Asymptotic Analysis of the Fourier Transform of a Probability Measure with Application to Quantum Zeno Effect

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Abstract

Let \( \mu \) be a probability measure on the set \( \mathbb{R} \) of real numbers and \( \hat{\mu}(t) := \int_{\mathbb{R}} e^{-it\lambda} d\mu(\lambda) \) \((t \in \mathbb{R})\) be the Fourier transform of \( \mu \) (\( i \) is the imaginary unit). Then, under suitable conditions, asymptotic formulae of \( |\hat{\mu}(t/x)|^2 x \) in \( 1/x \) as \( x \to \infty \) are derived. These results are applied to the so-called quantum Zeno effect to establish asymptotic formulae of its occurrence probability in the inverse of the number \( N \) of measurements made in a time interval as \( N \to \infty \).

Keywords: quantum Zeno effect, Hamiltonian, probability measure, asymptotic analysis

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1 Introduction

A series of measurements on a quantum system may hinder or inhibit transitions from the initial state to other different states. If such a phenomenon occurs, then it is called quantum Zeno effect (QZE) (see, e.g., [2, 3, 4, 5, 6]). Recently Arai and Fuda [1] reconsidered QZE from mathematical physics points of view and clarified some general mathematical features of it. But, in [1], a problem was left open, which is concerned with asymptotic behaviors of the occurrence probability of QZE in \( 1/N \) as \( N \to \infty \) with \( N \) being the number of the measurements made on a quantum system in a time interval. In this paper, we concentrate our attention on this problem and give a complete solution to it.

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To explain the problem concretely, let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot , \cdot \rangle$ (linear in the second variable) and norm $\| \cdot \|$, and $H$ be a self-adjoint operator on $\mathcal{H}$ with domain $D(H)$. In the context of QZE, $\mathcal{H}$ and $H$ are respectively the Hilbert space of state vectors and the Hamiltonian of the quantum system under consideration. By an axiom of quantum mechanics, the strongly continuous one-parameter unitary group $\{ e^{-itH} \}_{t \in \mathbb{R}}$ describes the time development of the quantum system: if the state at time $t = t_0 \in \mathbb{R}$ is a unit vector $\Psi \in \mathcal{H}$, then the state at time $t \in \mathbb{R}$ is $\Psi(t) := e^{-i(t-t_0)H} \Psi$, provided that no measurement is made during the time interval $(t_0, t]$. Moreover, the probability of finding by measurement a state $\Phi \in \mathcal{H}$ with $\| \Phi \| = 1$ at time $t$ is equal to $| \langle \Phi, \Psi(t) \rangle |^2$.

Suppose that, in a time interval $[0, t]$ ($t > 0$), $N$ measurements on the quantum system are made successively at times $t_1 = t/N, t_2 = 2t/N, \ldots, t_j = jt/N, \ldots, t_N = t$ ($j = 1, \ldots, N$) with initial state $\Psi \in \mathcal{H}$, the state at time $t_0 = 0$, satisfying $\| \Psi \| = 1$. Then the probability of finding the state $\Psi$ at each time $t_j$ ($j = 1, \ldots, N$) is given by

$$P_N(\Psi, t) := \prod_{j=1}^{N} \left| \langle \Psi, e^{-i(t_j-t_{j-1})H} \Psi \rangle \right|^2 = \left| \langle \Psi, e^{-itHN} \Psi \rangle \right|^{2N}. \quad (1.1)$$

It is proved [1, Theorem 2.1] that, if $\Psi$ is in $D(H)$, then

$$\lim_{N \to \infty} P_N(\Psi, t) = 1. \quad (1.2)$$

This corresponds to the occurrence of QZE in the present context. In this sense, we call $P_N(\Psi, t)$ the occurrence probability of QZE with respect to the initial state $\Psi$ and the time interval $[0, t]$.

