Hadamard variation for Electromagnetic Frequencies

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§1. Introduction

In this paper we deal with the harmonic oscillation in the Maxwell equation in a bounded domain (under a certain boundary condition) and consider the smooth dependency of the eigenfrequency under the domain perturbation. The set of the eigenfrequencies of the oscillation depends on the geometric feature of the domain and it is one of the important quantities of the domain. The purose of this paper is to consider the regular deformation of the domain and to give a perturbation formula for each eigenfrequency.

Such kind of studies are called a problem of Hadamard variation or a domain variation problem and there have been a lot of studies for the case of the Laplace operator (or other elliptic operator) after Hadamard's pioneering study (cf. Hadamard [6]). (cf. Garabedian-Schiffer [4]). J. Hadamard studied the eigenvalue problem of Laplacian and Green function for the regular domain variation (under the Dirichlet B.C.) and obtained the perturbation formula of the eigenvalue and Green function (Bi-Lapalacian was also studied). As well as regular pertubation of domain, there a lot of studeis on the eigenvalues of elliptic operators under singular domain perturbation in many different situations (see the References or those of Jimbo [8], Maz'ya-Nazarov-Plamenevskij [11]). We study the perturbation of the eigenvalues of the elliptic operator which arises in the Maxwell equation for regular perturbation of the domain.

The electric-magnetic phenomena is modelled by the Maxwell equation (with an appropriate boundary condition) in the classical theory of Electromagnetism, which is a coupled system of the electric field E and the magnetic field H. Let Ω be a bounded domain in $\mathbb{R}^3$ and consider the Maxwell equation

$$\epsilon_0 \frac{\partial E}{\partial t} - \mathrm{rot}\, H = \mathbf{0}, \quad \mu_0 \frac{\partial H}{\partial t} + \mathrm{rot}\, E = \mathbf{0}, \quad \mathrm{div}\, E = 0, \quad \mathrm{div}\, H = 0. \tag{1.1}$$

with some boundary condition (cf. (1.2)). Here $\epsilon_0 > 0$ is the dielectric constant and $\mu_0 > 0$ is the magnetic permeability of the space where the electric-magnetic wave occurs (cf. Hirakawa [6,7]). We impose the boundary so that the space is surrounded by a perfect conductor. It give the following condition

$$E \times \nu = \mathbf{0}, \quad H \cdot \nu = 0 \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

Here ν is the outward unit normal vector on the boundary and the dot $\cdot$ is the standard inner product.

Time harmonic solutions are very fundamental in the sense that general solutions can expressed by superposition (infinite linear combination) of the time harmonic solutions which are written in the following form

$$E(t,x) = \exp(i\omega t)\widetilde{E}(x), \quad H(t,x) = \exp(i\omega t)\widetilde{H}(x) \tag{1.3}$$

where $\omega > 0$ is a parameter. Substitute this function into (1.1) and get

$$i\,\epsilon_0\,\omega\,\widetilde{E} - \mathrm{rot}\,\widetilde{H} = \mathbf{0}, \quad i\,\mu_0\,\omega\,\widetilde{H} + \mathrm{rot}\,\widetilde{E} = \mathbf{0}, \quad \mathrm{div}\,\widetilde{E} = 0, \quad \mathrm{div}\,\widetilde{H} = 0. \tag{1.4}$$

Applying the operator rot on the second equation and use the first equation, we get

$$-\mu_0\epsilon_0\omega^2\,\widetilde{E} + \mathrm{rot}\,\mathrm{rot}\,\widetilde{E} = \mathbf{0} \quad \text{in} \quad \Omega, \quad \widetilde{E} \times \nu = \mathbf{0} \quad \text{on} \quad \partial\Omega. \tag{1.5}$$

The eigenfrequency is the value ω, for which (1.5) allows a nontrivial solution $widetildeE$. By a scale transform of the space variable, we can assume $\mu_0\epsilon_0 = 1$ without loss of a mathematical generality. Replace the symbol of the variable of the vector field $\widetilde{E}$ by Φ and put $\lambda = \omega^2$. We get the following eigenvalue problem,

$$\mathrm{rot}\,\mathrm{rot}\,\Phi - \lambda\,\Phi = \mathbf{0}, \quad \mathrm{div}\,\Phi = 0 \quad \text{in} \quad \Omega, \quad \Phi \times \nu = \mathbf{0} \quad \text{on} \quad \partial\Omega. \tag{1.6}$$

Here the unknown function Φ is $\mathbb{R}^3-$valued in Ω.

It is easty to see that any eigenvalue is a nonnegative real number. Actually, take the inner product of (1.6) and Φ, and integrate in Ω, we have

$$\begin{aligned} 0 &= \int_\Omega \langle \mathrm{rot}\,\mathrm{rot}\Phi, \Phi\rangle dx - \lambda \int_\Omega \langle \Phi, \Phi\rangle\, dx \\ &= \int_{\partial\Omega} \langle \nu \times \mathrm{rot}\Phi, \Phi\rangle dS + \int_\Omega \langle \mathrm{rot}\Phi, \mathrm{rot}\Phi\rangle dx - \lambda \int_\Omega \langle \Phi, \Phi\rangle\, dx \end{aligned}$$

$$= \int_{\partial\Omega} \langle \mathrm{rot}\Phi, \Phi \times \nu \rangle dS + \int_{\Omega} |\mathrm{rot}\,\Phi|^2 dx - \lambda \int_{\Omega} \langle \Phi, \Phi \rangle \, dx.$$

Using the boundary condition $\Phi \times \nu = \mathbf{0}$ on $\partial\Omega$, we get

$$\int_{\Omega} |\mathrm{rot}\,\Phi(x)|^2 dx = \lambda \int_{\Omega} |\Phi(x)|^2 \, dx. \tag{1.7}$$

This implies that λ is nonnegative if $\Phi \not\equiv \mathbf{0}$ in Ω. Hence we see that any eigenvalue is non-negative.

[**Zero eigenspace**]

We consider the equation (1.6) for $\lambda = 0$. From (1.7), if $\Phi \not\equiv \mathbf{0}$ in Ω, we have $\mathrm{rot}\,\Phi = 0$ in Ω. This implies that Φ has an expression $\Phi(x) = \nabla\eta$. Here we should note that the function η could be multi-valued function (in the case Ω is not simply connedted). From the boundary condition $\Phi \times \nu = \nabla\eta \times \nu = \mathbf{0}$, $\nabla\eta$ is paralell to the normal vector ν at any boundary point. This implies that η is constant in any connected component of $\partial\Omega$. So η necessarily single-valued function. On the other hand, take any function $\eta \in H^1(\Omega)$ which is constant on any connected component of $\partial\Omega$ and put Φ becomes an eigenfunction for $\lambda = 0$.

Summing up the above arguments, we know that the zero eigenspace is the following.

$$X_0 = \{\nabla\eta \mid \eta \in C^2(\overline{\Omega}),\ \Delta\eta = 0 \text{ in } \Omega,\ \eta \text{ is constant in each component of } \partial\Omega\} \tag{1.8}$$

[**Existence of eigenvalues**]

We define a basic function space for the argument to prove the existence of the eigenvalues. Put

$$X = \{\Phi \in H^1(\Omega; \mathbb{R}^3) \mid \mathrm{div}\,\Phi = 0 \text{ in } \Omega,\ \Phi \times \nu = \mathbf{0} \text{ on } \partial\Omega\}. \tag{1.9}$$

It is easy to see that $\dim X_0 = \sharp(\text{components of}\,\partial\Omega) - 1$. It is also known that X is a closed subspace of $H^1(\Omega;\mathbb{R}^3)$ and it is also closed in the sense of weak convergence in $H^1(\Omega;\mathbb{R}^3)$.

As we know about the zero eigenvalue and the corresponding eigenfunctions, we deal with the positive eigenvalues from now.

Proposition 1.1. The eigenvalue problem (1.6) has the positive eigenvalues $\{\Lambda_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \Lambda_k = \infty$.

(Proof of Proposition 1.1) To prove the existence of the eigenvalues, we can carry out a completely similar argument as the case of the Laplacian and the Schrödinger operator (cf. Edmunds-Evans [6]). So we only give a sketch of the argument. Hereafter the symbol $\perp$ means the orthogonality in $L^2(\Omega;\mathbb{R}^3)$. Put

$$\Lambda_1 = \inf\{\mathcal{R}(\phi) \mid \phi \in X,\ \phi \perp X_0\}, \quad \text{where} \quad \mathcal{R}(\phi) = \int_{\Omega} |\mathrm{rot}\phi|^2 / \int_{\Omega} |\phi|^2 dx.$$

$\mathcal{R}$ attains the minimum Λ_1 with a minimizer $\Phi_1 \in X$ which is an eigenfunction corresponding to the eigenvalue λ_1. This is proved as follows. Take a minimizing sequence $\{\phi_\ell\}_{\ell=1}^{\infty}$ with $\|\phi_\ell\|_{L^2(\Omega;\mathbb{R}^3)} = 1$. It is bounded also in $H^1(\Omega;\mathbb{R}^3)$ due to Lemma 1.2 and Lemma 1.3 below. This sequence contains a weakly converent subsequence in $H^1(\Omega;\mathbb{R}^3)$ which is also strongly convergent in $L^2(\Omega;\mathbb{R}^3)$. Since X is closed, the limit Φ_1 of the subsequence belongs to X and satisfies $\Phi_1 \perp X_0$. Taking the variation of $\mathcal{R}$ at Φ_1, we get

$$\mathrm{rot}\,\mathrm{rot}\,\Phi_1 - \Lambda_1\Phi_1 = \mathbf{0} \quad \text{in} \quad \Omega.$$

Carry out this argument in the space $X \cap (X_0 \oplus L.H.[\Phi_1])^{\perp}$, we get the second positive eigenvalue Λ_2 as the minimum of $\mathcal{R}$ with the eigenfunction $\Phi_2 \in X$ with $\Phi_2 \perp (X_0 \oplus L.H.[\Phi_1])^{\perp}$. We can repeat this argument and get the sequence $0 < \Lambda_1 \leqq \Lambda_2 \leqq \Lambda_3 \leqq \cdots$. We also note that this sequence is unbounded. The eigenfunctions obtained above are sufficiently regular if $\partial\Omega$ is regular. This can be proved by the arguments in the chapter 7 in Morrey [12], where the harmonic forms in the smooth manifold with boundary, are studied. The estimate of the eigenfunction is also obtain in this process. □

Lemma 1.2. For any $\eta > 0$, there exists $c(\eta) > 0$ such that

$$\int_{\partial\Omega} \phi(x)^2 dS \leqq \eta \int_{\Omega} |\nabla\phi(x)|^2 dx + c(\eta) \int_{\Omega} \phi(x)^2 dx \quad (\phi \in H^1(\Omega)).$$

This is sometimes referred as the Friedrichs inequality. See Mizohata [11; Chap.3] for the proof.

