The lightlike geometry of marginally trapped surfaces in Minkowski space-time

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Abstract

The lightlike geometry of codimension two spacelike submanifolds in Lorentz-Minkowski space has been developed in [15] which is a natural Lorentzian analogue of the classical Euclidean differential geometry of hypersurfaces. In this paper we investigate a special class of surfaces (i.e., *marginally trapped surfaces*) in Minkowski space-time from the view point of the lightlike geometry.

1 Introduction

The notion of trapped surfaces introduced by Roger Penrose [26] is one of the most important subjects in cosmology and general relativity. It plays an principal role for the singularity theorems, the analysis of gravitational collapse, the cosmic censorship hypothesis, Penrose inequality, etc. A marginally trapped surface separates the trapped surfaces from the untrapped one, which is defined to be a spacelike surface with lightlike mean curvature vector field. The surface of a black hole might be the marginally trapped surface. Mathematically, marginally trapped surfaces are viewed as space-time analogues of minimal surfaces in Riemannian manifolds. In Minkowski space-time (i.e., Lorentz-Minkowski 4-space), a marginally trapped surface is defined to be a spacelike surface with the isotropic (i.e, zero or lightlike) mean curvature vector field. If the mean curvature vector field is the zero vector, it is said to be minimal. However, we call it strongly marginally trapped in this paper. In [1] it has been given the Weierstrass-Bryant type representation of a marginally trapped surface in Minkowski space-time.

On the other hand, it has been claimed in [15] that codimension two spacelike submanifolds in Lorentz-Minkowski space play similar roles to hypersurfaces in Euclidean space. For a codimension two spacelike submanifold in Lorentz-Minkowski space, we have two lightlike

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normal directions along the submanifolds, if we use these lightlike normal directions as the unit normal vector field along a hypersurface in Euclidean space, we can define curvatures, etc. This is the basic idea of the lightlike geometry for codimension two spacelike submanifolds [14, 15, 16]. In this paper, inspired by the beautiful survey article [9], we apply the lightlike geometry to marginally trapped surfaces in Minkowski space-time. Then we observe natural geometric properties analogous to minimal surfaces in Euclidean 3-space. For example, a marginally trapped surface is a spacelike surface such that one of the lightcone mean curvatures vanishes (Proposition 3.2 and Corollary 4.4). We say that a spacelike surface is *strongly marginally trapped* if both of the lightcone mean curvatures vanish, which is said to be *minimal* in the previous researches. One of the consequence of the lightlike geometry is that a totally umbilical marginally trapped surface is given by a graph of a smooth function $f(u_1, u_2)$ (Corollary 3.5). Therefore, there are so many complete marginally trapped surfaces in Minkowski space-time compared with minimal surfaces in Euclidean space. Motivated by this fact, we obtain a partial differential equation for a marginally trapped graph surface of two functions $f(u_1, u_2), g(u_1, u_2)$ (Theorem 6.1). As a consequence, a graph surface of a single function $f(u_1, u_2)$ is strongly marginally trapped if and only if f is harmonic. Therefore, the Bernstein theorem [4] does not hold even for strongly marginally trapped surfaces. We also have a characterization of marginally trapped surfaces by the variational problem of the area functionals with respect to lightlike normal directions (Theorem 5.2). We remark that the class of marginally trapped surfaces includes a generalization of the notion of not only minimal surfaces in Euclidean 3-space, maximal surfaces in Lorentz-Minkowski 3-space, CMC-1 surfaces in Hyperbolic 3-space and CMC-1 spacelike surfaces in de Sitter 3-space but also intrinsic flat spacelike surfaces in the lightcone (Theorem 6.3).

In §2 we explain the basic facts and notations of Minkowski space-time. Basic properties of the lightlike geometry are given in §3 and §4. We consider the variation problem with respect to lightlike normal directions in §5. In §6, we consider partial differential equations which marginally trapped or strongly marginally trapped surfaces are satisfied. In §7 we explain how naturally interpret minimal surfaces in Euclidean 3-space, maximal surfaces in Lorentz-Minkowski 3-space, CMC-1 surfaces in Hyperbolic 3-space, CMC-1 spacelike surfaces in de Sitter 3-space and intrinsic flat spacelike surfaces in the lightcone as marginally trapped surfaces.

2 Basic facts and notations in Minkowski space-time

In this section we briefly introduce basic notations on Minkowski space-time. The detailed properties should be referred in [25]. Denote by $\mathbb{R}^4_1$ the Lorentz-Minkowski 4-dimensional space $(\mathbb{R}^4, \langle , \rangle_1)$, with the pseudo scalar product given by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_1 = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3,$$

where $\boldsymbol{x} = (x_0, x_1, x_2, x_3), \boldsymbol{y} = (y_0, y_1, y_2, y_3)$. The *norm* of a vector $\boldsymbol{x}$ is defined to be $\|\boldsymbol{x}\|_1 = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle_1|}$. A vector $\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \backslash \{\boldsymbol{0}\}$ is said to be *spacelike, timelike* or *lightlike* according to $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_1 > 0$, <0, or $= 0$, respectively. For any $\boldsymbol{v} \in \mathbb{R}^4_1 \setminus \{\boldsymbol{0}\}$ and $c \in \mathbb{R}$, the hyperplane with the pseudo-normal $\boldsymbol{v}$is given by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^4_1 \,|\, \langle \boldsymbol{x}, \boldsymbol{v} \rangle_1 = c\}.$$

We say that $HP(\boldsymbol{v}, c)$ is *spacelike, timelike* or *lightlike* provided $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively.

The *Hyperbolic* 3*-space* is defined in this context as the subset

$$H^3_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^4_1 \,|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle_1 = -1,\ x_0 > 0\}.$$

Other well known hypersurfaces in Lorentz-Minkowski space are the *de Sitter* 3*-space:*

$$S^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_1 \,|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle_1 = 1\ \},$$

and the *open lightcone (at the origin):*

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^4_1 \,|\, \boldsymbol{x} \neq \boldsymbol{0},\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle_1 = 0\}.$$

We shall also consider the *future lightcone:*

$$LC^*_+ = \{\boldsymbol{x} \in LC^* \,|\, x_0 > 0\}$$

and the *lightcone unit* 2*-sphere:*

$$S^2_+ = \{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle_1 = 0,\, x_0 = 1\}.$$

If $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$ is a non-zero lightlike vector, we have $x_0 \neq 0$ and denote

$$\widetilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) \in S^2_+.$$

If $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ represents the canonical basis of $\mathbb{R}^4_1$, we shall say that a timelike vector $\boldsymbol{v}$ is *future directed* if $\langle \boldsymbol{v}, \boldsymbol{e}_0 \rangle_1 < 0$.

Given 3 vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{R}^4_1$, we can consider their wedge product,

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x^1_0 & x^1_1 & x^1_2 & x^1_3 \\ x^2_0 & x^2_1 & x^2_2 & x^2_3 \\ x^3_0 & x^3_1 & x^3_2 & x^3_3 \end{vmatrix},$$

where $\boldsymbol{x}_i = (x^i_0, x^i_1, x^i_2, x^i_3),\ i = 1, 2, 3.$

Clearly $\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle_1 = \det(\boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$. So $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is pseudo orthogonal to $\boldsymbol{x}_i, i = 1, 2, 3.$

3 Lightcone curvatures of spacelike surfaces in Minkowski space-time

We introduce the basic geometrical tools for the study of spacelike surfaces in Minkowski space-time. Consider the orientation of $\mathbb{R}^4_1$ provided by the volume form $\boldsymbol{l}_0 \wedge \cdots \wedge \boldsymbol{l}_4$, where $\{\boldsymbol{l}_i\}^4_{i=0}$ is the dual basis of the canonical basis $\{\boldsymbol{e}_i\}^4_{i=0}$. We also give $\mathbb{R}^4_1$ a timelike orientation by choosing $\boldsymbol{e}_0 = (1, 0, 0, 0)$ as a future timelike vector field.

Given a surface M in $\mathbb{R}^4_1$ consider a local parametrization (embedding) $\boldsymbol{X} : U \to \mathbb{R}^4_1$ of M, where U is an open subset of $\mathbb{R}^2$. We write $M = \boldsymbol{X}(U)$ and identify M and U through the embedding $\boldsymbol{X}$. We say that $\boldsymbol{X}$ is *spacelike* if $\boldsymbol{X}_{u_i}$ $i = 1, 2$ are always spacelike vectors. Therefore, the tangent space T_pM of M at p is a spacelike subspace (i.e., consists of spacelike vectors) for any point $p \in M$. In this case, the pseudo-normal space N_pM is a timelike plane (i.e., Lorentz plane). We denote by $N(M)$ the pseudo-normal bundle over M. We can arbitrarily choose a future directed unit timelike normal section $\boldsymbol{n}^T(\underline{u}) \in N_p(M)$, where $p = \boldsymbol{X}(\underline{u})$. We remark that $\boldsymbol{n}^T$ always exists even globally. Here, we say that $\boldsymbol{n}^T$ is *future directed* if $\langle \boldsymbol{n}^T, \boldsymbol{e}_0\rangle_1 < 0$. Therefore we can construct a spacelike unit normal section $\boldsymbol{n}^S(\underline{u}) \in N_p(M)$ by

$$\boldsymbol{n}^S(\underline{u}) = \frac{\boldsymbol{n}^T(\underline{u}) \wedge \boldsymbol{X}_{u_1}(\underline{u}) \wedge \boldsymbol{X}_{u_2}(\underline{u})}{\|\boldsymbol{n}^T(\underline{u}) \wedge \boldsymbol{X}_{u_1}(\underline{u}) \wedge \boldsymbol{X}_{u_2}(\underline{u})\|_1}, \tag{1}$$

and we have

$$\langle \boldsymbol{n}^T, \boldsymbol{n}^T\rangle_1 = -1, \quad \langle \boldsymbol{n}^T, \boldsymbol{n}^S\rangle_1 = 0, \quad \langle \boldsymbol{n}^S, \boldsymbol{n}^S\rangle_1 = 1.$$

Although we could also choose $-\boldsymbol{n}^S(\underline{u})$ as a spacelike unit normal section with the above properties, we shall fix the direction $\boldsymbol{n}^S(\underline{u})$ throughout this section. In the global sense, $\boldsymbol{n}^S$ exists for an orientable surface M. We call $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ a *future directed normal frame* along $M = \boldsymbol{X}(U)$. Clearly, the vectors $\boldsymbol{n}^T(\underline{u}) \pm \boldsymbol{n}^S(\underline{u})$ are lightlike. We choose here $\boldsymbol{n}^T + \boldsymbol{n}^S$ as a lightlike normal vector field along M. Since $\{\boldsymbol{X}_{u_1}(\underline{u}), \boldsymbol{X}_{u_2}(\underline{u})\}$ is a basis of T_pM, the vectors $\{\boldsymbol{n}^T(\underline{u}), \boldsymbol{n}^S(\underline{u}), \boldsymbol{X}_{u_1}(\underline{u}), \boldsymbol{X}_{u_2}(\underline{u})\}$ provide a basis for $T_p\mathbb{R}^4_1$. In [15] it has been shown the following lemma.

Lemma 3.1 *Given two future directed unit timelike normal sections $\boldsymbol{n}^T(\underline{u}), \overline{\boldsymbol{n}}^T(\underline{u}) \in N_p(M)$, the corresponding lightlike normal sections $\boldsymbol{n}^T(\underline{u}) \pm \boldsymbol{n}^S(\underline{u}), \overline{\boldsymbol{n}}^T(\underline{u}) \pm \overline{\boldsymbol{n}}^S(\underline{u})$ are parallel. It follows that*

$$\boldsymbol{n}^T(\widetilde{\underline{u}) \pm \boldsymbol{n}^S}(\underline{u}) = \overline{\boldsymbol{n}}^T(\widetilde{\underline{u}) \pm \overline{\boldsymbol{n}}^S}(\underline{u})$$

Under the identification of M and U through $\boldsymbol{X}$, we have the linear mapping given by the derivative of the lightcone normal vector field $\boldsymbol{n}^T \pm \boldsymbol{n}^S$ at each point $p = \boldsymbol{X}(\underline{u}) \in M$,

$$d_p(\boldsymbol{n}^T \pm \boldsymbol{n}^S) : T_pM \to T_p\mathbb{R}^4_1 = T_pM \oplus N_p(M).$$

Consider the orthogonal projections

$$\pi^t : T_pM \oplus N_p(M) \to T_p(M)$$

and

$$\pi^n : T_p(M) \oplus N_p(M) \to N_p(M).$$

We define the $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$-*shape operator* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(\underline{u})$

$$-d_p(\boldsymbol{n}^T \pm \boldsymbol{n}^S)^t = -\pi^t \circ d_p(\boldsymbol{n}^T \pm \boldsymbol{n}^S) \tag{2}$$

and denote it by $S_p(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$.

