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# Upper bounds for fundamental solutions to non-local diffusion equations with divergence free drift

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## Abstract

We investigate some non-local diffusion equations in the presence of a divergence free drift term. We derive pointwise upper bounds for fundamental solutions under low regularity assumptions for the velocity of the drift term.

**Keywords:** non-local diffusion; divergence free drift; fundamental solutions; pointwise upper bound.

## 1 Introduction

We consider the following non-local diffusion equations in the presence of a given divergence free drift term:

$$\partial_t \theta + (-\Delta)^{\frac{\alpha}{2}} \theta + v \cdot \nabla \theta = 0, \quad \nabla \cdot v = 0, \quad t > 0, \quad x \in \mathbb{R}^d, \quad (1.1)$$

where  $d \geq 2$  is the dimension,  $\alpha \in (0, 2)$  is a constant, and  $(-\Delta)^{\alpha/2}$  is the fractional Laplacian formally defined by

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = C_{\alpha,d} P.V. \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+\alpha}} dy. \quad (1.2)$$

When there is no drift term (i.e.,  $v = 0$ ) this equation appears in the theory of Dirichlet forms of jump type as a special case, and it has been investigated mainly from the probabilistic viewpoint; see [1, 3, 4, 5, 9] and the references therein. Although these equations are expected to enjoy similar diffusion effects as usual diffusion equations, due to the non-local character of the fractional Laplacian, it is not always straightforward to obtain analytic properties such as regularity, asymptotic behavior of solutions. On the other hand in recent years the case with the drift term has also attracted much attention especially in the field of fluid mechanics, mathematical finance, biology, and so on. Among of them, the 2-dimensional dissipative surface quasi-geostrophic equations (QG) for the active scalar in the geophysical fluid introduced by [13] are extensively studied in the last decade; see, e.g., [7, 10, 11, 12, 16, 19, 20]. The equations (QG) are nonlinear equations of the form (1.1)

where the velocity in the drift term is related by  $v = (-R_2\theta, R_1\theta)$  via the Riesz transform  $R_i$ .

In [21], motivated by the equations (QG) we studied the existence and the continuity properties of fundamental solutions of (1.1) under weak regularity assumptions for the drift term. The purpose of the present paper is to derive pointwise upper bounds for the fundamental solutions. In [6, 18] they considered fundamental solutions when  $\alpha \in (1, 2)$  and  $v$  belongs to a suitable Kato class without assuming the divergence free condition. They proved the existence of fundamental solutions and showed pointwise estimates. However, there seems to be still few works on fundamental solutions for  $\alpha \in (0, 1]$ . In such cases the drift term formally becomes the leading term and is no longer regarded as a simple perturbation of the diffusion term. Moreover, for applications to nonlinear problems it is important to study the linear problem of the form (1.1) under weak assumption for  $v$  beyond the Kato class. In such situations the interplay between the diffusion term and the drift term makes problems more subtle and the divergence free structure for the velocity plays a crucial role.

To state our main result let us describe the regularity assumptions for  $v$ . For the purpose we recall the definition of the Campanato spaces:

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^d) = \left\{ f \in L_{loc}^p(\mathbb{R}^d) \mid \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^d)} = \sup_B \left( R^{-\lambda} \int_B |f(x) - \fint_B f|^p dx \right)^{\frac{1}{p}} < \infty \right\}. \quad (1.3)$$

Here the supremum is taken over all balls  $B = B_R(x)$  (the ball with radius  $R > 0$  centered at  $x \in \mathbb{R}^d$ ), the value  $\fint_B f$  is the average in  $B$  defined by  $\fint_B f = |B|^{-1} \int_B f(x) dx$ , and  $\|\cdot\|_{\mathcal{L}^{p,\lambda}}$  becomes a seminorm. It is easy to see that the continuous embedding

