gamma(t) \to \mathbb{R}^2$ is at each time instant a Cahn-Hoffman vector. If $M := \sup_{t\in[0,T)} \max\{L_1(t), R_1(t)\} + 1$, and if for $j = \Lambda$, R, we set

$$\begin{aligned}
\tilde{\xi}^R(t,x) &\in \begin{cases} \{(-\gamma(\mathbf{n}_\Lambda), \gamma(\mathbf{n}_R))\} & x \in [-M, -R_1(t)], \\ \{\xi(t, (x, d^R(t,x)))\} & x \in [-R_1(t), R_1(t)], \\ \{(\gamma(\mathbf{n}_\Lambda), \gamma(\mathbf{n}_R))\} & x \in [R_1(t), M]; \end{cases} \\
\tilde{\xi}^\Lambda(t,x) &\in \begin{cases} \{(-\gamma(\mathbf{n}_\Lambda), -\gamma(\mathbf{n}_R))\} & x \in [-M, -L_1(t)], \\ \{\xi(t, (d^\Lambda(t,x), x))\} & x \in [-L_1(t), L_1(t)], \\ \{(-\gamma(\mathbf{n}_\Lambda), \gamma(\mathbf{n}_R))\} & x \in [L_1(t), M]; \end{cases}
\end{aligned} \tag{2.2}$$

then we assume that $t \mapsto \tilde{\xi}^j(t,\cdot) \in L^\infty(0,T; L^2(-M,M))$, $j = \Lambda, R$;

This extension is made for the sake of simplification, so that our problem becomes independent of the parametrization of Γ. It works because the corners evolve as the intersection of facets.

(c) Equation (1.1) is satisfied in the L^2 sense for *a.e.* $t \geq 0$ after interpreting κ_γ as $-\mathrm{div}_S \xi$.

We note that on one hand condition (c) is quite broad, on the other hand it seems improper for the viscosity solution outside of facets. However, if the data of the Hamilton-Jacobi equations are sufficiently regular, *i.e.*, Lipschitz continuous, then the viscosity solution is Lipschitz continuous, hence a.e. differentiable. In fact, we will assume enough regularity of the data, making this definition meaningful.

In principle, the Cahn-Hoffman vector depends upon time t and $x = (x_1, x_2) \in \Gamma(t)$. However, we shall frequently suppress t and write $\xi(x)$, when the meaning of the spacial argument is clear from the context, *e.g.* on the sides.

We also distinguished variational solutions based on a specific way to select ξ. In order to define them, we introduce two convenient energy functionals,

$$\mathcal{E}_j(\xi) = \frac{1}{2}\int_{S_j} |\sigma - \mathrm{div}_S \xi|^2 \, d\mathcal{H}^1, \quad j = R, \Lambda. \tag{2.3}$$

Their natural domains of definition are the sets of Cahn-Hoffman vectors, satisfying all the above constraints,

$$\begin{aligned} \mathcal{D}_\Lambda &= \{\xi \in L^\infty(S_\Lambda) : \ \xi(x) \in \partial\gamma(\mathbf{n}(x)), \ \mathrm{div}_S \xi \in L^2(S_\Lambda), \ \text{(2.5) holds}\}, \\ \mathcal{D}_R &= \{\xi \in L^\infty(S_R) : \ \xi(x) \in \partial\gamma(\mathbf{n}(x)), \ \mathrm{div}_S \xi \in L^2(S_R), \ \text{(2.5) holds}\}. \end{aligned} \tag{2.4}$$

where

$$\xi(\pm R_1, \pm L_1) \in \partial\gamma(\pm\mathbf{n}_\Lambda) \cap \partial\gamma(\pm\mathbf{n}_R). \tag{2.5}$$

Strictly speaking $\mathbf{n}$ is defined only $\mathcal{H}^1$-a.e. but this has no influence on the above definition, partly because $\partial\gamma(\xi)$ is a constant singleton on each connected component of the curved part of Γ.

We recall (see also [23] for a discussion of this notion) that $\{(\Gamma(t), \xi(t))\}$, $t \in [0,T)$, a solution to (1.1), was called a *variational solution* if in addition for each $t \in [0,T)$ $\xi|_{S_j}(t) \in L^2(S_j)$ is a solution to

$$\mathcal{E}_j(\xi) = \min\{\mathcal{E}_j(\zeta) : \ \zeta \in \mathcal{D}_j\}, \quad j = R, \Lambda. \tag{2.6}$$

We also recall from [22] a number of auxiliary notions. Let us consider an open line segment I in the plane, *i.e.* $I = (a,b) \equiv \{x = at + b(1-t), \ t \in (0,1)\}$, where $a, b \in \mathbb{R}^2$. We shall say that $I \subset \Gamma$, having a normal equal to $\mathbf{n}_\Lambda$ or $\mathbf{n}_R$, is *a faceted region* of Γ if it is maximal (with respect to inclusion) and it satisfies

$$(\sigma - \mathrm{div}_S \xi)|_I = \text{const.}, \tag{2.7}$$

where ξ is a solution to (2.6).

We keep in mind that, $S^\pm_\Lambda(t)$ and $S^\pm_R(t)$ are graphs, *e.g.* S^+_R is the image of segment $[-R(t), R(t)]$ under the function

$$x \mapsto (x, d^R(t,x)) =: \tilde{d}^R(t,x). \tag{2.8}$$

Frequently, it is more convenient to work with the inverse image of a faceted region I, $(\alpha, \beta) = \tilde{d}^{-1}(I)$. We stress that this definition permits $S_j^{\pm}(t)$, $j = R, \Lambda$ being a line segment which has more than one faceted region.

Let us now introduce formally the class of bent rectangles we want to deal with.

Definition 2.1 We shall say that a bent Γ is *regular*, provided that the admissisible functions d^R, d^Λ, restricted to the closure of the complement of the preimages of facets are of class C^2.

In order to make the presentation more clear, we propose to use the notion of a *curved part* of side to denote the (relative) interior of the subset of Γ, where normal $\mathbf{n}$ is such that $\partial\gamma(\mathbf{n})$ is a singleton. However, this definition is not quite precise in case of Lipschitz curves, for the normal vector may not be everywhere defined, but nonetheless $\partial\gamma(\mathbf{n})$ is a singleton. In particular, it may happen that a line segment of Γ will be called a curved part if its normal is different from $\mathbf{n}_R$, $\mathbf{n}_\Lambda$.

At this point, we mention that there are other approaches to deal with the ill-defined operator $\mathrm{div}_S \nabla_\zeta \gamma(\mathbf{n})$. The first is the viscosity method developed by the M.-H. Giga and Y. Giga, [12]; see also [15] for graph-like functions when σ is non-uniform. This approach is applied for the evolution of closed curves [14] when σ is spatially constant. However, so far it is not yet adjusted to handle spatially inhomogeneous σ for curves. The second approach is by anisotropic distance function with variational principles e.g. [3]. It applies to higher dimensional problems but it does not allow σ. Another approach is the 'operator approach', which is based on a proper interpretation of the composition of multivalued operators, [25], [27]. However, we do not know whether this approach is applicable for closed curves.

2.2 The reduction to the local coordinate system

We want to write (1.1) in the local coordinate system for variational solutions. We proceed as in [22] without major changes while keeping in mind that we work with bent rectangles. A very important part of this process is solving a double obstacle problem. An inherent part of the solution is a free boundary, *i.e.,* the coincidence set. Its boundary defines the endpoints of the facets, and we pay special attention to the facets of our bent rectangles. It is however, more convenient to work with the pre-images of faceted regions. We restrict our attention to such variational solutions (Γ, ξ) of (1.1) that each facet S_j has exactly three faceted regions, whose pre-images are:

$$(-L_1, -l_1), \quad (-l_0, l_0), \quad (l_1, L_1), \quad (-R_1, -r_1), \quad (-r_0, r_0), \quad (r_1, R_1).$$

Moreover, the functions $d^\Lambda|_{[0,L_1]}$, $d^R|_{[0,R_1]}$ are increasing.

We noticed in [22, Proposition 3.1]) that under our typical assumptions for σ, if d^Λ, d^R are Lipschitz continuous with a finite number of non-differentiability points, then ξ is constant over each component of the curved parts. As a result, equation (1.1) takes the form of

$$\beta V = \sigma.$$

The curvature obviously vanishes.

In the present context (1.1) becomes

$$\begin{aligned} \sigma(t, d^\Lambda(t,x_2), x_2) &= \beta(\mathbf{n}) \tfrac{d_t^\Lambda(t,x_2)}{\sqrt{1+(d^\Lambda_{x_2}(t,x_2))^2}} \quad \text{on } [l_0(t), l_1(t)], \\ \sigma(t, x_1, d^R(t,x_1)) &= \beta(\mathbf{n}) \tfrac{d_t^R(t,x_1)}{\sqrt{1+(d^R_{x_1}(t,x_1))^2}} \quad \text{on } [r_0(t), r_1(t)], \end{aligned} \tag{2.9}$$

where $\mathbf{n}$ is the outer normal to $\Gamma(t)$. Let us notice that

$$\beta(\mathbf{n}) = \beta\left(-\frac{d^R_{x_1}}{\sqrt{1+(d^R_{x_1})^2}}, \frac{1}{\sqrt{1+(d^R_{x_1})^2}} \right) =: \tilde\beta^R(d^R_{x_1}) \quad \text{on } S^+_R \cap \{x_1 > 0\},$$

$$\beta(\mathbf{n}) = \beta\left(-\frac{1}{\sqrt{1+(d^\Lambda_{x_2})^2}}, -\frac{d^\Lambda_{x_2}}{\sqrt{1+(d^\Lambda_{x_2})^2}} \right) =: \tilde\beta^\Lambda(d^\Lambda_{x_2}) \quad \text{on } S^+_\Lambda \cap \{x_2 > 0\}.$$

This is why (2.9) can be written as

$$d^R_t - \sigma(t, d^R, x) m^R(d^R_x) = 0 \quad \text{and} \quad d^\Lambda_t - \sigma(t, x, d^\Lambda) m^\Lambda(d^\Lambda_x) = 0 \tag{2.10}$$

on respective intervals. Here,

$$m^i(p) = \frac{\sqrt{1+p^2}}{\tilde\beta^i(p)}. \tag{2.11}$$

We want to set minimal assumptions on m^Λ, m^R. They are:

$$\frac{1}{\beta_\Lambda} = m^\Lambda(0) \le m^\Lambda(p), \quad \frac{1}{\beta_R} = m^R(0) \le m^R(p), \tag{2.12}$$

$$m^i(p) = m^i(-p), \quad i = \Lambda, R, \tag{2.13}$$

$$m^i \text{ is Lipschitz continuous and } m^i \in C^2(\mathbb{R} \setminus \{0\}), \quad i = \Lambda, R, \tag{2.14}$$

$$m^i \text{ is convex for } |p| \le 1, \quad i = \Lambda, R, \tag{2.15}$$

$$m^i(p) \le C(1 + |p|) \quad i = \Lambda, R. \tag{2.16}$$

Here, we use the shorthands $\beta_R = \beta(\mathbf{n}_R)$, $\beta_\Lambda = \beta(\mathbf{n}_\Lambda)$.

In [3], the author considers the flow of (1.1) with $\beta = \frac{1}{\gamma}$. Our assumptions on β reflect our desire to consider a broad set of examples while giving us additional freedom. At the same time we want that the Wulff shapes of γ and $\frac{1}{\beta}$ be somehow close, but they need not coincide.

We may now write equation (1.1) in the local coordinates. It is a good idea to recall the conclusions of [23, Proposition 3.1]. Namely, what we showed in [23, (3.11)] amounts to saying that for a variational solution, (1.1) takes the following form,

$$\begin{aligned} \beta_R \dot L_0 &= \fint_0^{r_0} \sigma(t, s, L_0)\, ds + \frac{\gamma(\mathbf{n}_\Lambda)}{r_0} && \text{on} \quad [0, r_0], \\ d^R_t &= \sigma(t, x_1, d^R) m^R(d^R_x) && \text{on} \quad [r_0, r_1], \\ \beta_R \dot L_1 &= \fint_{r_1}^{R_1} \sigma(t, s, L_1)\, ds - \frac{2\gamma(\mathbf{n}_\Lambda)}{R_1 - r_1} && \text{on} \quad [r_1, R_1], \\ \beta_\Lambda \dot R_0 &= \fint_0^{l_0} \sigma(t, R_0, s)\, ds + \frac{\gamma(\mathbf{n}_R)}{l_0} && \text{on} \quad [0, l_0], \\ d^\Lambda_t &= \sigma(t, d^\Lambda, x_2) m^\Lambda(d^\Lambda_x) && \text{on} \quad [l_0, l_1], \\ \beta_\Lambda \dot R_1 &= \fint_{l_1}^{L_1} \sigma(t, R_1, s)\, ds - \frac{2\gamma(\mathbf{n}_R)}{L_1 - l_1} && \text{on} \quad [l_1, L_1], \end{aligned} \tag{2.17}$$

augmented with the following initial conditions,

$$
\begin{aligned}
l_0(0) = l_{00}, \quad l_1(0) = l_{10}, \quad r_0(0) = r_{00}, \quad r_1(0) = r_{10},\\
R_0(0) = R_{00}, \quad R_1(0) = R_{10}, \quad L_0(0) = L_{00}, \quad L_1(0) = L_{10},\\
d^R(0,x_1) = d_0^R(x_1), \quad d^\Lambda(0,x_2) = d_0^\Lambda(x_2).
\end{aligned} \tag{2.18}
$$

We keep in force the simplifying symmetry assumptions. The notation is illustrated in Figure 1.

An important observation is that in order to close this system we need information about the evolution of $l_i(\cdot)$, $r_i(\cdot)$, $i = 0, 1$. These points are zero-dimensional free boundaries, whose evolution is not determined by (2.17). As a result, we have a system of Hamilton-Jacobi equations with free boundaries.

The definition of a variational solution requires that a Cahn-Hoffman vector field ξ is presented along with the curve $\Gamma(t)$. We shall do this later in Theorem 3.5.