The *normal connection with respect to* $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$ of M at p is defined by the linear transformation

$$d_p(\boldsymbol{n}^T \pm \boldsymbol{n}^S)^n = \pi^n \circ d_p(\boldsymbol{n}^T \pm \boldsymbol{n}^S). \tag{3}$$

We also define $S_p(\boldsymbol{n}^T) = -\pi^t \circ d_p\boldsymbol{n}^T$ and $S_p(\boldsymbol{n}^S) = -\pi \circ d_p\boldsymbol{n}^S$. We respectively call these a $\boldsymbol{n}^T$-*shape operator* and a $\boldsymbol{n}^S$-*shape operator* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(\underline{u})$. The eigenvalues of $S_p(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$, denoted by $\{\kappa_i(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(p)\}_{i=1}^2$, are called the *lightcone principal curvatures* with respect to $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$ at p. We also define the $\boldsymbol{n}^T$-*principal curvature* $\kappa_i(\boldsymbol{n}^T)(p)$ (respectively, $\boldsymbol{n}^S$-*principal curvature* $\kappa_i(\boldsymbol{n}^S)(p)$) as the eigenvalues of $S_p(\boldsymbol{n}^T)$ (respectively, $S_p(\boldsymbol{n}^S)$).

Since $S_p(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) = S_p(\boldsymbol{n}^T) \pm S_p(\boldsymbol{n}^S)$, we have $\kappa_i(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(p) = \kappa_i(\boldsymbol{n}^T)(p) \pm \kappa_i(\boldsymbol{n}^S)(p)$. The *lightcone Gauss-Kronecker curvature* with respect to $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$ at p is defined as

$$K_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(p) = \det S_p(\boldsymbol{n}^T, \pm\boldsymbol{n}^S). \tag{4}$$

The *lightcone mean curvature* with respect to $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$ at p is defined to be

$$H_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(p) = \frac{1}{2}\mathrm{Trace}\, S_p(\boldsymbol{n}^T, \pm\boldsymbol{n}^S).$$

On the other hand, the *mean curvature vector* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(\underline{u})$ is given by

$$\mathfrak{H}(p) = \frac{1}{2}\mathrm{Trace}\, S_p(\boldsymbol{n}^T)\boldsymbol{n}^T(\underline{u}) + \frac{1}{2}\mathrm{Trace}\, S_p(\boldsymbol{n}^S)\boldsymbol{n}^S(\underline{u}).$$

We have the following proposition.

Proposition 3.2 *Under the above notations, the following conditions are equivalent.*
(1) $H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)(p) = 0$ *or* $H_\ell(\boldsymbol{n}^T, -\boldsymbol{n}^S)(p) = 0$.
(2) *The mean curvature vector* $\mathfrak{H}(p)$ *is an isotropic vector,*
where a vector is isotropic if it is a zero vector or a lightlike vector.

Proof. $S_p(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) = S_p(\boldsymbol{n}^T) \pm S_p(\boldsymbol{n}^S)$, we have

$$\mathrm{Trace}\, S_p(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) = \mathrm{Trace}\, S_p(\boldsymbol{n}^T) \pm \mathrm{Trace}\, S_p(\boldsymbol{n}^S).$$

On the other hand,

$$4\langle \mathfrak{H}(p), \mathfrak{H}(p)\rangle_1 = -\mathrm{Trace}\, S_p(\boldsymbol{n}^T)^2 + \mathrm{Trace}\, S_p(\boldsymbol{n}^S)^2.$$

Thus, $\langle \mathfrak{H}(p), \mathfrak{H}(p)\rangle_1 = 0$ if and only if $\mathrm{Trace}\, S_p(\boldsymbol{n}^T) = \pm\mathrm{Trace}\, S_p(\boldsymbol{n}^S)$. This condition is equivalent to the condition (1). $\square$

We denote that $H(\boldsymbol{n}^T)(p) = (\mathrm{Trace}\, S_p(\boldsymbol{n}^T))/2$ and $H(\boldsymbol{n}^S)(p) = (\mathrm{Trace}\, S_p(\boldsymbol{n}^S))/2$, which we respectively call the *mean curvature* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(\underline{u})$ *with respect to* $(\boldsymbol{n}^T, \boldsymbol{n}^S)$.

On the other hand, there is a concept of *trapped surfaces* in a space-time introduced by Penrose in [26] which plays an extremely important role in cosmology and general relativity. In terms of mean curvature vector, a spacelike surface in a space-time is *marginally trapped* if its mean curvature vector is isotropic at each point. The above proposition asserts that $M \subset \mathbb{R}^4_1$ is marginally trapped if and only if $H_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) \equiv 0$ for any future directed normal frame $(\boldsymbol{n}^T, \boldsymbol{n}^S)$. We also say that $M = \boldsymbol{X}(U)$ is a *strongly marginally trapped surface* if the mean curvature vector $\mathfrak{H}$ is zero at any point. In some articles, this notion is called *minimal.* However, this class of surfaces includes both the minimal surfaces in Euclidean space and the maximal surfaces in Minkowski 3-space (cf., §7). Therefore we call these strongly marginally trapped surfaces. By the theorem on space-time singularities in [26], there are no compact marginally trapped surfaces in Minkowski space-time. By Proposition 3.2, we have the following corollary.

Corollary 3.3 *Under the same notations as those in Proposition 3.2, we have the followings*:
(1) M *is marginally trapped if and only if* $H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S) \equiv 0$ *or* $H_\ell(\boldsymbol{n}^T, -\boldsymbol{n}^S) \equiv 0$.
(2) M *is strongly marginally trapped if and only if* $H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S) \equiv 0$ *and* $H_\ell(\boldsymbol{n}^T, -\boldsymbol{n}^S) \equiv 0$.

On the other hand, we say that a point p is a $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$*-umbilical point* if all the principal curvatures coincide at p and thus $S_p(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) = \kappa(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(p)1_{T_pM}$, for some function κ. We say that M is *totally* $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$*-umbilical* if all points on M are $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$-umbilical. In [15] we have shown that a totally $(\boldsymbol{n}^T, +\boldsymbol{n}^S)$-umbilical or $(\boldsymbol{n}^T, -\boldsymbol{n}^S)$-umbilical spacelike surface with the vanishing principal curvature in Minkowski space-time is a spacelike surface in a lightlike hyperplane. Therefore we have the following proposition.

Proposition 3.4 *A spacelike surface M is marginally trapped with $H_\ell(\boldsymbol{n}^T, \sigma\boldsymbol{n}^S) = 0$ and totally $(\boldsymbol{n}^T, \sigma\boldsymbol{n}^S)$-umbilical if and only if M is a spacelike surface in a lightlike hyperplane, where $\sigma = +$ or $\sigma = -$. Moreover, M is strongly marginally trapped and totally $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$-umbilical if and only if it is a spacelike plane.*

Proof. By definition, we have $H_\ell(\boldsymbol{n}^T, \sigma\boldsymbol{n}^S) = (\kappa_1(\boldsymbol{n}^T, \sigma\boldsymbol{n}^S)(p) + \kappa_2(\boldsymbol{n}^T, \sigma\boldsymbol{n}^S)(p))/2$. If M is marginally trapped with $H_\ell(\boldsymbol{n}^T, \sigma\boldsymbol{n}^S) = 0$ and totally $(\boldsymbol{n}^T, \sigma\boldsymbol{n}^S)$-umbilical, then $\kappa_1(\boldsymbol{n}^T, \sigma\boldsymbol{n}^S) = \kappa_2(\boldsymbol{n}^T, \sigma\boldsymbol{n}^S) = 0$. Therefore, M is a spacelike surface in a lightlike hyperplane. By the similar arguments to the above, other assertions hold. $\square$

By the above proposition, we have the following corollary.

Corollary 3.5 *A spacelike surface M is marginally trapped and totally $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilical if and only if up to rigid motions of $\mathbb{R}^4_1$, M is given by*

$$\boldsymbol{X}_f(u_1, u_2) = (f(u_1, u_2), f(u_1, u_2), u_1, u_2)),$$

for a smooth function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Proof. By a straight forward calculation, we can show that $\boldsymbol{X}_f$ is always spacelike. If we consider the lightlike vector $\boldsymbol{v} = (1, 1, 0, 0)$, then we have $\langle \boldsymbol{X}_f(u_1, u_2), \boldsymbol{v}\rangle_1 = 0$, so that $\boldsymbol{X}_f$ is a spacelike surface in the lightlike hyperplane $HP(\boldsymbol{v}, 0)$. For the converse, if M is a spacelike surface in a lightlike hyperplane, by a rigid motion of $\mathbb{R}^4_1$, M can be included in $HP(\boldsymbol{v}, 0)$. We consider a projection $\pi : HP(\boldsymbol{v}, 0) \longrightarrow \mathbb{R}^2$ defined by $\pi(x_0, x_0, x_1, x_2) = (x_1, x_2)$. Then the fiber of π is directed to the lightlike direction $\boldsymbol{v}$. Since M is spacelike, $\pi|_M : M \longrightarrow \mathbb{R}^2$ is a diffeomorphism, so that there exists a smooth function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $\boldsymbol{X}_f(\mathbb{R}^2) = M$. This completes the proof. $\square$

Remark 3.6 The above corollary indicates that there are so many complete marginally trapped surfaces in Minkowski space-time compared with minimal surfaces in Euclidean space etc. In [6], Chen and Ishikawa have shown that if M is a biharmonic surface in $\mathbb{R}^4_1$ with flat normal connection and marginally trapped, then up to rigid motions M is given by the surface $\boldsymbol{X}_f(\mathbb{R}^2)$ in the above corollary. We remark that if M is a subset of a lightlike hyperplane, the normal connection is flat. Related results and problems are presented in the survey article [7].

We deduce now the *lightcone Weingarten formula:* Since $\boldsymbol{X}_{u_i}$ $(i = 1, 2)$ are spacelike vectors, we have a Riemannian metric (the *hyperbolic first fundamental form*) on M defined by

$$ds^2 = g_{11}du_1^2 + 2g_{12}du_1du_2 + g_{22}du_2^2,$$

where $g_{ij}(\underline{u}) = \langle \boldsymbol{X}_{u_i}(\underline{u}), \boldsymbol{X}_{u_j}(\underline{u})\rangle_1$ for any $\underline{u} \in U$.

We also have the *lightcone second fundamental form*, which is defined as the second fundamental form associated to the normal vector field $\boldsymbol{n}^T + \boldsymbol{n}^S$. This is given by

$$h_{ij}(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(\underline{u}) = \langle -(\boldsymbol{n}^T \pm \boldsymbol{n}^S)_{u_i}(\underline{u}), \boldsymbol{X}_{u_j}(\underline{u})\rangle_1; i = 1, 2,$$

for any $\underline{u} \in U$. In [15], we have shown the following proposition.

Proposition 3.7 (The lightcone Weingarten formula) *Under the above notations, we have the following lightcone Weingarten formula with respect to* $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$*:*

a) $(\boldsymbol{n}^T \pm \boldsymbol{n}^S)_{u_i} = \langle \boldsymbol{n}^S, \boldsymbol{n}^T_{u_i}\rangle_1(\boldsymbol{n}^T \pm \boldsymbol{n}^S) - \sum_{j=1}^{2} h_i^j(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)\boldsymbol{X}_{u_j}; i = 1, 2,$

b) $\pi^t \circ (\boldsymbol{n}^T \pm \boldsymbol{n}^S)_{u_i} = -\sum_{j=1}^{2} h_i^j(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)\boldsymbol{X}_{u_j}; i = 1, 2,$

where $\left(h_i^j(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)\right) = \left(h_{ik}(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right) = \left(g_{kj}\right)^{-1}$.

The following corollary provides an explicit expression of the lightcone curvature in terms of the Riemannian metric and the lightcone second fundamental invariant.

