THE L^∞-STOKES SEMIGROUP IN EXTERIOR DOMAINS

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Abstract. The Stokes semigroup is extended to an analytic semigroup in spaces of bounded functions in an exterior domain with C^3 boundary. Some of these spaces include vector fields non-decaying at the space infinity. Moreover, uniform bounds by a sup-norm of initial velocity are established in finite time for second spacial derivatives of velocity and also for gradient of pressure to the Stokes equations.

1. Introduction and main results

1.1. Analyticity of the Stokes semigroup. In this paper as a continuation of [1] we study the Stokes equations, the linearlized Navier-Stokes equations:

$$v_t - \Delta v + \nabla q = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{1.1}$$

$$\operatorname{div} v = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{1.2}$$

$$v = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.3}$$

$$v = v_0 \quad \text{on} \quad \Omega \times \{t = 0\}, \tag{1.4}$$

in a domain $\Omega \subset \mathbf{R}^n$ with $n \geq 2$. There are a large literature on analyticity of the Stokes semigroup $S(t) : v_0 \mapsto v(\cdot,t)$ $(t \geq 0)$ in $L^r_\sigma(\Omega)$ for $r \in (1,\infty)$, L^r-solenoidal vector space in Ω for various kinds of domains including a smoothly bounded domain and an exterior domain (i.e. a domain whose complement is compact) [52], [26]. However, it had been long standing open problems whether or not $S(t)$ is an analytic semigroup in L^∞-type spaces even if the domain Ω is bounded until we gave an affirmative answer in [1] based on a blow-up argument which is a typical indirect argument to obtain an a priori upper bound for solutions; see [27], [30], [42], [43] (also [25]) for semilinear heat equations and [34], [32] for the Navier-Stokes equations. If a suitable explicit solution formula is available, the analyticity in L^∞ can be proved from a direct estimate for the solution formula. For example, when $\Omega = \mathbf{R}^n_+$, a half space, an explicit solution formula [56], [38] yields the analyticity in $C_{0,\sigma}(\Omega)$, the L^∞-closure of $C^\infty_{c,\sigma}(\Omega)$, the space of all smooth solenoidal vector fields with compact support in Ω. Here and in what follows, we do not distinguish the space of vector-valued and scalar functions. Appealing to an explicit formula for resolvent, the analyticity of $S(t)$ is known in several L^∞-type spaces including

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$C_{0,\sigma}(\Omega), L^\infty_\sigma(\Omega)$ where Ω is a half space [14]. The space $L^\infty_\sigma(\Omega)$ includes functions non-decaying at the space infinity. Note that if $\Omega = \mathbf{R}^n$, a whole space, the Stokes semigroup is nothing but a heat semigroup in a solenoidal space. If Ω is a general domain, no explicit solution formula is available. Nevertheless, in [1] we are able to prove that $S(t)$ is a C_0-analytic semigroup in $C_{0,\sigma}(\Omega)$ for what is called an admissible domain Ω. This notion relates to a solution of the Neumann problem for the Laplace equation and it is proved in [1] that a bounded domain with C^3 boundary is admissible.

In this paper we discuss the case when Ω is an exterior domain, a typical example of an unbounded domain. Our goal has two folds. First goal is to assert that an exterior domain is admissible so that our theory applies to conclude that $S(t)$ is an C_0-analytic semigroup in $C_{0,\sigma}(\Omega)$. The second goal is to extend the Stokes semigroup to spaces of bounded functions which may be non-decaying at the space infinity. Less is known even existence of a solution. We prove analyticity of $S(t)$ also in these spaces. We first recall the notion of an admissible domain [1] and a priori estimates for a solution of (1.1)-(1.4). Our a priori estimate is a bound for

$$N(v,q)(x,t) = \left|v(x,t)\right| + t^{1/2}\left|\nabla v(x,t)\right| + t\left|\nabla^2 v(x,t)\right| + t\left|v_t(x,t)\right| + t\left|\nabla q(x,t)\right| \tag{1.5}$$

of the form

$$\sup_{0\le t\le T_0} \left\|N(v,q)\right\|_\infty(t) \le C\|v_0\|_\infty \tag{1.6}$$

for some $T_0 > 0$ with a constant C depending only on the domain Ω, where $\|v_0\|_\infty = \|v_0\|_{L^\infty(\Omega)}$ denotes the sup-norm of $|v_0|$ in Ω. For a general domain the Helmholtz decomposition in L^r space may not hold in general so we appeal to $\tilde{L}^r$-theory instead of L^r-theory to apply a priori estimate (1.6). We invoke $\tilde{L}^r$-theory developed in [16],[17],[18]. If a domain has a uniform regularity of the boundary, the Helmholtz projection $\mathbf{P}$ acts as a bounded operator from $\tilde{L}^r$ to $\tilde{L}^r_\sigma$, $\tilde{L}^r$-solenoidal vector space and the Stokes semigroup $S(t)$ is a C_0-analytic semigroup in $\tilde{L}^r_\sigma$. Here we only use the space $\tilde{L}^r$ between $2 \le r < \infty$ with $\tilde{L}^r = L^r \cap L^2$. We simply call a solution $\tilde{L}^r$-solution provided by $\tilde{L}^r$-theory. To establish the estimate (1.6), a key is a pressure estimate by velocity

$$\sup_{x\in\Omega} d_\Omega(x)\left|\nabla q(x,t)\right| \le C_\Omega\|\nabla v\|_{L^\infty(\partial\Omega)}(t) \tag{1.7}$$

with some constant C_Ω depending only on Ω, where $d_\Omega(x)$ denotes the distance function from the boundary, i.e. $d_\Omega(x) = \inf_{y\in\partial\Omega}|x-y|$ for $x \in \Omega$. If a domain Ω is admissible, the pressure estimate (1.7) is available for all $\tilde{L}^r$-solutions ($r \ge n$). We recall the definition of admissible domain and an analyticity result of the Stokes semigroup in $C_{0,\sigma}$.

Definition 1.1 ([1]). Let Ω be a uniformly C^1-domain in $\mathbf{R}^n (n \ge 2)$ with $\partial\Omega \neq \emptyset$. Let $\mathbf{Q}$ denote $\mathbf{Q} = I - \mathbf{P}$. We call Ω *admissible* if there exists $r \ge n$ and a constant $C = C_\Omega > 0$ such that

$$\sup_{x\in\Omega} d_\Omega(x)\left|\mathbf{Q}[\nabla\cdot f](x)\right| \le C_\Omega\|f\|_{L^\infty(\partial\Omega)} \tag{1.8}$$

holds for all matrix-valued function $f = (f_{ij})_{1\leq i,j\leq n} \in C^1(\bar{\Omega})$ which satisfies $\nabla \cdot f = (\sum_{j=1}^n \partial_j f_{ij})_{1\leq i\leq n} \in \tilde{L}^r(\Omega)$,

$$\mathrm{tr}\, f = 0 \quad \text{and} \quad \partial_l f_{ij} = \partial_j f_{il} \tag{1.9}$$

for $i, j, l \in \{1, \cdots, n\}$, where $\partial_j = \partial/\partial x_j$.

Theorem 1.2 ([1]). *Let Ω be an admissible, uniformly C^3-domain. Then, there exists positive constants C and T_0 depending only on Ω such that (1.6) holds for all $\tilde{L}^r$-solutions $(r > n)$ with $v_0 \in C^\infty_{c,\sigma}(\Omega)$. Further, $S(t)$ is uniquely extendable to a C_0-analytic semigroup in $C_{0,\sigma}(\Omega)$ with (1.6) under a suitable choice of pressure q.*

Our theory implies that if a given domain Ω is admissible, it immediately implies that $S(t)$ is a C_0-analytic semigroup in $C_{0,\sigma}(\Omega)$ with the estimates (1.6). (The pressure q is also approximated in the uniform topology.) If the L^r-theory works, the assertion of Theorem 1.2 is still valid even if we replace $\tilde{L}^r$-solutions to L^r-solutions with $r > n$. Since an exterior domain is such an example, we apply the estimate (1.6) for L^r-solutions instead of $\tilde{L}^r$-solutions to extend the Stokes semigroup to the spaces of bounded functions. It is noted that the estimate (1.6) is extendable for an arbitrary time interval by the semigroup property of $S(t)$, so the time T_0 can be taken arbitrary. We only need to show that an exterior domain is admissible to apply Theorem 1.2 to conclude that $S(t)$ is a C_0-analytic semigroup in $C_{0,\sigma}$. So the first goal of this paper is to prove

Theorem 1.3. *An exterior domain in $\mathbf{R}^n (n \geq 2)$ with C^3 boundary is admissible.*

In this paper we introduce a stronger notion of admissibility called a *strictly admissible domain* to extend the Stokes semigroup to spaces of non-decaying solenoidal vector fields. Actually our a priori estimate (1.6) is extendable for non-decaying solutions in a strictly admissible domain as explained later in the introduction. We prove that an exterior domain is indeed strictly admissible.

1.2. **Non-decaying solenoidal spaces.** The second goal of this paper is to extend the Stokes semigroup to spaces of non-decaying bounded functions. We handle initial data in the space $L^\infty_\sigma(\Omega)$ which is larger than $C_{0,\sigma}(\Omega)$ defined by

$$L^\infty_\sigma(\Omega) = \left\{ f \in L^\infty(\Omega) \;\middle|\; \int_\Omega f \cdot \nabla\varphi dx = 0 \quad \text{for all} \quad \varphi \in \hat{W}^{1,1}(\Omega) \right\},$$

where $\hat{W}^{1,1}(\Omega)$ is the homogeneous Sobolev space of the form

$$\hat{W}^{1,1}(\Omega) = \left\{ \varphi \in L^1_{\mathrm{loc}}(\Omega) \;\middle|\; \nabla\varphi \in L^1(\Omega) \right\}.$$

If Ω is bounded, L^r-theory yields a solution for initial data $v_0 \in L^\infty_\sigma$ since $L^\infty_\sigma \subset L^r_\sigma$ for $r \in (1, \infty)$. Moreover, the estimate (1.6) holds for all $v_0 \in L^\infty_\sigma$ and $S(t)$ can be regarded as an analytic semigroup in L^∞_σ [1, Theorem 1.4]. However, for a general unbounded domain even existence of a solution is unknown for $v_0 \in L^\infty_\sigma$. When $\Omega = \mathbf{R}^n_+$, an explicit solution formula [56] yields a solution (v, q) satisfying initial data $v_0 \in L^\infty_\sigma$ $*$-weakly in $L^\infty(\Omega)$. (If

$v(\cdot,t) \to v_0$ as $t \downarrow 0$ $*$-weakly in L^∞, we simply say that v has initial data v_0 $*$-weakly in L^∞.) From a direct estimate, this solution (v,q) satisfies the estimates (1.6) and

$$\sup\left\{ t^{1/2} d_\Omega(x) \left|\nabla q(x,t)\right| \,\middle|\, x \in \Omega,\ t \in (0,T_0) \right\} \leq C\|v_0\|_\infty,$$

where $\Omega = \mathbf{R}^n_+$ with $d_{\mathbf{R}^n_+}(x) = x_n$. The uniqueness of this solution is also proved in [56] based on a duality argument. Note that above pressure estimate implies a decay condition for ∇q to the normal direction, i.e. $\nabla q \to 0$ as $x_n \to \infty$. Such a decay for ∇q is necessary to assert the uniqueness of a non-decaying solution to exclude Poiseuille flows in $\mathbf{R}^n_+$, non-trivial solutions which do not decay at the space infinity [1, Remark 4.2]. When Ω is an exterior domain, by reducing the problem with a compactly supported external force we are able to construct a solution for $v_0 \in L^\infty_\sigma$ with help of the stationary Stokes the boundary-value problem [40]. We prove both existence and uniqueness of a solution with $v_0 \in L^\infty_\sigma$ by appealing to a priori estimate (1.7) directly. Note that a bound for $d_\Omega(x)|\nabla q(x,t)|$ does not imply a decay for pressure at the space infinity. Such a bound still allows growing q as $|x| \to \infty$. We prove the uniqueness without assuming any spatial decay condition for velocity and also pressure not only for exterior domains but also for general strictly admissible domains. For a precise statement, we define a solution for $v_0 \in L^\infty_\sigma$ and simply call it L^∞-solution.

Definition 1.4 (L^∞-solution)**.** Let Ω be a domain in $\mathbf{R}^n (n \geq 2)$ with $\partial\Omega \neq \emptyset$. Let $(v, \nabla q) \in C^{2,1}(\bar{\Omega} \times (0,T]) \times C(\bar{\Omega} \times (0,T])$ solve (1.1)-(1.3) and satisfy (1.4) with $v_0 \in L^\infty_\sigma(\Omega)$ in the sense that $v(\cdot,t) \to v_0$ $*$-weakly in $L^\infty(\Omega)$ as $t \downarrow 0$. We call (v,q) L^∞-*solution* if quantities (1.5) and

$$t^{1/2} d_\Omega(x) \left|\nabla q(x,t)\right| \tag{1.10}$$

are bounded in $\Omega \times (0,T)$.

Here is our main result.

Theorem 1.5. *Let Ω be an exterior domain in $\mathbf{R}^n (n \geq 2)$ with C^3 boundary.*
(i) (Unique existence of an L^∞-solution)
For $v_0 \in L^\infty_\sigma(\Omega)$ there exists a unique L^∞-solution (v,q) satisfying (1.6) for any T_0 with some constant C depending only on T_0 and Ω.
(ii) (Analyticity in L^∞_σ)
The Stokes semigroup $S(t)$ is uniquely extendable to a (non C_0-) analytic semigroup in $L^\infty_\sigma(\Omega)$.

For a non-densely defined sectorial operator in a Banach space X it is known that it generates a semigroup in X satisfying the properties of a usual analytic semigroup except a continuity at time zero [50, 1.1.2]; see also [36] for basic properties of an analytic semigroup and generation results in L^∞ for general elliptic operators in various situations. Here we call a semigroup $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ analytic if $t\|dT(t)/dt\|_{\mathcal{L}}$ is bounded for $t \in (0,1]$, where $\mathcal{L}= \mathcal{L}(X)$ denotes the space of all bounded linear operators from the Banach space X onto itself equipped with a usual operator norm $\|\cdot\|_{\mathcal{L}}$. Note that $T(t)x \to x$ as $t \downarrow 0$ in X holds for x which belongs to the closure of $D(A)$ in X, where $D(A)$ is the domain of the generator A of $T(t)$. If an analytic semigroup $T(t)$ is not a C_0-semigroup, we say $T(t)$ is a non C_0-analytic semigroup. The analyticity of $S(t)$ in L^∞_σ follows from the estimate (1.6) for $v_0 \in L^\infty_\sigma$, so once the first assertion (i) is proved, the second assertion (ii) easily follows. Since $S(t)v_0 \to v_0$ as $t \downarrow 0$ in L^∞_σ is not always true for some $v_0 \in L^\infty_\sigma$, it is natural to restrict initial data v_0 to a space of uniformly continuous functions to assert that $S(t)$ is a C_0-analytic semigroup in that space. We shall discuss a continuity at time zero of $S(t)$ later in the introduction.

It is well-known that the Stokes semigroup $S(t)$ is a bounded analytic semigroup in L^r_σ [11], [33], [12] in the sense that both $\|S(t)\|_{\mathcal{L}}$ and $t\|dS(t)/dt\|_{\mathcal{L}}$ are bounded in $(0,\infty)$, where $X = L^r_\sigma$ for $r \in (1,\infty)$. Our estimate (1.6) here gives a local-in-time bound for $S(t)$. Recently, based on boundedness of $\|S(t)\|_{\mathcal{L}}$ proved in [1] for a bounded domain Maremonti [40] proved the uniform bound for $S(t)$, i.e. $\|S(t)v_0\|_\infty \leq C\|v_0\|_\infty$ for $t > 0$, $v_0 \in L^\infty_\sigma$ with some constant C independent of time $t > 0$. This type result is called a maximum modulus Theorem [59], first proved locally in time for exterior domains in [54] by a potential theoretic method. Note that it is unknown whether $t\|dS(t)/dt\|_{\mathcal{L}}$ is bounded in $(0,\infty)$.