2.3 The interfacial curves

We showed in [23] that the obstacle problem leads to two types of the interfacial curves l_i, r_j, $i, j = 0, 1$. Each of them may be: either tangency curve or matching curve. For the sake of definiteness, we shall concentrate on r_0. The variational problem (2.6) is of a double obstacle type. It may happen that for any $t \geq 0$ the point $r_0(t)$ is on the boundary of the coincidence set. In this case the solution ξ satisfies $\frac{\partial \xi}{\partial x_1}(r_0(t)) = 0$, see [26], and we say that the *tangency condition* is satisfied at $r_0(t)$. If the tangency condition is satisfies at $r_0(t)$ for all $t \in [0, T)$, then we call the curve $r_0(\cdot)$ a *tangency curve*. Equivalent and more convenient versions of the tangency condition for r_0 and r_1 are (see [22, Proposition 2.1] and [22, (3.10)]),

$$
\begin{aligned}
\sigma(t, r_0(t), L_0(t)) &= \fint_0^{r_0(t)} \sigma(t, s, L_0(t))\, ds + \frac{\gamma(\mathbf{n}_\Lambda)}{r_0(t)}, \ \xi_1(r_i(t)) = -\gamma(\mathbf{n}_\Lambda), \ i = 0, 1,\\
\sigma(t, r_1(t), L_1(t)) &= \fint_{r_1(t)}^{R_1(t)} \sigma(t, s, L_1(t))\, ds - \frac{2\gamma(\mathbf{n}_\Lambda)}{R_1(t) - r_1(t)}.
\end{aligned} \tag{2.19}
$$

We note that here ξ is a solution to (2.6). An obvious modification is required for curves l_0, l_1.

Let us notice that the definitions we adopted mean that the coordinate system (r, d^R), *i.e.*, (x_1, x_2) is positively oriented, while the orientation of (l, d^Λ) *i.e.*, (x_2, x_1) is negative, see Fig. 1. Thus, some care has to be exercised while replacing r_1 (resp. r_0) with l_1 (resp. l_0) in the statements of theorems here.

The coincidence set may be empty. However, we always demand that $\Gamma(t)$ is a Lipschitz curve, thus the solutions to (2.17) must satisfy

$$
d^R(t, r_i(t)) = L_i(t), \qquad d^\Lambda(t, l_i(t)) = R_i(t), \qquad i = 0, 1. \tag{2.20}
$$

We call (2.20) the *matching condition*, because R_i (resp. L_i) must match d^Λ (resp. d^R).

The tangency condition is not open. Thus, we expect in a typical case the interfacial curve will not satisfy it. On the other hand, (2.20) is more basic. Let us note

Proposition 2.1 *(cf. [23, Proposition 3.3]) Let us suppose that $\sigma \in C^1([0,T_*) \times \mathbb{R}^2)$ is given, it satisfies the Berg's effect (1.3) and (1.4), $(\Gamma(t), \xi(t))_{t\in[0,T)}$ is a variational solution. We assume that d^R is such that $r_0(t) < r_1(t)$. Moreover $r_0(\cdot)$ and $r_1(\cdot)$ are C^1 curves. In addition $d^R(\cdot, x)$ is a continuous piecewise C^1-function.*
(a) If the tangency as well as matching conditions are satisfied at $r_0(t)$ for all $t \in [0, \epsilon)$, then $r_0(\cdot)$ is decreasing.
(b) If the tangency as well as matching conditions are satisfied at $r_1(t)$ for all $t \in [0, \epsilon)$, then $r_1(\cdot)$ is increasing.

We proved this in [23] for a special β rendering m identical to 1. However, the original proof is valid, after modifications, also for a general β as long as $d(\cdot, x)$ (we suppress the superscript R or Λ) is a continuous piecewise C^1 function. We leave the details to the interested reader. □

We would like to gather more information on the interfacial curves.

Proposition 2.2 *Let us suppose that $(\Gamma(t), \xi(t))_{t\in[0,T)}$ is a variational solution to (1.1), $\sigma \in C^1([0,T) \times \mathbb{R}^2)$. We also assume that for each t, $\Gamma(t)$ is a regular bent rectangle and r_0 is the interfacial curve emanating from r_{00}.*
(a) If $\dot{L}_0(0) - \sigma(0, r_{00}, L_0(0))m(d^+_{0,x}(r_{00})) \neq 0$, the tangency condition (2.19) does not hold at $t = 0$ and $d^+_{0,x}(r_{00}) > 0$, then r_0 is a matching curve and

$$\dot{r}_0(t) = \frac{1}{d^+_x(t, r_0(t))}(\dot{L}_0(t) - \sigma(t, r_0(t), L_0(t))m(d^+_x(t, r_0(t))). \tag{2.21}$$

In particular, sgn $\dot{r}_0$ = sgn $(\dot{L}_0(t) - \sigma(t, r_0(t)), L_0(t))m(d^+_x(t, r_0(t))$. Moreover, the tangency condition (2.19) does not hold for $t > 0$.
(b) If the tangency condition (2.19) holds at $t = 0$, $d_x(t, r_0(t)) = 0$ and

$$\begin{aligned} 0 > \Sigma^R_0 \quad := \quad & \fint_0^{r_{00}} \sigma_t(0, y, L_{00})\, dy - \sigma_t(0, R_{00}, l_{00}) \\ & + \sigma(0, r_{00}, L_{00}) \left(\fint_0^{r_{00}} \sigma_{x_2}(0, y, L_{00})\, dy - \sigma_{x_2}(0, r_{00}, L_{00}) \right). \end{aligned} \tag{2.22}$$

Then, the tangency condition holds for all $t > 0$ and

$$\dot{r}_0(0) = \frac{\Sigma^R_0}{\sigma_{x_1}(0, r_{00}, L_{00})} < 0. \tag{2.23}$$

Remark. In fact, Σ^R_0 is a function of (t, r_0, L_0), here we are interested in the value of Σ^R_0 at $(0, r_{00}, L_{00})$. The notation Σ^R_0 was introduced in [22] and [23].

Proof. Our regularity assumptions permit us to take the time derivative of (2.20). Since $d_t = \sigma m(d_x)$, we conclude

$$d_x(t, r_0(t))\dot{r}_0(t) = (\dot{L}_0(t) - \sigma(t, r_0(t), L_0(t))m(d_x(t, r_0(t)))). \tag{2.24}$$

If $d^+_x(t, r_0(t)) > 0$, then we obtain (2.21) as we claimed in (a).

Since $(\Gamma(t), \xi(t))_{t\in[0,T)}$ is a variational solution, then $\xi(t, x) \in \partial\gamma(\mathbf{n})$. Let us suppose that for given t, L_0 the tangency condition holds at $r_{TC}(t, L_0)$. If $r_{00} < r_{TC}(0, L_{00})$, then the central facet expands as much as possible, on $[0, T)$.

If $r_{00} > r_{TC}$, then $[r_{TC}, r_{00}]$ is a subset of the coincidence set and the Hamilton-Jacobi equation (2.17$_2$) is considered not on $[r_0, r_1]$ but on its essential superset $[r_{TC}, r_1]$. If this is so, we have just Lipschitz continuous data, contrary to the regularity assumption.

Let us suppose now that the tangency condition holds for $t = 0$, $\dot{r}_0(t)$ exists and for all t, we have $d_x(t, r_0(t)) = 0$. Then, the LHS of (2.24) vanishes. Hence, $\dot{L}_0 = \sigma m(0)$, *i.e.*, the tangency condition holds for all $t \geq 0$. Then, after differentiating the tangency condition (2.19$_1$), we conclude that

$$\dot{r}_0 = \frac{\Sigma_0^R}{\sigma_{x_1}} < 0,$$

because Berg's effect implies $\sigma_{x_1} > 0$ and we know that $\Sigma_0^R < 0$. □

Remark. If $r_{00} = r_{TC}$, then the tangency condition is satisfied. This situation is not generic and we do not consider it. However, if $d_x(0, r_{00}) > 0$, then (2.21) implies $\dot{r}_0(0) < 0$.

We note that Σ_0^R was introduced in (see [23, eq. (3.14)]) for the purpose of studying the sign of $\dot{r}_0$, when $d_x^+(0, r_0(0)) = 0$ and the character of the interfacial curve had to be determined. Here, we consider $d_x^+(0, r_0(0)) = 0$ only along the tangency condition.

The statement of the above proposition for l_0 requires trivial change of notation. The same type of analysis leads to a corresponding statement for r_1 (hence, respectively for l_1). The proof is left to the interested reader.

Proposition 2.3 *Let us suppose that* $(\Gamma(t), \xi(t))_{t\in[0,T)}$ *is a variational solution to (1.1),* $\sigma \in C^1([0,T) \times \mathbb{R}^2)$. *We also assume that for each* t, $\Gamma(t)$ *is a regular bent rectangle and* r_1 *is the interfacial curve emanating from* r_{10}.
(a) If $\dot{L}_1(0) - \sigma(0, r_1(0), L_{10}) m(d^-_{0,x}(r_{10})) \neq 0$, *the tangency condition does not hold at* r_{10} *and* $d^-_{0,x}(r_{10}) > 0$, *then* r_1 *is a matching curve and*

$$\dot{r}_1(t) = \frac{1}{d_x^-(t, r_1(t))}(\dot{L}_1(t) - \sigma(t, r_1(t), L_1(t)) m(d_x^-(t, r_1(t))). \tag{2.25}$$

In particular sgn $\dot{r}_1 =$ *sgn* $(\dot{L}_1(t) - \sigma(t, L_1(t), r_1(t)) m(d_x^-(t, r_1(t)))$. *The tangency condition does not hold* $r_1(t)$ *for* $t > 0$.
(b) If the tangency condition holds for $t = 0$ *at* r_{10}, *for all* $t \geq 0$, *then we have* $d_x(t, r_1(t)) = 0$ *and*

$$\begin{aligned} 0 < \Sigma_1^R \quad := \quad & \fint_{r_{10}}^{R_1(0)} \sigma_t(0, y, L_{10})\, dy - \sigma_t(0, r_{10}, L_{10}) \\ & + \sigma(0, r_{10}, L_{10}) \left(\fint_{r_{10}}^{R_1(0)} \sigma_{x_2}(0, y, L_{10})\, dy - \sigma_{x_2}(0, r_{10}, L_{10}) \right) \\ & + \frac{\dot{R}_1(0)}{(R_1(0) - r_{10})}(\sigma(t, R_{10}, L_1(0)) - \dot{L}_1(0)), \end{aligned}$$

where

$$\dot{L}_1(0) = \fint_{r_{10}}^{R_{10}} \sigma(0, y, L_{10})\, dy - \frac{2\gamma(\mathbf{n}_\Lambda)}{R_{10} - r_{10}}, \quad \dot{R}_1(0) = \fint_{l_{10}}^{L_{10}} \sigma(0, R_{10}, y)\, dy - \frac{2\gamma(\mathbf{n}_R)}{L_{10} - l_{10}}.$$

Then the tangency condition holds for all $t > 0$.

□

2.4 Hamilton-Jacobi equations

The localized problem (2.17) for each of the sides is a coupled system of an ODE with a nonlocal nonlinearity and a Hamilton-Jacobi equation in a non-cylindrical domain. Since there are no tailor made existence results of viscosity solutions for such problems, we will construct them with the help of the method of characteristics.

In order to streamline the statements of the theorems in this section, we will gather in one place below the common hypotheses and we will call them *the standard set of assumptions*:

(S_1) (*conditions on* σ) We assume that σ is of class C^2 and it satisfies the symmetry relation (1.4) and Berg's effect (1.3);

(S_2) (*conditions on* m *i.e.,* β) the mobility coefficient m satisfies the assumptions (2.12–2.16);

(S_3) (*conditions on the initial curve*) d_0^Λ, d_0^R is an admissible pair of functions, which are of class C^2 on the closure of complement of the preimages of the facets.

We will collectively call (S_1), (S_2) and (S_3) by (S). We also note that (S_3) is a repetition of Definition 2.1 of a regular bent rectangle from subsection 2.1.

Proposition 2.4 *Let us suppose that the standard assumptions (S) hold and* $r_{00} \le \rho_0 < \rho_1 \le r_{10}$*. Then, there is* $T > 0$ *and a unique* C^2 *solution* d *to*

$$\begin{array}{ll} d_t - \sigma(t,x,d)m(d_x) = 0 & \text{in } G(\rho_0,\rho_1), \\ d(0,x) = d_0(x) & \text{for } x \in [\rho_0,\rho_1], \end{array} \tag{2.26}$$

where

$$G(\rho_0,\rho_1) = \{(t,x) \in (0,T) \times \mathbb{R} : \ x(t,\rho_0) \le x \le x(t,\rho_1)\}$$

and $x(t,\rho_i)$ *is the projected characteristic curve starting at* $(0,\rho_i)$, $i = 0,1$*, (see (2.27) below).*

Moreover, if Lip$(d_0) =: p_0 < 1$*, then:*
(a) there is a positive T_0*, such that for all* $t \in (0,T_0]$*, we have Lip*$(d(t,\cdot)) \le 1$ *and Lip*$(d(\cdot,x)) \le M := m(1)\sup\sigma$ *in* $G(\rho_0,\rho_1)$*;*
(b) if $\inf_{x\in(\rho_0,\rho_1)} d^+_{0,x}(x) = \delta > 0$*, then for all* (t,x)*,* $(t,y) \in G(\rho_0,\rho_1)$*, we have* $d(t,y) - d(t,x) \ge \delta(y-x)$*;*
(c) Let us suppose that d_{01}*,* d_{02} *are two pieces of initial data as above, satisfying* $(d_{01} - d_{02})|_{[\rho_0,\rho_1]} \equiv 0$ *and* d_1*,* d_2 *are the corresponding solutions to the characteristic system (2.27), associated to (2.26). Then, for any* $t < T_0$*, such that* $\rho_1 - \mu t > \rho_0$ *we have*

$$(d_1 - d_2)|_{[0,t]\times[\lambda_0,\lambda_1-\mu t]} \equiv 0,$$

where $\mu = \sup\sigma \cdot \sup m_p$*.*

Remark 2.1 The projected characteristic x depends not only on the initial position ζ, but also on p_0. It is understood that $x(\cdot,\zeta)$ means $x(\cdot,\zeta,d_{0,x}(\zeta))$. If this is not the case we will explicitly write $x(\cdot,\zeta,p_0)$.

