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Author(s)	IZUMIYA, Shyuichi; HANDAN, Yildirim
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Slant geometry on spacelike submanifolds of codimension two in Lorentz-Minkowski space

Shyuichi Izumiya and Handan Yıldırım

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Abstract

In this paper, we construct one-parameter families of new extrinsic differential geometries on spacelike submanifolds of codimension two in Lorentz-Minkowski space.

1 Introduction

Analysis of extrinsic differential geometries of the submanifolds in Lorentz-Minkowski space has great interest in the relativity theory. Consequently, recently, singularity theory techniques have been used to study extrinsic differential geometries of the submanifolds in the pseudo-spheres in Lorentz-Minkowski space (see [2], [7]-[11], [13]-[25]).

For this purpose, extrinsic differential geometries of spacelike hypersurfaces in Hyperbolic space and de Sitter space were given in [16] and [24], respectively. On the other hand, Legendrian dualities between pseudo-spheres in Lorentz-Minkowski space were shown in [8]. As an application of these Legendrian dualities, an extrinsic differential geometry of spacelike hypersurfaces in the lightcone was constructed in [8]. Moreover, these Legendrian dualities were extended in [12] for one-parameter families depending on a parameter $\phi \in [0, \pi/2]$ of pseudo-spheres in Lorentz-Minkowski space. As applications of these extended Legendrian dualities, one-parameter families depending on ϕ of extrinsic differential geometries on spacelike hypersurfaces in Hyperbolic space, de Sitter space and the lightcone were established in [2, 12, 13]. And it was obtained that the results in [16] and [24] and the results in [8] are respectively special cases of the results in [2] and [13]. We call the geometries depending on the parameter ϕ of spacelike submanifolds in Lorentz-Minkowski space *slant geometry*.

Here, we explain our original motivation for the study of slant geometry. Recently, the first author and his collaborators have constructed a new geometry which is called a *horospherical geometry* in Hyperbolic space (see [9], [16], [17], [19], [20], [23]). Traditionally, there is another geometry which is non-Euclidean geometry of Gauss-Bolyai-Lobachevski (i.e., Hyperbolic geometry) in Hyperbolic space. We explain both of the geometries when the dimension is two (i.e., Hyperbolic plane). We consider Poincaré disk model D^2 of Hyperbolic plane which is an

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open unit disk in the (x, y) plane with Riemannian metric : $ds^2 = 4(dx^2 + dy^2)/(1 - x^2 - y^2)^2$. It is conformally equivalent to Euclidean plane, so that a circle in Poincaré disk is also a circle in Euclidean plane. A geodesic in Poincaré disk is an Euclidean circle perpendicular to the ideal boundary (i.e., unit circle). If we adopt geodesics as lines in Poincaré disk, we have the model of Hyperbolic geometry. We have another class of curves in Poincaré disk which has an analogous property with lines in Euclidean plane. A *horocycle* is an Euclidean circle which is tangent to the ideal boundary. We remark that a line in Euclidean plane can be considered as a limit of circles when the radii tends to infinity. A horocycle is also a curve as a limit of circles when the radii tends to infinity in Poincaré disk. Therefore, horocycles are also an analogous notion of lines. If we adopt horocycles as lines, what kind of geometry do we obtain? We say that two horocycles are *parallel* if they have common tangent point at ideal boundary. Under this definition, the axiom of parallel is satisfied. However, for any two points in the disk, there are always two horocycles passing through the points, so that the axiom 1 of Euclidean Geometry is not satisfied. In the case of general dimensions, we call this geometry a *horospherical geometry.* However, we have another kind of curves with the properties similar to those of Euclidean lines. A curve in Poincaré disk is called an *equidistant curve* if it is a circle whose intersection with the ideal boundary consists of two points. Generally, the angle between an equidistant curve and ideal boundary is $\phi \in (0, \pi/2]$. A geodesic is a special case of the equidistant curves with $\phi = \pi/2$. A horocycle is not an equidistant curve, but it is a circle with $\phi = 0$. In [2], one-parameter families depending on ϕ of the extrinsic differential geometries on spacelike hypersurfaces in Hyperbolic space and de Sitter space were investigated. We call them *slant geometry* of spacelike hypersurfaces in Hyperbolic space and de Sitter space. Moreover, slant geometry of spacelike hypersurfaces in the lightcone was considered in [13].

In this paper, we study some local properties of slant geometry on spacelike submanifolds of codimension two in Lorentz-Minkowski space. Thus, the results which were given in [2], [12] and [13] are special cases of our results. On the other hand, we generalize some of the results which were obtained in [10].

2 Basic notions

In this section, we give some basic notions related with Lorentz-Minkowski space. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R},\ i = 0, 1, \dots, n\}$ be an $(n+1)$-dimensional real vector space. For any vectors $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_0, y_1, \dots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called *Lorentz-Minkowski* $(n+1)$*-space* and denoted by $\mathbb{R}^{n+1}_1$. A vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0, respectively. The norm of a vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$, (cf. [27]). Given $\boldsymbol{v} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ and $c \in \mathbb{R}$, a *hyperplane* $HP(\boldsymbol{v}, c)$ *with pseudo normal* $\boldsymbol{v}$ is defined as

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\ \}.$$

It is said to be *spacelike*, *timelike* or *lightlike* provided $\boldsymbol{v}$ is timelike, spacelike or lightlike, respectively. In $\mathbb{R}^{n+1}_1$, there are three kinds of pseudo-spheres which are called *Hyperbolic* n*-space* with center $\boldsymbol{a}$ and radius r, *de Sitter* n*-space* with center $\boldsymbol{a}$ and radius r and the *lightcone* with center $\boldsymbol{a}$ and radius 0 and defined by

$$H^n(\boldsymbol{a}, r) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \ \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = -r^2\},$$

$$S_1^n(\boldsymbol{a}, r) = \{\boldsymbol{x} \in \mathbb{R}_1^{n+1} |\ \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = r^2\}$$

and

$$LC_{\boldsymbol{a}} = \{\boldsymbol{x} \in \mathbb{R}_1^{n+1} |\ \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0\},$$

respectively, where $r \in \mathbb{R} \setminus \{0\}$. If the center $\boldsymbol{a}$ is $\mathbf{0}$, then we denote these pseudo-spheres by

$$H^n(-r^2) = \{\boldsymbol{x} \in \mathbb{R}_1^{n+1} |\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -r^2\},$$

$$S_1^n(r^2) = \{\boldsymbol{x} \in \mathbb{R}_1^{n+1} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = r^2\ \}$$

and

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}_1^{n+1} \setminus \{\mathbf{0}\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\ \},$$

respectively. Instead of $S_1^n(1)$, we usually write S_1^n. For $\phi \in [0, \pi/2]$, we call $H^n(-\sin^2\phi)$ (respectively, $S_1^n(\sin^2\phi)$) *ϕ-hyperbolic space* (respectively, *ϕ-de Sitter space*) (see [2, 12, 13]). We remark that $H^n(-\sin^2 0) \setminus \{\mathbf{0}\} = S_1^n(\sin^2 0) \setminus \{\mathbf{0}\} = LC^*$. Throughout our paper, if $\phi = 0$, we will deal with LC^*, that is we will not consider the vector $\mathbf{0} \in \mathbb{R}_1^{n+1}$.

Now, we construct the basic tools for the study of slant geometry on spacelike submanifolds of codimension two in $\mathbb{R}_1^{n+1}$. We consider the orientation of $\mathbb{R}_1^{n+1}$ by the volume form $\boldsymbol{l}_0 \wedge \boldsymbol{l}_1 \wedge \cdots \wedge \boldsymbol{l}_n$, where $\{\boldsymbol{l}_0, \boldsymbol{l}_1, \ldots, \boldsymbol{l}_n\}$ is the dual basis of the canonical basis $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$. And also we give $\mathbb{R}_1^{n+1}$ a timelike orientation by choosing $\boldsymbol{e}_0 = (1, 0, \ldots, 0)$ as a future timelike vector field.

Let $\boldsymbol{X} : U \longrightarrow \mathbb{R}_1^{n+1}$ be a spacelike embedding for an open subset $U \subset \mathbb{R}^{n-1}$. We identify M with U through the embedding $\boldsymbol{X}$ and write $M = \boldsymbol{X}(U)$. Since $\boldsymbol{X}$ is a spacelike embedding, for any point $p = \boldsymbol{X}(u) \in M$, the tangent space T_pM of M at p is a spacelike subspace. In this case, it follows that the normal space N_pM is a timelike plane, (see [27]). So, we can choose a future directed timelike unit normal vector $\boldsymbol{n}^T(u) \in N_pM$. We say that $\boldsymbol{n}^T(u)$ is *future directed* if $\langle \boldsymbol{n}^T(u), \boldsymbol{e}_0 \rangle < 0$. Now, we define a spacelike unit normal vector $\boldsymbol{n}^S(u) \in N_pM$ by

$$\boldsymbol{n}^S(u) = \frac{\boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)}{\|\boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\|}.$$

Here, $\langle \boldsymbol{n}^T(u), \boldsymbol{n}^T(u) \rangle = -1$, $\langle \boldsymbol{n}^T(u), \boldsymbol{n}^S(u) \rangle = 0$ and $\langle \boldsymbol{n}^S(u), \boldsymbol{n}^S(u) \rangle = 1$. We remark that $\boldsymbol{n}^T \pm \boldsymbol{n}^S$ are lightlike and the directions of $\boldsymbol{n}^T \pm \boldsymbol{n}^S$ are independent of the choice of $\boldsymbol{n}^T$ (cf.[10]). We could also choose $-\boldsymbol{n}^S(u)$ as a spacelike unit normal vector with the above properties. But throughout our paper, we fix the direction of $\boldsymbol{n}^S(u)$. Then $\{\boldsymbol{n}^T, \boldsymbol{n}^S\}$ is called a *future directed normal frame* along $M = \boldsymbol{X}(U)$, [10]. Taking into account these, the lightlike geometry on spacelike submanifolds of codimension two in Lorentz-Minkowski space was constructed in [10]. This geometry is an extrinsic geometry such that the lightlike vector fields $\boldsymbol{n}^T \pm \boldsymbol{n}^S$ play a similar role as the Gauss map of a hypersurface in Euclidean space. If we choose $\boldsymbol{n}^T$ (respectively, $\pm\boldsymbol{n}^S$) as the Gauss map, we can construct an extrinsic geometry according to $\boldsymbol{n}^T$ (respectively, $\pm\boldsymbol{n}^S$) which is analogous to Euclidean case [26]. In this paper, we construct one-parameter families of extrinsic geometries between the lightlike geometry depending on $\boldsymbol{n}^T + \boldsymbol{n}^S$ and the geometry according to $\boldsymbol{n}^S$ (respectively, $\boldsymbol{n}^T$).

Since $\{\boldsymbol{X}_{u_1}(u), \ldots, \boldsymbol{X}_{u_{n-1}}(u)\}$ is a basis of T_pM, $\{\boldsymbol{n}^T(u), \boldsymbol{n}^S(u), \boldsymbol{X}_{u_1}(u), \ldots, \boldsymbol{X}_{u_{n-1}}(u)\}$ is a basis of $T_p\mathbb{R}_1^{n+1}$.

3 Spacelike slant geometry on spacelike submanifolds of codimension two

In this section, we construct spacelike slant geometry on spacelike submanifolds of codimension two in Lorentz-Minkowski space.

Let take $\cos\phi\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u) \in S_1^n(\sin^2\phi)$ for $\phi \in [0, \pi/2]$. For $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$), $\cos\phi\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)$ (respectively, $\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)$) are spacelike (respectively, lightlike) normal vectors, where $u \in U$. We choose $\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S$ (respectively, $\boldsymbol{n}^T + \boldsymbol{n}^S$) as a spacelike (respectively, lightlike) normal vector field along M for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$). We call $\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S$ *ϕ-de Sitter Gauss map* of M.

At each point $p = \boldsymbol{X}(u) \in M$, we consider the following linear mapping

$$d_p(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S) : T_pM \longrightarrow T_p\mathbb{R}_1^{n+1} = T_pM \oplus N_pM$$

and the following orthogonal projections

$$\pi^t : T_pM \oplus N_pM \longrightarrow T_pM, \pi^n : T_pM \oplus N_pM \longrightarrow N_pM.$$

We define

$$d_p(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)^t = \pi^t \circ d_p(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)$$

and

$$d_p(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)^n = \pi^n \circ d_p(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S).$$

The linear transformations $S^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p) = -d_p(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)^t$ and $d_p(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)^n$ of T_pM are called respectively *ϕ-de Sitter $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-shape operator* and *$\phi$-de Sitter normal connection* of M at p. Moreover, unit normal vector field $\boldsymbol{n}^T$ is said to be parallel at p if $d_p(\boldsymbol{n}^T)^n = \pi^n \circ d_p(\boldsymbol{n}^T) = \boldsymbol{0}$. Furthermore, $\boldsymbol{n}^T$ is called parallel if it is parallel at all points of M.

The eigen values which are denoted by $\kappa_i^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p)$ $(i = 1, ..., n-1)$ of $S^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p)$ are called *ϕ-de Sitter principal curvatures* according to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ at p. And also *ϕ-de Sitter Gauss-Kronecker curvature* with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ at p is defined as

$$K^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p) = \det S^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p).$$

A point p is called *ϕ-de Sitter $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic* if $S^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p) = \kappa^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p)\,\mathrm{id}_{T_pM}$ for a function $\kappa^d[\phi]$. M is called *totally ϕ-de Sitter $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic* if all points of M are ϕ-de Sitter $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic. Moreover, a point p is called *ϕ-de Sitter $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-parabolic* if $K^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p) = 0$. Furthermore, p is called *ϕ-de Sitter $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-flat* if it is ϕ-de Sitter $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic and $K^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p) = 0$.

