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# Asymptotic Analysis of the Fourier Transform of a Probability Measure with Application to Quantum Zeno Effect

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## Abstract

Let  $\mu$  be a probability measure on the set  $\mathbb{R}$  of real numbers and  $\hat{\mu}(t) := \int_{\mathbb{R}} e^{-it\lambda} d\mu(\lambda)$  ( $t \in \mathbb{R}$ ) be the Fourier transform of  $\mu$  ( $i$  is the imaginary unit). Then, under suitable conditions, asymptotic formulae of  $|\hat{\mu}(t/x)|^{2x}$  in  $1/x$  as  $x \rightarrow \infty$  are derived. These results are applied to the so-called quantum Zeno effect to establish asymptotic formulae of its occurrence probability in the inverse of the number  $N$  of measurements made in a time interval as  $N \rightarrow \infty$ .

**Keywords:** quantum Zeno effect, Hamiltonian, probability measure, asymptotic analysis

**Mathematics Subject Classification 2010:** 47N50, 81Q10

## 1 Introduction

A series of measurements on a quantum system may hinder or inhibit transitions from the initial state to other different states. If such a phenomenon occurs, then it is called quantum Zeno effect (QZE) (see, e.g., [2, 3, 4, 5, 6]). Recently Arai and Fuda [1] reconsidered QZE from mathematical physics points of view and clarified some general mathematical features of it. But, in [1], a problem was left open, which is concerned with asymptotic behaviors of the occurrence probability of QZE in  $1/N$  as  $N \rightarrow \infty$  with  $N$  being the number of the measurements made on a quantum system in a time interval. In this paper, we concentrate our attention on this problem and give a complete solution to it.

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To explain the problem concretely, let  $\mathcal{H}$  be a complex Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  (linear in the second variable) and norm  $\|\cdot\|$ , and  $H$  be a self-adjoint operator on  $\mathcal{H}$  with domain  $D(H)$ . In the context of QZE,  $\mathcal{H}$  and  $H$  are respectively the Hilbert space of state vectors and the Hamiltonian of the quantum system under consideration. By an axiom of quantum mechanics, the strongly continuous one-parameter unitary group  $\{e^{-itH}\}_{t \in \mathbb{R}}$  describes the time development of the quantum system<sup>1</sup>: if the state at time  $t = t_0 \in \mathbb{R}$  is a unit vector  $\Psi \in \mathcal{H}$ , then the state at time  $t \in \mathbb{R}$  is  $\Psi(t) := e^{-i(t-t_0)H}\Psi$ , provided that no measurement is made during the time interval  $(t_0, t]$ . Moreover, the probability of finding by measurement a state  $\Phi \in \mathcal{H}$  with  $\|\Phi\| = 1$  at time  $t$  is equal to  $|\langle \Phi, \Psi(t) \rangle|^2$ .

Suppose that, in a time interval  $[0, t]$  ( $t > 0$ ),  $N$  measurements on the quantum system are made successively at times  $t_1 = t/N, t_2 = 2t/N, \dots, t_j = jt/N, \dots, t_N = t$  ( $j = 1, \dots, N$ ) with initial state  $\Psi \in \mathcal{H}$ , the state at time  $t_0 = 0$ , satisfying  $\|\Psi\| = 1$ . Then the probability of finding the state  $\Psi$  at each time  $t_j$  ( $j = 1, \dots, N$ ) is given by

$$P_N(\Psi, t) := \prod_{j=1}^N |\langle \Psi, e^{-i(t_j - t_{j-1})H}\Psi \rangle|^2 = |\langle \Psi, e^{-itH/N}\Psi \rangle|^{2N}. \quad (1.1)$$

It is proved [1, Theorem 2.1] that, if  $\Psi$  is in  $D(H)$ , then

$$\lim_{N \rightarrow \infty} P_N(\Psi, t) = 1. \quad (1.2)$$

This corresponds to the occurrence of QZE in the present context. In this sense, we call  $P_N(\Psi, t)$  the occurrence probability of QZE with respect to the initial state  $\Psi$  and the time interval  $[0, t]$ .