Although recently, we find a way to prove analyticity in L^∞-type spaces [2] by resolvent estimates, our approach here gives a priori estimate for higher derivative which does not follow resolvent estimate in direct way.

1.3. **Approximation for initial data.** Let us give ideas in proving the assertion (i) of Theorem 1.5. Our basic approach is to approximate a solution by appealing to a priori estimate (1.6) which is available in an admissible domain. We approximate a solution for initial data $v_0 \in L^\infty_\sigma$ by choosing a compactly supported initial sequence $\{v_{0,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}$ satisfying

$$\|v_{0,m}\|_\infty \leq C\|v_0\|_\infty$$

and

$$v_{0,m} \to v_0 \quad \text{a.e. in } \Omega$$

as $m \to \infty$ with a constant C depending only on Ω. When Ω is bounded, we are able to construct this sequence by localizing Ω into a star-shaped domain [1, Lemma 6.3]. We extend this approximation for an exterior domain by dividing the domain Ω to reduce it to the cases of a bounded domain and $\mathbf{R}^n$. By this approximation we are able to choose approximate solutions by L^r-solutions ($r > n$), (v_m, q_m) with initial data $v_{0,m}$. Since $v_{0,m} \to v_0$ as $m \to \infty$, it is natural to expect that (v_m, q_m) converges to a solution with initial data $v_0 \in L^\infty_\sigma$. In fact, the sequence of (v_m, q_m) (subsequently) converges to an L^∞-solution (v, q) with $v_0 \in L^\infty_\sigma$ satisfying (1.6) and (1.7). Note that the approximation topology is not uniform.

A key step is to solve the uniqueness problem for the limit (v, q). If an L^∞-solution is unique, the limit (v, q) is independent of a choice of approximation so $S(t)$ is uniquely extendable for all $v_0 \in L^\infty_\sigma$. When the problem is the heat equation, the uniqueness easily follows from a bound for the quantity (1.5) (disregarding the pressure term). However, it may not be enough in general to assert the uniqueness to the Stokes equations. We invoke a bound for pressure (1.10) besides (1.5) to solve the uniqueness problem.

1.4. **Uniqueness for non-decaying solutions.** We appeal to a blow-up argument as in [1] to show the estimate (1.6) for L^∞-solutions with $v_0 = 0$ which immediately implies the uniqueness. Actually, we are able to prove the estimate (1.6) even if $v_0 \neq 0$. We derive the uniqueness from the extension of a priori estimate (1.6). Although our blow-up argument still works for L^∞-solutions to extend the estimate (1.6), the pressure estimate (1.7) does not follow directly from the estimate (1.8) in an admissible domain. The estimate (1.7) implies a Hölder continuity for pressure in time which is a key to get a compactness for solutions of (1.1)-(1.4) to apply a blow-up argument. Because of the unboundedness of the projection $\mathbf{Q}$ in L^∞, the representation $\nabla q = \mathbf{Q}[\Delta v]$ is no longer available for L^∞-solutions. This is the reason why (1.8) does not follows imply (1.7) directly. To overcome this difficulty we go back to the Neumann problem for pressure q to give a suitable definition of an admissible domain without invoking the projection $\mathbf{Q}$.

The estimate (1.7) is a regularizing type estimate for the homogeneous Neumann problem for pressure q,

$$\Delta q = 0 \text{ in } \Omega, \quad \partial q/\partial n_\Omega = \Delta v \cdot n_\Omega \text{ on } \partial\Omega.$$

In fact, one can observe that q is harmonic in Ω (for each time) by taking a divergence of (1.1). Since a normal component of velocity is zero on $\partial\Omega$, the Neumann data of q equals $\Delta v \cdot n_\Omega$, where n_Ω denotes the outward unit normal vector field of $\partial\Omega$. We invoke the fact [1, Remark 2.7 (ii)] that the divergence-free condition (1.2) implies that the Neumann data can be transformed into a surface divergence form,

$$\Delta v \cdot n_\Omega = \operatorname{div}_{\partial\Omega} W(v) \text{ on } \partial\Omega, \quad \text{with } W(v) = -\Big(\nabla v - (\nabla v)^T\Big) \cdot n_\Omega,$$

where $f^T = (f_{ji})_{1\leq i,j\leq n}$ denotes the transpose of a matrix $f = (f_{ij})_{1\leq i,j\leq n}$ with $f_{ij} = \partial_j v^i$ and $f \cdot n_\Omega = (\sum_{j=1}^n f_{ij} n_\Omega^j)_{1\leq i\leq n}$. Here we do not invoke the Dirichlet boundary condition $v = 0$ on $\partial\Omega$ so this representation is still valid for the Robin-type boundary condition; see Remark 3.2 (iii). Then the problem is reduced to the estimate of solutions for the Neumann problem of the form,

$$\Delta P = 0 \text{ in } \Omega, \quad \partial P/\partial n_\Omega = \operatorname{div}_{\partial\Omega} W \text{ on } \partial\Omega, \tag{1.11}$$

for a tangential vector field W on $\partial\Omega$. Note that when $n = 3$, $W(v)$ is a tangential trace of vorticity, i.e. $W(v) = \operatorname{curl} v \times n_\Omega$. We observe that a priori estimate

$$\sup_{x\in\Omega} d_\Omega(x) \left|\nabla P(x)\right| \leq C\|W\|_{L^\infty(\partial\Omega)} \tag{1.12}$$

implies the estimate (1.7) for L^∞-solutions. In fact, if one takes $P = q$ and $W = W(v)$ for an L^∞-solution (v, q), (1.7) follows from (1.12), i.e.

$$\begin{aligned}\sup_{x\in\Omega} d_\Omega(x)\left|\nabla q(x,t)\right| &\leq C\|W(v)\|_{L^\infty(\partial\Omega)}(t)\\ &\leq 2C\|\nabla v\|_{L^\infty(\partial\Omega)}(t).\end{aligned}$$

Since (1.12) may not hold for a general domain, we call Ω *strictly admissible* if a priori estimate (1.12) holds for all solutions of (1.11) with a tangential vector field W. Of course, a strictly admissible domain is admissible. Although the converse does not hold in general, a half space and a bounded domain are still strictly admissible; see Remark 2.4. We prove that an exterior domain with C^3 boundary is also strictly admissible by a blow-up argument. Let us give a general uniqueness result which includes exterior domains as a particular case.

Theorem 1.6 (Uniqueness of an L^∞-solution)**.** *In Theorem 1.2, assume that Ω is strictly admissible. Then an L^∞-solution is unique. Moreover, a priori estimates (1.6) is extendable for all L^∞-solutions with $v_0 \in L^\infty_\sigma(\Omega)$ for any $T_0 > 0$ with some constant C depending only on T_0 and Ω.*

If the approximation for L^∞_σ is established, our argument can be adjusted to show the existence of an L^∞-solution in a general strictly admissible domain. Although L^r-theory works for various kinds of domains such as a perturbed half space [19], a layer domain [3] and an aperture domain [20] (even for variable viscosity coefficients [4], [5]), for a uniformly C^3-domain we have to appeal to $\tilde{L}^r$-theory instead of L^r-theory to approximate a solution. Note that L^r-theory works for domains of uniformly C^3 where the Helmholtz decomposition holds in L^r [24], so it is enough to use L^r-theory to approximate a solution in those domains for the analyticity of $S(t)$ in L^∞_σ.

1.5. **Strictly admissible domains.** We prove that an exterior domain with C^3 boundary is strictly admissible by a blow-up argument. Suppose that the estimate (1.12) were false for any choice of a constant C. Then there would exist a sequence of $\{P_m\}_{m=1}^\infty$ and a sequence of points $\{x_m\}_{m=1}^\infty$ such that

$$\frac{1}{2} \leq d_\Omega(x_m)\left|\nabla P_m(x_m)\right| \leq \sup_{x\in\Omega} d_\Omega(x)\left|\nabla P_m(x)\right| = 1. \tag{1.13}$$

If $\{x_m\}_{m=1}^\infty$ converges to an interior point $x_\infty \in \Omega$, we solve a uniqueness problem of (1.11) for a limit P with assuming a bound

$$\sup_{x\in\Omega} d_\Omega(x)\left|\nabla P(x)\right| < \infty. \tag{1.14}$$

By a duality argument we prove the uniqueness of a solution and conclude that $\nabla P \equiv 0$ which is a contradiction to $1/2 \leq d_\Omega(x_\infty)|\nabla P(x_\infty)|$. A compactness of $\{P_m\}_{m=1}^\infty$ easily follows since P_m is harmonic in Ω. If $\{x_m\}_{m=1}^\infty$ converges to $\partial\Omega$, a contradiction occurs by appealing to the same scaling argument for the case of a bounded domain as proved in [1]. Thus the first two cases do not occur neither.

A key step is to solve the case when $\{x_m\}_{m=1}^\infty$ goes to the space infinity. We reduce the problem to $\mathbf{R}^n$ by rescaling (downscaling) the solution P_m around the point x_m by setting as

$$Q_m(x) = P_m(x_m + d_m x) \quad \text{with } d_m = d_\Omega(x_m).$$

Since $d_m \to \infty$ as $m \to \infty$, a complement of the downscaled domain Ω_m, accumulates to some point $a \in \mathbf{R}^n$ ($a \neq 0$) so the limit domain is $\mathbf{R}^n\backslash\{a\}$. We remove a singularity at the point $x = a$ for a limit Q with a bound

$$\sup_{x\in\mathbf{R}^n\backslash\{a\}} \left|x-a\right|\left|\nabla Q(x)\right| < \infty. \tag{1.15}$$

Although for $n = 2$ the bound (1.15) is not enough to remove the singularity, appealing to a mean value of Q (inherited from P_m) around the point $x = a$, we conclude this singularity is still removable.

When a domain Ω has a non-compact boundary, a downscaled domain Ω_m does not go to $\mathbf{R}^n$ in general so this procedure is difficult to carry out for a general unbounded domain. We conjecture that an unbounded domain (with smooth boundary) is strictly admissible if and only if Ω is not quasicylindrical (see [6, 6.32]), i.e. $\overline{\lim}_{|x|\to\infty} d_\Omega(x) = \infty$. In fact, layer domains and cylinders are not strictly admissible since the uniqueness under the bound (1.14) is not valid; see Remark 2.4 (v).

1.6. **Continuity at time zero.** Finally we conclude that $S(t)$ is a C_0-analytic semigroup in a space of uniformly continuous functions. We prove a continuity at time zero of $S(t)$ in $BUC_\sigma(\Omega)$ which is vanishing on $\partial\Omega$ defined by

$$BUC_\sigma(\Omega) = \Big\{ f \in BUC(\Omega) \ \Big|\ \operatorname{div} f = 0 \text{ in } \Omega,\ f = 0 \text{ on } \partial\Omega \Big\},$$

where $BUC(\Omega)$ denotes the space of all uniformly continuous functions. When $\Omega = \mathbf{R}^n_+$, an explicit solution formula implies that $S(t)$ is a C_0-analytic semigroup in BUC_σ [14] (see also [44], [56]). Note that if Ω is bounded, the space $BUC_\sigma(\Omega)$ agrees with $C_{0,\sigma}(\Omega)$ [37],[1]. Although we do not discuss a domain of the Stokes operator in BUC_σ, we are able to prove continuity of $S(t)$ at $t = 0$ in BUC_σ. We reduce it to the uniqueness problem. Observe that for $v_0 \in C^\infty_{c,\sigma}$,

$$v_t(\cdot, t) = S(t)\Delta v_0 \tag{1.16}$$

holds with $v = S(t)v_0$ since $\mathbf{P}\Delta v_0 = \Delta v_0$. Actually, (1.16) is still valid even for non-compactly supported v_0 as long as the support of v_0 is away from $\partial\Omega$. Appealing to the uniqueness result for L^∞-solution we prove (1.16) for such initial data v_0 by approximation. The convergence $S(t)v_0 \to v_0$ as $t \downarrow 0$ in BUC_σ easily follows from (1.16) for this v_0. For a general $v_0 \in BUC_\sigma$ we divide v_0 into two terms, compactly supported $v_0^1 \in C_{0,\sigma}(\Omega)$ and $v_0^2 \in BUC_\sigma(\Omega)$ whose support is away from $\partial\Omega$. Since $S(t)$ can be regarded as a C_0-semigroup in $C_{0,\sigma}(\Omega)$, the convergence of $S(t)v_0^1$ easily follows. So one can claim a continuity of $S(t)$ at time zero for BUC_σ. In addition to the main result (Theorem 1.5), we further obtain

Theorem 1.7 (Continuity in BUC_σ)**.** *In Theorem 1.5, $S(t)$ is a C_0-analytic semigroup in $BUC_\sigma(\Omega)$.*

The analyticity as well as (1.6) is fundamental to study the Navier-Stokes equations with non-decaying initial data in an exterior domain. Although one can handle non-decaying initial data of Hölder class by reducing it to the boundary-value problem to the Navier-Stokes equations [22], a direct semigroup approach with L^∞_σ initial data is still unknown. So far L^∞-type theory is only established when $\Omega = \mathbf{R}^n$ [29] (see also [31], [45]) and $\mathbf{R}^n_+$ [56], [7].

This paper is organized as follows. In Section 2 we define a strictly admissible domain and verify that a strictly admissible domain is indeed admissible. We prove that an exterior domain is strictly admissible which implies Theorem 1.3. In Section 3 we prove the estimate (1.7) for L^∞-solutions in a strictly admissible domain which yields a necessary compactness result to prove Theorem 1.6. In Section 4 we prove Theorem 1.6 by a blow-up argument. In Section 5 we construct approximate initial sequence for initial data in L^∞_σ in an exterior domain and prove Theorem 1.5. In Section 6 we prove Theorem 1.7.

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2. Strictly admissible domains

In this section we give a definition of a strictly admissible domain which is simpler but more restrictive definition of an admissible domain. We shall prove that an exterior domain is indeed strictly admissible.

2.1. **Definition of a strictly admissible domain.** We first give a rigorous definition of a solution for the Neumann problem,

$$\Delta P = 0 \text{ in } \Omega, \quad \partial P/\partial n_\Omega = \mathrm{div}_{\partial\Omega} W \text{ on } \partial\Omega. \tag{2.1}$$

Let Ω be a domain in $\mathbf{R}^n (n \geq 2)$ with C^1 boundary. We define a gradient on $\partial\Omega$ for a (scalar-valued) C^1-function φ in $\bar{\Omega}$ by a tangential component of $\nabla\varphi$, i.e.

$$\nabla_{\partial\Omega}\varphi = \nabla\varphi - n_\Omega(\partial\varphi/\partial n_\Omega),$$

and also define a surface divergence on $\partial\Omega$ by

$$\mathrm{div}_{\partial\Omega} h = \mathrm{tr}\, \nabla_{\partial\Omega} h$$

for a vector-valued C^1-function h, where $\nabla_{\partial\Omega} h = (\nabla_{\partial\Omega} h^1, \cdots, \nabla_{\partial\Omega} h^n)$. If a support of φh is compact on $\partial\Omega$, the Gauss-Green formula on $\partial\Omega$

$$\int_{\partial\Omega} h \cdot \nabla_{\partial\Omega}\varphi d\mathcal{H}^{n-1}(x) = -\int_{\partial\Omega} (\mathrm{div}_{\partial\Omega} h + \kappa h \cdot n_\Omega)\varphi d\mathcal{H}^{n-1}(x)$$

holds (see e.g. [28], [49]), where $\kappa = \kappa(x)$ denotes the mean curvature of $\partial\Omega$ and $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure.