Proof. The existence of C^2 solution d follows immediately from solving the characteristics system,

$$\begin{cases} \dot{x} = -\sigma(t,x,d)m_p(p), \\ \dot{p} = (\sigma_{x_1}(t,x,d) + \sigma_{x_2}(t,x,d)p)\, m(p), \\ \dot{d} = \sigma(t,x,d)\,(m(p) - m_p(p)p)\,, \end{cases} \tag{2.27}$$

$$x(0,\zeta) = \zeta, \quad d(0,\zeta) = d_0(\zeta), \quad p(0,\zeta) = d_{0,x}(\zeta), \quad \zeta \in \mathbb{R}. \tag{2.28}$$

The uniqueness is the implied by the local Lipschitz continuity of the RHS of (2.27). For the method of characteristics to work, the assumed high regularity is indispensable.

Parts (a) – (c) were proved for (2.26) in $(0,T)\times\mathbb{R}$ instead of $G(\rho_0,\rho_1)$ in [16, Theorem 3.1], however the argument needs no changes.

□

In principle, existence of classical solutions to Hamilton-Jacobi equation in noncylindrical domains is not immediate. The question is not only whether or not the curve r_0 is non-characteristic, but rather if the characteristics emanate from r_0. Otherwise we cannot specify data on it.

Both curves $r_0(\cdot)$ and $x(\cdot, r_0(t_0))$, the characteristic from $(t_0, r_0(t_0))$, are in the (t,x) phase space. We notice that $x(\cdot, r_0(t_0))$ goes into $\{(t,x) \in [0,T) \times \mathbb{R} :\ r_0(t) \le x\}$, for sufficiently small $t_0 > 0$, provided that

$$\dot{r}_0(t) < \dot{x}(t, r_{00}, 0) \tag{2.29}$$

at $t = 0$. After restricting $T > 0$, if necessary, by the continuity of both sides of (2.29), this inequality will hold for all $t \in [0,T]$. If r_0 is a tangency curve, then by (2.23), we have $\dot{r}_0 = \Sigma^R/\sigma_{x_1}$. The tangency condition implies that the vertical speeds of $L_0(\cdot)$ and of $d(\cdot, r_0(\cdot))$ are equal, provided that $d_x(t, r_0(t)) = 0$. This is why, (2.27) implies that $\dot{x}(t, r_0(t)) = -\sigma(t, r_0(t), L_0(t))m_p(0)$ Thus, (2.29) is equivalent to

$$\Sigma_0^R(t, r_0(t), L_0(t)) + \frac{1}{2}\sigma_{x_1}(t, r_0(t), L_0(t))\sigma(t, r_0(t), L_0(t)) < 0 \qquad \text{for } t \in (0,T). \tag{2.30}$$

Corollary 2.1 *The interfacial curve r_0 is non-characteristic, provided that (2.30) holds.* □

Proposition 2.5 *Let us suppose that (S) is satisfied, r_0 is a non-characteristic tangency curve, i.e., (2.30) holds and*

$$G(r_0, x(\rho_1)) = \{(t,x) :\ r_0(t) \le x \le x(\rho_1)\}.$$

We consider (2.26) in $G(r_0, x(\rho_1))$ augmented with the initial condition $d(0,x) = d_0(x)$ and a boundary data $d(t, r_0(t)) = L_0(t)$, where L_0 is of class C^2. Then, there exists d, a unique C^2 solution to (2.26) in $G(r_0, x(\rho_1))$. In particular, $d_x^+(t, r_0(t)) = 0$.

Remark 2.2 In [16], we did not introduce (2.30), because after mollification of m at all points of the tangency curve the characteristics shoot vertically upward. However, this does not hold in the limit. This aspect was overlooked in [16]. However, [16, Theorem 4.1 and Theorem 4.2] are valid if (2.30) holds.

Proof. We use the method of characteristics, *i.e.*, we will solve (2.27, 2.28). We immediately obtain solution to the Hamilton-Jacobi equation in $G(x(r_{00}), x(\rho_1))$. In order to solve the Hamilton-Jacobi equation in $G(r_0, x(r_{00}))$, we have to specify d and p on $(t, r_0(t))$. We set $d(t, r_0(t)) = L_0(t)$ and $p(t, r_0(t)) = 0$. We do so, because we want to have $d_x(t, r_0(t)) = 0$.

The continuous dependence of solutions on initial data implies that d is of class C^2 in $G(r_0, x(\rho_1))$. □

3 Evolution of bent rectangles

Let us highlight the aspects of the problem. In principle, at each interfacial curve we have the following possibilities:
1) the sign of the interfacial velocity is either positive or negative;
2) the tangency condition either holds or is violated;
3) the one-sided derivative of d^R or d^Λ at the interfacial curve is zero or positive.

In principle, we would have to study eight possibilities of behavior of each of the interfaces. However, we consider here the regular cases of shrinking or expanding facets. These are:
a) The one-sided derivative of d at the end of the facet is positive, the facet shrinks or expands and the tangency condition is violated. This situation is typical, because it persists under small perturbations.
b) The facet shrinks, d_x and the tangency condition holds along the interfacial curve. In this case, we also have $d \in C^2$ in a neighborhood of the interface. This situation is not typical, but it is important because it appears at the onset of facet bending.

The central facet expands, when $\dot{r}_0(t) > 0$ for $t \geq 0$, then the tangency does not hold at $t = 0$, when $d_x(0, r_{00}) > 0$. On the other hand, when this facet shrinks, then the tangency condition must hold for all $t \geq 0$, provided that $d_x(t, r_0(t)) = 0$. This is backed by our uniqueness theorem. We studied these two situations in [16] for r_0 (resp. l_0). It may also happen that $d_x^+(t, r_0(t)) > 0$, the facet shrinks or expands and the tangency condition does not hold.

It turns out that geometrically the case of r_1 (resp. l_1) is much different, see 3.2.2 below. We do not any have tool to treat it in full generality, yet. This is why, we restrict our attention to regular cases.

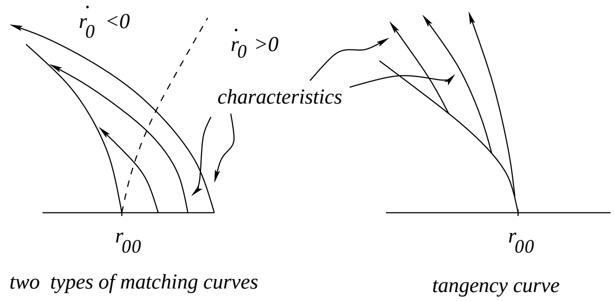

Fig. 2

We notice that, formally, system (2.17) decouples into three subsystems: two 'central facets' (this is a shorthand for systems for (i) r_0, L_0 and d^R, (2.17$_{1,2}$) (ii) a system for l_0, R_0 and d^Λ, (2.17$_{4,5}$)) and the 'corner system' *i.e.*, a system for R_1, l_1, L_1, r_1 and d^Λ, d^R. Formally, this is obvious, because equation (2.17$_{2,3,5,6}$) does not contain any unknown from the remaining system, as long as $r_0(t) < r_1(t)$ and $l_0(t) < l_1(t)$. The last condition is valid at least for some time provided that the interfacial curves are continuous.

When the central facet shrinks, then we have to solve the Hamilton-Jacobi equation (2.17_2) in a non-cylindrical domain. In order to do so, we have to make sure that we can specify the boundary data on the tangency curve r_0, this is why $\dot{r}_0 < \dot{x}$ or condition (2.30), see the right picture on Fig. 2, is necessary. There is no such problem on r_1 if $\dot{r}_1 > 0$ and it is a tangency curve. This is so, because we always have $\dot{x} < 0$, see the right picture on Fig. 3.

The matching curves behave differently. If the central facet expands, then the characteristics, which turn left impinge upon the facet, then we have a shock wave, see the left picture on Fig. 2. In [16], we succeeded in showing uniqueness of the matching curves not only if $d_x^+(t, r_0(t)) > 0$, but also a unique selection of interfacial curves in case $d_x^+(t, r_0(t)) = 0$. This is outside of the scope of this paper.

The case of the expanding outer facet, when $\dot{r}_1 < 0$ is more intriguing. The matching condition will specify the position of a curve provided d is known in the region bounded by r_1. This happens if $\dot{r}_1 < \dot{x}$, see condition (3.5). Otherwise we there is a rarefaction wave between $x(r_{10})$, the characteristic emanating from r_{10}, and r_1. We do not have right tools to study these phenomena.

We stress that in our considerations, the corner is defined as the point of intersection of the outer facets. In principle, we should evolve it by (1.1). We might say that this situation is similar to the rarefaction fan mentioned for the matching curve emanating from r_{10}. Under certain conditions, we can however, show that the evolution taken here is correct, this will be reported elsewhere.

The systems for central facets are first studied provided that $l_{00} < l_{10}$ and $r_{00} < r_{10}$. Systems (i) and (ii) above are similar, it is sufficient to study just one on them. If $l_{00} = l_{10}$ or $r_{00} = r_{10}$, then we are forced to make further consideration, see subsection 3.4.

We have already learned what are the component of these systems: Propositions 2.2 and 2.3 give us tools to detect the type of curve emanating from an interfacial point.

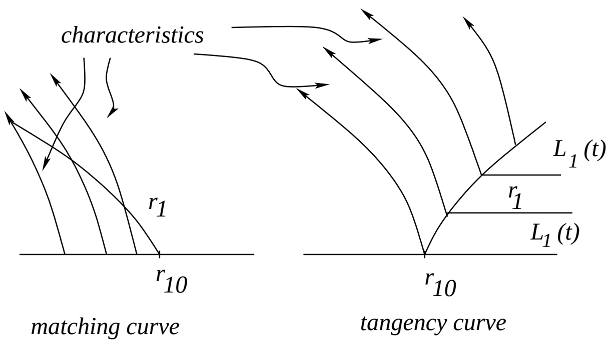

Fig. 3

3.1 The central facet system

Since both central facet systems have the same structure,it is sufficient to consider just one of them. The point is to study possible behavior of the interfacial curves.

Let us now turn to the problem of existence of tangency curves. Their names come from the fact that the tangency condition (2.19) holds along them. More importantly, the structure of (2.19) suggests that the Implicit Function Theorem can be used to construct them. Simultaneously, we will solve the equation for the evolution of the central facet. Only after that, we are in a position to construct d with initial data on the tangency curves.

Proposition 3.1 *Let us assume that* $\Sigma_0^R < 0$ *and the tangency condition is satisfied at* $t = 0$, $L_0 = L_{00}$ *and* $x = r_{00}$, *i.e.,*

$$\frac{1}{r_{00}} \int_0^{r_{00}} \sigma(0, s, L_{00})\, ds + \frac{\gamma(\mathbf{n}_\Lambda)}{r_{00}} = \sigma(0, r_{00}, L_{00}),$$

holds, see (2.19$_1$). If (S) holds, then there exists $(r_0, L_0) \in C^2([0,T]; \mathbb{R}^2)$ *a unique local in time solution to the following problem,*

$$\begin{aligned} \frac{1}{r_0(t)} \int_0^{r_0(t)} \sigma(t, s, L_0(t))\, ds + \frac{\gamma(\mathbf{n}_\Lambda)}{r_0(t)} &= \sigma(t, r_0(t), L_0(t)), \\ \beta_R \dot{L}_0(t) = \frac{1}{r_0(t)} \int_0^{r_0(t)} \sigma(t, s, L_0(t))\, ds &+ \frac{\gamma(\mathbf{n}_\Lambda)}{rl_0(t)}, \\ L_0(0) = L_{00}, \quad r_0(0) &= r_{00}. \end{aligned}$$

In addition, $\dot{r}_0$ *is given by (2.23), i.e.,*

$$\begin{aligned} \dot{r}_0(t) \;=\; & \frac{1}{\sigma_{x_1}(t, r_0(t), L_0(t))} \int_0^{r_0} (\sigma_t(t, y, L_0(t)) + \sigma_{x_2}(t, y, L_0(t)))\dot{L}_0)\, dy \\ & - \frac{1}{\sigma_{x_1}(t, r_0(t), L_0(t))} (\sigma_t(t, r_0(t), L_0(t)) + \sigma_{x_2}(t, r_0(t), L_0(t)))\dot{L}_0), \end{aligned} \tag{3.1}$$

where

$$\dot{L}_0 = \sigma(t, r_0(t), L_0(t))/\beta_R.$$

Proof. We start with preparations for the Implicit Function Theorem. We define a map $\mathcal{F} : \mathbb{R}^3_+ \to \mathbb{R}$ by the following formula

$$\mathcal{F}(t, r_0, L_0) = \int_0^{r_0} \sigma(t, s, L_0)\, ds + \gamma(\mathbf{n}_\Lambda) - r_0 \sigma(t, r_0, L_0).$$

We notice that $\mathcal{F}(0, r_{00}, L_{00}) = 0$ and

$$\frac{\partial \mathcal{F}}{\partial r_0}(0, r_{00}, L_{00}) = -r_{00}\sigma_{x_1}(0, r_{00}, L_{00}) < 0,$$

where the last inequality is implied by the Berg's effect (1.3). Thus, we may apply the Implicit Function Theorem. As a result, there exists $\mathcal{U}$, a neighborhood of $(0, L_{00})$, and a function $r_0 = r_0(t, L_0)$ such that

$$\mathcal{F}(t, r_0, L_0(t, L_0)) = 0 \quad \text{for } (t, L_0) \in \mathcal{U}.$$

Moreover, we can rewrite the equation for L_0 as an ODE

$$\dot{L}_0 = \sigma(t, r_0(t, L_0), L_0), \quad L_0(0) = L_{00}.$$

In order to prove existence and uniqueness of solutions, it is sufficient to check that the right hand side of the above formula is Lipschitz continuous with respect to L_0. Indeed, it is easy to check that

$$\frac{\partial r_0}{\partial L_0} = \frac{\int_0^{r_0} \sigma_{x_2}(t, s, L_0)\, ds - r_0 \sigma_{x_2}(t, r_0, R_0)}{r_0 \sigma_{x_1}(t, R_0, L_0)} \tag{3.2}$$

is bounded. Thus, by the standard ODE theory, we conclude the proof of the present Proposition. $\square$

This result yields the interfacial curve r_0 as well as the position of the central facet L_0. Time regularity of r_0 and L_0 correspond to the smoothness of the data.