Lemma 1.3. If $\Psi \in H^1(\Omega;\mathbb{R}^3)$ and $\Psi \times \nu = \mathbf{0}$ on $\partial\Omega$, then

$$\int_\Omega |\mathrm{rot}\,\Psi|^2 dx + \int_\Omega |\mathrm{div}\,\Psi|^2\,dx = \int_\Omega |\nabla\Psi|^2\,dx + \int_{\partial\Omega} H(x)|\Psi(x)|^2\,dS.$$

Here $H(x)$ is the mean curvature at $x \in \partial\Omega$ with respect to the unit outward normal vector ν.

(Proof of Lemma 1.3) The proof is carried out through the straightforward calculation. □

Proposition 1.4 (Max-Min principle). The k-th positive eigenvalue Λ_k is characterized by the following formula.

$$\Lambda_k = \sup_{E\subset X_0^\perp,\dim E \leqq k-1} \inf\{\mathcal{R}(\Phi) \mid \Phi \in X,\ \Phi \perp X_0,\ \Phi \perp E\} \tag{1.10}$$

Here E is a subspace of $L^2(\Omega;\mathbb{R}^3)$. For Max-Min principle for more general frame work of selfadjoint elliptic operators in Hilbert spaces, see Davies [2], Edmunds-Evans [3], Reed-Simon [16].

[**Formulation of the domain variation and Eigenvalue problem**]

Assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with C^3 boundary $\partial\Omega$. Let $\rho = \rho(\xi)$ be a C^1 function in $\partial\Omega$. Put the set

$$\partial\Omega(\epsilon) = \{\xi + \epsilon\rho(\xi)\nu(\xi) \in \mathbb{R}^3 \mid \xi \in \partial\Omega\} \tag{1.11}$$

when $|\epsilon|$ is small, there exists a unique bounded domain such that $\Omega(\epsilon)$ is homeomorphic to Ω and its boundary agrees to $\partial\Omega(\epsilon)$.

For this domain $\Omega(\epsilon)$, we consider the following eigenvalue problem,

$$\begin{cases} \mathrm{rot}\,\mathrm{rot}\,\Phi - \lambda\,\Phi = \mathbf{0}, \quad \mathrm{div}\,\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \\ \Phi \times \nu = \mathbf{0} \quad \text{on} \quad \partial\Omega(\epsilon). \end{cases} \tag{1.12}$$

The eigenvalue problem is also written as

$$\begin{cases} \Delta\Phi + \lambda\,\Phi = 0, \ \mathrm{div}\,\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \\ \Phi \times \nu = \mathbf{0} \qquad \text{on} \quad \partial\Omega(\epsilon). \end{cases} \tag{1.13}$$

The set of the eigenvalue is a discrete unbounded sequence of real values.

Definition. Let $\{\lambda_k(\epsilon)\}_{k=1}^\infty$ be the set of positive eigenvalues (of (1.12)) which are arranged in increasing order with counting multiplicity.

Definition. Let $\{\Phi_\epsilon^{(k)}\}_{k=1}^\infty$ be the corresponding system of the eigenfunctions, which is orthonormal as

$$(\Phi_\epsilon^{(p)}, \Phi_\epsilon^{(q)})_{L^2(\Omega(\epsilon);\mathbb{R}^3)} = \delta(p,q) \quad (p,q \geqq 1).$$

We note that $\Lambda_k = \lambda_k(0)$.

Theorem 1.5. Assume that the k-th eigenvalue Λ_k in (1.6) is simple. Then $\lambda_k(\epsilon)$ is differentiable at $\epsilon = 0$ and its derivative is given by the following formula.

$$\frac{d\lambda_k(\epsilon)}{d\epsilon}_{|\epsilon=0} = \int_{\partial\Omega} \left(|\nabla\Phi_0^{(k)}|^2 - 2|\frac{\partial\Phi_0^{(k)}}{\partial\nu}|^2 + (2K(x) - \Lambda_k)|\Phi_0^{(k)}(x)|^2 \right) \rho\,dS \tag{1.14}$$
$$+2\int_{\partial\Omega} \langle\Phi_0^{(k)}, \nu\rangle\,\langle \mathrm{rot}\Phi_0^{(k)} \times \nabla\rho, \nu\rangle\,dS$$

Here $K(x)$ is the Gaussian curvature of $\partial\Omega$ at x. $\nabla\rho$ is the gradient vector in the tangent space of $\partial\Omega$.

§2. Transformation of the problem

The method of the proof is to make a transformation (diffeomorphism) $\gamma_\epsilon : \Omega \longrightarrow \Omega(\epsilon)$ and to transplant the problem to the fixed domain Ω through the change of the variable $x = \gamma_\epsilon(y)$. So the problem on a ϵdependent variable domain reduces the problem which includes ϵ in a coefficient on a fixed domain.

So we prepare the transformation map and calculate the equation in a fixed domain Ω.

Lemma 2.1. There exists $\delta_0 > 0$ and a smooth diffeomorphism map

$$\gamma_\epsilon : \overline{\Omega} \longrightarrow \overline{\Omega(\epsilon)}$$

such that γ_ϵ depends smoothly on $\epsilon \in (-\delta_0, \delta_0)$ and

$$\gamma_\epsilon(\xi + t\nu(\xi)) = \xi + (t+\epsilon)\rho(\xi)\nu(\xi) \quad \text{for} \quad \xi \in \partial\Omega,\ |t| < \delta_0,\ |\epsilon| < \delta_0. \tag{2.1}$$

(Proof) Prepare a coordinate near the boundary $\partial\Omega$ and consider the map which moves a point $\xi \in \partial\Omega$ to $\xi + \epsilon\rho(\xi)\nu(\xi)$. We can construct a smooth map with this property using a cut-off in that neighbood. □

From the condition (2.1), we have

$$\frac{d\gamma_\epsilon}{d\epsilon}(\xi + t\nu(\xi))_{\epsilon=0} = \rho(\xi)\nu(\xi) \quad \text{for} \quad \xi \in \partial\Omega, |t| < \delta_0. \tag{2.2}$$

We denote the unknown variable by Φ and the transformed unknown variable by $\widetilde{\Phi}$. Their relation is

$$\widetilde{\Phi}(y) = (\Phi \circ \gamma_\epsilon)(y).$$

We express the unknown variable Φ by its components as follows.

$$\Phi(x) = (\Phi_1(x), \Phi_2(x), \Phi_3(x))^t, \quad \widetilde{\Phi}(y) = (\widetilde{\Phi}_1(y), \widetilde{\Phi}_2(y), \widetilde{\Phi}_3(y))^t$$

Accordingly we have

$$\widetilde{\Phi}_i(y) = (\Phi_i \circ \gamma_\epsilon)(y) \quad (y \in \Omega, \quad i = 1, 2, 3).$$

To transform the system (1.13) into the system for $\widetilde{\Phi}$, we calculate the equation and the boundary condition for $\widetilde{\Phi}$ in Ω. $\nu = (\nu_1, \nu_2, \nu_3)$ is the unit outward normal vector on $\partial\Omega$. We extend this field ν up to some neighborhood of $\partial\Omega$ for later convenience.

$$\nabla_y \widetilde{\Phi}_i(y) = \nabla_x \Phi_i(x)\ (\frac{\partial\gamma_\epsilon}{\partial y}(y)), \quad \nabla_x \Phi_i = (\frac{\partial\Phi_i}{\partial x_1}, \frac{\partial\Phi_i}{\partial x_2}, \frac{\partial\Phi_i}{\partial x_3}), \quad \nabla_y \widetilde{\Phi}_i = (\frac{\partial\widetilde{\Phi}_i}{\partial y_1}, \frac{\partial\widetilde{\Phi}_i}{\partial y_2}, \frac{\partial\widetilde{\Phi}_i}{\partial y_3}).$$

$$\gamma_\epsilon(y) = (\gamma_{1,\epsilon}(y), \gamma_{2,\epsilon}(y), \gamma_{3,\epsilon}(y))^t, \quad \frac{\partial\gamma_\epsilon}{\partial y}(y) = \begin{pmatrix} \partial\gamma_{1,\epsilon}/\partial y_1 & \partial\gamma_{1,\epsilon}/\partial y_2 & \partial\gamma_{1,\epsilon}/\partial y_3 \\ \partial\gamma_{2,\epsilon}/\partial y_1 & \partial\gamma_{2,\epsilon}/\partial y_2 & \partial\gamma_{2,\epsilon}/\partial y_3 \\ \partial\gamma_{3,\epsilon}/\partial y_1 & \partial\gamma_{3,\epsilon}/\partial y_2 & \partial\gamma_{3,\epsilon}/\partial y_3 \end{pmatrix}$$

We get the transformed equation.

$$\mathrm{div}_y \left(\det(\frac{\partial\gamma_\epsilon}{\partial y}) \nabla_y \widetilde{\Phi}_i [\frac{\partial\gamma_\epsilon}{\partial y}]^{-1} ([\frac{\partial\gamma_\epsilon}{\partial y}]^{-1})^t \right) + \lambda \det(\frac{\partial\gamma_\epsilon}{\partial y}) \widetilde{\Phi}_i = 0 \quad \text{in} \quad \Omega \quad (i = 1, 2, 3). \tag{2.3}$$

The "div-free" condition is written as

$$\sum_{k=1}^{3} \sum_{\ell=1}^{3} \frac{\partial\widetilde{\Phi}_k}{\partial y_\ell} ([\frac{\partial\gamma_\epsilon}{\partial y}(y)]^{-1})_{\ell k} = 0 \quad \text{in} \quad \Omega. \tag{2.4}$$

Here, for the matrix M, $M_{\ell k}$ is the (ℓ, k) component of M. M^t is the transpose of M.