To describe a solution for the Neumann problem (2.1) in a simple way we introduce function spaces $L^\infty_{\mathrm{tan}}(\partial\Omega)$ and $L^\infty_{d_\Omega}(\Omega)$. The space $L^\infty_{\mathrm{tan}}(\partial\Omega)$ denotes the closed subspace of all tangential vector fields in $L^\infty(\partial\Omega)$, the space of all essentially bounded functions on $\partial\Omega$ with respect to the $n-1$ dimensional Hausdorff measure $\mathcal{H}^{n-1}$ equipped with the norm $\|\cdot\|_{L^\infty(\partial\Omega)} = \|\cdot\|_{\infty,\partial\Omega}$. Here we say a vector field h is tangential if $h\cdot n_\Omega = 0$ on $\partial\Omega$. The space $L^\infty_{d_\Omega}(\Omega)$ denotes the space of all locally integrable functions f such that $d_\Omega f$ is essentially bounded in Ω equipped with a norm

$$|f|_{\infty,d_\Omega} = \sup_{x\in\Omega} d_\Omega(x)\,|f(x)|\,,$$

where $d_\Omega(x)$ denotes the distance function for $x \in \Omega$ from the boundary $\partial\Omega$. We define a solution for the Neumann problem (2.1) for $W \in L^\infty_{\mathrm{tan}}(\partial\Omega)$ in a weak sense.

Definition 2.1 (Weak solution)**.** Let Ω be a domain in $\mathbf{R}^n (n \geq 2)$ with C^1 boundary. We call $P \in L^1_{\mathrm{loc}}(\bar{\Omega})$ a *weak solution* of (2.1) for $W \in L^\infty_{\tan}(\partial\Omega)$ if $\nabla P \in L^\infty_{d_\Omega}(\Omega)$ satisfies

$$\int_\Omega P\Delta\varphi dx = \int_{\partial\Omega} W \cdot \nabla_{\partial\Omega}\varphi d\mathcal{H}^{n-1}(x) \tag{2.2}$$

for all $\varphi \in C^2_c(\bar{\Omega})$ satisfying $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$.

Remark 2.2. (i) Although $L^\infty_{d_\Omega}(\Omega) \not\subset L^r_{\mathrm{loc}}(\bar{\Omega})$ for any $r \in [1,\infty)$, $\nabla P \in L^\infty_{d_\Omega}(\Omega)$ implies that $P \in L^r_{\mathrm{loc}}(\bar{\Omega})$ for $r \in [1,\infty)$ so we are able to define the integral in the left-hand side of (2.2) for $\nabla P \in L^\infty_{d_\Omega}(\Omega)$. It is noted that if W is sufficiently smooth, the Gauss-Green formula implies that

$$\int_{\partial\Omega} W \cdot \nabla_{\partial\Omega}\varphi d\mathcal{H}^{n-1}(x) = -\int_{\partial\Omega} \mathrm{div}_{\partial\Omega} W\varphi d\mathcal{H}^{n-1}(x)$$

since $W \cdot n_\Omega = 0$ $\partial\Omega$.
(ii) For a general $W \in L^\infty_{\tan}(\partial\Omega)$ a weak solution of (2.1) exists at least when Ω has a compact boundary. Moreover, we are able to construct a solution operator $\mathbf{K} : L^\infty_{\tan}(\partial\Omega) \to L^\infty_{d_\Omega}(\Omega)$ for (2.1). We call this operator $\mathbf{K}$ harmonic-pressure operator; see Remark 2.10. The Neumann problem of the form (2.1) is studied in the literature [53], [55], [8] to construct a solution for the Dirichlet-boundary value problem to the Stokes equations.

We now define a strictly admissible domain.

Definition 2.3 (Strictly admissible domain)**.** Let Ω be a domain in $\mathbf{R}^n (n \geq 2)$ with C^1 boundary. We call Ω *strictly admissible* if there exists a constant $C = C_\Omega > 0$ such that a priori estimate

$$\left|\nabla P\right|_{\infty,d_\Omega} \leq C_\Omega \|W\|_{\infty,\partial\Omega} \tag{2.3}$$

holds for all weak solutions of (2.1) with $W \in L^\infty_{\tan}(\partial\Omega)$.

Remark 2.4. (i) The constant C_Ω in (2.3) is independent of dilation and translation of Ω, i.e. $C_{\lambda\Omega+x_0} = C_\Omega$ for $\lambda > 0$ and $x_0 \in \Omega$.
(ii) The uniqueness of a weak solution of (2.1) is a necessary condition to assert that a given domain Ω is strictly admissible. If $\nabla P \in L^\infty_{d_\Omega}(\Omega)$ solves (2.1) with $W = 0$ in a strictly admissible domain, $\nabla P = 0$ immediately follows from the estimate (2.3).
(iii) A half space $\mathbf{R}^n_+$ is strictly admissible. Since a weak solution is unique in $\mathbf{R}^n_+$ [1, Lemma 2.9], a weak solution P can be represented explicitly by the Poisson semigroup $\mathcal{P}_s$, i.e.

$$\begin{aligned}\nabla P &= \nabla \int_{x_n}^\infty \mathcal{P}_s[-\mathrm{div}_{\partial\mathbf{R}^n_+} W] ds \\ &= \nabla \int_{x_n}^\infty \int_{\mathbf{R}^{n-1}} \nabla_{\partial\mathbf{R}^n_+} P_s(x'-y') \cdot W(y') dy' ds\end{aligned}$$

with the kernel of the Poisson semigroup $P_s(x') = as/(|x'|^2 + s)^{n/2}$, $x' \in \mathbf{R}^{n-1}$, where $2/a$ is the surface area of the $n-1$ dimensional unit sphere. As in [1, Remark 2.4 (iv)] a direct estimate implies that $\mathbf{R}^n_+$ is strictly admissible.

(iv) In [1] we proved that a bounded domain is admissible by a blow-up argument. Our proof still works for weak solutions of (2.1) to derive the estimate (2.3) (see [1, Remark 2.7 (ii)].) Thus our proof in [1] can be adjusted to prove that a bounded C^3-domain is strictly admissible.
(v) Layer domains and cylindrical domains are not strictly admissible. For instance, in a layer domain $\Omega = \{a < x_n < b\}$, $P = x^1$ is a non-trivial weak solution with $W = 0$.

2.2. **Uniformly C^k-domains.** We shall verify that a strictly admissible domain is evidently admissible. Admissible domain is defined with the Helmholtz projection operator associated to the Helmholtz decomposition, a topological direct sum decomposition of the form

$$L^r(\Omega) = L^r_\sigma(\Omega) \oplus G^r(\Omega), G^r(\Omega) = \{\nabla p \in L^r(\Omega) \bigm| p \in L^r_{\mathrm{loc}}(\Omega)\},$$

and $L^r_\sigma(\Omega)$, the L^r-closure of $C^\infty_{c,\sigma}(\Omega)$ for $r \in (1,\infty)$. We suppress a subscript of $\mathbf{P} = \mathbf{P}_r : L^r(\Omega) \to L^r_\sigma(\Omega)$ and also $\mathbf{Q} = \mathbf{Q}_r$, $\mathbf{Q}_r = I - \mathbf{P}_r$. Although this decomposition is known to hold (see e.g. [21, III.1]) for various domains like a bounded or exterior domain with smooth boundary, in general there is a domain with (uniformly) smooth boundary such that the L^r-Helmholtz decomposition does not hold (cf. [10], [41]).

In [16] Farwig, Kozono and Sohr introduced an $\tilde{L}^r$ space and proved that Helmholtz decomposition is valid for any uniformly C^2-domain for $n = 3$. Later, it is generalized for arbitrary uniformly C^1-domain for $n \geq 2$ [17]. We set

$$\tilde{L}^r(\Omega) = \begin{cases} L^2(\Omega) \cap L^r(\Omega), 2 \leq r < \infty \\ L^2(\Omega) + L^r(\Omega), 1 < r < 2. \end{cases}$$

Note that $\tilde{L}^{r_1} \subset \tilde{L}^r$ for $r_1 > r$. We define $\tilde{L}^r_\sigma$ and $\tilde{G}^r$ in a similar way. In this paper we shall use $\tilde{L}^r$ space for $r \geq 2$ so $\tilde{L}^r$ norm is given as

$$\|f\|_{\tilde{L}^r} = \max(\|f\|_{L^r}, \|f\|_{L^2}).$$

We then recall a definition of uniformly C^k-domain for $k \geq 1$; see e.g. [51, I.3.2].

Definition 2.5 (Uniformly C^k-domain)**.** Let Ω be a domain in $\mathbf{R}^n$ with $n \geq 2$. Assume that there exists $\alpha, \beta, K > 0$ such that for each $x_0 \in \partial\Omega$, there exists C^k-function h of $n-1$ variable y' such that

$$\sup_{|l| \leq k, |y'| < \alpha} \left|\partial^l_{y'} h(y')\right| \leq K,\ \nabla' h(0) = 0,\ h(0) = 0$$

and denote a neighborhood of x_0 by

$$U_{\alpha,\beta,h}(x_0) = \{(y', y_n) \in \mathbf{R}^n \bigm| h(y') - \beta < y_n < h(y') + \beta, |y'| < \alpha\}.$$

Assume that up to rotation and translation we have

$$U_{\alpha,\beta,h}(x_0) \cap \Omega = \{(y', y_n) \bigm| h(y') < y_n < h(y') + \beta,\ |y'| < \alpha\}$$

and

$$U_{\alpha,\beta,h}(x_0) \cap \partial\Omega = \{(y', y_n) \bigm| y_n = h(y'), |y'| < \alpha\}.$$

Then we call Ω a uniformly C^k-domain of type α, β, K. Here $\partial^l_x = \partial^{l_1}_{x_1} \cdots \partial^{l_n}_{x_n}$ with multi-index $l = (l_1, \ldots, l_n)$ and $\partial_{x_j} = \partial/\partial x_j$ as usual and ∇' denotes the gradient in $y' \in \mathbf{R}^{n-1}$.

We now verify that a strictly admissible domain is indeed admissible.

Proposition 2.6. *Let Ω be a strictly admissible domain in $\mathbf{R}^n (n \geq 2)$ with uniformly C^1 boundary. Then Ω is admissible.*

Proof. Let $f = (f_{ij})_{1 \leq i,j \leq n} \in C^1(\bar{\Omega})$ be a matrix-valued function satisfying $\nabla \cdot f \in \tilde{L}^r(\Omega) (r \geq n)$ and the condition (1.9). We shall show that $\nabla P = \mathbf{Q}[\nabla \cdot f]$ is a weak solution of (2.1) with $W = -(f - f^T) \cdot n_\Omega$. This W is a tangential vector field on $\partial\Omega$, i.e.

$$\begin{aligned} W \cdot n_\Omega &= -\sum_{i,j=1}^{n} (f_{ij} - f_{ji}) n_\Omega^j n_\Omega^i \\ &= 0, \end{aligned}$$

so $W \in L^\infty_{\mathrm{tan}}(\partial\Omega)$. We first show that P satisfies (2.2), i.e.

$$\int_\Omega P \Delta\varphi dx = \int_{\partial\Omega} W \cdot \nabla_{\partial\Omega} \varphi d\mathcal{H}^{n-1}(x)$$

for all $\varphi \in C_c^2(\bar{\Omega})$ satisfying $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$. Since $\nabla\varphi$ is orthogonal to $\tilde{L}^r_\sigma$, it follows that

$$\begin{aligned} \int_\Omega \nabla P \cdot \nabla\varphi dx &= \int_\Omega (\nabla \cdot f) \cdot \nabla\varphi dx \\ &= \sum_{i,j=1}^{n} \int_\Omega \partial_j f_{ij} \partial_i \varphi dx. \end{aligned}$$

Integration by parts yields that

$$\begin{aligned} \int_\Omega \partial_j f_{ij} \partial_i \varphi dx &= -\int_\Omega f_{ij} \partial_j \partial_i \varphi dx + \int_{\partial\Omega} f_{ij} \partial_i \varphi n_\Omega^j d\mathcal{H}^{n-1}(x) \\ &= \int_\Omega \partial_i f_{ij} \partial_j \varphi dx + \int_{\partial\Omega} f_{ij} (\partial_i \varphi n_\Omega^j - \partial_j \varphi n_\Omega^i) d\mathcal{H}^{n-1}(x). \end{aligned}$$

Since $\partial_i f_{ij} = \partial_j f_{ii}$ and $\operatorname{tr} f = 0$, we have

$$\int_\Omega \nabla P \cdot \nabla\varphi dx = \sum_{i,j=1}^{n} \int_{\partial\Omega} f_{ij} (\partial_i \varphi n_\Omega^j - \partial_j \varphi n_\Omega^i) d\mathcal{H}^{n-1}(x).$$

By transposition of the indexes i and j of the last term, we have

$$\begin{aligned} \int_\Omega P \Delta\varphi dx &= -\sum_{i,j=1}^{n} \int_{\partial\Omega} (f_{ij} - f_{ji}) n_\Omega^j \partial_i \varphi d\mathcal{H}^{n-1}(x) \\ &= \int_{\partial\Omega} W \cdot \nabla_{\partial\Omega} \varphi dx \end{aligned}$$

since $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$. Thus P satisfies (2.2) with $W = -(f - f^T) \cdot n_\Omega$. We next show $\nabla P \in L^\infty_{d_\Omega}(\Omega)$. Since P is harmonic in Ω, the mean value formula yields that

$$\nabla P(x) = \fint_{B_x(\tau)} \nabla P(y) d\mathcal{H}^{n-1}(y) \quad \text{for } x \in \Omega \text{ with } \tau = d_\Omega(x).$$

Applying the Hölder inequality implies that

$$|\nabla P(x)| \leq \frac{C_s}{\tau^{n/s}} \|\nabla P\|_{L^s(\Omega)}$$

with the constant C_s depending on $s \in (1, \infty)$ but independent of $\tau = d_\Omega(x)$. If $d_\Omega(x) \leq 1$ take $s = r \geq n$. If $d_\Omega(x) > 1$, take $s = 2$. Since $\mathbf{Q}$ is bounded in $\tilde{L}^r(\Omega)$, the estimate

$$|\nabla P|_{\infty, d_\Omega} \leq C \|\nabla \cdot f\|_{\tilde{L}^r(\Omega)}$$

holds with the constant C depending on r. Thus P is a weak solution of (2.1). We now apply the estimate (2.3) for $\nabla P = \mathbf{Q}[\nabla \cdot f]$. Since Ω is strictly admissible, there exists a constant $C = C_\Omega$ depending only on Ω such that

$$\begin{aligned} \sup_{x \in \Omega} d_\Omega(x) \left| \mathbf{Q}[\nabla \cdot f](x) \right| &\leq C_\Omega \|(f - f^T) \cdot n_\Omega\|_{\infty, \partial\Omega} \\ &\leq 2C_\Omega \|f\|_{\infty, \partial\Omega} \end{aligned}$$

holds. The proof is now complete. □

2.3. **Uniqueness of the Neumann problem.** As we have seen in Remark 2.4 (ii), the uniqueness for the Neumann problem (2.1) is a necessary condition to assert that a given domain Ω is strictly admissible. To show that an exterior domain is strictly admissible, we first prove the uniqueness of a weak solution in an exterior domain by a duality argument.

Theorem 2.7 (Uniqueness for an exterior domain)**.** *Let Ω be an exterior domain with C^3 boundary. Then a weak solution of (2.1) is unique up to an additive constant.*

We first prepare the estimate near the space infinity for functions whose gradient belongs to $L^\infty_{d_\Omega}(\Omega)$.

Proposition 2.8. *Let Ω be an exterior domain in $\mathbf{R}^n (n \geq 2)$. Let $0 \in \Omega^c$ and $R_\Omega > diam\Omega^c$. For $\nabla \tilde{P} \in L^\infty_{d_\Omega}(\Omega)$, there are constants C_1 and C_2 depending on R_Ω and $|\nabla \tilde{P}|_{\infty, d_\Omega}$ such that*

$$\left| \tilde{P}(x) \right| \leq C_1 \log |x| + C_2 \quad \textit{for } |x| \geq 2R_\Omega \tag{2.4}$$

holds.