Since $\boldsymbol{X}$ is a spacelike embedding, the induced Riemannian metric (the *first fundamental form*) on M is given by $ds^2 = \sum_{i,j=1}^{n-1} g_{ij}du_idu_j$, where $g_{ij}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$. Moreover, *ϕ-de Sitter second fundamental invariant* with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ is defined by $h^d[\phi]_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) = \langle -(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$.

Proposition 3.1 *Under the above notations, we have the following:*

(1) $\left(\cos\phi \boldsymbol{n}^T + \boldsymbol{n}^S\right)_{u_i} = \left\langle \boldsymbol{n}^T_{u_i}, \boldsymbol{n}^S\right\rangle (\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S) - \sum_{j=1}^{n-1} h^d[\phi]_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\boldsymbol{X}_{u_j}$ *(ϕ-de Sitter Weingarten formula),*

(2) $\pi^t \circ (\cos\phi \boldsymbol{n}^T + \boldsymbol{n}^S)_{u_i} = -\sum_{j=1}^{n-1} h^d[\phi]_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\boldsymbol{X}_{u_j}$.

Here, $\left(h^d[\phi]_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\right) = \left(h^d[\phi]_{ik}(\boldsymbol{n}^T, \boldsymbol{n}^S)\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right) = (g_{kj})^{-1}$.

Proof. There exist real numbers λ, μ, Γ_i^j such that $(\cos\phi \boldsymbol{n}^T + \boldsymbol{n}^S)_{u_i} = \lambda \boldsymbol{n}^T + \mu \boldsymbol{n}^S + \sum_{j=1}^{n-1} \Gamma_i^j \boldsymbol{X}_{u_j}$. From the equations $\left\langle \boldsymbol{n}^T, \boldsymbol{n}^T\right\rangle = -1$ and $\left\langle \boldsymbol{n}^S, \boldsymbol{n}^S\right\rangle = 1$, it follows that $\left\langle \boldsymbol{n}^T_{u_i}, \boldsymbol{n}^T\right\rangle = 0$ and $\left\langle \boldsymbol{n}^S_{u_i}, \boldsymbol{n}^S\right\rangle = 0$, respectively. On the other hand, we have $\left\langle \boldsymbol{n}^T, \boldsymbol{n}^S\right\rangle = \left\langle \boldsymbol{n}^T, \boldsymbol{X}_{u_i}\right\rangle = \left\langle \boldsymbol{n}^S, \boldsymbol{X}_{u_i}\right\rangle = 0$. Therefore, we obtain $\lambda = -\left\langle \boldsymbol{n}^S_{u_i}, \boldsymbol{n}^T\right\rangle$ and $\mu = \cos\phi \left\langle \boldsymbol{n}^T_{u_i}, \boldsymbol{n}^S\right\rangle$. Moreover, it is found from the equation $\left\langle \boldsymbol{n}^T, \boldsymbol{n}^S\right\rangle = 0$ that $\left\langle \boldsymbol{n}^S_{u_i}, \boldsymbol{n}^T\right\rangle = -\left\langle \boldsymbol{n}^T_{u_i}, \boldsymbol{n}^S\right\rangle$. Furthermore, $\left\langle \lambda \boldsymbol{n}^T + \mu \boldsymbol{n}^S, \boldsymbol{X}_{u_j}\right\rangle = 0$ and so

$$-h^d[\phi]_{i\beta}(\boldsymbol{n}^T, \boldsymbol{n}^S) = \sum_{\alpha=1}^{n-1} \Gamma_i^\alpha \left\langle \boldsymbol{X}_{u_\alpha}, \boldsymbol{X}_{u_\beta}\right\rangle = \sum_{\alpha=1}^{n-1} \Gamma_i^\alpha g_{\alpha\beta}.$$

Thus, we get

$$-h^d[\phi]_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S) = -\sum_{\beta=1}^{n-1} h^d[\phi]_{i\beta}(\boldsymbol{n}^T, \boldsymbol{n}^S) g^{\beta j} = \sum_{\beta=1}^{n-1}\sum_{\alpha=1}^{n-1} \Gamma_i^\alpha g_{\alpha\beta} g^{\beta j} = \Gamma_i^j.$$

This completes the proof of the formula (1). The formula (2) follows from the formula (1). □

We remark that this proposition is a generalization of Proposition 3.2 in [10]. Moreover, as a corollary of this proposition, we have the following explicit expression of ϕ-de Sitter Gauss-Kronecker curvature with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ by means of Riemannian metric and ϕ-de Sitter second fundamental invariant.

Corollary 3.2 *Under the same notations as in the above proposition, ϕ-de Sitter Gauss-Kronecker curvature with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ is given by*

$$K^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S) = \frac{\det\left(h^d[\phi]_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)\right)}{\det\left(g_{\alpha\beta}\right)}.$$

Proof. In terms of the formula (2) in the previous proposition, the representation matrix of ϕ-de Sitter shape operator with respect to the basis $\left\{\boldsymbol{X}_{u_1}, ..., \boldsymbol{X}_{u_{n-1}}\right\}$ is $\left(h^d[\phi]_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\right)$. It is clear from this fact that

$$\begin{aligned} K^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S) &= \det S^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S) = \det\left(h^d[\phi]_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\right) \\ &= \det\left(\left(h^d[\phi]_{i\beta}(\boldsymbol{n}^T, \boldsymbol{n}^S)\right)\left(g^{\beta j}\right)\right) = \frac{\det\left(h^d[\phi]_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)\right)}{\det\left(g_{\alpha\beta}\right)}. \end{aligned}$$

□

Since $\langle -(\cos\phi \boldsymbol{n}^T + \boldsymbol{n}^S)(u), \boldsymbol{X}_{u_j}(u)\rangle = 0$, we obtain that $h^d[\phi]_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) = \langle(\cos\phi \boldsymbol{n}^T + \boldsymbol{n}^S)(u), \boldsymbol{X}_{u_i u_j}(u)\rangle$. Hence, ϕ-de Sitter second fundamental invariant at a point $p = \boldsymbol{X}(u) \in M$

depends only on the values $\cos\phi\boldsymbol{n}^T(u)+\boldsymbol{n}^S(u)$ and $\boldsymbol{X}_{u_iu_j}(u)$ of the vector fields $\cos\phi\boldsymbol{n}^T+\boldsymbol{n}^S$ and $\boldsymbol{X}_{u_iu_j}$. And also ϕ-de Sitter Gauss-Kronecker curvature with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ at p depends only on the vectors $\cos\phi\boldsymbol{n}^T(u)+\boldsymbol{n}^S(u)$, $\boldsymbol{X}_{u_i}(u)$ and $\boldsymbol{X}_{u_iu_j}(u)$.

Now, we define a *ϕ-de Sitter height function* on $M=\boldsymbol{X}(U)$ by

$$\begin{array}{rccl} H^d_\phi & : & U\times S^n_1(\sin^2\phi) & \longrightarrow & \mathbb{R} \\ & & (u,\boldsymbol{v}) & \longmapsto & \langle \boldsymbol{X}(u),\boldsymbol{v}\rangle. \end{array}$$

Besides, we denote the Hessian matrix of ϕ-de Sitter height function $h^d_{\phi,\boldsymbol{v}_0}(u)=H^d_\phi(u,\boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}\left(h^d_{\phi,\boldsymbol{v}_0}\right)(u_0)$.

Proposition 3.3 *Let $H^d_\phi: U\times S^n_1(\sin^2\phi)\longrightarrow\mathbb{R}$ be a ϕ-de Sitter height function on $M=\boldsymbol{X}(U)$. Then we have the following:*
(1) *$\partial H^d_\phi/\partial u_i(u_0,\boldsymbol{v_0})=0$ $(i=1,...,n-1)$ if and only if $\boldsymbol{v_0}=\lambda\boldsymbol{n}^T(u_0)+\mu\boldsymbol{n}^S(u_0)$ such that $-\lambda^2+\mu^2=\sin^2\phi$.*
Assume that $p_0=\boldsymbol{X}(u_0)$ and $\boldsymbol{v_0}=(\cos\phi\boldsymbol{n}^T+\boldsymbol{n}^S)(u_0)$.
(2) *p_0 is a ϕ-de Sitter $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-parabolic point if and only if* $\det\mathrm{Hess}\left(h^d_{\phi,\boldsymbol{v}_0}\right)(u_0)=0$.
(3) *p_0 is a ϕ-de Sitter $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-flat point if and only if* $\mathrm{rank}\,\mathrm{Hess}\left(h^d_{\phi,\boldsymbol{v}_0}\right)(u_0)=0$.

Proof. (1) As $\{\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0),\boldsymbol{X}_{u_1}(u_0),\ldots,\boldsymbol{X}_{u_{n-1}}(u_0)\}$ is a basis of $T_{p_0}\mathbb{R}^{n+1}_1$ for $p_0=\boldsymbol{X}(u_0)$, there exist real numbers λ, μ, ξ_j $(j=1,...,n-1)$ such that $\boldsymbol{v_0}=\lambda\boldsymbol{n}^T(u_0)+\mu\boldsymbol{n}^S(u_0)+\sum\limits_{j=1}^{n-1}\xi_j\boldsymbol{X}_{u_j}(u_0)$. Since $\partial H^d_\phi/\partial u_i(u_0,\boldsymbol{v_0})=\langle\boldsymbol{X}_{u_i}(u_0),\boldsymbol{v_0}\rangle$, we deduce that $\langle\boldsymbol{X}_{u_i}(u_0),\boldsymbol{v_0}\rangle=\sum\limits_{j=1}^{n-1}\xi_jg_{ij}(u_0)$. As g_{ij} is positive definite, it is obvious that $\xi_j=0$ $(j=1,...,n-1)$. Moreover, since $\boldsymbol{v_0}\in S^n_1(\sin^2\phi)$, it follows that $-\lambda^2+\mu^2=\sin^2\phi$. Thus, the proof of the first assertion is completed.
(2) From the definition, we have $h^d_{\phi,\boldsymbol{v}_0}(u_0)=\langle\boldsymbol{X}(u_0),\boldsymbol{v}_0\rangle$. Using this equation and Proposition 3.1, we obtain

$$\begin{aligned}
\frac{\partial^2 h^d_{\phi,\boldsymbol{v}_0}}{\partial u_i\partial u_j}(u_0) &= \left\langle\boldsymbol{X}_{u_iu_j}(u_0),\boldsymbol{v}_0\right\rangle \\
&= -\left\langle\boldsymbol{X}_{u_i}(u_0),\left(\cos\phi\boldsymbol{n}^T+\boldsymbol{n}^S\right)_{u_j}(u_0)\right\rangle \\
&= \left\langle\boldsymbol{X}_{u_i}(u_0),\sum_{k=1}^{n-1}h^d[\phi]^k_j(\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0))\boldsymbol{X}_{u_k}(u_0)\right\rangle \\
&= h^d[\phi]_{ij}(\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0)).
\end{aligned}$$

This means that $\mathrm{Hess}\left(h^d_{\phi,\boldsymbol{v}_0}\right)(u_0)=\left(h^d[\phi]_{ij}(\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0))\right)$. Consequently, we have

$$K^d[\phi](\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0))=\frac{\det\left(h^d[\phi]_{ij}(\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0))\right)}{\det\left(\left(g_{\alpha\beta}\right)(u_0)\right)}=\frac{\det\mathrm{Hess}\left(h^d_{\phi,\boldsymbol{v}_0}\right)(u_0)}{\det\left(\left(g_{\alpha\beta}\right)(u_0)\right)}.$$

So, the second assertion follows from these formulas.
(3) By means of ϕ-de Sitter Weingarten formula, $p_0=\boldsymbol{X}(u_0)$ is a ϕ-de Sitter $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ umbilic point if and only if there exists an orthogonal matrix A such that $A^t\left(h^d[\phi]^\alpha_i(\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0))\right)A=\kappa^d[\phi](u_0)I$. Therefore, we get

$$\left(h^d[\phi]^\alpha_i(\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0))\right)=A\,\kappa^d[\phi](u_0)\,A^t=\kappa^d[\phi](u_0)I,$$

so that

$$\begin{aligned}
\operatorname{Hess}\left(h^d_{\phi,\boldsymbol{v}_0}\right)(u_0) &= \left(h^d[\phi]_{ij}(\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0))\right)\\
&= \left(h^d[\phi]_i^\alpha(\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0))\right)((g_{\alpha j})(u_0))\\
&= \kappa^d[\phi](u_0)\,((g_{ij})(u_0)).
\end{aligned}$$

Thus, p_0 is a ϕ-de Sitter $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-flat point (i.e., $\kappa^d[\phi](u_0)=0$) if and only if $\operatorname{rank}\operatorname{Hess}\left(h^d_{\phi,\boldsymbol{v}_0}\right)(u_0)=0$. □

Theorem 3.4 *For a spacelike embedding $\boldsymbol{X}: U\to\mathbb{R}^{n+1}_1$, where U is an open subset of $\mathbb{R}^{n-1}$, the following conditions are equivalent:*
(1) $M=\boldsymbol{X}(U)$ is totally ϕ-de Sitter flat, where $\boldsymbol{n}^T$ is a parallel vector field.
(2) ϕ-de Sitter Gauss map $\cos\phi\boldsymbol{n}^T+\boldsymbol{n}^S$ is constant, where $\boldsymbol{n}^T$ is a parallel vector field.
(3) There exists a vector $\boldsymbol{v}\in S^n_1(\sin^2\phi)$ such that M is a subset of a timelike (respectively, lightlike) hyperplane $HP(\boldsymbol{v},c)$ for $\phi\in(0,\pi/2]$ (respectively, $\phi=0$).