It may be interesting to investigate an asymptotic behavior of $P_N(\Psi, t)$ in $1/N$, i.e.,

$$P_N(\Psi, t) = 1 + c_1(\Psi, t) \frac{1}{N} + c_2(\Psi, t) \left( \frac{1}{N} \right)^2 + \cdots + c_p(\Psi, t) \left( \frac{1}{N} \right)^p + o \left( \frac{1}{N^p} \right) \quad (N \to \infty), \quad (1.3)$$

with some $p \in \mathbb{N}$ (the set of natural numbers), where $c_j(\Psi, t)$ ($j = 1, \ldots, p$) are real numbers to be determined. In [1, Theorem 3.1], it is shown that (1.3) for $p = 1$ holds with

$$c_1(\Psi, t) = -t^2(\Delta H)^2_\Psi, \quad (1.4)$$

where

$$(\Delta H)_\Psi := \| (H - \langle \Psi, H \Psi \rangle) \Psi \| = \sqrt{\| H \Psi \|^2 - \langle \Psi, H \Psi \rangle^2}$$

is the uncertainty of $H$ in the state $\Psi$. But, to find higher order asymptotics of $P_N(\Psi, t)$ was left open. It is the goal of the present paper to derive an asymptotic formula of $P_N(\Psi, t)$ up to an arbitrary order of $1/N$.

The method used in [1], which is operator-theoretical, seems to be difficult to extend for higher order asymptotics of $P_N(\Psi, t)$ in $1/N$. This suggests that one has to seek another method. In this paper, we present a new and simple method. The idea of it is as follows.

\(^1\text{We use the physical unit system such that } h = h/2\pi \text{ (h is the Planck constant) is equal to 1.}\)
We first note that the quantity $\langle \Psi, e^{-isH}\Psi \rangle$ ($s \in \mathbb{R}$) is written as follows:

$$
\langle \Psi, e^{-isH}\Psi \rangle = \int_{\mathbb{R}} e^{-is\lambda} d\|E_H(\lambda)\|_2^2,
$$

where $E_H(\cdot)$ is the spectral measure of $H$. The measure

$$
\mu_{\Psi}(\cdot) := \|E_H(\cdot)\|_2^2
$$
on $\mathbb{R}$ is a probability measure. Putting

$$
\hat{\mu}_{\Psi}(s) := \int_{\mathbb{R}} e^{-is\lambda} d\mu_{\Psi}(\lambda), \quad s \in \mathbb{R},
$$

the Fourier transform of the probability measure $\mu_{\Psi}$, one has

$$
\langle \Psi, e^{-isH}\Psi \rangle = \hat{\mu}_{\Psi}(s), \quad s \in \mathbb{R}.
$$

Hence

$$
P_N(\Psi, t) = |\hat{\mu}_{\Psi}(t/N)|^{2N}.
$$

Thus the problem may be stated in a general form as follows:

**Problem:** Let $\mu$ be a probability measure on $\mathbb{R}$ and

$$
\hat{\mu}(s) := \int_{\mathbb{R}} e^{-is\lambda} d\mu(\lambda), \quad s \in \mathbb{R}.
$$

Then, for each $t \in \mathbb{R}$, find asymptotic formulae of $|\hat{\mu}(t/x)|^{2x}$ in $1/x$ as $x \to \infty$.

In our method, we first derive asymptotic formulae of $\log |\hat{\mu}(t/x)|^{2x}$ in $1/x$ as $x \to \infty$, instead of $|\hat{\mu}(t/x)|^{2x}$ itself. This is done in Section 2. Then we derive in Section 3 asymptotic formulae of $|\hat{\mu}(t/x)|^{2x}$ in $1/x$ as $x \to \infty$. In the last section we apply the results in Sections 2 and 3 to $P_N(\Psi, t)$ to obtain asymptotic formulae of $\log P_N(\Psi, t)$ and $P_N(\Psi, t)$ in $1/N$ as $N \to \infty$.

### 2 Asymptotic Formulae of $\log |\hat{\mu}(t/x)|^{2x}$

Let $\mu$ be a probability measure on $\mathbb{R}$. For each $k \in \mathbb{N}$, we define

$$
M_k := \int_{\mathbb{R}} \lambda^k d\mu(\lambda),
$$

the $k$-th moment of the random variable $\lambda$, provided that $\int_{\mathbb{R}} |\lambda|^k d\mu(\lambda) < \infty$. With these constants, for each $n \in \mathbb{N}$, we introduce a number $a_n$ by

$$
a_n := \sum_{r=1}^{n} \frac{(-1)^{r-1}}{r} \sum_{k_1 + \cdots + k_r = n} \frac{M_{k_1} \cdots M_{k_r}}{k_1! \cdots k_r!},
$$

provided that $\int_{\mathbb{R}} |\lambda|^n d\mu(\lambda) < \infty$. 