Corollary 3.8 *Under the same notations as in the above proposition, the lightcone Gauss-Kronecker curvature with respect to* $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$ *is given by*

$$K_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) = \frac{\det\left(h_{ij}(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)\right)}{\det\left(g_{\alpha\beta}\right)}. \tag{5}$$

Since $\langle -(\boldsymbol{n}^T \pm \boldsymbol{n}^S)(\underline{u}), \boldsymbol{X}_{u_j}(\underline{u})\rangle_1 = 0$, we have that

$$h_{ij}(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(\underline{u}) = \langle \boldsymbol{n}^T(\underline{u}) \pm \boldsymbol{n}^S(\underline{u}), \boldsymbol{X}_{u_iu_j}(\underline{u})\rangle_1.$$

So the lightcone second fundamental form at $p_0 = \boldsymbol{X}(\underline{u}_0)$ depends only on the values of the vector fields $\boldsymbol{n}^T + \boldsymbol{n}^S$ and $\boldsymbol{X}_{u_iu_j}$ at the point p_0. Consequently the lightcone curvature depends only on $\boldsymbol{n}^T(\underline{u}_0) \pm \boldsymbol{n}^S(\underline{u}_0)$, $\boldsymbol{X}_{u_i}(\underline{u}_0)$ and $\boldsymbol{X}_{u_iu_j}(\underline{u}_0)$ regardless the choice of the normal vector fields $\boldsymbol{n}^T$ and $\boldsymbol{n}^S$. We write as $K_\ell(\boldsymbol{n}^T_0, \pm\boldsymbol{n}^S_0)(\underline{u}_0)$ the lightcone curvature at p_0 with respect to $(\boldsymbol{n}^T_0, \pm\boldsymbol{n}^S_0) = (\boldsymbol{n}^T(\underline{u}_0), \pm\boldsymbol{n}^S(\underline{u}_0))$. It thus makes sense to say that a point p_0 is $(\boldsymbol{n}^T_0, \pm\boldsymbol{n}^S_0)$-*umbilic* for the lightcone $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$-shape operator at p_0 just depends on the normal vectors $(\boldsymbol{n}^T_0, \pm\boldsymbol{n}^S_0)$. Analogously, we say that the point p_0 is a $(\boldsymbol{n}^T_0, \pm\boldsymbol{n}^S_0)$-*parabolic point* of M if $K_\ell(\boldsymbol{n}^T_0, \pm\boldsymbol{n}^S_0)(\underline{u}_0) = 0$. And we say that p_0 is a $(\boldsymbol{n}^T_0, \pm\boldsymbol{n}^S_0)$-*flat point* if it is $(\boldsymbol{n}^T_0, \pm\boldsymbol{n}^S_0)$-umbilic and $K_\ell(\boldsymbol{n}^T_0, \pm\boldsymbol{n}^S_0)(\underline{u}_0) = 0$.

On the other hand, we define a normal vector field $\mathfrak{K}(p) = \det S_p(\boldsymbol{n}^T)\boldsymbol{n}^T(\underline{u}) + \det S_p(\boldsymbol{n}^S)\boldsymbol{n}^S(\underline{u})$, where $p = \boldsymbol{X}(\underline{u})$. We call $\mathfrak{K}$ the *Gaussian curvature* vector field along $M = \boldsymbol{X}(U)$. We also denote that $K(\boldsymbol{n}^T)(p) = \det S_p(\boldsymbol{n}^T)$ and $K(\boldsymbol{n}^S)(p) = \det S_p(\boldsymbol{n}^S)$. We respective call $K(\boldsymbol{n}^T)(p)$ and $K(\boldsymbol{n}^S)(p)$ the *Gauss-Kronecker curvature* of $M = \boldsymbol{X}(U)$ at p with respect to $\boldsymbol{n}^T$ and $\boldsymbol{n}^S$. The *mean curvature* of $M = \boldsymbol{X}(U)$ at p with respect to $\boldsymbol{n}^T$ (respectively, $\boldsymbol{n}^S$) is defined to be $H(\boldsymbol{n}^T)(p) = (\mathrm{Trace}\, S_p(\boldsymbol{n}^T))/2$ (respectively, $H(\boldsymbol{n}^S)(p) = (\mathrm{Trace}\, S_p(\boldsymbol{n}^S))/2$). We also define the *second fundamental invariants* $h_{ij}(\boldsymbol{n}^T) = -\langle \boldsymbol{n}^T_{u_i}, \boldsymbol{X}_{u_j}\rangle_1$ with respect to $\boldsymbol{n}^T$ and $h_{ij}(\boldsymbol{n}^S) = -\langle \boldsymbol{n}^S_{u_i}, \boldsymbol{X}_{u_j}\rangle_1$ with respect to $\boldsymbol{n}^S$ respectively. By the standard arguments, we have the following Weingarten formulae.

Proposition 3.9 *We have*

$$\pi^t \circ \boldsymbol{n}^T_{u_i} = -\sum_{j=1}^{2} h_i^j(\boldsymbol{n}^T)\boldsymbol{X}_{u_j} \text{ and } \pi^t \circ \boldsymbol{n}^S_{u_i} = -\sum_{j=1}^{2} h_i^j(\boldsymbol{n}^S)\boldsymbol{X}_{u_j},$$

where $(h_i^j(\boldsymbol{n}^T)) = (h_{ik}(\boldsymbol{n}^T))(g^{kj})$ *and* $(h_i^j(\boldsymbol{n}^S)) = (h_{ik}(\boldsymbol{n}^S))(g^{kj})$.

Thus, we have the following corollary.

Corollary 3.10 *We have*

$$K(\boldsymbol{n}^T) = \frac{\det(h_{ij}(\boldsymbol{n}^T))}{\det(g_{\alpha\beta})} \text{ and } K(\boldsymbol{n}^S) = \frac{\det(h_{ij}(\boldsymbol{n}^S))}{\det(g_{\alpha\beta})}.$$

We also get in this context the *lightcone Gauss equations* as we shall see next. Since $\boldsymbol{X}(U) = M$ is a Riemannian manifold, it makes sense to consider the *Christoffel symbols*:

$$\left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\} = \frac{1}{2} \sum_m g^{km} \left\{ \frac{\partial g_{jm}}{\partial u_i} + \frac{\partial g_{im}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_m} \right\}.$$

Proposition 3.11 *Let $\boldsymbol{X} : U \longrightarrow \mathbb{R}^4_1$ be a spacelike surface. Then we have the following lightcone Gauss equations:*

$$\boldsymbol{X}_{u_i u_j} = \sum_k \left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\} \boldsymbol{X}_{u_k} - h_{ij}(\boldsymbol{n}^T)\boldsymbol{n}^T + h_{ij}(\boldsymbol{n}^S)\boldsymbol{n}^S.$$

Proof. Since $\{\boldsymbol{n}^T, \boldsymbol{n}^S, \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_2}\}$ is a frame of $\mathbb{R}^4_1$, we can write $\boldsymbol{X}_{u_i u_j} = \sum_k \Gamma^k_{ij} \boldsymbol{X}_{u_k} + \Gamma_{ij}\boldsymbol{n}^T + \Lambda_{ij}\boldsymbol{n}^S$. We now have

$$\langle \boldsymbol{X}_{u_i u_j}, \boldsymbol{X}_{u_\ell} \rangle_1 = \sum_k \Gamma^k_{ij} \langle \boldsymbol{X}_{u_k}, \boldsymbol{X}_{u_\ell} \rangle_1 = \sum_k \Gamma^k_{ij} g_{k\ell}.$$

Since $\frac{\partial g_{i\ell}}{\partial u_j} = \langle \boldsymbol{X}_{u_i u_j}, \boldsymbol{X}_{u_\ell}\rangle_1 + \langle \boldsymbol{X}_{u_i}, \boldsymbol{X}_{u_\ell u_j}\rangle_1$ and $\boldsymbol{X}_{u_i u_j} = \boldsymbol{X}_{u_j u_i}$, we get $\Gamma^k_{ij} = \Gamma^k_{ji}$, $\Gamma_{ij} = \Gamma_{ji}$, $\Gamma^{ij} = \Gamma^{ji}$. Then by exactly the same calculation as those applied in the case of surfaces in Euclidean space, it follows $\Gamma^k_{ij} = \left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\}$.

On the other hand, $-\Gamma_{ij} = \langle \boldsymbol{X}_{u_i u_j}, \boldsymbol{n}^T \rangle_1 = h_{ij}(\boldsymbol{n}^T)$. Moreover $\Lambda_{ij} = \langle \boldsymbol{X}_{u_i u_j}, \boldsymbol{n}^S \rangle_1 = h_{ij}(\boldsymbol{n}^S)$. This completes the proof. □

Consider the Riemannian curvature tensor

$$R^\ell_{ijk} = \frac{\partial}{\partial u_k} \left\{ \begin{matrix} \ell \\ i\ j \end{matrix} \right\} - \frac{\partial}{\partial u_j} \left\{ \begin{matrix} \ell \\ i\ k \end{matrix} \right\} + \sum_m \left\{ \begin{matrix} m \\ i\ j \end{matrix} \right\} \left\{ \begin{matrix} \ell \\ m\ k \end{matrix} \right\} - \sum_m \left\{ \begin{matrix} m \\ i\ k \end{matrix} \right\} \left\{ \begin{matrix} \ell \\ m\ j \end{matrix} \right\}.$$

By the similar calculations to those of the classical differential geometry on surfaces in Euclidean space and the fact $\boldsymbol{X}_{u_i u_j u_k} = \boldsymbol{X}_{u_i u_k u_j}$ lead to the formula

$$R^\ell_{ijk} = \sum_a \left\{ -h_{ij}(\boldsymbol{n}^T) h_{ka}(\boldsymbol{n}^T) + h_{ij}(\boldsymbol{n}^S) h_{ka}(\boldsymbol{n}^S) + h_{ik}(\boldsymbol{n}^T) h_{ja}(\boldsymbol{n}^T) - h_{ik}(\boldsymbol{n}^S) h_{ja}(\boldsymbol{n}^S) \right\} g^{a\ell}.$$

We also consider the tensor $R_{ijk\ell} = \sum_m g_{im} R^m_{jk\ell}$. Then we have the following proposition.

Proposition 3.12 *We have*

$$R_{ijk\ell} = -h_{jk}(\boldsymbol{n}^T) h_{i\ell}(\boldsymbol{n}^T) + h_{j\ell}(\boldsymbol{n}^T) h_{ik}(\boldsymbol{n}^T) + h_{jk}(\boldsymbol{n}^S) h_{i\ell}(\boldsymbol{n}^S) - h_{j\ell}(\boldsymbol{n}^S) h_{ik}(\boldsymbol{n}^S).$$

The *intrinsic Gauss curvature* (or, *sectional curvature*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(\underline{u})$ is defined by $K_I(p) = -R_{1212}(\underline{u}) / \det(g_{\alpha\beta}(\underline{u}))$. We can show that the following "Theorema Egregium".

Theorem 3.13 *Let K_I be the intrinsic Gauss curvature of $M = \boldsymbol{X}(U)$. Then we have*

$$K_I(p) = -K(\boldsymbol{n}^T)(p) + K(\boldsymbol{n}^S)(p).$$

Proof. By Propositions 3.11, we have

$$R_{1212} = \det(h_{ij}(\boldsymbol{n}^T)) - \det(h_{ij}(\boldsymbol{n}^S)).$$

Thus, by Corollary 3.9, we have

$$K_I = -\frac{\det(h_{ij}(\boldsymbol{n}^T))}{\det(g_{\alpha\beta})} + \frac{\det(h_{ij}(\boldsymbol{n}^S))}{\det(g_{\alpha\beta})} = -K(\boldsymbol{n}^T) + K(\boldsymbol{n}^S).$$

This completes the proof. $\square$

We give the following characterization of intrinsic flat surfaces (i.e., $K_I \equiv 0$) by using the Gaussian curvature vector $\mathfrak{K}(p) = K(\boldsymbol{n}^T)(p)\boldsymbol{n}^T(p) + K(\boldsymbol{n}^S)(p)\boldsymbol{n}^S(p)$.