Proof. Suppose that $M=\boldsymbol{X}(U)$ is totally ϕ-de Sitter flat. In this case, $h^d[\phi]_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)(u)=0$ $(i,j=1,...,n-1)$ at any point $u\in U$ for any future directed normal frame $\{\boldsymbol{n}^T,\boldsymbol{n}^S\}$ along M. As a result, from Proposition 3.1 (1), it is clear that

$$\left(\cos\phi\boldsymbol{n}^T+\boldsymbol{n}^S\right)_{u_i}(u)=\left\langle \boldsymbol{n}^T_{u_i}(u),\boldsymbol{n}^S(u)\right\rangle(\boldsymbol{n}^T+\cos\phi\boldsymbol{n}^S)(u).$$

Moreover, as $\boldsymbol{n}^T$ is a parallel vector field, it follows that $\left(\cos\phi\boldsymbol{n}^T+\boldsymbol{n}^S\right)_{u_i}(u)=0$. This means that ϕ-de Sitter Gauss map $\cos\phi\boldsymbol{n}^T+\boldsymbol{n}^S$ is constant. Thus, condition (1) implies condition (2). By Proposition 3.1, condition (2) implies condition (1).

Now, we assume that ϕ-de Sitter Gauss map $\cos\phi\boldsymbol{n}^T+\boldsymbol{n}^S$ is constant. Then we write $(\cos\phi\boldsymbol{n}^T+\boldsymbol{n}^S)(u)=\boldsymbol{v}$ and consider the following function

$$\begin{array}{rccl} F & : & U & \longrightarrow & \mathbb{R}\\ & & u & \longmapsto & \langle\boldsymbol{X}(u),\boldsymbol{v}\rangle. \end{array}$$

Hence, we have

$$\partial F/\partial u_i(u)=\left\langle \boldsymbol{X}_{u_i}(u),(\cos\phi\boldsymbol{n}^T+\boldsymbol{n}^S)(u)\right\rangle=0$$

for $(i=1,\dots,n-1)$. Consequently, $F(u)=\langle\boldsymbol{X}(u),\boldsymbol{v}\rangle$ is constant. If we denote it by c, since $\boldsymbol{v}\in S^n_1(\sin^2\phi)$, M is a subset of a timelike (respectively, lightlike) hyperplane $HP(\boldsymbol{v},c)$ for $\phi\in(0,\pi/2]$ (respectively, $\phi=0$). This completes the proof that condition (2) implies condition (3). Now, we suppose that M is a subset of a timelike (respectively, lightlike) hyperplane $HP(\boldsymbol{v},c)$ for $\phi\in(0,\pi/2]$ (respectively, $\phi=0$), where $\boldsymbol{v}\in S^n_1(\sin^2\phi)$. So, we can write $\langle\boldsymbol{X}(u),\boldsymbol{v}\rangle=c$ for any $u\in U$. As $\langle\boldsymbol{X}_{u_i}(u),\boldsymbol{v}\rangle=0$, from Proposition 3.3 (1), it follows that $\boldsymbol{v}=\lambda\boldsymbol{n}^T(u)+\mu\boldsymbol{n}^S(u)$ such that $-\lambda^2+\mu^2=\sin^2\phi$. Since the vector $\boldsymbol{v}$ doesn't depend on the choices of λ and μ, we can take $\lambda=\cos\phi$ and $\mu=1$. This means that $\left(\cos\phi\boldsymbol{n}^T+\boldsymbol{n}^S\right)_{u_i}(u)=\boldsymbol{0}$. Therefore, we obtain $0=\left\langle\left(\cos\phi\boldsymbol{n}^T+\boldsymbol{n}^S\right)_{u_i}(u),\boldsymbol{n}^T(u)\right\rangle=\left\langle\boldsymbol{n}^S_{u_i}(u),\boldsymbol{n}^T(u)\right\rangle$. On the other hand, we have $\left\langle\boldsymbol{n}^T_{u_i}(u),\boldsymbol{n}^T(u)\right\rangle=0$ and $\left\langle\boldsymbol{n}^T_{u_i}(u),\boldsymbol{n}^S(u)\right\rangle=-\left\langle\boldsymbol{n}^T(u),\boldsymbol{n}^S_{u_i}(u)\right\rangle$. As a result, $\boldsymbol{n}^T$ is a parallel vector field. Thus, we prove that condition (3) implies condition (2). □

Theorem 3.5 *Suppose that $M = \boldsymbol{X}(U)$ is non-flat totally umbilical, $\boldsymbol{n}^T$ is a parallel vector field and $\phi \in [0, \pi/2]$ is fixed. Then $\kappa^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p)$ is non-zero constant $\kappa^d[\phi]$ and M is located in $S_1^n(\boldsymbol{a}, r)$, where $\boldsymbol{a} = \frac{1}{\kappa^d[\phi]}\boldsymbol{c}$ and $r = |\frac{\sin\phi}{\kappa^d[\phi]}|$.*

Proof. From the definition, we get $S^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(\boldsymbol{X}_{u_i}) = -d(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)^t(\boldsymbol{X}_{u_i})$. Since $\boldsymbol{n}^T$ is a parallel vector field, from ϕ-de Sitter Weingarten formula, we deduce that

$$(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_i} = -\sum_{j=1}^{n-1} h^d[\phi]_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\boldsymbol{X}_{u_j}.$$

Hence, we can write that

$$S^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(\boldsymbol{X}_{u_i}) = -(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_i} = \kappa^d[\phi]\boldsymbol{X}_{u_i}$$

for $(i = 1, ..., n-1)$. Consequently, it follows that

$$-(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_iu_j} = (\kappa^d[\phi])_{u_j}\boldsymbol{X}_{u_i} + \kappa^d[\phi]\boldsymbol{X}_{u_iu_j}.$$

On the other hand, we have

$$S^d[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(\boldsymbol{X}_{u_j}) = -(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_j} = \kappa^d[\phi]\boldsymbol{X}_{u_j}$$

for $(j = 1, ..., n-1)$. Therefore, we get

$$-(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_ju_i} = (\kappa^d[\phi])_{u_i}\boldsymbol{X}_{u_j} + \kappa^d[\phi]\boldsymbol{X}_{u_ju_i}.$$

As $(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_iu_j} = (\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_ju_i}$ and $\kappa^d[\phi]\boldsymbol{X}_{u_iu_j} = \kappa^d[\phi]\boldsymbol{X}_{u_ju_i}$, we obtain

$$(\kappa^d[\phi])_{u_j}\boldsymbol{X}_{u_i} = (\kappa^d[\phi])_{u_i}\boldsymbol{X}_{u_j}.$$

Since $\left\{\boldsymbol{X}_{u_1}, ..., \boldsymbol{X}_{u_{n-1}}\right\}$ is linearly independent, $\kappa^d[\phi]$ is constant. Moreover, from the equation $d(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S) = -\kappa^d[\phi]d\boldsymbol{X}$, it is obvious that $d(\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S + \kappa^d[\phi]\boldsymbol{X}) = \boldsymbol{0}$. So, there is a constant vector $\boldsymbol{c}$ such that

$$\boldsymbol{c} = \cos\phi\boldsymbol{n}^T(u) + \boldsymbol{n}^S(u) + \kappa^d[\phi]\boldsymbol{X}(u).$$

From this equation, we have

$$\boldsymbol{X}(u) - \frac{1}{\kappa^d[\phi]}\boldsymbol{c} = -\frac{1}{\kappa^d[\phi]}(\cos\phi\boldsymbol{n}^T(u) + \boldsymbol{n}^S(u)).$$

Hence, it is found that

$$\left\langle \boldsymbol{X}(u) - \frac{1}{\kappa^d[\phi]}\boldsymbol{c}, \boldsymbol{X}(u) - \frac{1}{\kappa^d[\phi]}\boldsymbol{c} \right\rangle = \frac{\sin^2\phi}{\kappa^d[\phi]^2}.$$

This completes the proof. □

Remark 3.6 *We remark that if $\phi = 0$ in the above theorem, then M is located in LC_a.*

4 Spacelike slant geometry from the viewpoint of Lagrangian singularity theory

In this section, we interpret the results of Proposition 3.3 from the viewpoint of Lagrangian singularity theory. We consider the relation between the contact of submanifolds with foliations and the $\mathcal{R}^+$-classification of functions. Let X_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and let $g_i : (X_i, \bar{x}_i) \to (\mathbb{R}^n, \bar{y}_i)$ be immersion germs and $f_i : (\mathbb{R}^n, \bar{y}_i) \to (\mathbb{R}, 0)$ be submersion germs. For a submersion germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, we define the regular foliation $\mathcal{F}_f$ by $\mathcal{F}_f = \{f^{-1}(c) | c \in (\mathbb{R}, 0)\}$. We say that *the contact of* X_1 *with* $\mathcal{F}_{f_1}$ at $\bar{y}_1$ is of the *same type* as *the contact of* X_2 *with* $\mathcal{F}_{f_2}$ at $\bar{y}_2$ if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, \bar{y}_1) \to (\mathbb{R}^n, \bar{y}_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1(c)) = Y_2(c)$, where $Y_i(c) = f_i^{-1}(c)$ for each $c \in (\mathbb{R}, 0)$. In this case, we write $K(X_1, \mathcal{F}_{f_1}; \bar{y}_1) = K(X_2, \mathcal{F}_{f_2}; \bar{y}_2)$. It is clear that in the definition, $\mathbb{R}^n$ could be replaced by any manifold. We apply Goryunov's method [6] for $\mathcal{R}^+$-equivalences of function germs.

Proposition 4.1 ([6]) *Let* X_i $(i = 1, 2)$ *be submanifolds of* $\mathbb{R}^n$ *with* $\dim X_1 = \dim X_2 = n - 1$ *(i.e., hypersurfaces),* $g_i : (X_i, \bar{x}_i) \to (\mathbb{R}^n, \bar{y}_i)$ *be immersion germs and* $f_i : (\mathbb{R}^n, \bar{y}_i) \to (\mathbb{R}, 0)$ *be submersion germs. Then* $K(X_1, \mathcal{F}_{f_1}; \bar{y}_1) = K(X_2, \mathcal{F}_{f_2}; \bar{y}_2)$ *if and only if* $f_1 \circ g_1$ *and* $f_2 \circ g_2$ *are* $\mathcal{R}^+$*-equivalent (i.e., there exists a diffeomorphism germ* $\phi : (X_1, \bar{x}_1) \to (X_2, \bar{x}_2)$ *such that* $(f_2 \circ g_2) \circ \phi = f_1 \circ g_1$*).*

Now, we denote the set of function germs $(X, 0) \to \mathbb{R}$ by $C_0^\infty(X)$. Let J_f be the Jacobian ideal in $C_0^\infty(X)$ (i.e., $J_f = \langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle_{C_0^\infty(X)}$). Let $\mathcal{R}_k(f) = C_0^\infty(X)/J_f^k$ and $\bar{f}$ be the image of f in this local ring. Here, J_f^k denotes the kth power of the Jacobian ideal J_f. We say that f satisfies the *Milnor condition* if $\dim_{\mathbb{R}} \mathcal{R}_1(f) < \infty$ (cf.[5]). Thus, we have the following algebraic characterization which was given by Golubitsky and Guillemin in [5] for the $\mathcal{R}^+$-equivalence of function germs.

Proposition 4.2 ([5]) *Let* f *and* g *be germs of functions at* 0 *in* X *satisfying the Milnor condition with* $df(0) = dg(0) = 0$*. Then* f *and* g *are* $\mathcal{R}^+$*-equivalent if and only if*

(1) *The ranks and signatures of* $\mathrm{Hess}(f)(0)$ *and* $\mathrm{Hess}(g)(0)$ *are equal.*

(2) *There is an isomorphism* $\gamma : \mathcal{R}_2(f) \to \mathcal{R}_2(g)$ *such that* $\gamma(\bar{f}) = \bar{g}$*.*

Now, we consider a function

$$\begin{array}{cccc} \mathfrak{h}^d_{\phi,\boldsymbol{v}_0} & : & \mathbb{R}^{n+1}_1 & \longrightarrow & \mathbb{R} \\ & & \boldsymbol{x} & \longmapsto & \langle \boldsymbol{x}, \boldsymbol{v}_0 \rangle, \end{array}$$

where $\boldsymbol{v_0} = \lambda \boldsymbol{n}^T(u_0) + \mu \boldsymbol{n}^S(u_0)$ such that $-\lambda^2 + \mu^2 = \sin^2 \phi$. In this case, we have $\mathfrak{h}^d_{\phi,\boldsymbol{v}_0} \circ \boldsymbol{X}(u) = H^d_\phi(u, \boldsymbol{v}_0)$. Therefore, from Proposition 3.3, we get

$$\frac{\partial \mathfrak{h}^d_{\phi,\boldsymbol{v}_0} \circ \boldsymbol{X}}{\partial u_i}(u_0) = \frac{\partial H^d_\phi}{\partial u_i}(u_0, \boldsymbol{v}_0) = 0$$

for $(i = 1, \ldots, n-1)$. This means that for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$), the timelike (respectively, lightlike) hyperplane ${\mathfrak{h}^d_{\phi,\boldsymbol{v}_0}}^{-1}(c) = HP(\boldsymbol{v}_0, c)$ is tangent to $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(u_0)$, where $c = \langle \boldsymbol{X}(u_0), \boldsymbol{v}_0\rangle$. So, we call $HP(\boldsymbol{v}_0, c)$ a *tangent timelike (respectively, lightlike) hyperplane* with the pseudo-normal $\boldsymbol{v}_0$ for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$). As we have spacelike (respectively, lightlike) directions $\cos\phi \boldsymbol{n}^T(u) + \boldsymbol{n}^S(u)$ (respectively, $\boldsymbol{n}^T(u) + \boldsymbol{n}^S(u)$) and $\cos\phi \boldsymbol{n}^T(u) - \boldsymbol{n}^S(u)$ (respectively, $\boldsymbol{n}^T(u) - \boldsymbol{n}^S(u)$) in the normal plane, we have two tangent timelike (respectively, lightlike) hyperplanes at the point p_0 depending on the direction of $\boldsymbol{v}_0$ for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$). Here, we choose only one of them. Let ε be a sufficiently small positive real number. For any $t \in I_\varepsilon = (c - \varepsilon, c + \varepsilon)$, we have a timelike (respectively, lightlike) hyperplane $HP(\boldsymbol{v}_0, t) = {\mathfrak{h}^d_{\phi,\boldsymbol{v}_0}}^{-1}(t)$. In this case, $\mathcal{F}_{\mathfrak{h}^d_{\phi,\boldsymbol{v}_0}}$ is a family of parallel timelike (respectively, lightlike) hyperplanes around p_0 such that ${\mathfrak{h}^d_{\phi,\boldsymbol{v}_0}}^{-1}(c)$ is the tangent timelike (respectively, lightlike) hyperplane of M at p_0. Let $\boldsymbol{X}_i : (U, \bar{u}_i) \to (\mathbb{R}^{n+1}_1, \boldsymbol{X}_i(\bar{u}_i))$ $(i = 1, 2)$ be spacelike embedding germs with codimension two. Then we have $h^d_{\phi i,\boldsymbol{v}_i}(u) = \mathfrak{h}^d_{\phi,\boldsymbol{v}_i} \circ \boldsymbol{X}_i(u)$. Hence, we obtain the following proposition as a corollary of Proposition 4.1 and Proposition 4.2.