3
Theorem 2.1 Assume that, for some $c > 0$,
\[ \int_{\mathbb{R}} e^{c|\lambda|} d\mu(\lambda) < \infty. \] (2.3)

Let
\[ K := \left\{ y \in \mathbb{R} \mid \int_{\mathbb{R}} e^{|y|\lambda} d\mu(\lambda) < 2 \right\}. \] (2.4)

Then, for all $x \in \mathbb{R} \setminus \{0\}$ and $t \in \mathbb{R}$ satisfying $t/x \in K$,
\[ \log |\hat{\mu}(t/x)|^{2x} = \sum_{n=1}^{\infty} (-1)^n a_{2n} \left( \frac{1}{x} \right)^{2n-1}, \] (2.5)
converging absolutely.

Remark 2.2 Under assumption (2.3), for all $k \in \mathbb{N}$, $\int_{\mathbb{R}} |\lambda|^k d\mu(\lambda) < \infty$ and there exists a constant $\varepsilon_0 > 0$ such that $(-\varepsilon_0, \varepsilon_0) \subset K$.

Remark 2.3 In the right hand side on (2.5), only even powers for $t$ and only odd powers for $1/x$ appear. This is natural, because $\log |\hat{\mu}(t/x)|^{2x}$ is even in $t$ and odd in $1/x$.

To prove Theorem 2.1, we first present an elementary lemma. Let
\[ u(x) := \int_{\mathbb{R}} (e^{-ix\lambda} - 1) d\mu(\lambda) = \hat{\mu}(x) - 1, \quad x \in \mathbb{R}. \] (2.6)

Lemma 2.4 Assume (2.3). Then, for all $x \in K$,
\[ u(x) = \sum_{k=1}^{\infty} \frac{(-ix)^k}{k!} M_k. \] (2.7)
where the right hand side is absolutely convergent.

Proof. Let $x \in K$ be fixed. Then we have $u(x) = \int_{\mathbb{R}} \lim_{N \to \infty} g_N(\lambda) d\mu(\lambda)$ with $g_N(\lambda) := \sum_{k=1}^{N} (-ix)^k \lambda^k/k!$, $\lambda \in \mathbb{R}$. It is easy to see that $|g_N(\lambda)| \leq e^{2|\lambda|}$. Since $x$ is in $K$, the right hand side is integrable independent of $N$. Hence, by the Lebesgue dominated convergence theorem, we obtain $u(x) = \lim_{N \to \infty} \int_{\mathbb{R}} g_N(\lambda) d\mu(\lambda)$, which gives (2.7). Moreover
\[ \sum_{k=1}^{\infty} \frac{|x|^k}{k!} |M_k| \leq \sum_{k=1}^{\infty} \int_{\mathbb{R}} \frac{|x||\lambda|^k}{k!} d\mu(\lambda) = \int_{\mathbb{R}} e^{x|\lambda|} d\mu(\lambda) - 1 < \infty. \]
Hence the infinite series on the right hand side of (2.7) is absolutely convergent. \qed
Proof of Theorem 2.1

By replacing $t/x$ in $|\hat{\mu}(t/x)|^{2x}$ by $x$, we need only to consider the behavior of the function

$$F(x) := |\hat{\mu}(x)|^{2t/x}$$

as $x \downarrow 0$. Since $\hat{\mu}(x) - 1 = \int_{\mathbb{R}} (e^{-ix\lambda} - 1)d\mu(\lambda)$ and $|e^{-ix\lambda} - 1| \leq e^{||x|\lambda|} - 1$, $\forall x \in \mathbb{R}$, it follows that, for all $x \in K$,

$$|\hat{\mu}(x) - 1| < 1.$$  \hfill (2.9)

Hence we can define

$$f(x) := \log \hat{\mu}(x), \quad x \in K.$$ \hfill (2.10)

We note that $|\hat{\mu}(x)|^2 = \hat{\mu}(x)\hat{\mu}(-x)$. Hence we have

$$\log F(x) = \frac{t}{x}(f(x) + f(-x)), \quad x \in K \setminus \{0\}$$ \hfill (2.11)

Assumption (2.3) implies that, for all $k \in \mathbb{N}$, $\hat{\mu}$ is $k$ times continuously differentiable on $\mathbb{R}$ with the $k$-th derivative equal to

$$\hat{\mu}^{(k)}(x) = (-i)^k \int_{\mathbb{R}} \lambda^k e^{-i\lambda x}d\mu(\lambda), \quad x \in \mathbb{R}.$$ \hfill (2.12)

In particular, we have

$$\hat{\mu}^{(k)}(0) = (-i)^k M_k.$$ \hfill (2.13)

Hence $f$ also is infinitely differentiable on $K$.