It may be interesting to investigate an asymptotic behavior of  $P_N(\Psi, t)$  in  $1/N$ , i.e.,

$$P_N(\Psi, t) = 1 + c_1(\Psi, t) \frac{1}{N} + c_2(\Psi, t) \left(\frac{1}{N}\right)^2 + \dots + c_p(\Psi, t) \left(\frac{1}{N}\right)^p + o\left(\frac{1}{N^p}\right) \quad (N \rightarrow \infty), \quad (1.3)$$

with some  $p \in \mathbb{N}$  (the set of natural numbers), where  $c_j(\Psi, t)$  ( $j = 1, \dots, p$ ) are real numbers to be determined. In [1, Theorem 3.1], it is shown that (1.3) for  $p = 1$  holds with

$$c_1(\Psi, t) = -t^2(\Delta H)_\Psi^2, \quad (1.4)$$

where

$$(\Delta H)_\Psi := \|(H - \langle \Psi, H\Psi \rangle)\Psi\| = \sqrt{\|H\Psi\|^2 - \langle \Psi, H\Psi \rangle^2}$$

is the uncertainty of  $H$  in the state  $\Psi$ . But, to find higher order asymptotics of  $P_N(\Psi, t)$  was left open. It is the goal of the present paper to derive an asymptotic formula of  $P_N(\Psi, t)$  up to an arbitrary order of  $1/N$ .

The method used in [1], which is operator-theoretical, seems to be difficult to extend for higher order asymptotics of  $P_N(\Psi, t)$  in  $1/N$ . This suggests that one has to seek another method. In this paper, we present a new and simple method. The idea of it is as follows.

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<sup>1</sup>We use the physical unit system such that  $\hbar = h/2\pi$  ( $h$  is the Planck constant) is equal to 1.

We first note that the quantity  $\langle \Psi, e^{-isH} \Psi \rangle$  ( $s \in \mathbb{R}$ ) is written as follows:

$$\langle \Psi, e^{-isH} \Psi \rangle = \int_{\mathbb{R}} e^{-is\lambda} d\|E_H(\lambda)\Psi\|^2, \quad (1.5)$$

where  $E_H(\cdot)$  is the spectral measure of  $H$ . The measure

$$\mu_{\Psi}(\cdot) := \|E_H(\cdot)\Psi\|^2 \quad (1.6)$$

on  $\mathbb{R}$  is a probability measure. Putting

$$\hat{\mu}_{\Psi}(s) := \int_{\mathbb{R}} e^{-is\lambda} d\mu_{\Psi}(\lambda), \quad s \in \mathbb{R}, \quad (1.7)$$

the Fourier transform of the probability measure  $\mu_{\Psi}$ , one has

$$\langle \Psi, e^{-isH} \Psi \rangle = \hat{\mu}_{\Psi}(s), \quad s \in \mathbb{R}. \quad (1.8)$$

Hence

$$P_N(\Psi, t) = |\hat{\mu}_{\Psi}(t/N)|^{2N}. \quad (1.9)$$

Thus the problem may be stated in a general form as follows:

**Problem:** Let  $\mu$  be a probability measure on  $\mathbb{R}$  and

$$\hat{\mu}(s) := \int_{\mathbb{R}} e^{-is\lambda} d\mu(\lambda), \quad s \in \mathbb{R}. \quad (1.10)$$

Then, for each  $t \in \mathbb{R}$ , find asymptotic formulae of  $|\hat{\mu}(t/x)|^{2x}$  in  $1/x$  as  $x \rightarrow \infty$ .

In our method, we first derive asymptotic formulae of  $\log |\hat{\mu}(t/x)|^{2x}$  in  $1/x$  as  $x \rightarrow \infty$ , instead of  $|\hat{\mu}(t/x)|^{2x}$  itself. This is done in Section 2. Then we derive in Section 3 asymptotic formulae of  $|\hat{\mu}(t/x)|^{2x}$  in  $1/x$  as  $x \rightarrow \infty$ . In the last section we apply the results in Sections 2 and 3 to  $P_N(\Psi, t)$  to obtain asymptotic formulae of  $\log P_N(\Psi, t)$  and  $P_N(\Psi, t)$  in  $1/N$  as  $N \rightarrow \infty$ .