Proposition 4.3 *Let $\boldsymbol{X}_i : (U, \bar{u}_i) \to (\mathbb{R}^{n+1}_1, \boldsymbol{X}_i(\bar{u}_i))$ $(i = 1, 2)$ be spacelike embedding germs of codimension two such that $h^d_{\phi i,\boldsymbol{v}_i}$ satisfy the Milnor condition, where $\boldsymbol{v}_i = \cos\phi \boldsymbol{n}^T(\bar{u}_i) + \boldsymbol{n}^S(\bar{u}_i)$ (respectively, $\boldsymbol{v}_i = \boldsymbol{n}^T(\bar{u}_i) + \boldsymbol{n}^S(\bar{u}_i)$) are pseudo-normals of the tangent timelike (respectively, lightlike) hyperplanes of $\boldsymbol{X}_i$ for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$). Then the following conditions are equivalent:*

(1) $K(\boldsymbol{X}_1(U), \mathcal{F}_{\mathfrak{h}^d_{\phi,\boldsymbol{v}_1}}; \boldsymbol{X}_1(\bar{u}_1)) = K(\boldsymbol{X}_2(U), \mathcal{F}_{\mathfrak{h}^d_{\phi,\boldsymbol{v}_2}}; \boldsymbol{X}_2(\bar{u}_2))$.

(2) $h^d_{\phi 1,\boldsymbol{v}_1}$ *and* $h^d_{\phi 2,\boldsymbol{v}_2}$ *are $\mathcal{R}^+$-equivalent.*

(3) (*a*) *The ranks and signatures of* $\mathrm{Hess}(h^d_{\phi 1,\boldsymbol{v}_1})(\bar{u}_1)$ *and* $\mathrm{Hess}(h^d_{\phi 2,\boldsymbol{v}_2})(\bar{u}_2)$ *are equal.*

(*b*) *There is an isomorphism* $\gamma : \mathcal{R}_2(h^d_{\phi 1,\boldsymbol{v}_1}) \to \mathcal{R}_2(h^d_{\phi 2,\boldsymbol{v}_2})$ *such that* $\gamma(\overline{h}^d_{\phi 1,\boldsymbol{v}_1}) = (\overline{h}^d_{\phi 2,\boldsymbol{v}_2})$.

Now, we can interpret the meaning of the above proposition from the viewpoint of Lagrangian singularity theory. We consider $\boldsymbol{v} = (v_0, v_1, \dots, v_n) \in S^n_1(\sin^2\phi) \setminus \{\boldsymbol{0}\}$. Then we have $(v_1, \dots, v_n) \neq (0, \dots, 0)$. Without the loss of generality, we assume that $v_1 > 0$. We choose the local coordinate neighbourhood system (V^1_+, U^1, ψ), where

$$V^1_+ = \{\boldsymbol{v} \in S^n_1(\sin^2\phi) \mid v_1 > 0\},\ U^1 = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 - \sum_{i=2}^n x_i^2 + \sin^2\phi > 0\ \}$$

and $\psi : V^1_+ \longrightarrow U^1$ is induced by the canonical projection.

Proposition 4.4 *The ϕ-de Sitter height function $H^d_\phi : U \times S^n_1(\sin^2\phi) \longrightarrow \mathbb{R}$ is a Morse family of hypersurfaces.*

Proof. We consider the local coordinate neighborhood V^1_+. For any $\boldsymbol{v} = (v_0, v_1, \dots, v_n) \in V^1_+$, it is obvious that $v_1 = \sqrt{v_0^2 - \sum_{i=2}^n v_i^2 + \sin^2\phi}$, so that

$$H^d_\phi(u, \boldsymbol{v}) = -x_0(u)v_0 + x_1(u)\sqrt{v_0^2 - \sum_{i=2}^n v_i^2 + \sin^2\phi} + x_2(u)v_2 + \cdots + x_n(u)v_n,$$

where $\boldsymbol{X}(u) = (x_0(u), \ldots, x_n(u))$. We define a mapping $\Delta H^d_\phi = \left(\partial H^d_\phi/\partial u_1, \ldots, \partial H^d_\phi/\partial u_{n-1}\right)$. We have to prove that ΔH^d_ϕ is non-singular at any point in $(\Delta H^d_\phi)^{-1}(0)$. If $(u, \boldsymbol{v}) \in \Delta H^{d^{-1}}_\phi(0)$, then from Proposition 3.3, we have $\boldsymbol{v} = \lambda \boldsymbol{n}^T(u) + \mu \boldsymbol{n}^S(u)$ such that $-\lambda^2 + \mu^2 = \sin^2\phi$. On the other hand, the Jacobian matrix of ΔH^d_ϕ at $(u, \boldsymbol{v}) \in \Delta H^{d^{-1}}_\phi(0)$ is given as follows:

$$\left(\begin{array}{ccc|c} \langle \boldsymbol{X}_{u_1u_1}(u), \boldsymbol{v}\rangle & \cdots & \langle \boldsymbol{X}_{u_1u_{n-1}}(u), \boldsymbol{v}\rangle & \\ \vdots & \vdots & \vdots & \text{A} \\ \langle \boldsymbol{X}_{u_{n-1}u_1}(u), \boldsymbol{v}\rangle & \cdots & \langle \boldsymbol{X}_{u_{n-1}u_{n-1}}(u), \boldsymbol{v}\rangle & \end{array}\right),$$

where

$$\text{A} = \begin{pmatrix} -x_{0u_1}(u) + \dfrac{v_0}{v_1}x_{1u_1}(u) & -\dfrac{v_2}{v_1}x_{1u_1}(u) + x_{2u_1}(u) & \cdots & -\dfrac{v_n}{v_1}x_{1u_1}(u) + x_{nu_1}(u) \\ \vdots & \vdots & \vdots & \vdots \\ -x_{0u_{n-1}}(u) + \dfrac{v_0}{v_1}x_{1u_{n-1}}(u) & -\dfrac{v_2}{v_1}x_{1u_{n-1}}(u) + x_{2u_{n-1}}(u) & \cdots & -\dfrac{v_n}{v_1}x_{1u_{n-1}}(u) + x_{nu_{n-1}}(u) \end{pmatrix}.$$

Now, we consider the following matrix

$$\text{B} = \begin{pmatrix} -n^T_0(u) + \dfrac{v_0}{v_1}n^T_1(u) & -\dfrac{v_2}{v_1}n^T_1(u) + n^T_2(u) & \cdots & -\dfrac{v_n}{v_1}n^T_1(u) + n^T_n(u) \\ -x_{0u_1}(u) + \dfrac{v_0}{v_1}x_{1u_1}(u) & -\dfrac{v_2}{v_1}x_{1u_1}(u) + x_{2u_1}(u) & \cdots & -\dfrac{v_n}{v_1}x_{1u_1}(u) + x_{nu_1}(u) \\ \vdots & \vdots & \vdots & \vdots \\ -x_{0u_{n-1}}(u) + \dfrac{v_0}{v_1}x_{1u_{n-1}}(u) & -\dfrac{v_2}{v_1}x_{1u_{n-1}}(u) + x_{2u_{n-1}}(u) & \cdots & -\dfrac{v_n}{v_1}x_{1u_{n-1}}(u) + x_{nu_{n-1}}(u) \end{pmatrix}$$

and use the following notations

$$\boldsymbol{a} = \begin{pmatrix} n^T_0(u) \\ x_{0u_1}(u) \\ \vdots \\ x_{0u_{n-1}}(u) \end{pmatrix}, \boldsymbol{b}_1 = \begin{pmatrix} n^T_1(u) \\ x_{1u_1}(u) \\ \vdots \\ x_{1u_{n-1}}(u) \end{pmatrix}, \ldots, \boldsymbol{b}_n = \begin{pmatrix} n^T_n(u) \\ x_{nu_1}(u) \\ \vdots \\ x_{nu_{n-1}}(u) \end{pmatrix}.$$

Then we get

$$\det B = \frac{v_0}{v_1}\det(\boldsymbol{b}_1 \ \ldots \ \boldsymbol{b}_n) - \frac{v_1}{v_1}\det(\boldsymbol{a} \ \boldsymbol{b}_2 \ \ldots \boldsymbol{b}_n) - \cdots - \frac{v_n}{v_1}\det(\boldsymbol{b}_1 \ \ldots \ \boldsymbol{b}_{n-1} \ \boldsymbol{a}).$$

On the other hand, we have

$$\boldsymbol{n}^T(u)\wedge\boldsymbol{X}_{u_1}(u)\wedge\cdots\wedge\boldsymbol{X}_{u_{n-1}}(u) = (-\det(\boldsymbol{b}_1 \ \ldots \ \boldsymbol{b}_n), -\det(\boldsymbol{a}\,\boldsymbol{b}_2 \ \ldots \boldsymbol{b}_n), \ldots, -\det(\boldsymbol{b}_1 \ \ldots \ \boldsymbol{b}_{n-1}\,\boldsymbol{a})).$$

Consequently, we obtain

$$\begin{aligned} \det B &= \left\langle \left(\frac{v_0}{v_1}, \ldots, \frac{v_n}{v_1}\right), \boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u) \right\rangle \\ &= \frac{1}{v_1}\left\langle (v_0, \ldots, v_n), \boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\right\rangle \\ &= \frac{1}{v_1}\left\langle \lambda\boldsymbol{n}^T(u) \pm \sqrt{\lambda^2 + \sin^2\phi}\,\boldsymbol{n}^S(u), \boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\right\rangle \\ &= \pm\frac{\sqrt{\lambda^2 + \sin^2\phi}}{v_1}\left\langle \boldsymbol{n}^S(u), \|\boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\|\boldsymbol{n}^S(u)\right\rangle \\ &= \pm\frac{\sqrt{\lambda^2 + \sin^2\phi}}{v_1}\|\boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\| \neq 0. \end{aligned}$$

As a result, the rank of the matrix A is $n-1$.

If we adopt the other local coordinates, we get the similar calculations to the above. This completes the proof. $\square$

Now, taking into account the method of constructing a Lagrangian immersion germ from a Morse family of functions (see [1] for the details), we can give the following corollary:

Corollary 4.5 *There exists a Lagrangian immersion $\mathcal{L}(H^d_\phi) : C(H^d_\phi) \longrightarrow T^*S^n_1(\sin^2\phi)$ such that ϕ-de Sitter height function $H^d_\phi : U \times S^n_1(\sin^2\phi) \longrightarrow \mathbb{R}$ of $M = \boldsymbol{X}(U)$ is a generating family of $\mathcal{L}(H^d_\phi)$.*

Here, $C(H^d_\phi) = (\Delta H^d_\phi)^{-1}(0) = \{\lambda \boldsymbol{n}^T(u) + \mu \boldsymbol{n}^S(u) | -\lambda^2 + \mu^2 = sin^2\phi\}$.

Thus, it follows that Lagrangian map of $\mathcal{L}(H^d_\phi)$ is ϕ-de Sitter Gauss map of $M = \boldsymbol{X}(U)$. We call $\mathcal{L}(H^d_\phi)$ *Lagrangian lift* of ϕ-de Sitter Gauss map of M. Using this, we can interpret Proposition 4.3 from the viewpoint of Lagrangian singularity theory.

Theorem 4.6 *Let $\boldsymbol{X}_i : (U, \bar{u}_i) \to (R^{n+1}_1, \boldsymbol{X}_i(\bar{u}_i))$ $(i = 1, 2)$ be spacelike embedding germs of codimension two such that Lagrangian lift germs $\mathcal{L}(H^d_{\phi i}) : (C(H^d_{\phi i}), (\bar{u}_i, \boldsymbol{v}_i)) \to (T^*S^n_1(\sin^2\phi), \bar{z}_i)$ of ϕ-de Sitter Gauss map germs $(\cos\phi \boldsymbol{n}^T + \boldsymbol{n}^S)_i$ are Lagrangian stable, where $\boldsymbol{v}_i = (\cos\phi\boldsymbol{n}^T + \boldsymbol{n}^S)_i(\bar{u}_i)$. Then the following conditions are equivalent:*

(1) $K(\boldsymbol{X}_1(U), \mathcal{F}_{\mathfrak{h}^d_{\phi,\boldsymbol{v}_1}}; \boldsymbol{X}_1(\bar{u}_1)) = K(\boldsymbol{X}_2(U), \mathcal{F}_{\mathfrak{h}^d_{\phi,\boldsymbol{v}_2}}; \boldsymbol{X}_2(\bar{u}_2))$.