Proof. For $x \in \Omega$ there is some point $z \in \partial\Omega$ such that $d_\Omega(x) = |x - z|$. Since $\Omega^c \subset B_0(R_\Omega)$, a closed ball centered at the origin with radius R_Ω, we have $|x| \leq |x - z| + |z| \leq d_\Omega(x) + R$. Since $R_\Omega \leq |x|/2$, $|x| \leq d_\Omega(x)$ holds for $|x| \geq 2R_\Omega$. Thus we have

$$\sup_{|x| \geq 2R_\Omega} |x| \left| \nabla \tilde{P}(x) \right| \leq 2 \left| \nabla \tilde{P} \right|_{\infty, d_\Omega}$$

for $\nabla \tilde{P} \in L^\infty_{d_\Omega}(\Omega)$. For $|x| > 2R_\Omega$ we connect x to $y = 2R_\Omega x/|x|$ by a straight line $c(t) = tx + (1 - t)y$ to estimate $|\tilde{P}(x) - \tilde{P}(y)|$. Since $|c(t)| \geq 2R_\Omega$ and y is parallel to x, it follows that

$$
\begin{aligned}
\left|\tilde{P}(x)-\tilde{P}(y)\right| &\leq |x-y| \int_0^1 \left|\nabla \tilde{P}(c(t))\right| dt \\
&\leq |x-y| \left(\int_0^1 \frac{dt}{|y|+t|x-y|} \right) \sup_{|x|\geq 2R_\Omega} |x| \left|\nabla \tilde{P}(x)\right| \\
&\leq 2(\log|x|-\log 2R_\Omega) \left|\nabla \tilde{P}\right|_{\infty,d_\Omega}.
\end{aligned}
$$

By taking $C_1 = 2|\nabla \tilde{P}|_{\infty,d_\Omega}$ and $C_2 = -C_1 \log 2R_\Omega + \sup_{|y|=2R_\Omega} |\tilde{P}(y)|$, the estimate (2.4) follows. The estimate (2.4) also holds for $|x| = 2R_\Omega$ with the same constant C_1 and C_2. □

Proof of Theorem 2.7. Let $P \in L^1_{\mathrm{loc}}(\bar{\Omega})$ be a weak solution of (2.1) with $W = 0$. We consider the dual problem,

$$\Delta \varphi = \operatorname{div} g \text{ in } \Omega, \quad \partial\varphi/\partial n_\Omega = 0 \text{ on } \partial\Omega, \tag{2.5}$$

to show that

$$\int_\Omega P \operatorname{div} g dx = 0 \tag{2.6}$$

for all $g \in C_c^\infty(\Omega)$. For $g \in C_c^\infty(\Omega)$ there exists a solution of (2.5), $\nabla\varphi \in L^r(\Omega)$ for $r \in (1,\infty)$ satisfying the estimate

$$\|\nabla \varphi\|_{L^r(\Omega)} \leq C_r \|g\|_{L^r(\Omega)}, \tag{2.7}$$

with C_r depending on r and Ω [47]. Since $g \in C_c^\infty(\Omega), \nabla\varphi$ belongs to $L^r(\Omega)$ for all $r \in (1,\infty)$ and (2.7) holds. To substitute φ into (2.2) with $W = 0$, we set $\varphi_R = \varphi\theta_R$ with a cut-off function θ_R so that φ_R is a compactly supported in $\bar{\Omega}$. Let θ be a smooth function in $[0,\infty)$ satisfying $\theta \equiv 1$ in $[0,1/2]$ and $\theta \equiv 0$ in $[1,\infty)$. Set $\theta_R(x) = \theta(|x|/R)$ with $R > 2R_\Omega,\ R_\Omega > \operatorname{diam} \Omega^c$ and observe that

$$\Delta \varphi_R = \operatorname{div} g\theta_R + 2\nabla\varphi \cdot \nabla\theta_R + \varphi\Delta\theta_R \quad \text{in } \Omega,$$

and $\partial\varphi_R/\partial n_\Omega = 0$ on $\partial\Omega$. Since Ω has C^3 boundary, the elliptic regularity theory [33, Lemma 2.3] implies that φ_R is a C^2 function in $\bar{\Omega}$. We substitute φ_R into (2.2) with $W = 0$ to get

$$\int_\Omega (P \operatorname{div} g\, \theta_R + 2P\nabla\varphi \cdot \nabla\theta_R + P\varphi\Delta\theta_R)\, dx = 0. \tag{2.8}$$

To complete the proof it suffices to show the last two terms vanish as $R \to \infty$. We first estimate the second term. By Proposition 2.8 and the estimate (2.7), it follows that

$$
\begin{aligned}
\left|\int_\Omega P\nabla\varphi \cdot \nabla\theta_R dx\right| &\leq \frac{(C_1 \log R + C_2)}{R} \|\nabla\theta\|_\infty \int_{R/2<|x|<R} |\nabla\varphi| dx \\
&\leq \frac{(C_1 \log R + C_2)}{R} \|\nabla\theta\|_\infty (C_n R^n)^{1-1/r} \|\nabla\varphi\|_{L^r(\Omega)} \\
&\leq C \frac{(C_1 \log R + C_2)}{R^{1+n/r-n}} \|g\|_{L^r(\Omega)}
\end{aligned}
$$

for $R > 4R_\Omega$ where the constant C_n is the volume of n dimensional unit ball and $C = {C_n}^{1-1/r}C_r\|\nabla\theta\|_\infty$. We now take $r \in (1, n/(n-1))$ to observe that the right-hand side vanishes as $R \to \infty$. It remains to estimate the last term of (2.8). Since P is harmonic in Ω and a support of $\Delta\theta_R$ is in an annulus $D_R = \operatorname{int} B_0(R)\backslash B_0(R/2)$, we may replace φ to

$$\tilde{\varphi}(x) = \varphi(x) - \fint_{D_R} \varphi(x)dx.$$

The Poincaré inequality [?, 5.8.1] implies that

$$\|\tilde{\varphi}\|_{L^r(D_R)} \leq C_0 R\|\nabla\varphi\|_{L^r(D_R)}$$

with the constant C_0 independent of R. Thus we have

$$\begin{aligned}\left|\int_\Omega P\tilde{\varphi}\Delta\theta_R dx\right| \leq& \frac{(C_1 \log R + C_2)}{R^2}\|\Delta\theta\|_\infty \int_{D_R} |\tilde{\varphi}|dx \\ \leq& \frac{(C_1 \log R + C_2)}{R^2}\|\Delta\theta\|_\infty (C_n R^n)^{1-1/r}\|\tilde{\varphi}\|_{L^r(D_R)} \\ \leq& C' \frac{(C_1 \log R + C_2)}{R^{1+n/r-n}}\|g\|_{L^r(\Omega)}\end{aligned}$$

with the constant $C' = {C_n}^{1-1/r}C_r C_0\|\Delta\theta\|_\infty$. Since $r \in (1, n/(n-1))$, letting $R \to \infty$ implies that the right-hand side goes to zero. Thus we have (2.6) for all $g \in C_c^\infty(\Omega)$ which implies that $\nabla P \equiv 0$. The proof is now complete. □

2.4. **Blow-up arguments.** We now prove that an exterior domain is strict admissible by a blow-up argument. To remove a singularity for a limit of downscaled solutions we apply a criterion on singularities for a harmonic function (Lemma 2.11 in the next subsection).

Theorem 2.9. *An exterior domain in $\mathbf{R}^n (n \geq 2)$ with C^3 boundary is strictly admissible.*

Proof. We argue by contradiction. Suppose that a priori estimate (2.3) were false, then there would exist a sequence of weak solutions $\{\tilde{P}_m\}_{m=1}^\infty$ such that

$$\left|\nabla\tilde{P}_m\right|_{\infty,d_\Omega} > m\|\tilde{W}_m\|_{\infty,\partial\Omega}. \tag{2.9}$$

We take a point $x_m \in \Omega$ such that

$$d_\Omega(x_m)\left|\nabla\tilde{P}_m(x_m)\right| \geq 1/2\left|\nabla\tilde{P}_m\right|_{\infty,d_\Omega},$$

and normalize the solution dividing by $M_m = |\nabla\tilde{P}_m|_{\infty,d_\Omega}$ to get

$$|\nabla P_m|_{\infty,d_\Omega} = 1, \tag{2.10}$$

$$d_\Omega(x_m)\left|\nabla P_m(x_m)\right| \geq 1/2, \tag{2.11}$$

$$\|W_m\|_{\infty,\partial\Omega} < 1/m,$$

with $P_m = 1/M_m\tilde{P}_m, W_m = 1/M_m\tilde{W}_m$. Then the situation can be divided into two cases depending on whether $d_m = d_\Omega(x_m)$ converges or not.

Case 1. $\overline{\lim}_{m\to\infty} d_m < \infty$. We may assume that the sequence $\{x_m\}_{m=1}^\infty$ converges to some point $x_\infty \in \bar{\Omega}$ by taking a subsequence. If $x_\infty \in \partial\Omega$, a contradiction occurs by rescaling P_m around the point x_m in the same way as proved when Ω is a bouded domain [1, Theorem

2.4]. Thus the points $\{x_m\}_{m=1}^{\infty}$ do not accumulate to $\partial\Omega$ so we may assume $x_\infty \in \Omega$. By (2.10), $\{P_m\}_{m=1}^{\infty}$ subsequently converges to a limit P locally uniformly in Ω with its all derivatives since P_m is harmonic in Ω. The bound (2.10) also implies that P_m is uniformly bounded in $L^r_{\mathrm{loc}}(\bar{\Omega})$ for any $r \in [1,\infty)$ so $\{P_m\}_{m=1}^{\infty}$ subsequently converges to P weakly in $L^r(D)$ for each subdomain $D \subset \Omega$. Since P_m satisfies (2.2) with W_m, i.e.

$$\int_\Omega P_m \Delta\varphi dx = \int_{\partial\Omega} W_m \cdot \nabla_{\partial\Omega}\varphi d\mathcal{H}^{n-1}(x)$$

for all $\varphi \in C^2_c(\bar{\Omega})$ satisfying $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$, we take a limit to observe that

$$\int_\Omega P\Delta\varphi dx = 0.$$

Since $\nabla P \in L^\infty_{d_\Omega}(\Omega)$ by (2.10), the limit P is also weak solution of (2.1) with $W = 0$. We now apply the uniqueness result for a weak solution (Theorem 2.7) which implies that $\nabla P \equiv 0$. Thus we have a contradiction to the fact that $d_\Omega(x_\infty)|\nabla P(x_\infty)| \geq 1/2$ so Case 1 does not occur.

Case 2. $\overline{\lim}_{m\to\infty} d_m = \infty$. We may assume that $\lim_{m\to\infty} d_m = \infty$ by choosing a subsequence. We appeal to the classical Liouville Theorem in the whole space by rescaling the solution P_m around the point x_m as

$$Q_m(x) = P_m(x_m + d_m x) \quad \text{for } x \in \Omega_m,$$

where $\Omega_m = (\Omega - x_m)/d_m$ is a downscaled domain. We observe that a complement of Ω_m accumulates to some point $a \in \mathbf{R}^n (a \neq 0)$ so the limit domain is $\mathbf{R}^n\backslash\{a\}$. In fact, take a point $y_m \in \partial\Omega$ such that $d_m = |x_m - y_m|$ and then Ω_m^c accumulates to a point $z_m = (y_m - x_m)/d_m$ with $|z_m| = 1$. Since the points $\{z_m\}_{m=1}^{\infty}$ subsequently converges to some point $a \in \mathbf{R}^n\backslash\{0\}$, any neighborhood of the point a includes Ω_m^c for sufficiently large $m \geq 1$. Here the estimates (2.10) and (2.11) are inherited to the estimates

$$|\nabla Q_m|_{\infty, d_{\Omega_m}} = 1, \tag{2.12}$$

$$\left|\nabla Q_m(0)\right| \geq 1/2. \tag{2.13}$$

Since Q_m is harmonic in Ω_m, by (2.12) Q_m subsequently converges to a limit Q locally uniformly in $\mathbf{R}^n\backslash\{a\}$ with its all derivatives. Then the limit Q is also harmonic in $\mathbf{R}^n\backslash\{a\}$ with a bound

$$\sup_{x\in\mathbf{R}^n\backslash\{a\}} \left|x-a\right|\left|\nabla Q(x)\right| \leq 1. \tag{2.14}$$

To solve the limit problem in $\mathbf{R}^n\backslash\{a\}$ we remove a singularity at $x = a$. If $n \geq 3$, the singularity is removable by the bound (2.14). (We apply Lemma 2.11 below which is proved in the next subsection for a rigorous proof.) Even if $n = 2$, the singularity is still removable by Lemma 2.11 because a mean value around the point a, i.e.

$$⨍_{\partial B_a(r)} Q(x) d\mathcal{H}^{n-1}(x) \tag{2.15}$$

is independent of $r > 0$. Since for a weak solution of (2.1) a mean value around Ω^c is independent of r (as discussed in Proposition 2.13 in the next subsection), for the harmonic function Q_m a mean value around Ω^c_m

$$\fint_{\partial B_a(r)} Q_m(x) d\mathcal{H}^{n-1}(x)$$

is independent of $r > \operatorname{diam} \Omega^c_m$, so is (2.15) for $r > 0$. We now apply the Liouville Theorem with (2.14) to conclude that $\nabla Q \equiv 0$ which contradicts the fact that $|\nabla Q(0)| \geq 1/2$. Both cases do not occur neither so we reach a contradiction. □

Proof of Theorem 1.3. Since an exterior domain has a compact boundary, it naturally posses a uniformly regularity of the boundary. By Proposition 2.6 and Theorem 2.9 the assertion follows. □

Remark 2.10. For bounded and exterior domains, we are able to construct a solution operator $\mathbf{K} : W \mapsto \nabla P$ for the Neumann problem (2.1). For smooth functions $W \in L^\infty_{\tan}(\partial\Omega)$, L^r-theory implies the existence of a weak solution for (2.1); for instance, see [47]. Strictly admissibility implies a bound

$$\Big|\mathbf{K}[W]\Big|_{\infty,d_\Omega} \leq C_\Omega \|W\|_{\infty,\partial\Omega} \tag{2.16}$$

for a weak solution $\nabla P = \mathbf{K}[W]$. For a general $W \in L^\infty_{\tan}(\partial\Omega)$ we are able to extend $\mathbf{K}$ by approximating $W \in L^\infty_{\tan}(\partial\Omega)$ with smooth tangential vector fields $\{W_m\}_{m=1}^\infty \subset C^1(\partial\Omega)$ by changing a coordinate to a flat boundary. The uniqueness of an weak solution (Theorem 2.7) implies that $\mathbf{K} : W \mapsto \nabla P$ is uniquely extendable to a bounded linear operator from $L^\infty_{\tan}(\partial\Omega)$ to $L^\infty_{d_\Omega}(\Omega)$. We here call $\mathbf{K}$ harmonic-pressure operator which is useful because pressure can be represented as $\nabla q = \mathbf{K}[W(v)]$ for solutions (v, q) with $W(v) = -(\nabla v - (\nabla v)^T) \cdot n_\Omega$ although $\nabla q = \mathbf{Q}[\Delta v]$ is no longer available for L^∞-solutions.

2.5. **Removable singularities for a harmonic function.** We here give a criterion for a harmonic function to remove an isolated singularity.