## 2 Asymptotic Formulae of $\log |\hat{\mu}(t/x)|^{2x}$

Let  $\mu$  be a probability measure on  $\mathbb{R}$ . For each  $k \in \mathbb{N}$ , we define

$$M_k := \int_{\mathbb{R}} \lambda^k d\mu(\lambda), \quad (2.1)$$

the  $k$ -th moment of the random variable  $\lambda$ , provided that  $\int_{\mathbb{R}} |\lambda|^k d\mu(\lambda) < \infty$ . With these constants, for each  $n \in \mathbb{N}$ , we introduce a number  $a_n$  by

$$a_n := \sum_{r=1}^n \frac{(-1)^{r-1}}{r} \sum_{\substack{k_1+\dots+k_r=n \\ k_1, \dots, k_r \geq 1}} \frac{M_{k_1} \cdots M_{k_r}}{k_1! \cdots k_r!}, \quad (2.2)$$

provided that  $\int_{\mathbb{R}} |\lambda|^n d\mu(\lambda) < \infty$ .

**Theorem 2.1** Assume that, for some  $c > 0$ ,

$$\int_{\mathbb{R}} e^{c|\lambda|} d\mu(\lambda) < \infty. \quad (2.3)$$

Let

$$K := \left\{ y \in \mathbb{R} \mid \int_{\mathbb{R}} e^{|y||\lambda|} d\mu(\lambda) < 2 \right\}. \quad (2.4)$$

Then, for all  $x \in \mathbb{R} \setminus \{0\}$  and  $t \in \mathbb{R}$  satisfying  $t/x \in K$ ,

$$\log |\hat{\mu}(t/x)|^{2x} = 2 \sum_{n=1}^{\infty} (-1)^n a_{2n} t^{2n} \left( \frac{1}{x} \right)^{2n-1}, \quad (2.5)$$

converging absolutely.

**Remark 2.2** Under assumption (2.3), for all  $k \in \mathbb{N}$ ,  $\int_{\mathbb{R}} |\lambda|^k d\mu(\lambda) < \infty$  and there exists a constant  $\varepsilon_0 > 0$  such that  $(-\varepsilon_0, \varepsilon_0) \subset K$ .

**Remark 2.3** In the right hand side on (2.5), only even powers for  $t$  and only odd powers for  $1/x$  appear. This is natural, because  $\log |\hat{\mu}(t/x)|^{2x}$  is even in  $t$  and odd in  $1/x$ .

To prove Theorem 2.1, we first present an elementary lemma. Let

$$u(x) := \int_{\mathbb{R}} (e^{-ix\lambda} - 1) d\mu(\lambda) = \hat{\mu}(x) - 1, \quad x \in \mathbb{R}. \quad (2.6)$$

**Lemma 2.4** Assume (2.3). Then, for all  $x \in K$ ,

$$u(x) = \sum_{k=1}^{\infty} \frac{(-ix)^k}{k!} M_k. \quad (2.7)$$

where the right hand side is absolutely convergent.

*Proof.* Let  $x \in K$  be fixed. Then we have  $u(x) = \int_{\mathbb{R}} \lim_{N \rightarrow \infty} g_N(\lambda) d\mu(\lambda)$  with  $g_N(\lambda) := \sum_{k=1}^N (-ix)^k \lambda^k / k!$ ,  $\lambda \in \mathbb{R}$ . It is easy to see that  $|g_N(\lambda)| \leq e^{|x||\lambda|}$ . Since  $x$  is in  $K$ , the right hand side is integrable independent of  $N$ . Hence, by the Lebesgue dominated convergence theorem, we obtain  $u(x) = \lim_{N \rightarrow \infty} \int_{\mathbb{R}} g_N(\lambda) d\mu(\lambda)$ , which gives (2.7). Moreover

$$\sum_{k=1}^{\infty} \frac{|x|^k}{k!} |M_k| \leq \sum_{k=1}^{\infty} \int_{\mathbb{R}} \frac{(|x||\lambda|)^k}{k!} d\mu(\lambda) = \int_{\mathbb{R}} e^{|x||\lambda|} d\mu(\lambda) - 1 < \infty.$$

