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# An area minimizing scheme for anisotropic mean curvature flow

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**Abstract.** We consider an area minimizing scheme for anisotropic mean curvature flow originally due to Chambolle (2004). We show the convergence of the scheme to anisotropic mean curvature flow in the sense of Hausdorff distance by the level set method provided that no fattening occurs.

## 1 Introduction

In this paper we study an approximation scheme to the interface moving by anisotropic mean curvature. It is an extension of the scheme proposed by Chambolle [17] and is related to the area minimizing scheme introduced by Almgren - Taylor - Wang [1].

A family  $\{\Gamma(t)\}_{t \geq 0}$  of interfaces is called a motion by anisotropic mean curvature or an anisotropic mean curvature flow (AMCF for short) provided that  $\Gamma(t)$  evolves by

$$(1.1) \quad V = -\gamma(\mathbf{n}) \operatorname{div}_{\Gamma(t)} \xi(\mathbf{n}) \quad \text{in } \Gamma(t), \quad t > 0,$$

where  $\mathbf{n}$  is the Euclidean outer unit normal vector field of  $\Gamma(t)$ ,  $V$  is the normal velocity in the direction of  $\mathbf{n}$ ,  $\operatorname{div}_{\Gamma(t)}$  denotes the surface divergence on  $\Gamma(t)$ ,  $\gamma = \gamma(p)$  is the *surface energy density* and  $\xi = \nabla \gamma := (\gamma_{p_1}, \dots, \gamma_{p_N})$  is called the *Cahn-Hoffman vector*. In particular, if  $\gamma(p) = |p|$ , then (1.1) is the usual mean curvature flow equation:

$$(1.2) \quad V = -\operatorname{div}_{\Gamma(t)} \mathbf{n} \quad \text{on } \Gamma(t), \quad t > 0.$$

These motions arise in geometry, the interface dynamics and the image processing etc.

The main mathematical characteristic of such evolutions as above is the development of singularities in finite time even if the initial interface is sufficiently smooth. Many works have been done for years to interpret the evolution past the singularities. A rather general approach to provide a weak formulation for the motion past the singularities, known as the level set approach, was introduced for numerical computations by Osher - Sethian [37] and was rigorously developed by Evans - Spruck [26] for (1.2) and independently by

Chen - Giga - Goto [20] for more general evolutions including (1.1) and (1.2). See also Barles - Soner - Souganidis [7], Soner [40], Ishii - Souganidis [33], Ambrosio - Soner [4] and Barles - Souganidis [9] for further developments. Giga [27] provides a self-contained introduction to the level set approach for various surface evolution equations.

The outcome of the aforementioned works has been the development of a weak notion of evolving interfaces called generalized motion. The generalized motion  $\{\Gamma(t)\}_{t \geq 0}$  by (1.1) or (1.2) is defined globally in time, although it may become the empty set in finite time, develop singularities, change topological types and so on. In spite of these peculiarities, the generalized motion  $\{\Gamma(t)\}_{t \geq 0}$  has been proven to be the right way to extend the classical motion.

With relation to the applications mentioned above, many people studied various algorithm to approximate the motion by (1.1) or (1.2). Especially, we focus on a variational approximation. In [1] Almgren, Taylor and Wang introduced the following area minimizing scheme: Let  $\mathcal{K}$  be the family of all bounded, Lebesgue measurable subsets of  $\mathbb{R}^N$  with finite perimeter. Given an initial set  $K_0 \in \mathcal{K}$  and a time step  $h > 0$ , they defined a new set  $\mathcal{J}_h(K_0)$  as a minimizer of the functional defined by

$$\mathcal{E}_h(L, K_0) := \Phi(\partial L) + \frac{1}{h} \int_{L \Delta K_0} \text{dist}(x, \partial K_0) dx \quad \text{for } L \in \mathcal{K}.$$

Here  $\Phi(\partial L)$  is the perimeter of  $L$ ,  $L \Delta K_0 := (L \cup K_0) \setminus (L \cap K_0)$  and  $\text{dist}(x, \partial K_0)$  denotes the Euclidean distance function to  $\partial K_0$ . They set

$$(1.3) \quad K^h(t) := \mathcal{J}_h^{\lfloor t/h \rfloor}(K_0) \quad \text{for } t \geq 0.$$

Here  $\lfloor \alpha \rfloor$  denotes the integer part of  $\alpha \in \mathbb{R}$ . In this way, they are able to construct an approximate flow  $\{K^h(t)\}_{t \geq 0}$  and proved the convergence of this flow to a smooth “flat  $\Phi$  curvature flow” (see [1] for the details). However, we should note that  $\{K^h(t)\}_{t \geq 0}$  cannot be uniquely determined because the main drawback of their approach is the lack of the uniqueness of minimizers of  $\mathcal{E}_h(\cdot, K_0)$ .

In order to resolve this drawback, Chambolle [17] proposed another scheme  $T_h$  to compute the interface moving by (1.2) and proved that  $T_h$  provides a monotonous selection of the discrete scheme by [1]. His algorithm is stated as follows: Let  $E_0 \subset \mathbb{R}^N$  be compact and fix a time step  $h > 0$ . Choose a bounded domain  $\Omega \subset \mathbb{R}^N$  so that  $E_0 \subset \Omega$  and define a function  $w_{E_0}^h \in L^2(\Omega) \cap BV(\Omega)$  as a unique minimizer of the following functional:

$$(1.4) \quad J_h(v) := \begin{cases} \int_{\Omega} |Dv| + \frac{1}{2h} \|v - d_{E_0}\|_{L^2(\Omega)}^2 & \text{if } v \in L^2(\Omega) \cap BV(\Omega), \\ +\infty & \text{if } v \in L^2(\Omega) \setminus BV(\Omega). \end{cases}$$

Here  $\int_{\Omega} |Dv|$  is the total variation of  $v$ ,  $Dv$  is the gradient of  $v$  in the sense of distribution and  $d_{E_0}$  denotes the Euclidean signed distance function to  $\partial E_0$ , namely,

$$d_{E_0}(x) := \text{dist}(x, E_0) - \text{dist}(x, \mathbb{R}^N \setminus E_0).$$

An important advantage over  $\mathcal{E}_h(\cdot, K_0)$  is that  $J_h(v)$  is strictly convex so that the minimizer is unique. Set

$$(1.5) \quad T_h(E_0) := \{w_{E_0}^h \leq 0\}.$$

Here and in the sequel we use the notations  $\{f \geq \mu\} := \{x \in \mathbb{R}^N \mid f(x) \geq \mu\}$ ,  $\{f \leq \mu\} := \{x \in \mathbb{R}^N \mid f(x) \leq \mu\}$  etc. We note that once  $T_h(E_0) \subset\subset \Omega$  the definition of  $T_h(E_0)$  does not depend on the choice of  $\Omega$  including  $E_0$  and that it is compact. Hence, by contrast with  $\mathcal{T}_h$  in (1.3)  $T_h$  surely defines a map from  $\mathcal{C}_0(\mathbb{R}^N)$  into itself, where  $\mathcal{C}_0(\mathbb{R}^N)$  stands for the family of all compact subsets of  $\mathbb{R}^N$ . In addition, Chambolle showed in [17] that the discrete evolution  $E^h(t) := T_h^{\lfloor t/h \rfloor}(E_0)$  converges to the continuous one  $E(t)$  by (1.2) starting from  $E_0$  in the  $L^1$ -topology, whenever no fattening occurs: Let  $u = u(t, x)$  be a unique viscosity solution of the level set equation for (1.2):

$$(1.6) \quad \begin{cases} u_t - |\nabla u| \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right) = 0 & \text{in } (0, T) \times \mathbb{R}^N, \\ u(0, x) = \chi_{E_0}(x) - \chi_{\mathbb{R}^N \setminus E_0}(x) & \text{for } x \in \mathbb{R}^N. \end{cases}$$

Then  $\chi_{\tilde{E}^h} \rightarrow \chi_{\tilde{E}}$  in  $L^1((0, T) \times \Omega)$  as  $h \rightarrow 0$ . Here  $\tilde{E}^h := \cup_{t \geq 0}(\{t\} \times E^h(t))$ ,  $\tilde{E} := \cup_{t \geq 0}(\{t\} \times E(t))$ ,  $E(t) := \{u(t, \cdot) \geq 0\}$  and  $\chi_A$  denotes the characteristic function for  $A \subset \mathbb{R}^N$ . We notice that the above convergence does not necessarily derive that of  $\partial E^h(t)$  since both of  $\partial E^h(t)$  and  $\partial E(t)$  are null sets in the sense of the  $N$ -dimensional Lebesgue measure for all  $t \in [0, T)$ .

Hence it is a natural question whether  $E^h(t)$  converges to  $E(t)$  as  $h \rightarrow 0$  in a stronger topology than the above one. For this direction Eto [22] has provided a new scheme, combining Chambolle's one and the mathematical morphology in image processing developed by Matheron [35] and Serra [39] (see also Cao [15]): Given  $u_0 \in UC(\mathbb{R}^N)$  and  $h > 0$ , we define a new function  $S_h u_0$  by

$$[S_h u_0](x) := \sup\{\mu \in \mathbb{R} \mid x \in T_h(\{u_0 \geq \mu\})\}.$$

where  $T_h$  is given by (1.5). The way to construct  $S_h$  by  $T_h$  is often appeared in the Bence - Merriman - Osher algorithm [13]. See Evans [24], Ishii [30], Ishii - Pires - Souganidis [32] and Ishii - Ishii [31] etc. Setting

$$u^h(t, x) := [S_h^{\lfloor t/h \rfloor} u_0](x) \quad \text{for } (t, x) \in [0, T) \times \mathbb{R}^N,$$

one is able to expect that  $u^h$  converges to a unique viscosity solution  $u$  of (1.6) with the initial data  $u(0, x) = u_0(x)$  and that the set  $\{u^h(t, \cdot) \geq 0\}$  for the above scheme converges to the set  $\{u(t, \cdot) \geq 0\}$  in the sense of Hausdorff distance, whenever no fattening occurs. Indeed, Eto has essentially already obtained in [22] such results in the case where  $N = 2$  and  $\gamma$  is isotropic although a complete proof is not given.

The main purpose of this paper is to extend the result of [22] to the case where  $N \geq 2$  and  $\gamma$  is anisotropic. Moreover, as discussed in [22], we extend the scheme so that the convergence result is still valid for the case where  $E_0$  is unbounded.

To define an anisotropic version of Chambolle's scheme, we utilize the elliptic differential inclusion which is the Euler - Lagrange equation for such a variational problem as (1.4). This idea is essentially given by Caselles - Chambolle [16, Proposition 3.1] at least for bounded convex sets. Since we would like to include the case of unbounded sets, we use the elliptic inclusion rather than variational problems. Since the set  $\{u \geq \mu\}$  may not be bounded, we need to extend the domain of  $T_{\gamma^\circ, h}$ . For this purpose it is convenient to

use the elliptic differential inclusion rather than such a problem as (1.4). In addition, we need to modify the results in subsection 4.1. Consequently,  $T_{\gamma^\circ, h}$  maps  $\mathcal{C}(\mathbb{R}^N)$  into itself and fulfills monotonicity, continuity, translation invariance and rotation property. Here  $\mathcal{C}(\mathbb{R}^N)$  is the family of all closed subsets of  $\mathbb{R}^N$ . By the theory due to [35] and [39] we will see that  $S_{\gamma^\circ, h}$  also has such properties as those of  $T_{\gamma^\circ, h}$ .

Finally we define a time discrete function  $u^h : [0, T) \times \mathbb{R}^N \rightarrow \mathbb{R}$  by

$$u^h(t, x) := [S_{\gamma^\circ, h}^{\lfloor t/h \rfloor} u_0](x).$$

As observed later, the set  $\{u^h(t, \cdot) \geq \mu\}$  coincides with  $T_{\gamma^\circ, h}^{\lfloor t/h \rfloor}(\{u_0 \geq \mu\})$ , which is supposed to have some relation with AMCF starting with  $\{u_0 \geq \mu\}$ . Thereby, we could expect that  $u^h$  will approximate a unique viscosity solution  $u = u(t, x)$  of the level set equation for (1.1):

$$(1.7) \quad \begin{cases} u_t - \gamma(\nabla u) \operatorname{div} \nabla \gamma(\nabla u) = 0 & \text{in } (0, T) \times \mathbb{R}^N, \\ u(0, x) = u_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases}$$

where  $\nabla u := (u_{x_1}, \dots, u_{x_N})$ . See Giga [27, Chapter 1] for the derivation of (1.7) from (1.1). To derive the convergence of  $u^h$  to  $u$ , we need to estimate  $S_{\gamma^\circ, h}$  (cf. Theorem 5.1 and 5.2 in section 5 below). For this purpose, we use the characterization of  $S_{\gamma^\circ, h}$  in Proposition 5.1. This characterization is important to our analysis for  $S_{\gamma^\circ, h}$  because it says that for  $u \in C(\mathbb{R}^N)$  and  $x \in \mathbb{R}^N$   $[S_{\gamma^\circ, h} u](x)$  is attained in a small ball with radius  $O(\sqrt{h})$ . Thus combining Taylor's theorem with this fact, we obtain the estimates of Theorems 5.1 and 5.2. Once we have such estimates for  $S_{\gamma^\circ, h}$ , we are able to show the convergence of  $u^h$  to  $u$ , essentially similar to those of monotone schemes due to Barles - Souganidis [8].

This paper is organized as follows. In section 2 we state the assumptions and some properties on  $\gamma$ . Also we prepare some definitions and results used in this paper. In section 3 we recall an anisotropic version of Chambolle's scheme for (1.1). Besides, we explain the relation between the level set of our scheme and the minimizers of the scheme due to [1]. Section 4 provides some results on the mathematical morphology and the definitions and properties of set operators and function operators. We will see in subsection 4.3 that a function operator  $S_{\gamma^\circ, h}$  is a morphological operator. Section 5 is devoted to the consistency of the scheme. The points of the proof of the consistency are the fact that the value of  $S_{\gamma^\circ, h} u$  is attained in a closed ball with radius  $O(\sqrt{h})$ , as mentioned before this paragraph, and a local approximation of a weak solution of (3.1) in section 3. In section 6 we state our main results. In section 7 we mention how to apply the results in sections 2 - 6 to the AMCF with a mobility different from  $\gamma$ . In section 8 we show the existence, uniqueness and stability of weak solutions of (2.6) in section 2 and a convergence property of the signed distance functions.

Recently, we learned that in [18] and [19] Chambolle and Novaga considered some approximation schemes to (1.1), the anisotropic/crystalline versions of the algorithms [13] and [1]. In their papers they proved that the discrete flow by their schemes converges to a regular flow of compact sets (cf. [18, Definition 2.1] and [19, Definition 2.1]) in the sense of Hausdorff distance for each  $t > 0$ . They used a very similar method to Goto - Ishii - Ogawa [29] and Ishii [34] for their versions of [13] and some variational techniques for those of [1]. We should note that our results are new even in the isotropic case since

$E_0$  is allowed to be unbounded and the convergence is locally uniformly with respect to the  $t$ -variable. Our methods are different from theirs.

The results in this paper have been announced in [23].

## 2 Preliminaries

### 2.1 Assumptions and some properties on $\gamma$

As for the surface energy density  $\gamma$ , we assume that

$$(\gamma 1) \quad \gamma \in C^2(\mathbb{R}^N \setminus \{0\}),$$

$$(\gamma 2) \quad \gamma(ap) = |a|\gamma(p) \text{ for all } p \in \mathbb{R}^N \text{ and } a \in \mathbb{R},$$

$$(\gamma 3) \quad \text{there are constants } 0 < \lambda \leq \Lambda \text{ satisfying } \lambda|p| \leq \gamma(p) \leq \Lambda|p| \text{ for all } p \in \mathbb{R}^N,$$

$$(\gamma 4) \quad \gamma \text{ is convex,}$$

$$(\gamma 5) \quad \nabla^2 \gamma^2(p) > O \text{ for all } p \in \mathbb{R}^N \setminus \{0\}.$$

Note that  $(\gamma 2)$  implies  $\gamma(0) = 0$  and  $\gamma(-p) = \gamma(p)$  for all  $p \in \mathbb{R}^N$ .

We recall some properties on  $\gamma$ . Let  $\partial\gamma(p)$  be the subdifferential of  $\gamma$  at  $p \in \mathbb{R}^N$ :

$$\partial\gamma(p) := \{\xi \in \mathbb{R}^N \mid \langle \xi, q - p \rangle \leq \gamma(q) - \gamma(p) \text{ for all } q \in \mathbb{R}^N\}.$$

If  $\gamma$  is differentiable at  $p$  so that  $\partial\gamma(p)$  is a singleton, then we simply write  $\nabla\gamma(p)$  in place of  $\partial\gamma(p)$ . Set  $B(x, r) := \{y \in \mathbb{R}^N \mid |y - x| < r\}$  for  $x \in \mathbb{R}^N$  and  $r > 0$ .

**Lemma 2.1.** *Assume  $(\gamma 2)$  -  $(\gamma 4)$ . Then  $\partial\gamma(p) \subset \partial\gamma(0) \subset \overline{B(0, \Lambda)}$  for all  $p \in \mathbb{R}^N$ .*

**Proof.** It follows from  $(\gamma 2)$  and  $(\gamma 4)$  that  $\gamma(p+q) \leq \gamma(p) + \gamma(q)$  for all  $p, q \in \mathbb{R}^N$ . Hence

$$(2.1) \quad |\gamma(p) - \gamma(q)| \leq \gamma(p - q) \quad \text{for all } p, q \in \mathbb{R}^N.$$

It is easy to see that for any  $p \in \mathbb{R}^N$  and  $\xi \in \partial\gamma(p)$

$$\langle \xi, q - p \rangle \leq \gamma(q) - \gamma(p) \leq \gamma(q - p) \quad \text{for all } q \in \mathbb{R}^N.$$

Replacing  $q - p$  with  $q$ , we get from  $\gamma(0) = 0$

$$(2.2) \quad \langle \xi, q \rangle \leq \gamma(q) - \gamma(0) \quad \text{for all } q \in \mathbb{R}^N.$$

Thus  $\partial\gamma(p) \subset \partial\gamma(0)$  for any  $p \in \mathbb{R}^N$ .

Setting  $q = \xi$  in (2.2), we have  $|\xi|^2 \leq \gamma(\xi) \leq \Lambda|\xi|$  for all  $\xi \in \partial\gamma(0)$  by  $(\gamma 3)$ . Therefore,  $\partial\gamma(0) \subset \overline{B(0, \Lambda)}$ .  $\square$

**Lemma 2.2.** ([27, Remark 1.7.5]) *Assume  $(\gamma 1)$  -  $(\gamma 5)$ . Then  $\text{Ker}(\nabla^2 \gamma(p)) = \{ap \mid a \in \mathbb{R}\}$  for all  $p \in \mathbb{R}^N \setminus \{0\}$ .*

In the following part of this paper we always assume that  $\gamma$  satisfies  $(\gamma 1)$  -  $(\gamma 5)$ .

## 2.2 Anisotropic signed distance function and anisotropic mean curvature

We define the support function  $\gamma^\circ$  of the convex set  $\{\gamma \leq 1\}$  by

$$\gamma^\circ(p) := \sup_{\gamma(q) \leq 1} \langle p, q \rangle.$$

It is verified from  $(\gamma 1)$  -  $(\gamma 5)$  that

$$(2.3) \quad \Lambda^{-1}|p| \leq \gamma^\circ(p) \leq \lambda^{-1}|p| \quad \text{for all } p \in \mathbb{R}^N$$

and that  $\gamma^\circ$  satisfies  $(\gamma 1)$ ,  $(\gamma 2)$ ,  $(\gamma 4)$  and  $(\gamma 5)$  (cf. [38, Section 2.5] and [28, Section 2.2]). See [11] for some important properties of  $\gamma$  and  $\gamma^\circ$ .

For any  $E \subset \mathbb{R}^N$  let  $d_{\gamma^\circ, E}$  be the anisotropic signed distance function to  $\partial E$ :

$$(2.4) \quad d_{\gamma^\circ, E}(x) := \inf_{y \in E} \gamma^\circ(x - y) - \inf_{y \in \mathbb{R}^N \setminus E} \gamma^\circ(x - y) \quad \text{for } x \in \mathbb{R}^N.$$

Then  $d_{\gamma^\circ, E}$  satisfies

$$|d_{\gamma^\circ, E}(x) - d_{\gamma^\circ, E}(y)| \leq \gamma^\circ(x - y) \quad \text{for all } x, y \in \mathbb{R}^N.$$

Hence  $d_{\gamma^\circ, E}$  is Lipschitz continuous in  $\mathbb{R}^N$ . In addition, we have the following lemma.

**Lemma 2.3.** ([12, Theorem 3.2])  $\gamma(\nabla d_{\gamma^\circ, E}(x)) = 1$  for a.e.  $x \in \mathbb{R}^N$ .

This yields that  $\nabla d_{\gamma^\circ, E}(x) = \mathbf{n}_E(x)/\gamma(\mathbf{n}_E(x))$  for each  $x \in \partial E$  where  $\nabla d_{\gamma^\circ, E}(x)$  exists. Here  $\mathbf{n}_E(x)$  denotes the Euclidean outer unit normal to  $\partial E$ .

Assume that  $\partial E$  is smooth. We define the anisotropic outer normal  $\mathbf{n}_{\gamma^\circ, E}(x)$  to  $\partial E$  by

$$\mathbf{n}_{\gamma^\circ, E}(x) := \nabla \gamma(\nabla d_{\gamma^\circ, E}(x)) \quad \text{for } x \in \partial E.$$

Then  $\gamma^\circ(\mathbf{n}_{\gamma^\circ, E}) = 1$  on  $\partial E$ . Following [10, Section 3 and 4], one is able to show several properties of the anisotropic signed distance function  $d_{\gamma^\circ, E}$  and the anisotropic normal vector  $\mathbf{n}_{\gamma^\circ, E}$ .

**Proposition 2.1.** *Let  $E \subset \mathbb{R}^N$  be an open set with the smooth boundary  $\partial E$  and let  $d_{\gamma^\circ, E}$  be defined by (2.4). Then there is a neighborhood  $V$  including  $\partial E$  such that*

- (1)  $d_{\gamma^\circ, E} \in C^2(V)$ ,
- (2)  $d_{\gamma^\circ, E}(x + r\mathbf{n}_{\gamma^\circ, E}(x)) = r$  for all  $x \in \partial E$  and  $r \in \mathbb{R}$  satisfying  $x + r\mathbf{n}_{\gamma^\circ, E}(x) \in V$ ,
- (3)  $\nabla d_{\gamma^\circ, E}(x + r\mathbf{n}_{\gamma^\circ, E}(x)) = \nabla d_{\gamma^\circ, E}(x)$  and  $\langle \nabla d_{\gamma^\circ, E}(x + r\mathbf{n}_{\gamma^\circ, E}(x)), \mathbf{n}_{\gamma^\circ, E}(x) \rangle = 1$  for all  $x \in \partial E$  and  $r \in \mathbb{R}$  satisfying  $x + r\mathbf{n}_{\gamma^\circ, E}(x) \in V$ ,
- (4)  $\nabla^2 d_{\gamma^\circ, E} \mathbf{n}_{\gamma^\circ, E} = 0$  in  $V$ ,
- (5)  $\operatorname{div}_{\partial E} \mathbf{n}_{\gamma^\circ, E} = \operatorname{div} \mathbf{n}_{\gamma^\circ, E}$  on  $\partial E$ .

We now define the anisotropic mean curvature.

**Definition 2.1.** Let  $E$  be an open set in  $\mathbb{R}^N$  with the smooth boundary  $\partial E$ . Then the anisotropic mean curvature  $\kappa_{\gamma^\circ, E}(x)$  of  $\partial E$  is defined by

$$\kappa_{\gamma^\circ, E}(x) := -\operatorname{div}_{\partial E} \mathbf{n}_{\gamma^\circ, E}(x) (= -\operatorname{div} \mathbf{n}_{\gamma^\circ, E}(x)) \quad \text{for } x \in \partial E.$$

We give an example to the anisotropic outer normal and the anisotropic mean curvature. Let  $B_{\gamma^\circ}(x, r)$  be the open ball of radius  $r > 0$  with respect to the norm  $\gamma^\circ$  centered at  $x \in \mathbb{R}^N$ , i.e.,  $B_{\gamma^\circ}(x, r) := \{y \in \mathbb{R}^N \mid \gamma^\circ(y - x) < r\}$  for  $x \in \mathbb{R}^N$ . The (closure of the) ball  $B_{\gamma^\circ}(0, 1)$  is often called the *Wulff shape* of  $\gamma$ .

**Example 2.1.** Set  $E := B_{\gamma^\circ}(0, r)$  for  $r > 0$ . Then  $d_{\gamma^\circ, E}(x) = \gamma^\circ(x) - r$ . Some calculations yield that for all  $x \in \partial E$ ,

$$\mathbf{n}_{\gamma^\circ, E}(x) = \frac{x}{\gamma^\circ(x)} = \frac{x}{r}, \quad \kappa_{\gamma^\circ, E}(x) = -\frac{N-1}{\gamma^\circ(x)} = -\frac{N-1}{r}.$$

## 2.3 Anisotropic total variation

Let  $\Omega \subset \mathbb{R}^N$  be an open set with Lipschitz boundary.

**Definition 2.2.** (1) We say that  $u \in L^1(\Omega)$  is a function of bounded variation if its gradient  $Du$  in the distribution sense is a (vector-valued) Radon measure with finite total variation in  $\Omega$ . We denote by  $BV(\Omega)$  the class of all functions of bounded variation.

(2) We say that  $u \in L^1_{loc}(\Omega)$  is a function of locally bounded variation if  $u \in BV(K)$  for any compact set  $K \subset \Omega$ . We denote by  $BV_{loc}(\Omega)$  the class of all functions of locally bounded variation.