(2) *$h^d_{\phi 1,\boldsymbol{v}_1}$ and $h^d_{\phi 2,\boldsymbol{v}_2}$ are $\mathcal{R}^+$-equivalent.*

(3) *$H^d_{\phi 1}$ and $H^d_{\phi 2}$ are stably P-$\mathcal{R}^+$-equivalent.*

(4) *$\mathcal{L}(H^d_{\phi 1})$ and $\mathcal{L}(H^d_{\phi 2})$ are Lagrangian equivalent.*

(5) *(a) The ranks and signatures of $\mathrm{Hess}(h^d_{\phi 1,\boldsymbol{v}_1})(\bar{u}_1)$ and $\mathrm{Hess}(h^d_{\phi 2,\boldsymbol{v}_2})(\bar{u}_2)$ are equal.*

(b) There is an isomorphism $\gamma : \mathcal{R}_2(h^d_{\phi 1,\boldsymbol{v}_1}) \to \mathcal{R}_2(h^d_{\phi 2,\boldsymbol{v}_2})$ such that $\gamma(\overline{h}^d_{\phi 1,\boldsymbol{v}_1}) = (\overline{h}^d_{\phi 2,\boldsymbol{v}_2})$.

Proof. Since the germs $\mathcal{L}(H^d_{\phi i})$ are Lagrangian stable, the generating families $H^d_{\phi i}$ are $\mathcal{R}^+$-versal unfoldings of $h^d_{\phi i,\boldsymbol{v}_i}$ for $(i = 1, 2)$, respectively, (cf. Appendix). As a result, it is obtained that $h^d_{\phi i,\boldsymbol{v}_i}$ $(i = 1, 2)$ satisfy the Milnor condition, [1,3]. It is given in Proposition 4.3 that conditions (1) and (2) are equivalent. Moreover, from Theorem A.1, the condition (3) is equivalent to the condition (4). Furthermore, in terms of Corollary A.2, the condition (2) is equivalent to the condition (4). If we use Proposition 4.3, it follows that the condition (2) is equivalent to the condition (5). $\square$

5 Timelike slant geometry on spacelike submanifolds of codimension two

In this section, we construct timelike slant geometry on spacelike submanifolds of codimension two in Lorentz-Minkowski space. Since most of the proofs are similar to those of the section 3, we omit them.

Let take $\boldsymbol{n}^T(u) \pm \cos\phi \boldsymbol{n}^S(u) \in H^n(-\sin^2\phi)$ for $\phi \in [0, \pi/2]$. For $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$), $\boldsymbol{n}^T(u) \pm \cos\phi \boldsymbol{n}^S(u)$ (respectively, $\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)$) are timelike (respectively, lightlike) normal vectors, where $u \in U$. We choose $\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S$ (respectively, $\boldsymbol{n}^T + \boldsymbol{n}^S$) as a timelike (respectively, lightlike) normal vector field along M for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$). We call $\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S$ *ϕ-hyperbolic Gauss map* of M.

At each point $p = \boldsymbol{X}(u) \in M$, we consider the following linear mapping

$$d_p(\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S) : T_pM \longrightarrow T_p\mathbb{R}^{n+1}_1 = T_pM \oplus N_pM.$$

We define

$$d_p(\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S)^t = \pi^t \circ d_p(\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S)$$

and

$$d_p(\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S)^n = \pi^n \circ d_p(\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S).$$

The linear transformations $S^h[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p) = -d_p(\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S)^t$ and $d_p(\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S)^n$ of T_pM are called respectively *ϕ-hyperbolic $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-shape operator* and *$\phi$-hyperbolic normal connection* of M at p.

The eigen values which are denoted by $\kappa^h_i[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)\,(p)$ $(i = 1, ..., n-1)$ of $S^h[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p)$ are called *ϕ-hyperbolic principal curvatures* according to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ at p. And also *ϕ-hyperbolic Gauss-Kronecker curvature* with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ at p is defined as

$$K^h[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)\,(p) = \det S^h[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p).$$

A point p is called *ϕ-hyperbolic $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic* if $S^h[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p) = \kappa^h[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)\,(p)\,\mathrm{id}_{T_pM}$ for a function $\kappa^h[\phi]$. M is called *totally ϕ-hyperbolic $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic* if all points of M are ϕ-hyperbolic $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic. Moreover, a point p is called *ϕ-hyperbolic $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-parabolic* if $K^h[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)\,(p) = 0$. Furthermore, p is called *ϕ-hyperbolic $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-flat* if it is ϕ-hyperbolic $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic and $K^h[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)\,(p) = 0$.

On the other hand, *ϕ-hyperbolic second fundamental invariant* with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ is defined by $h^h[\phi]_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)\,(u) = \left\langle -(\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S)_{u_i}(u), \boldsymbol{X}_{u_j}(u)\right\rangle$ for any $u \in U$.

Proposition 5.1 *Under the above notations, we have the following:*

(1) $\left(\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S\right)_{u_i} = \left\langle \boldsymbol{n}^T_{u_i}, \boldsymbol{n}^S\right\rangle (\cos\phi \boldsymbol{n}^T + \boldsymbol{n}^S) - \sum_{j=1}^{n-1} h^h[\phi]^j_i(\boldsymbol{n}^T, \boldsymbol{n}^S)\boldsymbol{X}_{u_j}$ *(ϕ-hyperbolic Weingarten formula),*

(2) $\pi^t \circ (\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S)_{u_i} = -\sum_{j=1}^{n-1} h^h[\phi]^j_i(\boldsymbol{n}^T, \boldsymbol{n}^S)\boldsymbol{X}_{u_j}$.

Here, $\left(h^h[\phi]^j_i(\boldsymbol{n}^T, \boldsymbol{n}^S)\right) = \left(h^h[\phi]_{ik}(\boldsymbol{n}^T, \boldsymbol{n}^S)\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right) = \left(g_{kj}\right)^{-1}$.

We remark that this proposition is a generalization of Proposition 3.2 in [10]. Moreover, as a corollary of this proposition, we have the following explicit expression of ϕ-hyperbolic Gauss-Kronecker curvature with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ by means of Riemannian metric and ϕ-hyperbolic second fundamental invariant.

Corollary 5.2 *Under the same notations as in the above proposition, ϕ-hyperbolic Gauss-Kronecker curvature with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ is given by*

$$K^h[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S) = \frac{\det\left(h^h[\phi]_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)\right)}{\det\left(g_{\alpha\beta}\right)}.$$

As $\langle -(\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S)(u), \boldsymbol{X}_{u_j}(u)\rangle = 0$, we get $h^h[\phi]_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) = \langle (\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S)(u), \boldsymbol{X}_{u_i u_j}(u)\rangle$. Hence, ϕ-hyperbolic second fundamental invariant at a point $p = \boldsymbol{X}(u)$ depends only on the values $\boldsymbol{n}^T(u) + \cos\phi \boldsymbol{n}^S(u)$ and $\boldsymbol{X}_{u_i u_j}(u)$ of the vector fields $\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S$ and $\boldsymbol{X}_{u_i u_j}$. And also ϕ-hyperbolic Gauss-Kronecker curvature with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ at p depends only on the vectors $\boldsymbol{n}^T(u) + \cos\phi \boldsymbol{n}^S(u)$, $\boldsymbol{X}_{u_i}(u)$ and $\boldsymbol{X}_{u_i u_j}(u)$.

Now, we define a *ϕ-hyperbolic height function* on $M = \boldsymbol{X}(U)$ by

$$\begin{array}{cccc} H^h_\phi & : & U \times H^n(-\sin^2\phi) & \longrightarrow & \mathbb{R} \\ & & (u, \boldsymbol{v}) & \longmapsto & \langle \boldsymbol{X}(u), \boldsymbol{v}\rangle. \end{array}$$

Besides, we denote Hessian matrix of ϕ-hyperbolic height function $h^h_{\phi,\boldsymbol{v}_0}(u) = H^h_\phi(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}\left(h^h_{\phi,\boldsymbol{v}_0}\right)(u_0)$.

Proposition 5.3 *Let $H^h_\phi : U \times H^n(-\sin^2\phi) \longrightarrow \mathbb{R}$ be a ϕ-hyperbolic height function on $M = \boldsymbol{X}(U)$. Then, we have the following:*
(1) *$\partial H^h_\phi/\partial u_i(u_0, \boldsymbol{v_0}) = 0$ $(i = 1, ..., n-1)$ if and only if $\boldsymbol{v_0} = \lambda \boldsymbol{n}^T(u_0) + \mu \boldsymbol{n}^S(u_0)$ such that $-\lambda^2 + \mu^2 = -\sin^2\phi$.*
Suppose that $p_0 = \boldsymbol{X}(u_0)$ and $\boldsymbol{v_0} = (\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S)(u_0)$.
(2) *p_0 is a ϕ-hyperbolic $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-parabolic point if and only if* $\det \mathrm{Hess}\left(h^h_{\phi,\boldsymbol{v}_0}\right)(u_0) = 0$.
(3) *p_0 is a ϕ-hyperbolic $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-flat point if and only if* $\mathrm{rank}\, \mathrm{Hess}\left(h^h_{\phi,\boldsymbol{v}_0}\right)(u_0) = 0$.

Theorem 5.4 *For a spacelike embedding $\boldsymbol{X} : U \to \mathbb{R}^{n+1}_1$, where U is an open subset of $\mathbb{R}^{n-1}$, the following conditions are equivalent:*
(1) *$M = \boldsymbol{X}(U)$ is totally ϕ-hyperbolic flat, where $\boldsymbol{n}^T$ is a parallel vector field.*
(2) *ϕ-hyperbolic Gauss map $\boldsymbol{n}^T + \cos\phi \boldsymbol{n}^S$ is constant, where $\boldsymbol{n}^T$ is a parallel vector field.*
(3) *There exists a vector $\boldsymbol{v} \in H^n(-\sin^2\phi)$ such that M is a subset of a spacelike (respectively, lightlike) hyperplane $HP(\boldsymbol{v}, c)$ for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$).*

Theorem 5.5 *Assume that $M = \boldsymbol{X}(U)$ is non-flat totally umbilical, $\boldsymbol{n}^T$ is a parallel vector field and $\phi \in [0, \pi/2]$ is fixed. Then $\kappa^h[\phi](\boldsymbol{n}^T, \boldsymbol{n}^S)(p)$ is non-zero constant $\kappa^h[\phi]$ and M is located in $H^n(\boldsymbol{a}, r)$, where $\boldsymbol{a} = \frac{1}{\kappa^h[\phi]}\boldsymbol{c}$ and $r = |\frac{\sin\phi}{\kappa^h[\phi]}|$.*

Remark 5.6 *We remark that if $\phi = 0$ in the above theorem, then M is located in LC_a.*

6 Timelike slant geometry from the viewpoint of Lagrangian singularity theory

In this section, we interpret the results of Proposition 5.3 from the viewpoint of Lagrangian singularity theory. We consider a function

$$\begin{array}{cccc} \mathfrak{h}^h_{\phi,\boldsymbol{v}_0} & : & \mathbb{R}^{n+1}_1 & \longrightarrow & \mathbb{R} \\ & & \boldsymbol{x} & \longmapsto & \langle \boldsymbol{x}, \boldsymbol{v}_0\rangle, \end{array}$$

where $\boldsymbol{v_0} = \lambda \boldsymbol{n}^T(u_0) + \mu \boldsymbol{n}^S(u_0)$ such that $-\lambda^2 + \mu^2 = -\sin^2\phi$. In this case, we can write $\mathfrak{h}^h_{\phi,\boldsymbol{v}_0} \circ \boldsymbol{X}(u) = H^h_\phi(u, \boldsymbol{v}_0)$. From Proposition 5.3, we obtain

$$\frac{\partial \mathfrak{h}^h_{\phi,\boldsymbol{v}_0} \circ \boldsymbol{X}}{\partial u_i}(u_0) = \frac{\partial H^h_\phi}{\partial u_i}(u_0, \boldsymbol{v}_0) = 0$$

for $(i = 1, \dots, n-1)$. This means that for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$), spacelike (respectively, lightlike) hyperplane ${\mathfrak{h}^h_{\phi,\boldsymbol{v}_0}}^{-1}(c) = HP(\boldsymbol{v}_0, c)$ is tangent to $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(u_0)$, where $c = \langle \boldsymbol{X}(u_0), \boldsymbol{v}_0 \rangle$. Hence, we call $HP(\boldsymbol{v}_0, c)$ a *tangent spacelike (respectively, lightlike) hyperplane* with the pseudo-normal $\boldsymbol{v}_0$ for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$). Since we have timelike (respectively, lightlike) directions $\boldsymbol{n}^T(u) + \cos\phi \boldsymbol{n}^S(u)$ (respectively, $\boldsymbol{n}^T(u) + \boldsymbol{n}^S(u)$) and $\boldsymbol{n}^T(u) - \cos\phi \boldsymbol{n}^S(u)$ (respectively, $\boldsymbol{n}^T(u) - \boldsymbol{n}^S(u)$) in the normal plane, we have two tangent spacelike (respectively, lightlike) hyperplanes at the point p_0 depending on the direction of $\boldsymbol{v}_0$ for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$). Here, we choose only one of them. Let ε be a sufficiently small positive real number. For any $t \in I_\varepsilon = (c - \varepsilon, c + \varepsilon)$, we have a spacelike (respectively, lightlike) hyperplane $HP(\boldsymbol{v}_0, t) = {\mathfrak{h}^h_{\phi,\boldsymbol{v}_0}}^{-1}(t)$. In this case, $\mathcal{F}_{\mathfrak{h}^h_{\phi,\boldsymbol{v}_0}}$ is a family of parallel spacelike (respectively, lightlike) hyperplanes around p_0 such that ${\mathfrak{h}^h_{\phi,\boldsymbol{v}_0}}^{-1}(c)$ is the tangent spacelike (respectively, lightlike) hyperplane of M at p_0. Let $\boldsymbol{X}_i : (U, \bar{u}_i) \to (\mathbb{R}^{n+1}_1, \boldsymbol{X}_i(\bar{u}_i))$ $(i = 1, 2)$ be spacelike embedding germs with codimension two. Then we have $h^h_{\phi i,\boldsymbol{v}_i}(u) = \mathfrak{h}^h_{\phi,\boldsymbol{v}_i} \circ \boldsymbol{X}_i(u)$. Therefore, we get the following proposition as a corollary of Proposition 4.1 and Proposition 4.2.