Hence the infinite series on the right hand side of (2.7) is absolutely convergent. ■

## Proof of Theorem 2.1

By replacing  $t/x$  in  $|\hat{\mu}(t/x)|^{2x}$  by  $x$ , we need only to consider the behavior of the function

$$F(x) := |\hat{\mu}(x)|^{2t/x} \quad (2.8)$$

as  $x \downarrow 0$ . Since  $\hat{\mu}(x) - 1 = \int_{\mathbb{R}} (e^{-ix\lambda} - 1) d\mu(\lambda)$  and  $|e^{-ix\lambda} - 1| \leq e^{x|\lambda|} - 1$ ,  $\forall x \in \mathbb{R}$ , it follows that, for all  $x \in K$ ,

$$|\hat{\mu}(x) - 1| < 1. \quad (2.9)$$

Hence we can define

$$f(x) := \log \hat{\mu}(x), \quad x \in K. \quad (2.10)$$

We note that  $|\hat{\mu}(x)|^2 = \hat{\mu}(x)\hat{\mu}(-x)$ . Hence we have

$$\log F(x) = \frac{t}{x}(f(x) + f(-x)), \quad x \in K \setminus \{0\} \quad (2.11)$$

Assumption (2.3) implies that, for all  $k \in \mathbb{N}$ ,  $\hat{\mu}$  is  $k$  times continuously differentiable on  $\mathbb{R}$  with the  $k$ -th derivative equal to

$$\hat{\mu}^{(k)}(x) = (-i)^k \int_{\mathbb{R}} \lambda^k e^{-i\lambda x} d\mu(\lambda), \quad x \in \mathbb{R}. \quad (2.12)$$

In particular, we have

$$\hat{\mu}^{(k)}(0) = (-i)^k M_k. \quad (2.13)$$

Hence  $f$  also is infinitely differentiable on  $K$ .

With  $u$  defined by (2.6), we can write

$$f(x) = \log(1 + u(x)).$$

By (2.9), for all  $x \in K$ ,  $|u(x)| < 1$ . Hence we have

$$f(x) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} u(x)^r, \quad x \in K,$$

where the infinite series is absolutely convergent. By Lemma 2.4, we have for all  $x \in K$

$$u(x)^r = \sum_{k_1, \dots, k_r=1}^{\infty} \frac{(-ix)^{k_1+\dots+k_r}}{k_1! \cdots k_r!} M_{k_1} \cdots M_{k_r} = \sum_{n=r}^{\infty} (-ix)^n \sum_{\substack{k_1+\dots+k_r=n \\ k_1, \dots, k_r \geq 1}} \frac{M_{k_1} \cdots M_{k_r}}{k_1! \cdots k_r!}.$$

Hence, for all  $x \in K$

$$f(x) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} \sum_{n=r}^{\infty} (-ix)^n \sum_{\substack{k_1+\dots+k_r=n \\ k_1, \dots, k_r \geq 1}} \frac{M_{k_1} \cdots M_{k_r}}{k_1! \cdots k_r!}. \quad (2.14)$$

It is easy to see that, for all  $x \in K$ ,

$$\sum_{r=1}^{\infty} \frac{1}{r} \sum_{n=r}^{\infty} |x|^n \sum_{\substack{k_1+\dots+k_r=n \\ k_1, \dots, k_r \geq 1}} \frac{|M_{k_1}| \cdots |M_{k_r}|}{k_1! \cdots k_r!}.$$

converges. Hence, in (2.14), we can interchange the sums on  $r$  and  $n$  to obtain

$$f(x) = \sum_{n=1}^{\infty} (-i)^n a_n x^n \quad (2.15)$$

where  $a_n$  is given by (2.2). Therefore

$$\log F(x) = 2t \sum_{n=1}^{\infty} (-1)^n a_{2n} x^{2n-1}, \quad x \in K, \quad (2.16)$$

converging absolutely. Replacing  $x$  by  $t/x$ , we obtain (2.5). ■

We next consider the case where (2.3) does not necessarily hold. In this case, we have the following result:

**Theorem 2.5** *Let  $n \in \mathbb{N}$  and suppose that*

$$\int_{\mathbb{R}} |\lambda|^n d\mu(\lambda) < \infty. \quad (2.17)$$

Let

$$p_n := \begin{cases} \frac{n}{2} & \text{for } n \geq 2 \text{ even} \\ \frac{n-1}{2} & \text{for } n \geq 2 \text{ odd} \end{cases} \quad (2.18)$$

Then

$$\log |\hat{\mu}(t/x)|^{2x} = 2 \sum_{k=1}^{p_n} (-1)^k a_{2k} t^{2k} \left(\frac{1}{x}\right)^{2k-1} + o\left(\frac{1}{x^{2p_n-1}}\right) \quad (x \rightarrow \infty). \quad (2.19)$$

*Proof.* Since  $\hat{\mu}(0) = 1$  and  $\hat{\mu}$  is continuous on  $\mathbb{R}$ , there exists a constant  $\delta > 0$  such that, for all  $x \in I_\delta := (-\delta, \delta)$ , inequality (2.9) holds. Hence we can define  $g : I_\delta \rightarrow \mathbb{R}$  by

$$g(x) := \log \hat{\mu}(x), \quad x \in I_\delta.$$

Then we have

$$F(x) = \frac{t}{x} (g(x) + g(-x)), \quad x \in I_\delta \setminus \{0\}. \quad (2.20)$$

Under the present assumption,  $\hat{\mu}$  is  $n$  times continuously differentiable on  $\mathbb{R}$ . Hence so is  $g$  on  $I_\delta$  with derivative  $g'$  satisfying

$$g' \hat{\mu} = \hat{\mu}'. \quad (2.21)$$

By Taylor's theorem, we have

$$g(x) = \sum_{k=1}^n \frac{g^{(k)}(0)}{k!} x^k + o(x^n) \quad (x \rightarrow 0).$$

Differentiating the both sides of (2.21)  $(k-1)$  times and applying the Leibniz formula, we obtain the following recursion relation on  $g^{(j)}(0)$ :

$$g'(0) = -iM_1, \quad g^{(k)}(0) = (-i)^k \left( M_k - \sum_{j=1}^{k-1} {}_{k-1}C_{j-1} i^j M_{k-j} g^{(j)}(0) \right) \quad (k = 2, \dots, n), \quad (2.22)$$

where  ${}_m C_l := m! / [(m-l)!l!]$  ( $m, l \in \{0\} \cup \mathbb{N}, m \geq l$ ).

It is obvious that the function  $f$  in the proof of Theorem 2.1 also satisfies (2.21) with  $g$  replaced by  $f$ . Hence (2.22) holds with  $g$  replaced by  $f$ . Therefore  $g^{(k)}(0) = f^{(k)}(0), k = 1, \dots, n$ . From the proof of Theorem 3.2, we see that  $f^{(k)}(0) = (-i)^k a_k k!$ . Hence  $g^{(k)}(0) = (-i)^k a_k k!$ . Thus

$$g(x) = \sum_{k=1}^n (-i)^k a_k x^k + o(x^n) \quad (x \rightarrow 0),$$

which implies that

$$F(x) = 2t \sum_{k=1}^{p_n} (-1)^k a_{2k} x^{2k-1} + o(x^{2p_n-1}).$$

Thus (2.19) holds. ■

### 3 Asymptotic Formulae of $|\hat{\mu}(t/x)|^{2x}$

To derive from (2.5) an asymptotic formula of  $|\mu(t/x)|^{2x}$  itself in  $1/x$ , we need only to note an elementary fact:

**Lemma 3.1** *Let  $\{c_m\}_{m=1}^{\infty}$  be a sequence of complex numbers such that the infinite series  $S := \sum_{m=1}^{\infty} c_m$  converges absolutely. Let*

$$\gamma_n := \sum_{k=1}^n \frac{1}{k!} \sum_{\substack{m_1 + \dots + m_k = n \\ m_1, \dots, m_k \geq 1}} c_{m_1} \cdots c_{m_k}. \quad (3.1)$$

Then

$$e^S = 1 + \sum_{n=1}^{\infty} \gamma_n, \quad (3.2)$$

converging absolutely.