We calculate the boundary condition for $\widetilde{\Phi}$ on $\partial\Omega$. The unit outward normal vector ν_ϵ at $x = \gamma_\epsilon(y)$ on $\partial\Omega(\epsilon)$ is given by

$$\nu_\epsilon(\gamma_\epsilon(y)) = [(\partial\gamma_\epsilon/\partial y)^t(y)]^{-1}\nu(y)/|[(\partial\gamma_\epsilon/\partial y)^t(y)]^{-1}\nu(y)| \quad \text{for} \quad \partial\Omega.$$

Using this formula, we get the following condition for $\widetilde{\Phi}$ from $\Phi \times \nu_\epsilon = \mathbf{0}$ on $\partial\Omega(\epsilon)$. We express each component of that condition.

$$\widetilde{\Phi}_1(-\nu_2(y)\frac{\partial\gamma_{1,\epsilon}}{\partial y_1} + \nu_1(y)\frac{\partial\gamma_{1,\epsilon}}{\partial y_2}) + \widetilde{\Phi}_2(-\nu_2(y)\frac{\partial\gamma_{2,\epsilon}}{\partial y_1} + \nu_1(y)\frac{\partial\gamma_{2,\epsilon}}{\partial y_2}) \tag{2.5}$$
$$+\widetilde{\Phi}_3(-\nu_2(y)\frac{\partial\gamma_{3,\epsilon}}{\partial y_1} + \nu_1(y)\frac{\partial\gamma_{3,\epsilon}}{\partial y_2}) = 0 \quad \text{on} \quad \partial\Omega,$$

$$\widetilde{\Phi}_1(\nu_3(y)\frac{\partial\gamma_{1,\epsilon}}{\partial y_1} - \nu_1(y)\frac{\partial\gamma_{1,\epsilon}}{\partial y_3}) + \widetilde{\Phi}_2(\nu_3(y)\frac{\partial\gamma_{2,\epsilon}}{\partial y_1} - \nu_1(y)\frac{\partial\gamma_{2,\epsilon}}{\partial y_3}) \tag{2.6}$$
$$+\widetilde{\Phi}_3(\nu_3(y)\frac{\partial\gamma_{3,\epsilon}}{\partial y_1} - \nu_1(y)\frac{\partial\gamma_{3,\epsilon}}{\partial y_3}) = 0 \quad \text{on} \quad \partial\Omega,$$

$$\widetilde{\Phi}_1(-\nu_3(y)\frac{\partial\gamma_{1,\epsilon}}{\partial y_2} + \nu_2(y)\frac{\partial\gamma_{1,\epsilon}}{\partial y_3}) + \widetilde{\Phi}_2(-\nu_3(y)\frac{\partial\gamma_{2,\epsilon}}{\partial y_2} + \nu_2(y)\frac{\partial\gamma_{2,\epsilon}}{\partial y_3}) \tag{2.7}$$

$$+\widetilde{\Phi}_3(-\nu_3(y)\frac{\partial\gamma_{3,\epsilon}}{\partial y_2}+\nu_2(y)\frac{\partial\gamma_{3,\epsilon}}{\partial y_3})=0 \quad \text{on} \quad \partial\Omega.$$

The above system (2.3), (2.4),(2.5),(2.6), (2.7) is the equation with the boundary condition in the domain Ω.

§3. The pertubation of the eigenvalue problem for the toy model

The main purpose of this paper is study the $\epsilon-$dependence of the system (2.3)-(2.7). We explain the line of the proof for the perturbation formula of the eigenvalue $\lambda_k(\epsilon)$ by using a simple toy model, which is a similar finite dimensional problem.

Let W be a finite dimensional space with an inner product. Let $\mathcal{A}(\epsilon)$ be a linear transformation in W which depends on the parameter $\epsilon \in \mathbb{R}$ smoothly. We assume that $\mathcal{A}(\epsilon)$ is self-adjoint for each ϵ. Assume also that $\mathcal{A}(0)$ has a simple eigenvalue $\mu(0)$ with an eigenvector $\boldsymbol{v}_0$. From Kato [9], there exists exactly one eigenvalue $\mu(\epsilon)$ of $\mathcal{A}(\epsilon)$ when $|\epsilon|$ is small and it approaches $\mu(0)$ for $\epsilon \to 0$. We study the differentiability of $\mu(\epsilon)$ at $\epsilon = 0$.

We study the eigenvalu problem

$$\mathcal{A}(\epsilon)\boldsymbol{v} + \mu\,\boldsymbol{v} = 0.$$

[Formal calculus]

First we carry out a heuristic argument. Denote $\boldsymbol{v}_\epsilon \in X$ be the eigenvector corresponding to the eigenvalue $\mu(\epsilon)$ and assume that $\mu(\epsilon)$ and $\boldsymbol{v}_\epsilon$ are differentiable in ϵ.

That is

$$\mathcal{A}(\epsilon)\,\boldsymbol{v}_\epsilon + \mu(\epsilon)\,\boldsymbol{v}_\epsilon = 0.$$

Differentiate the both sides and we get

$$\mathcal{A}(\epsilon)\dot{\boldsymbol{v}}_\epsilon + \dot{\mathcal{A}}(\epsilon)\boldsymbol{v}_\epsilon + \mu(\epsilon)\dot{\boldsymbol{v}}_\epsilon + \dot{\mu}(\epsilon)\boldsymbol{v}_\epsilon = 0$$

Put $\epsilon = 0$ and we have

$$(\mathcal{A}(0) + \mu(0))\dot{\boldsymbol{v}}_0 = -\dot{\mathcal{A}}(0)\boldsymbol{v}_0 - \dot{\mu}(0)\boldsymbol{v}_0.$$

Regard this equation for the unknown variable $\dot{\boldsymbol{v}}_0$ and then we get the condition for the right hand side term. We see that the kernel of $\mathcal{A}(0) + \mu(0)I$ is orthogonal to the right hand side from the self-adjointness of the operator. This is saying

$$(\dot{\mathcal{A}}(0)\boldsymbol{v}_0 + \dot{\mu}(0)\boldsymbol{v}_0) \perp \boldsymbol{v}_0 \;\Rightarrow\; \dot{\mu}(0)\|\boldsymbol{v}_0\|^2 = -(\dot{\mathcal{A}}(0)\boldsymbol{v}_0, \boldsymbol{v}_0)$$

This consideration give the candidate for $\dot{\mu}(0)$ which is obtained throgh the necessary condition of the existence of $\dot{\mu}(0)$ and $\dot{\boldsymbol{v}}_0$. The heuristic argument is finished.

[Justification]

First we put

$$d = -(\dot{\mathcal{A}}(0)\boldsymbol{v}_0, \boldsymbol{v}_0)/\|\boldsymbol{v}_0\|^2$$

and consider the following equation (with the unknown variable $\boldsymbol{w}$)

$$(\mathcal{A}(0) + \mu(0))\boldsymbol{w} = -\dot{\mathcal{A}}(0)\boldsymbol{v}_0 + d\,\boldsymbol{v}_0$$

We recall that $\mathcal{A}$ is self-adjoint. From the definition of d (orthogonality condition of the right hand side), it has a solution and it is unique if the condition $\boldsymbol{w} \perp \boldsymbol{v}_0$ is prescribed. We consider the following vector as an approximate eigenvector for "$\epsilon-$problem"

$$\widetilde{\boldsymbol{v}}_\epsilon = \boldsymbol{v}_0 + \epsilon\,\boldsymbol{w}.$$

Take the inner product of $\widetilde{\boldsymbol{v}}_\epsilon$ and the eigenvalue equation and get

$$(\mathcal{A}(\epsilon)\boldsymbol{v}_\epsilon + \mu(\epsilon)\boldsymbol{v}_\epsilon, \boldsymbol{v}_0 + \epsilon\,\boldsymbol{w}) = 0.$$

$$(\mathcal{A}(\epsilon)\boldsymbol{v}_\epsilon, \boldsymbol{v}_0) + \mu(\epsilon)(\boldsymbol{v}_\epsilon, \boldsymbol{v}_0) + \epsilon(\mathcal{A}(\epsilon)\boldsymbol{v}_\epsilon + \mu\,(\epsilon)\boldsymbol{v}_\epsilon, \boldsymbol{w}) = 0$$