We define the anisotropic total variation of  $u \in BV(\Omega)$  with respect to  $\gamma$  in  $\Omega$  as

$$\int_{\Omega} \gamma(Du) := \sup \left\{ \int_{\Omega} u \operatorname{div} \varphi \, dx \mid \varphi \in C_0^1(\Omega; \mathbb{R}^N), \gamma^\circ(\varphi) \leq 1 \text{ in } \Omega \right\}.$$

Recall that  $\gamma(Du)$  coincides with the nonnegative Radon measure in  $\mathbb{R}^N$  given by

$$(2.5) \quad \gamma(Du) = \gamma(\nabla u(x)) dx + \gamma \left( \frac{D^s u}{|D^s u|} \right) |D^s u|,$$

where  $\nabla u(x) dx$  and  $D^s u$  denote, respectively, the absolutely continuous part of  $Du$  and the singular part of  $Du$  with respect to the  $N$ -dimensional Lebesgue measure (cf. [3, Theorem 5.47]).

## 2.4 An elliptic differential inclusion

For  $g \in L^2_{loc}(\mathbb{R}^N)$  and  $h > 0$  we consider an elliptic differential inclusion:

$$(2.6) \quad w - h \operatorname{div} \partial \gamma(\nabla w) \ni g \quad \text{in } \mathbb{R}^N.$$



Let  $\Omega$  be an open set in  $\mathbb{R}^N$  with Lipschitz boundary. We set  $X(\Omega) := \{z \in L^\infty(\Omega; \mathbb{R}^N) \mid \operatorname{div} z \in L^2(\Omega)\}$ . For  $w \in L^2(\Omega) \cap BV(\Omega)$  and  $z \in X(\Omega)$ , we define a functional on  $C_0^1(\Omega)$  as

$$(2.7) \quad \int_{\Omega} (z, Dw)\psi := - \int_{\Omega} w \psi \operatorname{div} z \, dx - \int_{\Omega} w \langle z, \nabla \psi \rangle \, dx \quad \text{for } \psi \in C_0^1(\Omega).$$

We can extend this functional to a linear one on  $C_0(\Omega)$ . Hence  $(z, Dw)$  is a Radon measure. We recall Green's formula for  $w \in L^2(\Omega) \cap BV(\Omega)$  and  $z \in X(\Omega)$ .

**Theorem 2.1.** (cf. [5, Theorem 1.9]) *Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with Lipschitz boundary. Let  $w \in L^2(\Omega) \cap BV(\Omega)$  and  $z \in X(\Omega)$ . Then there exists  $[z \cdot \mathbf{n}_\Omega] \in L^\infty(\partial\Omega)$  such that  $\|[z \cdot \mathbf{n}_\Omega]\|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty(\Omega)}$  and*

$$\int_{\Omega} w \operatorname{div} z \, dx + \int_{\Omega} (z, Dw) = \int_{\partial\Omega} [z \cdot \mathbf{n}_\Omega] w \, d\mathcal{H}^{N-1},$$

where  $\mathcal{H}^{N-1}$  is the  $(N-1)$ -dimensional Hausdorff measure. In the case  $\Omega = \mathbb{R}^N$  we have

$$\int_{\mathbb{R}^N} w \operatorname{div} z \, dx + \int_{\mathbb{R}^N} (z, Dw) = 0$$

for all  $w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$  and  $z \in X(\mathbb{R}^N)$ .

We give the definition of weak solutions of (2.6).

**Definition 2.3.** *We say that  $w \in L_{loc}^2(\mathbb{R}^N) \cap BV_{loc}(\mathbb{R}^N)$  is a weak solution of (2.6) provided that there exists  $z \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ ,  $\operatorname{div} z \in L_{loc}^2(\mathbb{R}^N)$  such that*

- (1)  $z \in \partial\gamma(\nabla w)$  a.e. in  $\mathbb{R}^N$ ,
- (2)  $(z, Dw) = \gamma(Dw)$  locally as measures in  $\mathbb{R}^N$ ,
- (3)  $w - h \operatorname{div} z = g$  in  $\mathcal{D}'(\mathbb{R}^N)$ .

We also call a pair  $(w, z)$  a weak solution of (2.6) if it satisfies these conditions.

**Proposition 2.2.** ([16, Proposition 3.1]) *Let  $g \in L_{loc}^2(\mathbb{R}^N)$  and  $w \in L_{loc}^2(\mathbb{R}^N) \cap BV_{loc}(\mathbb{R}^N)$ . The following assertions are equivalent.*

- (1)  $w$  is a weak solution of (2.6).
- (2) Fix any compact set  $K \subset \mathbb{R}^N$ . Then  $w$  satisfies, for any  $\phi \in C_0^1(K)$ ,

$$\int_K \gamma(Dw) + \frac{1}{2h} \|w - g\|_{L^2(K)}^2 \leq \int_K \gamma(D(w + \phi)) + \frac{1}{2h} \|w + \phi - g\|_{L^2(K)}^2.$$

- (3) For any  $R > 0$   $w$  is a minimizer of

$$\min \left\{ \int_{B(0,R)} \gamma(Dv) + \frac{1}{2h} \|v - g\|_{L^2(B(0,R))}^2 \mid \begin{array}{l} v \in L^2(B(0,R)) \cap BV(B(0,R)) \\ \text{such that } v = w \text{ on } \partial B(0,R) \end{array} \right\}.$$

(4) For any  $R > 0$   $w$  is a minimizer of

$$\min \left\{ \int_{B(0,R)} \gamma(Dv) + \frac{1}{2h} \|v - g\|_{L^2(B(0,R))}^2 + \int_{\partial B(0,R)} \gamma(\mathbf{n}_{B(0,R)}) |v - w| d\mathcal{H}^{N-1} \mid v \in L^2(B(0,R)) \cap BV(B(0,R)) \right\}.$$

**Theorem 2.2.** For any  $g \in L_{loc}^2(\mathbb{R}^N)$  (2.6) admits a unique weak solution.

**Lemma 2.4.** Let  $g, g_n \in L_{loc}^2(\mathbb{R}^N)$  and let  $w$  and  $w_n$  be weak solutions of (2.6) with  $g, g_n$ , respectively. If  $g_n \rightarrow g$  in  $L_{loc}^2(\mathbb{R}^N)$  as  $n \rightarrow +\infty$ , then  $w_n \rightarrow w$  in  $L_{loc}^2(\mathbb{R}^N)$  as  $n \rightarrow +\infty$ .

The proofs of Theorem 2.2 and Lemma 2.4 will be given in Appendix.

In the case  $g = \gamma^\circ$  we have the explicit solution of (2.6). Set  $R_1 := 2N/\sqrt{N+1}$  and  $R_2 := \sqrt{N+1}$ .

**Lemma 2.5.** ([16, Appendix B]) For any  $h > 0$  let  $v^h$  be defined by

$$v^h(x) := \begin{cases} R_1\sqrt{h} & \text{if } \gamma^\circ(x) < R_2\sqrt{h}, \\ \gamma^\circ(x) + \frac{h(N-1)}{\gamma^\circ(x)} & \text{if } \gamma^\circ(x) \geq R_2\sqrt{h}. \end{cases}$$

Then  $v^h$  is a unique weak solution of (2.6) with  $g = \gamma^\circ$ .

We obtain a regularity for a weak solution of (2.6) with  $g = d_{\gamma^\circ, E}$ .

**Proposition 2.3.** For any  $E(\neq \emptyset, \mathbb{R}^N) \in \mathcal{C}(\mathbb{R}^N)$  and  $h > 0$ , let  $w$  be a weak solution of (2.6) with  $g = d_{\gamma^\circ, E}$  (that is, a weak solution of (3.1) below). Then it is Lipschitz continuous in  $\mathbb{R}^N$  and satisfies  $\gamma(\nabla w) \leq 1$  a.e. in  $\mathbb{R}^N$ .

**Proof.** This is known as [16, Lemma 5.8] when  $E$  is compact. We shall approximate general  $E$  by compact sets. Set  $n_0 := \inf\{n \in \mathbb{N} \mid E \cap \overline{B(0, n)} \neq \emptyset\}$  and  $E_n := E \cap \overline{B(0, n)}$  for  $n \geq n_0$ . Since  $\{E_n\}_{n=n_0}^{+\infty}$  is nondecreasing and  $\cup_{n=n_0}^{+\infty} E_n = E$ , we deduce from Theorem 8.2 in subsection 8.2 that

$$(2.8) \quad d_{\gamma^\circ, E_n} \searrow d_{\gamma^\circ, E} \quad \text{locally uniformly in } \mathbb{R}^N \text{ as } n \rightarrow \infty.$$

For each  $n \geq n_0$  let  $w_n$  be a weak solution of (2.6) with  $g = d_{\gamma^\circ, E_n}$ . Note by (2.8) and Theorem 8.1 in subsection 8.1 that  $\{w_n\}_{n=n_0}^{+\infty}$  is nonincreasing. It follows from (2.8) and Lemma 2.4 that

$$(2.9) \quad w_n \rightarrow w \quad \text{in } L_{loc}^2(\mathbb{R}^N) \quad \text{as } n \rightarrow \infty.$$

Since  $E_n$  is compact for each  $n \geq n_0$ , we have  $\gamma(\nabla w_n) \leq 1$  a.e. in  $\mathbb{R}^N$  by [16, Lemma 5.8]. Thus  $\{w_n\}_{n=n_0}^{+\infty}$  is equi-Lipschitz continuous in  $\mathbb{R}^N$ . Besides, it is easily seen from (2.4) that for fixed  $y_0 \in \mathbb{R}^N \setminus E$ ,

$$d_{\gamma^\circ, E_n}(x) \geq d_{\gamma^\circ, E}(x) \geq - \inf_{y \in \mathbb{R}^N \setminus E} \gamma^\circ(x - y) \geq -\gamma^\circ(x - y_0).$$

We observe from Lemma 2.5 that  $\underline{w} := -v^h(\cdot - y_0)$  is a weak solution of (2.6) with  $g(x) := -\gamma^\circ(\cdot - y_0)$ . Thus we obtain  $\underline{w} \leq w_n$  a.e. in  $\mathbb{R}^N$  for all  $n \geq n_0$  by Theorem 8.1. The nonincreasing property of  $\{w_n\}_{n=n_0}^{+\infty}$  and this inequality imply that  $\{w_n\}_{n=n_0}^{+\infty}$  is locally uniformly bounded in  $\mathbb{R}^N$ .

Hence, Ascoli-Arzelà's theorem allows us to extract a subsequence  $\{w_{n_j}\}_{j=1}^{+\infty}$  from  $\{w_n\}_{n=n_0}^{+\infty}$  which converges to a Lipschitz function  $\tilde{w} : \mathbb{R}^N \rightarrow \mathbb{R}$  locally uniformly in  $\mathbb{R}^N$  as  $j \rightarrow \infty$ . Thus we have  $\tilde{w} = w$  a.e. in  $\mathbb{R}^N$  since  $w_{n_j} \rightarrow w$  a.e. in  $\mathbb{R}^N$  as  $j \rightarrow +\infty$  by (2.9). Therefore, we conclude that  $w$  is Lipschitz continuous in  $\mathbb{R}^N$ . The estimate  $\gamma(\nabla w) \leq 1$  a.e. in  $\mathbb{R}^N$  follows from  $\gamma(\nabla w_n) \leq 1$  a.e. in  $\mathbb{R}^N$  for all  $n \in \mathbb{N}$ .  $\square$

## 2.5 Viscosity solutions for (1.7)

Set  $F(p, X) := -\gamma(p)\text{tr}(\nabla^2\gamma(p)X)$  and denote by  $\mathbb{S}^N$  the set of all real  $N \times N$  symmetric matrices. Then one is able to easily check that

- (1)  $F$  is degenerate elliptic, namely,  $F(p, X) \leq F(p, Y)$  for all  $p \in \mathbb{R}^N \setminus \{0\}$  and  $X, Y \in \mathbb{S}^N$  satisfying  $X \geq Y$ ,
- (2)  $F \in C(\mathbb{R}^N \setminus \{0\} \times \mathbb{S}^N)$ ,
- (3)  $\sup\{|F(p, X)| \mid 0 < |p| \leq R, \|X\| \leq R\} < +\infty$  for each  $R > 0$ ,
- (4)  $F$  is geometric (cf. [20] and [27]), namely,  $F(\lambda p, \lambda X + \sigma p \otimes p) = \lambda F(p, X)$  for all  $\lambda > 0, \sigma \in \mathbb{R}, p \in \mathbb{R}^N \setminus \{0\}$  and  $X \in \mathbb{S}^N$ .

Let  $U$  be a subset of a metric space  $(X, \rho)$  and let  $f$  be a function on  $U$ . The upper (resp., lower) semicontinuous envelope  $f^*$  (resp.,  $f_*$ ) is defined as follows: For each  $x \in \overline{U}$ ,

$$f^*(x) := \limsup_{r \rightarrow 0} \{f(y) \mid y \in U, \rho(y, x) < r\},$$

$$f_*(x) := \liminf_{r \rightarrow 0} \{f(y) \mid y \in U, \rho(y, x) < r\}.$$

Then we observe that  $f^*$  (resp.,  $f_*$ ) is upper (resp., lower) semicontinuous on  $\overline{U}$  and that  $f_* \leq f \leq f^*$  in  $U$ . In addition,  $f = f^* = f_*$  in  $U$  if and only if  $f$  is continuous in  $U$ .

We give the definition of viscosity solutions of (1.7).

**Definition 2.4.** Let  $u : [0, T) \times \mathbb{R}^N \rightarrow \mathbb{R}$ .

- (1) We say that  $u$  is a viscosity subsolution of (1.7) provided that  $u^*(t, x) < +\infty$  for all  $(t, x) \in [0, T) \times \mathbb{R}^N$  and for any  $\phi \in C^\infty((0, T) \times \mathbb{R}^N)$ , if  $u^* - \phi$  takes a local maximum at  $(\hat{t}, \hat{x})$ , then

$$\phi_t(\hat{t}, \hat{x}) + F_*(\nabla\phi(\hat{t}, \hat{x}), \nabla^2\phi(\hat{t}, \hat{x})) \leq 0.$$

- (2) We say that  $u$  is a viscosity supersolution of (1.7) provided that  $u_*(t, x) > -\infty$  for all  $(t, x) \in [0, T) \times \mathbb{R}^N$  and for any  $\phi \in C^\infty((0, T) \times \mathbb{R}^N)$ , if  $u_* - \phi$  takes a local minimum at  $(\hat{t}, \hat{x})$ , then

$$\phi_t(\hat{t}, \hat{x}) + F^*(\nabla\phi(\hat{t}, \hat{x}), \nabla^2\phi(\hat{t}, \hat{x})) \geq 0.$$

- (3) We say that  $u$  is a viscosity solution of (1.7) if  $u$  is a viscosity sub- and super-solution of (1.7).

The following proposition eliminates, at least partially, the difficulty in treating (1.7) in the case  $\nabla\phi(\hat{t}, \hat{x}) = 0$ .

**Proposition 2.4.** ([6, Proposition 2.2], [27, Chapter 2]) *A function  $u : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$  is a viscosity subsolution of (1.7) if and only if the following conditions are satisfied:  $u^*(t, x) < +\infty$  for all  $(t, x) \in [0, T] \times \mathbb{R}^N$  and for any  $\phi \in C^\infty((0, T) \times \mathbb{R}^N)$  if  $u^* - \phi$  takes a local maximum at  $(\hat{t}, \hat{x})$ , then*

$$\begin{aligned} \phi_t(\hat{t}, \hat{x}) + F_*(\nabla\phi(\hat{t}, \hat{x}), \nabla^2\phi(\hat{t}, \hat{x})) &\leq 0 \quad \text{if } \nabla\phi(\hat{t}, \hat{x}) \neq 0, \\ \phi_t(\hat{t}, \hat{x}) &\leq 0 \quad \text{if } \nabla\phi(\hat{t}, \hat{x}) = 0 \text{ and } \nabla^2\phi(\hat{t}, \hat{x}) = O. \end{aligned}$$

Similar assertions are valid for viscosity supersolutions of (1.7).

Denote by  $UC(\mathbb{R}^N)$  (resp.,  $UC([0, T] \times \mathbb{R}^N)$ ) the class of all uniformly continuous functions in  $\mathbb{R}^N$  (resp., in  $[0, T] \times \mathbb{R}^N$ ). We mention the uniqueness and existence of viscosity solutions of (1.7) and the well-definedness of a generalized AMCF, according to [21], [33] and [27].

**Theorem 2.3.** *Let  $u$  and  $v$  be, respectively, a viscosity subsolution and a viscosity supersolution of (1.7). If  $u^*(0, \cdot) \leq v_*(0, \cdot)$  in  $\mathbb{R}^N$  and  $u^*(t, x) \leq C(1 + |x|)$ ,  $v_*(t, x) \geq -C(1 + |x|)$  for all  $(t, x) \in [0, T] \times \mathbb{R}^N$  and some  $C > 0$ , then  $u^* \leq v_*$  in  $[0, T] \times \mathbb{R}^N$ . Moreover, for any  $u_0 \in UC(\mathbb{R}^N)$  there is a unique viscosity solution  $u \in UC([0, T] \times \mathbb{R}^N)$  of (1.7).*

**Theorem 2.4.** *Let  $\{\Gamma(t)\}_{t \geq 0}$  be a generalized AMCF defined by  $\Gamma(t) = \{u(t, \cdot) = 0\}$ . Here  $u$  is a unique viscosity solution of (1.7). Besides, define  $D^+(t) := \{u(t, \cdot) > 0\}$  and  $D^-(t) := \{u(t, \cdot) < 0\}$  for each  $t \in [0, T]$ . Then the triplet  $(\Gamma(t), D^+(t), D^-(t))$  is determined independently of the choice of the initial data  $u_0 \in UC(\mathbb{R}^N)$  in (1.7) satisfying  $\Gamma(0) = \{u_0 = 0\}$ ,  $D^+(0) := \{u_0 > 0\}$  and  $D^-(0) := \{u_0 < 0\}$ .*

### 3 An anisotropic version of Chambolle's scheme

To recall an anisotropic version of Chambolle's scheme, we use the following differential inclusion: For  $E \in \mathcal{C}(\mathbb{R}^N)$

$$(3.1) \quad w - h \operatorname{div} \partial\gamma(\nabla w) \ni d_{\gamma^\circ, E} \quad \text{in } \mathbb{R}^N,$$

where  $d_{\gamma^\circ, E}$  is the anisotropic signed distance function to  $\partial E$  defined by (2.4).

Fix  $E_0 \in \mathcal{C}(\mathbb{R}^N)$ . Let  $w_{\gamma^\circ, E_0}^h$  be a weak solution of (3.1) with  $E = E_0$ . We then define a new set  $T_{\gamma^\circ, h}(E_0)$  by

$$T_{\gamma^\circ, h}(E_0) := \{w_{\gamma^\circ, E_0}^h \leq 0\}.$$

Notice by Proposition 2.3 that  $T_{\gamma^\circ, h}(E_0) \in \mathcal{C}(\mathbb{R}^N)$ . Let  $w_{\gamma^\circ, T_{\gamma^\circ, h}(E_0)}^h$  be a weak solution of (3.1) with  $E = T_{\gamma^\circ, h}(E_0)$ . Again we define a new set  $T_{\gamma^\circ, h}^2(E_0)$  by

$$T_{\gamma^\circ, h}^2(E_0) := \{w_{\gamma^\circ, T_{\gamma^\circ, h}(E_0)}^h \leq 0\}.$$

Repeating this process, we have a sequence  $\{T_{\gamma^\circ, h}^k(E_0)\}_{k=0}^{+\infty}$  of closed subsets of  $\mathbb{R}^N$ . Here  $T_{\gamma^\circ, h}^k(E_0) := T_{\gamma^\circ, h}(T_{\gamma^\circ, h}^{k-1}(E_0))$  for  $k \in \mathbb{N}$  and  $T_{\gamma^\circ, h}^0(E_0) := E_0$ . Set

$$E^h(t) := T_{\gamma^\circ, h}^{\lfloor t/h \rfloor}(E_0) \quad \text{for } t \geq 0.$$

Letting  $h \rightarrow 0$ , we obtain a limit flow  $\{E(t)\}_{t \geq 0}$  and formally observe that  $\partial E(t)$  moves by (1.1).

We give an example to this scheme. Set  $R_h(r) := (r + \sqrt{r^2 - 4(N-1)h})/2$  and  $R(t) := \sqrt{r^2 - 2(N-1)t}$ . The Hausdorff distance  $d_H(A, B)$  for sets  $A, B \subset \mathbb{R}^N$  is defined by

$$(3.2) \quad d_H(A, B) := \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{x \in B} \text{dist}(x, A) \right\}.$$

Recall  $R_1 = 2N/\sqrt{N+1}$ .

**Example 3.1.** ([22, Sections 3 and 5]) Fix  $r > 0$ . It is seen that  $\{\partial B_{\gamma^\circ}(0, R(t))\}_{t \geq 0}$  is a smooth AMCF and shrinks to the origin at  $T_0 := r^2/2(N-1)$ .

We apply the above scheme to  $\{\partial B_{\gamma^\circ}(0, R(t))\}_{t \geq 0}$ . Then we see by Lemma 2.5 that

$$\begin{aligned} T_{\gamma^\circ, h}(\overline{B_{\gamma^\circ}(0, r)}) &= \begin{cases} \overline{B_{\gamma^\circ}(0, R_h(r))} & \text{if } r > R_1\sqrt{h}, \\ \emptyset & \text{otherwise,} \end{cases} \\ T_{\gamma^\circ, h}(\mathbb{R}^N \setminus B_{\gamma^\circ}(0, r)) &= \begin{cases} \mathbb{R}^N \setminus B_{\gamma^\circ}(0, R_h(r)) & \text{if } r > R_1\sqrt{h}, \\ \mathbb{R}^N & \text{otherwise.} \end{cases} \end{aligned}$$

We get

$$\lim_{h \rightarrow 0} \sup_{t \in [0, T_0 - \delta]} d_H(T_{\gamma^\circ, h}^{\lfloor t/h \rfloor}(\overline{B_{\gamma^\circ}(0, r)}), \overline{B_{\gamma^\circ}(0, R(t))}) = 0 \quad \text{for any } \delta \in [0, T_0].$$

We state the relations between the equation (3.1) and some variational problems. The equation (3.1) is the Euler - Lagrange equation for the following variational problem: Let  $E_0 \in \mathcal{C}_0(\mathbb{R}^N)$  and  $\Omega \subset \mathbb{R}^N$  a bounded domain such that  $E_0 \subset \Omega$ . Find a function  $w \in L^2(\Omega) \cap BV(\Omega)$  satisfying

$$(3.3) \quad \begin{aligned} J_h(w) &= \min_{v \in L^2(\Omega)} J_h(v), \\ J_h(v) &:= \begin{cases} \int_{\Omega} \gamma(Dv) + \frac{1}{2h} \|v - d_{\gamma^\circ, E_0}\|_{L^2(\Omega)}^2 & \text{if } v \in L^2(\Omega) \cap BV(\Omega), \\ +\infty & \text{if } v \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \end{aligned}$$

It follows from [16, Lemma 5.7] that for sufficiently large domain  $\Omega \subset \mathbb{R}^N$  satisfying  $E_0 \subset \Omega$ , the function  $w_{\gamma^\circ, E_0}^h|_{\Omega}$  is a unique minimizer of (3.3). In addition, we see by [16, Section 5.2] that the level sets  $\{w_{\gamma^\circ, E_0}^h \leq 0\}$  and  $\{w_{\gamma^\circ, E_0}^h < 0\}$  are minimizers of the following area minimization problem:

$$(3.4) \quad \min_{F \subset \Omega} \int_{\partial F} \gamma(\mathbf{n}_F) d\mathcal{H}^{N-1} + \frac{1}{h} \int_{F \Delta E_0} |d_{\gamma^\circ, E_0}(x)| dx.$$

This problem is suggested by [1] as an implicit time-discretization of (1.1). Moreover, it is shown in [2, Lemma 4] that any minimizer  $F$  of (3.4) satisfies  $\{w_{\gamma^\circ, E_0}^h < 0\} \subset F \subset \{w_{\gamma^\circ, E_0}^h \leq 0\}$ . Combining the property that  $T_{\gamma^\circ, h}(E) \subset T_{\gamma^\circ, h}(E')$  if  $E \subset E'$  with this fact, we say that Chambolle's scheme and the anisotropic version provide a monotonous selection of the scheme by [1] and choose the maximal element among all minimizers of (3.4) for each time step.

## 4 Set operators and Function operators

### 4.1 Some results on mathematical morphology

We briefly review some results of set operators and function operators, according to [15, Chapter 4]. See [35] and [39] for the details of the mathematical morphology.

**Definition 4.1.** (1) *Let  $\mathcal{B}$  be a family of subsets of  $\mathbb{R}^N$ . A map  $T : \mathcal{B} \rightarrow \mathcal{B}$  is called a set operator.*

(2) *Let  $\mathcal{F}$  be a class of functions defined in  $\mathbb{R}^N$ . A map  $S$  on  $\mathcal{F}$  is called a function operator provided that  $S$  maps  $u \in \mathcal{F}$  to a function  $Su \in \mathcal{F}$ .*

We assume that a set operator  $T$  satisfies the following properties: Let  $E, E', E_n \in \mathcal{B}$  ( $n \in \mathbb{N}$ ),  $x \in \mathbb{R}^N$  and  $U \in O(N)$ . Here we denote by  $O(N)$  the set of all  $N \times N$ -real orthogonal matrices and a matrix  $U \in O(N)$  is identified with an orthogonal transformation in  $\mathbb{R}^N$ .

(M) Monotonicity:  $T(E) \subset T(E')$  if  $E \subset E'$ .