We state the main results of this subsection.

Theorem 3.1 *Let us suppose that the standard set of assumptions (S) is satisfied. If* $d_{0x}(r_{00}) = 0$*, the tangency condition (2.19$_1$) holds and the tangency curve is non-characteristic, i.e., (2.30) is satisfied, then there exists a unique solution* L_0, d *to problem (2.17$_1$)–(2.17$_2$).*

Theorem 3.2 *Let us suppose that standard set of assumptions (S) is satisfied. If* $\dot{L}_0(0) - \sigma(0, r_{00}, L_{00})m(d_{0,x}(r_{00}^+)) \neq 0$*, the tangency condition does not hold and* $d_{0,x}(r_{00}^+) > 0$*, then there exists a unique matching curve* r_0 *and a unique solution* L_0, d *to problem (2.17$_1$)–(2.17$_2$).*

Remarks.
Of course, a similar condition holds, after replacing $r_0(\cdot)$ (resp. $L_0(\cdot)$) with $l_0(\cdot)$ (resp. $R_0(\cdot)$).

Condition (2.30) is stronger than $\Sigma_0^R < 0$. We will leave the gap between conditions (2.30) and $\Sigma_0^R < 0$ open.

Proof of Theorem 3.1. Since the tangency condition holds, then due to Proposition 3.1, we can construct a unique C^2 tangency curve $r_0(\cdot)$. Condition (2.30) is open and it will be satisfied for all $t \in [0, T)$, possibly after restricting $T > 0$. This guarantees that the characteristics will emanate from $r_0(\cdot)$. Hence, we may set initial conditions on this curve for the Hamilton-Jacobi equation (2.17$_2$). We apply Proposition 2.5, note that the C^2 regularity of σ yields r_0 of class C^2. This is of course true, if we solve (2.27)–(2.28) with $p(0, \zeta) = 0$. By Proposition 2.5, we obtain a unique solution to (2.17$_{1,2}$) in $G(r_0, x(r_{10}))$.

Now, we will offer a *proof of Theorem 3.2.* Due to our smoothness assumption (S) Proposition 2.4 yields d of class C^2 in $G(x(r_{00}), x(r_{10}))$, we may apply the method of characteristics to solve (2.17$_1$–2.17$_2$)for d. We note that by Proposition 2.4 (a) we have $|d_x| \leq 1$ and $d_x(t, x) > \delta$ on $[0, T_0]$. Once we constructed d we can invoke [16, Theorem 4.2] to conclude existence of a unique matching curve r_0. This is true when the facet expands, *i.e.*, $\dot{L}_0 - \sigma m(d_x) > 0$, however, we have also the case of shrinking facet to handle. In this situation, we have to check, if the interface is in the region filled with characteristics of the Hamilton-Jacobi equation (2.17$_2$). This happens provided that $\dot{r}_0 < \dot{x}(r_{00}) < 0$. This is equivalent to

$$\sigma(0, r_0, L_0)(m(d_x) - m_p(d_x)d_x) < \dot{L}_0. \tag{3.3}$$

Convexity of m for $|p| \leq 1$ implies that

$$m(d_x) - m_p(d_x)d_x = d_x \int_0^1 (m_p(sd_x) - m_p(d_x))ds < 0,$$

as long as $m_p \neq 0$. On the other hand $\dot{L}_0 > 0$, hence (3.3) is trivially satisfied. The point is, [16, Theorem 4.2] keeps holding also if the matching curve r_0 shrinks.

Hence, system (2.17$_1$–2.17$_3$) closes and we can uniquely solve for L_0 on $[0, T_0]$. $\square$

3.2 The corner system

We present a general existence result for regular data. We are left with just three basic cases:
(α) two tangency curves emanate from r_{10} and l_{10}; (β) two matching curves emanate from r_{10} and l_{10}; (γ) a tangency curve (resp. a matching curve) emanates from r_{10} and a matching curve (resp. a tangency curve) emanates from l_{10}. Analysis of these cases is the content of the next theorem. Its proof is conducted so that it will be easy to use it when we study the evolution of the corner itself.

Theorem 3.3 *Let us suppose that (S) holds and the assumptions of Theorems 3.1 of 3.2 are fulfilled.*
(a) If $\Sigma_1^\Lambda > 0$, $\Sigma_1^R > 0$ and the tangency conditions are satisfied at l_{10}, r_{10} as well as $d^\Lambda_{0,x}(l_{10}) = 0 = d^R_{0,x}(r_{10})$, then there are two tangency curves $l_1(\cdot)$, $r_1(\cdot)$, and a unique solution $(d^\Lambda, R_1, L_1, d^R)$ to the system (2.17$_{2,3,5,6}$).
(b) We assume that $\Sigma_1^R > 0$, the tangency condition is satisfied at r_{10}, as well as $d^R_{0,x}(r_{10}) = 0$. The tangency condition does not hold at l_{10}, $\dot L_1(0) - \sigma(0, R_{10}, l_{10}) m(d^{\Lambda,-}_{0,x}(l_{10})) \neq 0$ and $p_0 := d^{\Lambda,-}_{0,x}(l_{10}) > 0$. In addition, we need that the following relation be satisfied,

$$\begin{aligned} &m(p_0)\sigma(t, d_0, l_{10}) - \frac{m(0)}{L_{10} - l_{10}} \left(\int_{l_{10}}^{L_{10}} \sigma(t, R_{10}, s)\, ds - 2\gamma(\mathbf{n}_R) \right) \\ &< -p_0 m_p(p_0) \sigma(t, R_{10}, l_{10}) \end{aligned} \tag{3.4}$$

Then, there is a tangency $r_1(\cdot)$, and a matching curve $l_1(\cdot)$, and a solution $(d^\Lambda, R_1, L_1, d^R)$ to the system (2.17$_{2,3,5,6}$). A similar statements is valid if we change the roles of r_1 and l_1.
(c) Let us suppose that $d^{R,-}_{0,x}(r_{10}), m(d^{\Lambda,-}_{0,x}(l_{10}) > 0$,

$$\dot R_1(0) - \sigma(0, r_{10}, L_{10}) m(d^{R,-}_{0,x}(r_{10})) \neq 0 \quad \textit{and} \quad \dot L_1(0) - \sigma(0, R_{10}, l_{10}) m(d^{\Lambda,-}_{0,x}(l_{10})) \neq 0,$$

the tangency conditions do not hold neither at l_{10} nor at r_{10}. If in addition (3.5) and

$$\begin{aligned} &m(p_0)\sigma(t, r_{10}, d_0) - \frac{m(0)}{R_{10} - r_{10}} \left(\int_{r_{10}}^{R_{10}} \sigma(t, s, L_{10})\, ds - 2\gamma(\mathbf{n}_\Lambda) \right) \\ &< -p_0 m_p(p_0) \sigma(t, r_{10}, L_{10}). \end{aligned} \tag{3.5}$$

are satisfied, then there are two matching curves $l_1(\cdot)$, $r_1(\cdot)$ and a solution $(d^\Lambda, R_1, L_1, d^R)$ to the system (2.17$_2$)–(2.17$_5$).

The role of the strange looking conditions (3.5), (3.4) guaranteeing existence of the matching curves will be explained in subsection 3.2.2

Having this result at hand, we can show.

Theorem 3.4 *Let us suppose that, the standard set of assumptions (S) holds, in particular the initial curve Γ_0 is a regular bent rectangle and $l_{00} < l_{10}$, $r_{00} < r_{10}$. We assume that the initial data fulfill conditions (a) and (b) below:*
(a) One of the following conditions holds at the interface r_{00}:

(i) $d_{0,x}(r_{00}) = 0$, $\Sigma_0^R < 0$, the tangency condition holds at r_{00} and (2.30) is satisfied

or

(ii) $\dot{L}_0 - \sigma(0, r_{00}, L_{00})m(d^+_{0,x}(r_{00})) \neq 0$*, the tangency condition is violated at* r_{00} *and* $d^+_{0,x}(r_{00}) > 0$*.*

Moreover, a respective version of (i) and (ii) holds for l_{00}*.*

(b) One of the following conditions hold at the interface r_{10}*:*

(iii) $d_{0,x}(r_{10}) = 0$*,* $\Sigma^R_1 > 0$*, the tangency condition holds at* r_{10} *and a condition corresponding to (2.30) for* r_{10}*, i.e.,*

or

(iv) $\dot{L}_1 - \sigma(0, r_{10}, L_{10})m(d^-_{0,x}(r_{10})) \neq 0$*, the tangency condition is violated at* r_{10} *and* $d^-_{0,x}(l_{10}) > 0$*.*

Moreover, a respective version of (iii) and (iv) holds for l_{10}*.*

Then, there is a solution, i.e. $(d^\Lambda, d^R, L_0, L_1, R_0, R_1)$ *to the system (2.17). In addition, if conditions (i) for* l_{00} *and* r_{00}*, (iii) for* l_{10} *and* r_{10} *are satisfied, then* d^Λ*,* $d^R \in C^2$*.*

For the sake of clarity of the presentation the proof will be relegated to separated paragraphs.

3.2.1 The case of two tangency curves

Proof of Theorem 3.3, part (a). We shall show more a general result. Namely, we allow the right hand end of the facet to be governed by a given unspecified law. This will be useful later, when we decide to study evolution of the corner itself, *e.g.*, its rounding off.

Lemma 3.1 *Let* $L_2, R_2 \in C^1(\mathbb{R}^4; \mathbb{R})$ *and we introduce a shorthand for their value at* $(R_{10}, L_{10}, r_{10}, l_{10})$*, i.e.,* $L_{20} = L_2(R_{10}, L_{10}, r_{10}, l_{10})$ *and* $R_{20} = R_2(R_{10}, L_{10}, r_{10}, l_{10})$*. Let us assume that the tangency conditions are satisfied at* R_{10}, r_{10}, L_{10} *and* l_{10}*, i.e.,*

$$\sigma(0, R_{10}, l_{10}) = \frac{1}{L_{20} - l_{10}} \int_{l_{10}}^{L_{20}} \sigma(0, R_{10}, s)\, ds - \frac{2\gamma(\mathbf{n}_R)}{L_{20} - l_{10}},$$
$$\sigma(0, r_{10}, L_{10}) = \frac{1}{R_{20} - r_{10}} \int_{r_{10}}^{R_{20}} \sigma(0, s, L_{10})\, ds - \frac{2\gamma(\mathbf{n}_\Lambda)}{R_{20} - r_{10}}.$$

In addition, we have

$$\begin{aligned}
&(\sigma(0, R_{10}, l_{10}) - \sigma(0, R_{10}, L_{20})) \left.\frac{\partial L_2}{\partial y_3}\right|_{y_3 = r_{10}} (\sigma(0, r_{10}, L_{10}) - \sigma(0, R_{20}, L_{10})) \left.\frac{\partial R_2}{\partial y_4}\right|_{y_4 = l_{10}} \neq \\
&\left((L_{20} - l_{10})\sigma_{x_2}(0, R_{10}, L_{10}) + (\sigma(0, R_{10}, l_{10}) - \sigma(0, R_{10}, L_{20})) \left.\frac{\partial L_2}{\partial y_4}\right|_{y_4 = l_{10}} \right) \cdot \\
&\left((R_{20} - r_{10})\, \sigma_{x_1}(0, r_{10}, l_{10}) + (\sigma(0, r_{10}, L_{10}) - \sigma(0, R_{20}, L_{10})) \left.\frac{\partial R_2}{\partial y_3}\right|_{y_3 = r_{10}} \right).
\end{aligned} \tag{3.6}$$

Then, there exists a unique local in time solution to the problem:

$$
\begin{aligned}
&\dot{R}_1 = \frac{1}{L_2(R_1,L_1,r_1,l_1)-l_1}\left(\int_{l_1}^{L_2(R_1,L_1,r_1,l_1)} \sigma(t,R_1(t),s)\,ds - 2\gamma(\mathbf{n}_R)\right),\\
&\sigma(t,R_1(t),l_1(t)) = \frac{1}{L_2(R_1,L_1,r_1,l_1)-l_1}\left(\int_{l_1}^{L_2(R_1,L_1,r_1,l_1)} \sigma(t,R_1(t),s)\,ds - 2\gamma(\mathbf{n}_R)\right),\\
&\dot{L}_1 = \frac{1}{R_2(R_1,L_1,r_1,l_1)-r_1}\left(\int_{r_1}^{R_2(R_1,L_1,r_1,l_1)} \sigma(t,s,L_1(t))\,ds - 2\gamma(\mathbf{n}_\Lambda)\right),\\
&\sigma(t,r_1(t),L_1(t)) = \frac{1}{R_2(R_1,L_1,r_1,l_1)-r_1}\left(\int_{r_1}^{R_2(R_1,L_1,r_1,l_1)} \sigma(t,s,L_1(t))\,ds - 2\gamma(\mathbf{n}_\Lambda)\right),\\
&R_1(0)=R_{10},\quad L_1(0)=L_{10},\quad r_1(0)=r_{10},\quad l_1(0)=l_{10}.
\end{aligned}
\tag{3.7}
$$

Remark 3.1 If the position of the corners is determined by the intersection of the outer facets, then $L_2 = L_1$ and $R_2 = R_1$. Hence, condition (3.6) takes the following form

$$(L_{10}-l_{10})(R_{10}-r_{10})\sigma_{x_2}(0,R_{10},L_{10})\sigma_{x_1}(0,r_{10},l_{10}) \neq 0.$$

It is always satisfied since $L_{10} > l_{10}$ and $R_{10} > r_{10}$.