Here we assume that the eigenvector is normalized $\|\boldsymbol{v}_\epsilon\| = 1$. Consider the Taylor expansion of $\mathcal{A}(\epsilon)$ in ϵ at 0

$$\mathcal{A}(\epsilon) = \mathcal{A}(0) + \epsilon\dot{\mathcal{A}}(0) + o(\epsilon) \quad \text{(Taylor expansion)},$$

and we calculate the quantity

$$\begin{aligned}(\mathcal{A}(\epsilon)\boldsymbol{v}_\epsilon, \boldsymbol{v}_0) &= (\boldsymbol{v}_\epsilon, \mathcal{A}(\epsilon)\boldsymbol{v}_0) = (\boldsymbol{v}_\epsilon, \mathcal{A}(0)\boldsymbol{v}_0) + \epsilon(\boldsymbol{v}_\epsilon, \dot{\mathcal{A}}(0)\boldsymbol{v}_0) + (\boldsymbol{v}_\epsilon, o(\epsilon)\boldsymbol{v}_0)\\ &= -\mu(0)(\boldsymbol{v}_\epsilon, \boldsymbol{v}_0) + \epsilon(\boldsymbol{v}_\epsilon, \dot{\mathcal{A}}(0)\boldsymbol{v}_0) + (\boldsymbol{v}_\epsilon, o(\epsilon)\boldsymbol{v}_0)\end{aligned}$$

$$(\mathcal{A}(\epsilon)\boldsymbol{v}_\epsilon, \boldsymbol{w}) = (\boldsymbol{v}_\epsilon, \mathcal{A}(\epsilon)\boldsymbol{w}) = (\boldsymbol{v}_\epsilon, \mathcal{A}(0)\boldsymbol{w}) + \epsilon(\boldsymbol{v}_\epsilon, \dot{\mathcal{A}}(0)\boldsymbol{w}) + (\boldsymbol{v}_\epsilon, o(\epsilon)\boldsymbol{w})$$

$$= -(\boldsymbol{v}_\epsilon, (\dot{\mathcal{A}}(0) + d)\boldsymbol{v}_0) - \mu(0)(\boldsymbol{v}_\epsilon, \boldsymbol{w}) + \epsilon(\boldsymbol{v}_\epsilon, \dot{\mathcal{A}}(0)\boldsymbol{w}) + (\boldsymbol{v}_\epsilon, o(\epsilon)\boldsymbol{w})$$

Substitute it to the equation and get

$$\frac{\mu(\epsilon) - \mu(0)}{\epsilon}(\boldsymbol{v}_\epsilon, \boldsymbol{v}_0) = -(\boldsymbol{v}_\epsilon, \dot{\mathcal{A}}(0)\boldsymbol{v}_0) - (\boldsymbol{v}_\epsilon, o(1)\boldsymbol{v}_0)$$

$$-(\mu(\epsilon) - \mu(0))(\boldsymbol{v}_\epsilon, \boldsymbol{w}) + (\boldsymbol{v}_\epsilon, (\dot{\mathcal{A}}(0) + d)\boldsymbol{v}_0) - \epsilon(\boldsymbol{v}_\epsilon, \dot{\mathcal{A}}(0)\boldsymbol{w}) - (\boldsymbol{v}_\epsilon, o(\epsilon)\boldsymbol{w})$$

Take any sequence $\{\epsilon(m)\}_{m=1}^\infty$ which approaches 0 for $m \to \infty$. Then there exists a subsequence $\{\epsilon(m(p))\}_{p=1}^\infty$ and a unit vector $\boldsymbol{v}' \in X$ such that

$$\lim_{p\to\infty} \boldsymbol{v}_{\epsilon(m(p))} = \boldsymbol{v}', \quad \mathcal{A}(0)\boldsymbol{v}' + \mu(0)\boldsymbol{v}' = 0.$$

As we assumed that the eigenvalue $\mu(0)$ is simple, $\boldsymbol{v}' = \boldsymbol{v}_0$ or $\boldsymbol{v}' = -\boldsymbol{v}_0$.

$$\lim_{p\to\infty} \frac{\mu(\epsilon(m(p))) - \mu(0)}{\epsilon(m(p))}(\boldsymbol{v}', \boldsymbol{v}_0) = d\,(\boldsymbol{v}', \boldsymbol{v}_0).$$

The sequence $\{\epsilon(m)\}_{m=1}^\infty$ was arbitrary and d is defined independently of the choice of this sequence and so we conclude

$$\lim_{\epsilon\to 0} \frac{\mu(\epsilon) - \mu(0)}{\epsilon} = d = -\frac{(\dot{\mathcal{A}}(0)\boldsymbol{v}_0, \boldsymbol{v}_0)}{\|\boldsymbol{v}_0\|^2}.$$

This implies that $\mu(\epsilon)$ is differentiable in ϵ at 0 and the derivative agrees exactly to the value d.

§4. Analysis of the perturbation of the electromagnetic eigenvalue

We consider the positive eigenvalues $\{\lambda_k(\epsilon)\}_{k=1}^\infty$ and the corresponding eigenfunctions $\{\Phi_\epsilon^{(k)}\}_{k=1}^\infty$ of (1.12), which are orthonormal in $L^2(\Omega(\epsilon); \mathbb{R}^3)$ and satisfy

$$(4.1) \qquad (\Phi_\epsilon^{(p)}, \Phi_\epsilon^{(q)})_{L^2(\Omega(\epsilon);\mathrm{R}^3)} = \delta(p,q), \quad (\mathrm{rot}\,\Phi_\epsilon^{(p)}, \mathrm{rot}\,\Phi_\epsilon^{(q)})_{L^2(\Omega(\epsilon);\mathbb{R}^3)} = \delta(p,q)\lambda_p(\epsilon) \quad (p, q \geqq 1).$$

By the aid of the max-min principle (Lemma 1.4) for $\lambda_k(\epsilon)$ in (1.12) with the (almost) test functions $\Phi_0^{(k)}(\gamma_\epsilon^{-1}(x))$ $(k \geqq 1)$, we can derive an **upper estimate**

$$(4.2) \qquad \lambda_k(\epsilon) \leqq \Lambda_k + O(\epsilon).$$

To obtain a lower estimate, we first note that there exists a constant $\delta_k > 0$ and $c_k > 0$ (from (4.1),(4.2)) such that

$$(4.3) \qquad \|\Phi_\epsilon^{(k)}\|_{H^1(\Omega(\epsilon);\mathbb{R}^3)} \leqq c_k \quad \text{for} \quad |\epsilon| \leqq \delta_k.$$

Recall $\widetilde{\Phi}_\epsilon^{(k)}(y) = (\Phi_\epsilon^{(k)} \circ \gamma_\epsilon)(y)$. As the tranformation $x = \gamma_\epsilon(y)$ smoothly approach the identity map, we have the following estimates (with the aid of Lemma 1.2 and Lemma 1.3).

Lemma 4.1. For each $k \in \mathbb{N}$, there exists a constant $c(k) > 0$ such that

$$\|\widetilde{\Phi}_\epsilon^{(k)}\|_{H^1(\Omega;\mathbb{R}^3)} \leqq c(k) \quad \text{for small} \quad \epsilon > 0.$$

As $\Omega(\epsilon)$ depends smoothly on ϵ, we can apply the regularity arugument for $\Phi_\epsilon^{(k)}$ in the boundary value problem (1.13) which is developed in the book Morrey [12]. We can have the following regularity.

Lemma 4.2. For each $k \in \mathbb{N}$, there exists a constant $c'(k) > 0$ such that

$$\|\widetilde{\Phi}_\epsilon^{(k)}\|_{C^2(\overline{\Omega};\mathbb{R}^3)} \leqq c'(k) \quad \text{for small} \quad \epsilon > 0.$$

Take an arbitrary sequence $\{\epsilon(p)\}_{p\geqq 1}$ such that $\lim_{p\to\infty} \epsilon(p) = 0$. Then, there exists a subsequence $\{\epsilon(p(m))\}_{m=1}^\infty$ and an orthonormal system $\{\mathcal{M}^{(k)}\}_{k=1}^\infty$ in $L^2(\Omega; \mathbb{R}^3)$ such that

$$(4.4) \qquad \widetilde{\Phi}_{\epsilon(p(m))}^{(k)} \longrightarrow \mathcal{M}^{(k)} \quad (m \to \infty)$$

strongly in $L^2(\Omega; \mathbb{R}^3)$ and weakly in $H^1(\Omega; \mathbb{R}^3)$ and $\mathrm{div}\mathcal{M}^{(k)} = 0$ in Ω, $\mathcal{M}^{(k)} \times \nu = \mathbf{0}$ on $\partial\Omega$. From (4.2), (4.4), we have

$$(4.5) \qquad \Lambda_k \geqq \liminf_{m\to\infty} \lambda_k(\epsilon(p(m))) = \liminf_{m\to\infty} \int_{\Omega(\epsilon(p(m)))} |\mathrm{rot}\Phi_{\epsilon(p(m))}^{(k)}(x)|^2 dx$$

$$= \liminf_{m\to\infty} \int_\Omega |\mathrm{rot}\, \widetilde{\Phi}^{(k)}_{\epsilon(p(m))}(y)|^2 dy \geqq \int_\Omega |\mathcal{M}^{(k)}(y)|^2 dy.$$

From the orghogonality of $\{\mathcal{M}^{(k)}\}_{k=1}^\infty$ in $L^2(\Omega;\mathbb{R}^3) \cap X_0^\perp$ with (4.5), we have $\int_\Omega |\mathcal{M}^{(k)}|^2 dy = \Lambda_k$ for $k \geqq 1$. This implies $\Psi^{(k)}$ is necessarily a $k-$th eigenfunction. Eventually we get the convergence $\lim_{m\to\infty} \lambda_k(\epsilon(p(m))) = \Lambda_k$. Since $\{\epsilon(p)\}$ was arbitrary, we have the following result.

Proposition 4.3. $\lim_{\epsilon\to 0} \lambda_k(\epsilon) = \Lambda_k$ $(k \geqq 1)$.