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^d) \hookrightarrow \mathcal{L}^{1,\mu}(\mathbb{R}^d) \quad \text{if} \quad p \geq 1 \quad \text{and} \quad \mu = \frac{\lambda - d}{p} + d \quad (1.4)$$

holds. In the case of  $\lambda < d$ , the function in  $\mathcal{L}^{p,\lambda}$  is uniformly locally integrable, and  $\mathcal{L}^{p,\lambda}$  is identified by the Morrey space  $L^{p,\lambda}$  modulo constant. Moreover, it is known that

$$\begin{aligned} L_{\text{w}}^{\frac{pd}{d-\lambda}}(\mathbb{R}^d) &\hookrightarrow \mathcal{L}^{p,\lambda}(\mathbb{R}^d) && \text{if } 0 < \lambda < d, \\ \mathcal{L}^{p,\lambda}(\mathbb{R}^d) &= BMO(\mathbb{R}^d) && \text{if } \lambda = d, \\ \mathcal{L}^{p,\lambda}(\mathbb{R}^d) &= \dot{C}^{\frac{\lambda-d}{p}}(\mathbb{R}^d) && \text{if } d < \lambda \leq d + p, \end{aligned}$$

hold: See, e.g., [17, 23]. Here  $L_{\text{w}}^p(\mathbb{R}^d)$  is the weak  $L^p$  space and  $\dot{C}^\beta(\mathbb{R}^d)$ ,  $\beta \in (0, 1]$ , is the homogeneous Hölder space of the order  $\beta$ , i.e.,

$$\dot{C}^\beta(\mathbb{R}^d) = \left\{ f \in C(\mathbb{R}^d) \mid \|f\|_{\dot{C}^\beta} = \sup_{x,y \in \mathbb{R}^d} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty \right\}. \quad (1.5)$$

Next we introduce the Morrey type spaces of  $\mathcal{L}^{p,\lambda}$ -valued functions:

$$\begin{aligned} L^{p,\lambda_1}(0, \infty; \mathcal{L}^{q,\lambda_2}(\mathbb{R}^d)) &= \left\{ f \in L_{loc}^p(0, \infty; \mathcal{L}^{q,\lambda_2}(\mathbb{R}^d)) \mid \right. \\ &\left. \|f\|_{L^{p,\lambda_1}(0, \infty; \mathcal{L}^{q,\lambda_2}(\mathbb{R}^d))} = \sup_{t>0} \sup_{0 < s < t} \left( (t-s)^{-\lambda_1} \int_s^t \|f(\tau)\|_{\mathcal{L}^{q,\lambda_2}}^p d\tau \right)^{\frac{1}{p}} < \infty \right\}. \end{aligned} \quad (1.6)$$

For  $\alpha \in (0, 2)$  we impose the following conditions on  $v$ :

(C) There are  $\lambda \in [2d/\alpha - d, 2d/\alpha + d)$  and  $q \in (1, \infty]$  such that

$$\begin{cases} v \in L^{2, \frac{2}{\alpha} - \frac{\lambda}{d}}(0, \infty; (\mathcal{L}^{\frac{2d}{\alpha}, \lambda}(\mathbb{R}^d))^d) \cap L_{loc}^q(0, \infty; (L_{loc}^1(\mathbb{R}^d))^d) & \text{when } \lambda \in [\frac{2d}{\alpha} - d, d], \\ v \in L^{1, \frac{1}{2} + \frac{1}{\alpha} - \frac{\lambda}{2d}}(0, \infty; (\mathcal{L}^{\frac{2d}{\alpha}, \lambda}(\mathbb{R}^d))^d) \cap L_{loc}^q(0, \infty; (L_{loc}^\infty(\mathbb{R}^d))^d) & \text{when } \lambda \in (d, \frac{2d}{\alpha} + d). \end{cases}$$

For simplicity of notations we set

$$\|v\|_{X_\lambda} = \begin{cases} \|v\|_{L^{2, \frac{2}{\alpha} - \frac{\lambda}{d}}(0, \infty; \mathcal{L}^{\frac{2d}{\alpha}, \lambda})} & \text{when } \lambda \in [\frac{2d}{\alpha} - d, d], \\ \|v\|_{L^{1, \frac{1}{2} + \frac{1}{\alpha} - \frac{\lambda}{2d}}(0, \infty; \mathcal{L}^{\frac{2d}{\alpha}, \lambda})} & \text{when } \lambda \in (d, \frac{2d}{\alpha} + d]. \end{cases} \quad (1.7)$$

We now state our main result on the upper bound for the fundamental solutions denoted by  $P_{\alpha,v}(t,x;s,y)$ . The precise definition of the fundamental solutions will be given in the next section.