Lemma 2.11 (Removable singularities)**.** *Let Ω be a domain in $\mathbf{R}^n$ with $n \geq 2$. Let P be a harmonic function in Ω except a point $a \in \Omega$ with a bound*

$$\sup_{|x-a|\leq\delta} \Big|x-a\Big|\Big|\nabla P(x)\Big| < \infty \tag{2.17}$$

for some small $\delta > 0$. If $n \geq 3$, the singularity is $x = a$ removable. If $n = 2$, assume in addition that

$$\fint_{\partial B_a(r)} P(x) d\mathcal{H}^1(x) \tag{2.18}$$

is independent of sufficiently small $r > 0$. Then the singularity $x = a$ is removable.

Proof. We may assume that $a = 0$ and $\delta = 1$ by translation and dilation of Ω. From (2.17) we observe that P is locally integrable around $a = 0 \in \Omega$. By connecting a point

$x \in \operatorname{int} B_0(1)$ and $y = x/|x|$, it holds that

$$\begin{aligned}|P(x) - P(y)| &\leq |x - y| \int_0^1 |\nabla P(\tau x + (1-\tau)y)|\, d\tau \\ &\leq M(1 - |x|) \int_0^1 \frac{d\tau}{1 - \tau(1 - |x|)} \\ &= -M \log |x|\end{aligned}$$

with the constant M, larger that the quantity (2.17). Thus we have a bound

$$|P(x)| \leq M \log |x| + C_P \quad \text{for } x \in B_0(1) \tag{2.19}$$

with the constant C_P, the supremum of $|P(y)|$ on $|y| = 1$. To prove that $x = 0$ is removable for P, it suffices to show that

$$\int_{B_0(1)} P \Delta \varphi dx = 0 \tag{2.20}$$

for all $\varphi \in C_c^\infty(B_0(1))$. If (2.20) holds, the assertion easily follows since the estimate (2.19) and (2.20) implies that a mollified function (i.e. $P_\varepsilon = P * \eta_\varepsilon$ with a standard mollifier η_ε) is harmonic in $B_0(1)$ and uniformly bounded around the origin, so P can be understood as a limit of smooth functions in the uniform topology.

We shall show (2.20). Since P is harmonic in $B_0(1)\backslash\{0\}$, integration by parts yields that

$$\int_{\varepsilon<|x|<1} P \Delta \varphi dx = \int_{|x|=\varepsilon} \left(P \frac{\partial \varphi}{\partial n_{B_0(\varepsilon)}} - \frac{\partial P}{\partial n_{B_0(\varepsilon)}} \varphi \right) d\mathcal{H}^{n-1}(x) \tag{2.21}$$

for $\varepsilon > 0$. By (2.19) the first term of the right-hand side vanishes as $\varepsilon \downarrow 0$. We estimate the second term. By (2.17) we have

$$\left| \int_{|x|=\varepsilon} \frac{\partial P}{\partial n_{B_0(\varepsilon)}} \varphi d\mathcal{H}^{n-1}(x) \right| \leq C_n \varepsilon^{n-2} \|\varphi\|_\infty \tag{2.22}$$

with the constant C_n independent of $\varepsilon > 0$. If $n \geq 3$ letting $\varepsilon \downarrow 0$ implies (2.20) so the first assertion follows. It remains to show the case $n = 2$. Differentiate (2.18) with respect to r to observe that

$$\frac{d}{dt} \fint_{\partial B_0(r)} P(x) d\mathcal{H}^1(x) = \fint_{\partial B_0(r)} \frac{\partial P(x)}{\partial n_{B_0(r)}} d\mathcal{H}^1(x)$$

for $r < 1$. Since the mean value (2.18) is independent of r, we have

$$\int_{\partial B_0(r)} \frac{\partial P(x)}{\partial n_{B_0(r)}} d\mathcal{H}^1(x) = 0.$$

We may replace φ up to additive constant to estimate the left-hand side of (2.22). We connect $x \in \operatorname{int} B_0(1)$ and the origin to get

$$|\varphi(x) - \varphi(0)| \leq |x| \|\nabla \varphi\|_\infty.$$

Replacing φ to $\tilde{\varphi} = \varphi - \varphi(0)$ in (2.22), we have

$$\left| \int_{|x|=\varepsilon} \frac{\partial P}{\partial n_{B_0(\varepsilon)}} \tilde{\varphi} d\mathcal{H}^{n-1}(x) \right| \leq C_n \varepsilon \|\nabla \varphi\|_\infty.$$

We then take a limit $\varepsilon \downarrow 0$ which implies (2.20) for $n = 2$ so the proof is now complete. □

Remark 2.12. (i)The fundamental solution of the Laplace equation, $\log|x-a|$ satisfies (2.17) when $n = 2$. However, (2.18) excludes such a function which has a singularity at $x = a$.
(ii)We state Lemma 2.11 simply to apply in the proof of Theorem 2.9. If we assume that the mean value (2.18) is independent of r also for $n \geq 3$, the assertion of Lemma 2.11 is still valid by replacing (2.17) to

$$\left|P(x)\right| = \mathrm{O}\left(\frac{\log|x-a|}{|x-a|^{n-2}}\right) \quad \text{as} \quad x \to a.$$

For instance, see [48, Chapter I, Theorem 3.2].

In fact, a mean value around Ω^c of a weak solution is independent of $r >$diam Ω^c. We shall give a short proof for

Proposition 2.13. *Let Ω be an exterior domain with C^1 boundary such that $0 \in \Omega^c$. Then a mean value of a weak solution of (2.1) around Ω^c, i.e.*

$$\fint_{\partial B_0(r)} P(x)d\mathcal{H}^{n-1}(x) \tag{2.23}$$

is independent of $r >$ diam Ω^c.

Proof. Differentiate (2.23) with respect to r to observe that

$$\frac{d}{dr}\fint_{\partial B_0(r)} P(x)d\mathcal{H}^{n-1}(x) = \fint_{\partial B_0(r)} \frac{\partial P}{\partial n_{B_0(r)}}(x)d\mathcal{H}^{n-1}(x). \tag{2.24}$$

We shall show that the right-hand side equals zero. We take a smooth function φ satisfying $\varphi \equiv 1$ for $|x| \leq r$ and $\varphi \equiv 0$ for $|x| \geq 2r$ for each $r >$ diam Ω^c. Since P is a weak solution of (2.1), substituting φ into (2.2) and integration by parts yields that

$$\begin{aligned} -\int_{r<|x|<2r} \nabla P \cdot \nabla\varphi dx &= \int_{\partial\Omega} W \cdot \nabla_{\partial\Omega}\varphi d\mathcal{H}^{n-1}(x) \\ &= 0. \end{aligned}$$

Since P is harmonic in Ω and $\varphi(x) = 0$ on $|x| = 2r$, $\varphi(x) = 1$ on $|x| = r$, integration by parts yields that

$$\begin{aligned} \int_{r<|x|<2r} \nabla P \cdot \nabla\varphi dx &= \int_{\partial B_0(r)} \frac{\partial P}{\partial n_{B_0(r)}}\varphi d\mathcal{H}^{n-1}(x) + \int_{\partial B_0(2r)} \frac{\partial P}{\partial n_{B_0(2r)}}\varphi d\mathcal{H}^{n-1}(x) \\ &= \int_{\partial B_0(r)} \frac{\partial P}{\partial n_{B_0(r)}} d\mathcal{H}^{n-1}(x). \end{aligned}$$

Thus the right-hand side of (2.24) equals zero so the mean value (2.23) is independent of r. The proof is now complete. □

3. Uniform Hölder estimates for pressure gradient

In this section for the proof of Theorem 1.6 we prepare local Hölder estimates for solutions to the Stokes equations (1.1)-(1.4) both interior and up to boundary. The pressure estimate (1.7) is a key to get those Hölder estimates. We invoke a priori estimate (2.3) for weak solutions of (2.1) in a strictly admissible domain.

3.1. **Pressure gradient estimates for L^∞-solutions.** We shall show the estimate (1.7) in a strictly admissible domain which is essential to establish the Hölder estimates for solutions of (1.1)-(1.4). Although the proof is similar to the proof of Proposition 2.6, we here give a rigorous proof.

Lemma 3.1. *Let Ω be a domain in $\mathbf{R}^n (n \geq 2)$ with C^1 boundary. Let (v,q) be an L^∞-solution of (1.1)-(1.4) in $\Omega \times (0,T)$. Then the pressure q is a weak solution of (2.1) with*

$$W(v) = -\Big(\nabla v - (\nabla v)^T\Big) \cdot n_\Omega$$

for each time. If Ω is strictly admissible, there exists a constant C depending only on Ω such that

$$\Big|\nabla q\Big|_{\infty, d_\Omega}(t) \leq C\|\nabla v\|_{\infty,\partial\Omega}(t) \tag{3.1}$$

and also

$$\Big|\nabla q(\cdot,t) - \nabla q(\cdot,s)\Big|_{\infty,d_\Omega} \leq C\|\nabla v(\cdot,t) - \nabla v(\cdot,s)\|_{\infty,\partial\Omega} \tag{3.2}$$

holds for $t, s \in (0,T)$. The constant C is independent of dilation and translation of Ω.

Proof. Let (v,q) be an L^∞-solution in $\Omega \times (0,T)$. We first show that q is a weak solution of (2.1) with $W(v) = -(\nabla v - (\nabla v)^T) \cdot n_\Omega$. This $W(v)$ is a tangential vector field on $\partial\Omega$, i.e.

$$\begin{aligned} W(v) \cdot n_\Omega &= -\sum_{i,j=1}^n (\partial_j v^i - \partial_i v^j) n_\Omega^j n_\Omega^i \\ &= 0. \end{aligned}$$

Since the quantity (1.5) for (v,q) is bounded in $\Omega \times (0,T)$, $t^{1/2}W(v)$ is bounded in $\partial\Omega \times (0,T)$. In particular, $W(v) \in L^\infty_{\mathrm{tan}}(\partial\Omega)$ for each $t \in (0,T)$. We now observe that q satisfies (2.2), i.e.

$$\int_\Omega q\Delta\varphi dx = \int_{\partial\Omega} W \cdot \nabla_{\partial\Omega}\varphi d\mathcal{H}^{n-1}(x)$$

for all $\varphi \in C_c^2(\bar{\Omega})$ satisfying $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$. Since $v_t \in L^\infty_\sigma(\Omega)$ for $t \in (0,T)$, multiplying $\nabla\varphi$ to (1.1) and integration by parts yields that

$$\begin{aligned} \int_\Omega \nabla q \cdot \nabla\varphi dx &= -\int_\Omega (v_t - \Delta v) \cdot \nabla\varphi dx \\ &= \sum_{i,j=1}^n \int_\Omega \partial_j^2 v^i \partial_i \varphi dx. \end{aligned}$$

Since

$$\begin{aligned}\int_\Omega \partial_j^2 v^i \partial_i \varphi dx &= -\int_\Omega \partial_j v^i \partial_j \partial_i \varphi dx + \int_{\partial\Omega} \partial_j v^i \partial_i \varphi n_\Omega^j d\mathcal{H}^{n-1}(x)\\ &= \int_\Omega \partial_j \partial_i v^i \partial_j \varphi dx + \int_{\partial\Omega} \partial_j v^i (\partial_i \varphi n_\Omega^j - \partial_j \varphi n_\Omega^i) d\mathcal{H}^{n-1}(x)\end{aligned}$$

and div $v = 0$, we have

$$\int_\Omega \nabla q \cdot \nabla \varphi dx = \sum_{i,j=1}^n \int_{\partial\Omega} \partial_j v^i (\partial_i \varphi n_\Omega^j - \partial_j \varphi n_\Omega^i) d\mathcal{H}^{n-1}(x).$$

We transpose indexes i and j of the last term to observe that

$$\begin{aligned}\sum_{i,j=1}^n \int_{\partial\Omega} \partial_j v^i (\partial_i \varphi n_\Omega^j - \partial_j \varphi n_\Omega^i) d\mathcal{H}^{n-1}(x) &= \sum_{i,j=1}^n \int_{\partial\Omega} (\partial_j v^i - \partial_i v^j) n_\Omega^j \partial_i \varphi d\mathcal{H}^{n-1}(x)\\ &= -\int_{\partial\Omega} W(v) \cdot \nabla \varphi d\mathcal{H}^{n-1}(x)\\ &= -\int_{\partial\Omega} W(v) \cdot \nabla_{\partial\Omega} \varphi d\mathcal{H}^{n-1}(x).\end{aligned}$$

Since $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$, integration by parts yields that

$$\int_\Omega q \Delta \varphi dx = \int_{\partial\Omega} W(v) \cdot \nabla_{\partial\Omega} \varphi d\mathcal{H}^{n-1}(x).$$

Thus q satisfies (2.2) with $W(v)$. Since $t^{1/2} d_\Omega(x) |\nabla q(x,t)|$ is bounded in $\Omega \times (0,T)$, $\nabla q(x,t) \in L^\infty_{d_\Omega}(\Omega)$ for each $t \in (0,T)$. Thus q is a weak solution of (2.1). It is noted that $q(\cdot,t) - q(\cdot,s)$ is also a weak solution of (2.1) with $W(v(\cdot,t) - v(\cdot,s))$ for $t, s \in (0,T)$. If Ω is strictly admissible, there exists a dilation invariant constant C_Ω such that

$$\left|\nabla q\right|_{\infty,d_\Omega}(t) \leq C_\Omega \|W(v)\|_{\infty,\partial\Omega}(t)$$

and also

$$\left|\nabla q(\cdot,t) - \nabla q(\cdot,s)\right|_{\infty,d_\Omega} \leq C_\Omega \|W(v(\cdot,t) - v(\cdot,s))\|_{\infty,\partial\Omega}(t)$$

holds for $t, s \in (0,T)$. Since $\|W(v)\|_{\infty,\partial\Omega} \leq 2\|\nabla v\|_{\infty,\partial\Omega}$, the estimates (3.1) and (3.2) follows with the constant $C = 2C_\Omega$. □

Remark 3.2. (i) The fact that q is a weak solution of (2.1) with $W(v)$ is essentially proved by [1, Remark 2.7 (ii)] where we use the projection $\mathbf{Q}$. Although the proof is essentially the same, we prove the statement without using $\mathbf{Q}$.
(ii) The estimate (3.2) implies a Hölder continuity for pressure in time for L^∞-solutions. For a bounded and exterior domain, the estimates (3.1) and (3.2) automatically follow from a bound for the Harmonic-pressure operator $\mathbf{K} : L^\infty_{\tan}(\partial\Omega) \to L^\infty_{d_\Omega}(\Omega)$ with $\nabla q = \mathbf{K}[W(v)]$ by Remark 2.10.

(iii) We here only use the boundary condition for velocity, $v \cdot n_\Omega = 0$ on $\partial\Omega$ which implies $v_t \cdot n_\Omega = 0$ on $\partial\Omega$ to apply the estimate (2.3). We observe that the estimates (3.1) and (3.2) are still valid under the Robin boundary condition [46] (see also [44]), i.e.

$$\alpha v + \beta \{T(v,q) \cdot n_\Omega - n_\Omega (T(v,q) \cdot n_\Omega) \cdot n_\Omega\} = h \quad \text{on } \partial\Omega$$

for a tangential vector field h with $\alpha + \beta = 1$, $\alpha, \beta \geq 0$, where $T(v,q) = ((\nabla v) + (\nabla v)^T - qI)$ is the stress tensor.

3.2. **Uniform Hölder estimates.** We here state the local Hölder estimates for second derivatives of solutions to (1.1)-(1.4) which we already established for $\tilde{L}^r$-solutions ($r > n$) in an admissible domain [1, Proposition 3.2, Theorem 3.4]. By the estimates (3.1) and (3.2) we extend these estimates for L^∞-solutions in a strictly admissible domains.