We study the detailed asymptotics of $\lambda_k(\epsilon)$ for $\epsilon \to 0$. We follow the line of argument given in the previous section. By a formal perturbation argument, we first find a candidate of $(d\lambda_k/d\epsilon)_{|\epsilon=0}$ by a formal calculus.

We prepare some notation. The variation of the map γ_ϵ under perturbation by ϵ is given by a vector field g which is defined as follows,

$$g(y) = (g_1(y), g_2(y), g_3(y))^t = \frac{\partial \gamma_\epsilon(y)}{\partial \epsilon}_{|\epsilon=0} \quad (y \in \Omega).$$

Note that $g(\xi) = \rho(\xi)\nu(\xi)$ for $\xi \in \partial\Omega$ (cf. (2.1)).

To calculate the derivative of the equation of (2.3)-(2.4) and the boundary condition (2.5)-(2.7), we prepare some formulas.

Lemma 4.4. Let $A(\epsilon)$ be a invertible square matrix which is differentiable in ϵ. Then we have

$$\frac{d}{d\epsilon} A(\epsilon)^{-1} = -A(\epsilon)^{-1} \frac{d}{d\epsilon} A(\epsilon) A(\epsilon)^{-1}. \tag{4.6}$$

Moreover, if $A(0) = I$ (Identity matrix), then

$$\frac{d}{d\epsilon} \det A(\epsilon)_{|\epsilon=0} = \mathrm{Tr}\left(\frac{dA(\epsilon)}{d\epsilon}_{|\epsilon=0} \right). \tag{4.7}$$

(Proof) This is proved by a direct calculation.

Since $\gamma_0(y) = y$ (Identity map), it follows $(\partial\gamma_0/\partial y) = I$. Hence we can apply the above formulas (4.2), (4.3) to the Jacobian matrix $\partial\gamma_\epsilon/\partial y$, we have

$$\frac{d}{d\epsilon} (\frac{\partial \gamma_\epsilon}{\partial y})^{-1}_{|\epsilon=0} = -\frac{\partial g(y)}{\partial y}. \tag{4.8}$$

$$\frac{d}{d\epsilon} \det(\frac{\partial\gamma_\epsilon}{\partial y})_{|\epsilon=0} = \mathrm{div}_y\, g(y) = \sum_{j=1}^3 \frac{\partial g_j(y)}{\partial y_j}. \tag{4.9}$$

[Variational equation]

Fix a natural number k hereafter. Drop the index k and denote $\Phi_\epsilon = \Phi_\epsilon^{(k)}$, $\widetilde{\Phi}_\epsilon = \widetilde{\Phi}_\epsilon^{(k)}$, $\lambda(\epsilon) = \lambda_k(\epsilon)$. Note that $\widetilde{\Phi}_0 = \Phi_0$ because γ_0 is the identity map. Assume that $\widetilde{\Phi}_\epsilon$, $\lambda(\epsilon)$ is differentiable in ϵ at 0 and put

$$\Psi(y) = (\Psi_1(y), \Psi_2(y), \Psi_3(y)) = (\partial\widetilde{\Phi}_\epsilon^{(k)}/\partial\epsilon)_{\epsilon=0}, \quad \boldsymbol{\kappa} = (d\lambda_k/d\epsilon)(0). \tag{4.10}$$

We seek for the relation which Ψ and $\boldsymbol{\kappa}$ should satisfy if they exists. Take the derivative of (2.3),(2.4),(2.5),(2.6), (2.7) and put $\epsilon = 0$ and calculate by the formula (4.8) and (4.9) and substitute $\epsilon = 0$, we get

$$\begin{aligned} &\mathrm{div}(\nabla\Psi_i) + \mathrm{div}_y((\mathrm{div} g)\nabla_y \Phi_{0i}) - \mathrm{div}(\nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t)) \\ &+\boldsymbol{\kappa}\Phi_{0i} + \lambda(0)(\mathrm{div} g)\Phi_{0i} + \lambda(0)\Psi_i = 0 \quad (y \in \Omega, i = 1,2,3), \end{aligned} \tag{4.11}$$

$$\mathrm{div}\,\Psi = \sum_{i=1}^3 \sum_{\ell=1}^3 \frac{\partial\Phi_{0i}}{\partial y_\ell} \frac{\partial g_\ell}{\partial y_i} \quad \text{in} \quad \Omega. \tag{4.12}$$

From (2.5), (2.6), (2.7), we have the boundary condition for Ψ which gives the values of $\nu\times\Psi$ on $\partial\Omega$,

$$\Psi_2\nu_1 - \Psi_1\nu_2 = \Phi_{01}(\nu_2\frac{\partial g_1}{\partial y_1} - \nu_1\frac{\partial g_1}{\partial y_2}) + \Phi_{02}(\nu_2\frac{\partial g_2}{\partial y_1} - \nu_1\frac{\partial g_2}{\partial y_2}) + \Phi_{03}(\nu_2\frac{\partial g_3}{\partial y_1} - \nu_1\frac{\partial g_3}{\partial y_2}), \tag{4.13}$$

$$\Psi_1\nu_3 - \Psi_3\nu_1 = \Phi_{01}(\nu_1\frac{\partial g_1}{\partial y_3} - \nu_3\frac{\partial g_1}{\partial y_1}) + \Phi_{02}(\nu_1\frac{\partial g_2}{\partial y_3} - \nu_3\frac{\partial g_2}{\partial y_1}) + \Phi_{03}(\nu_1\frac{\partial g_3}{\partial y_3} - \nu_3\frac{\partial g_3}{\partial y_1}), \tag{4.14}$$

$$\Psi_3\nu_2 - \Psi_2\nu_3 = \Phi_{01}(\nu_3\frac{\partial g_1}{\partial y_2} - \nu_2\frac{\partial g_1}{\partial y_3}) + \Phi_{02}(\nu_3\frac{\partial g_2}{\partial y_2} - \nu_2\frac{\partial g_2}{\partial y_3}) + \Phi_{03}(\nu_3\frac{\partial g_3}{\partial y_2} - \nu_2\frac{\partial g_3}{\partial y_3}). \tag{4.15}$$

For the domain derivative of solution of poisson equation, we can learn many things in Murat-Simon [13,14]. For later convenience we define the vector field ψ_0 by

$$\psi_0 = -(\frac{\partial g}{\partial y})^t\Phi_0 \quad \text{in} \quad \Omega.$$

Using ψ_0, the boundary condition for Ψ (i.e. (4.13),(4.14),(4.15)) is equivalently written by

$$\Psi \times \nu = \psi_0 \times \nu \quad \text{on} \quad \partial\Omega. \tag{4.16}$$

We multiply both sides of the equation (4.11) by Φ_{0i} and sum for $i = 1, 2, 3$.

$$\sum_{i=1}^{3}\int_\Omega \left\{\Phi_{0i}\Delta\Psi_i + \Phi_{0i}\mathrm{div}((\mathrm{div}g)\nabla\Phi_{0i}) - \Phi_{0i}\mathrm{div}(\nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t))\right\} dy$$

$$+\sum_{i=1}^{3}\int_\Omega(\boldsymbol{\kappa}\Phi_{0i}^2 + \lambda(0)(\mathrm{div}g)\Phi_{0i}^2 + \lambda(0)\Psi_i\Phi_{0i})dy = 0.$$

Denote the left hand side by J. Substitute $\Delta\Psi = \mathrm{rot}\,\mathrm{rot}\Psi - \nabla\,\mathrm{div}\Psi$ into J with (4.12) and integrate by parts, we get

$$J = \int_\Omega\langle\Phi_0, \nabla(\sum_{i,\ell=1}^{3}\frac{\partial\Phi_{0i}}{\partial y_\ell}\frac{\partial g_\ell}{\partial y_i})\rangle dy + \int_{\partial\Omega}\langle\Phi_0, (-\nu)\times\mathrm{rot}\Psi\rangle dS - \int_\Omega\langle\mathrm{rot}\Phi_0, \mathrm{rot}\Psi\rangle dy$$

$$+\sum_{i=1}^{3}\int_\Omega\{\Phi_{0i}\mathrm{div}((\mathrm{div}g)\nabla\Phi_{0i}) - \Phi_{0i}\mathrm{div}(\nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t))\}dy$$

$$+\sum_{i=1}^{3}\int_\Omega(\boldsymbol{\kappa}\Phi_{0i}^2 + \lambda(0)(\mathrm{div}g)\Phi_{0i}^2 + \lambda(0)\Psi_i\Phi_{0i})dy$$

$$= \int_{\partial\Omega}\langle\Phi_0,\nu\rangle(\sum_{i,\ell=1}^{3}\frac{\partial\Phi_{0i}}{\partial y_\ell}\frac{\partial g_\ell}{\partial y_i})dS - \int_\Omega(\mathrm{div}\Phi_0)(\sum_{i,\ell=1}^{3}\frac{\partial\Phi_{0i}}{\partial y_\ell}\frac{\partial g_\ell}{\partial y_i})dy$$

$$-\int_{\partial\Omega}\langle\mathrm{rot}\Psi, \Phi_0\times\nu\rangle dS - \int_{\partial\Omega}\langle\mathrm{rot}\Phi_0, \nu\times\Psi\rangle dS - \int_\Omega\langle\mathrm{rot}\,\mathrm{rot}\Phi_0, \Psi\rangle dy$$

$$+\sum_{i=1}^{3}\int_{\partial\Omega}\Phi_{0i}(\mathrm{div}g)\langle\nu, \nabla\Phi_{0i}\rangle dS - \sum_{i=1}^{3}\int_\Omega(\mathrm{div}g)|\nabla\Phi_{0i}|^2dy$$

$$-\sum_{i=1}^{3}\int_{\partial\Omega}\Phi_{0i}\langle\nu, \nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t)\rangle dS + \sum_{i=1}^{3}\int_\Omega\langle\nabla\Phi_{0i}, \nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t)\rangle dy$$

$$+\sum_{i=1}^{3}\int_\Omega(\boldsymbol{\kappa}\Phi_{0i}^2 + \lambda(0)(\mathrm{div}g)\Phi_{0i}^2 + \lambda(0)\Psi_i\Phi_{0i})dy$$