With $u$ defined by (2.6), we can write

$$f(x) = \log(1 + u(x)).$$

By (2.9), for all $x \in K$, $|u(x)| < 1$. Hence we have

$$f(x) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} u(x)^r, \quad x \in K,$$

where the infinite series is absolutely convergent. By Lemma 2.4, we have for all $x \in K$

$$u(x)^r = \sum_{k_1, \ldots, k_r=1}^{\infty} \frac{(-ix)^{k_1+\cdots+k_r}}{k_1! \cdots k_r!} M_{k_1} \cdots M_{k_r} = \sum_{n=r}^{\infty} (-ix)^n \sum_{k_1+\cdots+k_r=n, \ k_1, \ldots, k_r \geq 1} \frac{M_{k_1} \cdots M_{k_r}}{k_1! \cdots k_r!}.$$  

Hence, for all $x \in K$

$$f(x) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} \sum_{n=r}^{\infty} (-ix)^n \sum_{k_1+\cdots+k_r=n, \ k_1, \ldots, k_r \geq 1} \frac{M_{k_1} \cdots M_{k_r}}{k_1! \cdots k_r!}. \hfill (2.14)$$

It is easy to see that, for all $x \in K$,

$$\sum_{r=1}^{\infty} \frac{1}{r} \sum_{n=r}^{\infty} \frac{|M_{k_1} \cdots M_{k_r}|}{k_1! \cdots k_r!}.$$  

5
converges. Hence, in (2.14), we can interchange the sums on $r$ and $n$ to obtain

\[ f(x) = \sum_{n=1}^{\infty} (-i)^n a_n x^n \]  \hspace{1cm} (2.15)

where $a_n$ is given by (2.2). Therefore

\[ \log F(x) = 2t \sum_{n=1}^{\infty} (-1)^n a_{2n} x^{2n-1}, \quad x \in K, \]  \hspace{1cm} (2.16)

converging absolutely. Replacing $x$ by $t/x$, we obtain (2.5).

We next consider the case where (2.3) does not necessarily hold. In this case, we have the following result:

**Theorem 2.5** Let $n \in \mathbb{N}$ and suppose that

\[ \int_{\mathbb{R}} |\lambda|^n d\mu(\lambda) < \infty. \]  \hspace{1cm} (2.17)

Let

\[ p_n := \begin{cases} \frac{n}{2} & \text{for } n \geq 2 \text{ even} \\ \frac{n-1}{2} & \text{for } n \geq 2 \text{ odd} \end{cases} \]  \hspace{1cm} (2.18)

Then

\[ \log \left| \hat{\mu}(t/x) \right|^{2x} = 2t \sum_{k=1}^{p_n} (-1)^k a_{2k} t^{2k} \left( \frac{1}{x} \right)^{2k-1} + o \left( \frac{1}{x^{2p_n-1}} \right) \quad (x \to \infty). \]  \hspace{1cm} (2.19)

**Proof.** Since $\hat{\mu}(0) = 1$ and $\hat{\mu}$ is continuous on $\mathbb{R}$, there exists a constant $\delta > 0$ such that, for all $x \in I_\delta := (-\delta, \delta)$, inequality (2.9) holds. Hence we can define $g : I_\delta \to \mathbb{R}$ by

\[ g(x) := \log \hat{\mu}(x), \quad x \in I_\delta. \]

Then we have

\[ F(x) = \frac{x}{t} (g(x) + g(-x)), \quad x \in I_\delta \setminus \{0\}. \]  \hspace{1cm} (2.20)

Under the present assumption, $\hat{\mu}$ is $n$ times continuously differentiable on $\mathbb{R}$. Hence so is $g$ on $I_\delta$ with derivative $g'$ satisfying

\[ g' \hat{\mu} = \hat{\mu}'. \]  \hspace{1cm} (2.21)

By Taylor’s theorem, we have

\[ g(x) = \sum_{k=1}^{n} \frac{g^{(k)}(0)}{k!} x^k + o(x^n) \quad (x \to 0). \]
Differentiating the both sides of (2.21) \((k - 1)\) times and applying the Leibniz formula, we obtain the following recursion relation on \(g^{(j)}(0)\):

\[
g'(0) = -iM_1, \quad g^{(k)}(0) = (-i)^k \left( M_k - \sum_{j=1}^{k-1} C_{j-1} i^j M_{k-j} g^{(j)}(0) \right) \quad (k = 2, \ldots, n),
\]

where \(mC_l := m!/[(m - l)!l!]\) \((m, l \in \{0\} \cup \mathbb{N}, m \geq l)\).