Proposition 3.14 *For a spacelike surface $M = \boldsymbol{X}(U)$ in $\mathbb{R}^4_1$, the following conditions are equivalent*:
1) $K_I(p) = 0$ *at* $p = \boldsymbol{X}(\underline{u})$.
2) $\mathfrak{K}(p) = \boldsymbol{0}$ *or* $\mathfrak{K}(p)$ *is parallel to the lightlike vector* $\boldsymbol{n}^T(p) + \boldsymbol{n}^S(p)$.

Proof. Since $K_I = -K(\boldsymbol{n}^T) + K(\boldsymbol{n}^S)$, $K_I(p) = 0$ if and only if $K(\boldsymbol{n}^T)(p) = K(\boldsymbol{n}^S)(p)$. Therefore, we have

$$\mathfrak{K}(p) = K(\boldsymbol{n}^T)(p)\boldsymbol{n}^T(p) + K(\boldsymbol{n}^S)(p)\boldsymbol{n}^S(p) = K(\boldsymbol{n}^T)(\boldsymbol{n}^T(p) + \boldsymbol{n}^S(p)).$$

For the converse, if $\mathfrak{K}(p) = \lambda(\boldsymbol{n}^T(p) + \boldsymbol{n}^S(p))$, then $K(\boldsymbol{n}^T)(p) = \lambda = K(\boldsymbol{n}^S)(p)$, so that $K_I(p) = 0$. $\square$

We also say that $M = \boldsymbol{X}(U)$ is an *extrinsic flat surface* if $\mathfrak{K} \equiv \boldsymbol{0}$.

4 Normalized lightcone curvatures

Given a spacelike embedding $\boldsymbol{X} : U \to \mathbb{R}^4_1$ from an open subset $U \subset \mathbb{R}^2$, and a point $p = \boldsymbol{X}(\underline{u})$, consider a future directed unit timelike normal section $\boldsymbol{n}^T(\underline{u}) \in N_p(M)$ and the corresponding spacelike unit normal section $\boldsymbol{n}^S(\underline{u}) \in N_p(M)$ constructed in the previous section. For given another future directed unit timelike normal section $\overline{\boldsymbol{n}}^T(\underline{u})$, we have $\widetilde{(\boldsymbol{n}^T \pm \boldsymbol{n}^S)}(\underline{u}) = \widetilde{(\overline{\boldsymbol{n}}^T \pm \overline{\boldsymbol{n}}^S)}(\underline{u}) \in S^2_+$, it is possible to define a *lightcone Gauss map* of $M = \boldsymbol{X}(U)$ as

$$\begin{array}{rccl} \widetilde{\mathbb{L}}^{\pm} : & U & \longrightarrow & S^2_+ \\ & \underline{u} & \longmapsto & \widetilde{(\boldsymbol{n}^T \pm \boldsymbol{n}^S)}(\underline{u}). \end{array}$$

This induces a linear mapping $d\widetilde{\mathbb{L}}^{\pm}_p : T_pM \to T_p\mathbb{R}^4_1$ under the identification of U and M, where $p = \boldsymbol{X}(\underline{u})$. The following proposition has been shown in [15].

Proposition 4.1 *Under the above notation, we have the following normalized lightcone Weingarten formula:*

$$\pi^t \circ \widetilde{\mathbb{L}}^{\pm}_{u_i} = -\sum_{j=1}^{2} \frac{1}{\ell_0^{\pm}(\underline{u})} h_i^j(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)\boldsymbol{X}_{u_j}, \tag{6}$$

where $\mathbb{L}^{\pm}(\underline{u}) = (\ell_0^{\pm}(\underline{u}), \ell_1^{\pm}(\underline{u}), \ell_2^{\pm}(\underline{u}), \ell_3^{\pm}(\underline{u}))$.

We now define the *normalized lightcone second fundamental invariant* by

$$\widetilde{h}[\pm]_{ij}(\underline{u}) = \frac{1}{\ell_0(\underline{u})} h_{ij}(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(\underline{u}).$$

Since $h_{ij}(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(\underline{u}) = \langle(\boldsymbol{n}^T \pm \boldsymbol{n}^S)(\underline{u}), \boldsymbol{X}_{u_iu_j}(\underline{u})\rangle_1$, we have

$$\widetilde{h}[\pm]_{ij}(\underline{u}) = \frac{1}{\ell_0(\underline{u})}\langle(\boldsymbol{n}^T \pm \boldsymbol{n}^S)_{u_i}(\underline{u}), \boldsymbol{X}_{u_j}(\underline{u})\rangle_1 = \langle\widetilde{\mathbb{L}}^{\pm}(\underline{u}), \boldsymbol{X}_{u_iu_j}(\underline{u})\rangle_1 = \langle -\widetilde{\mathbb{L}}^{\pm}_{u_i}(\underline{u}), \boldsymbol{X}_{u_j}(\underline{u})\rangle_1.$$

We need the following detailed formula.

Lemma 4.2 *Under the above notations, we have*

$$\widetilde{\mathbb{L}}^{\pm}_{u_i} = \left(\frac{-\ell_{0u_i}}{\ell_0^2} + \frac{\langle \boldsymbol{n}^S, \boldsymbol{n}^T_{u_i}\rangle_1}{\ell_0}\right)(\boldsymbol{n}^T \pm \boldsymbol{n}^S) - \sum_{j=1}^{2} \widetilde{h}[\pm]_i^j \boldsymbol{X}_{u_j},$$

where $(\widetilde{h}[\pm]_i^j) = (\widetilde{h}[\pm]_{ik})(g^{kj})$.

Proof.
In the proof of Proposition 4.1, we have $\ell_0\widetilde{\mathbb{L}}^{\pm}_{u_i} = (\boldsymbol{n}^T \pm \boldsymbol{n}^S)_{u_i} - \ell_{0u_i}\widetilde{\mathbb{L}}^{\pm}$. Thus the above formula directly follows form the assertion a) of Proposition 3.3 and the relation $\ell_0\widetilde{\mathbb{L}}^{\pm} = \boldsymbol{n}^T \pm \boldsymbol{n}^S$. $\square$

We call the linear transformation $\widetilde{S}^{\pm}_p = -\pi^t \circ d\widetilde{\mathbb{L}}^{\pm}_p$ the *normalized lightcone shape operator* of $M = \boldsymbol{X}(U)$ at p. The *normalized lightcone Gauss-Kronecker curvature* of $M = \boldsymbol{X}(U)$ is defined to be $\widetilde{K}^{\pm}_\ell(\underline{u}) = \det \widetilde{S}^{\pm}_p$. We say that $p = \boldsymbol{X}(\underline{u})$ is a *lightlike parabolic point* if $\widetilde{K}^{\pm}_\ell(\underline{u}) = 0$. We also define the *normalized lightcone mean curvature* of $M = \boldsymbol{X}(U)$ by $\widetilde{H}^{\pm}_\ell(p) = \operatorname{Trace}\widetilde{S}^{\pm}_p/2$. The eigenvalues $\{\widetilde{\kappa}^{\pm}_i(p)\}_{i=1}^2$ of $\widetilde{S}^{\pm}_p$ are called *normalized lightcone principal curvatures*. It follows from the above formula that $\widetilde{\kappa}^{\pm}_i(p) = (1/\ell_0)\kappa_i(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(p)$. Clearly, the eigenvectors of $\widetilde{S}^{\pm}_p$ coincide with the lightcone principal directions with respect to $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$, for any future directed frame $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$ on M, therefore, we can refer to the $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$-lightcone principal configuration, simply as the *lightcone principal configuration* on M. The $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$-umbilics shall be called *lightlike umbilics*. We say that $M = \boldsymbol{X}(U)$ is *totally lightlike umbilical* if all points on M are lightlike umbilic. The point p is called a *lightlike flat point* if p is both lightlike umbilic and parabolic. The spacelike submanifold $M = \boldsymbol{X}(U)$ is called *lightlike flat* provided every point of M is lightlike flat. As observed in the previous section, the lightcone principal configuration is preserved by Lorentz transformations, although the normalized lightcone principal curvatures are not. We remark that the normalized principal curvatures are in variant under the $SO(3)$-action and parallel translations, where $SO(3)$ is the canonical subgroup of $SO_0(1,3)$.

By Proposition 4.1, we have the following relations:

$$\widetilde{K}_\ell^{\pm}(p) = \left(\frac{1}{\ell_0^{\pm}(p)}\right)^2 K_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(p), \quad \widetilde{H}_\ell^{\pm}(p) = \frac{1}{\ell_0^{\pm}(p)} H_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(p).$$

Therefore, we have the following proposition.

Proposition 4.3 *Let $M = \boldsymbol{X}(U)$ be a spacelike surface in $\mathbb{R}^4_1$. Then*

1) $\widetilde{K}_\ell^{\pm}(p) = 0$ if and only if $K_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(p) = 0$ for any future directed normal frame $(\boldsymbol{n}^T, \boldsymbol{n}^S)$.

2) $\widetilde{H}_\ell^{\pm}(p) = 0$ if and only if $H_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(p) = 0$ for any future directed normal frame $(\boldsymbol{n}^T, \boldsymbol{n}^S)$.

We have the following corollary.

Corollary 4.4 *Let $M = \boldsymbol{X}(U)$ be a spacelike surface in $\mathbb{R}^4_1$. Then M is a marginally trapped surface if and only if $\widetilde{H}_\ell^{+} \equiv 0$ or $\widetilde{H}_\ell^{-} \equiv 0$. Moreover, $M = \boldsymbol{X}(U)$ is strongly marginally trapped surface if and only if $\widetilde{H}_\ell^{+} \equiv \widetilde{H}_\ell^{-} \equiv 0$.*

On the other hand, we say that $(u_1, u_2) \in U$ is an *isothermal parameter* of $M = \boldsymbol{X}(U)$ if $g_{11} = g_{22}$ and $g_{12} = 0$. For any two dimensional Riemannian manifold, there exists an isothermal parameter at any point [8].

Proposition 4.5 *Let $\boldsymbol{X} : U \longrightarrow \mathbb{R}^4_1$ be a spacelike surface with an isothermal parameter $(u_1, u_2) \in U$. Then we have*

$$\boldsymbol{X}_{u_1u_1} + \boldsymbol{X}_{u_2u_2} = g_{11}\left(-\ell_0^{+}(\widetilde{H}_\ell^{+} + \widetilde{H}_\ell^{-})\boldsymbol{n}^T + \ell_0^{-}(\widetilde{H}_\ell^{+} - \widetilde{H}_\ell^{-})\boldsymbol{n}^S\right).$$

Proof. Since (u_1, u_2) is an isothermal parameter, we have

$$\langle \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_1}\rangle_1 = \langle \boldsymbol{X}_{u_2}, \boldsymbol{X}_{u_2}\rangle_1, \; \langle \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_2}\rangle_1 = 0.$$

It follows that

$$\langle \boldsymbol{X}_{u_1u_1}, \boldsymbol{X}_{u_1}\rangle_1 = \langle \boldsymbol{X}_{u_1u_2}, \boldsymbol{X}_{u_2}\rangle_1 = -\langle \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_2u_2}\rangle_1.$$

Therefore, we have $\langle \boldsymbol{X}_{u_1u_1} + \boldsymbol{X}_{u_2u_2}, \boldsymbol{X}_{u_1}\rangle_1 = 0$. By the similar arguments to the above, we have $\langle \boldsymbol{X}_{u_1u_1} + \boldsymbol{X}_{u_2u_2}, \boldsymbol{X}_{u_2}\rangle_1 = 0$. Thus the vector $\boldsymbol{X}_{u_1u_1} + \boldsymbol{X}_{u_2u_2}$ is a normal to $M = \boldsymbol{X}(U)$, so that there exist $\lambda, \mu \in \mathbb{R}$ such that $\boldsymbol{X}_{u_1u_1} + \boldsymbol{X}_{u_2u_2} = \lambda\boldsymbol{n}^T + \mu\boldsymbol{n}^S$. Then we have

$$\begin{aligned} -\lambda &= \langle \boldsymbol{X}_{u_1u_1} + \boldsymbol{X}_{u_2u_2}, \boldsymbol{n}^T\rangle_1 = h_{11}(\boldsymbol{n}^T) + h_{22}(\boldsymbol{n}^T) \\ \mu &= \langle \boldsymbol{X}_{u_1u_1} + \boldsymbol{X}_{u_2u_2}, \boldsymbol{n}^S\rangle_1 = h_{11}(\boldsymbol{n}^S) + h_{22}(\boldsymbol{n}^S). \end{aligned}$$