**Theorem 1.1** *Assume that (C) holds. Then there exists a fundamental solution  $P_{\alpha,v}(t,x;s,y)$  to (1.1) such that for all  $t > s \geq 0$  and  $x, y \in \mathbb{R}^d$ ,*

$$P_{\alpha,v}(t,x;s,y) \leq C_1(t-s)^{-\frac{d}{\alpha}}, \quad (1.8)$$

$$P_{\alpha,v}(t,x;s,y) \leq C_2(t-s)^{-\frac{d}{\alpha}} \left(1 + \frac{(|x-y| - CF[v](t,s,x,y))_+}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha}, \quad (1.9)$$

where

$$F[v](t,s,x,y) := \sup_{s < r < t} \left| \int_s^r \int_{B_{|x-y|}(x)} v(\tau) d\tau \right|. \quad (1.10)$$

Here  $C_1$  depend only on  $d$  and  $\alpha$ ,  $C_2$  depends only on  $d$ ,  $\alpha$ , and  $\|v\|_{X_\lambda}$ , and  $C > 1$  is some absolute constant.

**Remark 1.2** The norm  $\|\cdot\|_{X_\lambda}$  is invariant under the scaling

$$v_\lambda(x,t) = \lambda^{\alpha-1}v(\lambda^\alpha t, \lambda x). \quad (1.11)$$

This scaling is natural in the following sense: If  $\theta(t,x)$  is a solution to (1.1) then the rescaled function  $\theta(\lambda^\alpha t, \lambda x)$  satisfies (1.1) with the velocity  $v_\lambda$ , instead of  $v$ . Heuristically, in order to ensure a smoothing effect by the diffusion term it is essential to assume that  $v$  belongs to a scale-invariant function space; see, e.g., [8, 7, 19, 22, 24]. The space  $X_\lambda$  covers the following classes as special cases:  $L^\infty(0, \infty; (BMO(\mathbb{R}^d))^d)$  for  $\alpha = 1$ ;  $L^\infty(0, \infty; (\dot{C}^{1-\alpha}(\mathbb{R}^d))^d)$  for  $\alpha \in (0, 1)$ . Moreover it also allows a singularity at some  $t_0 \geq 0$ :  $|t - t_0|^{\frac{\lambda}{2a} + \frac{1}{2} - \frac{1}{\alpha}}v(t) \in L^\infty(0, \infty; (\mathcal{L}^{\frac{2d}{\alpha}, \lambda}(\mathbb{R}^d))^d)$ . One of the advantages to use the Campanato spaces (1.3) is that they contain certain homogeneous functions. This fact is important for the study of the self-similar solutions in some nonlinear problems. Another advantage is that in the case of  $\lambda \geq d$  they contain growing functions at spatial infinity. Except some special cases, e.g., the fractional Ornstein-Uhlenbeck operators, such velocity fields seem not to be studied.

We note that the assumption  $v \in L^q_{loc}(0, \infty; (L^1_{loc}(\mathbb{R}^d))^d)$  or  $L^q_{loc}(0, \infty; (L^\infty_{loc}(\mathbb{R}^d))^d)$  in (C) is used only to guarantee the existence of the fundamental solution in [21]. It is weaker than the assumption  $v \in X_\lambda$  in view of the scaling.