It is obvious that the function \(f\) in the proof of Theorem 2.1 also satisfies (2.21) with \(g\) replaced by \(f\). Hence (2.22) holds with \(g\) replaced by \(f\). Therefore \(g^{(k)}(0) = (-i)^k a_k k!\). Hence \(g^{(k)}(0) = (-i)^k a_k k!\). Thus

\[
g(x) = \sum_{k=1}^{n} (-i)^k a_k x^k + o(x^n) \quad (x \rightarrow 0),
\]

which implies that

\[
F(x) = 2t \sum_{k=1}^{p_n} (-1)^k a_{2k} x^{2k-1} + o(x^{2p_n-1}).
\]

Thus (2.19) holds.

\section{Asymptotic Formulae of \(|\hat{\mu}(t/x)|^{2x}\)}

To derive from (2.5) an asymptotic formula of \(|\mu(t/x)|^{2x}\) itself in \(1/x\), we need only to note an elementary fact:

\textbf{Lemma 3.1} Let \(\{c_m\}_{m=1}^{\infty}\) be a sequence of complex numbers such that the infinite series \(S := \sum_{m=1}^{\infty} c_m\) converges absolutely. Let

\[
\gamma_n := \frac{1}{k!} \sum_{m_1 + \cdots + m_k = n, \ m_1, \ldots, m_k \geq 1} c_{m_1} \cdots c_{m_k}.
\]

Then

\[
e^S = 1 + \sum_{n=1}^{\infty} \gamma_n,
\]

converging absolutely.

\textit{Proof.} An easy exercise. \hfill \blacksquare

For each \(t \in \mathbb{R}\), we define a sequence \(\{\alpha_n(t)\}_{n=1}^{\infty}\) as follows:

\[
\alpha_{2n-1}(t) := 2(-1)^n a_{2n} t^{2n}, \quad \alpha_{2n}(t) := 0.
\]
Theorem 3.2 Assume (2.3) and let
\[ A_n(t) := \sum_{k=1}^{n} \frac{1}{k!} \sum_{m_1 + \cdots + m_k = n \atop m_1, \ldots, m_k \geq 1} \alpha_{m_1}(t) \cdots \alpha_{m_k}(t), \quad n \in \mathbb{N}. \] (3.4)

Then, for all \( x \in \mathbb{R} \setminus \{0\} \) and \( t \in \mathbb{R} \) satisfying \( t/x \in K \),
\[ |\hat{\mu}(t/x)|^{2x} = 1 + \sum_{n=1}^{\infty} A_n(t) \left( \frac{1}{x} \right)^n, \] (3.5)
converging absolutely.

Proof. By Theorem 2.1, we have
\[ |\hat{\mu}(t/x)|^{2x} = \exp \left( \sum_{m=1}^{\infty} \alpha_m(t)x^{-m} \right). \]
Hence, by Lemma 3.1, we obtain (3.5).

A finite sum version of Lemma 3.1 is given as follows, which also is easy to prove:

Lemma 3.3 Let \( c_m, m = 1, \ldots, p, \) be complex numbers, \( p \in \mathbb{N} \), and
\[ S_p := \sum_{m=1}^{p} c_m x^m + o(x^p) \quad (x \to 0). \]
Then
\[ e^{S_p} = 1 + \sum_{n=1}^{p} \gamma_n x^n + o(x^p) \quad (x \to 0), \] (3.6)
where \( \gamma_n \) is defined by (3.1).

Theorem 3.4 Assume (2.17). Then, for all \( t \in \mathbb{R} \),
\[ |\hat{\mu}(t/x)|^{2x} = 1 + \sum_{n=1}^{2p_n-1} A_n(t) \left( \frac{1}{x} \right)^n + o \left( \frac{1}{x^{2p_n-1}} \right) \quad (x \to \infty). \] (3.7)

Proof. Similar to the proof of Theorem 3.2.