(C) Continuity:  $T(E_n) \searrow T(E)$  as  $n \rightarrow +\infty$  if  $E_n \searrow E$  as  $n \rightarrow +\infty$ .

(T) Translation invariance:  $T(x + E) = x + T(E)$ .

(R) Rotation property:  $T(UE) = UT(E)$ .

We set  $T(\emptyset) := \emptyset$  and  $T(\mathbb{R}^N) := \mathbb{R}^N$ . Here and in the sequel, the convergence  $E_n \searrow E$  as  $n \rightarrow +\infty$  means that  $\{E_n\}_{n=1}^{+\infty}$  is nonincreasing and  $\bigcap_{n=1}^{+\infty} E_n = E$ .

To define a function operator from a set operator, we recall some relations between functions and their level sets. Let  $\{X_\mu\}_{\mu \in \mathbb{R}}$  be a family of subsets of  $\mathbb{R}^N$  satisfying the following conditions:

$$(4.1) \quad \bigcap_{\mu \in \mathbb{R}} X_\mu = \emptyset \text{ and } \bigcup_{\mu \in \mathbb{R}} X_\mu = \mathbb{R}^N,$$

$$(4.2) \quad X_\nu \subset X_\mu \text{ if } \nu > \mu,$$

$$(4.3) \quad \bigcap_{\nu < \mu} X_\nu = X_\mu \text{ for all } \mu \in \mathbb{R}.$$

**Lemma 4.1.** *For any  $u : \mathbb{R}^N \rightarrow \mathbb{R}$ , the family  $\{\{u \geq \mu\}\}_{\mu \in \mathbb{R}}$  satisfies (4.1) - (4.3).*

We omit the proof since it is elementary.

One is able to show a converse to Lemma 4.1.

**Lemma 4.2.** *Assume that  $\{X_\mu\}_{\mu \in \mathbb{R}}$  satisfies (4.1) - (4.3). Let  $u : \mathbb{R}^N \rightarrow \mathbb{R}$  be defined by*

$$u(x) := \sup\{\mu \in \mathbb{R} \mid x \in X_\mu\} \quad \text{for } x \in \mathbb{R}^N.$$

*Then  $\{u \geq \mu\} = X_\mu$  for all  $\mu \in \mathbb{R}$ .*

**Proof.** Fix any  $\mu \in \mathbb{R}$ . The inclusion  $X_\mu \subset \{u \geq \mu\}$  is obvious. To show  $\{u \geq \mu\} \subset X_\mu$ , we assume that  $x \notin X_\mu$ . Then, by (4.3) we choose  $\nu < \mu$  so that  $x \notin X_\nu$ . As  $\mu - \nu > 0$ , we can find  $\nu' \in \mathbb{R}$  such that  $u(x) - (\mu - \nu) < \nu' < u(x)$ . Then  $x \in X_{\nu'}$  by (4.2). Moreover, we notice by  $x \notin X_\nu$  that  $\nu' \leq \nu$ . Hence we have  $u(x) < \mu - \nu + \nu' \leq \mu$ , which implies that  $x \notin \{u \geq \mu\}$ . Therefore, we get  $\{u \geq \mu\} \subset X_\mu$ .  $\square$

For any function  $u : \mathbb{R}^N \rightarrow \mathbb{R}$  we define

$$[Su](x) := \sup\{\mu \in \mathbb{R} \mid x \in \{u \geq \mu\}\} \quad \text{for } x \in \mathbb{R}^N.$$

This  $S$  is an identical operator by Lemma 4.2. However, this formula suggests a way to define a function operator in terms of a set operator. Actually, we define a function operator  $S$  in the following way:

$$(4.4) \quad [Su](x) := \sup\{\mu \in \mathbb{R} \mid x \in T(\{u \geq \mu\})\} \quad \text{for } x \in \mathbb{R}^N.$$

The next proposition will play an important role for our study.

**Proposition 4.1.** *Let  $T$  be a set operator satisfying (M), (C) and (T). Let  $\mathcal{F}(\mathbb{R}^N)$  be a class of functions on  $\mathbb{R}^N$  such that  $\bigcap_{\mu \in \mathbb{R}} T(\{u \geq \mu\}) = \emptyset$  and  $\bigcup_{\mu \in \mathbb{R}} T(\{u \geq \mu\}) = \mathbb{R}^N$ . Then the function operator  $S$  on  $\mathcal{F}(\mathbb{R}^N)$  defined by (4.4) satisfies the following properties.*

- (1)  $Su \leq Sv$  in  $\mathbb{R}^N$  if  $u \leq v$  in  $\mathbb{R}^N$ .
- (2)  $S(g \circ u) = g \circ Su$  for any nondecreasing function  $g \in C(\mathbb{R}^N)$ .
- (3) For fixed  $y \in \mathbb{R}^N$ ,  $[Su(\cdot + y)](x) = [Su](x + y)$  for all  $x \in \mathbb{R}^N$ .

A operator  $S$  satisfying the above properties is called a morphological operator.

**Proof of Proposition 4.1.** We skip the proofs of (1) and (2) since they are given in the proof of [15, Proposition 4.12].

We confirm the property (3). Fix  $u \in \mathcal{F}(\mathbb{R}^N)$  and  $y \in \mathbb{R}^N$ . Then we compute that for any  $x \in \mathbb{R}^N$ ,

$$\begin{aligned} [Su](x + y) &= \sup\{\mu \in \mathbb{R} \mid x + y \in T(\{u \geq \mu\})\} \\ &= \sup\{\mu \in \mathbb{R} \mid x \in T(\{u \geq \mu\}) - y\} \\ &= \sup\{\mu \in \mathbb{R} \mid x \in T(\{u \geq \mu\} - y)\} \\ &= \sup\{\mu \in \mathbb{R} \mid x \in T(\{u(\cdot + y) \geq \mu\})\} \\ &= [Su(\cdot + y)](x). \end{aligned}$$

Here we have used (T) to derive the third equality. Hence the proof is completed.  $\square$

Denote by  $C_K(\mathbb{R}^N)$  the class of all continuous functions whose upper level sets are compact.

**Proposition 4.2.** ([15, Proposition 4.16]) *Let  $u, v \in C_K(\mathbb{R}^N)$ . Assume that  $\{u \geq 0\} = \{v \geq 0\}$ . Let  $S$  be a morphological operator on  $C_K(\mathbb{R}^N)$ . Then  $\{Su \geq 0\} = \{Sv \geq 0\}$ .*

This means that the zero super-level set of  $Su$  does not depend on the choice of  $u \in C_K(\mathbb{R}^N)$ .

We give the sup-inf representation for a morphological operator  $S$ , due to [35].

**Theorem 4.1.** ([15, Theorem 4.19]) *Let  $S$  be a morphological operator defined on  $C_K(\mathbb{R}^N)$ . Then*

$$[Su](x) = \sup_{B \in \mathcal{B}} \inf_{y \in B} u(x + y) \quad \text{for } u \in C_K(\mathbb{R}^N) \text{ and } x \in \mathbb{R}^N,$$

where  $\mathcal{B} := \{X \in \mathcal{C}_0(\mathbb{R}^N) \mid [S(-d_X)](0) \geq 0\}$ .

## 4.2 The set operator associated with our scheme

For  $E \in \mathcal{C}(\mathbb{R}^N)$  and  $h > 0$  we define a set operator  $T_{\gamma^\circ, h}$  by

$$T_{\gamma^\circ, h}(E) := \{w_{\gamma^\circ, E}^h \leq 0\},$$

where  $w_{\gamma^\circ, E}^h$  is a weak solution of (3.1).

In the case  $\gamma(p) = |p|$  Chambolle [17] treated  $T_{\gamma^\circ, h}$  as an operator from  $\mathcal{C}_0(\mathbb{R}^N)$  into itself. However, in our case we need to extend the domain of  $T_{\gamma^\circ, h}$  since the set  $\{u \geq \mu\}$  may not be bounded.

It is easily seen by Proposition 2.3 that  $T_{\gamma^\circ, h}(E) \in \mathcal{C}(\mathbb{R}^N)$  for each  $E \in \mathcal{C}(\mathbb{R}^N)$ . Thus  $T_{\gamma^\circ, h}$  maps from  $\mathcal{C}(\mathbb{R}^N)$  into itself. Moreover,  $T_{\gamma^\circ, h}$  has the following properties, which are almost the same as to (M), (C), (T) and (R) in subsection 4.1 except one should take the anisotropy into account.

**Theorem 4.2.** *Let  $E, E', E_n \in \mathcal{C}(\mathbb{R}^N)$  ( $n \in \mathbb{N}$ ),  $x \in \mathbb{R}^N$  and  $U \in O(N)$ .*

- (1)  $T_{\gamma^\circ, h}(E) \subset T_{\gamma^\circ, h}(E')$  if  $E \subset E'$ .
- (2)  $T_{\gamma^\circ, h}(E_n) \searrow T_{\gamma^\circ, h}(E)$  as  $n \rightarrow +\infty$  if  $E_n \searrow E$  as  $n \rightarrow +\infty$ .
- (3)  $T_{\gamma^\circ, h}(x + E) = x + T_{\gamma^\circ, h}(E)$ .
- (4)  $T_{\gamma_U^\circ, h}(UE) = UT_{\gamma^\circ, h}(E)$ . Here  $\gamma_U^\circ(p) := \gamma^\circ(U^*p)$  and  $U^*$  is the transposed matrix of  $U$ .

**Proof.** (1) Assume  $E \subset E'$ . Then  $d_{\gamma^\circ, E} \geq d_{\gamma^\circ, E'}$  in  $\mathbb{R}^N$  and hence  $w_{\gamma^\circ, E}^h \geq w_{\gamma^\circ, E'}^h$  in  $\mathbb{R}^N$  by Theorem 8.1 in subsection 8.1. This implies that  $T_{\gamma^\circ, h}(E) \subset T_{\gamma^\circ, h}(E')$ .

(2) Assume that  $\{E_n\}_{n=1}^{+\infty} \subset \mathcal{C}(\mathbb{R}^N)$  satisfies  $E_n \searrow E$  as  $n \rightarrow +\infty$ . We have only to show that

$$(4.5) \quad \bigcap_{n=1}^{\infty} T_{\gamma^\circ, h}(E_n) = T_{\gamma^\circ, h}(E).$$



since the nonincreasing property of  $\{T_{\gamma^\circ, h}(E_n)\}_{n=1}^{+\infty}$  follows from the assertion (1). We divide our consideration into two cases.

*Case 1.*  $E \neq \emptyset, \mathbb{R}^N$ .

Thanks to Theorem 8.2 in subsection 8.2, we see that  $d_{\gamma^\circ, E_n} \nearrow d_{\gamma^\circ, E}$  locally uniformly in  $\mathbb{R}^N$  as  $n \rightarrow +\infty$ . The proof of Proposition 2.3 yields that  $w_{\gamma^\circ, E_n}^h \nearrow w_{\gamma^\circ, E}^h$  locally uniformly in  $\mathbb{R}^N$  as  $n \rightarrow +\infty$ .

Assume  $x \notin T_{\gamma^\circ, h}(E)$ . Then we get  $w_{\gamma^\circ, E}^h(x) > 0$  and thus  $w_{\gamma^\circ, E_n}^h(x) > 0$  for large  $n \in \mathbb{N}$ . This means that  $x \notin \bigcap_{n=1}^{+\infty} T_{\gamma^\circ, h}(E_n)$ . Therefore, we obtain  $\bigcap_{n=1}^{+\infty} T_{\gamma^\circ, h}(E_n) \subset T_{\gamma^\circ, h}(E)$ . Since the reverse inclusion is trivial by (1), the equality (4.5) is derived.

*Case 2.*  $E = \emptyset$  or  $\mathbb{R}^N$ .

If  $E = \mathbb{R}^N$ , then the assertion is obvious as  $E_n = \mathbb{R}^N$  for all  $n \in \mathbb{N}$ . Hence we may consider the case  $E = \emptyset$ .

Fix any  $r > R_1\sqrt{h}$ . We claim that there exists  $n_r \in \mathbb{N}$  such that

$$(4.6) \quad E_{n_r} \subset \mathbb{R}^N \setminus B_{\gamma^\circ}(0, r).$$

Indeed, if  $E_n \cap B_{\gamma^\circ}(0, r) \neq \emptyset$  for all  $n \in \mathbb{N}$ , then there exist  $x_n \in E_n \cap B_{\gamma^\circ}(0, r)$  for each  $n \in \mathbb{N}$ . Since  $\{x_n\}_{n=1}^{+\infty}$  is bounded, we extract a subsequence  $\{x_{n_k}\}_{k=1}^{+\infty} \subset \{x_n\}_{n=1}^{+\infty}$  such that  $x_{n_k} \rightarrow x \in \overline{B_{\gamma^\circ}(0, r)}$  as  $k \rightarrow +\infty$ . We see by  $E_n \searrow E$  as  $n \rightarrow +\infty$  that for any fixed  $n \in \mathbb{N}$ , there exists  $k_0 \in \mathbb{N}$  such that  $n_k \geq n$  and  $x_{n_k} \in E_{n_k} \subset E_n$  for all  $k \geq k_0$ . Letting  $k \rightarrow +\infty$ , we get  $x \in E_n$ . As  $n$  is arbitrary, we have  $x \in E = \bigcap_{n=1}^{+\infty} E_n$ . However, this contradicts to  $E = \emptyset$ . Hence we obtain (4.6). Applying  $T_{\gamma^\circ, h}$  on both sides of (4.6), we deduce from (1) and Example 3.1 that

$$T_{\gamma^\circ, h}(E_{n_r}) \subset T_{\gamma^\circ, h}(\mathbb{R}^N \setminus B_{\gamma^\circ}(0, r)) = \mathbb{R}^N \setminus B_{\gamma^\circ}(0, R_h(r)).$$

Taking the intersection over all  $r > R_1\sqrt{h}$  gives

$$\bigcap_{n=1}^{\infty} T_{\gamma^\circ, h}(E_n) \subset \bigcap_{r > R_1\sqrt{h}} T_{\gamma^\circ, h}(E_{n_r}) \subset \emptyset = T_{\gamma^\circ, h}(E).$$

Hence we have derived (4.5).

(3) Fix any  $x_0 \in \mathbb{R}^N$ . Let a pair  $(w_{\gamma^\circ, E}^h, z)$  be a weak solution of (3.1). Set  $\tilde{w}(x) := w_{\gamma^\circ, E}^h(x - x_0)$ ,  $\tilde{z}(x) := z(x - x_0)$  and  $\tilde{d}(x) := d_{\gamma^\circ, E}(x - x_0)(= d_{\gamma^\circ, x_0+E}(x))$ . We then observe that  $(\tilde{w}, \tilde{z})$  is a weak solution of (2.6) with  $g = \tilde{d}$ . Thus we get  $w_{\gamma^\circ, x_0+E}^h = \tilde{w}$  from Theorem 2.2. Therefore, we see that

$$\begin{aligned} T_{\gamma^\circ, h}(x_0 + E) &= \{w_{\gamma^\circ, x_0+E}^h(x) \leq 0\} = \{w_{\gamma^\circ, E}^h(\cdot - x_0) \leq 0\} \\ &= x_0 + \{w_{\gamma^\circ, E}^h \leq 0\} = x_0 + T_{\gamma^\circ, h}(E). \end{aligned}$$

(4) Fix any  $U \in O(N)$ . Let  $(w_{\gamma^\circ, E}^h, z)$  be a weak solution of (3.1). Put  $\tilde{w}(y) := w_{\gamma^\circ, E}^h(U^*y)$  and  $\tilde{z}(y) := Uz(U^*y)$ . It is easy to see that  $\nabla_y \tilde{w}(y) = U \nabla w_{\gamma^\circ, E}^h(U^*y)$  and  $\tilde{z}(y) \in \partial \gamma_U(\nabla \tilde{w}(y))$  for a.e.  $y \in \mathbb{R}^N$  since  $w_{\gamma^\circ, E}^h$  is Lipschitz continuous. Hence we observe that  $(\tilde{w}, \tilde{z})$  is a weak solution of (2.6) with  $g = d_{\gamma_U^\circ, UE}$ . Thus we have  $w_{\gamma_U^\circ, UE}^h = w_{\gamma^\circ, E}^h(U^*\cdot)$  by the uniqueness. Consequently, we obtain

$$T_{\gamma_U^\circ, h}(UE) = \{w_{\gamma_U^\circ, UE}^h \leq 0\} = \{w_{\gamma^\circ, E}^h(U^*\cdot) \leq 0\} = U\{w_{\gamma^\circ, E}^h \leq 0\} = UT_{\gamma^\circ, h}(E).$$

Therefore, we have completed the proof of Theorem 4.2.  $\square$

In addition to Theorem 4.2 we have a scaling property.

**Theorem 4.3.** Fix  $E \in \mathcal{C}(\mathbb{R}^N)$  and  $h > 0$ . Then  $T_{\gamma^\circ, \theta^2 h}(\theta E) = \theta T_{\gamma^\circ, h}(E)$  for any  $\theta > 0$ .

**Proof.** Let a pair  $(w_{\gamma^\circ, E}^h, z)$  be a weak solution of (3.1). Set  $\tilde{w} := \theta w_{\gamma^\circ, E}^h(\cdot/\theta)$  and  $\tilde{z} := z(\cdot/\theta)$ . We then observe that  $(\tilde{w}, \tilde{z})$  is a weak solution of

$$w - \theta^2 h \operatorname{div} \partial \gamma(\nabla w) \ni d_{\gamma^\circ, \theta E} \quad \text{in } \mathbb{R}^N.$$

We have  $w_{\gamma^\circ, \theta E}^{\theta^2 h} = \theta w_{\gamma^\circ, E}^h(\cdot/\theta)$  from the uniqueness. Hence we get

$$T_{\gamma^\circ, \theta^2 h}(\theta E) = \{w_{\gamma^\circ, \theta E}^{\theta^2 h} \leq 0\} = \{w_{\gamma^\circ, E}^h(\cdot/\theta) \leq 0\} = \theta \{w_{\gamma^\circ, E}^h \leq 0\} = \theta T_{\gamma^\circ, h}(E).$$

□

**Proposition 4.3.** Let  $E, E' \in \mathcal{C}(\mathbb{R}^N)$  and  $h > 0$ . Assume  $E \cap E' = \emptyset$  and  $\inf_{x \in E, y \in E'} |x - y| > 0$ . Then  $T_{\gamma^\circ, h}(E) \cap T_{\gamma^\circ, h}(E') = \emptyset$ .

**Proof.** Let  $w_{\gamma^\circ, E}^h, w_{\gamma^\circ, E'}^h$  be a weak solution of (2.6) with  $g = d_{\gamma^\circ, E}, g = d_{\gamma^\circ, E'}$ , respectively. Set  $\rho := -d_{\gamma^\circ, E} (= d_{\gamma^\circ, \overline{\mathbb{R}^N \setminus E}})$ . We easily verify by use of  $(\gamma 3)$  that  $\tilde{w} := -w_{\gamma^\circ, E}^h$  is a weak solution of (2.6) with  $g = \rho$ . Thus it readily follows that  $E = \{\rho \geq 0\}$  and  $T_{\gamma^\circ, h}(E) = \{\tilde{w} \geq 0\}$ .

It follows from the assumptions on  $E$  and  $E'$  that there is a small  $\delta > 0$  such that  $\rho < d_{\gamma^\circ, E'} - \delta$  in  $\mathbb{R}^N$ . Since  $w_{\gamma^\circ, E'}^h - \delta$  is a weak solution of (2.6) with  $g = d_{\gamma^\circ, E'} - \delta$ , we get  $\tilde{w} \leq w_{\gamma^\circ, E'}^h - \delta$  in  $\mathbb{R}^N$  by Theorem 8.1. Hence we have  $w_{\gamma^\circ, E'}^h(x) \geq \delta$  for all  $x \in T_{\gamma^\circ, h}(E)$  and therefore  $T_{\gamma^\circ, h}(E) \subset \mathbb{R}^N \setminus T_{\gamma^\circ, h}(E')$ . □

To apply the results in subsection 4.1, we show the following theorem.

**Theorem 4.4.** For each  $h > 0, u \in C(\mathbb{R}^N)$  and  $n \in \mathbb{N}$ , the family  $\{T_{\gamma^\circ, h}^n(\{u \geq \mu\})\}_{\mu \in \mathbb{R}}$  satisfies (4.1) - (4.3).

**Proof.** *Step 1.* The monotonicity (4.2) directly follows from Theorem 4.2 (1) and induction.

*Step 2.* We prove (4.3).

Set  $n = 1$ . Since we see by Lemma 4.1 and the continuity of  $u$  that

$$\bigcap_{\nu < \mu} \{u \geq \nu\} = \bigcap_{k=1}^{+\infty} \{u \geq \mu_k\} = \{u \geq \mu\}, \quad \mu_k := \mu - \frac{1}{k},$$

we get from Theorem 4.2 (2) and (4.2) with  $X_\mu := T_{\gamma^\circ, h}(\{u \geq \mu\})$ ,

$$\begin{aligned} \bigcap_{\nu < \mu} T_{\gamma^\circ, h}(\{u \geq \nu\}) &\subset \bigcap_{k=1}^{+\infty} T_{\gamma^\circ, h}(\{u \geq \mu_k\}) \\ &= T_{\gamma^\circ, h}(\{u \geq \mu\}) \subset \bigcap_{\nu < \mu} T_{\gamma^\circ, h}(\{u \geq \nu\}). \end{aligned}$$

Hence the (4.3) holds in the case  $n = 1$ .

Assume that  $\{T_{\gamma^\circ, h}^n(\{u \geq \mu\})\}_{\mu \in \mathbb{R}}$  satisfies (4.3). Fix any  $\mu \in \mathbb{R}$ . Since  $T_{\gamma^\circ, h}^n(\{u \geq \nu\}) \searrow T_{\gamma^\circ, h}^n(\{u \geq \mu\})$  as  $\nu \nearrow \mu$  by the assumption of induction, we deduce from the continuity of  $T_{\gamma^\circ, h}$  that

$$\begin{aligned} \bigcap_{\nu < \mu} T_{\gamma^\circ, h}^{n+1}(\{u \geq \nu\}) &= \bigcap_{\nu < \mu} T_{\gamma^\circ, h}(T_{\gamma^\circ, h}^n(\{u \geq \nu\})) \\ &= T_{\gamma^\circ, h}(T_{\gamma^\circ, h}^n(\{u \geq \mu\})) = T_{\gamma^\circ, h}^{n+1}(\{u \geq \mu\}). \end{aligned}$$

Thus  $\{T_{\gamma^\circ, h}^n(\{u \geq \mu\})\}_{\mu \in \mathbb{R}}$  satisfies (4.3) for all  $n \in \mathbb{N}$ .

*Step 3.* We confirm  $\bigcup_{\mu \in \mathbb{R}} T_{\gamma^\circ, h}^n(\{u \geq \mu\}) = \mathbb{R}^N$  for all  $n \in \mathbb{N}$ .

Recall  $R_h(r) = (r + \sqrt{r^2 - 4(N-1)h})/2$ . Then it is easily observed that  $R_h(r) \geq (3/4)r$  for all  $r \geq 3\sqrt{h(N-1)}$ . Besides, we calculate that

$$R_h^2(r) := R_h(R_h(r)) \geq \frac{3}{4}R_h(r) \quad \text{for all } r > 0 \text{ satisfying } R_h(r) \geq 3\sqrt{h(N-1)}.$$

Thus we get  $R_h^2(r) \geq (3/4)^2 r$  for all  $r \geq 4\sqrt{h(N-1)}$ . Repeating the above calculations inductively, we have for each  $n \in \mathbb{N}$

$$R_h^n(r) := R_h(R_h^{n-1}(r)) \geq \left(\frac{3}{4}\right)^n r \quad \text{for all } r \geq 3\left(\frac{4}{3}\right)^{n-1} \sqrt{h(N-1)}.$$

Fix  $n \in \mathbb{N} \cup \{0\}$  and select  $r > 0$  so large that  $r \geq 3(4/3)^{n-1} \sqrt{h(N-1)}$ . Set  $\underline{\mu}_r := \min_{x \in \overline{B_{\gamma^\circ}(0, r)}} u(x)$ . Then we obtain  $\{u \geq \underline{\mu}_r\} \supset \overline{B_{\gamma^\circ}(0, r)}$ . Applying  $T_{\gamma^\circ, h}$  on both sides of this inclusion  $n$ -times gives

$$T_{\gamma^\circ, h}^n(\{u \geq \underline{\mu}_r\}) \supset T_{\gamma^\circ, h}^n(\overline{B_{\gamma^\circ}(0, r)}) = \overline{B_{\gamma^\circ}(0, R_h^n(r))} \supset \overline{B_{\gamma^\circ}(0, (3/4)^n r)}.$$

Taking sum over all  $r > 0$  yields that

$$\bigcup_{\mu \in \mathbb{R}} T_{\gamma^\circ, h}^n(\{u \geq \mu\}) \supset \bigcup_{r > 0} T_{\gamma^\circ, h}^n(\{u \geq \underline{\mu}_r\}) = \mathbb{R}^N.$$

*Step 4.* We derive  $\bigcap_{\mu \in \mathbb{R}} T_{\gamma^\circ, h}^n(\{u \geq \mu\}) = \emptyset$  for all  $n \in \mathbb{N}$ .