**Remark 1.3** Est. (1.9) shows that  $P_{\alpha,v}(t,x;s,y)$  is bounded by the modification of  $C(t-s)^{-d/\alpha}(1 + |x-y|(t-s)^{-1/\alpha})^{-d-\alpha}$ , which means that  $P_{\alpha,v}(t,x;s,y)$  possesses the similar decay estimate for the fractional heat equations

$$\partial_t \theta + (-\Delta)^{\frac{\alpha}{2}} \theta = 0, \quad t > 0, \quad x \in \mathbb{R}^d. \quad (1.12)$$

$F[v]$  in (1.9) represents the transport effect by the drift term. Since  $\mathcal{L}^{p,\lambda}$  includes some growing functions, the term  $F[v]$  is not necessarily bounded in space variables. More precisely, from our assumption (C) one can see that  $F[v]$  grows no faster than linearly, thus (1.9) shows that the fundamental solution decays with order  $-d - \alpha$  when  $|x - y|$  is large. On the other hand, in the case of  $\alpha \in [1, 2)$  if we assume  $v \in L^{1,1/\alpha}(0, \infty; (L^\infty(\mathbb{R}^d))^d)$ , then it is easy to see from Theorem 1.1 that  $P_{\alpha,v}(t,x;s,y)$  is bounded by a constant multiple of the fundamental solution to (1.12). Instead, if we impose in addition to (C) that

$$(C') \quad v \in L^{1, \frac{1}{\alpha}}(0, \infty; (L^1_{uloc}(\mathbb{R}^d))^d),$$

where  $L^1_{uloc}(\mathbb{R}^d) = \{f \in L^1_{loc}(\mathbb{R}^d) \mid \|f\|_{L^1_{uloc}} = \sup_{x \in \mathbb{R}^d} \|f\|_{L^1(B_1(x))} < \infty\}$ , then we get the following:

**Corollary 1.4** *Let  $\alpha \in [1, 2)$ . Assume that **(C)** and **(C')** hold. Then*

(i) *if  $\lambda \in [2d/\alpha - d, d)$  we have*

$$P_{\alpha,v}(t, x; s, y) \leq C(t-s)^{-\frac{d}{\alpha}} \left(1 + \frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha}, \quad (1.13)$$

(ii) *if  $\lambda = d$  we have*

$$P_{\alpha,v}(t, x; s, y) \leq \begin{cases} C(t-s)^{-\frac{d}{\alpha}} (1 + |\log(t-s)|)^{d+\alpha} \left(1 + \frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha}, \\ \quad \text{when } (t-s)^{\frac{1}{\alpha}} \leq |x-y| \leq 1, \\ C(t-s)^{-\frac{d}{\alpha}} \left(1 + \frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha}, \quad \text{otherwise,} \end{cases} \quad (1.14)$$

(iii) *if  $\lambda \in (d, 2d/\alpha + d)$  we have*

$$P_{\alpha,v}(t, x; s, y) \leq \begin{cases} C(t-s)^{-\frac{d}{\alpha} - (d+\alpha)(\frac{\lambda}{2d} - \frac{1}{2})} \left(1 + \frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha}, \\ \quad \text{when } (t-s)^{\frac{1}{\alpha}} \leq |x-y| \leq 1, \\ C(t-s)^{-\frac{d}{\alpha}} \left(1 + \frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha}, \quad \text{otherwise.} \end{cases} \quad (1.15)$$

Here  $C$  depends only on  $d, \alpha, \|v\|_{X_\lambda}$ , and  $\|v\|_{L^{1,1/\alpha}(0,\infty;L^1_{uloc})}$ .

The above corollary shows that  $P_{\alpha,v}$  is bounded by the fundamental solution of (1.12) for the case  $\lambda < d$ , while we need the additional modification factor when  $(t-s)^{1/\alpha} \leq |x-y| \leq 1$  for the case  $\lambda \geq d$ .