Since $\ell_0^{\pm}\widetilde{h}[\pm]_{ij} = h_{ij}(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) = h_{ij}(\boldsymbol{n}^T) \pm h_{ij}(\boldsymbol{n}^S)$, we have

$$\ell_0^{\pm}(\widetilde{h}[\pm]_{22} + \widetilde{h}[\pm]_{11}) = h_{22}(\boldsymbol{n}^T) + h_{11}(\boldsymbol{n}^T) \pm (h_{22}(\boldsymbol{n}^S) + h_{11}(\boldsymbol{n}^S)).$$

Therefore we have

$$\begin{aligned} 2(h_{11}(\boldsymbol{n}^T) + h_{22}(\boldsymbol{n}^T)) &= \ell_0^{+}(\widetilde{h}[+]_{22} + \widetilde{h}[+]_{11} + \widetilde{h}[-]_{22} + \widetilde{h}[-]_{11}) = 2\ell_0^{+} g_{11}(\widetilde{H}_\ell^{+} + \widetilde{H}_\ell^{-}) \\ 2(h_{11}(\boldsymbol{n}^S) + h_{22}(\boldsymbol{n}^S)) &= \ell_0^{-}(\widetilde{h}[+]_{22} + \widetilde{h}[+]_{11} - \widetilde{h}[-]_{22} - \widetilde{h}[-]_{11}) = 2\ell_0^{-} g_{11}(\widetilde{H}_\ell^{+} - \widetilde{H}_\ell^{-}) \end{aligned}$$

This means that

$$\boldsymbol{X}_{u_1u_1} + \boldsymbol{X}_{u_2u_2} = -\ell_0^+ g_{11}(\widetilde{H}_\ell^+ + \widetilde{H}_\ell^-)\boldsymbol{n}^T + \ell_0^- g_{11}(\widetilde{H}_\ell^+ - \widetilde{H}_\ell^-)\boldsymbol{n}^S.$$

This completes the proof. □

Corollary 4.6 *Let $\boldsymbol{X} : U \longrightarrow \mathbb{R}^4_1$ be a spacelike surface with an isothermal parameter $(u_1, u_2) \in U$. Then $\boldsymbol{X}_{u_1u_1} + \boldsymbol{X}_{u_2u_2} = \boldsymbol{0}$ if and only if $M = \boldsymbol{X}(U)$ is a strongly marginally trapped surface.*

Proof. By the above proposition, $\boldsymbol{X}_{u_1u_1} + \boldsymbol{X}_{u_2u_2} = \boldsymbol{0}$ if and only if $\widetilde{H}_\ell^+ + \widetilde{H}_\ell^- = \widetilde{H}_\ell^+ - \widetilde{H}_\ell^- = 0$. The last conditions mean that $\widetilde{H}_\ell^+ = \widetilde{H}_\ell^- = 0$. Since $\ell_0^\pm \widetilde{H}_\ell^\pm = H_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) = H(\boldsymbol{n}^T) \pm H(\boldsymbol{n}^S)$, the last condition is equivalent to the condition $H(\boldsymbol{n}^T) = H(\boldsymbol{n}^S) = 0$ which means $\mathfrak{H} = 0$. □

5 Variation formula for marginally trapped surfaces

In this section, we prove the first and second variation formula of area (Proposition 5.1 and Theorem 5.2). Let U be a bounded compact domain in $\mathbb{R}^2$. The area $A(\boldsymbol{X})$ of a spacelike embedding $\boldsymbol{X} : U \to \mathbb{R}^4_1$ is given by

$$A(\boldsymbol{X}) = \int_U dM, \qquad \left(dM = \sqrt{g_{11}g_{22} - g_{22}^2}\, du_1 du_2\right).$$

We denote by $\{\boldsymbol{n}^T, \boldsymbol{n}^S\}$ an orthonormal frame of the normal bundle NM of $\boldsymbol{X}$, where $\boldsymbol{n}^T$ and $\boldsymbol{n}^S$ are timelike and spacelike, respectively. Then, if we set the lightlike vectors $\boldsymbol{\ell}^\pm$ as $\boldsymbol{\ell}^\pm = \boldsymbol{n}^T \pm \boldsymbol{n}^S$, the pair $\{\boldsymbol{\ell}^+, \boldsymbol{\ell}^-\}$ is a basis of NM which satisfies the following relations

$$\left\langle \boldsymbol{\ell}^+, \boldsymbol{\ell}^+ \right\rangle_1 = \left\langle \boldsymbol{\ell}^-, \boldsymbol{\ell}^- \right\rangle_1 = 0, \qquad \left\langle \boldsymbol{\ell}^+, \boldsymbol{\ell}^- \right\rangle_1 = -2.$$

5.1 First variation formula

Let $\boldsymbol{X}^\varepsilon$ be a smooth variation of a spacelike embedding $\boldsymbol{X} : U \to \mathbb{R}^4_1$, where ε denotes the variation parameter. That is, $\boldsymbol{X}^\varepsilon : U \to \mathbb{R}^4_1$ is a smooth spacelike surface for each ε, and it satisfies $\boldsymbol{X}^0 = \boldsymbol{X}$.

We assume that the variation vector field $V^\pm$ of the variation $\boldsymbol{X}^\varepsilon$ is given by

$$V^\pm = \left.\frac{\partial}{\partial\varepsilon}\right|_0 \boldsymbol{X}^\varepsilon = \alpha\boldsymbol{\ell}^\pm \tag{7}$$

where α is a smooth function on U.

Proposition 5.1 (First variation formula) *For a spacelike surface $\boldsymbol{X} : \Sigma \to \mathbb{R}^4_1$, let $\boldsymbol{X}^\varepsilon$ be a variation whose variation vector field $V^\pm$ of $\boldsymbol{X}^\varepsilon$ is given by (7). Then the first variation of area is given by*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} A(\boldsymbol{X}^\varepsilon) = -2\int_U \alpha H_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) dM.$$

Proof. For the sake of simplicity, we will give the proof in the case of $V = V^+ = \alpha\boldsymbol{\ell}^+$ here. Similar proof is valid for the case of V^-. Since the area $A(\boldsymbol{X}^\varepsilon)$ of $\boldsymbol{X}^\varepsilon$ is given by

$$A(\boldsymbol{X}^\varepsilon) = \int_U \sqrt{g_{11}^\varepsilon g_{22}^\varepsilon - (g_{22}^\varepsilon)^2}\, du_1 du_2,$$

we have

$$\frac{d}{d\varepsilon} A(\boldsymbol{X}^\varepsilon) = \int_U \frac{\partial}{\partial\varepsilon}\left(\sqrt{g_{11}^\varepsilon g_{22}^\varepsilon - (g_{22}^\varepsilon)^2}\right) du_1 du_2 = \int_U \frac{\partial_\varepsilon\left(g_{11}^\varepsilon g_{22}^\varepsilon - (g_{22}^\varepsilon)^2\right)}{2\left(g_{11}^\varepsilon g_{22}^\varepsilon - (g_{22}^\varepsilon)^2\right)} dM$$

hold, where $\partial_\varepsilon = \partial/\partial\varepsilon$. Since

$$\frac{\partial}{\partial\varepsilon}\left(g_{11}^\varepsilon g_{22}^\varepsilon - (g_{22}^\varepsilon)^2\right) = \left(\frac{\partial}{\partial\varepsilon} g_{11}^\varepsilon\right) g_{22}^\varepsilon + g_{11}^\varepsilon\left(\frac{\partial}{\partial\varepsilon} g_{22}^\varepsilon\right) - 2g_{12}^\varepsilon\left(\frac{\partial}{\partial\varepsilon} g_{12}^\varepsilon\right),$$

and

$$\begin{aligned}
\left.\frac{\partial}{\partial\varepsilon}\right|_0 g_{ij}^\varepsilon &= \left.\frac{\partial}{\partial\varepsilon}\right|_0 \left\langle \boldsymbol{X}_{u_i}^\varepsilon, \boldsymbol{X}_{u_j}^\varepsilon \right\rangle_1 \\
&= \left\langle V_{u_i}, \boldsymbol{X}_{u_j}\right\rangle_1 + \left\langle \boldsymbol{X}_{u_i}, V_{u_j}\right\rangle_1 \\
&= \left\langle \alpha\boldsymbol{\ell}_{u_i}^+, \boldsymbol{X}_{u_j}\right\rangle_1 + \left\langle \boldsymbol{X}_{u_i}, \alpha\boldsymbol{\ell}_{u_j}^+ \right\rangle_1 = -2\alpha h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S),
\end{aligned}$$

we have

$$\frac{\partial_\varepsilon|_{\varepsilon=0}\left(g_{11}^\varepsilon g_{22}^\varepsilon - (g_{22}^\varepsilon)^2\right)}{2\left(g_{11} g_{22} - g_{22}^2\right)} = -\alpha \sum_{i,j} g^{ij} h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S) = -2\alpha H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S).$$

This completes the proof. □

5.2 Second variation formula

Now, we prove the second variation formula of area.

Theorem 5.2 (Second variation formula) *Let* $\boldsymbol{X} : \Sigma \to \mathbb{R}_1^4$ *be a marginally trapped surface, that is,* $\boldsymbol{X}$ *satisfies* $H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S) = 0$ *(resp.* $H_\ell(\boldsymbol{n}^T, -\boldsymbol{n}^S) = 0$*). If* $\boldsymbol{X}^\varepsilon$ *is a variation whose variation vector field* V^+ *of* $\boldsymbol{X}^\varepsilon$ *is given by*

$$V^+ = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \boldsymbol{X}^\varepsilon = \alpha\boldsymbol{\ell}^+ \qquad \left(\textit{resp. } V^- = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \boldsymbol{X}^\varepsilon = \alpha\boldsymbol{\ell}^-\right),$$

then the second variation of area satisfies

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} A(\boldsymbol{X}^\varepsilon) = 2\int_U \alpha^2 K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S) dM$$

$$\left(\textit{resp. } \left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} A(\boldsymbol{X}^\varepsilon) = 2\int_U \alpha^2 K_\ell(\boldsymbol{n}^T, -\boldsymbol{n}^S) dM\right).$$

For the proof of Theorem 5.2, we prepare Lemma 5.3 and Lemma 5.4. The following Lemma 5.3 is obtained by the Cayley-Hamilton theorem.

Lemma 5.3 *For spacelike surfaces in $\mathbb{R}^4_1$, we have the followings.*

(1) $\sum_k h^i_k(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) h^k_j(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) = 2H_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) h^i_j(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) - K_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)\delta^i_j$,

(2) $\left\langle \boldsymbol{\ell}^\pm_{u_i}, \boldsymbol{\ell}^\pm_{u_j} \right\rangle_1 = 2H_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) h_{ij}(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) - K_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S) g_{ij}$.