*Proof.* An easy exercise. ■

For each  $t \in \mathbb{R}$ , we define a sequence  $\{\alpha_n(t)\}_{n=1}^{\infty}$  as follows:

$$\alpha_{2n-1}(t) := 2(-1)^n a_{2n} t^{2n}, \quad \alpha_{2n}(t) := 0. \quad (3.3)$$



**Theorem 3.2** Assume (2.3) and let

$$A_n(t) := \sum_{k=1}^n \frac{1}{k!} \sum_{\substack{m_1+\dots+m_k=n \\ m_1, \dots, m_k \geq 1}} \alpha_{m_1}(t) \cdots \alpha_{m_k}(t), \quad n \in \mathbb{N}. \quad (3.4)$$

Then, for all  $x \in \mathbb{R} \setminus \{0\}$  and  $t \in \mathbb{R}$  satisfying  $t/x \in K$ ,

$$|\hat{\mu}(t/x)|^{2x} = 1 + \sum_{n=1}^{\infty} A_n(t) \left(\frac{1}{x}\right)^n, \quad (3.5)$$

converging absolutely.

*Proof.* By Theorem 2.1, we have

$$|\hat{\mu}(t/x)|^{2x} = \exp\left(\sum_{m=1}^{\infty} \alpha_m(t) x^{-m}\right).$$

Hence, by Lemma 3.1, we obtain (3.5). ■

A finite sum version of Lemma 3.1 is given as follows, which also is easy to prove:

**Lemma 3.3** Let  $c_m, m = 1, \dots, p$ , be complex numbers,  $p \in \mathbb{N}$ , and

$$S_p := \sum_{m=1}^p c_m x^m + o(x^p) \quad (x \rightarrow 0).$$

Then

$$e^{S_p} = 1 + \sum_{n=1}^p \gamma_n x^n + o(x^p) \quad (x \rightarrow 0), \quad (3.6)$$

where  $\gamma_n$  is defined by (3.1).

**Theorem 3.4** Assume (2.17). Then, for all  $t \in \mathbb{R}$ ,

$$|\hat{\mu}(t/x)|^{2x} = 1 + \sum_{n=1}^{2p_n-1} A_n(t) \left(\frac{1}{x}\right)^n + o\left(\frac{1}{x^{2p_n-1}}\right) \quad (x \rightarrow \infty). \quad (3.7)$$

*Proof.* Similar to the proof of Theorem 3.2. ■

## 4 Applications to QZE

To apply the results in Sections 2 and 3 to QZE, for each  $k \in \mathbb{N}$  and a unit vector  $\Psi \in D(|H|^{k/2})$ , we introduce

$$\langle H^k \rangle := \int_{\mathbb{R}} \lambda^k d\|E_H(\lambda)\Psi\|^2, \quad (4.1)$$

the  $k$ -th expectation value of the Hamiltonian  $H$  in the state  $\Psi$ , and, for each  $n \in \mathbb{N}$  and a unit vector  $\Psi \in D(|H|^{n/2})$ , we define

$$b_n(\Psi) := \sum_{r=1}^n \frac{(-1)^{r-1}}{r} \sum_{\substack{k_1+\dots+k_r=n \\ k_1, \dots, k_r \geq 1}} \frac{\langle H^{k_1} \rangle \cdots \langle H^{k_r} \rangle}{k_1! \cdots k_r!}. \quad (4.2)$$

**Theorem 4.1** *Let  $t \in \mathbb{R}$  be fixed. Suppose that, for some  $c > 0$ ,  $\Psi \in D(e^{c|H|})$  with  $\|\Psi\| = 1$  and that  $N$  obeys the following condition:*

$$\int_{\mathbb{R}} e^{t|\lambda|/N} d\|E_H(\lambda)\Psi\|^2 < 2. \quad (4.3)$$

Then

$$\log P_N(\Psi, t) = 2 \sum_{k=1}^{\infty} (-1)^k b_{2k}(\Psi) t^{2k} \left( \frac{1}{N} \right)^{2k-1}, \quad (4.4)$$

converging absolutely.