Proof. We notice that ($3.7_{2,4}$) are functional, not differential equations. Thus, in order to determine r_1 and l_1, we will use the Implicit Function Theorem. For that purpose, we define the map $\mathcal{J} : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$
\begin{aligned}
&\mathcal{J}(t,R_1,L_1,r_1,l_1) = \\
&\begin{pmatrix} \sigma(t,R_1,l_1)(L_2(R_1,L_1,r_1,l_1)-l_1) - \int_{l_1}^{L_2(R_1,L_1,r_1,l_1)} \sigma(t,R_1,s)\,ds + 2\gamma(\mathbf{n}_R) \\ \sigma(t,r_1,L_1)(R_2(R_1,L_1,r_1,l_1)-r_1) - \int_{r_1}^{R_2(R_1,L_1,r_1,l_1)} \sigma(t,s,L_1)\,ds + 2\gamma(\mathbf{n}_\Lambda) \end{pmatrix}.
\end{aligned}
$$

By straightforward calculation, we obtain

$$D_{r_1,l_1}\mathcal{J}(0,R_{10},L_{10},r_{10},l_{10}) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix},$$

where

$$
\begin{aligned}
j_{11} &= \sigma(0,R_{10},l_{10}) - \sigma(0,R_{10},L_{20}))\left.\frac{\partial L_2}{\partial y_3}\right|_{y_3=r_{10}},\\
j_{12} &= (L_{20}-l_{10})\sigma_z(0,R_{10},L_{10}) + (\sigma(0,R_{10},l_{10}) - \sigma(0,R_{10},L_{20}))\left.\frac{\partial L_2}{\partial y_4}\right|_{y_4=l_{10}},\\
j_{21} &= (R_{20}-r_{10})\,\sigma_y(0,r_{10},l_{10}) + (\sigma(0,r_{10},L_{10}) - \sigma(0,R_{20},L_{10}))\left.\frac{\partial R_2}{\partial y_3}\right|_{y_3=r_{10}},\\
j_{22} &= (\sigma(0,r_{10},L_{10}) - \sigma(0,R_{20},L_{10}))\left.\frac{\partial R_2}{\partial y_4}\right|_{y_4=l_{10}}.
\end{aligned}
$$

By (3.6) the map $D_{r_1,l_1}\mathcal{J}(0,R_{10},L_{10},r_{10},l_{10})$ is an isomorphism and we can apply implicit function theorem. Therefore, there exists a neighborhood of $(0,R_{10},L_{10})$ and functions

$$r_1 = r_1(t,R_1,L_1),\quad l_1 = l_1(t,R_1,L_1)$$

such that

$$\mathcal{J}(t, R_1, L_1, r_1, l_1) = 0.$$

Moreover, we can rewrite the equation for R_1 and L_1, as follows

$$\begin{aligned}
\dot{R}_1 &= \sigma(t, R_1, l_1(t, R_1, L_1)), \qquad R_1(0) = R_{10}, \\
\dot{L}_1 &= \sigma(t, r_1(t, R_1, L_1), L_1), \qquad L_1(0) = L_{10}.
\end{aligned}$$

The right hand side of the above equation is Lipschitz continuous with respect to R_1 and L_1. □

Remark 3.2 Let us consider the flat initial data such that $l_{00} = l_{10}$ or $r_{00} = r_{10}$. A scrutiny of the proofs of Lemma 3.1 and Proposition 3.1 reveals that the two tangency curves r_0 and r_1 (respectively, l_0 and l_1) may be constructed by the Implicit Function Theorem. The coalescence $r_{00} = r_{10}$ does not interfere with its applicability.

We proceed with a *proof of Theorem 3.3 (a).* We use Lemma 3.1 when $L_2 = L_1$, $R_2 = R_1$. By Remark 3.1 condition (3.6) is satisfied, then we obtain two tangency curves l_1 and r_1. By Proposition 2.4 we can solve (2.17$_2$) in $G(l_0, l_1)$ (resp. in $G(r_0, r_1)$ and the solution d^Λ (resp. d^R) is of class C^2 there.

Moreover, Theorem 3.1 or Theorem 3.2 yield interfacial curves l_0 and r_0. In addition, if necessary, we invoke Proposition 2.5 to construct d over the tangency curves r_0 or l_0. □

3.2.2 Analysis of a single interfacial curve starting at r_{10}

When we study the outer facet, we take into account positions and behavior of both of its endpoints. It is not quite appropriate to talk about shrinking nor expanding solely upon the position of the endpoints. What really matters is the vertical speed of d to the left and right of the interfacial point r_{10}, *i.e.*, $\dot{L}_1$ and d_t. We notice that if $d_t > \dot{L}_1$, then the curved part collides with a slower outer facet, hence $\dot{r}_1 < 0$ and the facet slows down even more. On the other hand, if $d_t < \dot{L}_1$, then the facet is rapidly advancing, it leaves the curved part behind, *i.e.*, $\dot{r}_1 > 0$. The case $d_t = \dot{L}_1$ corresponds to the tangency curve, provided that $d_x(t, r_1(t)) = 0$.

These considerations give an interpretation to the following calculations, which are rigorous, if we take into account the smoothness of solutions, we are interested in. Taking the time derivative of the matching condition (2.20) yields

$$\dot{L}_1(t) = d_t(t, r_1(t)) + d_x^-(t, r_1(t))\dot{r}_1(t). \tag{3.8}$$

Since the case of a tangency curve has already been studied, we subsequently assume that $d_x^-(t, r_1(t)) > 0$. We have two obvious situations to consider:

$$d_t = m(d_{0,x})\sigma > \dot{L}_1 \iff \dot{r}_1 < 0, \tag{3.9}$$

$$d_t = m(d_{0,x})\sigma < \dot{L}_1 \iff \dot{r}_1 > 0. \tag{3.10}$$

Let us look more closely at (3.10). We know how to proceed if the interface r_1 is a tangency curve.

On the other hand, if r_1 is not a tangency curve, we notice that the characteristics, always turn left, $\dot{x}(t, p) < 0$ see (2.27). Thus, a rarefaction region forms. This situation is much similar to the problem of the corner evolution. It will be studied elsewhere.

When $\dot{r}_1 < 0$, then can we expect only a matching curve. However, each curve emanating from r_{10} (or from l_{10}) are distinctively different from matching curve r_0 (or l_0), see the left pictures on Figures 2 and 3. The former one is the result of the collision of the central facet with the characteristics of the Hamilton-Jacobi equation (2.17$_2$). This is so, because the characteristics run against the facet. This is no longer true for curves at r_{10}, they turn away from the outer facet and in principle a rarefaction region may form again. Indeed, if $\dot{x}(0, d_x(0, r_{10})) < \dot{r}_1(0) < 0$, then a rarefaction region forms. We will not deal with it. If,

$$\dot{x}(t, d_x(t, r_{10})) > \dot{r}_1(t), \tag{3.11}$$

at $t = 0$, then the interface collides with the slower characteristics. We notice that, if we recall that $\dot{x}(0, d_x(0, r_{10})) = -m_p(d_x(0, r_{10}))\sigma(0, r_{10}, L_{10})$, then (3.11) is nothing but (3.5).

Indeed, if (3.11) holds, then we can construct a matching curve.

Proposition 3.2 *Let us suppose that R_1 is a given Lipschitz continuous function and $d^-_{0,x}(r_{10}) > 0$. If (3.9) is satisfied, i.e.,*

$$\dot{r}_1(0) = \frac{1}{d^-_{0,x}(r_{10})}(m(d^-_{0,x}(r_{10}))\sigma(0, r_{10}, L_{10}) - \dot{L}_1(0)) < 0$$

and (3.11) holds, then there is a unique solution to

$$\begin{aligned} \dot{r}_1 \quad &= \tfrac{1}{d_x}\left(\sigma m - \tfrac{m(0)}{R_1 - r_1}\int_{r_1}^{R_1} \sigma\, ds - \tfrac{2\gamma(\mathbf{n}_\Lambda)}{R_1 - r_1}\right) =: f(r_1, t), \\ r_1(0) \quad &= r_{10}. \end{aligned} \tag{3.12}$$

Let us also observe that since f depends in a Lipschitz continuous way upon R_1, we conclude that,

Proposition 3.3 *Let us suppose that Lipschitz continuous R_1 and R_2 are given, then there exists $\Lambda > 0$ such that*

$$\|r_1(R_1) - r_1(R_2)\|_{C^0} \leq \Lambda\|R_1 - R_2\|_{C^0}.$$

Proof. This is an immediate consequence of the Lipschitz dependence of the fixed point upon the parameter, provided that F is Lipschitz with respect to R, which is indeed the case. □

Now, we will present a *proof of Proposition 3.2.* Having assumed regular data, we can invoke Proposition 2.4 to claim existence of solutions to (2.17$_2$) in $G(x(r_{00}), x(r_{10}))$. Due to Proposition 2.4 (b), we know that $d_x(t, x) > 0$ there.

Due to continuity of f there is a neighborhood of $(r_{10}, L_1(0), d_{0,x}(r_{10}))$ of the form

$$(r_{10} - \eta, r_{10} + \eta) \times (L_1(0) - \eta, L_1(0) + \eta) \times (d_{0,x}(r_{10}) - \eta, d_{0,x}(r_{10}) + \eta) =: \mathcal{U},$$

and such that $\dot{r}_1 < \dot{x}$ holds in $\mathcal{U}$.

We conclude that for any $K > 1$ there is $t_0 > 0$ such that the solutions to the characteristic system (2.27) satisfies

$$|x(t)|,\ |d(t)|,\ |p(t)| \leq K \quad t \in [0, t_0].$$

If this is so, then there is another $0 < t_1 \leq t_0$ and such that

$$(x(t), d(t), p(t)) \in \mathcal{U} \quad t \in [0, t_1].$$

In other words, the function $f - \dot{x}$ restricted to $\mathcal{U}$ is negative.

If we restrict the behavior of R_1 by requiring that $R_1 \geq r_{10} + e$, then $\sup_{\mathcal{U}} |f| = M < \infty$. Thus, there exists $t_2 \in (0, t_1)$ such that for any selection of arguments of f, then $r_1(t) \in \bar{B} \subset \mathcal{U}$ for $t \in (0, t_2)$, where $\bar{B}$ is a closed ball centered at r_{10}. At the same time $\dot{r}_1 < \dot{x}$ in $\bar{B}$.

Let us set for $0 < T \leq t_2$

$$X_T = \{\tilde{r} \in C^0[0,T] : \ \tilde{r}(t) \in \bar{B}\}$$

This is a complete metric space with the C^0 norm. We set $F : X_T \to X_T$ by formula

$$F(\tilde{r}) = r_{10} + \int_0^t f(\tilde{r}(s), s)\, ds.$$

Since $\tilde{r} \in X_T$, then by the very definition of $\bar{B}$.

$$\frac{d}{dt} F(\tilde{r})(t) = f(\tilde{r}(t), t) < \dot{x}(t).$$

As a result

$$F(\tilde{r})(t) < x(t) \quad \text{and} \quad x(t) \quad \text{for } t \in [0, t_2].$$

This shows that indeed we can show the Banach contraction principle to F, after possibly taking smaller T. However, we can further extend the solution to the interval $[0, t_2]$, this is due to the abstract argument, based on the fact that r_1 stays in the set where $\dot{r}_1 < \dot{x}$, here f is well-defined. □

3.2.3 The case of two matching curves

Also in this case, we admit that the dynamics of the corner is not defined as the intersection of the outer facets. Thus, positions of the endpoints R_2 and L_2 is given. Existence of the interfacial curve will be shown assuming continuity of L_2 and R_2. The uniqueness of the interfaces is guaranteed provided that L_2 and R_2 are locally Lipschitz continuous.

Proof of Theorem 3.3, part (c). Let $L_2, R_2 \in C(\mathbb{R}^4; \mathbb{R})$ and let us assume that d^Λ and d^R are solutions to the Hamilton-Jacobi equations (2.17$_2$) and (2.17$_5$).

If $(d^\Lambda_{0,x_2})^-(l_{10}) > 0$, $(d^R_{0,x_1})^-(r_{10}) > 0$ and $\dot{l}_1(0) < \dot{x}(0, l_{10})$, $\dot{r}_1(0) < \dot{x}(0, r_{10})$, where $x(0, l_{10})$ and $x(0, r_{10})$ are the characteristics starting from l_{10} and r_{10} respectively, then, there exists unique local in time solution to the problem:

$$\begin{aligned}
\dot{R}_1 &= \frac{1}{L_2(R_1, L_1, r_1, l_1) - l_1} \left(\int_{l_1}^{L_2(R_1,L_1,r_1,l_1)} \sigma(t, R_1(t), s)\, ds - 2\gamma(\mathbf{n}_R) \right), \\
&\qquad\qquad R_1 = d^\Lambda(t, l_1), \\
\dot{L}_1 &= \frac{1}{R_2(R_1, L_1, r_1, l_1) - r_1} \left(\int_{r_1}^{R_2(R_1,L_1,r_1,l_1)} \sigma(t, L_1(t), s)\, ds - 2\gamma(\mathbf{n}_\Lambda) \right), \\
&\qquad\qquad L_1 = d^R(t, r_1), \\
&R_1(0) = R_{10}, \quad L_1(0) = L_{10}, \quad r_1(0) = r_{10}, \quad l_1(0) = l_{10}.
\end{aligned} \tag{3.13}$$

Since $L_2, R_2 : \mathbb{R}^4 \to \mathbb{R}$ are continuous then there exists δ_1 such that if

$$\max\left(|R - R_{10}|, |L - L_{10}|, |r - r_{10}|, |l - l_{10}|\right) \leq \delta_1,$$

then

$$\max\left(|R_2(R,L,r,l)-R_{20}|,|L_2(R,L,r,l)-L_{20}|\right)\leq\frac{1}{4}\min\left(L_{20}-l_{10},R_{20}-r_{10}\right).$$

If we differentiate (3.13_2) and (3.13_4) with respect to t, then we obtain the equivalent problem:

$$\begin{aligned}\dot{l}_1 &= G_1(R_1,L_1,r_1,l_1), \quad R_1=d^\Lambda(t,l_1),\\ \dot{r}_1 &= G_2(R_1,L_1,r_1,l_1), \quad L_1=d^R(t,r_1),\\ R_1(0)=R_{10}, \quad L_1(0)&=L_{10}, \quad r_1(0)=r_{10}, \quad l_1(0)=l_{10}.\end{aligned}$$

Here:

$$G_1(R_1,L_1,r_1,l_1,t)=$$
$$\frac{1}{d_x^\Lambda(t,l_1)}\frac{\int_{l_1}^{L_2(R_1,L_1,r_1,l_1)}\sigma(t,R_1(t),s)\,ds-2\gamma(\mathbf{n}_R)}{L_2(R_1,L_1,r_1,l_1)-l_1}-\frac{m(d_x^\Lambda(t,l_1))\sigma(t,l_1,d^\Lambda(t,l_1))}{d_x^\Lambda(t,l_1)},$$

$$G_2(R_1,L_1,r_1,l_1,t)=$$
$$\frac{1}{d_x^R(t,r_1)}\frac{\int_{r_1}^{R_2(R_1,L_1,r_1,l_1)}\sigma(t,L_1(t),s)\,ds-2\gamma(\mathbf{n}_\Lambda)}{R_2(R_1,L_1,r_1,l_1)-r_1}-\frac{m(d_x^R(t,r_1))\sigma(t,l_1,d^R(t,r_1))}{d_x^R(t,r_1)}.$$

Since G_2 and G_1 are continuous, there exist T_2 and δ_2 such that $G_1<\dot{x}(t,l_{10})$ and $G_2<\dot{x}(t,r_{10})$ on the set $\overline{B}\left((R_{10},L_{10},r_{10},l_{10}),\delta_2\right)\times[0,T_2]$. Moreover, since d_x^R and d_x^Λ are continuous and $d_x^R(0,r_{10})>0$ $d_x^\Lambda(0,l_{10})>0$, there exist T_3,δ_3,ϵ such that if $t\in[0,T_3]$, $\max\{|l_1-l_{10}|,|r_1-r_{10}|\}\leq\delta_3$, then $\min(d_x^\Lambda(t,l_1),d_x^R(t,r_1))>\epsilon$.