Using $\Phi_0\times\nu = \mathbf{0}$ on $\partial\Omega$ and $\mathrm{rot}\,\mathrm{rot}\widetilde{\Phi}_0 - \lambda(0)\widetilde{\Phi}_0 = 0$ and $\mathrm{div}\Phi_0 = 0$ in Ω, we can simplify this expression and get

$$J = \int_{\partial\Omega}\langle\Phi_0,\nu\rangle(\sum_{i,\ell=1}^{3}\frac{\partial\Phi_{0i}}{\partial y_\ell}\frac{\partial g_\ell}{\partial y_i})dS - \int_{\partial\Omega}\langle\mathrm{rot}\Phi_0, \nu\times\Psi\rangle dS$$

$$+\sum_{i=1}^{3}\int_{\partial\Omega}\Phi_{0i}(\mathrm{div}g)\langle\nu, \nabla\Phi_{0i}\rangle dS - \sum_{i=1}^{3}\int_\Omega(\mathrm{div}g)|\nabla\Phi_{0i}|^2dy$$

$$-\sum_{i=1}^{3}\int_{\partial\Omega}\Phi_{0i}\langle\nu, \nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t)\rangle dS + \sum_{i=1}^{3}\int_\Omega\langle\nabla\Phi_{0i}, \nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t)\rangle dy$$

$$+\sum_{i=1}^{3}\int_\Omega(\boldsymbol{\kappa}\Phi_{0i}^2 + \lambda(0)(\mathrm{div}g)\Phi_{0i}^2)dy$$

$$J=-\int_{\partial\Omega} A\,dS+\int_{\partial\Omega} B\,dS-\sum_{i=1}^{3}\int_{\partial\Omega}\langle g,\nu\rangle|\nabla\Phi_{0i}|^2dS+2\sum_{i=1}^{3}\int_{\partial\Omega}\langle g,\nabla\Phi_{0i}\rangle\langle\nu,\nabla\Phi_{0i}\rangle dS$$
$$+\lambda(0)\sum_{i=1}^{3}\int_{\partial\Omega}\langle g,\nu\rangle\Phi_{0i}^2dS+\boldsymbol{\kappa}\sum_{i=1}^{3}\int_{\Omega}\Phi_{0i}^2\,dy$$

Here A,B are given as follows. Note that the expression of $\nu\times\Psi$ is substitued.

$$A=\langle\mathrm{rot}\Phi_0,\nu\times\Psi\rangle=(\frac{\partial\Phi_{03}}{\partial y_2}-\frac{\partial\Phi_{02}}{\partial y_3})[\Phi_{01}(\nu_3\frac{\partial g_1}{\partial y_2}-\nu_2\frac{\partial g_1}{\partial y_3})+\Phi_{02}(\nu_3\frac{\partial g_2}{\partial y_2}-\nu_2\frac{\partial g_2}{\partial y_3})+\Phi_{03}(\nu_3\frac{\partial g_3}{\partial y_2}-\nu_2\frac{\partial g_3}{\partial y_3})]$$
$$+(\frac{\partial\Phi_{01}}{\partial y_3}-\frac{\partial\widetilde{\Phi}_3}{\partial y_1})[\Phi_{01}(\nu_1\frac{\partial g_1}{\partial y_3}-\nu_3\frac{\partial g_1}{\partial y_1})+\Phi_{02}(\nu_1\frac{\partial g_2}{\partial y_3}-\nu_3\frac{\partial g_2}{\partial y_1})+\Phi_{03}(\nu_1\frac{\partial g_3}{\partial y_3}-\nu_3\frac{\partial g_3}{\partial y_1})]$$
$$+(\frac{\partial\Phi_{02}}{\partial y_1}-\frac{\partial\widetilde{\Phi}_1}{\partial y_2})[\Phi_{01}(\nu_2\frac{\partial g_1}{\partial y_1}-\nu_1\frac{\partial g_1}{\partial y_2})+\Phi_{02}(\nu_2\frac{\partial g_2}{\partial y_1}-\nu_1\frac{\partial g_2}{\partial y_2})+\Phi_{03}(\nu_2\frac{\partial g_3}{\partial y_1}-\nu_1\frac{\partial g_3}{\partial y_2})]$$
$$B=\langle\Phi_0,\nu\rangle\sum_{i,\ell=1}^{3}\frac{\partial\Phi_{0i}}{\partial y_\ell}\frac{\partial g_\ell}{\partial y_i}+\sum_{i=1}^{3}(\mathrm{div}g)\Phi_{0i}\frac{\partial\Phi_{0i}}{\partial\nu}-\sum_{i,j,\ell=1}^{3}\nu_\ell(\frac{\partial g_\ell}{\partial y_j}+\frac{\partial g_j}{\partial y_\ell})\Phi_{0i}\frac{\partial\widetilde{\Phi}_i}{\partial y_j}$$

We mention some useful property for the boundary condition of $\mathrm{rot}\Phi_0$.

Lemma 4.5. We have $\langle\mathrm{rot}\Phi_0,\nu\rangle=0$ on $\partial\Omega$.

(Proof) From the direct calculation near $\partial\Omega$, the boundary condition $\Phi\times\nu=\mathbf{0}$ on $\partial\Omega$ gives this property of $\mathrm{rot}\Phi$. □

[**Evaluation of** A,B]

We see the values A and B in terms of Ω, Φ_0, ρ. For that purpose, we take an arbitray point of $\partial\Omega$ and a special coordinate around the point to calculate A and B. Take any point $O\in\partial\Omega$ and take the orthogonal coordinate $y=(y_1,y_2,y_3)$ centered at O such that $\nu(O)=(1,0,0)$. We express $\partial\Omega$ by a graph $y_1=h(y_2,y_3)$ near O. There exists a $\delta>0$ and C^2 function such that

$$\Omega\cap U(O,\delta)=\{(y_1,y_2,y_3)\in\mathbb{R}^3\mid |y|<\delta,\,y_1<h(y_2,y_3)\}.$$

It holds that $(\partial h/\partial y_2)(0,0)=0$, $(\partial h/\partial y_2)(0,0)=0$. We can assume that two vectors $(0,1,0)$ and $(0,0,1)$ are principal direction in the tangent space of $\partial\Omega$ at O. In this case

$$\frac{\partial\nu}{\partial y_2}(O)=\alpha\,(0,1,0),\quad \frac{\partial\nu}{\partial y_3}(O)=\beta\,(0,0,1).$$

where α and β are the principal curvatures at O.

We note that

$$\nu_1(O)=1,\,\nu_2(O)=0,\,\nu_3(O)=0,\,\Phi_1(O)=\phi(O),\,\Phi_2(O)=0,\,\Phi_3(O)=0,$$
$$\frac{\partial g_1}{\partial y_1}(O)=0,\ \frac{\partial g_2}{\partial y_1}(O)=0,\ \frac{\partial g_3}{\partial y_1}(O)=0,\ \frac{\partial g_1}{\partial y_2}(O)=\frac{\partial\rho}{\partial y_2}(O),$$
$$\frac{\partial g_1}{\partial y_3}(O)=\frac{\partial\rho}{\partial y_3}(O),\ \frac{\partial g_2}{\partial y_2}(O)=\rho(O)\frac{\partial\rho}{\partial y_2}(O)=\rho(O)\alpha,$$
$$\frac{\partial g_2}{\partial y_3}(O)=\frac{\partial g_3}{\partial y_2}(O)=0,\ \frac{\partial g_3}{\partial y_3}(O)=\rho(O)\frac{\partial\nu_3}{\partial y_3}(O)=\rho(O)\,\beta,$$

From the condition $\Phi_0\times\nu=0$ on the boundary, we have

$$\Phi_{0i}(\xi)\nu_j(\xi)-\Phi_{0j}(\xi)\nu_i(\xi)=0\quad(\xi\in\partial\Omega,1\leqq i,j\leqq 3).$$

We can operate $\partial/\partial y_2$, $\partial/\partial y_3$ (tangential derivative) on the above equations at O and get the following properties,

$$\frac{\partial\Phi_{02}}{\partial y_2}(O)=\alpha\,\Phi_{01}(O),\quad \frac{\partial\Phi_{03}}{\partial y_3}(O)=\beta\,\Phi_{01}(O),\quad \frac{\partial\Phi_{01}}{\partial y_1}(O)=-(\alpha+\beta)\Phi_{01}(O),$$

$$\frac{\partial \Phi_{02}}{\partial y_3}(O) = \frac{\partial \Phi_{03}}{\partial y_2}(O) = 0.$$

Substituting these quantities into A and B, we have

$$A(O) = (\frac{\partial \Phi_{01}}{\partial y_3} - \frac{\partial \Phi_{03}}{\partial y_1})\phi(O)\frac{\partial \rho}{\partial y_3}(O) + (\frac{\partial \Phi_{02}}{\partial y_1} - \frac{\partial \Phi_{01}}{\partial y_2})\phi(O)(-1)\frac{\partial \rho}{\partial y_2}(O) = \langle \mathrm{rot}\Phi_0 \times \nabla\rho, \nu\rangle$$

$$\begin{aligned} B(O) &= \frac{\partial \rho}{\partial y_2}(O)\phi(O)(\frac{\partial \Phi_{02}}{\partial y_1} - \frac{\partial \Phi_{01}}{\partial y_2}) + \frac{\partial \rho}{\partial y_3}(O)\phi(O)(\frac{\partial \Phi_{03}}{\partial y_1} - \frac{\partial \Phi_{01}}{\partial y_3}) \\ &\quad +\alpha^2\phi(O)\rho(O) + \beta^2\phi(O)\rho(O) - \rho(O)\phi(O)(\alpha+\beta)^2 \\ &= \phi(O)\langle \nabla\rho \times \mathrm{rot}\Phi_0, \nu\rangle - 2K(O)\rho(O)\phi(O) \end{aligned}$$

Note that $K(O) = \alpha\,\beta$ is the Gaussian curvature of $\partial\Omega$ at O.