To state the estimates in a precise way, we recall the notation for Hölder (semi)norms for space-time functions [35]. Let $f = f(x,t)$ be a real-valued or an $\mathbf{R}^n$-valued function defined in $Q = \Omega \times (0,T]$, where Ω is a domain in $\mathbf{R}^n$. For $\mu \in (0,1)$ we set several Hölder semi-norms

$$[f]^{(\mu)}_{(0,T]}(x) = \sup \left\{ \left| f(x,t) - f(x,s) \right| / |t-s|^\mu \;\middle|\; t, s \in (0,T], t \neq s \right\}$$

$$[f]^{(\mu)}_{\Omega}(t) = \sup \left\{ \left| f(x,t) - f(y,t) \right| / |x-y|^\mu \;\middle|\; x, y \in \Omega, x \neq y \right\}$$

and

$$[f]^{(\mu)}_{t,Q} = \sup_{x \in \Omega} [f]^{(\mu)}_{(0,T]}(x), \;\; [f]^{(\mu)}_{x,Q} = \sup_{t \in (0,T]} [f]^{(\mu)}_{\Omega}(t)$$

In parabolic scale for $\gamma \in (0,1)$ we set

$$[f]^{(\gamma,\gamma/2)}_{Q} = [f]^{(\gamma/2)}_{t,Q} + [f]^{(\gamma)}_{x,Q}$$

If $l = [l] + \gamma$ where $[l]$ is nonnegative integer and $\gamma \in (0,1)$, we set

$$[f]^{(l,l/2)}_{Q} = \sum_{\alpha + 2\beta = [l]} [\partial_x^\alpha \partial_t^\beta f]^{(\gamma,\gamma/2)}_{Q}$$

and the parabolic Hölder norm

$$|f|^{(l,l/2)}_{Q} = \sum_{\alpha + 2\beta \leq [l]} \|\partial_x^\alpha \partial_t^\beta f\|_{L^\infty(Q)} + [f]^{(l,l/2)}_{Q}$$

The estimates (3.1) and (3.2) implies the uniform Hölder estimates for pressure gradient in time as stated below in Lemma 3.3 which is a key for our localization argument to estimate local Hölder norms for solutions of (1.1)-(1.4).

Lemma 3.3. *Let Ω be a strictly admissible, uniformly C^2 domain in $\mathbf{R}^n (n \geq 2)$. Then there exists a constant $M(\Omega) > 0$ such that a priori estimate*

$$[d_\Omega(\cdot)\nabla q]^{(1/2)}_{t,Q_\delta} \leq \frac{M}{\delta} \sup \left\{ \left(\|v_t\|_\infty(t) + \|\nabla^2 v\|_\infty(t) \right) t \;\middle|\; \delta \leq t \leq T \right\}$$

holds for all L^∞-solution (v,q) of (1.1)-(1.4) and all $\delta \in (0,T)$, where $Q_\delta = \Omega \times (\delta, T]$. The constant M can be taken uniform with respect to translation and dilation, i.e. $M(\lambda\Omega + x_0) = M(\Omega)$ for all $\lambda > 0$ and $x_0 \in \mathbf{R}^n$.

Lemma 3.3 is proved for $\tilde{L}^r$-solutions ($r \geq n$) in an admissible domain [1, Lemma 3.1]. One can extend this estimates also for L^∞-solutions in a strictly admissible domain by the estimates (3.1) and (3.2). By the same localization argument in [1] as proved for $\tilde{L}^r$-solutions based on Solonnikov's Hölder estimates [52], [57], [58], applying Lemma 3.3 implies the local Hölder estimates for L^∞-solutions both interior and up to boundary with uniform constants. We here only state results and omit the proof.

Theorem 3.4 (Interior Hölder estimates)**.** *Let Ω be a strictly admissible, uniformly C^2-domain in $\mathbf{R}^n$. Take $\gamma \in (0,1)$, $\delta > 0$, $T > 0$, $R > 0$. Then there exists a constant $C = C(M(\Omega), \delta, R, d, \gamma, T)$ such that a priori estimate*

$$[\nabla^2 v]_{Q'}^{(\gamma,\gamma/2)} + [v_t]_{Q'}^{(\gamma,\gamma/2)} + [\nabla q]_{Q'}^{(\gamma,\gamma/2)} \leq CN_T \tag{3.3}$$

holds for all L^∞-solution (v,q) of (1.1)-(1.4) provided that $B_R(x_0) \subset \Omega$ and $x_0 \in \Omega$, where $Q' = \operatorname{int} B_R(x_0) \times (\delta, T]$ and d denotes the distance of $B_R(x)$ and $\partial\Omega$. Here

$$N_T = \sup_{0 \leq t \leq T} \left\| N(v,q) \right\|_\infty (t) < \infty$$

and $M(\Omega)$ is the constant in Lemma 3.3.

Theorem 3.5 (Estimates near the boundary)**.** *Let Ω be a strictly admissible, uniformly C^3-domain of type (α, β, K) in $\mathbf{R}^n$. Then there exists $R_0 = R_0(\alpha, \beta, K) > 0$ such that for any $\gamma \in (0,1)$, $\delta \in (0,T)$ and $R \leq R_0/2$ there exists a constant*

$$C = C(M(\Omega), \delta, \gamma, T, R, \alpha, \beta, K)$$

such that (3.3) is valid for all L^∞-solution (v,q) of (1.1)-(1.4) with

$$Q' = Q'_{x_0,R,\delta} = \Omega_{x_0,R} \times (\delta, T], \ \Omega_{x_0,R} = \operatorname{int} B_R(x_0) \cap \Omega$$

provided that $x_0 \in \partial\Omega$.

4. Uniqueness in a strictly admissible domain

In this section we prove Theorem 1.6. Appealing to a blow-up argument, we observe that there exists a time $T_0 > 0$ such that a priori estimate (1.6) holds in $[0, T_0]$, which in particular, implies the uniqueness.

Proposition 4.1 (Uniqueness of an L^∞-solution)**.** *Let Ω be a strictly admissible, uniformly C^3-domain. Then a priori estimate (1.6) holds with some constant T_0 and C for all L^∞-solutions with $v_0 \in L^\infty_\sigma(\Omega)$. In particular, an L^∞-solution is unique.*

The local Hölder estimates (Theorem 3.4 and Theorem 3.5) guarantees a compactness of a blow-up sequence of L^∞-solutions both interior and up to boundary. All other parts of our proof [1, Proposition 5.1] still works for L^∞-solutions in a strictly admissible domain, so the proof is omitted.

We shall extend (1.6) for an arbitrary time interval. We appeal to a blow-up argument again. Although for a fixed time interval a blow-up time does not converges to zero, the uniqueness implies that a blow-up does not occur.

Proof of Theorem 1.6. We take an arbitrary time $T_0 > 0$. Suppose that (1.6) ware false for any choice of a constant C. Then there would exist a sequence of L^∞-solutions $(\tilde{v}_m, \tilde{q}_m)$ such that

$$\sup_{0\le t\le T_0} \left\|N(\tilde{v}_m, \tilde{q}_m)\right\|_\infty(t) > m\|\tilde{v}_{0,m}\|_\infty.$$

Take a point $t_m \in (0, T_0]$ such that

$$\left\|N(\tilde{v}_m, \tilde{q}_m)\right\|_\infty(t_m) \ge \frac{1}{2}M_m, \quad M_m = \sup_{0\le t\le T_0} \left\|N(\tilde{v}_m, \tilde{q}_m)\right\|_\infty(t)$$

and normalize $(\tilde{v}_m, \tilde{q}_m)$ dividing by M_m to observe that

$$\sup_{0\le t\le T_0} \left\|N(v_m, q_m)\right\|_\infty(t) \le 1, \tag{4.1}$$

$$\left\|N(v_m, q_m)\right\|_\infty(t_m) \ge 1/2, \tag{4.2}$$

$$\|v_{0,m}\|_\infty < 1/m, \tag{4.3}$$

with $v_m = \tilde{v}_m/M_m$, $q_m = \tilde{q}_m/M_m$. Note that the estimate

$$\sup_{0\le t\le T_0} t^{1/2}\left|\nabla q_m\right|_{\infty,d_\Omega}(t) \le C \tag{4.4}$$

holds by Lemma 3.3 and (4.1) with the constant C independent of translation of Ω. By (4.2) we are able to take a point $x_m \in \Omega$ such that

$$N(v_m, q_m)(x_m, t_m) \ge 1/4. \tag{4.5}$$

We may assume $\{t_m\}_{m=1}^\infty \subset (0, T_0]$ converges to some $t_\infty \in [0, T_0]$ by choosing a subsequence. Since (1.6) holds up to some finite time by Proposition 4.1, we may assume $t_\infty \neq 0$. Then the situation can be divided into two cases depending on whether $d_m = d_\Omega(x_m)$ converges or not.

Case 1. $\overline{\lim}_{m\to\infty} d_m < \infty$. We may assume that $\{x_m\}_{m=1}^\infty$ converges to some point $x_\infty \in \bar{\Omega}$ by taking a subsequence. It is reduced to the uniqueness of an L^∞-solution. Integration by parts yields that

$$\int_0^{T_0}\int_\Omega \{v_m \cdot (\varphi_t + \Delta\varphi) - \nabla q_m \cdot \varphi\}\, \mathrm{d}x\mathrm{d}t = -\int_\Omega v_m(x, 0)\cdot\varphi(x, 0)\mathrm{d}x \tag{4.6}$$

for all $\varphi \in C_c^\infty(\Omega \times [0, T_0))$. We apply Theorem 3.4 and Theorem 3.5 to find a subsequence of (v_m, q_m) (still denoted by (v_m, q_m)) which converges to a limit (v, q) locally uniformly in $\bar{\Omega} \times (0, T_0]$ together with ∇v_m, $\nabla^2 v_m$, $\partial_t v_m$, ∇q_m. Then the limit $(v, \nabla q) \in C^{2,1}(\bar{\Omega}\times(0, T_0])\times C(\bar{\Omega}\times(0, T_0])$ solves the Stokes equations (1.1)-(1.3) and the quantities $N(v, q)(x, t)$ and $t^{1/2}d_\Omega(x)\nabla q(x, t)$ are bounded in $\Omega \times (0, T_0)$ by (4.1) and (4.4). Note that $N(v_m, q_m)$ converges to $N(v, q)$ locally uniformly so that $N(v, q)(x_\infty, t_\infty) \ge 1/4$. We now observe that $v(\cdot, t)$ converges to zero $*$-weakly in $L^\infty(\Omega)$ as $t \downarrow 0$. By (4.3) taking a limit to (4.6) implies that

$$\int_0^{T_0}\int_\Omega \{v \cdot (\varphi_t + \Delta\varphi) - \nabla q \cdot \varphi\}\, \mathrm{d}x\mathrm{d}t = 0.$$

We apply Proposition 4.2 (stated below the end of this proof) to observe that $v(\cdot,t)\to 0$ $*$-weakly in $L^\infty(\Omega)$ as $t\downarrow 0$, so (v,q) is an L^∞-solution with initial data zero. Since an L^∞-solution is unique by Proposition 4.1, we conclude that $v\equiv 0, \nabla q\equiv 0$ which is a contradiction to the fact that $N(v,q)(x_\infty,t_\infty)\geq 1/4$, so Case 1 does not occur.
Case 2. $\overline{\lim}_{m\to\infty} d_m=\infty$. We may assume $\lim_{m\to\infty} d_m=\infty$. By translation of Ω, it is reduced to the uniqueness of the heat equation in $\mathbf{R}^n$. We translate a solution by setting as $u_m(x,t)=v_m(x+x_m,t), p_m(x,t)=q_m(x+x_m,t)$ for $x\in\Omega_m$ with $\Omega_m=\{x\in\mathbf{R}^n \mid x=y-x_m,\ y\in\Omega\}$. Then Ω_m goes to $\mathbf{R}^n$ and the estimates (4.1), (4.3), (4.4) and (4.6) are inherited to the estimates

$$\begin{aligned}
\sup_{0\leq t\leq T_0}\Big\|N(u_m,p_m)\Big\|_\infty(t)&\leq 1,\\
\sup_{0\leq t\leq T_0} t^{1/2}\Big|\nabla p_m\Big|_{\infty,d_{\Omega_m}}(t)&\leq C,\\
\|u_{0,m}\|_\infty&<1/m,\\
N(u_m,p_m)(0,t_m)&\geq 1/4.
\end{aligned}\tag{4.6}$$

We apply Theorem 3.4 to choose a subsequence of (u_m,p_m) which converges to a limit (u,p) locally uniformly in $\mathbf{R}^n\times(0,T_0]$ together with $\nabla u_m, \nabla^2 u_m, \partial_t u_m, \nabla p_m$. Since for each $R>0$, $B_0(R)\subset\Omega_m$ for sufficiently large $m\geq 1$ and

$$\inf\{d_{\Omega_m}(x)\mid |x|\leq R\}\to\infty\quad \text{as } m\to\infty,$$

by (4.6) we have $\nabla p\equiv 0$. We now observe that the limit $u\in C(\mathbf{R}^n\times(0,T_0])$ solves the heat equation with initial data zero. Integration by parts for (u_m,p_m) yields that

$$\int_0^{T_0}\int_{\mathbf{R}^n}\{u_m\cdot(\varphi_t+\Delta\varphi)-\nabla p_m\cdot\varphi\}\,\mathrm{d}x\mathrm{d}t=-\int_{\mathbf{R}^n}u_m(x,0)\cdot\varphi(x,0)\mathrm{d}x$$

for all $\varphi\in C_c^\infty(\mathbf{R}^n\times[0,T_0))$. Since $\|u_{0,m}\|_\infty\to 0$ as $m\to\infty$, we have

$$\int_0^{T_0}\int_{\mathbf{R}^n}u\cdot(\varphi_t+\Delta\varphi)\mathrm{d}x\mathrm{d}t=0.$$

We apply the uniqueness of the heat equation in $\mathbf{R}^n$ [1, Lemma 4.5] to conclude that $u\equiv 0$ which contradicts to the fact that $N(u,p)(0,t_\infty)\geq 1/4$, so Case 2 does not occur neither.

We reach a contradiction. The proof is now complete. □

We here give a short proof for the fact that convergence to initial data $*$-weakly in L^∞ can be understood in a weak form which is useful to interpret initial data for a limit of a solution sequence.

Proposition 4.2. *Let Ω be a domain in $\mathbf{R}^n (n\geq 2)$ with $\partial\Omega\neq\emptyset$. Let $(v,\nabla q)\in C^{2,1}(\Omega\times(0,T])\times C(\Omega\times(0,T])$ solve (1.1) with a bound*

$$\sup_{0\leq t\leq T}\Big\{\|v\|_\infty(t)+t^{1/2}\Big|\nabla q\Big|_{\infty,d_\Omega}(t)\Big\}<\infty.\tag{4.7}$$

If (v,q) satisfies

$$\int_0^T \int_\Omega \{v\cdot(\varphi_t+\Delta\varphi)-\nabla q\cdot\varphi\}\, dxdt = -\int_\Omega v(x,0)\cdot\varphi(x,0)dx$$

with $v_0\in L^\infty_\sigma(\Omega)$ for all $\varphi\in C_c^\infty(\Omega\times[0,T))$, then $v\to v_0$ $*$-weakly in $L^\infty(\Omega)$ as $t\downarrow 0$. The converse is also valid.

Proof. Since (v,q) solves (1.1) in the classical sense, for $\varepsilon>0$ integration by parts yields that

$$\int_\varepsilon^T \int_\Omega \{v\cdot(\varphi_t+\Delta\varphi)-\nabla q\cdot\varphi\}\, \mathrm{d}x\mathrm{d}t = -\int_\Omega v(x,\varepsilon)\cdot\varphi(x,\varepsilon)\mathrm{d}x \tag{4.8}$$

for all $\varphi\in C_c^\infty(\Omega\times[0,T))$. Note that ∇q is integrable near $t=0$ by (4.7). Letting $\varepsilon\downarrow 0$ implies that

$$\int_\Omega v(x,\varepsilon)\cdot\varphi(x,\varepsilon)\mathrm{d}x \to \int_\Omega v(x,0)\cdot\varphi(x,0)\mathrm{d}x.$$

In particular $\int_\Omega v(\cdot,t)\cdot\psi\mathrm{d}x\to\int_\Omega v(\cdot,0)\cdot\psi\mathrm{d}x$ as $t\downarrow 0$ for all $\psi\in C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $L^1(\Omega)$, the assertion follows. The converse also follows by taking a limit $\varepsilon\downarrow 0$ to (4.8). □

5. Approximation in an exterior domain

The goal of this section is to prove the assertions both (i) and (ii) of Theorem 1.5. Since an L^∞-solution is unique in an exterior domain by Proposition 4.1, we are able to construct a solution for $v_0\in L^\infty_\sigma$ as a limit of approximate solutions. We first prepare the approximation Lemma for L^∞_σ in an exterior domain.