Take  $n \in \mathbb{N} \cup \{0\}$  and  $r > 0$  so large that  $r \geq 3(4/3)^{n-1} \sqrt{h(N-1)}$ . Set  $\bar{\mu}_r := \max_{x \in \overline{B_{\gamma^\circ}(0, r)}} u(x)$ . Then we have  $\{u \geq \mu_r\} \subset \mathbb{R}^N \setminus B_{\gamma^\circ}(0, r)$  for any  $\mu > \bar{\mu}_r$ . Applying  $T_{\gamma^\circ, h}$  on both sides of this inclusion  $n$ -times provides

$$T_{\gamma^\circ, h}^n(\{u \geq \mu\}) \subset T_{\gamma^\circ, h}^n(\mathbb{R}^N \setminus B_{\gamma^\circ}(0, r)) = \mathbb{R}^N \setminus B_{\gamma^\circ}(0, R_h^n(r)) \subset \mathbb{R}^N \setminus B_{\gamma^\circ}(0, (3/4)^n r).$$

We take intersection over all  $r > 0$  to deduce

$$\bigcap_{\mu \in \mathbb{R}} T_{\gamma^\circ, h}^n(\{u \geq \mu\}) \subset \bigcap_{r > 0} T_{\gamma^\circ, h}^n(\{u \geq \mu_r\}) = \emptyset.$$

Consequently, we have completed the proof of (4.1) by the results of Step 3 and 4.  $\square$

### 4.3 Function operator defined by $T_{\gamma^\circ, h}$

Following (4.4), we define  $S_{\gamma^\circ, h}u$  for  $u \in C(\mathbb{R}^N)$  by

$$(4.7) \quad \begin{aligned} [S_{\gamma^\circ, h}u](x) &:= \sup\{\mu \in \mathbb{R} \mid x \in T_{\gamma^\circ, h}(\{u \geq \mu\})\} \quad \text{for } x \in \mathbb{R}^N \\ & (= \sup\{\mu \in \mathbb{R} \mid w_{\gamma^\circ, \{u \geq \mu\}}^h(x) \leq 0\}). \end{aligned}$$

**Remark 4.1.** Eto's idea in [22] is to make use of the operator (4.7), which is an application of the mathematical morphology in image processing. To our best knowledge, his paper is the first one where the mathematical morphology is applied to the variational approximations to (1.1). Using the operator (4.7), he has essentially obtained in [22] the convergence of Chambolle's scheme (the case  $\gamma(p) = |p|$  and  $N = 2$ ) to the level set flow by (1.2) in the sense of Hausdorff distance provided that no fattening occurs.

At first, we check that  $S_{\gamma^\circ, h}$  is well-defined.

**Proposition 4.4.** *Let  $u \in C(\mathbb{R}^N)$ . Then  $[S_{\gamma^\circ, h}u](x)$  is finite for each  $x \in \mathbb{R}^N$  and  $h > 0$ .*

**Proof.** Fix  $x \in \mathbb{R}^N$  and  $r > R_1\sqrt{h}$  with  $R_1 = 2N/\sqrt{N+1}$ .

*Step 1.* Set  $\mu_0 := \inf_{y \in \overline{B_{\gamma^\circ}(x, r)}} u(y)$ . We prove  $[S_{\gamma^\circ, h}u](x) \geq \mu_0$ .

It is obvious that  $\{u \geq \mu_0\} \supset \overline{B_{\gamma^\circ}(x, r)}$ . Applying  $T_{\gamma^\circ, h}$  on both sides of this inclusion, we have by Example 3.1 and Theorem 4.2 (1), (3)

$$x \in \overline{B_{\gamma^\circ}(x, R_h(r))} = T_{\gamma^\circ, h}(\overline{B_{\gamma^\circ}(x, r)}) \subset T_{\gamma^\circ, h}(\{u \geq \mu_0\}).$$

Thus we have proved the assertion of this step.

*Step 2.* Set  $\mu_1 := \sup_{y \in \overline{B_{\gamma^\circ}(x, r)}} u(y)$ . We show  $[S_{\gamma^\circ, h}u](x) \leq \mu_1$ .

Take  $\mu > \mu_1$  arbitrarily. The inclusion  $\{u \geq \mu\} \subset \mathbb{R}^N \setminus B_{\gamma^\circ}(x, r)$ , Example 3.1 and Theorem 4.2 (1) yield that

$$(4.8) \quad T_{\gamma^\circ, h}(\{u \geq \mu\}) \subset T_{\gamma^\circ, h}(\mathbb{R}^N \setminus B_{\gamma^\circ}(x, r)) = \mathbb{R}^N \setminus B_{\gamma^\circ}(x, R_h(r)).$$

By (4.8) we get  $x \notin T_{\gamma^\circ, h}(\{u \geq \mu\})$  and hence  $[S_{\gamma^\circ, h}u](x) \leq \mu$ . Sending  $\mu \searrow \mu_1$ , we obtain the result of this step.

The results of Step 1 and 2 imply that  $[S_{\gamma^\circ, h}u](x)$  is finite.  $\square$

**Theorem 4.5.** *Let  $u, v \in C(\mathbb{R}^N)$ . Then we have*

- (1)  $S_{\gamma^\circ, h}u \leq S_{\gamma^\circ, h}v$  in  $\mathbb{R}^N$  if  $u \leq v$  in  $\mathbb{R}^N$ ,
- (2)  $S_{\gamma^\circ, h}(g \circ u) = g \circ S_{\gamma^\circ, h}u$  for any nondecreasing  $g \in C(\mathbb{R})$ ,
- (3) For fixed  $y \in \mathbb{R}^N$ ,  $[S_{\gamma^\circ, h}u(\cdot + y)](x) = [S_{\gamma^\circ, h}u](x + y)$  for all  $x \in \mathbb{R}^N$ .

We omit the proof since this theorem is a consequence from Proposition 4.1 and Theorem 4.4. Note by (2) that  $[S_{\gamma^\circ, h}(u + c)](x) = [S_{\gamma^\circ, h}u](x) + c$  for  $u \in C(\mathbb{R}^N)$  and  $c \in \mathbb{R}$ .

**Theorem 4.6.** *Let  $u \in UC(\mathbb{R}^N)$  and  $\omega$  a modulus of continuity of  $u$ . Then we have*

$$(4.9) \quad |[S_{\gamma^\circ, h}u](x) - [S_{\gamma^\circ, h}u](y)| \leq \omega(|x - y|) \quad \text{for all } x, y \in \mathbb{R}^N \text{ and } h > 0.$$

*That is,  $S_{\gamma^\circ, h}u \in UC(\mathbb{R}^N)$  for any  $u \in UC(\mathbb{R}^N)$ .*

**Proof.** Fix  $y \in \mathbb{R}^N$ . We have from  $u \in UC(\mathbb{R}^N)$

$$|u(x+y) - u(x)| \leq \omega(|y|) \quad \text{for all } x \in \mathbb{R}^N.$$

It is seen by Theorem 4.5 (1) that for all  $x \in \mathbb{R}^N$

$$[S_{\gamma^\circ, h}(u(\cdot) - \omega(|y|))](x) \leq [S_{\gamma^\circ, h}u(\cdot + y)](x) \leq [S_{\gamma^\circ, h}(u(\cdot) + \omega(|y|))](x).$$

The inequality (4.9) follows from Theorem 4.5 (2) and (3).  $\square$

The relation between the sets  $\{S_{\gamma^\circ, h}u \geq \mu\}$  and  $T_{\gamma^\circ, h}(\{u \geq \mu\})$  is stated as follows. Set  $S_{\gamma^\circ, h}^n u := S_{\gamma^\circ, h}(S_{\gamma^\circ, h}^{n-1}u)$  for  $n \in \mathbb{N}$  and  $S_{\gamma^\circ, h}^0 u := u$ .

**Theorem 4.7.** *Let  $u \in C(\mathbb{R}^N)$ . Then  $\{S_{\gamma^\circ, h}^n u \geq \mu\} = T_{\gamma^\circ, h}^n(\{u \geq \mu\})$  for all  $n \in \mathbb{N}$  and  $\mu \in \mathbb{R}$ .*

**Proof.** In the case  $n = 1$  the result is clear because of Lemma 4.2. To show the desired result for  $n \geq 2$ , we assume that

$$\{S_{\gamma^\circ, h}^{n-1}u \geq \mu\} = T_{\gamma^\circ, h}^{n-1}(\{u \geq \mu\}) \quad \text{for all } \mu \in \mathbb{R}.$$

It is seen by Theorem 4.4 that the family  $\{T_{\gamma^\circ, h}^{n-1}(\{u \geq \mu\})\}_{\mu \in \mathbb{R}}$  satisfies (4.1) - (4.3). Hence we deduce that

$$\begin{aligned} [S_{\gamma^\circ, h}^n u](x) &= [S_{\gamma^\circ, h}(S_{\gamma^\circ, h}^{n-1}u)](x) = \sup\{\mu \in \mathbb{R} \mid x \in T_{\gamma^\circ, h}(\{S_{\gamma^\circ, h}^{n-1}u \geq \mu\})\} \\ &= \sup\{\mu \in \mathbb{R} \mid x \in T_{\gamma^\circ, h}(T_{\gamma^\circ, h}^{n-1}(\{u \geq \mu\}))\} \\ &= \sup\{\mu \in \mathbb{R} \mid x \in T_{\gamma^\circ, h}^n(\{u \geq \mu\})\}. \end{aligned}$$

Using Lemma 4.2 again, we have  $\{S_{\gamma^\circ, h}^n u \geq \mu\} = T_{\gamma^\circ, h}^n(\{u \geq \mu\})$ .  $\square$

**Proposition 4.5.** *Let  $u, v \in C(\mathbb{R}^N)$  and assume that  $\{u \geq 0\} = \{v \geq 0\}$ . Then,  $\{S_{\gamma^\circ, h}u \geq 0\} = \{S_{\gamma^\circ, h}v \geq 0\}$ .*

**Theorem 4.8.** *For  $u \in C(\mathbb{R}^N)$  and  $x \in \mathbb{R}^N$ , it holds that*

$$(4.10) \quad [S_{\gamma^\circ, h}u](x) = \sup_{X \in \mathcal{B}_{\gamma^\circ, h}} \inf_{y \in X} u(x+y),$$

where  $\mathcal{B}_{\gamma^\circ, h} := \{X \in \mathcal{C}(\mathbb{R}^N) \mid [S_{\gamma^\circ, h}(-d_{\gamma^\circ, X})](0) \geq 0\}$ .

In the case  $u, v \in C_K(\mathbb{R}^N)$  Proposition 4.5 and Theorem 4.8 have already been obtained as, respectively, Proposition 4.2 and Theorem 4.1 in subsection 4.1. However, we give the proofs of Proposition 4.5 and Theorem 4.8 since we cannot apply the proofs of these results because the set  $\{u \geq \mu\}$  may be unbounded.

**Proof of Proposition 4.5.** *Step 1.* We show that  $[S_{\gamma^\circ, h}u](0) \geq 0$  if and only if  $[S_{\gamma^\circ, h}v](0) \geq 0$ .

First we note that the supremum of  $[S_{\gamma^\circ, h}u](0)$  is attained by  $[S_{\gamma^\circ, h}u](0)$  itself. Indeed, it follows from Lemma 4.2 that

$$T_{\gamma^\circ, h}(\{u \geq [S_{\gamma^\circ, h}u](0)\}) = \{S_{\gamma^\circ, h}u \geq [S_{\gamma^\circ, h}u](0)\}.$$

By the fact  $0 \in \{S_{\gamma^\circ, h}u \geq [S_{\gamma^\circ, h}u](0)\}$  we have

$$[S_{\gamma^\circ, h}u](0) = \max\{\mu \in \mathbb{R} \mid 0 \in T_{\gamma^\circ, h}(\{u \geq \mu\})\}.$$

Therefore, we see that  $[S_{\gamma^\circ, h}u](0) \geq 0$  holds if and only if there exists  $\mu \geq 0$  for which  $0 \in T_{\gamma^\circ, h}(\{u \geq \mu\})$ .

Notice by Theorem 4.2 (1) that  $T_{\gamma^\circ, h}(\{u \geq 0\}) = T_{\gamma^\circ, h}(\{v \geq 0\})$ . Combining the above observation with this formula, we have the desired result of this step.

*Step 2.* We show the assertion of Proposition 4.5.

Fix any  $x \in \mathbb{R}^N$ . Then we get  $\{u(\cdot + x) \geq 0\} = \{v(\cdot + x) \geq 0\}$  from  $\{u \geq 0\} = \{v \geq 0\}$ . Using the result of the previous step, we conclude that  $[S_{\gamma^\circ, h}u(\cdot + x)](0) \geq 0$  if and only if  $[S_{\gamma^\circ, h}v(\cdot + x)](0) \geq 0$ . Thus we apply Theorem 4.5 (3) to obtain the desired result.  $\square$

**Proof of Theorem 4.8.** We first note that if  $X \in \mathcal{B}_{\gamma^\circ, h}$  and  $Y \in \mathcal{C}(\mathbb{R}^N)$  includes  $X$ , then  $Y \in \mathcal{B}_{\gamma^\circ, h}$  by Theorem 4.2 (1). For  $\mu \in \mathbb{R}$  we get the following equivalences.

$$\begin{aligned} [S_{\gamma^\circ, h}u](x) \geq \mu &\iff \forall \nu < \mu; [S_{\gamma^\circ, h}u](x) \geq \nu \\ &\iff \forall \nu < \mu; [S_{\gamma^\circ, h}(u(\cdot + x) - \nu)](0) \geq 0. \end{aligned}$$

Here the second equivalence is deduced from Theorem 4.5 (2) and (3). Since  $\{-d_{\gamma^\circ, \{u \geq \nu\} - x} \geq 0\} = \{u(\cdot + x) - \nu \geq 0\}$ , we obtain from Proposition 4.5 the equivalence between  $[S_{\gamma^\circ, h}(u(\cdot + x) - \nu)](0) \geq 0$  and  $[S_{\gamma^\circ, h}(-d_{\gamma^\circ, \{u \geq \nu\} - x})](0) \geq 0$ . Moreover, we see that

$$\begin{aligned} [S_{\gamma^\circ, h}u](x) \geq \mu &\iff \forall \nu < \mu; [S_{\gamma^\circ, h}(-d_{\gamma^\circ, \{u \geq \nu\} - x})](0) \geq 0 \\ &\iff \forall \nu < \mu; \{u \geq \nu\} - x \in \mathcal{B}_{\gamma^\circ, h} \\ &\iff \forall \nu < \mu; \exists X \in \mathcal{B}_{\gamma^\circ, h} \text{ such that } X \subset \{u \geq \nu\} - x \\ &\iff \forall \nu < \mu; \exists X \in \mathcal{B}_{\gamma^\circ, h} \text{ such that } \inf_{y \in X} u(x + y) \geq \nu \\ &\iff \sup_{X \in \mathcal{B}_{\gamma^\circ, h}} \inf_{y \in X} u(x + y) \geq \mu. \end{aligned}$$

Since  $\mu \in \mathbb{R}$  is arbitrary, (4.10) holds.

$\square$

We give some properties of  $\mathcal{B}_{\gamma^\circ, h}$ .

**Proposition 4.6.** *Let  $U \in O(N)$ . Then  $X \in \mathcal{B}_{\gamma^\circ, h}$  if and only if  $UX \in \mathcal{B}_{\gamma_U^\circ, h}$ .*

**Proof.** It follows from the definition of  $d_{\gamma^\circ, X}$  that  $d_{\gamma^\circ, X}(\cdot) = d_{\gamma_U^\circ, UX}(U\cdot)$ . We observe that

$$\begin{aligned} [S_{\gamma^\circ, h}(-d_{\gamma^\circ, X})](0) &= \sup\{\mu \in \mathbb{R} \mid 0 \in T_{\gamma^\circ, h}(\{-d_{\gamma^\circ, X} \geq \mu\})\} \\ &= \sup\{\mu \in \mathbb{R} \mid 0 \in T_{\gamma^\circ, h}(\{-d_{\gamma_U^\circ, UX}(U\cdot) \geq \mu\})\} \\ &= \sup\{\mu \in \mathbb{R} \mid 0 \in T_{\gamma^\circ, h}(U^*\{-d_{\gamma_U^\circ, UX} \geq \mu\})\} \\ &= \sup\{\mu \in \mathbb{R} \mid 0 \in U^*T_{\gamma_U^\circ, h}(\{-d_{\gamma_U^\circ, UX} \geq \mu\})\} \\ &= \sup\{\mu \in \mathbb{R} \mid 0 \in T_{\gamma_U^\circ, h}(\{-d_{\gamma_U^\circ, UX} \geq \mu\})\} \\ &= [S_{\gamma_U^\circ, h}(-d_{\gamma_U^\circ, UX})](0). \end{aligned}$$

The fourth equality is obtained from Theorem 4.2 (4) and  $\gamma^\circ = (\gamma_U^\circ)_{U^*}$ . Hence the proof is completed.  $\square$

**Proposition 4.7.**  $\mathcal{B}_{\gamma^\circ, h} = \sqrt{h}\mathcal{B}_{\gamma^\circ, 1} (:= \{\sqrt{h}X \mid X \in \mathcal{B}_{\gamma^\circ, 1}\})$  for  $h > 0$ .

**Proof.** We observe that for any  $X \in \mathcal{B}_{\gamma^\circ, h}$ ,

$$\begin{aligned}
[S_{\gamma^\circ, 1}(-d_{\gamma^\circ, X/\sqrt{h}})](0) &= \sup \left\{ \mu \in \mathbb{R} \mid 0 \in T_{\gamma^\circ, 1}(\{-d_{\gamma^\circ, X/\sqrt{h}} \geq \mu\}) \right\} \\
&= \sup \left\{ \mu \in \mathbb{R} \mid 0 \in T_{\gamma^\circ, 1} \left( \left\{ -\frac{1}{\sqrt{h}}d_{\gamma^\circ, X}(\sqrt{h}\cdot) \geq \mu \right\} \right) \right\} \\
&= \frac{1}{\sqrt{h}} \sup \left\{ \mu \in \mathbb{R} \mid 0 \in T_{\gamma^\circ, 1} \left( \frac{1}{\sqrt{h}} \{-d_{\gamma^\circ, X} \geq \mu\} \right) \right\} \\
&= \frac{1}{\sqrt{h}} \sup \left\{ \mu \in \mathbb{R} \mid 0 \in \frac{1}{\sqrt{h}} T_{\gamma^\circ, h}(\{-d_{\gamma^\circ, X} \geq \mu\}) \right\} \\
&= \frac{1}{\sqrt{h}} \sup \{ \mu \in \mathbb{R} \mid 0 \in T_{\gamma^\circ, h}(\{-d_{\gamma^\circ, X} \geq \mu\}) \} \\
&= \frac{1}{\sqrt{h}} [S_{\gamma^\circ, h}(-d_{\gamma^\circ, X})](0).
\end{aligned}$$

Here we have used Theorem 4.3 to get the fourth equality. Thus we get

$$[S_{\gamma^\circ, 1}(-d_{\gamma^\circ, X/\sqrt{h}})](0) \geq 0 \iff [S_{\gamma^\circ, h}(-d_{\gamma^\circ, X})](0) \geq 0.$$

Therefore, the equality  $\mathcal{B}_{\gamma^\circ, h} = \sqrt{h}\mathcal{B}_{\gamma^\circ, 1}$  follows.  $\square$

Set  $U_i := \overline{B_{\gamma^\circ}(0, R_i\sqrt{h})}$  for  $i = 1, 2$ .

**Proposition 4.8.** For any  $h > 0$  and  $X \in \mathcal{C}(\mathbb{R}^N)$ , if  $T_{\gamma^\circ, h}(X) \cap U_2 \neq \emptyset$ , then  $X \cap U_1 \neq \emptyset$ . Especially,  $U_1 \in \mathcal{B}_{\gamma^\circ, h}$  and  $X \cap U_1 \neq \emptyset$  for all  $X \in \mathcal{B}_{\gamma^\circ, h}$  and  $h > 0$ .

**Proof.** For any  $h > 0$  and  $X \in \mathcal{C}(\mathbb{R}^N)$ , assume  $X \cap U_1 = \emptyset$ . Then we get  $\inf_{x \in X, y \in U_1} |x - y| > 0$ . From Proposition 4.5, we have  $T_{\gamma^\circ, h}(X) \cap T_{\gamma^\circ, h}(U_1) = \emptyset$ . Since Lemma 2.5 implies that  $T_{\gamma^\circ, h}(U_1) = U_2$ , the assertion is proved.  $\square$

## 5 Consistency of $S_{\gamma^\circ, h}$

We first note that the supremum of (4.10) is attained by a set in the ball  $U_1$ . This is crucial in evaluating the value of  $S_{\hat{\gamma}^\circ, h}\psi$  in Theorem 5.2 below.

**Proposition 5.1.** Let  $u \in C(\mathbb{R}^N)$ . For any  $h > 0$  and  $x \in \mathbb{R}^N$  there exists  $X_0 \in \mathcal{B}_{\gamma^\circ, h}$  such that

$$(5.1) \quad [S_{\gamma^\circ, h}u](x) = \inf_{y \in X_0} u(x + y) = \inf_{y \in X_0 \cap U_1} u(x + y).$$

**Proof.** Fix  $h > 0$  and  $x \in \mathbb{R}^N$ . Set  $a := [S_{\gamma^\circ, h}u](x)$ .

*Step 1.* We show that the first equality of (5.1) holds for some  $X_0 \in \mathcal{B}_{\gamma^\circ, h}$ .

For each  $n \in \mathbb{N}$  there exists  $X_n \in \mathcal{B}_{\gamma^\circ, h}$  satisfying

$$a - \frac{1}{n} < a_n := \inf_{y \in X_n} u(x + y) \leq a.$$

We may consider that  $\{a_n\}_{n=1}^{+\infty}$  is nondecreasing. Moreover, we may also assume that  $\{X_n\}_{n=1}^{+\infty}$  is nonincreasing, replacing  $X_n$  with  $\overline{\cup_{k=n}^{+\infty} X_k}$ .

Set  $X_0 := \bigcap_{n=1}^{+\infty} X_n$ . It follows from Theorem 4.2 (2) and  $\{X_n\}_{n=1}^{+\infty} \subset \mathcal{B}_{\gamma^\circ, h}$  that

$$T_{\gamma^\circ, h}(X_0) = \bigcap_{n=1}^{+\infty} T_{\gamma^\circ, h}(X_n) \ni 0.$$

We get  $X_0 \cap U_1 \neq \emptyset$  by Proposition 4.8 and this formula. Therefore, we obtain  $X_0 \in \mathcal{B}_{\gamma^\circ, h}$  and we have the result of this step by use of this  $X_0$  and  $u \in C(\mathbb{R}^N)$ .

*Step 2.* We prove that the second equality of (5.1) holds.

Suppose that  $\inf_{y \in X_0 \cap U_1} u(x+y) > a$ . Since the inequality  $\inf_{y \in U_1} u(x+y) \leq a$  readily follows from  $U_1 \in \mathcal{B}_{\gamma^\circ, h}$ , there exists  $y_1 \in U_1$  such that  $u(x+y_1) = a$ . Set

$$Z_0 := \overline{\{y \in U_1 \mid u(x+y) > a\}}.$$

Then  $X_0 \cap U_1 \not\subseteq Z_0$  and there exists  $y_2 \in Z_0$  such that  $u(x+y_2) = a$ . Hence replacing  $X_0$  with  $X_0 \cup Z_0$ , we obtain the second equality of (5.1). Therefore we have the result.  $\square$

We derive the generator of  $S_{\gamma^\circ, h}$ . Let  $\hat{\gamma}$  satisfy  $(\gamma 1) - (\gamma 5)$ . Let  $\psi (= \psi(y)) \in C^\infty(\mathbb{R}^N)$  be a function satisfying

$$(5.2) \quad \psi(0) = 0, \quad \nabla \psi(0) = |\nabla \psi(0)| e_N (\neq 0).$$

We take  $\delta > 0$  so small that for all  $y \in B(0, 6\delta)$

$$(5.3) \quad \frac{1}{2} |\nabla \psi(0)| \leq |\nabla \psi(y)| \leq 2 |\nabla \psi(0)|.$$

It follows from (5.2) and (5.3) that  $[S_{\hat{\gamma}^\circ, h} \psi](0) \leq L_1 \sqrt{h}$  ( $L_1 := 2 |\nabla \psi(0)| R_1$ ). To refine this estimate, we give a local approximation of a weak solution  $w_{\hat{\gamma}^\circ, E}^h$  of (3.1) with  $\gamma = \hat{\gamma}$ .

**Proposition 5.2.** *Let  $\hat{\gamma}$  satisfy  $(\gamma 1) - (\gamma 5)$ . Let  $\psi \in C^\infty(\mathbb{R}^N)$  satisfy (5.2). We take  $\delta > 0$  so that (5.3) holds. For  $h > 0$  and  $|\mu| \leq L_1 \sqrt{h}$  define  $E_\mu := \{\psi \geq \mu\}$  and let  $w_{\hat{\gamma}^\circ, E_\mu}^h$  be a weak solution of (3.1) with  $\gamma^\circ = \hat{\gamma}^\circ$  and  $E = E_\mu$ . Then for any  $\varepsilon > 0$  there exist  $r, h_1 > 0$  such that*

$$(5.4) \quad |w_{\hat{\gamma}^\circ, E_\mu}^h - (d_{\hat{\gamma}^\circ, E_\mu} - h \kappa_{\hat{\gamma}^\circ, E_\mu})| \leq \varepsilon h \quad \text{on } \overline{U_{\delta, r}} \quad \text{for all } h \in (0, h_1),$$

where  $U_{\delta, r} := \{y \in B(0, \delta) \mid |d_{\hat{\gamma}^\circ, E_\mu}(y)| < r\}$ .