Summing up these quantities $A(O), B(O)$ into we get

$$\boldsymbol{\kappa}\int_\Omega |\Phi_0|^2\,dx = \int_{\partial\Omega}\left(|\nabla\Phi_0^{(k)}|^2 - 2|\frac{\partial\Phi_0^{(k)}}{\partial\nu}|^2 + (2K(x) - \lambda_k(0))|\Phi_0^{(k)}(x)|^2\right)\rho\,dS \tag{4.17}$$
$$+2\int_{\partial\Omega}\langle\Phi_0^{(k)},\nu\rangle\langle\mathrm{rot}\Phi_0^{(k)} \times \nabla\rho, \nu\rangle\;dS$$

Thus we have obtained the candidate of $(d\lambda_k/d\epsilon)(0)$ which is the value $\boldsymbol{\kappa}$.

§5. Justification of the formula

In this section, we justifies the formula $\lim_{\epsilon\to 0}(\lambda_k(\epsilon) - \lambda_k(0))/\epsilon = \boldsymbol{\kappa}$. First we prepare some perturbation properties of γ_ϵ for the later calculation.

Lemma 5.1.

$$\frac{\partial\gamma_\epsilon}{\partial y}(y) = I + \frac{\partial g}{\partial y}(y)\epsilon + O(\epsilon^2), \quad \left(\frac{\partial\gamma_\epsilon}{\partial y}(y)\right)^{-1} = I - \frac{\partial g}{\partial y}(y)\epsilon + O(\epsilon^2) \tag{5.1}$$

$$\det\left(\frac{\partial\gamma_\epsilon}{\partial y}\right) = 1 + (\mathrm{div} g)\epsilon + O(\epsilon^2) \tag{5.2}$$

(Proof) These formulas are just the Taylor expansion. □

Lemma 5.2. The map $\gamma_\epsilon : \Omega \longrightarrow \Omega(\epsilon)$ (sufficiently smooth up to $\partial\Omega$) induces the following transformation of the surface element and the unit outward normal vector field as follows,

$$dS_\epsilon = (1 + \epsilon\rho H + O(\epsilon^2))dS, \quad \nu_\epsilon(x) = (\frac{\partial\gamma_\epsilon}{\partial y}^{-1})^t\nu(y)/|(\frac{\partial\gamma_\epsilon}{\partial y}^{-1})^t\nu(y)|. \tag{5.2}$$

See Murat-Simon [13,14] for the proofs of these formulas.

(Proof of the main result)

We consider the variational equation (4.11)-(4.15). First define the value $\boldsymbol{\kappa}$ by the relation (4.17).

We consider an approximate eigenfunction in the following form

$$\widetilde{\Phi}_*(y) = \Phi_0(y) + \epsilon\Psi_0(y) \quad \text{in} \quad \Omega \tag{5.3}$$

where Ψ_0 is a solution of (4.11)-(4.15) (the variational equation). Existence of Ψ_0 is seen from the following arguments. Define the new unknown variable Γ by

$$\Gamma(y) = \Psi(y) - \Xi(y) - \Theta(y) \tag{5.4}$$

where $\Xi(y) := -\Phi_0(y)(\partial g/\partial y)$ and Θ is a solution of the equation,

$$\begin{cases} \mathrm{div}\Theta = \displaystyle\sum_{n,\ell=1}^3 \frac{\partial\Phi_{0n}}{\partial y_\ell}\frac{\partial g_\ell}{\partial y_n} + \sum_{\ell=1}^3\frac{\partial}{\partial y_\ell}(\sum_{n=1}^3\Phi_{0n}\frac{\partial g_n}{\partial y_\ell}) \quad \text{in} \quad \Omega, \\ \Theta(y) = \mathbf{0} \quad \text{on} \quad \partial\Omega. \end{cases} \tag{5.5}$$

This equation has a solution because $\int_\Omega$(the right hand side the first equation)$dy = 0$, which follows the boundary condition of Φ_0 and g. Temam [18] applies to this equation and the existence of Θ follows.

Throught this change of variable from Ψ to Γ, we have

$$\mathrm{rot}\,\mathrm{rot}\,\Gamma - \lambda(0)\Gamma = F,\ \mathrm{div}\,\Gamma = 0 \quad \text{in} \quad \Omega, \quad \Gamma\times\nu = \mathbf{0} \quad \text{on} \quad \partial\Omega. \tag{5.6}$$

where

$$F = \boldsymbol{\kappa}\Phi_0 - \mathrm{div}(\nabla\Phi_0(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t)) + \nabla(\sum_{i,\ell=1}^{3}\frac{\partial\Phi_{0i}}{\partial y_\ell}\frac{\partial g_\ell}{\partial y_i}) \tag{5.7}$$

$$+\nabla(\mathrm{div} g)\cdot\nabla\Phi_0 - \mathrm{rot}\,\mathrm{rot}(\Xi+\Theta) + \lambda(0)(\Xi+\Theta).$$

It suffices to show that F is orthogonal to the kernel of the operator rot rot (with the conditions : "div-free" and the boundary condition). That is to show $F\perp\Phi_0$ and its calculation agrees to the one of the definition of $\boldsymbol{\kappa}$. So we know the existenc of Ψ_0.

Put $\widehat{\Phi}_\epsilon(x) = \Phi_0(\gamma_\epsilon^{-1}(x))$, $\widehat{\Psi}_\epsilon(x) = \Psi_0(\gamma_\epsilon^{-1}(x))$.

As the assumption that the eigenvalue $\lambda(0)$ is simple, $\widetilde{\Phi}_\epsilon$ approaches Φ_0 for $\epsilon\to 0$ (strongly in L^2 and weakly in H^1) because we can multiply $\widetilde{\Phi}_\epsilon$ by -1 for each ϵ if necessary. As in the standard regularity argument of elliptic equation theory (cf. Morrey [12]), we can prove the convergence in higher norm (like C^m). Multiply both sides of the eigenvalue equation (1.12) by $\widehat{\Phi}_\epsilon(x) + \epsilon\,\widehat{\Psi}_\epsilon(x)$ and integrate, we have

$$\int_{\Omega(\epsilon)}\langle \mathrm{rotrot}\,\Phi_\epsilon - \lambda(\epsilon)\Phi_\epsilon, (\widehat{\Phi}_\epsilon + \epsilon\,\widehat{\Psi})\rangle dx = 0 \tag{5.8}$$

By the partial integration, we have

$$\int_{\partial\Omega(\epsilon)}\langle \nu\times\mathrm{rot}\Phi_\epsilon, \widehat{\Phi}_\epsilon + \epsilon\,\widehat{\Psi}_\epsilon\rangle dS_\epsilon + \int_{\partial\Omega(\epsilon)}\langle \nu\times\Phi_\epsilon, \mathrm{rot}(\widehat{\Phi}_\epsilon + \epsilon\,\widehat{\Psi}_\epsilon)\rangle dS_\epsilon \tag{5.9}$$

$$+\int_{\Omega(\epsilon)}\langle\Phi_\epsilon, \mathrm{rot}\,\mathrm{rot}(\widehat{\Phi}_\epsilon + \epsilon\,\widehat{\Psi}_\epsilon)\rangle\, dx - \lambda(\epsilon)\int_{\Omega(\epsilon)}\langle\Phi_\epsilon, (\widehat{\Phi}_\epsilon + \epsilon\,\widehat{\Psi}_\epsilon)\rangle dx = 0$$

Use the boundary of $\Phi_\epsilon\times\nu = 0$ on $\partial\Omega(\epsilon)$ and $\mathrm{rot}\,\mathrm{rot} = \mathrm{grad}\,\mathrm{div} - \Delta$, we have

$$J_1(\epsilon) + J_2(\epsilon) - J_4(\epsilon) - \lambda(\epsilon)J_3(\epsilon) = 0, \tag{5.10}$$

where

$$J_1(\epsilon) := \int_{\partial\Omega(\epsilon)}\langle \mathrm{rot}\,\Phi_\epsilon, (\widehat{\Phi}_\epsilon + \epsilon\,\widehat{\Psi}_\epsilon)\times\nu\rangle dS_\epsilon, \quad J_2(\epsilon) := \int_{\Omega(\epsilon)}\langle \mathrm{rot}\,\Phi_\epsilon, \nabla\mathrm{div}\,(\widehat{\Phi}_\epsilon + \epsilon\,\widehat{\Psi}_\epsilon)\rangle dx$$

$$J_4(\epsilon) := \int_{\Omega(\epsilon)}\langle\Phi_\epsilon, \Delta(\widehat{\Phi}_\epsilon + \epsilon\,\widehat{\Psi}_\epsilon)\rangle dx, \quad J_3(\epsilon) := \int_{\Omega(\epsilon)}\langle\Phi_\epsilon, \widehat{\Phi}_\epsilon + \epsilon\,\widehat{\Psi}_\epsilon\rangle dx$$