**Remark 1.5** For the endpoint case  $\lambda = 2d/\alpha + d$  in **(C)**, the estimate (1.9) holds if  $\|v\|_{X_{2d/\alpha+d}}$  or  $|t-s|$  is sufficiently small; see Theorem 3.2 and Remark 3.3. We note that  $\mathcal{L}^{2d/\alpha, 2d/\alpha+d}(\mathbb{R}^d)$  coincides with  $\text{Lip}(\mathbb{R}^d)$ , the space of all Lipschitz functions.

After the pioneering work of [2], there are a lot of results on the pointwise upper bounds for the fundamental solutions of the second order parabolic equations. In particular, for the drift diffusion equation (1.1) with  $\alpha = 2$ , the Gaussian upper bounds are obtained in [22, 8] under the scale-invariant assumptions; see also [25] for recent related works. In contrast with the case  $\alpha = 2$ , the fundamental solution for  $\alpha < 2$  is expected to decay only with polynomial order: In the case  $v = 0$  it is not difficult to see the fundamental solution satisfies the estimate (1.13) via the Fourier transform. If  $v$  is regarded as a simple perturbation of the diffusion term, it is possible to obtain the same upper bound as well. However, under our assumptions for  $v$  (and  $\alpha$ ), the perturbation argument is no longer applicable to handle with our problem. To overcome the difficulty we will develop the idea of Carlen-Kusuoka-Stroock [9], where they derived pointwise upper bounds for the fundamental solution for certain non-local diffusion equations without the drift term based on Davies' method [15]. In our proof, the  $L^1 - L^\infty$  estimate for a certain weighted semigroup plays an important role as in [9, 15], and we have to choose an appropriate weight function to reflect the behavior of the fundamental solutions. The key idea to take the drift term into account for the choice of the weight function is the introduction of a trajectory determined by a local average of  $v$ . This idea is motivated by the work of [7, 19] where the authors studied the regularity of the weak solution of the equation (QG). Another ingredient of the proof is the use of the logarithmic Sobolev inequality of the fractional order recently proved in [14], which plays a crucial role to estimate the diffusion term. We note that our proof can be applied also for more general non-local diffusion equations associated with nonsmooth integral kernels as in [9]. We will formulate the class of the non-local diffusion in the next section, and the pointwise estimates in Theorem 1.1 will be proved for this class of equations.

Rest of this paper is organized as follows. In the next section, we will define notions related to the non-local diffusion equations and collect some fundamental estimates. Section 3 is devoted to the proof of our results.

## 2 Preliminaries

In this section we give a definition of fundamental solutions for non-local diffusion equations including (1.1) as a special case. We also prepare several inequalities which will be used in the proof of our results.

### 2.1 Definition of fundamental solutions

Let  $K(t, x, y)$  be a positive measurable function in  $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ . We assume that the function  $K$  satisfies

$$K(t, x, y) = K(t, y, x), \quad C_0^{-1}|x - y|^{-d-\alpha} \leq K(t, x, y) \leq C_0|x - y|^{-d-\alpha}. \quad (2.1)$$

Then the Dirichlet forms  $\mathcal{E}_K$  and  $\mathcal{E}_v$  are defined by

$$\mathcal{E}_K^{(t)}(f, g) = \frac{1}{2} \int_{\mathbb{R}^{2d}} [f][g](x, y) K(t, x, y) dx dy, \quad [f](x, y) = f(x) - f(y), \quad (2.2)$$

$$\mathcal{E}_{v(t)}(f, g) = - \langle f, v(t) \cdot \nabla g \rangle := - \int_{\mathbb{R}^d} f(x) v(t, x) \cdot \nabla g(x) dx, \quad (2.3)$$

for some positive constant  $C_0 > 0$ . The non-local diffusion operator  $A_K(t)$  associated with the Dirichlet form  $\mathcal{E}_K^{(t)}$  is then formally given by

$$(A_K(t)f)(x) = P.V. \int_{\mathbb{R}^d} [f](x, y) K(t, x, y) dy = \lim_{\epsilon \downarrow 0} \int_{|x-y| \geq \epsilon} [f](x, y) K(t, x, y) dy. \quad (2.4)$$

which makes sense at least when  $f$  is smooth and bounded.