We prepare a lemma to prove this proposition. Since the left-hand side of (3.1) is multi-valued, we shall approximate  $\hat{\gamma}$  by a smooth and strictly convex function  $\hat{\gamma}_n$  so that the equation (3.1) done by a smooth and uniformly elliptic equation. Since  $\hat{\gamma}$  satisfies  $(\gamma 1) - (\gamma 5)$ , there is a sequence  $\{\hat{\gamma}_n\}_{n=1}^{+\infty} \subset C^\infty(\mathbb{R}^N)$  satisfying

$$(5.5) \quad \hat{\gamma}_n \searrow \hat{\gamma} \quad \text{uniformly in } \mathbb{R}^N \text{ as } n \rightarrow +\infty,$$

$$(5.6) \quad \nabla^k \hat{\gamma}_n \longrightarrow \nabla^k \hat{\gamma} \quad \text{locally uniformly in } \mathbb{R}^N \setminus \{0\} \text{ as } n \rightarrow +\infty \quad (k = 1, 2),$$

$$(5.7) \quad \nabla^2 \hat{\gamma}_n \geq \frac{1}{n} I \quad \text{in } \mathbb{R}^N \text{ for all } n \in \mathbb{N},$$



where  $I$  denotes the identity matrix. We consider an approximate elliptic equation: For  $n \in \mathbb{N}$  and  $F \in \mathcal{C}(\mathbb{R}^N)$

$$(5.8) \quad w - h \operatorname{div} \nabla \hat{\gamma}_n(\nabla w) = d_{\gamma^\circ, F} \quad \text{in } \mathbb{R}^N.$$

We observe by the proofs of [17, Proposition A.4] and [16, Theorem 3] that this equation has a unique classical solution  $w_n$  satisfying  $\hat{\gamma}(\nabla w_n) \leq 1$  in  $\mathbb{R}^N$ .

**Lemma 5.1.** *For each  $n \in \mathbb{N}$  let  $v_n$  be a classical solution of (5.8) with  $F := \overline{B_{\hat{\gamma}^\circ}(0, r)}$  ( $r \in (0, 1)$ ). Then for any small  $h > 0$  there exists  $n_0 \in \mathbb{N}$  such that  $|v_n(x)| \leq L_2 h$  for all  $x \in \partial B_{\hat{\gamma}^\circ}(0, r)$  and  $n \geq n_0$ , where  $L_2 := (N - 1)/r + 1$ .*

**Proof.** The proof of Proposition 2.3 yields that for any small  $h > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $\|v_n - (v^h - r)\|_{C(\overline{B_{\hat{\gamma}^\circ}(0, 2r)}}) \leq h$  for all  $n \geq n_0$ , where  $v^h$  is given by Lemma 2.5. As  $v^h - r = h(N - 1)/r$  on  $\partial B_{\hat{\gamma}^\circ}(0, r)$ , we have for each  $n \geq n_0$ ,

$$|v_n(x)| \leq \frac{h(N - 1)}{r} + h = L_2 h \quad \text{for all } x \in \partial B_{\hat{\gamma}^\circ}(0, r).$$

□

**Proof of Proposition 5.2.** *Step 1.* Note by (5.3) that  $\partial E_\mu \cap B(0, 6\delta)$  is a smooth surface for all  $|\mu| \leq L_1 \sqrt{h}$ . For each  $|\mu| \leq L_1 \sqrt{h}$  we take open sets  $V_1$  and  $V_2$  satisfying

$$\begin{aligned} V_1 &\subset B(0, 4\delta) \cap E_\mu, \quad V_2 \subset B(0, 4\delta) \cap \mathbb{R}^N \setminus E_\mu, \\ \partial V_1 \cap B(0, 3\delta) &= \partial V_2 \cap B(0, 3\delta) = \partial E_\mu \cap B(0, 3\delta), \quad \operatorname{int} V_1 \cap \operatorname{int} V_2 = \emptyset. \end{aligned}$$

Then there exists  $r_1 \in (0, \delta/4)$  such that  $d_{\hat{\gamma}^\circ, V_i}$  ( $i = 1, 2$ ) satisfies

$$\begin{aligned} d_{\hat{\gamma}^\circ, V_i} &\in C^2(\overline{V_3}), \quad d_{\hat{\gamma}^\circ, V_1} = d_{\hat{\gamma}^\circ, \mathbb{R}^N \setminus V_2} = d_{\hat{\gamma}^\circ, E_\mu} \quad \text{on } U_{2\delta, 2r_1}, \\ d_{\hat{\gamma}^\circ, \mathbb{R}^N \setminus V_2} &\leq d_{\hat{\gamma}^\circ, E_\mu} \leq d_{\hat{\gamma}^\circ, V_1} \quad \text{in } \mathbb{R}^N. \end{aligned}$$

Here  $V_3 := \{|d_{\hat{\gamma}^\circ, V_i}| < 3r_1\}$ .

*Step 2.* We approximate  $w_{\hat{\gamma}^\circ, E_\mu}^h$  by a smoother function on  $\overline{B(0, 6\delta)}$ .

Denote by  $\{\hat{\gamma}_n\}_{n=1}^{+\infty} \subset C^\infty(\mathbb{R}^N)$  a sequence satisfying (5.5) - (5.7). Let  $w_n$  be a classical solution of (5.8) with  $F = F_n := E_\mu \cap \overline{B(0, n)}$ . It follows from the proof of Proposition 2.3 that

$$(5.9) \quad w_n \longrightarrow w_{\hat{\gamma}^\circ, E_\mu}^h \quad \text{uniformly on } \overline{B(0, 6\delta)} \text{ as } n \rightarrow +\infty.$$

Thus we derive a local estimate for  $w_n$  and pass to the limit. Set  $w = w_{\hat{\gamma}^\circ, E_\mu}^h$ ,  $d = d_{\hat{\gamma}^\circ, E_\mu}$  and  $\tilde{d} = d_{\hat{\gamma}^\circ, V_1}$  for notational simplicity.

*Step 3.* Set  $h_{1,1} := r_1^2 R_1^2 / 5$ . For each  $h \in (0, h_{1,1})$  and  $n \in \mathbb{N}$  we construct two barriers  $\hat{v}_n$  and  $\bar{v}_n$  for  $w_n$  near  $\partial V_1$  from above such that  $\hat{v}_n$  and  $\bar{v}_n$  are viscosity supersolutions of (5.8) with  $F = \overline{V_1}$ .

Replacing  $r_1$  with a smaller one if necessary, we observe that for each  $y_0 \in \{\tilde{d} = -3r_1\}$  there is a point  $\pi(y_0) \in \{\tilde{d} < -3r_1\}$  satisfying

$$B_{\hat{\gamma}^\circ}(\pi(y_0), r_1) \subset \{\tilde{d} < -3r_1\}, \quad \partial B_{\hat{\gamma}^\circ}(\pi(y_0), r_1) \cap \{\tilde{d} = -3r_1\} = \{y_0\}.$$

Applying Lemma 5.1 with  $r = r_1$ , we see that for any  $h \in (0, h_{1,1})$  there exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  a classical solution  $v_n(\cdot; y_0)$  of (5.8) with  $F := \overline{B_{\widehat{\gamma}^\circ}(\pi(y_0), r_1)}$  satisfies

$$(5.10) \quad |v_n(x; y_0)| \leq L_2 h \quad \text{for all } x \in \partial B_{\widehat{\gamma}^\circ}(\pi(y_0), r_1).$$

Note that the choice of  $n_0$  is independent of  $y_0 \in \{\widetilde{d} = -3r_1\}$ . Hence set  $n_0 = 1$  for simplicity. Since  $v_n(\cdot; y_0) - 3r_1$  is a classical solution of (5.8) with replacing  $d_{\widehat{\gamma}^\circ, F}$  with  $d_{\widehat{\gamma}^\circ, \overline{B_{\widehat{\gamma}^\circ}(\pi(y_0), r_1)}} - 3r_1$ , setting

$$\widehat{v}_n(x) := \inf_{y_0 \in \{\widetilde{d} = -3r_1\}} \{v_n(x; y_0) - 3r_1\},$$

we see that  $\widehat{v}_n$  is a viscosity supersolution of (5.8) with  $F = \overline{V_1}$  since  $d_{\widehat{\gamma}^\circ, \overline{B_{\widehat{\gamma}^\circ}(\pi(y_0), r_1)}} - 3r_1 \geq \widetilde{d}$  in  $\mathbb{R}^N$ . Moreover, we observe by (5.10) that  $\widehat{v}_n - \widetilde{d} \leq L_2 h$  on  $\{\widetilde{d} = -3r_1\}$  for all  $n \in \mathbb{N}$  and  $h \in (0, h_{1,1})$ .

Similarly we can construct a viscosity supersolution  $\overline{v}_n$  of (5.8) with  $F = \overline{V_1}$  satisfying  $\overline{v}_n - \widetilde{d} \leq L_1 h$  on  $\{\widetilde{d} = 3r_1\}$  for all  $n \geq n_0$ . Therefore, we have the desired functions  $\widehat{v}_n$  and  $\overline{v}_n$ .

*Step 4.* Fix  $\varepsilon > 0$ . We select a small  $h_1 > 0$  independent of  $n$  and construct a global barrier  $\overline{w}_n$  for  $w_n$  approximating  $w_n$  in  $\overline{U_{\delta, r}}$  such that  $\overline{w}_n$  is a viscosity supersolution of (5.8) with  $F := \overline{V_1}$  for all  $h \in (0, h_1)$ .

We observe by geometry that  $\|\kappa_{\widehat{\gamma}^\circ, \overline{V_1}}\|_{C(\overline{V_3})} \leq L_{3,1}$ . Here and in the sequel,  $L_{3,j}$  ( $j \in \mathbb{N}$ ) is a positive constant independent of  $\varepsilon, h > 0$ . Let  $\rho_\tau$  ( $\tau > 0$ ) be a mollifying kernel and set  $\kappa_\tau := \rho_\tau * \kappa_{\widehat{\gamma}^\circ, \overline{V_1}}$ . We take  $\tau = \tau(\varepsilon) \searrow 0$  as  $\varepsilon \rightarrow 0$  satisfying

$$(5.11) \quad \|\kappa_\tau - \kappa_{\widehat{\gamma}^\circ, \overline{V_1}}\|_{C(\overline{V_3})} \leq \varepsilon, \quad \|\kappa_\tau\|_{C(\overline{V_3})} \leq 2L_{3,1},$$

$$(5.12) \quad \tau^k \|\nabla^k \kappa_\tau\|_{C(\overline{V_3})} \leq L_{3,2} \quad \text{for } k = 1, 2.$$

Let  $\eta$  be a smooth cut-off function such that

$$(5.13) \quad \begin{aligned} 0 \leq \eta \leq 1 \quad \text{in } \mathbb{R}^N, \quad \eta &= \begin{cases} 1 & \text{on } \{|\widetilde{d}| \leq r_1\}, \\ 0 & \text{on } \{|\widetilde{d}| > 2r_1\}, \end{cases} \\ \|\nabla \eta\|_{L^\infty(\mathbb{R}^N)} + \|\nabla^2 \eta\|_{L^\infty(\mathbb{R}^N)} &\leq L_{3,3}. \end{aligned}$$

Define

$$\widetilde{w}_1 := \widetilde{d} - h\eta\kappa_\tau + L_{3,4}h(1 - \eta) + 2\varepsilon h \quad \text{on } \overline{V_3}, \quad L_{3,4} := 2 \max\{L_2, L_{3,1}\}.$$

Since  $\widetilde{w}_1$  is smooth on  $\overline{V_3}$ , we calculate that

$$\begin{aligned} \nabla \widetilde{w}_1 &= \nabla \widetilde{d} - h\nabla(\eta\kappa_\tau) - L_{3,4}h\nabla\eta, \\ \nabla^2 \widetilde{w}_1 &= \nabla^2 \widetilde{d} - h\nabla^2(\eta\kappa_\tau) - L_{3,4}h\nabla^2\eta. \end{aligned}$$

It follows from (5.6), (5.11), (5.12) and (5.13) that for some  $h_{1,2} > 0$

$$\frac{\lambda}{2} \leq \widehat{\gamma}_n(\nabla \widetilde{w}_1) \leq 2\Lambda \quad \text{on } \overline{V_3} \quad \text{for all } n \in \mathbb{N} \text{ and } h \in (0, h_{1,2}).$$

We see by (5.6) and the continuity of  $\nabla^2\gamma$  that for any  $h \in (0, h_{1,2})$  there exists  $n_1 \in \mathbb{N}$  such that

$$(5.14) \quad \begin{aligned} & \|\nabla^2\hat{\gamma}_n(\nabla\tilde{w}_1) - \nabla^2\hat{\gamma}_n(\nabla\tilde{d})\|_{C(\overline{V_3})} \\ & \leq \|\nabla^2\hat{\gamma}_n(\nabla\tilde{w}_1) - \nabla^2\hat{\gamma}(\nabla\tilde{w}_1)\|_{C(\overline{V_3})} + \|\nabla^2\hat{\gamma}(\nabla\tilde{w}_1) - \nabla^2\hat{\gamma}(\nabla\tilde{d})\|_{C(\overline{V_3})} \\ & \quad + \|\nabla^2\hat{\gamma}(\nabla\tilde{d}) - \nabla^2\hat{\gamma}_n(\nabla\tilde{d})\|_{C(\overline{V_3})} \\ & \leq 2h + \omega_1(L_{3,5}h(\tau^{-1} + 1)), \end{aligned}$$

$$(5.15) \quad \begin{aligned} & \|\kappa_{\hat{\gamma}_n, \overline{V_3}} - \kappa_{\hat{\gamma}, \overline{V_3}}\|_{C(\overline{V_3})} \\ & = \|\text{tr}\{\nabla^2\hat{\gamma}_n(\nabla\tilde{d})\nabla^2\tilde{d}\} - \text{tr}\{\nabla^2\hat{\gamma}(\nabla\tilde{d})\nabla^2\tilde{d}\}\|_{C(\overline{V_3})} \leq h, \end{aligned}$$

for all  $n > n_1$ , some  $L_{3,5}$  and  $\omega_1 \in C([0, +\infty))$ . Here  $\omega_1$  denotes the modulus of continuity of  $\nabla^2\gamma$  on  $\{\lambda/2 \leq \gamma^\circ \leq 2\lambda\}$ . Further calculations with using (5.14) and (5.15) yield that

$$\begin{aligned} \text{div}\nabla\hat{\gamma}_n(\nabla\tilde{w}_1) & = \text{tr}\{\nabla^2\hat{\gamma}_n(\nabla\tilde{w}_1)\nabla^2\tilde{w}_1\} \\ & \leq -\kappa_{\hat{\gamma}, V_3} + h(3 + L_{3,6}\tau^{-2}) + \omega_1(L_{3,5}h(\tau^{-1} + 1)) \end{aligned}$$

for some  $L_{3,6} > 0$ . Hence we observe by (5.11),  $d \leq \tilde{d}$  in  $\mathbb{R}^N$  and the choice of  $L_{3,4}$  that on  $\overline{V_3}$ ,

$$\begin{aligned} \tilde{w}_1 - h \text{div}\nabla\hat{\gamma}_n(\nabla\tilde{w}_1) & \geq \tilde{d} - h(\kappa_\tau - \kappa_{\hat{\gamma}^\circ, V_3}) + 2\varepsilon h \\ & \quad - h\{h(3 + L_{3,6}\tau^{-2}) + \omega_1(L_{3,5}h(\tau^{-1} + 1))\} \\ & \geq d + \varepsilon h - h\{h(3 + L_{3,6}\tau^{-2}) + \omega_1(L_{3,5}h(\tau^{-1} + 1))\}. \end{aligned}$$

Choosing  $h_1 \leq \min\{h_{1,1}, h_{1,2}\}$  so small that  $h(3 + L_{3,6}\tau^{-2}) + \omega_1(L_{3,5}h(\tau^{-1} + 1)) \leq \varepsilon$  for all  $h \in (0, h_1)$ , we conclude that  $\tilde{w}_1$  is a classical supersolution of (5.8) in  $V_3$ .

Define

$$\overline{w}_n := \begin{cases} \min\{\tilde{w}_1, \hat{v}_n, \overline{v}_n\} & \text{in } V_3, \\ \min\{\hat{v}_n, \overline{v}_n\} & \text{on } \mathbb{R}^N \setminus V_3. \end{cases}$$

Noting  $\tilde{w}_1 > \max\{\hat{v}_n, \overline{v}_n\}$  on  $\{|d| = 3r_1\}$ , we see that  $\overline{w}_n$  is a viscosity supersolution of (5.8) with  $F = V_1$  for all  $n \geq n_1$  (cf. [21]). Therefore, we have the desired function  $\overline{w}_n$ .

*Step 5.* We derive (5.4).

The comparison principle for viscosity solutions of (5.8) yields that  $w_n \leq \overline{w}_n$  in  $\mathbb{R}^N$  for large  $n > n_1$ . Consequently, we obtain  $w_n \leq \tilde{w}_1$  on  $\overline{U_{\delta,r}}$  for  $n > n_1$  and  $r = r_1$ . Hence letting  $n \rightarrow +\infty$ , we get by  $d = \tilde{d}$  on  $\overline{U_{\delta,r}}$  and (5.9),

$$w \leq d - h\kappa_\tau + 2\varepsilon h \leq d - h\kappa_{\hat{\gamma}^\circ, E_\mu} + 3\varepsilon h \quad \text{on } \overline{U_{\delta,r}}$$

for all  $h \in (0, h_1)$ . A similar argument to the above gives

$$w \geq d - h\kappa_{\hat{\gamma}^\circ, E_\mu} - 3\varepsilon h \quad \text{on } \overline{U_{\delta,r}} \quad \text{for all } h \in (0, h_1).$$

Therefore we have obtained the result.  $\square$

Recall by (5.3) that the level set  $\{\psi = \mu\} \cap B(0, 6\delta)$  is a smooth surface for each  $|\mu| \leq L_1\sqrt{h}$ . We approximate this level set as follows. Taking (5.3) into account, we observe by Implicit function theorem that for each  $|\mu| \leq L_1\sqrt{h}$  there is a smooth function  $y_N = f_\mu(y')$  for  $y' \in B(0, 6\delta) \cap \mathbb{R}^{N-1}$  satisfying  $\psi(y', f_\mu(y')) = \mu$  for  $y' \in B(0, 6\delta) \cap \mathbb{R}^{N-1}$ . Besides, we see that at  $y = (y', y_N) = (0, f_\mu(0))$

$$(5.16) \quad \psi_{y_i} + \psi_{y_N} f_{\mu, y_i} = 0,$$

$$(5.17) \quad \psi_{y_i y_j} + \psi_{y_i y_N} f_{\mu, y_j} + \psi_{y_N y_j} f_{\mu, y_i} + \psi_{y_N y_N} f_{\mu, y_i} f_{\mu, y_j} + \psi_{y_N} f_{\mu, y_i y_j} = 0,$$

for  $i, j = 1, 2, \dots, N-1$ .

We estimate  $f_\mu(0)$  in terms of  $\mu$  and the derivatives of  $\psi$ . It follows from Taylor's theorem and (5.2) that

$$\mu = \psi(y) = \psi(0) + \langle \nabla \psi(0), y \rangle + \frac{1}{2} \langle \nabla^2 \psi(\theta y) y, y \rangle = |\nabla \psi(0)| y_N + \frac{1}{2} \langle \nabla^2 \psi(\theta y) y, y \rangle$$

for some  $\theta \in (0, 1)$ . Substituting  $y = (0, f_\mu(0))$ , we get

$$(5.18) \quad f_\mu(0) = \frac{2\mu}{|\nabla \psi(0)| + \sqrt{|\nabla \psi(0)|^2 + 2\alpha\mu}}, \quad \alpha := \psi_{y_N y_N}(0, \theta f_\mu(0)).$$

Some calculations yield that

$$\left| f_\mu(0) - \frac{\mu}{|\nabla \psi(0)|} \right| \leq \frac{L_{4,1}\mu^2}{|\nabla \psi(0)|} \quad \text{for all } |\mu| \leq L_1\sqrt{h}.$$

Here and in the sequel  $L_{4,j}$  ( $j \in \mathbb{N}$ ) is a positive constant depending on  $|\nabla \psi(0)|^{-1}$  and  $\|\nabla^k \psi\|_{C(B(0,6\delta))}$  ( $k = 2, 3$ ), but not on  $\mu$ . Thus it follows from (5.16), (5.17) and this estimate that for  $i, j = 1, 2, \dots, N$

$$|f_{\mu, y_i}(0)| \leq \frac{L_{4,2}\mu}{|\nabla \psi(0)|}, \quad \left| f_{\mu, y_i y_j}(0) - \frac{\psi_{y_i y_j}(0)}{|\nabla \psi(0)|} \right| \leq \frac{L_{4,3}\mu}{|\nabla \psi(0)|}.$$

Thus we use Taylor's theorem, Young's inequality and these estimates to obtain

$$\left| f_\mu(y') - \left( \frac{\mu}{|\nabla \psi(0)|} + \frac{1}{2|\nabla \psi(0)|} \langle \nabla^2 \psi(0) y', y' \rangle \right) \right| \leq \frac{L_4}{|\nabla \psi(0)|} (|y'|^3 + |\mu|^{3/2})$$

for all  $y' \in B(0, 6\delta) \cap \mathbb{R}^{N-1}$  and some  $L_4 > 0$  independent of  $|\mu| \leq L_1\sqrt{h}$ . Consequently, we obtain

$$(5.19) \quad \begin{aligned} \{\psi_- \geq \mu - L_4|\mu|^{3/2}\} \cap B(0, 6\delta) &\subset \{\psi = \mu\} \cap B(0, 6\delta) \\ &\subset \{\psi_+ \leq \mu + L_4|\mu|^{3/2}\} \cap B(0, 6\delta), \end{aligned}$$

$$(5.20) \quad \psi_\pm = \psi_\pm(y) := |\nabla \psi(0)| y_N - \left( \frac{1}{2} \langle \nabla^2 \psi(0) y', y' \rangle \pm L_4 |y'|^3 \right),$$

$$\nabla^2 \psi := (\psi_{y_i y_j})_{1 \leq i, j \leq N-1}.$$

Figure 1 roughly shows the inclusions (5.19).

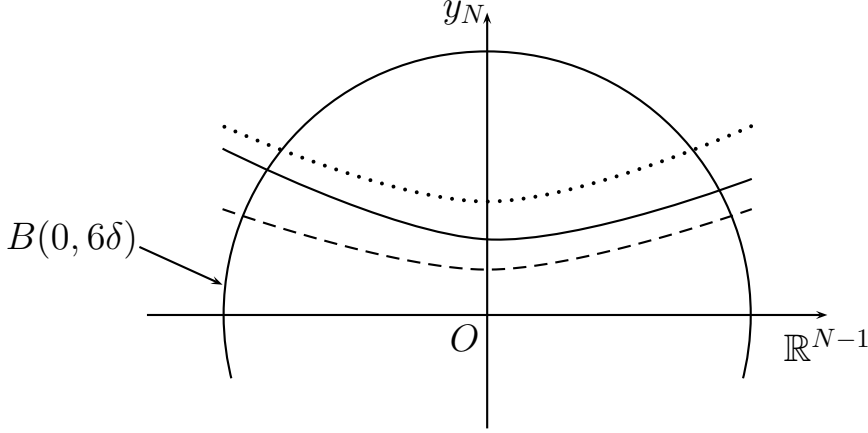


Figure 1: The level sets  $\{\psi = \mu\}$ ,  $\{\psi_+ = \mu + L_4|\mu|^{3/2}\}$  (dotted) and  $\{\psi_- = \mu - L_4|\mu|^{3/2}\}$  (dashed)

In this setting we obtain the following estimate for  $S_{\hat{\gamma}^\circ, h}\psi$ .

**Theorem 5.1.** *Let  $\hat{\gamma}$ ,  $\psi$  and  $E$  be the same as in Proposition 5.2. Then for any  $\varepsilon > 0$  there exists  $h_2 > 0$  such that*

$$|[S_{\hat{\gamma}^\circ, h}\psi](0) - h\hat{\gamma}(\nabla\psi(0))\kappa_{\hat{\gamma}^\circ, E}(0)| \leq L_5\varepsilon h \quad \text{for all } h \in (0, h_2),$$

where  $L_5$  is independent of  $\varepsilon$ ,  $h > 0$ .

**Proof.** Note by (4.9) and (5.3) that  $|[S_{\hat{\gamma}^\circ, h}\psi](0)| \leq L_1\sqrt{h}$  for any small  $h > 0$ . Hence we can apply Proposition 5.2 to the set  $\{\psi \geq \mu\}$  for  $|\mu| \leq L_1\sqrt{h}$ .

*Step 1.* Set  $L_{5,1} := \|\kappa_{\hat{\gamma}^\circ, E_\mu}\|_{C(\bar{V}_3)} + 1$  and  $L_{5,2} := 2\Lambda L_{5,1}|\nabla\psi(0)|$ . We show  $|[S_{\hat{\gamma}^\circ, h}\psi](0)| \leq L_{5,2}h$  for small  $h > 0$ .

Set  $\mu_0 := [S_{\hat{\gamma}^\circ, h}\psi](0)$  and  $E_{\mu'} := \{\psi \geq \mu'\}$  for  $|\mu'| \leq L_1\sqrt{h}$ . Applying (5.4), we have  $w_{\hat{\gamma}^\circ, E_{\mu'}} \geq d_{\hat{\gamma}^\circ, E_{\mu'}} - L_{5,1}h$  on  $\bar{U}_{\delta, r}$ . We see by (2.3) and  $0 \notin E_{\mu'}$  that for small  $h > 0$ ,

$$w_{\hat{\gamma}^\circ, E_{\mu'}}(0) \geq d_{\hat{\gamma}^\circ, E_{\mu'}}(0) - L_{5,1}h = \inf_{y \in E_{\mu'}} \hat{\gamma}^\circ(y) - L_{5,1}h \geq \frac{1}{\Lambda} \inf_{y \in E_{\mu'}} |y| - L_{5,1}h.$$

Note by (5.19) that  $\inf_{y \in E_{\mu'}} |y| \geq y_{0N}$ , where  $y_0 := (0, \dots, 0, y_{0N}) \in \{\psi_- = \mu' - L_4|\mu'|^{3/2}\}$ . It follows from the definition (5.20) of  $\psi_-$  that

$$y_{0N} \geq \frac{(1 - L_4h^{1/4})\mu'}{|\nabla\psi(0)|}.$$

Therefore, choosing small  $h_{2,1} > 0$  such that  $L_4h^{1/4} \leq 1/2$  for all  $h \in (0, h_{2,1})$ , we get

$$\frac{y_{0N}}{\Lambda} - L_{5,1}h \geq \frac{\mu'}{2\Lambda|\nabla\psi(0)|} - L_{5,1}h \geq \frac{1}{4\Lambda|\nabla\psi(0)|}(\mu' - L_{5,2}h).$$

Taking  $\mu' = L_{5,2}h$ , we obtain  $\mu_0 \leq L_{5,2}h$  for small  $h \in (0, h_{2,1})$ .