We change the variable from x to y by $x = \gamma_\epsilon(y)$ and express $J_j(\epsilon)$ $(j = 1,2,3,4)$ in the form of integration in the domain Ω and calculate the $\epsilon-$expansion to the first order. Later we use $O(\epsilon)$, $O(\epsilon^2)$ for function h_ϵ, they imply $\|h_\epsilon\|_{C^0(\overline{\Omega})} = O(\epsilon)$, $\|h_\epsilon\|_{C^0(\overline{\Omega})} = O(\epsilon^2)$, respectively.

$$J_3(\epsilon) = \int_\Omega\langle\widetilde{\Phi}_\epsilon, \Phi_0 + \epsilon\,\Psi_0\rangle\,\det(\frac{\partial\gamma_\epsilon}{\partial y})dy = \int_\Omega\langle\widetilde{\Phi}_\epsilon, \Phi_0 + \epsilon\,\Psi_0\rangle\,(1 + (\mathrm{div} g)\epsilon + O(\epsilon^2))dy \tag{5.11}$$

$$= \int_\Omega\langle\widetilde{\Phi}_\epsilon, \Phi_0\rangle\, dy + \epsilon\int_\Omega(\langle\widetilde{\Phi}_\epsilon, \Psi_0\rangle + \langle\widetilde{\Phi}_\epsilon, \Phi_0\rangle(\mathrm{div} g))\, dy + O(\epsilon^2)$$

$$J_4(\epsilon) = \int_\Omega\langle\widetilde{\Phi}_\epsilon, \mathrm{div}(\nabla(\Phi_0 + \epsilon\,\Psi_0)(\frac{\partial\gamma_\epsilon}{\partial y})^{-1}[(\frac{\partial\gamma_\epsilon}{\partial y})^{-1}]^t\det(\frac{\partial\gamma_\epsilon}{\partial y}))\rangle dy \tag{5.12}$$

$$= \int_\Omega\langle\widetilde{\Phi}_\epsilon, \mathrm{div}[\nabla(\Phi_0 + \epsilon\,\Psi_0)(I - \frac{\partial g}{\partial y}\epsilon + O(\epsilon^2))(I - (\frac{\partial g}{\partial y})^t\epsilon + O(\epsilon^2))(1 + (\mathrm{div} g)\epsilon + O(\epsilon^2))]\rangle dy$$

$$= \int_\Omega\langle\widetilde{\Phi}_\epsilon, \Delta\Phi_0\rangle dy + \epsilon\int_\Omega\left\{\langle\widetilde{\Phi}_\epsilon, \Delta\Psi_0\rangle - \langle\widetilde{\Phi}_\epsilon, \mathrm{div}[\nabla\Phi_0(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t)]\rangle + \langle\widetilde{\Phi}_\epsilon, \mathrm{div}(\nabla\Phi_0(\mathrm{div} g))\rangle\right\} dy + O(\epsilon^2)$$

$$J_2(\epsilon) = \sum_{\ell=1}^{3} \int_\Omega \widetilde{\Phi}_{\epsilon,\ell} \sum_{p,j,i=1}^{3} \frac{\partial}{\partial y_p} \left(\frac{\partial \Phi_{0i}}{\partial y_j} + \epsilon \frac{\partial \Psi_{0i}}{\partial y_j})[(\frac{\partial \gamma_\epsilon}{\partial y})^{-1}]_{ji} \right) [(\frac{\partial \gamma_\epsilon}{\partial y})^{-1}]_{pl} dy \tag{5.13}$$

$$= \sum_{\ell=1}^{3} \int_\Omega \widetilde{\Phi}_{\epsilon,\ell} \sum_{p,j,i=1}^{3} \frac{\partial}{\partial y_p} \left((\frac{\partial \Phi_{0i}}{\partial y_j} + \epsilon \frac{\partial \Psi_{0i}}{\partial y_j})(\delta(j,i) - \frac{\partial g_j}{\partial y_i}\epsilon + O(\epsilon^2)) \right) (\delta(p,\ell) - \frac{\partial g_p}{\partial y_\ell}\epsilon + O(\epsilon^2)) dy$$

$$= \int_\Omega \langle \widetilde{\Phi}_\epsilon, \nabla \mathrm{div} \Phi_0 \rangle dy + \epsilon \int_\Omega \langle \widetilde{\Phi}_\epsilon, \nabla (\mathrm{div} \Psi_0 - \sum_{j,i=1}^{3} \frac{\partial \Phi_{0i}}{\partial y_j} \frac{\partial g_j}{\partial y_i}) \rangle \, dy$$

$$+ \epsilon \int_\Omega \sum_{\ell,r=1}^{3} \widetilde{\Phi}_{\epsilon\ell} \frac{\partial (\mathrm{div} \Phi_0)}{\partial y_r} (\delta(r,\ell)(\mathrm{div} g) - \frac{\partial g_r}{\partial y_\ell}) dy + O(\epsilon^2) = O(\epsilon^2).$$

We used the condition div $\Phi_0 = 0$ in Ω and the expression for div Ψ_0.

$$J_1(\epsilon) = \int_{\partial\Omega} \langle \Upsilon_\epsilon, (\Phi_0 + \epsilon\, \Psi_0) \times \nu_\epsilon(\gamma_\epsilon(y)) \rangle (1 + H\rho\epsilon + O(\epsilon^2)) dS \tag{5.14}$$

where $\nu_\epsilon(\gamma_\epsilon(y))$ and $\Upsilon_\epsilon(y)$ are given as follows.

$$\nu_\epsilon(\gamma_\epsilon(y)) = [(\frac{\partial \gamma_\epsilon}{\partial y})^{-1}]^t \nu(y) / |[(\frac{\partial \gamma_\epsilon}{\partial y})^{-1}]^t \nu(y)| = \nu + \epsilon[-(\frac{\partial g}{\partial y})^t \nu + \langle (\frac{\partial g}{\partial y})^t \nu, \nu \rangle \nu] + O(\epsilon^2)$$

$$\Upsilon_\epsilon(y) = (\mathrm{rot}_x \widetilde{\Phi}_\epsilon)(\gamma_\epsilon(y)) = \begin{pmatrix} \sum_{j=1}^3 \frac{\partial \widetilde{\Phi}_{\epsilon 3}}{\partial y_j} [(\frac{\partial \gamma_\epsilon}{\partial y})^{-1}]_{j2} - \sum_{j=1}^3 \frac{\partial \widetilde{\Phi}_{\epsilon 2}}{\partial y_j} [(\frac{\partial \gamma_\epsilon}{\partial y})^{-1}]_{j3} \\ \sum_{j=1}^3 \frac{\partial \widetilde{\Phi}_{\epsilon 1}}{\partial y_j} [(\frac{\partial \gamma_\epsilon}{\partial y})^{-1}]_{j3} - \sum_{j=1}^3 \frac{\partial \widetilde{\Phi}_{\epsilon 3}}{\partial y_j} [(\frac{\partial \gamma_\epsilon}{\partial y})^{-1}]_{j1} \\ \sum_{j=1}^3 \frac{\partial \widetilde{\Phi}_{\epsilon 2}}{\partial y_j} [(\frac{\partial \gamma_\epsilon}{\partial y})^{-1}]_{j1} - \sum_{j=1}^3 \frac{\partial \widetilde{\Phi}_{\epsilon 1}}{\partial y_j} [(\frac{\partial \gamma_\epsilon}{\partial y})^{-1}]_{j2} \end{pmatrix}$$

$$(\Phi_0 + \epsilon\, \Psi_0) \times \nu_\epsilon(\gamma_\epsilon(y)) = \Phi_0 \times \nu + \epsilon(\Psi_0 \times \nu) + \epsilon(\Phi_0 \times [-(\frac{\partial g}{\partial y})^t \nu + \langle (\frac{\partial g}{\partial y})^t \nu, \nu \rangle \nu]) + O(\epsilon^2) \tag{5.15}$$

$$= \epsilon(\Psi_0 \times \nu) - \epsilon(\Phi_0 \times (\frac{\partial g}{\partial y})^t \nu) + O(\epsilon^2)$$

We used $\Phi_0 \times \nu = \mathbf{0}$ on $\partial\Omega$ above. Simple calculation gives

$$\Psi_0 \times \nu - \Phi_0 \times (\frac{\partial g}{\partial y})^t \nu = 0 \quad \text{on} \quad \partial\Omega$$

Substitute this and (5.15) into (5.14), we have

$$J_1(\epsilon) = O(\epsilon^2). \tag{5.16}$$

Now we calculate $J_4(\epsilon) + \lambda(\epsilon) J_3(\epsilon)$ with the equations for Ψ and Φ_0 and get the

$$J_4(\epsilon) + \lambda(\epsilon) J_3(\epsilon) = (\lambda(\epsilon) - \lambda(0)) \int_\Omega \langle \widetilde{\Phi}_\epsilon, \Phi_0 \rangle \, dy$$

$$+ (\lambda(\epsilon) - \lambda(0))\epsilon \int_\Omega (\langle \widetilde{\Phi}_\epsilon, \Psi_0 \rangle + \langle \widetilde{\Phi}_\epsilon, \Phi_0 \rangle (\mathrm{div} g) \, dy - \boldsymbol{\kappa}\epsilon \int_\Omega \langle \widetilde{\Phi}_\epsilon, \Phi_0 \rangle \, dy + O(\epsilon^2)$$

Substitute this relation into (5.10) and divide it ϵ and take the limit ϵ with the properties $J_1(\epsilon) = O(\epsilon^2)$, $J_2(\epsilon) = O(\epsilon^2)$, we get

$$\lim_{\epsilon \to 0} \frac{\lambda(\epsilon) - \lambda(0)}{\epsilon} = \boldsymbol{\kappa}.$$

This completes the proof of the main result Theorem 1.5.

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