4 Applications to QZE

To apply the results in Sections 2 and 3 to QZE, for each \( k \in \mathbb{N} \) and a unit vector \( \Psi \in D(|H|^{k/2}) \), we introduce
\[ \langle H^k \rangle := \int_{\mathbb{R}} \lambda^k d\|E_H(\lambda)\Psi\|^2, \] (4.1)
the $k$-th expectation value of the Hamiltonian $H$ in the state $\Psi$, and, for each $n \in \mathbb{N}$ and a unit vector $\Psi \in D(|H|^{n/2})$, we define

$$b_n(\Psi) := \sum_{r=1}^{n} \frac{(-1)^{r-1}}{r} \sum_{k_1, \ldots, k_r \geq 1, k_1 + \cdots + k_r = n} \frac{\langle H^{k_1} \rangle \cdots \langle H^{k_r} \rangle}{k_1! \cdots k_r!}. \quad (4.2)$$

**Theorem 4.1** Let $t \in \mathbb{R}$ be fixed. Suppose that, for some $c > 0$, $\Psi \in D(e^{c|H|})$ with $\|\Psi\| = 1$ and that $N$ obeys the following condition:

$$\int_{\mathbb{R}} e^{\|\lambda\|/N} d\|E_H(\lambda)\| \Psi \|^2 < 2. \quad (4.3)$$

Then

$$\log P_N(\Psi, t) = 2 \sum_{k=1}^{\infty} (-1)^k b_{2k}(\Psi) t^{2k} \left( \frac{1}{N} \right)^{2k-1}, \quad (4.4)$$

converging absolutely.

**Proof.** Let $\mu_{\Psi}$ be given by (1.6). Then we need only to show that $\mu = \mu_{\Psi}$ satisfies the assumption of Theorem 2.1. The assumption $\Psi \in D(e^{c|H|})$ is equivalent to that

$$\int_{\mathbb{R}} e^{2c|\lambda|} d\mu_{\Psi}(\lambda) < \infty.$$ 

Hence (2.3) holds with $\mu = \mu_{\Psi}$. In the present case, we have $M_k = \langle H^k \rangle$. Thus (2.5) gives (4.4).

In the case where $\Psi$ is not necessarily in $D(e^{c|H|})$, we have the following result:

**Theorem 4.2** Let $n \in \mathbb{N}$ and suppose that $\Psi \in D(|H|^n)$ with $\|\Psi\| = 1$. Then, for all $t \in \mathbb{R}$,

$$\log P_N(\Psi, t) = 2 \sum_{k=1}^{n} (-1)^k b_{2k}(\Psi) t^{2k} \left( \frac{1}{N} \right)^{2k-1} + o(1/N^{2p_n-1}) \quad (N \to \infty). \quad (4.5)$$

**Proof.** A simple application of Theorem 2.5.

Finally we derive asymptotic formulae of $P_N(\Psi, t)$ itself. For this purpose, we define a sequence $\{\beta_n(\Psi, t)\}_{n=1}^{\infty}$ ($t \in \mathbb{R}$) as follows:

$$\beta_{n-1}(\Psi, t) := 2(-1)^n b_{2n}(\Psi) t^{2n}, \quad (4.6)$$

$$\beta_{2n}(\Psi, t) := 0. \quad (4.7)$$

**Theorem 4.3** Suppose that the same assumption as in Theorem 4.1 holds. Let

$$\gamma_n(\Psi, t) := \sum_{k=1}^{n} \frac{1}{k!} \sum_{m_1 + \cdots + m_k = n, m_1, \ldots, m_k \geq 1} \beta_{m_1}(\Psi, t) \cdots \beta_{m_k}(\Psi, t), \quad n \in \mathbb{N}. \quad (4.8)$$

Then

$$P_N(\Psi, t) = 1 + \sum_{n=1}^{\infty} \gamma_n(\Psi, t) \left( \frac{1}{N} \right)^n, \quad (4.9)$$

converging absolutely.
Proof. A simple application of Theorem 3.2. \hfill\Box

**Theorem 4.4** Let $n \in \mathbb{N}$ and suppose that $\Psi \in D(|H|^n)$ with $\|\Psi\| = 1$. Then, for all $t \in \mathbb{R}$,

$$P_N(\Psi, t) = 1 + \sum_{n=1}^{2^{p_n}-1} \gamma_n(\Psi, t) \left( \frac{1}{N} \right)^n + o \left( \frac{1}{N^{2p_n-1}} \right) \quad (N \to \infty). \quad (4.10)$$

Proof. This follows from an application of Theorem 3.4. \hfill\Box

**Example 4.5** By direct computations, we have

$$\gamma_1(\Psi, t) = - (\Delta H) \psi t^2,$$

which coincides with $c_1(\Psi, t)$ is given by (1.4), and

$$\gamma_2(\Psi, t) = \frac{1}{2} (\Delta H)^4 \psi t^4.$$

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