Since we show by the same way as above that  $\mu_0 \geq -L_{5,2}h$  for all  $h \in (0, h_{2,2})$  and some  $h_{2,2} > 0$ , we obtain the claim of this step.

*Step 2.* Define  $\widehat{X} := \{\psi \geq [S_{\widehat{\gamma}^\circ, h}\psi](0)\}$ . We prove  $\widehat{X} \in \mathcal{B}_{\widehat{\gamma}^\circ, h}$ .

Notice by (4.7) and the proof of Proposition 4.5 that the supremum of  $[S_{\widehat{\gamma}^\circ, h}\psi](0)$  is attained by  $[S_{\widehat{\gamma}^\circ, h}\psi](0)$  itself. Thus  $w_{\widehat{\gamma}^\circ, \widehat{X}}^h(0) \leq 0$ . Moreover, we observe that  $w_{\widehat{\gamma}^\circ, \widehat{X}}^h(0) = 0$ . Therefore, it follows from  $\{-d_{\widehat{\gamma}^\circ, \widehat{X}} \geq 0\} = \widehat{X}$  and this inequality that  $w_{\widehat{\gamma}^\circ, \{-d_{\widehat{\gamma}^\circ, \widehat{X}} \geq 0\}}^h(0) = w_{\widehat{\gamma}^\circ, \widehat{X}}^h(0) \leq 0$ . Hence  $[S_{\widehat{\gamma}^\circ, h}(-d_{\widehat{\gamma}^\circ, \widehat{X}})](0) \geq 0$  and this means  $\widehat{X} \in \mathcal{B}_{\widehat{\gamma}^\circ, h}$ .

*Step 3.* We prove the assertion of Theorem 5.1.

We observe by Proposition 5.1, (5.19) and the result of Step 2 that

$$(5.21) \quad \inf_{X_- \cap U_1} \psi(y) \leq [S_{\widehat{\gamma}^\circ, h}\psi](0) = \inf_{y \in \widehat{X} \cap U_1} \psi(y) \leq \inf_{X_+ \cap U_1} \psi(y),$$

for small  $h > 0$ , where  $X_\pm := \{\psi_\pm \geq \mu_0 \pm L_4|\mu_0|^{3/2}\}$ . Note by (5.19) and (5.20) that  $d_H(X_\pm \cap B(0, 6\delta), \widehat{X} \cap B(0, 6\delta)) \leq L_4|\mu_0|^{3/2}/|\nabla\psi(0)|$ . By suitable modifications of  $X_\pm \cap (\mathbb{R}^N \setminus U_1)$ , we may consider  $X_+ \subset \widehat{X} \subset X_-$  and

$$(5.22) \quad d_H(X_\pm, \widehat{X}) \leq \frac{L_4|\mu_0|^{3/2}}{|\nabla\psi(0)|}.$$

Here  $d_H$  denotes the Hausdorff distance defined by (3.2). It is easily seen that  $\inf_{y \in X_+ \cap U_1} \psi(y)$  (resp.,  $\inf_{y \in X_- \cap U_1} \psi(y)$ ) is attained on  $\partial X_+ \cap U_1$  (resp.,  $\partial X_- \cap U_1$ ).

We estimate  $\widehat{\mu} := d_{\widehat{\gamma}^\circ, X_+}(0)$  by use of Proposition 5.2. We may assume  $0 \notin X_+$  because if otherwise, the desired result is derived by the same way. From the result of Step 1 we can find a point  $\widehat{y} \in \{\psi = [S_{\widehat{\gamma}^\circ, h}\psi](0)\} \cap U_1$  satisfying  $d_{\widehat{\gamma}^\circ, X_+}(0) = \widehat{\gamma}^\circ(\widehat{y})$  (cf. Figure 2 below). Since  $\mathbf{n}_{\widehat{\gamma}^\circ, X_+}(y)$  is an outward vector for each  $y \in \partial X_+$ , it follows from the smoothness of  $\{\psi = [S_{\widehat{\gamma}^\circ, h}\psi](0)\}$  and Proposition 2.1 that  $\widehat{y} = -\widehat{\mu}\mathbf{n}_{\widehat{\gamma}^\circ, X_+}(\widehat{y})$ .

We get  $|\widehat{y}| \leq 4\Lambda L_{5,1}h/\lambda$ . Indeed, since  $(-2\mu/|\nabla\psi(0)|)e_N \in X_+$  for small  $h > 0$ , we see by (2.3) that

$$\widehat{\mu} = d_{\widehat{\gamma}^\circ, X_+}(0) = \widehat{\gamma}^\circ(\widehat{y}) \leq \widehat{\gamma}^\circ \left( \frac{2L_{5,2}h}{|\nabla\psi(0)|} \mathbf{n}_{\widehat{\gamma}^\circ, X_+}(0) \right) \leq \frac{4\Lambda L_{5,1}h}{\lambda}.$$

Hence  $|\widehat{y}| \leq 4\Lambda L_{5,1}h/\lambda$ .

Since the inequality  $d_{\widehat{\gamma}^\circ, X_+} \leq d_{\widehat{\gamma}^\circ, \widehat{X}} + L_4|\mu_0|^{3/2}/|\nabla\psi(0)|$  in  $\mathbb{R}^N$  follows from (5.22), we get  $w_{\widehat{\gamma}^\circ, X_+}^h \leq w_{\widehat{\gamma}^\circ, \widehat{X}}^h + L_4|\mu_0|^{3/2}/|\nabla\psi(0)|$  in  $\mathbb{R}^N$  by Theorem 8.1. Especially,  $w_{\widehat{\gamma}^\circ, X_+}^h(0) \leq L_4|\mu_0|^{3/2}/|\nabla\psi(0)|$  because  $w_{\widehat{\gamma}^\circ, \widehat{X}}^h(0) = 0$ . Hence we have from (5.4) and  $|\mu_0| \leq L_{5,2}h$

$$\widehat{\mu} = d_{\widehat{\gamma}^\circ, \widehat{X}}(0) \leq h\kappa_{\widehat{\gamma}^\circ, \widehat{X}}(0) + \varepsilon h + \frac{L_{5,3}h^{3/2}}{|\nabla\psi(0)|}.$$

Here and hereafter  $L_{5,j}$  ( $j \geq 3$ ) denotes a positive constant independent of  $\varepsilon$ ,  $h > 0$ . Using (5.2) and Taylor's theorem, we observe that

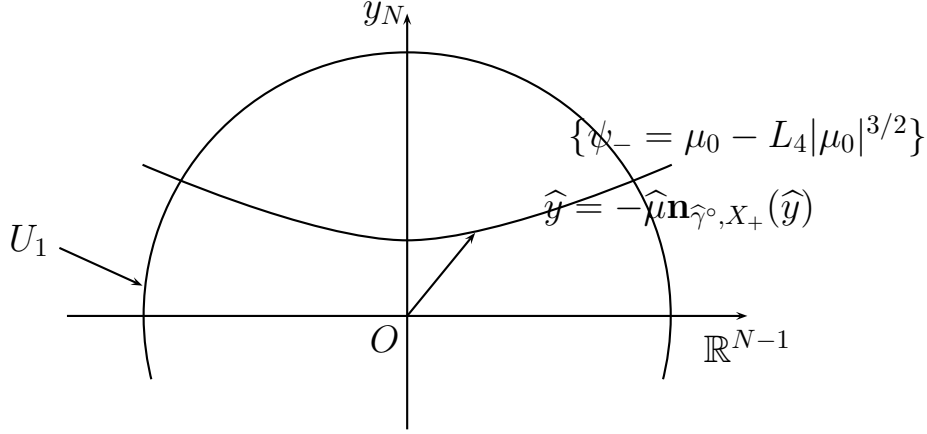


Figure 2: The level set  $\{\psi_- = \mu_0 - L_4 |\mu_0|^{3/2}\}$  and the point  $\hat{y}$

$$\begin{aligned} \inf_{y \in X_+ \cap U_1} \psi(y) &= \psi(-\hat{\mu} \mathbf{n}_{\hat{\gamma}^\circ, X_+}(\hat{y})) = -\hat{\mu} \langle \nabla \psi(0), \mathbf{n}_{\hat{\gamma}^\circ, X_+}(\hat{y}) \rangle + \frac{1}{2} \langle \nabla^2 \psi(\theta \hat{y}) \hat{y}, \hat{y} \rangle \\ &\leq -\hat{\mu} \langle \nabla \psi(0), \mathbf{n}_{\hat{\gamma}^\circ, X_+}(\hat{y}) \rangle + L_{5,4} h^2 \end{aligned}$$

for some  $\theta \in (0, 1)$  and some  $L_{5,4} > 0$ . Moreover, since the identity  $\langle \nabla \hat{\gamma}(p), p \rangle = \hat{\gamma}(p)$  for  $p \in \mathbb{R}^N \setminus \{0\}$  holds from  $(\gamma 1)$  and  $(\gamma 2)$ , we observe by  $(\gamma 3)$ , Lemma 2.1 and this identity that

$$\begin{aligned} |\langle \mathbf{n}_{\hat{\gamma}^\circ, X_+}(\hat{y}) - \mathbf{n}_{\hat{\gamma}^\circ, E}(0), e_N \rangle| &= |\langle \nabla \hat{\gamma}(\nabla \rho_{X_+}(\hat{y})) - \nabla \hat{\gamma}(\nabla \rho_{X_+}(0)), -\nabla \rho_E(0) \rangle| \\ &= |\langle \nabla \hat{\gamma}(\nabla \rho_{X_+}(\hat{y})), \nabla \rho_{X_+}(\hat{y}) - \nabla \rho_{X_+}(0) \rangle \\ &\quad + \hat{\gamma}(\nabla \rho_{X_+}(0)) - \hat{\gamma}(\nabla \rho_{X_+}(\hat{y}))| \\ &\leq 2\Lambda \|\nabla^2 \rho_{X_+}\|_{C(\overline{B(0, \delta)})} |\hat{y}|. \end{aligned}$$

where  $\rho_E$  (resp.,  $\rho_{X_+}$ ) is the signed distance functions to  $E = \{\psi \geq 0\}$  (resp.,  $X_+$ ) in the Euclidean sense and  $e_N = -\nabla \rho_E(0) = -\nabla \rho_{X_+}(0)$ . Note by geometry that  $\sup_{\varepsilon, h > 0} \|\nabla^2 \rho_{E_\mu}\|_{C(\overline{B(0, \delta)})} < +\infty$ . Using  $|\hat{y}| \leq 4\Lambda L_{5,1} h / \lambda$  and the above estimate, we have

$$|\langle \mathbf{n}_{\hat{\gamma}^\circ, X_+}(\hat{y}) - \mathbf{n}_{\hat{\gamma}^\circ, E}(0), e_N \rangle| \leq L_{5,5} h.$$

Therefore, noting that  $\mathbf{n}_{\hat{\gamma}^\circ, E}(0) = \nabla \hat{\gamma}(\nabla d_{\hat{\gamma}^\circ, E}(0)) = \nabla \hat{\gamma}(-\nabla \psi(0))$  and  $\nabla \psi(0) = |\nabla \psi(0)| e_N$ , we have

$$\begin{aligned} (5.23) \quad -\hat{\mu} \langle \nabla \psi(0), \mathbf{n}_{\hat{\gamma}^\circ, X_+}(y_0) \rangle &\leq \hat{\mu} \langle -\nabla \psi(0), \nabla \hat{\gamma}(-\nabla \psi(0)) \rangle \\ &\quad + |\hat{\mu}| L_{5,5} |\nabla \psi(0)| h \\ &\leq h \hat{\gamma}(\nabla \psi(0)) \kappa_{\hat{\gamma}^\circ, X_+} + \varepsilon h + \Lambda L_{5,3} h^{3/2} + L_{5,6} h^2. \end{aligned}$$

Since the equality  $\kappa_{\hat{\gamma}^\circ, X_+} = \kappa_{\hat{\gamma}^\circ, E}$  follows from the definition of  $\psi_+$ , choosing  $h_2 \leq \min\{h_{2,1}, h_{2,2}\}$  and  $L_5 > 0$ , we obtain from (5.21) and (5.23)  $[S_{\hat{\gamma}^\circ, h} \psi](0) \leq h \hat{\gamma}(\nabla \psi(0)) \kappa_{\hat{\gamma}^\circ, E}(0) + L_5 \varepsilon h$  for any  $h \in (0, h_2)$ .

The same argument as above yields that  $[S_{\hat{\gamma}^\circ, h} \psi](0) \geq h \hat{\gamma}(\nabla \psi(0)) \kappa_{\hat{\gamma}^\circ, E}(0) - L_5 \varepsilon h$  for all  $h \in (0, h_2)$ .  $\square$

Now we obtain the generator of  $S_{\gamma^\circ, h}$ .

**Theorem 5.2.** *Let  $\phi \in C^2(\mathbb{R}^N)$ ,  $z \in \mathbb{R}^N$  and  $\varepsilon > 0$ . Assume that  $\nabla\phi(z) \neq 0$ . Then there exist  $\delta > 0$  and  $h_0 > 0$  such that for all  $x \in B(z, \delta)$  and  $h \in (0, h_0)$ ,*

$$\begin{aligned} [S_{\gamma^\circ, h}\phi](x) &\leq \phi(x) + \{-F(\nabla\phi(x), \nabla^2\phi(x)) + L_5\varepsilon\}h, \\ [S_{\gamma^\circ, h}\phi](x) &\geq \phi(x) + \{-F(\nabla\phi(x), \nabla^2\phi(x)) - L_5\varepsilon\}h, \end{aligned}$$

where  $F(p, X) := -\gamma(p)\text{tr}(\nabla^2\gamma(p)X)$  and  $L_5$  is the constant in Theorem 5.1.

**Proof.** We prove only the first inequality because the second one is similarly obtained.

*Step 1.* In view of  $|\nabla\phi(z)| \neq 0$ , there exists  $\delta > 0$  such that for all  $x \in B(z, 3\delta)$

$$\frac{1}{2}|\nabla\phi(z)| \leq |\nabla\phi(x)| \leq 2|\nabla\phi(z)|.$$

It is easily seen that

$$[S_{\gamma^\circ, h}\phi](x) = \phi(x) + \sup_{X \in \mathcal{B}_{\gamma^\circ, h}} \inf_{y \in X} \{\phi(x+y) - \phi(x)\}.$$

Let  $\{U(x)\}_{x \in B(z, \delta)} \subset O(N)$  be a continuous family satisfying  $U(x)\nabla\phi(x) = |\nabla\phi(x)|e_N$  for all  $x \in B(z, \delta)$ . Replacing  $y$  with  $U^*(x)y$  in the above formula, we have

$$[S_{\gamma^\circ, h}\phi](x) = \phi(x) + \sup_{X \in \mathcal{B}_{\gamma_{U(x)}, h}} \inf_{y \in X} \{\phi(x + U^*(x)y) - \phi(x)\}.$$

Define

$$\begin{aligned} \psi(y) &:= \phi(x + U^*(x)y) - \phi(x), \quad E := \{\psi \geq 0\}, \\ d_{\gamma_{U(x), E}^\circ} &:= \gamma_{U(x)}^\circ\text{-signed distance function to } \partial E. \end{aligned}$$

Then  $\psi$  satisfies (5.2) and (5.3).

*Step 2.* Applying Theorem 5.1 with  $\hat{\gamma} = \gamma_{U(x)}$ , we observe that for any  $\varepsilon > 0$  there exists  $h_2 > 0$  such that

$$[S_{\gamma_{U(x)}^\circ, h}\psi](0) \leq h\gamma_{U(x)}(\nabla\psi(0))\kappa_{\gamma_{U(x), E}^\circ}(0) + L_5\varepsilon h.$$

We derive

$$(5.24) \quad \kappa_{\gamma_{U(x), E}^\circ}(0) = \text{tr}\{\nabla^2\gamma(\nabla\phi(x))\nabla^2\phi(x)\}.$$

It is clear that  $\gamma_{U(x)}(\nabla\psi(0)) = \gamma(\nabla\phi(x))$  and that

$$\kappa_{\gamma_{U(x), E}^\circ}(0) = \text{tr}\{\nabla^2\gamma_{U(x)}(\nabla d_{\gamma_{U(x), E}^\circ}(0))\nabla^2 d_{\gamma_{U(x), E}^\circ}(0)\}.$$

Some calculations yield that

$$\begin{aligned} \nabla d_{\gamma_{U(x), E}^\circ}(y) &= \frac{\nabla\psi(y)}{\gamma_{U(x)}(\nabla\psi(y))} \\ \nabla^2 d_{\gamma_{U(x), E}^\circ}(y) &= \frac{\nabla^2\psi(y)}{\gamma_{U(x)}(\nabla\psi(y))} - \frac{\nabla\psi(y) \otimes \nabla\gamma_{U(x)}(\nabla\psi(y))\nabla^2\psi(y)}{(\gamma_{U(x)}(\nabla\psi(y)))^2}. \end{aligned}$$



Set  $p := \nabla\psi(0)$  and  $A := \nabla^2\psi(0)$ . Then we have

$$\operatorname{tr}\{\nabla^2\gamma_{U(x)}(\nabla d_{\gamma^\circ,E}(0))\nabla^2 d_{\gamma^\circ,E}(0)\} = \operatorname{tr}\left[\nabla^2\gamma_{U(x)}(p)\left\{A - \frac{p \otimes \nabla\gamma(p)A}{\gamma_{U(x)}(p)}\right\}\right].$$

Here we have used  $\nabla^2\gamma_{U(x)}(p/a) = a\nabla^2\gamma_{U(x)}(p)$  for all  $p \in \mathbb{R}^N \setminus \{0\}$  and  $a > 0$ . We see by the choice of  $\{U(x)\}_{x \in B(z,\delta)}$  that

$$\begin{aligned}\nabla^2\gamma_{U(x)}(p) &= U(x)\nabla^2\gamma(p)U^*(x), \\ p = \nabla\psi(0) &= |\nabla\psi(0)|e_N = U(x)\nabla\phi(x), \\ A = \nabla^2\psi(0) &= U(x)\nabla^2\phi(x)U^*(x).\end{aligned}$$

Thus we get

$$\begin{aligned}\operatorname{tr}\{\nabla^2\gamma_{U(x)}^\circ(p)A\} &= \operatorname{tr}\{U(x)\nabla^2\gamma^\circ(\nabla\phi(x))U^*(x)U(x)\nabla^2\phi(x)U^*(x)\} \\ &= \operatorname{tr}\{\nabla^2\gamma^\circ(\nabla\phi(x))\nabla^2\phi(x)\}.\end{aligned}$$

On the other hand, it follows from  $\nabla^2\gamma(\nabla\phi(x))\nabla\phi(x) = 0$  (cf. Lemma 2.2) that

$$\begin{aligned}\operatorname{tr}\{\nabla^2\gamma_{U(x)}(p)p \otimes \nabla\gamma(p)A\} &= \langle A\nabla^2\gamma_{U(x)}(p)p, \nabla\gamma_{U(x)}(p) \rangle \\ &= \langle U(x)\nabla^2\phi(x)\nabla^2\gamma(\nabla\phi(x))\nabla\phi(x), U(x)\nabla\gamma(\phi(x)) \rangle \\ &= 0.\end{aligned}$$

Therefore, we obtain (5.24).

*Step 3.* Taking  $h_0 \leq \min\{h_1, h_2\}$  sufficiently small, we obtain

$$[S_{\gamma_{U(x)}^\circ, h\psi}](0) \leq -F(\nabla\phi(x), \nabla^2\phi(x)) + L_5\varepsilon h.$$

for all  $h \in (0, h_0)$ . Consequently, we have the first inequality of Theorem 5.2 for all  $h \in (0, h_0)$ .  $\square$

We use the following lemma to estimate the continuity of  $S_{\gamma^\circ, h}^n u$  for  $u \in UC(\mathbb{R}^N)$  and  $n \in \mathbb{N}$ .

**Lemma 5.2.** *For any  $x \in \mathbb{R}^N$ ,  $h > 0$  and  $n \in \mathbb{N}$  we have*

$$(5.25) \quad [S_{\gamma^\circ, h}^n(\gamma^\circ)^2](x) \leq \begin{cases} (\gamma^\circ(x))^2 + L_6nh & \text{if } \gamma^\circ(x) > R_2\sqrt{h}, \\ R_2^2h + L_6nh & \text{if } \gamma^\circ(x) \leq R_2\sqrt{h}, \end{cases}$$

where  $L_6 := (3N^2 - 2N - 1)/(N + 1)(= R_1^2 - R_2^2)$ .

**Remark 5.1.** Since it is seen that  $S_{\gamma^\circ, h}(-(\gamma^\circ)^2) = -S_{\gamma^\circ, h}(\gamma^\circ)^2$ , we have

$$[S_{\gamma^\circ, h}^n(-(\gamma^\circ)^2)](x) \geq \begin{cases} -\{(\gamma^\circ(x))^2 + L_6nh\} & \text{if } \gamma^\circ(x) > R_2\sqrt{h}, \\ -(R_2^2h + L_6nh) & \text{if } \gamma^\circ(x) \leq R_2\sqrt{h}. \end{cases}$$

**Proof of Lemma 5.2.** Recall  $R_1 = 2N/\sqrt{N+1}$  and  $R_2 = \sqrt{N+1}$ .

*Step 1.* We treat the case  $n = 1$ .

It is clear that  $[S_{\gamma^\circ, h}(\gamma^\circ)^2](x) \geq 0$  for all  $x \in \mathbb{R}^N$ . Set  $E := \{(\gamma^\circ)^2 \geq \mu\}$  and let  $w_{\gamma^\circ, E}^h$  be a weak solution of (3.1). We may assume  $\mu \geq 0$ . Since  $d_{\gamma^\circ, E}(x) = -\gamma^\circ(x) + \sqrt{\mu}$ , we have from Lemma 2.5  $w_{\gamma^\circ, E}^h(x) = -v^h(x) + \sqrt{\mu}$ . Hence the inequality  $w_{\gamma^\circ, E}^h(x) \geq 0$  is equivalent to

$$\mu \geq \begin{cases} \left\{ \gamma^\circ(x) + \frac{h(N-1)}{\gamma^\circ(x)} \right\}^2 & \text{if } \gamma^\circ(x) > R_2\sqrt{h}, \\ R_1^2 h & \text{if } \gamma^\circ(x) \leq R_2\sqrt{h}. \end{cases}$$

As  $R_1^2 = R_2^2 + L_6$ , it is clear that  $[S_{\gamma^\circ, h}^n(\gamma^\circ)^2](x) \leq R_2^2 h + L_6 n h$  for  $\gamma^\circ(x) \leq R_2\sqrt{h}$ .

In the case  $\gamma^\circ(x) > R_2\sqrt{h}$ , we easily estimate

$$\left\{ \gamma^\circ(x) + \frac{h(N-1)}{\gamma^\circ(x)} \right\}^2 \leq (\gamma^\circ(x))^2 + L_6 h.$$

Taking  $\mu \geq (\gamma^\circ(x))^2 + L_3 h$ , we get  $w_{\gamma^\circ, E}^h(x) \geq 0$ . Thus we have (5.25) with  $n = 1$ .

*Step 2.* We consider the case  $n \geq 2$ .

Let us argue by induction. Define

$$W_h(x) := \begin{cases} (\gamma^\circ(x))^2 & \text{if } \gamma^\circ(x) > R_2\sqrt{h}, \\ R_2^2 h & \text{if } \gamma^\circ(x) \leq R_2\sqrt{h}. \end{cases}$$

Suppose that  $[S_{\gamma^\circ, h}^{n-1}(\gamma^\circ)^2](x) \leq W_h(x) + L_6(n-1)h$  for all  $x \in \mathbb{R}^N$ . From Theorem 4.5 (1) and (3) we have only to show that  $S_{\gamma^\circ, h} W_h \leq W_h + L_6 h$  in  $\mathbb{R}^N$ .

We may consider  $\mu > R_2^2 h$  since it is easily seen that

$$E := \{W_h \geq \mu\} = \begin{cases} \mathbb{R}^N & \text{if } \mu \leq R_2^2 h, \\ \mathbb{R}^N \setminus B_{\gamma^\circ}(0, \sqrt{\mu}) & \text{if } \mu > R_2^2 h, \end{cases}$$

Then we have  $d_{\gamma^\circ, E} = -\gamma^\circ + \sqrt{\mu}$ . The same argument as in Step 1 yields that  $S_{\gamma^\circ, h} W_h \leq W_h + L_6 h$  in  $\mathbb{R}^N$ . Therefore we have completed the proof.  $\square$

## 6 Convergence of our scheme

Let  $u_0 \in UC(\mathbb{R}^N)$ . Set  $u^h(t, x) := [S_{\gamma^\circ, h}^{\lfloor t/h \rfloor} u_0](x)$  for  $(t, x) \in [0, T) \times \mathbb{R}^N$  and  $h > 0$ . Then we show the following theorem.