5.1. Approximation for initial data.

Lemma 5.1 (Approximation in an exterior domain)**.** *Let Ω be an exterior domain with Lipschitz boundary. There exists a constant $C=C_\Omega$ such that for any $v\in L^\infty_\sigma(\Omega)$ there exist a sequence $\{v_m\}_{m=1}^\infty\subset C^\infty_{c,\sigma}(\Omega)$ such that*

$$\|v_m\|_{L^\infty(\Omega)}\le C\|v\|_{L^\infty(\Omega)} \tag{5.1}$$

and

$$v_m\to v\quad \text{a.e. in } \Omega \tag{5.2}$$

as $m\to\infty$. If in addition $v\in C(\bar\Omega)$ and $v=0$ on $\partial\Omega$, the above convergence can be replaced by local uniform convergence in $\bar\Omega$. If in addition $v(x)$ vanishes at space infinity, i.e. $\lim_{|x|\to\infty}v(x)=0$, then the convergence can be replaced by uniform convergence in $\bar\Omega$. In particular, $C_{0,\sigma}(\Omega)$ agrees with the space $\{v\in C(\bar\Omega)\mid \lim_{|x|\to\infty}v(x)=0,\ div\ v=0 \text{ in } \Omega, v=0 \text{ on } \Omega\}$.

We start with the case when $\Omega=\mathbf{R}^n$.

Proposition 5.2. *The statement of Lemma 5.1 holds when $\Omega=\mathbf{R}^n$. If in addition $v\in C(\mathbf{R}^n)$, the convergence in (5.2) can be replaced by local uniform convergence in $\mathbf{R}^n$.*

If we do not care about divergence-free condition for an approximate sequence, it is easy to constract such a sequence by just cutting off the function v with standard mollification. To recover the divergence-free condition we recall the Bogovskiĭ operator [9],[21]. Let D be a bounded domain with Lipschitz boundary. The Bogovskiĭ operator B_D is a bounded linear operator from $L^r_{\mathrm{av}}(D)$ to the Sobolev space $W^{1,r}(D)$ $(1 < r < \infty)$ where $L^r_{\mathrm{av}}(D)$ denotes the space of all average zero function $g \in L^r(D)$ i.e. $\int_D g dx = 0$. This operator B_D selects a vector field $v = B_D(g)$ satisfying div $v = g$ in D and if spt $g \subset D$, spt $B_D(g) \subset D$. The operator B_D fulfills the estimate

$$\left\| B_D(g) \right\|_{W^{1,r}(D)} \leq C_B \|g\|_{L^r(D)} \quad \text{for all } g \in L^p_{\mathrm{av}}(D) \tag{5.3}$$

with the constant C_B depends on r and Lipschitz regularity of ∂D but independent of g. In the next proof we apply the estimate for the Bogovskiĭ opeator,

$$\left\| B_D(g) \right\|_{L^\infty(D)} \leq C_D \|g\|_{L^r(D)} \tag{5.4}$$

with $r > n$. This estimate easily follows from the Sobolev inequality [6, 4.12] and (5.3). The constant C_D is independent of g.

Proof of Proposition 5.2. Let θ be a smooth cut-off function in $[0,\infty)$ satisfying $\theta \equiv 1$ in $[0, 1/2]$, $\theta \equiv 0$ in $[1,\infty)$ and set $\theta_m(x) = \theta(|x|/m)$ for $x \in \mathbf{R}^n$ with $m \geq 1$. Observe that the support of $\nabla\theta_m$ is included in an annulus $D_m = \mathrm{int}\, B_0(m)\backslash B_0(m/2)$ expanding as m goes to infinity. For $v \in L^\infty_\sigma(\mathbf{R}^n)$ we rescale $g_m = v \cdot \nabla\theta_m$ as $f_m(x) = g_m(mx)$ for $x \in D_1$ and set $u^*_m(x) = B_{D_1}(f_m)(x)$ for $x \in D_1$. Then spt $u^*_m \subset D_1$ and div $u^*_m = f_m$ in D_1. Applying the estimate (5.4) in D_1 implies the estimate

$$\|u^*_m\|_{L^\infty(D_1)} \leq C_{D_1} \|f_m\|_{L^r(D_1)} \tag{5.5}$$

with $r > n$ and the constant C_{D_1} independent of m. We set

$$v^*_m(x) = m u^*_m(x/m) \quad \text{for } x \in D_m$$

and observe that spt $v^*_m \subset D_m$ and div $v^*_m = g_m$ in D_m. Since $\|f_m\|_{L^r(D_1)} = m^{-n/r}\|g_m\|_{L^r(D_m)}$, it follows form (5.5) that

$$\begin{aligned}
\|v^*_m\|_{L^\infty(D_m)} &\leq C_{D_1} m \|f_m\|_{L^r(D_1)} \\
&\leq C_{D_1} m^{1-n/r} \|g_m\|_{L^r(D_m)} \\
&\leq C_{D_1} \|\nabla\theta\|_{L^r(D_1)} \|v\|_{L^\infty(D_m)}.
\end{aligned}$$

We now set $\tilde{v}_m = v\theta_m - v^*_m$ with zero extension of v^*_m to $\mathbf{R}\backslash\bar{D}_m$ (still denoted by v^*_m). Then the desired sequence is obtained by mollifying $\tilde{v}_m$, i.e. $v_m = \tilde{v}_m * \eta_{1/m}$ with the standard mollifier $\eta_{1/m}$. If $v \in C(\mathbf{R}^n)$, then $v_m \in C(\mathbf{R}^n)$ so v_m converges to v locally uniformly in $\mathbf{R}^n$. If $\lim_{|x|\to\infty} v(x) = 0$, v^*_m converges to zero uniformly in $\mathbf{R}^n$ so the converges is uniform. □

Remark 5.3. We observe that (5.1) and (5.2) are still valid even for higher derivatives for $v \in W^{k,\infty}(\mathbf{R}^n) \cap L^\infty_\sigma(\mathbf{R}^n)$ for $k \geq 1$, i.e.

$$\|v_m\|_{W^{k,\infty}(\mathbf{R}^n)} \leq C\|v\|_{W^{k,\infty}(\mathbf{R}^n)} \tag{5.6}$$

and

$$\partial_x^l v_m \to \partial_x^l v \quad \text{a.e. in } \mathbf{R}^n \tag{5.7}$$

as $m \to \infty$ for all $|l| \leq k$. We recall that B_D is a bounded operator from $W_0^{k,r}(D)$ to $W_0^{k+1,r}(D)$ for $k \geq 1$[21], [24]. For example $k = 1$, analogously with (5.4) the estimate

$$\|\nabla B_D(g)\|_{L^\infty(D)} \leq C'_D \|g\|_{W^{1,r}(D)}$$

holds for $g \in L^p_{\mathrm{av}}(D) \cap W^{1,r}(D)$ with the constant C'_D depending on r and D. Applying this estimate for $v_m^* = m u_m^*(x/m)$ yields that

$$\|\nabla v_m^*\|_{L^\infty(D_m)} \leq C'_D \|\nabla\theta\|_{L^r(D_1)} \|\nabla v\|_{L^r(D_1)} + C/m \|v\|_{L^\infty(D_m)}$$

with $C = C'_{D_1}(\|\nabla^2\theta\|_{L^r(D_1)} + \|\nabla\theta\|_{L^r(D_1)})$, so v_m satisfies (5.6) and (5.7) with some constant C independent of m and v.

Here is a rough idea of the proof of Lemma 5.1. In an exterior domain Ω, a solenoidal vector field can be divided into two vector fields - one is compactly supported in Ω and the other is supported in $\mathbf{R}^n$ away from $\partial\Omega$ keeping a divergence-free condition by using a cut-off technique and the Bogovskiĭ operator. We shall reduce our problem to the case of $\mathbf{R}^n$ (Proposition 5.2) and a bounded domain. For a bounded domain, we already constructed the corresponding approximate sequences [1, Lemma 6.3].

Proof of Lemma 5.1. We may assume $0 \in \Omega^c$. Let θ be a smooth cut-off function in $[0, \infty)$ satisfying $\theta \equiv 1$ in $[0, 1/2]$, $\theta \equiv 0$ in $[1, \infty)$ and set $\theta_R(x) = \theta(|x|/R)$ for $x \in \mathbf{R}^n$ with $R > \operatorname{diam} \Omega^c$. We divide a support of $v \in L^\infty_\sigma(\Omega)$ into two parts, near the boundary and away by setting as

$$v_1 = v\theta_R - B_{D_R}(v \cdot \nabla\theta_R),$$
$$v_2 = v(1 - \theta_R) + B_{D_R}(v \cdot \nabla\theta_R),$$

where $D_R = \operatorname{int} B_0(R) \backslash B_0(R/2)$. By (5.4) we have

$$\|v_i\|_{L^\infty(\Omega)} \leq C\|v\|_{L^\infty(\Omega)} \quad \text{for} \quad i = 1, 2$$

with the constant C depending on R but independent of v. We construct the desired approximate sequence combining approximate sequences for v_1 and v_2.

Since the support of v_1 is bounded, it can be regarded as a solenoidal vector field in $\Omega_R = \Omega \cap \operatorname{int} B_0(R)$. We apply the approximation Lemma for $L^\infty_\sigma(\Omega_R)$ [1, Lemma 6.3] for a bounded domain, which guarantees the existence of an approximate sequence $\{v_{1,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\Omega_R)$ satisfying (5.1) and (5.2) in Ω_R with a constant C_{Ω_R} independent of m, i.e. $v_{1,m} \to v_1$ a.e. in Ω_R, $\|v_{1,m}\|_\infty \leq C_{\Omega_R}\|v_1\|_\infty$. We consider zero extensions of $\{v_{1,m}\}_{m=1}^\infty$ to $\Omega\backslash\bar{\Omega}_R$ (still denoted by $\{v_{1,m}\}_{m=1}^\infty$) so $v_{1,m}$ is a compactly supported smooth solenoidal vector field in Ω.

We next consider approximation for v_2, regarded as a solenoidal vector field in $\mathbf{R}^n$. We apply Proposition 5.2 which implies that there exists a sequence of functions $\{v_{2,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\mathbf{R}^n)$ satisfying (5.1) and (5.2) in $\mathbf{R}^n$ with a constant C_W independent of m, i.e. $v_{2,m} \to v_2$ a.e. in $\mathbf{R}^n$, $\|v_{2,m}\|_\infty \leq C_W\|v_2\|_\infty$. We restrict a domain of $v_{2,m}$ to Ω. From the proof of Proposition 5.2 we observe that the support of $v_{2,m}$ is included in Ω since the support of v_2 is away from $\partial\Omega$. So $v_{2,m}$ is a compactly supported smooth solenoidal in Ω.

We now set $v_m = v_{1,m} + v_{2,m}$. Clearly the sequence $\{v_m\}_{m=1}^\infty \subset C_{c,\sigma}^\infty(\Omega)$ satisfies (5.1) and (5.2). If $v \in C(\bar{\Omega})$ and $v = 0$ on $\partial\Omega$, it is easy to observe that the sequences $\{v_{1,m}\}_{m=1}^\infty$ and $\{v_{2,m}\}_{m=1}^\infty$ converge locally uniformly in $\bar{\Omega}$. Since the convergence of $\{v_{2,m}\}_{m=1}^\infty$ is uniform if $v_2(x) \to 0$ as $|x| \to \infty$, the last assertion follows. □

Remark 5.4. (i) The characterization of $C_{0,\sigma}(\Omega)$ in Lemma 5.1 is proved in [37, Lemma 3.1] ([39, Lemma A.1]) when Ω is a bounded domain or exterior domain with $C^{1,\alpha}$ boundary. The proof depends on the maximum modulus theorem of the stationary Stokes problem. Our proof is a natural extension of [1, Lemma 6.3] and the proof is direct via the Bogovskiĭ operator without appealing the Stokes equations.
(ii) Recently, an approximate sequence for L^∞_σ is constructed in [40, Lemma 2.6] with non-decaying smooth solenoidal vector fields in an exterior domain. Although the construction procedure is similar, we give an approximation with compactly supported functions to apply a priori estimate (1.6) for L^r-solutions.

5.2. **Regularity for L^r-solutions.** To apply the estimates (1.6) and (1.7) for L^r-solutions, we show boundedness of the quantities (1.5) and (1.10) with assuming an extra regularity for initial data which implies that an L^r-solution naturally can be regarded as an L^∞-solution. Although our assumption for initial data can be weakened, the following statement (Proposition 5.5) is sufficient for our purpose.

Proposition 5.5. *Let Ω be an exterior domain with C^3 boundary. Let (v, q) be an L^r-solution with $r > n$. Assume that $v_0 \in D(A_r)$, where $-A_r$ is the Stokes operator in $L^r_\sigma(\Omega)$. Then $(v, \nabla q) \in C^{2,1}(\bar{\Omega}\times(0, T])\times C(\bar{\Omega}\times(0, T])$ and the quantity (1.5) is bounded in $\Omega\times(0, T)$. If in addition $v_0 \in D(A_2)$, (1.10) is also bounded. In particular, (v, q) is an L^∞-solution and the estimate*

$$\sup_{0\le t\le T_0} \left\{ \left\| N(v, q) \right\|_\infty (t) + t^{1/2} \left| \nabla q \right|_{\infty, d_\Omega}(t) \right\} \le C \|v_0\|_\infty \tag{5.8}$$

holds for any $T_0 > 0$ with some constant C depending only on T_0 and Ω.

In an exterior domain an L^r-solution for (1.1)-(1.4) is provided by $v = S(t)v_0$ and $\nabla q = \mathbf{Q}[\Delta v]$ for $v_0 \in L^r_\sigma$ with the Stokes semigroup $S(t)$ which is an analytic semigroup in L^r_σ [52], [26]. This means that (v, q) is an L^r-solution. The Stokes semigroup is denoted by $S(t) = e^{-tA_r}$ with the Stokes operator $-A_r$ defined by $-A_r v = \mathbf{P}\Delta v$ for $v \in D(A_r)$, where $D(A_r) = W^{2,r} \cap W^{1,r}_0 \cap L^r_\sigma$ equipped with the norm

$$\|v\|_{D(A_r)} = \|v\|_{L^r(\Omega)} + \|A_r v\|_{L^r(\Omega)}.$$

As proved in [13], [52], [26] (see also [21]), the norm of $D(A_r)$ is equivalent to the norm

$$\|v\|_{W^{2,r}(\Omega)} = \sum_{|l|\le 2} \|\partial_x^l v\|_{L^r(\Omega)},$$

so spatial (also time) derivatives of $v = S(t)v_0$ can be estimated by the norm $\|\cdot\|_{D(A_r)}$ for initial data v_0.

Proof of Proposition 5.5. The first assertion is rather spacial case of [1, Proposition 5.2]. It is proved for $\tilde{L}^r$-solutions ($r > n$) in a uniformly C^3-domain. The proof also works for L^r-solutions. We here omit the proof.