Proposition 6.1 *Let $\boldsymbol{X}_i : (U, \bar{u}_i) \to (\mathbb{R}^{n+1}_1, \boldsymbol{X}_i(\bar{u}_i))$ $(i = 1, 2)$ be spacelike embedding germs of codimension two such that $h^h_{\phi i,\boldsymbol{v}_i}$ satisfy the Milnor condition, where $\boldsymbol{v}_i = \boldsymbol{n}^T(\bar{u}_i) + \cos\phi \boldsymbol{n}^S(\bar{u}_i)$ (respectively, $\boldsymbol{v}_i = \boldsymbol{n}^T(\bar{u}_i) + \boldsymbol{n}^S(\bar{u}_i)$) are pseudo-normals of the tangent spacelike (respectively, lightlike) hyperplanes of $\boldsymbol{X}_i$ for $\phi \in (0, \pi/2]$ (respectively, $\phi = 0$). Then the following conditions are equivalent:*

(1) $K(\boldsymbol{X}_1(U), \mathcal{F}_{\mathfrak{h}^h_{\phi,\boldsymbol{v}_1}}; \boldsymbol{X}_1(\bar{u}_1)) = K(\boldsymbol{X}_2(U), \mathcal{F}_{\mathfrak{h}^h_{\phi,\boldsymbol{v}_2}}; \boldsymbol{X}_2(\bar{u}_2))$.

(2) *$h^h_{\phi 1,\boldsymbol{v}_1}$ and $h^h_{\phi 2,\boldsymbol{v}_2}$ are $\mathcal{R}^+$-equivalent.*

(3) *(a) The ranks and signatures of* $\mathrm{Hess}(h^h_{\phi 1,\boldsymbol{v}_1})(\bar{u}_1)$ *and* $\mathrm{Hess}(h^h_{\phi 2,\boldsymbol{v}_2})(\bar{u}_2)$ *are equal,*

(b) There is an isomorphism $\gamma : \mathcal{R}_2(h^h_{\phi 1,\boldsymbol{v}_1}) \to \mathcal{R}_2(h^h_{\phi 2,\boldsymbol{v}_2})$ such that $\gamma(\overline{h}^h_{\phi 1,\boldsymbol{v}_1}) = (\overline{h}^h_{\phi 2,\boldsymbol{v}_2})$.

Now, we can interpret the meaning of the above proposition from the viewpoint of Lagrangian singularity. We consider $\boldsymbol{v} = (v_0, v_1, \dots, v_n) \in H^n(-\sin^2\phi) \setminus \{\boldsymbol{0}\}$. Without the loss of generality, we assume that $v_0 > 0$. Thus, we adopt the coordinate system $(v_1, \dots, v_n)$.

Proposition 6.2 *The ϕ-hyperbolic height function $H^h_\phi : U \times H^n(-\sin^2\phi) \longrightarrow \mathbb{R}$ is a Morse family of hypersurfaces.*

Proof. For any $\boldsymbol{v} = (v_0, v_1, \dots, v_n) \in H^n(-\sin^2\phi) \setminus \{\boldsymbol{0}\}$, we have $v_0 = \sqrt{\sum_{i=1}^n v_i^2 + \sin^2\phi}$, so that

$$H^h_\phi(u, \boldsymbol{v}) = -x_0(u)\sqrt{\sum_{i=1}^n v_i^2 + \sin^2\phi} + x_1(u)v_1 + \dots + x_n(u)v_n,$$

where $\boldsymbol{X}(u) = (x_0(u), \ldots, x_n(u))$. We define a mapping $\Delta H^h_\phi = \big(\partial H^h_\phi/\partial u_1, \ldots, \partial H^h_\phi/\partial u_{n-1}\big)$. We have to prove that ΔH^h_ϕ is non-singular at any point in $(\Delta H^h_\phi)^{-1}(0)$. If $(u, \boldsymbol{v}) \in \Delta H^{h^{-1}}_\phi(0)$, then from Proposition 5.3, we have $\boldsymbol{v} = \lambda \boldsymbol{n}^T(u) + \mu \boldsymbol{n}^S(u)$ such that $-\lambda^2 + \mu^2 = -\sin^2\phi$. On the other hand, the Jacobian matrix of ΔH^h_ϕ at $(u, \boldsymbol{v}) \in \Delta H^{h^{-1}}_\phi(0)$ is given as follows:

$$\left(\begin{array}{ccc|c} \langle \boldsymbol{X}_{u_1u_1}(u), \boldsymbol{v}\rangle & \cdots & \langle \boldsymbol{X}_{u_1u_{n-1}}(u), \boldsymbol{v}\rangle & \\ \vdots & \vdots & \vdots & \mathrm{A} \\ \langle \boldsymbol{X}_{u_{n-1}u_1}(u), \boldsymbol{v}\rangle & \cdots & \langle \boldsymbol{X}_{u_{n-1}u_{n-1}}(u), \boldsymbol{v}\rangle & \end{array}\right),$$

where

$$\mathrm{A} = \begin{pmatrix} -\dfrac{v_1}{v_0}x_{0u_1}(u) + x_{1u_1}(u) & -\dfrac{v_2}{v_0}x_{0u_1}(u) + x_{2u_1}(u) & \cdots & -\dfrac{v_n}{v_0}x_{0u_1}(u) + x_{nu_1}(u) \\ \vdots & \vdots & \vdots & \vdots \\ -\dfrac{v_1}{v_0}x_{0u_{n-1}}(u) + x_{1u_{n-1}}(u) & -\dfrac{v_2}{v_0}x_{0u_{n-1}}(u) + x_{2u_{n-1}}(u) & \cdots & -\dfrac{v_n}{v_0}x_{0u_{n-1}}(u) + x_{nu_{n-1}}(u) \end{pmatrix}.$$

Now, we take into account the following matrix

$$\mathrm{B} = \begin{pmatrix} -\dfrac{v_1}{v_0}n^S_0(u) + n^S_1(u) & -\dfrac{v_2}{v_0}n^S_0(u) + n^S_2(u) & \cdots & -\dfrac{v_n}{v_0}n^S_0(u) + n^S_n(u) \\ -\dfrac{v_1}{v_0}x_{0u_1}(u) + x_{1u_1}(u) & -\dfrac{v_2}{v_0}x_{0u_1}(u) + x_{2u_1}(u) & \cdots & -\dfrac{v_n}{v_0}x_{0u_1}(u) + x_{nu_1}(u) \\ \vdots & \vdots & \vdots & \vdots \\ -\dfrac{v_1}{v_0}x_{0u_{n-1}}(u) + x_{1u_{n-1}}(u) & -\dfrac{v_2}{v_0}x_{0u_{n-1}}(u) + x_{2u_{n-1}}(u) & \cdots & -\dfrac{v_n}{v_0}x_{0u_{n-1}}(u) + x_{nu_{n-1}}(u) \end{pmatrix}$$

and use the following notations

$$\boldsymbol{a} = \begin{pmatrix} n^S_0(u) \\ x_{0u_1}(u) \\ \vdots \\ x_{0u_{n-1}}(u) \end{pmatrix}, \boldsymbol{b}_1 = \begin{pmatrix} n^S_1(u) \\ x_{1u_1}(u) \\ \vdots \\ x_{1u_{n-1}}(u) \end{pmatrix}, \ldots, \boldsymbol{b}_n = \begin{pmatrix} n^S_n(u) \\ x_{nu_1}(u) \\ \vdots \\ x_{nu_{n-1}}(u) \end{pmatrix}.$$

Therefore, we obtain

$$\det B = \frac{v_0}{v_0}\det(\boldsymbol{b}_1 \ \ldots \ \boldsymbol{b}_n) - \frac{v_1}{v_0}\det(\boldsymbol{a}\ \boldsymbol{b}_2 \ \ldots \boldsymbol{b}_n) - \cdots - \frac{v_n}{v_0}\det(\boldsymbol{b}_1 \ \ldots \ \boldsymbol{b}_{n-1}\ \boldsymbol{a}).$$

On the other hand, we have

$$\boldsymbol{n}^S(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u) = (-\det(\boldsymbol{b}_1 \ \ldots \ \boldsymbol{b}_n), -\det(\boldsymbol{a}\ \boldsymbol{b}_2 \ \ldots \boldsymbol{b}_n), \ldots, -\det(\boldsymbol{b}_1 \ \ldots \ \boldsymbol{b}_{n-1}\ \boldsymbol{a})).$$

Consequently, we get

$$\begin{aligned} \det B &= \left\langle \left(\frac{v_0}{v_0}, \ldots, \frac{v_n}{v_0}\right), \boldsymbol{n}^S(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u) \right\rangle \\ &= \frac{1}{v_0}\big\langle (v_0, \ldots, v_n), \boldsymbol{n}^S(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\big\rangle \\ &= \frac{1}{v_0}\left\langle \pm\sqrt{\mu^2 + \sin^2\phi}\,\boldsymbol{n}^T(u) + \mu \boldsymbol{n}^S(u), \boldsymbol{n}^S(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u) \right\rangle \\ &= \pm\frac{\sqrt{\mu^2 + \sin^2\phi}}{v_0}\big\langle \boldsymbol{n}^T(u), \|\boldsymbol{n}^S(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\|\boldsymbol{n}^T(u)\big\rangle \\ &= \mp\frac{\sqrt{\mu^2 + \sin^2\phi}}{v_0}\|\boldsymbol{n}^S(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\| \neq 0. \end{aligned}$$

So, the rank of the matrix A is $n-1$. This completes the proof. $\square$

Now, by means of the method of constructing a Lagrangian immersion germ from a Morse family of functions (see [1] for the details), we can give the following corollary:

Corollary 6.3 *There exists a Lagrangian immersion* $\mathcal{L}(H^h_\phi) : C(H^h_\phi) \longrightarrow T^*H^n(-\sin^2\phi)$ *such that* ϕ*-hyperbolic height function* $H^h_\phi : U \times H^n(-\sin^2\phi) \longrightarrow \mathbb{R}$ *of* $M = \boldsymbol{X}(U)$ *is a generating family of* $\mathcal{L}(H^h_\phi)$.

Here, $C(H^h_\phi) = (\Delta H^h_\phi)^{-1}(0) = \{\lambda \boldsymbol{n}^T(u) + \mu \boldsymbol{n}^S(u) | -\lambda^2 + \mu^2 = -sin^2\phi\}$.

Thus, it follows that Lagrangian map of $\mathcal{L}(H^h_\phi)$ is ϕ-hyperbolic Gauss map of $M = \boldsymbol{X}(U)$. We call $\mathcal{L}(H^h_\phi)$ *Lagrangian lift* of ϕ-hyperbolic Gauss map of M. By using this terminology, we can interpret Proposition 6.1 from the viewpoint of Lagrangian singularity theory.

Theorem 6.4 *Let* $\boldsymbol{X}_i : (U, \bar{u}_i) \to (R^{n+1}_1, \boldsymbol{X}_i(\bar{u}_i))$ $(i = 1, 2)$ *be spacelike embedding germs of codimension two such that Lagrangian lift germs* $\mathcal{L}(H^h_{\phi}i) : (C(H^h_{\phi}i), (\bar{u}_i, \boldsymbol{v}_i)) \to (T^*H^n(-\sin^2\phi), \bar{z}_i)$ *of* ϕ*-hyperbolic Gauss map germs* $(\boldsymbol{n}^T + \cos\phi\boldsymbol{n}^S)_i$ *are Lagrangian stable, where* $\boldsymbol{v}_i = (\boldsymbol{n}^T + \cos\phi\boldsymbol{n}^S)_i(\bar{u}_i)$. *Then the following conditions are equivalent:*

(1) $K(\boldsymbol{X}_1(U), \mathcal{F}_{\mathfrak{h}^h_{\phi,\boldsymbol{v}_1}}; \boldsymbol{X}_1(\bar{u}_1)) = K(\boldsymbol{X}_2(U), \mathcal{F}_{\mathfrak{h}^h_{\phi,\boldsymbol{v}_2}}; \boldsymbol{X}_2(\bar{u}_2))$.