**Theorem 6.1.** *For  $u_0 \in UC(\mathbb{R}^N)$  let  $u$  be a unique viscosity solution of (1.7). Then  $u^h$  converges to  $u$  locally uniformly in  $[0, T) \times \mathbb{R}^N$  as  $h \rightarrow 0$ .*

Define

$$(6.1) \quad \begin{aligned} \bar{u}(t, x) &:= \limsup_{r \rightarrow 0} \{u^h(s, y) \mid |y - x| + |s - t| < r, 0 < h < r\}, \\ \underline{u}(t, x) &:= \liminf_{r \rightarrow 0} \{u^h(s, y) \mid |y - x| + |s - t| < r, 0 < h < r\}. \end{aligned}$$

**Theorem 6.2.** *Let  $u_0 \in UC(\mathbb{R}^N)$ . Then  $\bar{u}$  (resp.,  $\underline{u}$ ) is a viscosity subsolution (resp., a viscosity supersolution) of (1.7).*

**Proof.** We prove only the subsolution case because the other one can be similarly proved.

Fix any  $\phi \in C^\infty((0, T) \times \mathbb{R}^N)$ . Assume that  $\bar{u} - \phi$  has a strict maximum at  $(\hat{t}, \hat{x}) \in (0, T) \times \mathbb{R}^N$ . Taking Proposition 2.4 into account, we derive the following inequalities:

$$(6.2) \quad \phi_t(\hat{t}, \hat{x}) + F_*(\nabla\phi(\hat{t}, \hat{x}), \nabla^2\phi(\hat{t}, \hat{x})) \leq 0 \quad \text{if } \nabla\phi(\hat{t}, \hat{x}) \neq 0,$$

$$(6.3) \quad \phi_t(\hat{t}, \hat{x}) \leq 0 \quad \text{if } \nabla\phi(\hat{t}, \hat{x}) = 0 \text{ and } \nabla^2\phi(\hat{t}, \hat{x}) = O.$$

By some modifications we may consider  $\phi(t, x) = \phi_1(t) + \phi_2(x) + \phi(\hat{t}, \hat{x})$  for  $\phi_1 \in C^\infty(0, T)$  and  $\phi_2 \in C^\infty(\mathbb{R}^N)$  satisfying  $\phi_1(\hat{t}) = \phi_2(\hat{x}) = 0$ ,  $\phi_{1,t}(\hat{t}) = \phi_t(\hat{t}, \hat{x})$  and  $\nabla^k\phi_2(\hat{x}) = \nabla^k\phi(\hat{t}, \hat{x})$  ( $k = 1, 2$ ). In addition, we may assume by further modifications that  $\phi_2$  grows linearly at infinity since  $\bar{u}$  does so (cf. Theorem 6.3 below).

*Step 1.* From (6.1) we take a sequence  $\{h_n\}_{n=1}^{+\infty}$  and  $\{(t_n, x_n)\}_{n=1}^{+\infty} \subset (0, T) \times \mathbb{R}^N$  such that  $h_n \searrow 0$ ,  $(t_n, x_n, u^{h_n}(t_n, x_n)) \rightarrow (\hat{t}, \hat{x}, \bar{u}(\hat{t}, \hat{x}))$  as  $n \rightarrow +\infty$  and  $u^{h_n} - \phi$  has a maximum at  $(t_n, x_n)$  for each  $n \in \mathbb{N}$ . Then we obtain

$$u^{h_n}(t_n - h_n, x) - u^{h_n}(t_n, x_n) \leq \phi(t_n - h_n, x) - \phi(t_n, x_n) \quad \text{for all } x \in \mathbb{R}^N.$$

Applying  $S_{\gamma^\circ, h_n}$  on both sides and setting  $x = x_n$ , we get from Theorem 4.5 (1) and (3)

$$(6.4) \quad [S_{\gamma^\circ, h_n} u^{h_n}(t_n - h_n, \cdot)](x_n) - u^{h_n}(t_n, x_n) \leq [S_{\gamma^\circ, h_n} \phi(t_n - h_n, \cdot)](x_n) - \phi(t_n, x_n).$$

The left-hand side of this inequality vanishes since it follows from the definition of  $u^{h_n}$  that

$$\begin{aligned} [S_{\gamma^\circ, h_n} u^{h_n}(t_n - h_n, \cdot)](x_n) &= [S_{\gamma^\circ, h_n} (S_{\gamma^\circ, h_n}^{[t_n/h_n - 1]} u_0)](x_n) \\ &= [S_{\gamma^\circ, h_n}^{[t_n/h_n]} u_0](x_n) = u^{h_n}(t_n, x_n). \end{aligned}$$

Thereby, (6.4) turns to

$$(6.5) \quad \frac{\phi_1(t_n) - \phi_1(t_n - h_n)}{h_n} \leq \frac{[S_{\gamma^\circ, h_n} \phi_2](x_n) - \phi_2(x_n)}{h_n}.$$

*Step 2.* We suppose  $\nabla\phi_2(\hat{x}) \neq 0$  and derive (6.2).

Fix  $\varepsilon > 0$ . We estimate from Theorem 5.2 the right-hand side of (6.5) as follows:

$$\frac{[S_{\gamma^\circ, h_n} \phi_2](x_n) - \phi_2(x_n)}{h_n} \leq -F(\nabla\phi_2(x_n), \nabla^2\phi_2(x_n)) + L_5\varepsilon$$

for large  $n \in \mathbb{N}$ . Sending  $n \rightarrow +\infty$  and then  $\varepsilon \rightarrow 0$ , we have (6.2).

*Step 3.* We suppose  $\nabla\phi_2(\hat{x}) = 0$  and  $\nabla^2\phi_2(\hat{x}) = O$  and derive (6.3).

We apply the proof of [32, Theorem 3.3]. Since  $\phi_2(x) = o(|x - \hat{x}|^2)$  as  $x \rightarrow \hat{x}$ , for any  $\varepsilon > 0$ , there is a constant  $C_\varepsilon > 0$  such that  $\phi_2 \leq C_\varepsilon(\gamma^\circ(\cdot - \hat{x}))^4 + \varepsilon$  in  $B(\hat{x}, 1)$ . Thus, we may assume that  $\phi_2 \leq C_\varepsilon(\gamma^\circ(\cdot - \hat{x}))^4 + \varepsilon$  in  $\mathbb{R}^N$ , replacing  $C_\varepsilon$  with a larger one if necessary, because  $\phi_2$  grows linearly as  $|x| \rightarrow +\infty$ . Hence we are able to replace  $\phi_2$  with

$C_\varepsilon(\gamma^\circ(\cdot - \hat{x}))^4 + \varepsilon$  since it is seen by  $(\gamma 3)$  that  $(\gamma^\circ)^4 \in C^2(\mathbb{R}^N)$  and that  $\nabla(\gamma^\circ)^4(0) = 0$  and  $\nabla^2(\gamma^\circ)^4(0) = O$ . Thereby, (6.5) turns to

$$(6.6) \quad \frac{\phi_1(t_n) - \phi_1(t_n - h_n)}{h_n} \leq \frac{C_\varepsilon[S_{\gamma^\circ, h_n}(\gamma^\circ)^4](x_n - \hat{x}) - C_\varepsilon(\gamma^\circ(x_n - \hat{x}))^4}{h_n}.$$

Here we have used Theorem 4.5 (2), (3).

Theorem 4.5 (2) with  $g(r) := (r^+)^2$  ( $r^+ := \max\{r, 0\}$ ) and Lemma 5.2 yield that

$$\begin{aligned} [S_{\gamma^\circ, h_n}(\gamma^\circ)^4](x_n - \hat{x}) &= \{[S_{\gamma^\circ, h_n}(\gamma^\circ)^2](x_n - \hat{x})\}^2 \\ &\leq \begin{cases} \{(\gamma^\circ(x_n - \hat{x}))^2 + L_6 h_n\}^2 & \text{if } \gamma^\circ(x_n - \hat{x}) > R_2 \sqrt{h_n}, \\ R_1^2 h_n^2 & \text{if } \gamma^\circ(x_n - \hat{x}) \leq R_2 \sqrt{h_n}. \end{cases} \end{aligned}$$

Combining (6.6) with this estimate, we have (6.3) by letting  $n \rightarrow +\infty$  and  $\varepsilon \rightarrow 0$ .  $\square$

**Theorem 6.3.** *Assume  $u_0 \in UC(\mathbb{R}^N)$ . Then  $\underline{u}(0, \cdot) = \bar{u}(0, \cdot) = u_0$  in  $\mathbb{R}^N$  and  $|\underline{u}(t, x)| + |\bar{u}(t, x)| \leq L_7(1 + |x|)$  for all  $(t, x) \in [0, T] \times \mathbb{R}^N$  and some  $L_7 > 0$ .*

**Proof.** *Step 1.* We prove  $\underline{u}(0, \cdot) = \bar{u}(0, \cdot) = u_0$  in  $\mathbb{R}^N$ .

Let  $\omega$  be a modulus of continuity of  $u$ , that is,

$$|u_0(x) - u_0(y)| \leq \omega(|x - y|) \quad \text{for all } x, y \in \mathbb{R}^N.$$

Fix  $x \in \mathbb{R}^N$ . For any  $\varepsilon > 0$  there exists  $C_\varepsilon > 0$  such that

$$u_0(x) - \varepsilon - C_\varepsilon|y - x|^2 \leq u_0(y) \leq u_0(x) + \varepsilon + C_\varepsilon|y - x|^2.$$

(see, e.g., [22, Lemma 6.10]) By  $(\gamma 3)$  we replace  $C_\varepsilon/\lambda^2$  with  $C_\varepsilon$  to obtain

$$u_0(x) - \varepsilon - C_\varepsilon(\gamma^\circ(y - x))^2 \leq u_0(y) \leq u_0(x) + \varepsilon + C_\varepsilon(\gamma^\circ(y - x))^2.$$

Applying  $S_{\gamma^\circ, h}$  on both sides  $[t/h]$ -times and evaluating at  $y$ , we derive

$$\begin{aligned} u_0(x) - \varepsilon + C_\varepsilon[S_{\gamma^\circ, h}^{[t/h]}(-(\gamma^\circ)^2)](y - x) \\ \leq u^h(t, y) \leq u_0(x) + \varepsilon + C_\varepsilon[S_{\gamma^\circ, h}^{[t/h]}(\gamma^\circ)^2](y - x). \end{aligned}$$

Here we have invoked Theorem 4.5. We deduce from Lemma 5.2 and Remark 5.1 that

$$\begin{aligned} u_0(x) - \varepsilon - C_\varepsilon\{(\gamma^\circ(y - x))^2 + R_2^2 h + L_6(t + h)\} \\ \leq u^h(t, y) \leq u_0(x) + \varepsilon + C_\varepsilon\{(\gamma^\circ(y - x))^2 + R_2^2 h + L_6(t + h)\}. \end{aligned}$$

Letting  $(t, y) \rightarrow (0, x)$ ,  $h \rightarrow 0$  and  $\varepsilon \rightarrow 0$ , we obtain the desired result of this step.

*Step 2.* We show  $|\bar{u}(t, x)| + |\underline{u}(t, x)| \leq L_7(1 + |x|)$  for all  $(t, x) \in [0, T] \times \mathbb{R}^N$  and some  $L_7 > 0$ .

It is easily seen by  $u_0 \in UC(\mathbb{R}^N)$  and (2.3) that there exists  $L_{7,1} > 0$  such that

$$-L_{7,1}(1 + \gamma^\circ(x)) \leq u_0(x) \leq L_{7,1}(1 + \gamma^\circ(x)) \quad \text{for all } x \in \mathbb{R}^N.$$

Applying  $S_{\gamma^\circ, h}$   $[t/h]$ -times, we get

$$(6.7) \quad -L_{7,1}(1 + [S_{\gamma^\circ, h}^{[t/h]} \gamma^\circ](x)) \leq u^h(t, x) \leq L_{7,1}(1 + [S_{\gamma^\circ, h}^{[t/h]} \gamma^\circ](x)).$$

Here we have used Theorem 4.5 and the fact  $S_{\gamma^\circ, h}(-\gamma^\circ) = -S_{\gamma^\circ, h} \gamma^\circ$ . Choosing  $g(r) = \sqrt{r^+}$ , we observe from (2.3), Theorem 4.5 (2) and Lemma 5.2 that

$$\begin{aligned} [S_{\gamma^\circ, h}^{[t/h]} \gamma^\circ](x) &= \sqrt{[S_{\gamma^\circ, h}^{[t/h]} (\gamma^\circ)^2](x)} \leq \sqrt{(\gamma^\circ(x))^2 + L_6 T} \\ &\leq \gamma^\circ(x) + \sqrt{L_6 T} \leq \lambda^{-1}|x| + \sqrt{L_6 T}. \end{aligned}$$

Hence taking  $L_7 := L_{7,1}(1 + \lambda^{-1} + \sqrt{L_6 T})$ , we obtain from (6.7) and this inequality

$$|u^h(t, x)| \leq L_7(1 + |x|) \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^N.$$

Sending  $h \rightarrow 0$ , we have the result.  $\square$

We are now in a position to give

**Proof of Theorem 6.1.** It follows from Theorems 2.3, 6.2 and 6.3 that  $\bar{u} = \underline{u} (= u)$  in  $[0, T] \times \mathbb{R}^N$ . Hence  $u$  is a unique viscosity solution of (1.7). Applying the stability result (cf. [21, Remark 6.4]), we have the desired convergence.  $\square$

As a result of Theorem 6.1, one is able to obtain the convergence of the discrete evolution of sets to the continuous one by (1.1). More precisely, let  $E_0 \in \mathcal{C}(\mathbb{R}^N)$  and let  $u_0 \in UC(\mathbb{R}^N)$  be a function satisfying

$$u_0(x) \begin{cases} > 0 & (x \in \text{int } E_0), \\ = 0 & (x \in \partial E_0), \\ < 0 & (x \in \mathbb{R}^N \setminus E_0). \end{cases}$$

Let  $u$  be a unique viscosity solution of (1.7). Set  $E(t) := \{u(t, \cdot) \geq 0\}$  and  $E^h(t) := T_{\gamma^\circ, h}^{[t/h]}(E_0)$ . Then the locally uniform convergence of the auxiliary function  $u^h$  implies the convergence of its super level sets under no fattening assumptions. See [27, Lemma 4.6.5].

**Theorem 6.4.** *Assume that  $(\gamma 1)$  -  $(\gamma 5)$  and that  $\{u(t, \cdot) \geq 0\} = \overline{\{u(t, \cdot) > 0\}}$  for each  $t \in [0, T]$ . Define  $\mathcal{E} := \cup_{0 \leq t < T} (\{t\} \times E(t))$  and  $\mathcal{E}^h := \cup_{0 \leq t < T} (\{t\} \times E^h(t))$ . Then for any compact set  $\mathcal{K} \subset [0, T] \times \mathbb{R}^N$  we have*

$$\lim_{h \rightarrow 0} d_H(\mathcal{E}^h \cap \mathcal{K}, \mathcal{E} \cap \mathcal{K}) = 0,$$

where  $d_H$  is the Hausdorff distance defined by (3.2).

This theorem implies that for any  $\delta \in (0, T)$  and any compact set  $K \subset \mathbb{R}^N$

$$\lim_{h \rightarrow 0} \sup_{t \in [0, T - \delta]} d_H(\partial E(t) \cap K, \partial E^h(t) \cap K) = 0.$$

**Remark 6.1.** In [19] Chambolle and Novaga studied an anisotropic version of [17], which is similar to ours. They proved the Hausdorff convergence of their scheme to a regular flow of compact sets to (1.1) (see [19, Definition 2.1] for the definition of regular flows) in the pointwise sense with respect to the  $t$ -variable. Their proof is based on some variational techniques which does not apply to the flow of unbounded sets. Our results apply for unbounded sets and our methods are different from theirs on these points.

**Remark 6.2.** Since the Allen-Cahn approximation is uniform with respect to  $\gamma$  satisfying ( $\gamma 3$ ) (cf. [28]), it is expected that our convergence in Theorem 6.1 is also uniform with respect to  $\gamma$  (independent of derivatives of  $\gamma$ ). If so, our schema also would give a way to construct a crystalline flow. Thus the uniformity of the convergence is an important issue.

If one examines the proof, it turns out that the bound of second derivatives is only invoked for (5.6) and (5.14) and the estimate just below (5.15). There are two approximations. One is mollification of  $\gamma$ . The other one is approximation of  $\tilde{d}$  by  $\tilde{w}_1$ . The second part is more serious. One has to compare  $\operatorname{div} \nabla \gamma(\nabla \tilde{w}_1)$  by the quantity replaced by  $\tilde{w}_1$  by  $\tilde{d}$ . Although the difference between  $\tilde{w}_1$  and  $\tilde{d}$  is small and smooth, it is not clear whether the curvatures of its level sets are uniformly close with respect to  $\gamma$ .

## 7 The case of general mobilities

The motion of the interface  $\Gamma(t)$  we consider here is governed by

$$(7.1) \quad V = -\beta(\mathbf{n}) \operatorname{div}_{\Gamma(t)} \xi(\mathbf{n}) \quad \text{on } \Gamma(t), \quad t > 0,$$

where  $\beta$  is called the mobility and may be different from  $\gamma$ . We assume that  $\beta$  and  $\gamma$  satisfy ( $\gamma 1$ ) - ( $\gamma 5$ ). Then we easily see that

$$(7.2) \quad \frac{\lambda}{\Lambda} \gamma(p) \leq \beta(p) \leq \frac{\Lambda}{\lambda} \gamma(p), \quad \frac{\lambda}{\Lambda} \gamma^\circ(p) \leq \beta^\circ(p) \leq \frac{\Lambda}{\lambda} \gamma^\circ(p) \quad \text{for all } p \in \mathbb{R}^N,$$

where  $\beta^\circ$  is the support function of the convex set  $\{\beta \leq 1\}$ .

To define an approximation scheme to (7.1), we use a differential inclusion: Fix  $E_0 \in \mathcal{C}(\mathbb{R}^N)$ . Let  $w_{\beta^\circ, E_0}^h$  be a weak solution of

$$(7.3) \quad w - h \operatorname{div} \partial \gamma(\nabla w) \ni d_{\beta^\circ, E_0} \quad \text{in } \mathbb{R}^N,$$

where  $d_{\beta^\circ, E_0}$  is the anisotropic distance function defined by (2.4) with  $\gamma^\circ = \beta^\circ$  and  $E = E_0$ . By the same way as in section 3 we have a family  $\{E^h(t)\}_{t \geq 0}$  of closed sets in  $\mathbb{R}^N$ . Letting  $h \rightarrow 0$ , we formally obtain a limit flow  $\{E(t)\}_{t \geq 0}$  whose boundary  $\partial E(t)$  moves by (7.1).

We are able to prove by similar methods to those in sections 4 and 5 the convergence of the above scheme to the level set flow by (7.1), that is, the zero level set of a unique viscosity solution of

$$\begin{cases} u_t - \beta(\nabla u) \operatorname{div} \nabla \gamma(\nabla u) = 0 & \text{in } (0, T) \times \mathbb{R}^N, \\ u(0, x) = u_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases}$$

provided that no fattening occurs. Here  $u_0 \in UC(\mathbb{R}^N)$ .

In the case of general mobilities many results in this paper hold, replacing  $d_{\gamma^\circ, E}$  (resp.,  $T_{\gamma^\circ, h}$ ,  $S_{\gamma^\circ, h}$ ) with  $d_{\beta^\circ, E}$  (resp.,  $T_{\beta^\circ, h}$ ,  $S_{\beta^\circ, h}$ ), where  $T_{\beta^\circ, h}$  and  $S_{\beta^\circ, h}$  are defined by

$$\begin{aligned} T_{\beta^\circ, h}(\{u \geq \mu\}) &:= \{w_{\beta^\circ, \{u \geq \mu\}}^h \leq 0\}, \\ w_{\beta^\circ, \{u \geq \mu\}}^h &: \text{a weak solution of (7.3) with } E_0 := \{u \geq \mu\}, \\ [S_{\beta^\circ, h}u](x) &:= \sup\{\mu \in \mathbb{R} \mid x \in T_{\beta^\circ, h}(\{u \geq \mu\})\} \\ &\quad \text{for } u \in C(\mathbb{R}^N) \text{ and } x \in \mathbb{R}^N. \end{aligned}$$

The main difference is that we cannot use Lemma 2.5 to prove some propositions and lemmas, e.g., Propositions 2.3, 5.2 and Lemma 5.2. However, we are able to apply the following result: Fix  $r > 0$ . Let  $E_0 := \overline{B_{\gamma^\circ}(0, r)}$  and let  $w_{\beta^\circ, E_0}^h$  be a weak solution of (7.3) with  $E_0 := \overline{B_{\gamma^\circ}(0, r)}$ . We easily check by (7.2) that

$$(7.4) \quad \begin{aligned} d_{1,r} &\leq d_{\beta^\circ, E_0} \leq d_{2,r} \quad \text{in } \mathbb{R}^N, \\ d_{1,r} &:= M_1(\gamma^\circ - r)^+ + M_1^{-1}(\gamma^\circ - r)^-, \\ d_{2,r} &:= M_1^{-1}(\gamma^\circ - r)^+ + M_1(\gamma^\circ - r)^-, \\ M_1 &:= \frac{\lambda}{\Lambda}, \quad a^+ := \max\{a, 0\}, \quad a^- := \min\{a, 0\}. \end{aligned}$$

Besides, the weak solution  $v_i^h$  ( $i = 1, 2$ ) of  $w - h \operatorname{div} \partial \gamma(\nabla w) \ni d_{i,r}$  in  $\mathbb{R}^N$  is given by

$$(7.5) \quad v_i^h(x) = \begin{cases} d_{i,r}(x) + \frac{h(N-1)}{\gamma^\circ(x)} & \text{if } \gamma^\circ(x) \geq R_2 \sqrt{\frac{h}{M_1}}, \\ R_1 \sqrt{M_1 h} - M_1 r & \text{if } \gamma^\circ(x) < R_2 \sqrt{\frac{h}{M_1}}, \end{cases}$$

for  $h \leq M_1^{-1} r^2 / R_2^2$ , where  $R_1 = 2N / \sqrt{N+1}$  and  $R_2 = \sqrt{N+1}$  (cf. [16, Appendix D]). Hence  $v_2^h \leq w_{\beta^\circ, E_0}^h \leq v_1^h$  in  $\mathbb{R}^N$  by Theorem 8.1.

We use the above inequality to prove some propositions and lemmas mentioned before. For example, we apply the above inequality to obtain a substitute for Lemma 5.2:

$$[S_{\beta^\circ, h}^n(\gamma^\circ)^2](x) \leq \begin{cases} (\gamma^\circ(x))^2 + L_8 n h & \text{if } \gamma^\circ(x) > R_2 \sqrt{\frac{h}{M_1}}, \\ \frac{R_2^2 h}{M_1} + L_8 n h & \text{if } \gamma^\circ(x) \leq R_2 \sqrt{\frac{h}{M_1}}. \end{cases}$$

Here  $L_8 := \{3N^2 - 2N - 1\} / M_1(N+1)$ .

## 8 Appendix

### 8.1 Weak solutions of (2.6)

This subsection is devoted to the proofs of Theorem 2.2 and Lemma 2.4.

At first we show the comparison theorem for the elliptic inclusion (2.6).

**Theorem 8.1.** Assume  $\underline{g}, \bar{g} \in L^2_{loc}(\mathbb{R}^N)$ . Let  $(\underline{w}, \underline{z}), (\bar{w}, \bar{z}) \in L^2_{loc}(\mathbb{R}^N) \cap BV_{loc}(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N; \mathbb{R}^N)$  be weak solutions of (2.6) with  $g = \underline{g}, \bar{g}$ , respectively. If  $\underline{g} \leq \bar{g}$  a.e. in  $\mathbb{R}^N$ , then  $\underline{w} \leq \bar{w}$  a.e. in  $\mathbb{R}^N$ .