We shall prove the second assertion by showing a bound

$$\left|\nabla q\right|_{\infty,d_\Omega}(t) \leq C_1(\|v_0\|_{D(A_r)} + \|v_0\|_{D(A_2)}) \tag{5.9}$$

for $t \in (0,T)$. By the same way as we proved Proposition 2.6, a mean value formula for ∇q and the boundedness of the projection $\mathbf{Q}$ in L^r space implies

$$\left|\nabla q\right|_{\infty,d_\Omega}(t) \leq C_2(\|\Delta v\|_{L^r(\Omega)} + \|\Delta v\|_{L^2(\Omega)}). \tag{5.10}$$

Since the norm $\|v\|_{W^{2,r}(\Omega)}$ is equivalent to the norm $\|v\|_{D(A_r)}$ and $A_r e^{-tA_r} v_0 = e^{-tA_r} A_r v_0$, applying the estimate $\|e^{-tA_r} v_0\|_{L^r} \leq C_3 \|v_0\|_{L^r}$ for $t \in (0,T)$ implies

$$\begin{aligned} \|v\|_{W^{2,r}(\Omega)} &\leq C_3 \|v\|_{D(A_r)} \\ &\leq C_4 \|v_0\|_{D(A_r)}. \end{aligned} \tag{5.11}$$

The constant C_4 depends on r and Ω independent of t. Since $v_0 \in D(A_2)$, combining (5.10) and (5.11), (5.9) follows. The convergence $v(\cdot,t) \to v_0$ $*$-weakly in L^∞ as $t \downarrow 0$ easily follows from the convergence in L^r space. We thus conclude that (v,q) is L^∞-solution. By Theorem 1.6 and Lemma 3.1 we obtain

$$\sup_{0 \leq t \leq T_0} \left\|N(v,q)\right\|_\infty(t) \leq C_{T_0} \|v_0\|_\infty$$

for any $T_0 > 0$ with C_{T_0} depending on T_0 and Ω and

$$\left|\nabla q\right|_{\infty,d_\Omega}(t) \leq C_\Omega \|\nabla v\|_{\infty,\partial\Omega}(t)$$

for $t \in (0,T)$ with C_Ω depending only on Ω. Combining these estimates, the estimate (5.8) follows. □

5.3. **Analyticity of the Stokes semigroup in L^∞_σ.** We observe that a limit of approximate solutions is also an L^∞-solution by (5.1) and (5.8). It is noted that for the limit solution the weak $*$-convergence to initial data in L^∞ can be understood in a weak form by applying Proposition 4.2 again.

Proof of Theorem 1.5. We first prove the assertion (i). By Lemma 5.1 for $v_0 \in L^\infty_\sigma(\Omega)$ we are able to choose a compactly supported sequence $\{v_{0,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\Omega)$ such that

$$\|v_{0,m}\|_{L^\infty(\Omega)} \leq C_\Omega \|v_0\|_{L^\infty(\Omega)} \tag{5.12}$$

and

$$v_{0,m} \to v_0 \quad \text{a.e. in } \Omega$$

as $m \to \infty$ with the constant C_Ω depending only on Ω. Let (v_m, q_m) be an L^r-solution ($r > n$) with $v_{0,m}$. Since $v_{0,m} \in C^\infty_{c,\sigma} \subset D(A_r) \cap D(A_2)$, by Proposition 5.5 an (v_m, q_m) is L^∞-solution and the estimate (5.8) holds. Combining (5.8) for (v_m, q_m) with (5.12), we get a uniform bound

$$\sup_{0 \leq t \leq T_0} \left\{ \left\|N(v_m,q_m)\right\|_\infty(t) + t^{1/2} \left|\nabla q_m\right|_{\infty,d_\Omega}(t) \right\} \leq C\|v_0\|_\infty,$$

for any $T_0 > 0$ with the constant C depending only on T_0 and Ω. We apply Theorem 3.4 and Theorem 3.5 to choose a subsequence still denoted by (v_m, q_m) which converges to a limit (v, q) locally uniformly in $\bar{\Omega} \times (0, T_0]$ together with ∇v_m, $\nabla^2 v_m$, $\partial_t v_m$, ∇q_m. Then the limit $(v, \nabla q) \in C^{2,1}(\bar{\Omega} \times (0, T_0]) \times C(\bar{\Omega} \times (0, T_0])$ solves (1.1)-(1.3) satisfying

$$\sup_{0\leq t\leq T_0} \left\{ \left\| N(v,q) \right\|_\infty (t) + t^{1/2} \left| \nabla q \right|_{\infty, d_\Omega}(t) \right\} \leq C \|v_0\|_\infty$$

which is stronger than (1.6). We now observe that v has initial data v_0. Integration by parts yields that (v_m, q_m) fulfills

$$\int_0^{T_0} \int_\Omega \{ v_m \cdot (\varphi_t + \Delta\varphi) - \nabla q_m \cdot \varphi \} \,\mathrm{d}x\mathrm{d}t = -\int_\Omega v_m(x, 0) \cdot \varphi(x, 0)\mathrm{d}x$$

for all $\varphi \in C_c^\infty(\Omega \times [0, T_0))$. Since $v_{0,m} \to v_0$ a.e. in Ω as $m \to \infty$, we take a limit to observe that

$$\int_0^{T_0} \int_\Omega \{ v \cdot (\varphi_t + \Delta\varphi) - \nabla q \cdot \varphi \} \,\mathrm{d}x\mathrm{d}t = -\int_\Omega v(x, 0) \cdot \varphi(x, 0)\mathrm{d}x.$$

Applying Proposition 4.2 implies that $v(\cdot, t) \to v_0$ $*$-weakly in L^∞ as $t \downarrow 0$. We thus conclude that the limit (v, q) is an L^∞-solution with initial data $v_0 \in L^\infty_\sigma$. Since an L^∞-solution is unique by Proposition 4.1, we have the assertion (i).

We now extend the Stokes semigroup $S(t) : v_0 \mapsto v(\cdot, t)$ $(t \geq 0)$ for $v_0 \in L^\infty_\sigma(\Omega)$. Since an L^∞-solution is unique, $S(t)v_0$ is well-defined for all $v_0 \in L^\infty_\sigma$. It remains to show the semigroup property for $\{S(t)\}_{t\geq 0}$ in $L^\infty_\sigma(\Omega)$, i.e. $S(t + s) = S(t)S(s)$ for $t, s \geq 0$. Since $S(0) = I$, we may assume $s > 0$. Suppose that (v^1, q^1) is an L^∞-solution with $v_0 \in L^\infty_\sigma(\Omega)$. We consider an L^∞-solution (v^2, q^2) with initial data $v^1(\cdot, s)$ for each $s > 0$. From the uniqueness of an L^∞-solution, $(v^1, \nabla q^1) \equiv (v^2, \nabla q^2)$ holds for $t > s$. In other words, $S(t)v_0 = S(t - s)S(s)v_0$, $t > s$ for each fixed $s > 0$, so $S(\tau + s) = S(\tau)S(s) = S(s)S(\tau)$ for $\tau, s > 0$ by taking $t = \tau + s$. We have the assertion (ii) so the proof is now complete. □

6. Continuity at time zero

In this section we prove Theorem 1.7. To show that $S(t)v_0 \to v_0$ in BUC_σ as $t \downarrow 0$ for $v_0 \in BUC_\sigma(\Omega)$, we start with initial data v_0 whose support is away from $\partial\Omega$.

Lemma 6.1. *Let Ω be an exterior domain with C^3 boundary. Let $S(t)$ be the Stokes semigroup in $L^\infty_\sigma(\Omega)$. Then $S(t)v_0 \to v_0$ in BUC_σ as $t \downarrow 0$ for all $v_0 \in BUC_\sigma(\Omega)$ satisfying* $\mathrm{dist}(\mathrm{spt}\ v_0, \partial\Omega) > 0$.

We first prove Lemma 6.1 with assuming an extra regularity for v_0.

Proposition 6.2. *The assertion of Lemma 6.1 holds for $v_0 \in W^{2,\infty}(\Omega) \cap BUC_\sigma(\Omega)$ satisfying* $\mathrm{dist}(\mathrm{spt}\ v_0, \partial\Omega) > 0$.

Proof. Let (v, q) be an L^∞-solution with initial data v_0 satisfying the assumption. We shall show

$$v_t(\cdot, t) = S(t)\Delta v_0 \quad \text{for } t \geq 0. \tag{6.1}$$

The assertion easily follows from (6.1). Since

$$\begin{aligned}\|v(\cdot,t)-v_0\|_\infty &\leq \int_0^t \|v_s\|_\infty(s)ds \\ &\leq Ct\|\Delta v_0\|_\infty,\end{aligned}$$

letting $t\downarrow 0$ implies that $v(\cdot,t)\to v_0$ uniformly in $\bar{\Omega}$ as $t\downarrow 0$, so $S(t)v_0\to v_0$ in $BUC_\sigma(\Omega)$ as $t\downarrow 0$.

We first show (6.1) for compactly supported v_0 in Ω. Since Δv_0 is also compactly supported in Ω and $\Delta v_0=0$ on $\partial\Omega$, it follows that

$$\begin{aligned}-A_r v_0 &= \mathbf{P}\Delta v_0 \\ &= \Delta v_0.\end{aligned}$$

For compactly supported v_0, $S(t)$ can be regarded as the Stokes semigroup in L^r_σ space. We thus have

$$\begin{aligned}v_t(\cdot,t) &= -A_r S(t)v_0 \\ &= S(t)(-A_r v_0) \\ &= S(t)\Delta v_0.\end{aligned}$$

For non-compactly supported v_0 in Ω, we approximate v_0 by a sequence of compactly supported functions. By Lemma 5.1 there exists a sequence $\{v_{0,m}\}_{m=1}^\infty\subset C^\infty_{c,\sigma}(\Omega)$ such that $\|v_{0,m}\|_\infty\leq C\|v_0\|_\infty$ and $v_{0,m}\to v_0$ locally uniformly in $\bar{\Omega}$ as $m\to\infty$. Moreover, $\nabla^2 v_{0,m}\to\nabla^2 v_0$ locally uniformly in $\bar{\Omega}$ satisfying

$$\|\nabla^2 v_{0,m}\|_{L^\infty(\Omega)}\leq C\|v_0\|_{W^{2,\infty}(\Omega)} \tag{6.2}$$

with the constant C depending only on Ω by Remark 5.3. By dist(spt $v_0,\partial\Omega)>0$ we do not distinguish the zero extension of v_0 to $\mathbf{R}^n\backslash\bar{\Omega}$ from v_0, respectively for $v_{0,m}$. Let (u_m,p_m) be an L^∞-solution with initial data $\Delta v_{0,m}$. By (6.2) we have

$$\sup_{0\leq t\leq T_0}\left\|N(u_m,p_m)\right\|_\infty(t)\leq C\|v_0\|_{W^{2,\infty}(\Omega)}$$

with some constant C independent of m. Applying Theorem 3.4 and Theorem 3.5 implies that (u_m,p_m) subsequently converges to a limit (u,p) locally uniformly in $\bar{\Omega}\times(0,T]$ together with $\nabla u_m,\nabla^2 u_m,\partial_t u_m,\nabla p_m$. By Proposition 4.1 and Proposition 4.2 (u,p) is a unique L^∞-solution with initial data Δv_0. In particular, $u=S(t)\Delta v_0$. On the other hand, by (6.1) for compactly supported $v_{0,m}$ we have

$$\begin{aligned}\partial_t v_m(\cdot,t) &= S(t)\Delta v_{0,m} \\ &= u_m(\cdot,t).\end{aligned}$$

Since u_m subsequently converges to u locally uniformly in $\bar{\Omega}\times(0,T]$, we thus conclude that

$$\begin{aligned}\partial_t v(\cdot,t) &= \lim_{m\to\infty}\partial_t v_m(\cdot,t) \\ &= u(\cdot,t) \\ &= S(t)\Delta v_0.\end{aligned}$$

The proof is now complete. □

Proof of Lemma 6.1. We apply Proposition 6.2 to a mollified function $v_{0,\varepsilon}$ for v_0 (i.e. $v_{0,\varepsilon} = v_0 * \eta_\varepsilon$ with the standard mollifier η_ε.) Since v_0 is equal to zero near $\partial\Omega$ and uniformly continuous in $\bar{\Omega}$, $v_{0,\varepsilon}$ also equals zero near $\partial\Omega$ for sufficiently small $\varepsilon > 0$ and converges to v_0 uniformly in $\bar{\Omega}$, i.e. $\lim_{\varepsilon\downarrow 0}\|v_{0,\varepsilon} - v_0\|_\infty = 0$. Applying Proposition 6.2 to $v_{0,\varepsilon}$ implies that

$$\lim_{t\downarrow 0}\|S(t)v_{0,\varepsilon} - v_{0,\varepsilon}\|_\infty = 0,$$

we thus have

$$\begin{aligned}\overline{\lim_{t\downarrow 0}}\|S(t)v_0 - v_0\|_\infty &\leq \overline{\lim_{t\downarrow 0}}\|S(t)v_{0,\varepsilon} - S(t)v_0\|_\infty + \|v_{0,\varepsilon} - v_0\|_\infty \\ &\leq (C+1)\|v_{0,\varepsilon} - v_0\|_\infty.\end{aligned}$$

Letting $\varepsilon \downarrow 0$ yields that $S(t)v_0 \to v_0$ in BUC_σ as $t \downarrow 0$. The proof is now complete. □

We now prove Theorem 1.7.

Proof of Theorem 1.7. We may assume $0 \in \Omega^c$. Let θ be a smooth cut-off function in $[0,\infty)$ satisfying $\theta \equiv 1$ in $[0, 1/2]$, $\theta \equiv 0$ in $[1,\infty)$ and set $\theta_R(x) = \theta(|x|/R)$ for $x \in \mathbf{R}^n$ with $R > 2\mathrm{diam}\ \Omega^c$. We divide $v_0 \in BUC_\sigma(\Omega)$ into two terms v_0^1 and v_0^2 so that v_0^1 is compactly supported in Ω and v_0^2 is supported in $\mathbf{R}^n$ whose support is away from $\partial\Omega$ by setting as

$$\begin{aligned}v_0^1 &= v_0\theta_R - B_{D_R}(v_0\cdot\nabla\theta_R),\\ v_0^2 &= v_0(1-\theta_R) + B_{D_R}(v_0\cdot\nabla\theta_R),\end{aligned}$$

with the Bogovskiĭ operator B_{D_R} in $D_R = \mathrm{int}\ B_0(R)\backslash B_0(R/2)$. From the proof of Lemma 5.1 we observe that both v_0^1 and v_0^2 belong to $BUC_\sigma(\Omega)$. We shall prove the uniform convergence for $S(t)v_0^1$ and $S(t)v_0^2$ as $t \downarrow 0$.

We first show $\lim_{t\downarrow 0}\|S(t)v_0^1 - v_0^1\|_\infty = 0$. Since v_0^1 is compactly supported in Ω and $v_0^1 = 0$ on $\partial\Omega$, by Lemma 5.1 v_0^1 belongs to $C_{0,\sigma}(\Omega)$. By Theorem 1.2 $S(t)$ is a C_0-semigroup in $C_{0,\sigma}(\Omega)$ so $S(t)v_0^1 \to v_0^1$ in $BUC_\sigma(\Omega)$ as $t \downarrow 0$ follows. It remains to show $\lim_{t\downarrow 0}\|S(t)v_0^2 - v_0^2\|_\infty = 0$. Since $\mathrm{dist}(\mathrm{spt}\ v_0^2, \partial\Omega) > R/2$ with $R > 2\mathrm{diam}\ \Omega^c$, we apply Lemma 6.1 which implies that $S(t)v_0^2 \to v_0^2$ in BUC_σ as $t \downarrow 0$. We thus have

$$\begin{aligned}\overline{\lim_{t\downarrow 0}}\|S(t)v_0 - v_0\|_\infty &\leq \lim_{t\downarrow 0}\|S(t)v_0^1 - v_0^1\|_\infty + \lim_{t\downarrow 0}\|S(t)v_0^2 - v_0^2\|_\infty \\ &= 0.\end{aligned}$$

The proof is now complete. □

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