(2) $h^h_{\phi 1,\boldsymbol{v}_1}$ *and* $h^h_{\phi 2,\boldsymbol{v}_2}$ *are* $\mathcal{R}^+$*-equivalent.*

(3) $H^h_{\phi 1}$ *and* $H^h_{\phi 2}$ *are stably* P*-*$\mathcal{R}^+$*-equivalent.*

(4) $\mathcal{L}(H^h_{\phi 1})$ *and* $\mathcal{L}(H^h_{\phi 2})$ *are Lagrangian equivalent.*

(5) (a) *The ranks and signatures of* $\mathrm{Hess}(h^h_{\phi 1,\boldsymbol{v}_1})(\bar{u}_1)$ *and* $\mathrm{Hess}(h^h_{\phi 2,\boldsymbol{v}_2})(\bar{u}_2)$ *are equal.*

(b) *There is an isomorphism* $\gamma : \mathcal{R}_2(h^h_{\phi 1,\boldsymbol{v}_1}) \to \mathcal{R}_2(h^h_{\phi 2,\boldsymbol{v}_2})$ *such that* $\gamma(\overline{h}^h_{\phi 1,\boldsymbol{v}_1}) = (\overline{h}^h_{\phi 2,\boldsymbol{v}_2})$.

The proof is given by exactly same way as the proof of Theorem 4.6.

7 Spacial Cases

In this section, we give some special cases of our results.

The following four Legendrian double fibrations for the pseudo-spheres in Lorentz-Minkowski space were defined in [8]:

(1) (a) $H^n(-1) \times S^n_1 \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0\}$,

(b) $\pi_{11} : \Delta_1 \longrightarrow H^n(-1)$, $\pi_{12} : \Delta_1 \longrightarrow S^n_1$,

(c) $\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_1$, $\theta_{12} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_1$.

(2) (a) $H^n(-1) \times LC^* \supset \Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = -1\}$,

(b) $\pi_{21} : \Delta_2 \longrightarrow H^n(-1)$, $\pi_{22} : \Delta_2 \longrightarrow LC^*$,

(c) $\theta_{21} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_2$, $\theta_{22} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_2$.

(3) (a) $LC^* \times S^n_1 \supset \Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = +1 \}$,

(b) $\pi_{31} : \Delta_3 \longrightarrow LC^*$, $\pi_{32} : \Delta_3 \longrightarrow S^n_1$,

(c) $\theta_{31} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_3$, $\theta_{32} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_3$.

(4) (a) $LC^* \times LC^* \supset \Delta_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = -2 \}$,

(b) $\pi_{41} : \Delta_4 \longrightarrow LC^*$, $\pi_{42} : \Delta_4 \longrightarrow LC^*$,

(c) $\theta_{41} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_4$, $\theta_{42} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_4$.

Here, $\pi_{i1}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$ and $\pi_{i2}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$ (i=1,2,3,4). Moreover, $\langle d\boldsymbol{v}, \boldsymbol{w}\rangle = -w_0dv_0 + \sum_{i=1}^n w_idv_i$ and $\langle \boldsymbol{v}, d\boldsymbol{w}\rangle = -v_0dw_0 + \sum_{i=1}^n v_idw_i$ are one-forms on $\mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1$. We remark that ${\theta_{i1}}^{-1}(0)$ and ${\theta_{i2}}^{-1}(0)$) define the same tangent hyperplane field denoted by K_i over Δ_i (i=1,2,3,4).

In terms of these Legendrian dualities, the following duality theorem was given in [8]:

Theorem 7.1 *Under the above notations, (Δ_i, K_i) $(i = 1, 2, 3, 4)$ are contact manifolds such that π_{1j} and π_{ij} $(j = 1, 2)$ are Legendrian fibrations. Moreover, these contact manifolds are contact diffeomorphic to each other.*

On the other hand, above Legendrian dualities were generalized into the pseudo-spheres in general semi-Euclidean space in [4]. And then, in [12], they have been extended for one-parameter families depending on the parameter $\phi \in [0, \pi/2]$ of pseudo-spheres in Lorentz-Minkowski space as follows:

(1) (a) $H^n(-1) \times S^n_1(\sin^2\phi) \supset \Delta^{\pm}_{21}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm\cos\phi \}$,

(b) $\pi[\phi]^{\pm}_{(21)1} : \Delta^{\pm}_{21}(\phi) \longrightarrow H^n(-1)$, $\pi[\phi]^{\pm}_{(21)2} : \Delta^{\pm}_{21}(\phi) \longrightarrow S^n_1(\sin^2\phi)$,

(c) $\theta[\phi]^{\pm}_{(21)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta^{\pm}_{21}(\phi)$, $\theta[\phi]^{\pm}_{(21)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta^{\pm}_{21}(\phi)$.

(2) (a) $H^n(-\sin^2\phi) \times S^n_1 \supset \Delta^{\pm}_{31}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm\cos\phi \}$,

(b) $\pi[\phi]^{\pm}_{(31)1} : \Delta^{\pm}_{31}(\phi) \longrightarrow H^n(-\sin^2\phi)$, $\pi[\phi]^{\pm}_{(31)2} : \Delta^{\pm}_{31}(\phi) \longrightarrow S^n_1$,

(c) $\theta[\phi]^{\pm}_{(31)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta^{\pm}_{31}(\phi)$, $\theta[\phi]^{\pm}_{(31)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta^{\pm}_{31}(\phi)$.

(3) (a) $H^n(-\sin^2\phi) \times S^n_1(\sin^2\phi) \supset \Delta^{\pm}_{41}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm 2\cos\phi \}$,

(b) $\pi[\phi]^{\pm}_{(41)1} : \Delta^{\pm}_{41}(\phi) \longrightarrow H^n(-\sin^2\phi)$, $\pi[\phi]^{\pm}_{(41)2} : \Delta^{\pm}_{41}(\phi) \longrightarrow S^n_1(\sin^2\phi)$,

(c) $\theta[\phi]^{\pm}_{(41)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta^{\pm}_{41}(\phi)$, $\theta[\phi]^{\pm}_{(41)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta^{\pm}_{41}(\phi)$.

(4) (a) $H^n(-\sin^2\phi) \times S^n_1(\cos^2\phi) \supset \Delta^{\pm}_{32}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm(\cos\phi + \sin\phi) \}$,

(b) $\pi[\phi]^{\pm}_{(32)1} : \Delta^{\pm}_{32}(\phi) \longrightarrow H^n(-\sin^2\phi)$, $\pi[\phi]^{\pm}_{(32)2} : \Delta^{\pm}_{32}(\phi) \longrightarrow S^n_1(\cos^2\phi)$,

(c) $\theta[\phi]^{\pm}_{(32)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle | \Delta^{\pm}_{32}(\phi)$, $\theta[\phi]^{\pm}_{(32)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle | \Delta^{\pm}_{32}(\phi)$.

(5) (a) $H^n(-\sin^2\phi) \times LC^* \supset \Delta^{\pm}_{42}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm(\cos\phi + 1)\ \}$,

(b) $\pi[\phi]^{\pm}_{(42)1} : \Delta^{\pm}_{42}(\phi) \longrightarrow H^n(-\sin^2\phi)$, $\pi[\phi]^{\pm}_{(42)2} : \Delta^{\pm}_{42}(\phi) \longrightarrow LC^*$,

(c) $\theta[\phi]^{\pm}_{(42)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle | \Delta^{\pm}_{42}(\phi)$, $\theta[\phi]^{\pm}_{(42)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle | \Delta^{\pm}_{42}(\phi)$.

(6) (a) $LC^* \times S^n_1(\sin^2\phi) \supset \Delta^{\pm}_{43}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = \pm(\cos\phi + 1)\ \}$,

(b) $\pi[\phi]^{\pm}_{(43)1} : \Delta^{\pm}_{43}(\phi) \longrightarrow LC^*$, $\pi[\phi]^{\pm}_{(43)2} : \Delta^{\pm}_{43}(\phi) \longrightarrow S^n_1(\sin^2\phi)$,

(c) $\theta[\phi]^{\pm}_{(43)1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle | \Delta^{\pm}_{43}(\phi)$, $\theta[\phi]^{\pm}_{(43)2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle | \Delta^{\pm}_{43}(\phi)$.

Here, we define the tangent hyperplane field $K[\phi]^{\pm}_{ij}$ over $\Delta^{\pm}_{ij}(\phi)$ by $K[\phi]^{\pm}_{ij} = {\theta[\phi]^{\pm}_{(ij)1}}^{-1}(0) = {\theta[\phi]^{\pm}_{(ij)2}}^{-1}(0)$. Using these Legendrian dualities, the following duality theorem was obtained in [12].

Theorem 7.2 *Under the same notations as those of the previous paragraphs,* (Δ_1, K_1) *and* $(\Delta^{\pm}_{ij}(\phi), K[\phi]^{\pm}_{ij})$ $((i,j) = (1,2),(1,3),(1,4),(2,3),(2,4),(3,4))$ *are contact manifolds such that* π_{1k} *and* $\pi[\phi]^{\pm}_{(ij)k}$ $(k = 1,2)$ *are Legendrian fibrations. Moreover, these contact manifolds are contact diffeomorphic to each other.*

Let

$$\begin{array}{rccl} \mathcal{L}_{ij}[\phi] & : & U & \longrightarrow & \Delta^{\pm}_{ij}(\phi) \\ & & u & \longmapsto & (L_1(u), L_2(u)) \end{array}$$

be a Legendrian embedding. Then $L_1(u)$ and $L_2(u)$ are called $\Delta^{\pm}_{ij}(\phi)$*-dual*. Especially, $L_2(u)$ (respectively, $L_1(u)$) is said to be ϕ*-de Sitter dual* (respectively, ϕ*-hyperbolic dual*) of $L_1(u)$ (respectively, $L_2(u)$) if $L_1(u)$ and $L_2(u)$ are $\Delta^{-}_{21}(\phi)$-dual (respectively, $\Delta^{+}_{31}(\phi)$-dual).

The above two duality theorems are fundamental tools for the study of spacelike hypersurfaces in the pseudo-spheres in Lorentz-Minkowski space from the viewpoint of Legendrian singularity theory. As a result,taking into account these duality theorems, we have recently investigated slant geometry of spacelike hypersurfaces in the pseudo-spheres in Lorentz-Minkowski space in [2, 12, 13].

On the other hand, it is known that a hypersurface in the pseudo-spheres in Lorentz-Minkowski space is a hypersurface of codimension two in Lorentz-Minkowski space. Consequently, we can mention about our previous results in [2, 12, 13] as special cases of this paper in the following subsections.

7.1 Hypersurface case in Hyperbolic space

Let $\boldsymbol{X}^h : U \longrightarrow H^n(-1)$ be an embedding, where $U \subset \mathbb{R}^{n-1}$ is an open subset, $M^H = \boldsymbol{X}^h(U)$ and $\boldsymbol{X}^d = \dfrac{\boldsymbol{X}^h \wedge \boldsymbol{X}^h_{u_1} \wedge \cdots \wedge \boldsymbol{X}^h_{u_{n-1}}}{\|\boldsymbol{X}^h \wedge \boldsymbol{X}^h_{u_1} \wedge \cdots \wedge \boldsymbol{X}^h_{u_{n-1}}\|}$ is a unit normal vector field along M^H. It was shown in [2, 12] that ϕ-de Sitter dual $\mathbb{N}^d_\pm[\phi](u) = \cos\phi \boldsymbol{X}^h(u) \pm \boldsymbol{X}^d(u)$ can be taken as a normal vector of M^H at the point $p = \boldsymbol{X}^h(u)$, where

$$\begin{array}{rcccl} \mathcal{L}_{21}[\phi] & : & U & \longrightarrow & \Delta^-_{21}(\phi) \\ & & u & \longmapsto & (\boldsymbol{X}^h(u), \mathbb{N}^d_\pm[\phi](u)) \end{array}$$

is a Legendrian embedding for $\phi \in [0, \pi/2]$. We remark that $\mathbb{N}^d_\pm[0](u) = \boldsymbol{X}^h(u) \pm \boldsymbol{X}^d(u)$ (respectively, $\mathbb{N}^d_\pm[\pi/2](u) = \pm\boldsymbol{X}^d(u)$) is the hyperbolic Gauss indicatrix (respectively, de Sitter Gauss indicatrix) introduced in [15]. If we write $\boldsymbol{X}^h$, $\boldsymbol{X}^d$, $\boldsymbol{X}^h_{u_i}$ and M^H instead of (respectively, $\boldsymbol{n}^T$, $\boldsymbol{n}^S$, $\boldsymbol{X}_{u_i}$ and M), we obtain the results given in [2, 12] as special cases of some of our results in this paper.

7.2 Spacelike hypersurface case in de Sitter space

Let $\boldsymbol{X}^d : U \longrightarrow S^n_1$ be an embedding, where $U \subset \mathbb{R}^{n-1}$ is an open subset, $M^D = \boldsymbol{X}^d(U)$ and $\boldsymbol{X}^h = \frac{\boldsymbol{X}^d \wedge \boldsymbol{X}^d_{u_1} \wedge \cdots \wedge \boldsymbol{X}^d_{u_{n-1}}}{\|\boldsymbol{X}^d \wedge \boldsymbol{X}^d_{u_1} \wedge \cdots \wedge \boldsymbol{X}^d_{u_{n-1}}\|}$ is a unit normal vector field along M^D. It has been shown in [2, 12] that ϕ-hyperbolic dual $\mathbb{N}^h_\pm[\phi](u) = \pm\boldsymbol{X}^h(u) + \cos\phi \boldsymbol{X}^d(u)$ can be taken as a normal vector of M^D at the point $p = \boldsymbol{X}^d(u)$, where

$$\begin{array}{rcccl} \mathcal{L}_{31}[\phi] & : & U & \longrightarrow & \Delta^+_{31}(\phi) \\ & & u & \longmapsto & (\mathbb{N}^h_\pm[\phi](u), \boldsymbol{X}^d(u)) \end{array}$$

is a Legendrian embedding for $\phi \in [0, \pi/2]$. We remark that $\mathbb{N}^h_\pm[0](u) = \boldsymbol{X}^d(u) \pm \boldsymbol{X}^h(u)$ was investigated in [24]. And it is clear that $\mathbb{N}^h_\pm[\pi/2](u) = \pm\boldsymbol{X}^h(u)$. If we write $\boldsymbol{X}^h$, $\boldsymbol{X}^d$, $\boldsymbol{X}^d_{u_i}$ and M^D instead of (respectively, $\boldsymbol{n}^T$, $\boldsymbol{n}^S$, $\boldsymbol{X}_{u_i}$ and M), we get the results given in [2, 12] as special cases of some of our results in this paper.