Next, for $T<\min(T_2,T_3)$, $\tilde{\delta}=\min(\delta_1,\delta_2)$ and $\delta\leq\min\left(\frac{L_{20}-l_{10}}{4},\frac{R_{20}-r_{10}}{4},\tilde{\delta},\delta_3\right)$, we define the set

$$\begin{aligned}\tilde{Y}_{T,\delta} &= B_{C[0,T]}(R_{10},\tilde{\delta})\times B_{C[0,T]}(L_{10},\tilde{\delta})\times B_{C[0,T]}(r_{10},\delta)\times B_{C[0,T]}(l_{10},\delta),\\ Y_{T,\delta} &= \tilde{Y}_{T,\delta}\cap\{(R,L,r,l)\in C([0,T];\mathbb{R}^4):(R,L,r,l)(0)=(R_{10},L_{10},r_{10},l_{10},)\}.\end{aligned}$$

Subsequently, we define a map $\mathcal{Z}:Y_{T,\delta}\to C([0,T];\mathbb{R}^4)$,

$$\mathcal{Z}(R,L,r,l)=(\tilde{R},\tilde{L},\tilde{r},\tilde{l}),$$

as follows

$$\begin{aligned}\tilde{l}(t)=l_{10}+\int_0^t G_1(R_1,L_1,r_1,l_1,z)dz,\\ \tilde{R}(t)=d^\Lambda(t,\tilde{l}(t)),\\ \tilde{r}(t)=r_{10}+\int_0^t G_2(R_1,L_1,r_1,l_1,t)dz,\\ \tilde{L}(t)=d^R(t,\tilde{r}(t)).\end{aligned}$$

Then for an appropriate choice of δ and T, we establish that $\mathcal{Z}:Y_{T,\delta}\to Y_{T,\delta}$. After introducing the following shorthand

$$A(z)=L_2(R(z),L(z),r(z),l(z))-l(z),$$

we arrive at

$$
\begin{aligned}
&\left|\tilde{l}(t) - l_{10}\right| \\
&= \left| \int_0^t \frac{1}{d_x^\Lambda(z, l(z)) A(z)} \left(\int_{l(z)}^{L_2(R(z),L(z),r(z),l(z))} \sigma(z, R(z), s)\, ds - 2\gamma(\mathbf{n}_R) \right) dz \right| \\
&\quad + \left| \int_0^t \frac{1}{d_x^\Lambda(z, l(z))} m(d_x^\Lambda(z, l_1)) \sigma(t, l_1, d^\Lambda(z, l_1)) dz \right| \le \\
&\frac{T}{\epsilon} \left(\frac{\sup |L_2(R(z), L(z), r(z), l(z)) - l(s)| \sup |\sigma| + 2\gamma(\mathbf{n}_R)}{\inf |L_2(R(z), L(z), r(z), l(z)) - l(s)|} + \sup|\sigma| \sup |m(d^\Lambda)| \right) \\
&\le \frac{T}{\epsilon} \left(\frac{2}{L_{20} - l_{10}} \left(\frac{3}{2} |L_{20} - l_{10}| \sup |\sigma| + 2\gamma(\mathbf{n}_R) \right) + \sup |\sigma| \sup |m(d^\Lambda)| \right).
\end{aligned}
$$

In the same manner one can show the inequality

$$
|\tilde{r}(t) - r_{10}| \le \frac{T}{\epsilon} \left(\frac{2}{R_{20} - r_{10}} \left(\frac{3}{2} |R_{20} - r_{10}| \sup |\sigma| + 2\gamma(\mathbf{n}_\Lambda) \right) + \sup |\sigma| \sup |m(d^R)| \right).
$$

Hence, after having taken T small enough, we obtain

$$
\max \left(\|\tilde{r} - r_{10}\|_{C([0,T])}, \|\tilde{l} - l_{10}\|_{C([0,T])} \right) \le \delta.
$$

Next,

$$
|\tilde{R}(t) - R_{10}| = |d^\Lambda(t, \tilde{l}(t)) - R_{10}| \le \sup_{t \in [0,T], |x - l_{10}| \le \delta} |d^\Lambda(t, x) - l_{10}|.
$$

Since, the map $(t, x) \to d^\Lambda(t, x) - R_{10}$ is continuous, we can choose T and δ such that

$$
\|\tilde{R} - R_{10}\|_{C([0,T])} \le \tilde{\delta}.
$$

Using a similar argument, we can show that

$$
\|\tilde{L} - L_{10}\|_{C([0,T])} \le \tilde{\delta}.
$$

Now, we shall show that the map $\mathcal{Z}$ is compact. For this purpose, we take a sequence $(R_n, L_n, r_n, l_n) \in Y_{T,\delta}$. Thus, the sequences $\tilde{r}_n$ and $\tilde{l}_n$ are bounded in C^1-topology (see [16] for similar considerations). Thanks to Arzela-Ascoli Theorem, we extract subsequences $\tilde{r}_{n_k}$ of $\tilde{r}_n$ and $\tilde{l}_{n_k}$ of $\tilde{l}_n$ converging uniformly to $\tilde{r}$ and $\tilde{l}$ respectively. Hence, we get $\tilde{R}_{n_k} \to \tilde{R}$ and $\tilde{L}_{n_k} \to \tilde{L}$ in C. This finishes the proof of compactness. In the same way one can show that the map $\mathcal{Z}$ is continuous. Finally, by Schauder Theorem, we conclude that the map $\mathcal{Z}$ has a fixed point.

Now, we assume that L_2 and R_2 are locally Lipschitz continuous. We shall show that the solution is unique. For this purpose, we shall assume that we have two solutions (R_a, L_a, r_a, l_a) and (R_b, L_b, r_b, l_b). Since

$$
\min \left(d_y^\Lambda(t, y),\ d_y^R(t, y) \right) \ge \eta > 0,
$$

we can write:

$$
|R_a(t) - R_b(t)| = \left| d^\Lambda(t, l_a(t)) - d^\Lambda(t, l_b(t)) \right| \ge \left| \int_{l_b(t)}^{l_a(t)} \eta dy \right| = \eta \left| l_a(t) - l_b(t) \right|.
$$

Hence

$$\|l_a - l_b\|_{C([0,T])} \leq \frac{1}{\eta}\|R_a - R_b\|_{C([0,T])},$$
$$\|r_a - r_b\|_{C([0,T])} \leq \frac{1}{\eta}\|L_a - L_b\|_{C([0,T])}.$$

Subsequently, we introduce the notation

$$L_{2a}(z) = L_2(R_a(z), L_a(z), r_a(z), l_a(z)),$$
$$R_{2a}(z) = R_2(R_a(z), L_a(z), r_a(z), l_a(z)).$$

Let $T_a = \min\left(\sup\{t \in (0,T) | L_{2a}(t) > l_a(t)\}, \sup\{t \in (0,T) | R_{2a}(t) > r_a(t)\}\right)$. In the same fashion, we define T_b. Let us denote $\tilde{T} = \min(T_a, T_b)$. Then, for each $\delta > 0$, there exists $\epsilon(\delta)$ such that $\min\left(L_{2a}(t) - l_a(t), L_{2b}(t) - l_b(t), R_{2a}(t) - r_a(t), R_{2b}(t) - r_b(t)\right) > \epsilon(\delta)$ on the set $[0, \tilde{T} - \delta]$. Now, we estimate

$$\begin{aligned}
& |R_a(t) - R_b(t)| \\
\leq\ & \left|\int_0^t \left(\frac{1}{L_{2a}(z) - l_a(z)} - \frac{1}{L_{2b}(z) - l_b(z)}\right)\left(-2\gamma + \int_{l_a(z)}^{L_{2a}(z)} \sigma(z, R_a(z), s)\, ds\right) dz\right| \\
& + \left|\int_0^t \frac{1}{L_{2b}(z) - l_b(z)} \int_{l_a(z)}^{L_{2a}(z)} \left(\sigma(z, R_a(z), s) - \sigma(z, R_b(z), s)\right) ds\, dz\right| \\
& + \left|\int_0^t \left(\frac{1}{L_{2b}(z) - l_b(z)}\left(\int_{l_a(z)}^{L_{2a}(z)} \sigma(z, R_b(z), s) ds - \int_{l_b(z)}^{L_{2b}(z)} \sigma(z, R_b(z), s) ds\right)\right) dz\right| \\
\leq\ & \frac{C_1}{\epsilon(\delta)^2}\int_0^t \left(\|l_a - l_b\|_{C([0,z])} + \|L_{2a} - L_{2b}\|_{C([0,z])}\right) dz + \\
& \frac{C_2}{\epsilon(\delta)}\int_0^t \|R_a - R_b\|_{C([0,z])} dz + \frac{C_3}{\epsilon(\delta)}\int_0^t \left(\|l_a - l_b\|_{C([0,z])} + \|L_{2a} - L_{2b}\|_{C([0,z])}\right) dz,
\end{aligned}$$

where

$$C_1 = 2\gamma + \sup|L_{2a} - l_a| \sup|\sigma|, \quad C_2 = \sup|L_{2a} - l_a| \sup|\sigma_y|, \quad C_3 = \sup|\sigma|.$$

In the same manner we obtain the estimate for $|L_a(t) - L_b(t)|$. Since L_2 and R_2 are locally Lipschitz continuous we obtain the estimate:

$$\begin{aligned}
& \|R_a - R_b\|_{C([0,t])} + \|L_a - L_b\|_{C([0,t])} \leq \\
H(\epsilon(\delta)) & \int_0^t \left(\|L_a - L_b\|_{C([0,z])} + \|R_a - R_b\|_{C([0,z])}\right) dz.
\end{aligned}$$

Hence, by Gronwall inequality we obtain that $\|R_a - R_b\|_{C([0,t])} + \|L_a - L_b\|_{C([0,t])} = 0$ on $[0, \tilde{T} - \delta]$. Since, δ is arbitrary, we obtain that $\|R_a - R_b\|_{C([0,\tilde{T}))} + \|L_a - L_b\|_{C([0,\tilde{T}))} = 0$. This finishes the proof of the uniqueness. □

We notice that this construction yields a unique solution if $(d^{\Lambda}_{0,x_2})^-(l_{10}) > 0$ and $(d^R_{0,x_1})^-(r_{10}) > 0$.

The point of using the Schauder theorem is to give existence for a given time interval, without necessity of shrinking it as it is in the case of Banach contraction principle.

3.2.4 The case of a matching and a tangency curve

Proof of Theorem 3.3, part (b). We will proceed in several steps. Since the role of l_1 and r_1 is interchangeable in Theorem 3.3, part (c), for the sake of definiteness, we assume that a tangency curve emanates from r_{10} and matching curve from l_{10}.

First, we notice that Proposition 2.4 yields existence of d^Λ in $G(x(l_{00}), x(l_{10}))$. Once we have it, we will simultaneously construct the tangency curve emanating from r_{10}, the matching curve and $R_1(\cdot)$, $L_1(\cdot)$. After that, we will find remaining d^R.

Now, we can present the main result of this paragraph.

Lemma 3.2 *Let $L_2 \in C(\mathbb{R}^4;\mathbb{R})$ and $R_2 \in C^1(\mathbb{R}^4;\mathbb{R})$ be given functions. We assume that $\Sigma_1^R > 0$ initial conditions satisfy the tangency condition:*

$$\sigma(0, r_{10}, L_{10}) = \frac{1}{R_{20} - r_{10}} \left(\int_{r_{10}}^{R_{20}} \sigma(0, s, L_{10}) ds - 2\gamma(\mathbf{n}_\Lambda) \right)$$

and

$$(r_{10} - R_{20})\sigma_{x_1}(0, r_{10}, L_{10}) \neq (\sigma(0, r_{10}, L_{10}) - \sigma(0, R_{20}, L_{10})) \left. \frac{\partial R_2}{\partial y_3} \right|_{y_3 = r_{00}}. \quad (3.14)$$

If $(d^\Lambda_{x_2})^-(l_{10}) > 0$ and condition (3.5) hold, i.e.,

$$\frac{1}{d^\Lambda_{0,x}(l_{10})} \left(\sigma(t, d^\Lambda_0(l_{10}), x) m(d^\Lambda_{0,x}(l_{10})) - \frac{m(0)}{L_1 - l_1} \int_{l_1}^{L_1} \sigma(t, s, d^\Lambda_0(l_{10}))\, ds - \frac{2\gamma(\mathbf{n}_R)}{L_1 - l_1} \right)$$
$$< -m_p(d^\Lambda_{0,x}(l_{10}))\sigma(t, d^\Lambda_0(l_{10}), x),$$

then, there exists a local in time solution to the problem

$$\dot{R}_1 = \frac{1}{L_2(R_1, L_1, r_1, l_1) - l_1} \int_{l_1}^{L_2(R_1, L_1, r_1, l_1)} \sigma(t, R_1(t), s)\, ds - \frac{2\gamma(\mathbf{n}_R)}{L_2(R_1, L_1, r_1, l_1) - l_1},$$
$$R_1 = d^\Lambda(t, l_1),$$
$$\dot{L}_1 = \frac{1}{R_2(R_1, L_1, r_1, l_1) - r_1} \int_{r_1}^{R_2(R_1, L_1, r_1, l_1)} \sigma(t, L_1(t), s)\, ds - \frac{2\gamma(\mathbf{n}_\Lambda)}{R_2(R_1, L_1, r_1, l_1) - r_1},$$
$$\sigma(t, r_1(t), L_1(t)) = \frac{1}{R_2(R_1, L_1, r_1, l_1) - r_1} \left(\int_{r_1}^{R_2(R_1, L_1, r_1, l_1)} \sigma(t, s, L_1(t))\, ds - 2\gamma(\mathbf{n}_\Lambda) \right),$$
$$R_1(0) = R_{10}, \quad L_1(0) = L_{10}, \quad r_1(0) = r_{10}, \quad l_1(0) = l_{10}.$$

Moreover, if L_2 is locally Lipschitz continuous, then the solution is unique.