Lemma 5.4 *For a variation vector field $V = \alpha\boldsymbol{\ell}^+$, we have*

$$\left.\frac{\partial}{\partial\varepsilon}\right|_0 H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S) = \alpha\left(2H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)^2 - K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)\right) - \frac{1}{2}\left\langle \left.\frac{\partial}{\partial\varepsilon}\right|_0 \boldsymbol{\ell}^+, \boldsymbol{\ell}^- \right\rangle_1 H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S). \tag{8}$$

Proof. We have

$$\begin{aligned}
2\left.\frac{\partial}{\partial\varepsilon}\right|_0 H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S) &= \left.\frac{\partial}{\partial\varepsilon}\right|_0 \left(g^{ij} h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)\right) \\
&= \left(\left.\frac{\partial}{\partial\varepsilon}\right|_0 g^{ij}\right) h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S) + g^{ij}\left(\left.\frac{\partial}{\partial\varepsilon}\right|_0 h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)\right) \\
&= -g^{ib}\left(\left.\frac{\partial}{\partial\varepsilon}\right|_0 g_{ab}\right) g^{aj} h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S) + g^{ij} \left.\frac{\partial}{\partial\varepsilon}\right|_0 \langle \boldsymbol{\ell}^+, \boldsymbol{X}_{u_i u_j}\rangle_1 \\
&= -g^{ib} g^{aj} h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)\left(\left.\frac{\partial}{\partial\varepsilon}\right|_0 g_{ab}\right) + g^{ij}\left(\left\langle \left.\frac{\partial}{\partial\varepsilon}\right|_0 \boldsymbol{\ell}^+, \boldsymbol{X}_{u_i u_j}\right\rangle_1 + \langle \boldsymbol{\ell}^+, V_{u_i u_j}\rangle_1\right) \\
&=: P_1 + P_2 + P_3,
\end{aligned}$$

where

$$P_1 = -g^{ib} g^{aj} h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)\left(\left.\frac{\partial}{\partial\varepsilon}\right|_0 \langle \boldsymbol{X}_{u_a}, \boldsymbol{X}_{u_b}\rangle_1\right), \qquad P_2 = g^{ij}\left\langle \left.\frac{\partial}{\partial\varepsilon}\right|_0 \boldsymbol{\ell}^+, \boldsymbol{X}_{u_i u_j}\right\rangle_1,$$

and $P_3 = g^{ij}\langle \boldsymbol{\ell}^+, V_{u_i u_j}\rangle_1$. With respect to P_1, it follows that

$$\begin{aligned}
P_1 &= g^{ib} g^{aj} h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)\left(\langle V_{u_a}, \boldsymbol{X}_{u_b}\rangle_1 + \langle \boldsymbol{X}_{u_a}, V_{u_b}\rangle_1\right) \\
&= 2g^{ib} g^{aj} h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)\langle V, \boldsymbol{X}_{u_a u_b}\rangle_1 \\
&= 2\alpha g^{aj} g^{ib} h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S) h_{ab}(\boldsymbol{n}^T, \boldsymbol{n}^S) = 2\alpha h^j_b(\boldsymbol{n}^T, \boldsymbol{n}^S) h^b_j(\boldsymbol{n}^T, \boldsymbol{n}^S).
\end{aligned}$$

By (1) in Lemma 5.3, we have $h^j_b(\boldsymbol{n}^T, \boldsymbol{n}^S) h^b_j(\boldsymbol{n}^T, \boldsymbol{n}^S) = 4H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)^2 - 2K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)$, and hence

$$P_1 = 4\alpha\left(2H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)^2 - K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)\right) \tag{9}$$

holds. For P_3, we have

$$\begin{aligned}
P_3 &= g^{ij}\left\langle \boldsymbol{\ell}^+, \alpha_{u_i u_j}\boldsymbol{\ell}^+ + 2\alpha_{u_i}\boldsymbol{\ell}^+_{u_j} + \alpha\boldsymbol{\ell}^+_{u_i u_j}\right\rangle_1 \\
&= \alpha g^{ij}\left\langle \boldsymbol{\ell}^+, \boldsymbol{\ell}^+_{u_i u_j}\right\rangle_1 = -\alpha g^{ij}\left\langle \boldsymbol{\ell}^+_{u_i}, \boldsymbol{\ell}^+_{u_j}\right\rangle_1.
\end{aligned}$$

Then, by (2) in Lemma 5.3, it follows that

$$\begin{aligned}
P_3 &= -\alpha g^{ij}\left(2H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S) h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S) - K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S) g_{ij}\right) \\
&= -2\alpha\left(2H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)^2 - K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)\right).
\end{aligned} \tag{10}$$

Finally, for P_2, we shall calculate $\partial_\varepsilon|_{\varepsilon=0}\,\boldsymbol{\ell}^+$. If we put $\partial_\varepsilon|_{\varepsilon=0}\,\boldsymbol{\ell}^+ = p^j \boldsymbol{X}_{u_j} + \Phi\boldsymbol{\ell}^+$, we have $\left\langle \partial_\varepsilon|_{\varepsilon=0}\,\boldsymbol{\ell}^+, \boldsymbol{X}_{u_k}\right\rangle_1 = \left\langle p^j \boldsymbol{X}_{u_j}, \boldsymbol{X}_{u_k}\right\rangle_1 = p^j g_{jk}$. Hence

$$g^{kl}\left\langle \frac{\partial}{\partial\varepsilon}\bigg|_0 \boldsymbol{\ell}^+, \boldsymbol{X}_{u_k}\right\rangle_1 = g^{kl}p^j g_{jk} = p^j\delta^l_j = p^l$$

holds. On the other hand, it is checked that

$$\left\langle \frac{\partial}{\partial\varepsilon}\bigg|_0 \boldsymbol{\ell}^+, \boldsymbol{X}_{u_k}\right\rangle_1 = -\left\langle \boldsymbol{\ell}^+, V_{u_k}\right\rangle_1 = -\left\langle \boldsymbol{\ell}^+, \alpha_{u_k}\boldsymbol{\ell}^+ + \alpha\boldsymbol{\ell}^+_{u_k}\right\rangle_1 = 0,$$

and thus we have $p^j = 0$. Since $\left\langle \partial_\varepsilon|_{\varepsilon=0}\,\boldsymbol{\ell}^+, \boldsymbol{\ell}^-\right\rangle_1 = -2\Phi$, it follows that

$$\frac{\partial}{\partial\varepsilon}\bigg|_0 \boldsymbol{\ell}^+ = -\frac{1}{2}\left\langle \frac{\partial}{\partial\varepsilon}\bigg|_0 \boldsymbol{\ell}^+, \boldsymbol{\ell}^-\right\rangle_1 \boldsymbol{\ell}^+.$$

Thus we have

$$P_2 = -\frac{1}{2}\left\langle \frac{\partial}{\partial\varepsilon}\bigg|_0 \boldsymbol{\ell}^+, \boldsymbol{\ell}^-\right\rangle_1 g^{ij}h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S) = -\left\langle \frac{\partial}{\partial\varepsilon}\bigg|_0 \boldsymbol{\ell}^+, \boldsymbol{\ell}^-\right\rangle_1 H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S). \tag{11}$$

By (9), (10) and (11), we obtain (8). □

Proof of Theorem 5.2. We shall give the proof in the case of $H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S) = 0$. Similar proof is valid for the case of $H_\ell(\boldsymbol{n}^T, -\boldsymbol{n}^S) = 0$. In this situation, the variation vector field is given by $V = V^+ = \alpha\boldsymbol{\ell}^+$. By Proposition 5.1, the first variation formula of the area is

$$\frac{d^2}{d\varepsilon^2}A(\boldsymbol{X}^\varepsilon) = \int_U H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)\left\langle \frac{\partial}{\partial\varepsilon}\boldsymbol{X}^\varepsilon, \boldsymbol{\ell}^-\right\rangle_1 dM.$$

Since $H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S) = 0$ holds at $\varepsilon = 0$, the second variation of area is given by

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} A(\boldsymbol{X}^\varepsilon) &= \int_U \left\{\left(\frac{\partial}{\partial\varepsilon}H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)\right)\left\langle \frac{\partial}{\partial\varepsilon}\boldsymbol{X}^\varepsilon, \boldsymbol{\ell}^-\right\rangle_1 dM \right.\\ &\qquad\left. + H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)\frac{\partial}{\partial\varepsilon}\left(\left\langle \frac{\partial}{\partial\varepsilon}\boldsymbol{X}^\varepsilon, \boldsymbol{\ell}^-\right\rangle_1 dM\right)\right\}\bigg|_{\varepsilon=0}\\ &= \int_U \left(\frac{\partial}{\partial\varepsilon}\bigg|_0 H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)\right)\left\langle V, \boldsymbol{\ell}^-\right\rangle_1 dM.\end{aligned}$$

Then, by Lemma 5.4, we have

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S) = -\alpha K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S).$$

Since $\left\langle V, \boldsymbol{\ell}^-\right\rangle_1 = -2\alpha$, we get the proof. □

Remark 5.5 It should be remarked that the quantity $\partial_\varepsilon H_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)$ which we calculated in Lemma 5.4 is also calculated by Andersson and Metzger [2, Lemma 4.1] in the case of a general Lorentzian 4-manifold.

6 Marginally trapped graphs and the Bernstein-type problem

In this section, we consider a partial differential equation for graph of functions such that marginally trapped graph surface satisfy the equation. Then we have the following theorem.

Theorem 6.1 *Let U be a domain of $\mathbb{R}^2$, and f, g be smooth functions on U. Consider a graph immersion $\boldsymbol{X} : U \to \mathbb{R}^4_1$ given by $\boldsymbol{X}(u_1, u_2) = (f(u_1, u_2), g(u_1, u_2), u_1, u_2)$. If we set functions ϕ_1, ϕ_2 and Δ as*

$$\phi_1 = (1 - f_{u_2}^2 + g_{u_2}^2) f_{u_1 u_1} - 2(-f_{u_1} f_{u_2} + g_{u_1} g_{u_2}) f_{u_1 u_2} + (1 - f_{u_1}^2 + g_{u_1}^2) f_{u_2 u_2}, \tag{12}$$
$$\phi_2 = (1 - f_{u_2}^2 + g_{u_2}^2) g_{u_1 u_1} - 2(-f_{u_1} f_{u_2} + g_{u_1} g_{u_2}) g_{u_1 u_2} + (1 - f_{u_1}^2 + g_{u_1}^2) g_{u_2 u_2}, \tag{13}$$
$$\Delta = (f_{u_1} g_{u_1} + f_{u_2} g_{u_2})^2 - (1 + g_{u_1}^2 + g_{u_2}^2)(-1 + f_{u_1}^2 + f_{u_2}^2), \tag{14}$$

then, $\boldsymbol{X}$ is strongly marginally trapped if and only if

$$\Delta > 0 \qquad \text{and} \qquad \phi_1 = \phi_2 = 0 \tag{15}$$

hold. Moreover, $\boldsymbol{X}$ is marginally trapped if and only if the functions f, g and Δ satisfies

$$\Delta > 0 \quad \text{and} \quad (1 + g_{u_1}^2 + g_{u_2}^2)\phi_1^2 - 2(f_{u_1} g_{u_1} + f_{u_2} g_{u_2})\phi_1 \phi_2 + (-1 + f_{u_1}^2 + f_{u_2}^2)\phi_2^2 = 0. \tag{16}$$

For the proof of this theorem, we calculate the mean curvature vector of graph immersion.

Lemma 6.2 *For a spacelike surface $\boldsymbol{X} : U \to \mathbb{R}^4_1$ defined by $\boldsymbol{X}(u_1, u_2) = (f(u_1, u_2), g(u_1, u_2), u_1, u_2)$, let ϕ_1, ϕ_2 and Δ be functions defined as in* (12), (13) *and* (14)*, respectively. Setting $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as*

$$\boldsymbol{v}_1 = (1,\, 0,\, f_u,\, f_v), \qquad \boldsymbol{v}_2 = (0,\, 1,\, -g_u,\, -g_v),$$

and τ_{ij} $(i, j = 1, 2)$ as

$$\tau_{11} = 1 + g_{u_1}^2 + g_{u_2}^2, \qquad \tau_{12} = \tau_{21} = f_{u_1} g_{u_1} + f_{u_2} g_{u_2}, \qquad \tau_{22} = -1 + f_{u_1}^2 + f_{u_2}^2,$$

then the mean curvature vector $\mathfrak{H}$ of $\boldsymbol{X}$ is given by

$$\mathfrak{H} = \frac{(\tau_{11}\phi_1 - \tau_{12}\phi_2)\,\boldsymbol{v}_1 + (\tau_{12}\phi_1 - \tau_{22}\phi_2)\,\boldsymbol{v}_2}{2\Delta^2}. \tag{17}$$

Proof. The mean curvature vector $\mathfrak{H}$ is given by

$$\mathfrak{H} = \frac{g_{22} II(\boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_1}) - 2 g_{12} II(\boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_2}) + g_{11} II(\boldsymbol{X}_{u_2}, \boldsymbol{X}_{u_2})}{2(g_{11} g_{22} - g_{12}^2)}, \tag{18}$$

where

$$II(\boldsymbol{X}_{u_i}, \boldsymbol{X}_{u_j}) = \boldsymbol{X}_{u_i u_j} - \sum_k \left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\} \boldsymbol{X}_{u_k}, \tag{19}$$

for $i, j = 1, 2$, and $\left\{ \begin{smallmatrix} k \\ i\ j \end{smallmatrix} \right\}$ is the Christoffel symbols of the induced metric $g = g_{ij} du_i du_j$. The coefficients g_{ij} of the metric g is calculated as

$$g_{11} = \langle \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_1} \rangle_1 = 1 - f_{u_1}^2 + g_{u_1}^2,$$
$$g_{12} = \langle \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_2} \rangle_1 = -f_{u_1} f_{u_2} + g_{u_1} g_{u_2},$$
$$g_{22} = \langle \boldsymbol{X}_{u_2}, \boldsymbol{X}_{u_2} \rangle_1 = 1 - f_{u_2}^2 + g_{u_2}^2,$$

and hence we have

$$\begin{aligned} g_{11}g_{22} - g_{12}^2 &= (1 - f_{u_1}^2 + g_{u_1}^2)(1 - f_{u_2}^2 + g_{u_2}^2) - (-f_{u_1}f_{u_2} + g_{u_1}g_{u_2})^2 \\ &= \tau_{12}^2 - \tau_{11}\tau_{22} = \Delta. \end{aligned} \tag{20}$$

Substituting these into (18) and (19), we obtain (17). □

Proof of Theorem 6.1. By (20), we have that the graph immersion $\boldsymbol{X}$ is spacelike if and only if $\Delta > 0$. Moreover, by Lemma 6.2 and the linearly independentness of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, it follows that the spacelike graph immersion $\boldsymbol{X}$ is strongly marginally trapped if and only if

$$\tau_{11}\phi_1 - \tau_{12}\phi_2 = 0, \qquad \tau_{12}\phi_1 - \tau_{22}\phi_2 = 0 \tag{21}$$

hold. Since $\Delta > 0$ and by (20), we have that (21) is equivalent to $\phi_1 = \phi_2 = 0$. This completes the proof of (15).