7.3 Spacelike hypersurface case in the lightcone

For an open subset $U \subset \mathbb{R}^{n-1}$, let

$$\begin{array}{rcccl} \mathcal{L}_4 & : & U & \longrightarrow & \Delta_4 \\ & & u & \longmapsto & (\boldsymbol{X}^\ell_+(u), \boldsymbol{X}^\ell_-(u)) \end{array}$$

be a Legendrian embedding and $\boldsymbol{X}^\ell_+ : U \longrightarrow LC^*$ be a spacelike embedding. By means of Legendrian embedding $\mathcal{L}_4$, $\boldsymbol{X}^\ell_-(u)$ was defined in [8] as a lightlike normal vector which is called lightcone normal vector of the spacelike hypersurface $M^L_+ = \boldsymbol{X}^\ell_+(U)$ at the point $p = \boldsymbol{X}^\ell_+(u)$. In terms of these two vectors $\boldsymbol{X}^\ell_+(u)$ and $\boldsymbol{X}^\ell_-(u)$, also the following two vectors

$$\boldsymbol{X}^h(u) = \frac{\boldsymbol{X}^\ell_-(u) + \boldsymbol{X}^\ell_+(u)}{2} \quad \text{and} \quad \boldsymbol{X}^d(u) = \frac{\boldsymbol{X}^\ell_-(u) - \boldsymbol{X}^\ell_+(u)}{2}$$

were defined in [8]. Here, it is obvious that $\boldsymbol{X}^h(u) \in H^n(-1)$ and $\boldsymbol{X}^d(u) \in S^n_1$. Moreover, it was shown in [12, 13] that ϕ-de Sitter dual $\mathbb{N}^d_\ell[\phi](u) = \frac{1}{2}((\cos\phi - 1)\boldsymbol{X}^\ell_+(u) + (\cos\phi + 1)\boldsymbol{X}^\ell_-(u))$ can be taken as a normal vector of M^L_+ at p, where

$$\begin{array}{rcccl} \mathcal{L}_{43}[\phi] & : & U & \longrightarrow & \Delta^-_{43}(\phi) \\ & & u & \longmapsto & (\boldsymbol{X}^\ell_+(u), \mathbb{N}^d_\ell[\phi](u)) \end{array}$$

is a Legendrian embedding for $\phi \in [0, \pi/2]$. Here, it is clear that $\mathbb{N}^d_\ell[0](u) = \boldsymbol{X}^\ell_-(u)$ and $\mathbb{N}^d_\ell[\pi/2](u) = \boldsymbol{X}^d(u)$. If we write $\boldsymbol{X}^h$, $\boldsymbol{X}^d$, $\boldsymbol{X}^\ell_{+u_i}$ and M^L_+ instead of (respectively, $\boldsymbol{n}^T$, $\boldsymbol{n}^S$, $\boldsymbol{X}_{u_i}$ and M), we get the results given in [12, 13] as special cases of some of our results in this paper.

Let $\boldsymbol{X}^\ell_- : U \longrightarrow LC^*$ be a spacelike embedding for an open subset $U \subset \mathbb{R}^{n-1}$. In this case, $\boldsymbol{X}^\ell_+(u)$ can be defined as a lightlike normal vector which is called lightcone normal vector of the spacelike hypersurface $M^L_- = \boldsymbol{X}^\ell_-(U)$ at the point $p = \boldsymbol{X}^\ell_-(u)$. It was shown in [12, 13] that ϕ-hyperbolic dual $\mathbb{N}^h_\ell[\phi](u) = \frac{1}{2}((1+\cos\phi)\boldsymbol{X}^\ell_+(u) + (1-\cos\phi)\boldsymbol{X}^\ell_-(u))$ can be taken as a normal vector of M^L_- at p, where

$$\begin{array}{rcrcl} \mathcal{L}_{42}[\phi] & : & U & \longrightarrow & \Delta^-_{42}(\phi) \\ & & u & \longmapsto & (\mathbb{N}^h_\ell[\phi](u), \boldsymbol{X}^\ell_-(u)) \end{array}$$

is a Legendrian embedding for $\phi \in [0, \pi/2]$. Here, it is obvious that $\mathbb{N}^h_\ell[0](u) = \boldsymbol{X}^\ell_+(u)$ and $\mathbb{N}^h_\ell[\pi/2](u) = \boldsymbol{X}^h(u)$. If we write $\boldsymbol{X}^h$, $\boldsymbol{X}^d$, $\boldsymbol{X}^\ell_{-u_i}$ and M^L_- instead of (respectively, $\boldsymbol{n}^T$, $\boldsymbol{n}^S$, $\boldsymbol{X}_{u_i}$ and M), we get the results given in [12, 13] as special cases of some of our results in this paper.

A The theory of Lagrangian singularities

In this appendix, we give a brief review on the theory of Lagrangian singularities in [1]. We consider the cotangent bundle $\pi : T^*\mathbb{R}^r \to \mathbb{R}^r$ over $\mathbb{R}^r$. Let $(u,p) = (u_1, \ldots, u_r, p_1, \ldots, p_r)$ be the canonical coordinates on $T^*\mathbb{R}^r$. Then the canonical symplectic structure on $T^*\mathbb{R}^r$ is given by the *canonical two form* $\omega = \sum_{i=1}^r dp_i \wedge du_i$. Let $i : L \to T^*\mathbb{R}^r$ be an immersion. We say that i is a *Lagrangian immersion* if $\dim L = r$ and $i^*\omega = 0$. In this case, the critical value set of $\pi \circ i$ is called the *caustic* of $i : L \to T^*\mathbb{R}^r$ and it is denoted by C_L. The main result in the theory of Lagrangian singularities is the description of Lagrangian immersion germs by means of the families of function germs. Let $F : (\mathbb{R}^n \times \mathbb{R}^r, (\mathbf{0},\mathbf{0})) \to (\mathbb{R}, 0)$ be an r-parameter unfolding of function germs. We call

$$C(F) = \left\{ (x,u) \in (\mathbb{R}^n \times \mathbb{R}^r, (\mathbf{0},\mathbf{0})) \middle| \frac{\partial F}{\partial x_1}(x,u) = \cdots = \frac{\partial F}{\partial x_n}(x,u) = 0 \right\}$$

the *catastrophe set* of F and

$$\mathcal{B}_F = \left\{ u \in (\mathbb{R}^r, 0) \middle| \text{ there exsist } (x,u) \in C(F) \text{ such that rank} \left(\frac{\partial^2 F}{\partial x_i \partial x_j}(x,u) \right) < n \right\}$$

the *bifurcation set* of F.

Let $\pi_r : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^r, 0)$ be the canonical projection. Then the bifurcation set of F is the critical value set of $\pi_r|_{C(F)}$. We say that F is a *Morse family of functions* if the map germ

$$\Delta F = \left(\frac{\partial F}{\partial u_1}, \ldots, \frac{\partial F}{\partial u_r} \right) : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^r, 0)$$

is non-singular, where $(x,u) = (x_1, \ldots, x_n, u_1, \ldots, u_r) \in (\mathbb{R}^n \times \mathbb{R}^r, 0)$. In this case, we have a smooth submanifold germ $C(F) \subset (\mathbb{R}^n \times \mathbb{R}^r, 0)$ and a map germ $\mathcal{L}(F) : (C(F), 0) \to T^*\mathbb{R}^r$ defined by

$$\mathcal{L}(F)(x,u) = \left(u, \frac{\partial F}{\partial u_1}, \ldots, \frac{\partial F}{\partial u_r} \right).$$

We can show that $\mathcal{L}(F)$ is a Lagrangian immersion. It is known that all Lagrangian submanifold germs in $T^*\mathbb{R}^r$ can be constructed by the above method (cf.,[1]). We call F a *generating family* of $\mathcal{L}(F)$.

We define an equivalence relation among Lagrangian immersion germs. Let $i : (L, x) \to (T^*\mathbb{R}^r, p)$ and $i' : (L', x') \to (T^*\mathbb{R}^r, p')$ be Lagrangian immersion germs. Then i and i' are said to be *Lagrangian equivalent* if there exist a diffeomorphism germ $\sigma : (L, x) \to (L', x')$, a symplectic diffeomorphism germ $\tau : (T^*\mathbb{R}^r, p) \to (T^*\mathbb{R}^r, p')$ and a diffeomorphism germ $\bar{\tau} : (\mathbb{R}^r, \pi(p)) \to (\mathbb{R}^r, \pi(p'))$ such that $\tau \circ i = i' \circ \sigma$ and $\pi \circ \tau = \bar{\tau} \circ \pi$, where $\pi : (T^*\mathbb{R}^r, p) \to (\mathbb{R}^r, \pi(p))$ denotes the canonical projection and a symplectic diffeomorphism germ means a diffeomorphism germ which preserves the symplectic structure on $T^*\mathbb{R}^r$. In this case, the caustic C_L is diffeomorphic to the caustic $C_{L'}$ through the diffeomorphism germ $\bar{\tau}$. We can interpret this equivalence relation by means of the generating families. Denote by $\mathcal{E}_m$ the local ring of function germs $(\mathbb{R}^m, 0) \to \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_m = \{h \in \mathcal{E}_m | h(0) = 0\}$. Let $F, G : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and G are *P-$\mathcal{R}^+$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^n \times \mathbb{R}^r, 0)$ of the form $\Phi(x, u) = (\Phi_1(x, u), \phi(u))$ and a function germ $h : (\mathbb{R}^r, 0) \to \mathbb{R}$ such that $G(x, u) = F(\Phi(x, u)) + h(u)$. Given $F_1 \in \mathfrak{M}_{n+r}$ and $F_2 \in \mathfrak{M}_{n'+r}$, we say that $F_1,\ F_2$ are *stably P-$\mathcal{R}^+$-equivalent* if they become P-$\mathcal{R}^+$-equivalent after the addition of some new arguments y_i to the arguments x_i and of some nondegenerate quadratic forms Q_i in the new arguments y_i to the functions F_i (i.e., $F_1 + Q_1$ and $F_2 + Q_2$ are P-$\mathcal{R}^+$-equivalent). Let $F : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ be a function germ. We say that F is an *$\mathcal{R}^+$-versal unfolding* of $f = F|_{\mathbb{R}^n \times \{0\}}$ if

$$\mathcal{E}_n = J_f + \left\langle \frac{\partial F}{\partial u_1} | \mathbb{R}^n \times \{0\}, \dots, \frac{\partial F}{\partial u_r} | \mathbb{R}^n \times \{0\} \right\rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}},$$

where

$$J_f = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle_{\mathcal{E}_n}.$$

We say that the Lagrangian immersion germ $\mathcal{L}(F)$ is *Lagrangian stable* if the generating family F is $\mathcal{R}^+$-versal unfolding of $f = F|_{\mathbb{R}^n \times \{0\}}$. Actually, this definition is an infinitesimal version of the Lagrangian stablity [1]. However, we only need the above definition.

Theorem A.1 ([1]) *Let $F_1 \in \mathfrak{M}_{n+r}$ and $F_2 \in \mathfrak{M}_{n'+r}$ be Morse families of functions. Then $\mathcal{L}(F_1)$ and $\mathcal{L}(F_2)$ are Lagrangian equivalent if and only if F_1 and F_2 are stably P-$\mathcal{R}^+$-equivalent.*

We have the following corollary.

Corollary A.2 *Let $F_1 \in \mathfrak{M}_{n+r}$ and $F_2 \in \mathfrak{M}_{n'+r}$ be Morse families of functions. Suppose that $\mathcal{L}(F_1)$ and $\mathcal{L}(F_2)$ are Lagrangian stable. Then $\mathcal{L}(F_1)$ and $\mathcal{L}(F_2)$ are Lagrangian equivalent if and only if $f_1 = F_1|_{\mathbb{R}^n \times \{0\}}$ and $f_2 = F_2|_{\mathbb{R}^n \times \{0\}}$ are $\mathcal{R}^+$-equivalent.*

Here we say that f_1 and f_2 are $\mathcal{R}^+$-equivalent if there exists a diffeomorphism germ $\phi : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^n, \mathbf{0})$ such that $f_1 \circ \phi = f_2$.

Proof. Suppose that $f_1 = F_1|_{\mathbb{R}^n \times \{0\}}$ and $f_2 = F_2|_{\mathbb{R}^n \times \{0\}}$ are $\mathcal{R}^+$-equivalent. By the uniqueness theorem of the $\mathcal{R}^+$-versal unfolding [1, 3], F_1 and F_2 are stably P-$\mathcal{R}^+$-equivalent. Thus, from Theorem A.1, $\mathcal{L}(F_1)$ and $\mathcal{L}(F_2)$ are Lagrangian equivalent. The converse assertion is trivial by definition. □

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Shyuichi Izumiya, Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan.
e-mail:`izumiya@math.sci.hokudai.ac.jp`
Handan Yıldırım, Department of Mathematics, Faculty of Science, Istanbul University, 34134, Vezneciler, Istanbul, Turkey.
e-mail:`handanyildirim@istanbul.edu.tr`