Proof. Let us define the following mapping $\mathcal{F} : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$,

$$\mathcal{F}(t, R_1, L_1, r_1, l_1) =$$
$$(R_2(R_1, L_1, r_1, l_1) - r_1)\, \sigma(t, r_1, L_1) - \int_{r_1}^{R_2(R_1, L_1, r_1, l_1)} \sigma(t, s, L_1)\, ds - 2\gamma(\mathbf{n}_\Lambda).$$

Since by(3.14)

$$\frac{\partial \mathcal{F}}{\partial r_1}(0, R_{10}, L_{10}, r_{10}, l_{10}) =$$
$$(\sigma(0, r_{10}, L_{10}) - \sigma(0, R_{20}, L_{10})) \left. \frac{\partial R_2}{\partial y_3} \right|_{y_3 = r_{10}} + (R_{20} - r_{10})\sigma_{x_2}(0, r_{10}, L_{10}) \neq 0,$$

we can apply the Implicit Function Theorem. Therefore, there exists a neighborhood of $(0, R_{10}, L_{10}, l_{10})$ and the map $r_1 = r_1(t, R_1, L_1, l_1)$ such that

$$\mathcal{F}(t, R_1, r_1, L_1, l_1) = 0 \quad \text{and} \quad r_1(0, R_{10}, L_{10}, l_{10}) = r_{10}.$$

Now, we can rewrite our system as follows:

$$\begin{aligned}
\dot{R}_1 &= \frac{1}{L_2(R_1, L_1, r_1, l_1) - l_1}\left(\int_{l_1}^{L_2(R_1,L_1,r_1,l_1)} \sigma(t, R_1(t), s)\, ds - 2\gamma(\mathbf{n}_R)\right), \\
R_1 &= d^{\Lambda}(t, l_1), \\
\dot{L}_1 &= \sigma(t, r_1, L_1(t)), \\
R_1(0) &= R_{10}, \quad L_1(0) = L_{10}, \quad l_1(0) = l_{10}.
\end{aligned} \tag{3.15}$$

Next, we differentiate equation (3.15$_3$) with respect to t and we obtain the following system:

$$\dot{l}_1 = G_3, \quad R_1 = d^{\Lambda}(t, l_1), \quad \dot{L}_1 = \sigma(t, r_1, L_1(t)),$$
$$R_1(0) = R_{10}, \quad L_1(0) = L_{10}, \quad l_1(0) = l_{10},$$

where

$$G_3(R_1, L_1, r_1, l_1, t) =$$
$$\frac{1}{d_x^{\Lambda,-}(t, l_1)} \frac{\int_{l_1}^{L_2(R_1,L_1,r_1,l_1)} \sigma(t, R_1(t), s)\, ds - 2\gamma(\mathbf{n}_R)}{L_2(R_1, L_1, r_1, l_1) - l_1} - \frac{m(d_x^{\Lambda,-}(t, l_1))\sigma(t, l_1, d^{\Lambda}(t, l_1))}{d_x^{\Lambda,-}(t, l_1)}.$$

Subsequently, for $T > 0$, we consider the Banach space $C([0, T]; \mathbb{R}^3)$ with the norm

$$\|(u_1, u_2, u_3)\|_{C([0,T];\mathbb{R}^3)} = \max_{i=1,2,3} \|u_i\|_{C([0,T])}.$$

Since $L_2 : \mathbb{R}^4 \to \mathbb{R}$ is continuous, there is δ_1 such that if

$$\max\left(|R - R_{10}|, |L - L_{10}|, |l - l_{10}|, t\right) \leq \delta_1,$$

then

$$|L_2(R, L, r_1(t, R, L, l), l) - L_{20}| \leq \frac{1}{4} \min\left(L_{20} - l_{10}\right).$$

Continuity of G_3 implies existence T_2 and δ_2 such that $G_3 > \dot{x}_a$ in the cylinder

$$\overline{B}\left((R_{10}, L_{10}, l_{10}), \delta_2\right) \times [0, T_2].$$

Moreover, by Proposition 2.4 there exist T_3, δ_3, ϵ such that if $t \in [0, T_3]$, $|l_1 - l_{10}| \leq \delta_3$, then $d_x^{\Lambda}(t, l_1) > \epsilon$.

Next, for $T < \min(T_2, \delta_1)$, and $\delta \leq \min\left(\frac{L_{20} - l_{10}}{4}, \delta_1, \delta_2, \tilde{\delta}\right)$ we define the set

$$\begin{aligned}
\tilde{Y}_{T,\delta} &= B_{C[0,T]}(R_{10}, \delta) \times B_{C[0,T]}(L_{10}, \delta) \times B_{C[0,T]}(l_{10}, \tilde{\delta}), \\
Y_{T,\delta} &= \tilde{Y}_{T,\delta} \cap \{(R, L, l) \in C([0, T]; \mathbb{R}^3) : (R, L, l)(0) = (R_{10}, L_{10}, l_{10},)\}.
\end{aligned}$$

Subsequently, we define the mapping $\mathcal{Z} : Y_{T,\delta} \to C([0, T]; \mathbb{R}^3)$,

$$\mathcal{Z}(R, L, l) = (\tilde{R}, \tilde{L}, \tilde{l}),$$

as follows

$$
\begin{aligned}
\tilde{l}(t) &= l_{10} + \int_0^t G_3(R_1, L_1, r_1, l_1, z)dz, \\
\tilde{L}(t) &= L_{10} + \int_0^t \sigma(z, r_1, L)dz, \\
\tilde{R}(t) &= d^{\Lambda}(t, \tilde{l}(t)),
\end{aligned}
$$

where $r_1 = r_1(t, R, L, l)$.

In the similar way as in the proof performed in subsection 3.2.2, one can show that for δ and T small enough, we have $\mathcal{Z} : Y_{T,\delta} \to Y_{T,\delta}$. Moreover, map $\mathcal{Z}$ is compact and continuous. As a result, we apply Schauder fixed point theorem. Finally, if L_2 is locally Lipschitz continuous, then the Gronwall lemma yields uniqueness of a solution. $\square$

3.3 Existence of variational solutions to (1.1)

We combine the information about the solutions.

Theorem 3.5 *Let us suppose that all the assumptions of Theorem 3.4 hold, in particular $l_{00} < l_{10}$ and $r_{00} < r_{10}$. Then, there exist corresponding variational solutions. In addition, they are unique provided that all the interfacial curves are tangency curves.*

Proof. In the previous section, we constructed the bent rectangles $\{\Gamma(t)\}_{t\in[0,T_1)}$. Now, we construct the corresponding Cahn-Hoffman vector fields ξ on $S^{\pm}_{\Lambda}$, $S^{\pm}_{R}$. Actually, on S^{+}_{Λ} we use formula [22, eq. (2.28)] and [23, eq. 3.2] and [23, Proposition 3.2], $\xi = (\xi_1, \gamma(\mathbf{n}_R))$, where

$$
\xi_1(t,x) = \begin{cases}
x\left(\fint_0^x \sigma(t,s,L_0)\,ds - \fint_0^{r_0} \sigma(t,s,L_0)\,ds\right) - \dfrac{x}{r_0}\,\gamma(\mathbf{n}_\Lambda) & x \in [0, r_0] \\
-\gamma(\mathbf{n}_\Lambda) & x \in (r_0, r_1) \\
(R_1 - x)\left(\fint_{r_1}^{R_1} \sigma(t,s,L_1)\,ds - \fint_x^{R_1} \sigma(t,s,L_1)\,ds\right) + \gamma(\mathbf{n}_\Lambda) & \\
\qquad -2\dfrac{R_1 - x}{R_1 - r_1}\gamma(\mathbf{n}_\Lambda) & x \in [r_1, R_1].
\end{cases}
\tag{3.16}
$$

We extend it by symmetry to the whole S^{+}_{Λ},

$$
\xi_1(t,x) = -\xi_1(t,-x) \quad x \in [-R_1, 0]
$$

and to S^{-}_{Λ}. Namely, on S^{-}_{Λ} we set $\xi(t,x) = (\xi_1(t,x), -\gamma(\mathbf{n}_R))$.

In an analogous manner we have $\xi = (\gamma(\mathbf{n}_\Lambda), \xi_2)$ on S^{+}_{R}. The above formula may be reused for ξ_2, if we keep in mind that we have to replace the L's by the R's (and vice versa) and to change $\gamma(\mathbf{n}_\Lambda)$ to $\gamma(\mathbf{n}_R)$.

The proof that this ξ is a minimizer of $\mathcal{E}$ follows the lines of [23, Theorem 3.5 and Theorem 3.6], since some changes are required we will present its sketch.

We notice that it is sufficient to restrict our attention to S^{+}_{Λ}. We have to show that if $\xi + h \in \mathcal{D}_\Lambda$, then

$$
\mathcal{E}_\Lambda(\xi + h) \geq \mathcal{E}_\Lambda(\xi).
$$

It is easy to see that $\xi + h \in \mathcal{D}_\Lambda$ implies continuity of h and $h(\pm R_1) = 0$. Since $\pm r_0$, $\pm r_1$ form the boundary of the coincidence set of the variational obstacle problem, then $h(\pm r_i) = 0$, $i = 0, 1$, too.

We notice that for all $t > 0$ Proposition 2.4 (b) implies that

$$d_x(t, x) > 0 \quad \text{for all } r_0(t) < |x| < r_1(t).$$

Hence, $\partial\gamma(\mathbf{n}(x))$ is a singleton and then $h \equiv 0$ on $[-r_1, -r_0] \cup [r_0, r_1]$.

Once we checked that h_1 is continuous and vanishes on $[-r_1(t), -r_0(t)] \cup [r_0(t), r_1(t)]$ it is easy to deduce, by the method of [23, Theorem 3.5 and Theorem 3.6] that

$$\begin{aligned}
&\mathcal{E}_R(\xi + h) - \mathcal{E}_R(\xi) = \\
&\quad \int_{-r_0}^{r_0} [\frac{1}{2}(\frac{\partial h_1}{\partial x_1})^2 - \beta_R \dot{L}_0 \frac{\partial h_1}{\partial x_1}]\, dx + \left(\int_{-R_1}^{-r_1} + \int_{r_1}^{R_1}\right) [\frac{1}{2}(\frac{\partial h_1}{\partial x_1})^2 - \beta_R \dot{L}_1 \frac{\partial h_1}{\partial x_1}]\, dx \\
&\geq 0.
\end{aligned}$$

The method used in the above mentioned theorems yields r_1 depending in a Lipschitz continuous manner on R_1 depending precisely on the endpoints of the outer facets. The same conclusion holds, as in Lemma 3.2 above, when the position of one of the endpoints is not determined by the corner, but differently. $\square$

3.4 Evolution of rectangles

We know (see [23, Section 3.2]) that not all configurations are possible if $d_0 \equiv L_{00}$. In fact, the only viable option is that the tangency condition is satisfied at $r_{00} = r_{10}$. In this case, we have two tangency curves emanating from r_{00}. The problem is to construct the solution to the Hamilton-Jacobi equations in the region above them,.

Theorem 3.6 *Let us suppose that (S) holds. The initial data are such that $r_{00} = r_{10}$ and (2.30) are satisfied. In addition, one of the hypotheses of Theorem 3.4 yielding unique l_0, l_1 holds. Then, there is $T > 0$ such a variational solution to (1.1) exists for $t \in [0, T)$. Moreover, the interfacial curves $r_i(\cdot)$, $i = 0, 1$, are the tangency curves and $r_0(t) < r_1(t)$ for $t > 0$.*

Proof. We assume, that we have already decided what is the nature of the interface l_1 and we have determined its evolution as well as that of R_1. By Remark 3.1 Lemma 3.1 keeps holding even if $r_{00} = r_{10}$, then we may use the results of Lemma 3.1 to conclude existence of r_0 and r_1. As a result, theory developed for bent rectangles is applicable. In particular we deduce that $\dot{r}_0 < 0$ and $\dot{r}_1 > 0$ for $t > 0$. This permits us to construct the corresponding Cahn-Hoffman vector ξ uniquely, see Theorem 3.5. $\square$

Of course we know, that the only possibility is that two tangency curves begin at $r_{00} = r_{10}$. The conditions (2.30) are to make sure the data are correct and we can set up boundary data for the Hamilton-Jacobi equation. We notice that the above Theorem is valid also in $l_{00} = l_{10}$, provided that the version of (2.30) holds for them.

3.5 Uniqueness of variational solutions

We will show uniqueness using the approach of the proof of [23, Theorem 4.1]. We will do this only for special cases, when we can show that d has bounded second derivative, otherwise the method breaks down. We note that we guarantee higher smoothness away from the interfacial curves. By the very nature of the problem $d_x(t,\cdot)$ is discontinuous across all matching curves, which are not tangency curves. This problem was absent in [23], because there we could additionally use the gradient flow structure of our equations. This is no longer true here.

Theorem 3.7 *Let us assume (S) and consider a regular bent rectangle Γ_0 as an initial condition. If $(\Gamma^i(t),\xi^i(t))$, $i=1,2$, are two regular variational solutions to (1.1) with the initial condition Γ_0, i.e., all interfacial curves are tangency curves, then $(\Gamma^1(t),\xi^1(t)) = (\Gamma^2(t),\xi^2(t))$ for all $t\geq 0$.*

Proof. We will consider both solution in a common local coordinate system. Thus we have two systems like (2.17) for $d^{j,1}$, $d^{j,2}$, $j=\Lambda, R$. One of the difficulties is that $d^{\Lambda,1}$ and $d^{\Lambda,2}$ (resp. $d^{R,1}$ and $d^{R,2}$) are defined on different domains. We will consider in detail $d^{R,1}$ and $d^{R,2}$. The analysis of $d^{\Lambda,1}$ and $d^{\Lambda,2}$ goes along the same lines.