With respect to (16), by (17) in Lemma 6.2, we have that the spacelike graph immersion $\boldsymbol{X}$ is marginally trapped if and only if

$$\langle (\tau_{11}\phi_1 - \tau_{12}\phi_2)\,\boldsymbol{v}_1 + (\tau_{12}\phi_1 - \tau_{22}\phi_2)\,\boldsymbol{v}_2, (\tau_{11}\phi_1 - \tau_{12}\phi_2)\,\boldsymbol{v}_1 + (\tau_{12}\phi_1 - \tau_{22}\phi_2)\,\boldsymbol{v}_2\rangle_1 = 0. \tag{22}$$

Substituting the following equations

$$\langle \boldsymbol{v}_1, \boldsymbol{v}_1\rangle_1 = \tau_{22}, \qquad \langle \boldsymbol{v}_1, \boldsymbol{v}_2\rangle_1 = -\tau_{12}, \qquad \langle \boldsymbol{v}_2, \boldsymbol{v}_2\rangle_1 = \tau_{11}$$

into (22), we have that (22) is equivalent to

$$-\Delta(\tau_{11}\phi_1^2 - 2\tau_{12}\phi_1\phi_2 + \tau_{22}\phi_2^2) = 0.$$

Since $\Delta > 0$, we obtain (16). □

As a corollary of Theorem 6.1, we have the following.

Proposition 6.3 *A spacelike surface M is strongly marginally trapped and totally $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilical if and only if up to rigid motions of $\mathbb{R}^4_1$, M is given by*

$$\boldsymbol{X}_f(u_1, u_2) = (f(u_1, u_2), f(u_1, u_2), u_1, u_2)), \tag{23}$$

for a harmonic function $f : \mathbb{R}^2 \to \mathbb{R}$.

Proof. By Proposition 3.5, a totally $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilical marginally trapped surface is given by $\boldsymbol{X}_f$ as in (23) for a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$. Then, the functions ϕ_1, ϕ_2 defined by (12) and (13) are calculated as

$$\phi_1 = \phi_2 = f_{u_1u_1} + f_{u_2u_2}.$$

Thus, we have that $\boldsymbol{X}_f$ is strongly marginally trapped if and only if f is harmonic. □

Remark 6.4 Classical Berntein's theorem [4] says that *any minimal graph in $\mathbb{R}^3$ defined on the whole plane must be a plane.* By Proposition 6.3, we have that Bernstein-type theorem for strongly marginally trapped surfaces does not holds, that is, *a strongly marginally trapped graph in $\mathbb{R}^4_1$ defined on the whole plane need not to be a spacelike plane.*

7 Special cases

In this section we naturally interpret minimal surfaces in Euclidean 3-space, maximal surfaces in Lorentz-Minkowski 3-space, CMC-1 surfaces in Hyperbolic 3-space, CMC-1 spacelike surfaces in de Sitter 3-space as marginally trapped surfaces. These surfaces have been well-investigated in the previous researches. We also show that intrinsic flat spacelike surfaces in the lightcone can be interpreted as marginally trapped surfaces.

7.1 Surfaces in Euclidean space

Let $\boldsymbol{X} : U \to \mathbb{R}^3$ be a surface in the Euclidean space $\mathbb{R}^3_0 = \{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \mid x_0 = 0\ \}$. Then we can take $\boldsymbol{n}^T = \boldsymbol{e}_0 = (1,0,0,0)$, and have

$$\boldsymbol{n}^S(\underline{u}) = \frac{\boldsymbol{e}_0 \wedge \boldsymbol{X}_{u_1}(\underline{u}) \wedge \boldsymbol{X}_{u_2}(\underline{u})}{\|\boldsymbol{e}_0 \wedge \boldsymbol{X}_{u_1}(\underline{u}) \wedge \boldsymbol{X}_{u_2}(\underline{u})\|_1} \in S^2 \subset \mathbb{R}^3_0.$$

Therefore, $\boldsymbol{n}^S(\underline{u})$ is the Euclidean unit normal of $M = \boldsymbol{X}(U) \subset \mathbb{R}^3_0$ at $p = \boldsymbol{X}(\underline{u})$. In this case, the lightcone Gauss map is given by $\mathbb{L}^{\pm}(\underline{u}) = \boldsymbol{e}_0 \pm \boldsymbol{n}^S(\underline{u})$. So, the lightcone shape operator is $\widetilde{S}^{\pm}_p = S_p(\boldsymbol{e}_0, \pm\boldsymbol{n}^S) = -d(\boldsymbol{e}_0 \pm \boldsymbol{n}^S)(\underline{u}) = \mp d\boldsymbol{n}^S(\underline{u})$ which is the Weingarten map for the surface in the Euclidean space. It follows that $\widetilde{K}^{\pm}_{\ell}(\underline{u}) = K_{\ell}(\boldsymbol{e}_0, \pm\boldsymbol{n}^S)(\underline{u}) = K(\underline{u})$ (i.e., the Gauss curvature) and $\widetilde{H}^{\pm}_{\ell}(\underline{u}) = H_{\ell}(\boldsymbol{e}_0, \pm\boldsymbol{n}^S)(\underline{u}) = \pm H(\underline{u})$ (i.e., the mean curvature). Therefore, a lightcone flat spacelike surface is a developable surface and a marginally trapped surface is a minimal surface. In this case, a marginally trapped surface is always a strongly marginally trapped surface.

We remark that if $\boldsymbol{n}^T(\underline{u}) = \boldsymbol{v}$ is a constant timelike unit vector, the spacelike surface M is a surface in the spacelike hyperplane $HP(\boldsymbol{v}, c)$. Since $HP(\boldsymbol{v}, c)$ is isometric to the Euclidean space $\mathbb{R}^3_0$, all the results for $\mathbb{R}^3_0$ hold in this case.

7.2 Spacelike surfaces in Minkowski 3-space

Let $\boldsymbol{X} : U \to \mathbb{R}^3_1$ be a spacelike surface in the Minkowski space $\mathbb{R}^3_1 = \{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \mid x_3 = 0\ \}$. Then we can choose $\boldsymbol{n}^S = \boldsymbol{e}_3 = (0,0,0,1)$, and have

$$\boldsymbol{n}^T(\underline{u}) = \frac{\boldsymbol{X}_{u_1}(\underline{u}) \wedge \boldsymbol{X}_{u_2}(\underline{u}) \wedge \boldsymbol{e}_3}{\|\boldsymbol{X}_{u_1}(\underline{u}) \wedge \boldsymbol{X}_{u_2}(\underline{u}) \wedge \boldsymbol{e}_3\|_1} \in H^2(-1) \subset \mathbb{R}^3_1.$$

Therefore, $\boldsymbol{n}^T(\underline{u})$ is the timelike unit normal of $M = \boldsymbol{X}(U) \subset \mathbb{R}^3_1$ at $p = \boldsymbol{X}(\underline{u})$. In this case, the lightcone Gauss map is given by $\mathbb{L}^{\pm}(\underline{u}) = \boldsymbol{n}^T(\underline{u}) \pm \boldsymbol{e}_3$. So, the lightcone shape operator is $\widetilde{S}^{\pm}_p = S_p(\boldsymbol{n}^T, \pm\boldsymbol{e}_3)(\underline{u}) = -d(\boldsymbol{n}^T(\underline{u}) \pm \boldsymbol{e}_3) = -d\boldsymbol{n}^T(\underline{u})$ which is the *spacelike shape operator* for the spacelike surface in the Minkowski space. It follows that $\widetilde{K}^{\pm}_{\ell}(\underline{u}) = K(\boldsymbol{n}^T, \pm\boldsymbol{e}_3)(\underline{u}) = K(\underline{u})$ (i.e., the Gauss-Kronecker curvature) and $\widetilde{H}^{\pm}_{\ell}(\underline{u}) = H_{\ell}(\boldsymbol{n}^T, \pm\boldsymbol{e}_3)(\underline{u}) = H(\underline{u})$ (i.e., the mean curvature). Therefore, a lightcone flat spacelike surface is a spacelike developable surface and a marginally trapped surface is a maximal surface. In this case a marginally trapped surface is a strongly marginally trapped surface. Of course, if $\boldsymbol{n}^S(\underline{u}) = \boldsymbol{v}$ is constant spacelike unit vector, the spacelike surface M is surface in the timelike hyperplane $HP(\boldsymbol{v}, c)$. Since $HP(\boldsymbol{v}, c)$ is isometric to the Minkowski space $\mathbb{R}^3_1$, all the results for $\mathbb{R}^3_1$ hold in this case.

7.3 Surfaces in the hyperbolic 3-space

Let $\boldsymbol{X} : U \to H^3_+(-1)$ be a surface in the hyperbolic space. Then we can take $\boldsymbol{n}^T = \boldsymbol{X}$, and

$$\boldsymbol{n}^S(\underline{u}) = \frac{\boldsymbol{X}(\underline{u}) \wedge \boldsymbol{X}_{u_1}(\underline{u}) \wedge \boldsymbol{X}_{u_2}(\underline{u})}{\|\boldsymbol{X}(\underline{u}) \wedge \boldsymbol{X}_{u_1}(\underline{u}) \wedge \boldsymbol{X}_{u_2}(\underline{u})\|_1} \in S^3_1$$

is univocally defined. We denote $\boldsymbol{n}^S$ by $\boldsymbol{e}$ and call it the *de Sitter normal vector field along M*. The *de Sitter Gauss image* is defined as the map $\mathbb{E} : U \to S^3_1$ given by $\mathbb{E}(\underline{u}) = \boldsymbol{e}(\underline{u})$.

We also have the *hyperbolic Gauss image*

$$\begin{array}{rcccl} \mathbb{L}^{\pm} & : & U & \longrightarrow & LC^*_+ \\ & & \underline{u} & \longmapsto & \boldsymbol{X}(\underline{u}) \pm \mathbb{E}(\underline{u}). \end{array}$$

In this particular case of surfaces in $\mathbb{R}^4_1$, the lightcone Gauss map on M coincides with the *hyperbolic Gauss map*, introduced in [11] which is given by

$$\widetilde{\mathbb{L}}^{\pm}(\underline{u}) = \boldsymbol{X}\widetilde{(\underline{u}) \pm \mathbb{E}}(\underline{u}).$$

We observe that this notion of hyperbolic Gauss map is equivalent to the one defined [5, 10, 22, 23] in other models.