**Proof.** The proof is a modification of that of [16, Theorem 6]. For  $\alpha > 1, k \geq 0$  and  $r \in \mathbb{R}$  define a truncation function  $T_k^+(r) := \max\{\min\{r, k\}, 0\}$  and truncated power functions  $j_{k,\alpha}(r) := r(T_k^+(r))^{\alpha-1}$  and  $p_{k,\alpha}(r) := \alpha T_k^+(r)^{\alpha-1}$ . From the proof of [16, Theorem 6], we obtain

$$(8.1) \quad \int_{\mathbb{R}^N} (\underline{w} - \bar{w}) p_{k,\alpha}(\underline{w} - \bar{w}) \psi dx \leq \int_{\mathbb{R}^N} (\underline{g} - \bar{g}) p_{k,\alpha}(\underline{w} - \bar{w}) \psi dx + \int_{\mathbb{R}^N} \langle \bar{z} - \underline{z}, \nabla \psi \rangle p_{k,\alpha}(\underline{w} - \bar{w}) dx,$$

for all  $\psi \in C_0^\infty(\mathbb{R}^N)$  satisfying  $\psi \geq 0$  on  $\mathbb{R}^N$ . Let us replace  $\psi$  with  $\psi^\alpha$ . We divide both sides of (8.1) by  $\alpha$  and use  $\underline{g} \leq \bar{g}$  and  $p(\underline{w} - \bar{w})\psi^\alpha \geq 0$  on  $\mathbb{R}^N$  in the above inequality. Then we have

$$(8.2) \quad \int_{\mathbb{R}^N} j_{k,\alpha}(\underline{w} - \bar{w}) \psi^\alpha dx \leq 2\Lambda \int_{\mathbb{R}^N} \psi^{\alpha-1} |\nabla \psi| \{T_k^+(\underline{w} - \bar{w})\}^{\alpha-1} dx.$$

Here we have used the fact  $\|\underline{z}\|_{L^\infty(\mathbb{R}^N)}, \|\bar{z}\|_{L^\infty(\mathbb{R}^N)} \leq \Lambda$  from Lemma 2.1. Since we easily see by Hölder's inequality that

$$\begin{aligned} & \int_{\mathbb{R}^N} \psi^{\alpha-1} |\nabla \psi| \{T_k^+(\underline{w} - \bar{w})\}^{\alpha-1} dx \\ & \leq \left( \int_{\mathbb{R}^N} \{T_k^+(\underline{w} - \bar{w})\}^\alpha \psi^\alpha dx \right)^{(\alpha-1)/\alpha} \left( \int_{\mathbb{R}^N} |\nabla \psi|^\alpha dx \right)^{1/\alpha} \end{aligned}$$

and that  $\{T_k^+(r)\}^\alpha \leq j_{k,\alpha}(r)$  for all  $r \in \mathbb{R}$ , we obtain from (8.2) and this inequality

$$(8.3) \quad \int_{\mathbb{R}^N} j_{k,\alpha}(\underline{w} - \bar{w}) \psi^\alpha dx \leq 2\Lambda \int_{\mathbb{R}^N} |\nabla \psi|^\alpha dx.$$

We define a sequence  $\{\psi_n\}_{n=1}^{+\infty}$ , according to [16, Corollary C.2]. Let  $\psi_0 \in C_0^\infty(\mathbb{R}^N)$  be a function satisfying  $0 \leq \psi_0 \leq 1$  on  $\mathbb{R}^N$ ,  $\psi_0 \equiv 1$  on  $B(0, 1)$ ,  $\psi_0 \equiv 0$  outside  $B(0, 2)$ . Set  $\psi_n(x) := \psi_0(x/n)$ . Setting  $\psi = \psi_n$  in (8.3), we get

$$\int_{\mathbb{R}^N} j_{k,\alpha}(\underline{w} - \bar{w}) \psi_n^\alpha dx \leq 2\Lambda n^{-\alpha} \int_{\mathbb{R}^N} \left| \nabla \psi_n \left( \frac{x}{n} \right) \right|^\alpha dx = 2\Lambda n^{N-\alpha} \int_{\mathbb{R}^N} |\nabla \psi_0(x)|^\alpha dx.$$

Taking  $\alpha > N$  and sending  $n \rightarrow +\infty$ , we have

$$\int_{\mathbb{R}^N} j_{k,\alpha}(\underline{w} - \bar{w}) dx \leq 0$$

and we conclude that  $\underline{w} \leq \bar{w}$  a.e. in  $\mathbb{R}^N$ .  $\square$



Next we prove the existence of a weak solution of (2.6). Fix  $g \in L^2_{loc}(\mathbb{R}^N)$  and take a sequence  $\{g_n\}_{n=1}^{+\infty} \subset L^2(\mathbb{R}^N)$  such that  $g_n \rightarrow g$  in  $L^2_{loc}(\mathbb{R}^N)$  as  $n \rightarrow +\infty$ . Let  $\Phi$  be defined by

$$\Phi(v) := \begin{cases} \int_{\mathbb{R}^N} \gamma^\circ(Dv) & \text{if } v \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N), \\ +\infty & \text{if } v \in L^2(\mathbb{R}^N) \setminus BV(\mathbb{R}^N). \end{cases}$$

It follows from [14] that for each  $n \in \mathbb{N}$  there is a unique solution  $w_n$  of  $w + \partial\Phi(w) \ni g_n$  in  $\mathbb{R}^N$ . Applying Proposition 2.2 (see also [36]), we observe that for each  $n \in \mathbb{N}$  there exists  $z_n \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ ,  $\operatorname{div} z_n \in L^2(\mathbb{R}^N)$  such that a pair  $(w_n, z_n)$  is a weak solution of (2.6) with  $g = g_n$ .

We derive some locally uniform estimates on  $\{w_n\}_{n=1}^{+\infty}$ .

**Lemma 8.1.** *For each  $R > 0$  we have  $\sup_{n \in \mathbb{N}} \|w_n\|_{L^2(B(0,R))} \leq K_1$  for some  $K_1 > 0$ .*

Here and in the sequel, we denote by  $K_i$  ( $i = 1, 2, \dots$ ) the constants independent of  $n \in \mathbb{N}$ .

**Proof of Lemma 8.1.** Let  $\psi \in C_0^\infty(\mathbb{R}^N)$  be a function such that

$$(8.4) \quad 0 \leq \psi \leq 1 \text{ on } \mathbb{R}^N, \quad \psi \equiv \begin{cases} 1 & \text{on } B(0, R), \\ 0 & \text{on } \mathbb{R}^N \setminus B(0, R+1), \end{cases} \quad |\nabla\psi| \leq 2 \quad \text{on } \mathbb{R}^N.$$

We apply [16, Theorem 6] to obtain

$$\left( \int_{\mathbb{R}^N} j_{k,2}(w_n) \psi^2 dx \right)^{1/2} \leq \left( \int_{\mathbb{R}^N} (g_n^+)^2 \psi^2 dx \right)^{1/2} + 2\Lambda \left( \int_{\mathbb{R}^N} |\nabla\psi|^2 dx \right)^{1/2}.$$

Letting  $k \rightarrow +\infty$ , we get

$$\left( \int_{\mathbb{R}^N} (w_n^+)^2 \psi^2 dx \right)^{1/2} \leq \left( \int_{\mathbb{R}^N} (g_n^+)^2 \psi^2 dx \right)^{1/2} + 2\Lambda \left( \int_{\mathbb{R}^N} |\nabla\psi|^2 dx \right)^{1/2},$$

where  $v^+ := \max\{v, 0\}$  ( $v = w_n, g_n$ ). Thus we have

$$\sup_{n \geq 1} \|w_n^+\|_{L^2(B(0,R))} \leq \sup_{n \geq 1} \|g_n\|_{L^2(B(0,R+1))} + 4\Lambda \mathcal{L}^N(B(0, R+1) \setminus B(0, R)),$$

where  $\mathcal{L}^N(A)$  is the  $N$ -dimensional Lebesgue measure of  $A \subset \mathbb{R}^N$ . Since we give a similar estimate for  $\|(-w_n)^+\|_{L^2(B(0,R))}^2$ , we obtain the result.  $\square$

**Lemma 8.2.** *For each  $R > 0$  we get  $\sup_{n \in \mathbb{N}} \|\operatorname{div} z_n\|_{L^2(B(0,R))} \leq K_2$  for some  $K_2 > 0$ .*

**Proof.** Fix  $R > 0$ . Multiplying  $w_n - \operatorname{div} z_n = g_n$  by  $\varphi \in C_0^1(B(0, R))$  and integrating on  $B(0, R)$ , we have

$$\int_{B(0,R)} \varphi \operatorname{div} z_n dx = \int_{B(0,R)} (w_n - g_n) \varphi dx.$$

Denote by  $F(\varphi)$  the left-hand side of this formula. Since we are able to extend  $F(\varphi)$  as a linear functional on  $L^2(B(0, R))$ , we apply Riesz' representation theorem and Lemma 8.1 to have

$$\|\operatorname{div} z_n\|_{L^2(B(0,R))} = \|F\|_{(L^2(B(0,R)))'} \leq \sup_{n \geq 1} \|w_n - g_n\|_{L^2(B(0,R))} =: K_2.$$

$\square$

**Lemma 8.3.** For each  $R > 0$  we have  $\sup_{n \in \mathbb{N}} \int_{B(0,R)} \gamma(Dw_n) \leq K_3$  for some  $K_3 > 0$ .

**Proof.** Let  $\psi$  be defined by (8.4). Recall that for each  $n \in \mathbb{N}$   $\gamma(Dw_n)$  coincides with the nonnegative Radon measure (cf. (2.5)). Hence we get

$$(8.5) \quad \int_{B(0,R)} \gamma(Dw_n) \leq \int_{\mathbb{R}^N} \gamma(Dw_n)\psi = \int_{\mathbb{R}^N} (z_n, Dw_n)\psi.$$

The second equality follows from Definition 2.3 (2).

Multiplying  $w_n - \operatorname{div} z_n = g_n$  by  $w_n\psi$  and integrating by parts (cf. Theorem 2.1), we have

$$\int_{\mathbb{R}^N} w_n^2\psi dx + \int_{\mathbb{R}^N} \gamma(Dw_n)\psi + \int_{\mathbb{R}^N} w_n \langle z_n, \nabla\psi \rangle dx = \int_{\mathbb{R}^N} g_n w_n \psi dx.$$

Combining (8.5) and this formula, we obtain

$$\begin{aligned} \int_{B(0,R)} \gamma(Dw_n) &\leq \int_{\mathbb{R}^N} g_n w_n \psi dx - \int_{\mathbb{R}^N} w_n \langle z_n, \nabla\psi \rangle dx \\ &\leq \|g_n\|_{L^2(B(0,R+1))} \|w_n\|_{L^2(B(0,R+1))} \\ &\quad + \Lambda \|w_n\|_{L^2(B(0,R+1))} \|\nabla\psi\|_{L^2(B(0,R+1))}. \end{aligned}$$

Using Lemma 8.1, we get the result.  $\square$

**Proof of Theorem 2.2.** We show only the existence of a weak solution of (2.6) since the uniqueness follows from Theorem 8.1. By Lemmas 8.1 - 8.3 and the diagonal argument, there are a subsequence  $\{w_{n_l}\}_{l=1}^{+\infty} \subset \{w_n\}_{n=1}^{+\infty}$  and a function  $w \in L^2_{loc}(\mathbb{R}^N) \cap BV_{loc}(\mathbb{R}^N)$  such that for each  $j \in \mathbb{N}$

$$(8.6) \quad \lim_{l \rightarrow +\infty} w_{n_l} = w \quad \text{weakly in } L^2(B(0, j))$$

and strongly in  $L^1(B(0, j))$ ,

$$(8.7) \quad \lim_{l \rightarrow +\infty} z_{n_l} = z \quad \text{weakly in } L^2(B(0, j); \mathbb{R}^N)$$

and weakly  $-*$  in  $L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ ,

$$(8.8) \quad \lim_{l \rightarrow +\infty} \operatorname{div} z_{n_l} = \operatorname{div} z \quad \text{weakly in } L^2(B(0, j)),$$

$$(8.9) \quad \int_{B(0,j)} \gamma(Dw) \leq \liminf_{l \rightarrow +\infty} \int_{B(0,j)} \gamma(Dw_{n_l}) < +\infty,$$

$$(8.10) \quad \lim_{l \rightarrow +\infty} \int_{B(0,j)} \gamma(Dw_{n_l})\varphi = \int_{B(0,j)} \gamma(Dw)\varphi$$

for any  $\varphi \in C_0^\infty(B(0, j))$ .

See [25, Section 1.9, 5.2] for the convergences (8.6), (8.9) and (8.10). Set  $n_l = l$  for notational simplicity.

*Step 1.* We show that for each  $j \in \mathbb{N}$

$$(8.11) \quad w_l \longrightarrow w \quad \text{strongly in } L^2(B(0, j)) \text{ as } l \rightarrow +\infty.$$

Let  $\psi$  be defined by (8.4) with  $R = j$ . Applying (8.1) with  $\alpha = 2$  and Lemma 8.1, we get

$$\begin{aligned}
& \int_{\mathbb{R}^N} (w_l - w_m) p_{k,2}(w_l - w_m) \psi^2 dx \\
& \leq \int_{\mathbb{R}^N} (g_l - g_m) p_{k,2}(w_l - w_m) \psi^2 dx \\
& \quad + 2 \int_{\mathbb{R}^N} \langle z_m - z_l, \nabla \psi \rangle p_{k,2}(w_l - w_m) \psi dx \\
& \leq \|g_l - g_m\|_{L^2(B(0,j))} \|w_l - w_m\|_{L^2(B(0,j))} + 8\Lambda \|(w_l - w_m)^+\|_{L^1(B(0,j))} \\
& \leq 2K_1 \|g_l - g_m\|_{L^2(B(0,j))} + 8\Lambda \|w_l - w_m\|_{L^1(B(0,j))}.
\end{aligned}$$

Letting  $k \rightarrow +\infty$ , we get by the monotone convergence theorem

$$\|(w_l - w_m)^+\|_{L^2(B(0,j))}^2 \leq 2K_1 \|g_l - g_m\|_{L^2(B(0,j))} + 8\Lambda \|w_l - w_m\|_{L^1(B(0,j))}.$$

Similarly we have

$$\|(w_l - w_m)^-\|_{L^2(B(0,j))}^2 \leq 2K_1 \|g_m - g_l\|_{L^2(B(0,j))} + 8\Lambda \|w_m - w_l\|_{L^1(B(0,j))},$$

where  $a^- := \min\{a, 0\}$ . Hence

$$\|w_l - w_m\|_{L^2(B(0,j))}^2 \leq 8K_1 \|g_l - g_m\|_{L^2(B(0,j))} + 32\Lambda \|w_l - w_m\|_{L^1(B(0,j))}.$$

Hence it follows from (8.6) and  $g_n \rightarrow g$  in  $L^2_{loc}(\mathbb{R}^N)$  as  $n \rightarrow +\infty$  that  $\{w_l\}_{l=1}^{+\infty}$  is a Cauchy sequence in  $L^2(B(0,j))$  for each  $j \in \mathbb{N}$ . Thus we get (8.11).

*Step 2.* We prove that  $(z, Dw) = \gamma(Dw)$  locally as measures in  $\mathbb{R}^N$ .

Since the formula  $(z_l, Dw_l) = \gamma(Dw_l)$  holds locally as measures for all  $l \in \mathbb{N}$ , we see by (2.7) that for any  $\varphi \in C_0^\infty(\mathbb{R}^N)$  there exists  $j \in \mathbb{N}$  such that for all  $l \in \mathbb{N}$ ,

$$\begin{aligned}
\int_{B(0,j)} (z_l, Dw_l) \varphi &= - \int_{B(0,j)} w_l \varphi \operatorname{div} z_l dx - \int_{B(0,j)} w_l \langle z_l, \nabla \varphi \rangle dx \\
&= \int_{B(0,j)} \gamma(Dw_l) \varphi.
\end{aligned}$$

Taking (8.7) - (8.11) into account, letting  $l \rightarrow +\infty$ , we get

$$\int_{\mathbb{R}^N} (z, Dw) \varphi = \int_{\mathbb{R}^N} \gamma(Dw) \varphi \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^N).$$

Hence we have the result of this step.

*Step 3.* We can easily observe from (8.7) - (8.11) and the fact  $g_n \rightarrow g$  in  $L^2_{loc}(\mathbb{R}^N)$  as  $n \rightarrow +\infty$  that  $w$  satisfies Definition 2.3 (3). In addition, applying the proof of [36, Proposition 3], we have  $z(x) \in \partial\gamma(\nabla w(x))$  for a.e.  $x \in \mathbb{R}^N$ . Therefore, we conclude that a weak solution of (2.6) is the function  $w$  obtained in the beginning of this proof.  $\square$

**Proof of Lemma 2.4.** For each  $n \in \mathbb{N}$  we can choose  $\tilde{g}_n \in L^2(\mathbb{R}^N)$  and a weak solution  $\tilde{w}_n \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$  of (2.6) with  $g = \tilde{g}_n$  such that

$$(8.12) \quad \|\tilde{g}_n - g_n\|_{L^2(B(0,n))} < \frac{1}{2^n} \quad \text{and} \quad \|\tilde{w}_n - w_n\|_{L^2(B(0,n))} < \frac{1}{2^n}.$$

To prove the assertion, fix any compact set  $K \subset \mathbb{R}^N$  and choose  $n \in \mathbb{N}$  so large that  $K \subset B(0, n)$ . Then we obtain

$$\|w_n - w\|_{L^2(K)} \leq \|w_n - \tilde{w}_n\|_{L^2(B(0,n))} + \|\tilde{w}_n - w\|_{L^2(K)} < \frac{1}{2^n} + \|\tilde{w}_n - w\|_{L^2(K)}.$$

We note that  $\|\tilde{w}_n - w\|_{L^2(K)} \rightarrow 0$  as  $n \rightarrow +\infty$ . Indeed, we observe by (8.12) that

$$\begin{aligned} \|\tilde{g}_n - g\|_{L^2(K)} &\leq \|\tilde{g}_n - g_n\|_{L^2(B(0,n))} + \|g_n - g\|_{L^2(K)} \\ &< \frac{1}{2^n} + \|g_n - g\|_{L^2(K)} \rightarrow 0 \text{ as } n \rightarrow +\infty. \end{aligned}$$

Therefore, we get  $\|\tilde{w}_n - w\|_{L^2(K)} \rightarrow 0$  as  $n \rightarrow +\infty$  by a similar way to the proof of Theorem 8.1. The proof is completed since  $K \subset \mathbb{R}^N$  is arbitrary.  $\square$

## 8.2 A convergence property of the signed distance functions

**Theorem 8.2.** *Let  $\{C_n\}_{n=1}^{+\infty}$  be a family of closed sets in  $\mathbb{R}^N$  and  $C (\neq \emptyset, \mathbb{R}^N)$  a closed subset of  $\mathbb{R}^N$ . Let  $d_{\gamma^\circ, C_n}$  and  $d_{\gamma^\circ, C}$  be the anisotropic signed distance function defined by (2.4) with  $E = C_n, C$ , respectively. Assume that as  $n \rightarrow +\infty$ ,  $C_n \searrow C$  or  $C_n \nearrow C$  holds. Then  $d_{\gamma^\circ, C_n}$  converges to  $d_{\gamma^\circ, C}$  monotonously and locally uniformly in  $\mathbb{R}^N$  as  $n \rightarrow \infty$ .*

The notation  $C_n \nearrow C$  means that  $\{C_n\}_{n=1}^{+\infty}$  is nondecreasing and  $\cup_{n=1}^{+\infty} C_n = C$ .

We prepare a set-theoretic version of Dini's theorem to show the above theorem.

**Lemma 8.4.** *Let  $\{C_n\}_{n=1}^{+\infty}$  be a family of subsets of  $\mathbb{R}^N$  and  $C$  a nonempty subset of  $\mathbb{R}^N$ . Suppose that the one of the following conditions holds:*

- (1)  $\{C_n\}_{n=1}^{+\infty}$  is a sequence of compact subsets and  $C_n \searrow C$ .
- (2)  $C_n \nearrow C$  and  $C$  is bounded.

Then  $\lim_{n \rightarrow \infty} d_H(C_n, C) = 0$ .

**Proof.** *Step 1.* We show the assertion under the condition (1).

Notice that  $d_H(C_n, C) = \sup_{x \in C_n} \text{dist}(x, C)$ . For each  $n \in \mathbb{N}$ , we take  $x_n \in C_n$  such that  $\text{dist}(x_n, C) = d_H(C_n, C)$ . Since  $C_1$  is bounded, so is the sequence  $\{x_n\}_{n=1}^{+\infty}$ . Hence, there is a subsequence  $\{x_{n_k}\}_{k=1}^{+\infty} \subset \{x_n\}_{n=1}^{+\infty}$  such that  $x_{n_k} \rightarrow x \in \mathbb{R}^N$  as  $k \rightarrow \infty$ .

Fix any  $n \in \mathbb{N}$ . Then there exists  $k_0 \in \mathbb{N}$  such that  $x_{n_k} \in C_n$  for all  $k > k_0$ . Letting  $k \rightarrow +\infty$ , we get  $x \in C_n$ . We have  $x \in C$  as  $n$  is arbitrary. Thus

$$\lim_{k \rightarrow +\infty} d_H(C_{n_k}, C) = \lim_{k \rightarrow +\infty} \text{dist}(x_{n_k}, C) = \text{dist}(x, C) = 0.$$

As  $C_n \searrow C$ ,  $\{d_H(C_n, C)\}_{n=1}^{+\infty}$  is nonincreasing. Therefore we have the assertion from the above convergence.

*Step 2.* We prove the assertion under the condition (2).

Note that  $d_H(C_n, C) = \sup_{x \in C} \text{dist}(x, C_n)$ . For each  $n \in \mathbb{N}$  we take  $x_n \in C$  satisfying  $d_H(C_n, C) = \text{dist}(x_n, C_n)$ . Since  $C$  is bounded, there is a subsequence  $\{x_{n_k}\}_{k=1}^{+\infty} \subset \{x_n\}_{n=1}^{+\infty}$  such that  $x_{n_k} \rightarrow x \in \overline{C}$  as  $k \rightarrow +\infty$ .

Fix  $\varepsilon > 0$  and take  $z \in B(x, \varepsilon) \cap C$ . As  $C_n \nearrow C$ , there exists  $n_0 \in \mathbb{N}$  such that  $z \in C_n$  for all  $n > n_0$ . Hence, we see that for any  $n_k > n_0$

$$\text{dist}(x_{n_k}, C_{n_k}) \leq |x_{n_k} - z| \leq |x_{n_k} - x| + |x - z| < |x_{n_k} - x| + \varepsilon.$$

Sending  $k \rightarrow +\infty$  and then  $\varepsilon \rightarrow 0$ , we obtain

$$\lim_{k \rightarrow +\infty} d_H(C_{n_k}, C) = \lim_{k \rightarrow +\infty} \text{dist}(x_{n_k}, C_{n_k}) = 0.$$

We have the assertion from this convergence, since  $\{d_H(C_n, C)\}_{n=1}^{+\infty}$  is nonincreasing.  $\square$

**Proof of Theorem 8.2.** It follows from the condition (1) or (2) that  $\{d_{\gamma^\circ, C_n}\}_{n=1}^{+\infty}$  is monotonous. Thus once we show that

$$(8.13) \quad d_{\gamma^\circ, C_n}(x) \longrightarrow d_{\gamma^\circ, C}(x) \quad \text{as } n \rightarrow +\infty \quad \text{for each } x \in \mathbb{R}^N,$$

then the desired result follows from Dini's Theorem. Therefore, it suffices to prove (8.13).

*Step 1.* We assume that  $C_n \nearrow C$  as  $n \rightarrow +\infty$ . We divide our consideration into two cases.

*Case 1.*  $x \notin C$ .

Then  $x \notin C_n$  for all  $n \in \mathbb{N}$ . For each  $n \in \mathbb{N}$  there exists  $x_n \in C_n$  such that  $d_{\gamma^\circ, C_n}(x) = \gamma^\circ(x - x_n)$ . The sequence  $\{x_n\}_{n=1}^{+\infty}$  must be bounded. Indeed, if it is not bounded, then we extract a subsequence  $\{x_{n_k}\}_{k=1}^{+\infty} \subset \{x_n\}_{n=1}^{+\infty}$  such that  $\gamma^\circ(x - x_{n_k}) > k$  for any  $k \in \mathbb{N}$ , which leads to

$$\mathbb{R}^N \setminus \overline{B_{\gamma^\circ}(x, k)} \supset C_{n_k} \quad \text{for any } k \in \mathbb{N}.$$

We obtain

$$(\emptyset \neq) C = \bigcap_{k=1}^{\infty} C_{n_k} \subset \bigcap_{k=1}^{\infty} \mathbb{R}^N \setminus \overline{B_{\gamma^\circ}(x, k)} = \mathbb{R}^N \setminus \bigcup_{k=1}^{\infty} \overline{B_{\gamma^\circ}(x, k)} = \emptyset,$$

which is a contradiction. Thus  $\{x_n\}_{n=1}^{+\infty}$  is bounded and we can choose  $M > 0$  so large that  $\{x_n\}_{n=1}^{+\infty} \subset \overline{B_{\gamma^\circ}(x, M)}$ .

We set the compact sets  $\tilde{C}$ ,  $\tilde{C}_n$  as follows:

$$\tilde{C} := C \cap \overline{B_{\gamma^\circ}(x, M)}, \quad \tilde{C}_n := C_n \cap \overline{B_{\gamma^\circ}(x, M)} \quad (n \in \mathbb{N}).$$

Then,  $\tilde{C}_n \nearrow \tilde{C}$  as  $n \rightarrow +\infty$ ,  $d_{\gamma^\circ, C_n}(x) = d_{\gamma^\circ, \tilde{C}_n}(x)$  for each  $n \in \mathbb{N}$  and  $d_{\gamma^\circ, C}(x) = d_{\gamma^\circ, \tilde{C}}(x)$ . Note that  $\{d_{\gamma^\circ, \tilde{C}_n}\}_{n=1}^{+\infty}$  is also nondecreasing.

For each  $n \in \mathbb{N}$  let us take points  $x_n \in \tilde{C}_n$ ,  $x_n^* \in \tilde{C}$  such that  $d_{\gamma^\circ, \tilde{C}_n}(x) = \gamma^\circ(x - x_n)$  and  $\text{dist}(x_n, C) = |x_n - x_n^*|$ . We see by  $(\gamma 3)$  and (2.1) that

$$0 \leq d_{\gamma^\circ, \tilde{C}}(x) - d_{\gamma^\circ, \tilde{C}_n}(x) \leq \gamma^\circ(x - x_n^*) - \gamma^\circ(x - x_n) \leq \gamma^\circ(x_n^* - x_n) \leq \Lambda d_H(\tilde{C}_n, \tilde{C}).$$

Letting  $n \rightarrow +\infty$ , we have (8.13) by Lemma 8.4.

*Case 2.  $x \in C$*

In this case,  $d_{\gamma^\circ, C}(x) \leq 0$ . Henceforth we set

$$D := \mathbb{R}^N \setminus C, \quad D_n := \mathbb{R}^N \setminus C_n.$$

Then  $D_n \nearrow D$  as  $n \rightarrow \infty$  and  $x \notin D$ . We get by a similar way to Case 1

$$d_{\gamma^\circ, D_n}(x) \longrightarrow d_{\gamma^\circ, D}(x) \quad \text{as } n \rightarrow +\infty.$$

Noting that  $d_{\gamma^\circ, D_n} = -d_{\gamma^\circ, C_n}$  and  $d_{\gamma^\circ, D} = -d_{\gamma^\circ, C}$ , we have (8.13) in the case  $x \in C$ .

*Step 2.* We assume that  $C_n \searrow C$  as  $n \rightarrow +\infty$ . Then we can prove (8.13) by the same way as in Step 1.

Therefore, the proof of Theorem 8.2 is now complete.  $\square$

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