Let us set

$$\rho^R(t) = \min\{R^1_1(t), R^2_1(t)\} \qquad \rho^\Lambda(t) = \min\{L^1_1(t), L^2_1(t)\}.$$

We take the difference of systems ($2.17_{x,x}$) for $d^{R,1}$ and $d^{R,2}$ over $[-\rho^R(t), \rho^R(t)]$.

We note, that contrary to [23], we consider the difference of solutions in a time varying domain.

Thus, we obtained the following equation for $p^R := d^2 - d^1$ on $[-\rho,\rho]$,

$$p^R_t = Ap^R_x + Bp^R - \left(m^2\frac{\partial\xi^2}{\partial x} - m^1\frac{\partial\xi^1}{\partial x}\right), \tag{3.17}$$

where

$$A(t,x) = \sigma^2\frac{m(d^{R,2}_x) - m(d^{R,1}_x)}{d^{R,2}_x - d^{R,1}_x}, \quad B = m(d^{R,1}_x)\frac{\sigma^2-\sigma^1}{d^{R,2}-d^{R,1}}. \tag{3.18}$$

We notice that

$$A, B \in L^\infty(0,T; L^\infty(-\rho,\rho)).$$

Multiplication of (3.17) by p an integration over $[-\rho,\rho]$ yields

$$\frac{1}{2}\int_{-\rho}^{\rho}(p^R)^2_t = \frac{1}{2}\int_{-\rho}^{\rho}(A(p^R)^2_x + B(p^R)^2 - \int_{-\rho}^{\rho}\left(m^2\frac{\partial\xi^2}{\partial x} - m^1\frac{\partial\xi^1}{\partial x}\right)p, \tag{3.19}$$

We will estimate the second term using monotonicity of $\partial\gamma$,

$$\int_{-\rho}^{\rho} -\left(m^2\frac{\partial\xi^2}{\partial x} - m^1\frac{\partial\xi^1}{\partial x}\right)p \leq -\int_{-\rho^R}^{\rho^R} m^2\left(\frac{\partial\xi^2}{\partial x} - \frac{\partial\xi^1}{\partial x}\right)p - \int_{-\rho}^{\rho}(m^2-m^1)\frac{\partial\xi^1}{\partial x}.$$

Monotonicity of $\partial\gamma$ implies the pointwise estimate

$$(\frac{\partial\xi^2}{\partial x} - \frac{\partial\xi^1}{\partial x})\cdot(d^2_x - d^1_x) \geq 0,$$

thus

$$-\int_{-\rho^R}^{\rho^R} m^2\left(\frac{\partial\xi^2}{\partial x} - \frac{\partial\xi^1}{\partial x}\right)p \leq 0.$$

On the other hand, Lipschitz continuity of m yields

$$\int_{-\rho}^{\rho} -\left(m^2\frac{\partial \xi^2}{\partial x} - m^1\frac{\partial \xi^1}{\partial x}\right) p \le M\int_{-\rho}^{\rho} p^2.$$

Let us write A as follows,

$$A(t,x) = \int_0^1 m_p'((d_x^{R,2} - d_x^{R,1})s + d_x^{R,1})\, ds.$$

Our assumption that all the interfaces are tangency curves implies that d_x^R, d_x^Λ are continuous and differentiable. Regularity of $\Gamma(t)$ yields boundedness of d_{xx}^R and d_{xx}^Λ then we immediately see that $A_x \in L^\infty(-\rho,\rho)$. Thus, integration by part yields

$$\int_{-\rho}^{\rho} A(p^R)_x^2 = -\int_{-\rho}^{\rho} A_x(p^R)^2 + A(p^R)^2|_{x=-\rho^R}^{x=\rho^R}.$$

We notice, that the boundary term is easy to estimate,

$$(p^R)^2(\rho^R) = (L_1^2 - L_1^1)^2 = \frac{\min\{R_1^2, R_1^1\} - \max\{r_1^2, r_1^1\}}{\min\{R_1^2, R_1^1\} - \max\{r_1^2, r_1^1\}}(L_1^2 - L_1^1)^2 \le M\int_{-\rho}^{\rho} A(p^R)^2$$

Finally,

$$\int_{-\rho^R}^{\rho^R} (p^R)_t^2 \le \int_{-\rho^R}^{\rho^R} K(p^R)^2 + \int_{-\rho^\Lambda}^{\rho^\Lambda} K(p^\Lambda)^2.$$

So far, we have succeeded in showing the following inequality

$$\begin{aligned}
&\int_{-\rho^R(t)}^{\rho^R(t)} \frac{d}{dt}(d^{R,2} - d^{R,1})^2 + \int_{-\rho^\Lambda(t)}^{\rho^\Lambda(t)} \frac{d}{dt}(d^{\Lambda,2} - d^{\Lambda,1})^2 \\
\le\ & M\int_{-\rho^R(t)}^{\rho^R(t)} (d^{R,2} - d^{R,1})^2 + M\int_{-\rho^\Lambda(t)}^{\rho^\Lambda(t)} (d^{\Lambda,2} - d^{\Lambda,1})^2.
\end{aligned}$$

We notice that

$$\begin{aligned}
\frac{d}{dt}\int_{-\rho^R(t)}^{\rho^R(t)} (d^{R,2} - d^{R,1})^2 &= \int_{-\rho^R(t)}^{\rho^R(t)} \frac{d}{dt}(d^{R,2} - d^{R,1})^2 \\
&\quad + (d^{R,2} - d^{R,1})^2(\rho(t))\frac{d}{dt}\rho^R(t) + (d^{R,2} - d^{R,1})^2(-\rho^R(t))\frac{d}{dt}\rho^R(t) \\
&= \int_{-\rho^R(t)}^{\rho^R(t)} \frac{d}{dt}(d^{R,2} - d^{R,1})^2 \\
&\quad + (L_1^2 - L_1^1)(-\rho^R(t))\frac{d}{dt}\rho^R(t) + (L_1^2 - L_1^1)(\rho^R(t))\frac{d}{dt}\rho^R(t) \\
&\le \int_{-\rho^R(t)}^{\rho^R(t)} \frac{d}{dt}(d^{R,2} - d^{R,1})^2 \\
&\quad + M\int_{-\rho^R(t)}^{\rho^R(t)} (d^{R,2} - d^{R,1})^2 + M\int_{-\rho^\Lambda(t)}^{\rho^\Lambda(t)} (d^{\Lambda,2} - d^{\Lambda,1})^2.
\end{aligned}$$

The same inequality is valid for $\frac{d}{dt}\int_{-\rho^R(t)}^{\rho^R(t)}(d^{\Lambda,2}-d^{\Lambda,1})^2$. Hence, after summing them up and introducing,

$$u(t)=\int_{-\rho^R(t)}^{\rho^R(t)}(d^{R,2}-d^{R,1})^2+M\int_{-\rho^\Lambda(t)}^{\rho^\Lambda(t)}(d^{\Lambda,2}-d^{\Lambda,1})^2,$$

we will obtain the following estimate

$$\frac{d}{dt}u\le Ku.$$

This inequality takes into account the fact that $\frac{d}{dt}\rho^R(t)\in L^\infty(0,T)$.

Since $u(0)=0$, then Gronwall inequality implies that $u(t)\equiv 0$, i.e. $d^{R,2}(t)=d^{R,1}(t)$ on interval $[-\rho^R(t),\rho^R(t)]$ and $d^{\Lambda,2}-d^{\Lambda,1}$ on interval $[-\rho^\Lambda(t),\rho^\Lambda(t)]$. $\square$

Remarks.
We see that this argument can be adapted to solutions which possibly break the symmetry.

The problem of uniqueness of solutions to a system like

$$u_t-b\nabla u+cu=0\qquad\text{in }(0,T)\times\mathbb{R}^n$$

has been studied extensively, let us just mention [9] or [1]. Commonly, the authors need there div$b\in L^1(0,T;L^p_{loc}(\mathbb{R}^n))$, where $p\le\infty$. We could hope for this only, when the interfaces are tangency curves. If at least one of the interfacial curves is just a matching curve, then d_x is discontinuous at the endpoint of the facet set of arguments. Hence, if the coefficient A in (3.18) will have jumps, so A_x will be a measure. As a result, the assumptions of [9] and [1] are violated.

In fact, the problem we study (3.17) is not exactly in the form indicated above, thus we use here a simpler, direct method.

Acknowledgment. The research of Y.G. has been partly supported by Japan Society for the Promotion of Sciences (JSPS) through grant for scientific research 21224001. The research of P.G. and P.R. has been partly supported by MN grant No. N N201 268935. Moreover, Y.G. and P.R. thank Hokkaido University for hospitality, because part of work was performed there. P.G. and P.R. also thank Instituto de Matemática y Física, Universidad de Talca for support. Part of the research was performed there.

References

[1] Luigi Ambrosio, Transport equation and Cauchy problem for BV vector fields, *Invent. math.*, **158**, (2004), 227–260.

[2] F.Andreu-Vaillo, V.Caselles, J.Mazon, “Parabolic quasilinear equations minimizing linear growth functionals”, Progress in Mathematics, 223. Birkhäuser Verlag, Basel, 2004.

[3] G. Bellettini, V. Caselles, A. Chambolle and M. Novaga, Crystalline mean curvature flow of convex sets, *Arch. Ration. Mech. Anal.*, **179** (2006), 109-152.

[4] G.Bellettini, V.Caselles, A.Chambolle and M.Novaga, The volume preserving crystalline mean curvature flow of convex sets in $\mathbb{R}^N$, *J. Math. Pures Appl.*, **92**, (2009), 499–527.

[5] G.Bellettini, M.Novaga, M.Paolini, Characterization of facet breaking for nonsmooth mean curvature flow in the convex case, *Interfaces and Free Boundaries*, **3**, (2001), 415–446.

[6] G.Bellettini, M.Novaga, M.Paolini, On a crystalline variational problem, part I: First variation and global L^∞ regularity, *Arch. Rational Mech. Anal.*, **157**, (2001), 165–191.

[7] G.Bellettini, M.Novaga, M.Paolini, On a crystalline variational problem, part II: BV regularity and structure of minimizers on facets, *Arch. Rational Mech. Anal.*, **157**, (2001), 193–217.

[8] W.F.Berg, Crystal growth from solutions, *Proc. Roy. Soc. London A*, **164**, (1938), 79–95.

[9] R.J. DiPerna P.L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, *Invent. math.*, **98**, (1989), 511–547.

[10] M. Elbaum, J.S. Wettlaufer, Relation of growth and equilibrium crystal shapes, *Phys. Rev. E* , **48**, (1993), 3180–3183.

[11] T. Fukui and Y. Giga, Motion of a graph by nonsmooth weighted curvature, *in:* Proc. First World Congress of Nonlinear Analysis (ed. V. Lakshmikantham), vol.I 47 - 56, Walter de Gruyter, Berlin, 1996.

[12] M.-H. Giga Y. Giga, Evolving graphs by singular weighted curvature, *Arch. Rational Mech. Anal.*, **141**, (1998), 117-198.

[13] M.-H. Giga, Y. Giga, Crystalline and level set flow - convergence of a crystalline algorithm for a general anisotropic curvature flow in the plane, *in:* Proc. FBP '99 (ed. N. Kenmochi), Gakuto Internat. Ser. Math. Sci. Appl. 13. 64–79, Gakuto, Tokyo, 2000.

[14] M.-H. Giga, Y. Giga, Generalized motion by nonlocal curvature in the plane, *Arch. Rational Mech. Anal.*, **159** (2001), 295-333.

[15] M.H. Giga, Y. Giga, P.Rybka, A comparison principle for singular diffusion equations with spatially inhomogeneous driving force, *preprint*.

[16] Y.Giga, P.Górka, P.Rybka, Nonlocal spatially inhomogeneous Hamilton-Jacobi equation with unusual free boundary, *Discrete Contin. Dynam. Systems A*, **26**, (2010), 493-519.

[17] Y.Giga, P.Górka, P.Rybka, A Comparison Principle for Hamilton-Jacobi equations with discontinuous Hamiltonians, *Proc. AMS.*, **139**, (2011), 1777–1785.

[18] Y.Giga, R.Kobayashi, On Anisotropy and Curvature Effects for Growing Crystals, *Jpn. J. Ind. Appl. Math.*, **18**, (2001), 207–230.

[19] Y.Giga, P.Rybka, Quasi-static evolution of 3-D crystals grown from supersaturated vapor, *Diff. Integral Eqs.*, **15**, (2002), 1–15.

[20] Y.Giga, P.Rybka, Berg's Effect, *Adv. Math. Sci. Appl.*, **13**, (2003), 625-637.

[21] Y.Giga, P.Rybka, Stability of facets of crystals growing from vapor, *Discrete Contin. Dyn. Syst.*., **14** (2006), no. 4, 689–706.

[22] Y.Giga, P.Rybka, Facet bending in the driven crystalline curvature flow in the plane, *J. Geom. Anal.*, **18**, (2008), 109–147.

[23] Y.Giga, P.Rybka, Facet bending driven by the planar crystalline curvature with a generic nonuniform forcing term, *J. Diff. Eqs.*, **246**, (2009), 2264–2303.

[24] M. Gurtin, “Thermomechanics of Evolving Phase Boundaries in the Plane”, Clarendon Press, Oxford, 1993.

[25] K.Kielak, P.B.Mucha, P.Rybka, Almost classical solutions to the total variation flow, *preprint*.

[26] D.Kinderlehrer, G.Stampacchia, “An Introduction to Variational Inequalities and their Applications”, Academic Press, New York, 1980.

[27] P.Mucha, P.Rybka, A caricature of a singular curvature flow in the plane, *Nonlinearity*, **21**, (2008), no. 10, 2281–2316.

[28] A.Seeger, Diffusion problems associated with the growth of crystals from dilute solution, *Philos. Mag.* ser. 7 **44**, no 348, (1953), 1–13.