It is well-known that the hyperbolic space is a model for the non-Euclidean geometry of Gauss-Bolyai-Lobachevsky (i.e., the hyperbolic geometry). Recently, it has been discovered another geometry in the hyperbolic space which is called the horospherical geometry (cf., [11, 12, 13, 18]). In this case $K_\ell(\boldsymbol{X}, \mathbb{E})(\underline{u}) = K^{\pm}_h(\underline{u})$ (i.e., the *hyperbolic Gauss-Kronecker curvature* [11]), $\widetilde{K}^{\pm}_\ell(\underline{u}) = \widetilde{K}^{\pm}_h(\underline{u})$ (i.e., the *horospherical Gauss-Kronecker curvature* [13]) and $H_\ell(\boldsymbol{X}, \pm\mathbb{E})(\underline{u}) = H^{\pm}_h(\underline{u})$, where $H^{\pm}_h = -1 \pm H$ is the *hyperbolic mean curvature* [11] and H is the mean curvature in the ordinary sense (i.e., the *de Sitter mean curvature* [11]). Therefore, a lightcone flat spacelike surface is a horospherical flat surface [18] and a marginally trapped surface is a CMC$\pm$1 surface. Since $\boldsymbol{n}^T = \boldsymbol{X}$, $H(\boldsymbol{n}^T) \equiv -1$, so that there are no strongly marginally trapped surfaces in the hyperbolic space. Let $V = \alpha\mathbb{L}^{\pm}$ be the variation vector filed. Then the first variational formula is

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} A(\boldsymbol{X}^\varepsilon) = -2\int_U \alpha H^{\pm}_h dM = -2\int_U \alpha(-1 \pm H) dM$$

and the second variation formula is

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} A(\boldsymbol{X}^\varepsilon) = 2\int_U \alpha^2 K^{\pm}_h dM.$$

7.4 Spacelike surfaces in de Sitter 3-space

Let $\boldsymbol{X} : U \to S^3_1$ be a spacelike surface in the de Sitter 3-space. Then we can take $\boldsymbol{n}^S = \boldsymbol{X}$, and

$$\boldsymbol{n}^T(\underline{u}) = \frac{\boldsymbol{X}_{u_1}(\underline{u}) \wedge \boldsymbol{X}_{u_2}(\underline{u}) \wedge \boldsymbol{X}(\underline{u})}{\|\boldsymbol{X}_{u_1}(\underline{u}) \wedge \boldsymbol{X}_{u_2}(\underline{u}) \wedge \boldsymbol{X}(\underline{u})\|_1} \in H^3(-1)$$

is defined. We denote $\boldsymbol{n}^S$ by $\boldsymbol{e}$ and call it the *hyperbolic normal vector field along M*. The *hyperbolic Gauss image* is defined as the map $\mathbb{E} : U \to H^3(-1)$ given by $\mathbb{E}(\underline{u}) = \boldsymbol{e}(\underline{u})$.

We also have the *lightcone Gauss image*

$$\begin{array}{rcl} \mathbb{L}^{\pm} \ : \ U & \longrightarrow & LC^*_+ \\ \underline{u} & \longmapsto & \boldsymbol{X}(\underline{u}) \pm \mathbb{E}(\underline{u}). \end{array}$$

In this particular case of spacelike surfaces in $\mathbb{R}^4_1$, the lightcone Gauss image on M was introduced in [20]. We only give the following proposition as a corollary of Proposition 8.4.

Proposition 7.1 *Let $\boldsymbol{X} : U \to S^3_1$ be a spacelike surface in the de Sitter 3-space. Then the lightcone Gauss image $\mathbb{L}^{\pm}(\underline{u})$ is constant $\boldsymbol{v}$if and only if M is a parabola defined by $HP(\boldsymbol{v}, c) \cap S^3_1$.*

The parabola $HP(\boldsymbol{v}, c) \cap S^3_1$ for a lightlike vector $\boldsymbol{v}$is called a *de Sitter horosphere.* The geometry related to the hyperbolic Gauss image might be called *de Sitter horospherical geometry.* Further results on de Sitter horospherical geometry are referred in the articles [20, 21].

A marginally trapped surface in the de Sitter space is also a CMC± 1 surface and there are no strongly marginally trapped surfaces. In this case, we have the similar formulae for the variations to the case of surfaces in Hyperbolic space.

7.5 Spacelike surfaces in the lightcone

The induced metric is degenerate on the lightcone, so that the ordinary submanifolds theory cannot work for surfaces in the lightcone. In [17] we constructed the basic tools for the study of the extrinsic geometry on spacelike surfaces in the lightcone LC^* (see, also [24]) as one of the applications of the mandala of Legendrian dualitites between pseudo-spheres in Minkowski space-time [17, 19]. We define one-forms $\langle d\boldsymbol{v}, \boldsymbol{w}\rangle_1 = -w_0 dv_0 + \sum_{i=1}^3 w_i dv_i$, $\langle \boldsymbol{v}, d\boldsymbol{w}\rangle_1 = -v_0 dw_0 + \sum_{i=1}^3 v_i dw_i$ on $\mathbb{R}^4_1 \times \mathbb{R}^4_1$ and consider the following double fibration with one-forms

(a) $LC^* \times LC^* \supset \Delta_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle_1 = -2 \}$,

(b) $\pi_{41} : \Delta_4 \longrightarrow LC^*$,$\pi_{42} : \Delta_4 \longrightarrow LC^*$,

(c) $\theta_{41} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle_1 | \Delta_4$, $\theta_{42} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle_1 | \Delta_4$,

where $\pi_{41}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$, $\pi_{42}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$ are the canonical projections. Moreover, $\theta_{41} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle_1 | \Delta_4$ and $\theta_{42} = \langle d\boldsymbol{w}, \boldsymbol{v}\rangle_1 | \Delta_4$ are the restrictions of the one-forms $\langle d\boldsymbol{v}, \boldsymbol{w}\rangle_1$ and $\langle d\boldsymbol{w}, \boldsymbol{v}\rangle_1$ on Δ_4. It has been shown in [17] that (Δ_4, K_4) is a contact manifold and both of π_{4j} $(j = 1, 2)$ are Legendrian fibrations, where $K_4 = \theta_{41}^{-1}(0) = \theta_{42}^{-1}(0)$. In [17] we defined four Legendrian fibrations (Δ_i, K_i) $(i = 1, 2, 3, 4)$ such that these are contact diffeomorphic to each other. Here, we only use (Δ_4, K_4). For definitions and basic results of Legendrian fibrations, see [3, 17].

Let $\boldsymbol{X} : U \longrightarrow LC^*$ be a spacelike surface in $LC^* \subset \mathbb{R}^4_1$. In [17] it has been shown that there exists a unique map $\boldsymbol{X}^\ell : U \longrightarrow LC^*$ such that $\mathscr{L}_4 : U \longrightarrow \Delta_4$ defined by $\mathscr{L}_4(\underline{u}) = (\boldsymbol{X}(\underline{u}), \boldsymbol{X}^\ell(\underline{u}))$ is a Legendrian embedding. We call $\boldsymbol{X}^\ell$ the *lightcone Gauss image* of $M = \boldsymbol{X}(U)$. Applying the basic properties of $\mathscr{L}_4$ as a Legendrian embedding , we defined curvatures of $M = \boldsymbol{X}(U)$ in [17] by using $\boldsymbol{X}^\ell$ as a normal vector field. The lightcone shape operator is defined to be $S^\ell_p = -d\boldsymbol{X}^\ell(\underline{u})$ for $p = \boldsymbol{X}(\underline{u})$. The *lightcone Gauss-Kronecker curvature* is $K_\ell(p) = \det S^\ell_p$ and the *lightcone mean curvature* is $H_\ell(p) = (\mathrm{Trace}\, S^\ell_p)/2$. Since $\langle \boldsymbol{X}(\underline{u}), \boldsymbol{X}^\ell(\underline{u})\rangle_1 = -2$, we have

$$\boldsymbol{n}^T(\underline{u}) = \frac{\boldsymbol{X}(\underline{u}) + \boldsymbol{X}^\ell(\underline{u})}{2} \text{ and } \boldsymbol{n}^S(\underline{u}) = \frac{\boldsymbol{X}(\underline{u}) - \boldsymbol{X}^\ell(\underline{u})}{2}.$$

It follows that

$$\kappa_i(\boldsymbol{n}^T)(p) = \frac{1}{2}(-1 + \kappa_i^\ell(p)), \ \ \kappa_i(\boldsymbol{n}^S)(p) = \frac{1}{2}(-1 - \kappa_i^\ell(p)), \tag{24}$$

where $\kappa^\ell_i(p)$ $(i=1,2)$ are the lightcone principal curvature of $M=\boldsymbol{X}(U)$ at $p=\boldsymbol{X}(\underline{u})$ (i.e., the eigenvalues of S^ℓ_p). One of the interesting properties of these curvatures is the following "Theorema Egregium"([17, 24]).

Theorem 7.2 (Theorem 9.3 in [17]) *Let K_I be the intrinsic Gauss curvature of $M=\boldsymbol{X}(U)$. Then we have the relation*

$$K_I = H_\ell.$$

It follows that a marginally trapped surface is an intrinsic flat surface as the following theorem shows.

Theorem 7.3 *Let $\boldsymbol{X}: U \longrightarrow LC^*$ be a spacelike surface and $\boldsymbol{X}^\ell$ the Δ_4-dual of $\boldsymbol{X}$. Then the following conditions are equivalent*:
1) *$M=\boldsymbol{X}(U)$ is a marginally trapped surface.*
2) $H_\ell \equiv 0$.
3) *The mean curvature vector $\mathfrak{H}(p)$ is zero or parallel to $\boldsymbol{X}(\underline{u})$ at any $p=\boldsymbol{X}(\underline{u})$.*
4) $K_I \equiv 0$.
5) *The Gauss curvature vector $\mathfrak{K}(p)$ is zero or parallel to $\boldsymbol{X}(\underline{u})$ at any $p=\boldsymbol{X}(\underline{u})$.*

Proof. By (24), we have

$$\begin{aligned} H(\boldsymbol{n}^T)(\underline{u}) &= \frac{1}{4}(-2+\kappa^\ell_1(p)+\kappa^\ell_2(p)) = \frac{1}{2}H_\ell(\underline{u}) - \frac{1}{2}, \\ H(\boldsymbol{n}^S)(\underline{u}) &= \frac{1}{4}(-2-\kappa^\ell_1(p)-\kappa^\ell_2(p)) = -\frac{1}{2}H_\ell(\underline{u}) - \frac{1}{2}. \end{aligned}$$

It follows that

$$\begin{aligned} H_\ell(\boldsymbol{n}^T,\boldsymbol{n}^S)(\underline{u}) &= H(\boldsymbol{n}^T)(\underline{u}) + H(\boldsymbol{n}^S)(\underline{u}) = -1, \\ H_\ell(\boldsymbol{n}^T,-\boldsymbol{n}^S)(\underline{u}) &= H(\boldsymbol{n}^T)(\underline{u}) - H(\boldsymbol{n}^S)(\underline{u}) = H_\ell(\underline{u}). \end{aligned}$$

Therefore, $H_\ell(\underline{u})=0$ if and only if $H(\boldsymbol{n}^T)(\underline{u}) = H(\boldsymbol{n}^S)(\underline{u})$. This means that $\mathfrak{H}(p)$ is parallel to $\boldsymbol{X}(\underline{u}) = \boldsymbol{n}^T(\underline{u}) + \boldsymbol{n}^S(\underline{u})$. By definition, the condition 3) implies the condition 1). By Corollary 4.4, $M=\boldsymbol{X}(U)$ is marginally trapped if and only if $H_\ell \equiv 0$. Since $H_\ell = K_I$, the conditions 2) and 4) are equivalent. By Proposition 3.13, the condition 4) (i.e., $K_I(p)=0$) is equivalent to the condition that $\mathfrak{K}(p)=\boldsymbol{0}$ or $\mathfrak{K}(p)$ is parallel to $\boldsymbol{n}^T(\underline{u}) + \boldsymbol{n}^S(\underline{u}) = \boldsymbol{X}(\underline{u})$. □

Since $H_\ell(\boldsymbol{n}^T,\boldsymbol{n}^S)(\underline{u}) = -1$, there are no strongly marginally trapped surfaces in the lightcone.

Let $V=\alpha\boldsymbol{X}^\ell$ be the variation vector filed. Then the first variation formula is

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} A(\boldsymbol{X}^\varepsilon) = -2\int_U \alpha H_\ell dM = -2\int_U \alpha K_I dM$$

and the second variation formula is

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} A(\boldsymbol{X}^\varepsilon) = 2\int_U \alpha^2 K_\ell dM.$$

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