*Proof.* Let  $\mu_{\Psi}$  be given by (1.6). Then we need only to show that  $\mu = \mu_{\Psi}$  satisfies the assumption of Theorem 2.1. The assumption  $\Psi \in D(e^{c|H|})$  is equivalent to that

$$\int_{\mathbb{R}} e^{2c|\lambda|} d\mu_{\Psi}(\lambda) < \infty.$$

Hence (2.3) holds with  $\mu = \mu_{\Psi}$ . In the present case, we have  $M_k = \langle H^k \rangle$ . Thus (2.5) gives (4.4).  $\blacksquare$

In the case where  $\Psi$  is not necessarily in  $D(e^{c|H|})$ , we have the following result:

**Theorem 4.2** *Let  $n \in \mathbb{N}$  and suppose that  $\Psi \in D(|H|^n)$  with  $\|\Psi\| = 1$ . Then, for all  $t \in \mathbb{R}$ ,*

$$\log P_N(\Psi, t) = 2 \sum_{k=1}^{p_n} (-1)^k b_{2k}(\Psi) t^{2k} \left( \frac{1}{N} \right)^{2k-1} + o(1/N^{2p_n-1}) \quad (N \rightarrow \infty). \quad (4.5)$$

*Proof.* A simple application of Theorem 2.5.  $\blacksquare$

Finally we derive asymptotic formulae of  $P_N(\Psi, t)$  itself. For this purpose, we define a sequence  $\{\beta_n(\Psi, t)\}_{n=1}^{\infty}$  ( $t \in \mathbb{R}$ ) as follows:

$$\beta_{2n-1}(\Psi, t) := 2(-1)^n b_{2n}(\Psi) t^{2n}, \quad (4.6)$$

$$\beta_{2n}(\Psi, t) := 0. \quad (4.7)$$

**Theorem 4.3** *Suppose that the same assumption as in Theorem 4.1 holds. Let*

$$\gamma_n(\Psi, t) := \sum_{k=1}^n \frac{1}{k!} \sum_{\substack{m_1+\dots+m_k=n \\ m_1, \dots, m_k \geq 1}} \beta_{m_1}(\Psi, t) \cdots \beta_{m_k}(\Psi, t), \quad n \in \mathbb{N}. \quad (4.8)$$

Then

$$P_N(\Psi, t) = 1 + \sum_{n=1}^{\infty} \gamma_n(\Psi, t) \left( \frac{1}{N} \right)^n, \quad (4.9)$$

converging absolutely.

*Proof.* A simple application of Theorem 3.2. ■

**Theorem 4.4** *Let  $n \in \mathbb{N}$  and suppose that  $\Psi \in D(|H|^n)$  with  $\|\Psi\| = 1$ . Then, for all  $t \in \mathbb{R}$ ,*

$$P_N(\Psi, t) = 1 + \sum_{n=1}^{2p_n-1} \gamma_n(\Psi, t) \left(\frac{1}{N}\right)^n + o\left(\frac{1}{N^{2p_n-1}}\right) \quad (N \rightarrow \infty). \quad (4.10)$$

*Proof.* This follows from an application of Theorem 3.4. ■

**Example 4.5** By direct computations, we have

$$\gamma_1(\Psi, t) = -(\Delta H)_\Psi t^2,$$

which coincides with  $c_1(\Psi, t)$  is given by (1.4), and

$$\gamma_2(\Psi, t) = \frac{1}{2}(\Delta H)_\Psi^4 t^4.$$

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