We also set

$$\mathcal{B}_K^{(t)}(f, g) = \mathcal{E}_K^{(t)}(f, g) + \mathcal{E}_{v(t)}(f, g). \quad (2.5)$$

Let  $T > s \geq 0$ . A function  $\theta(t, x)$  is said to be a weak solution to the non-local diffusion equation

$$\partial_t \theta + A_K(t)\theta + v \cdot \nabla \theta = 0, \quad \nabla \cdot v = 0 \quad (2.6)$$

for  $t \in (s, T)$  with initial data  $\theta_s$  at  $t = s$  if  $\theta \in L^\infty(s, T; L^2(\mathbb{R}^d))$ , and

$$\int_s^T \mathcal{B}_K^{(t)}(\theta(t), \theta(t)) dt < \infty$$

and  $\theta$  satisfies

$$- \int_s^T \langle \theta(t), \partial_t \varphi(t) \rangle dt + \int_s^T \mathcal{B}_K^{(t)}(\theta(t), \varphi(t)) dt = \langle \theta(s), \varphi(s) \rangle, \quad \forall \varphi \in C_0^\infty([s, T] \times \mathbb{R}^d). \quad (2.7)$$

Then a measurable function  $P_{K,v}(t, x; s, y)$  on  $\{(t, s, x, y) \mid t > s \geq 0, x, y \in \mathbb{R}^d\}$  is said to be a fundamental solution of (2.6) if for each  $T > s \geq 0$  and  $f \in L^2(\mathbb{R}^d)$  the function

$$\theta(t, x) = \int_{\mathbb{R}^d} P_{K,v}(t, x; s, y) f(y) dy \quad (2.8)$$

is a weak solution of (2.6) for  $t \in (s, T)$  with initial data  $f$  at  $t = s$ .

For  $M \geq 0$ , we set  $K_M = K_M(t, x, y)$ ,  $K_{M^c} = K_{M^c}(t, x, y)$  as

$$K_M(t, x, y) = K(t, x, y) \chi_{\{|x-y| < M\}}(x, y), \quad K_{M^c}(t, x, y) = K(t, x, y) \chi_{\{|x-y| \geq M\}}(x, y),$$

where  $\chi_A$  is the characteristic function of the set  $A \subset \mathbb{R}^d$ .  $\mathcal{E}_{K_M}^{(t)}$  is defined by

$$\mathcal{E}_{K_M}^{(t)}(f, g) = \frac{1}{2} \int_{\mathbb{R}^{2d}} [f][g](x, y) K_M(t, x, y) dx dy. \quad (2.9)$$

$\mathcal{E}_{K_{M^c}}^{(t)}$ ,  $A_{K_M}(t)$ ,  $A_{K_{M^c}}(t)$ ,  $\mathcal{B}_{K_M}^{(t)}$ ,  $P_{K_M, v}(t, x; s, y) \dots$  are defined in the same manner. It is clear that  $\mathcal{E}_K^{(t)}(f, g) = \mathcal{E}_{K_M}^{(t)}(f, g) + \mathcal{E}_{K_{M^c}}^{(t)}(f, g)$  for each  $M \geq 0$ .

**Remark 2.1** In our previous work [21], more general class of the kernel  $K$  than (2.1) is treated. More precisely, in [21] the kernel  $K$  is assumed to satisfy

$$\begin{cases} K(t, x, y) = K(t, y, x), \\ \text{ess. sup}_{t>0, x \in \mathbb{R}^d} \int_{|x-y| \leq M} |x-y|^2 K(t, x, y) dy \leq C_0 M^{2-\alpha} \quad \text{for each } M \in (0, \infty) \\ \text{ess. inf}_{t>0, x, y \in \mathbb{R}^d} |x-y|^{d+\alpha} K(t, x, y) \geq C_0^{-1}. \end{cases} \quad (2.10)$$

In fact, it is possible to deal with the kernels of the class (2.10), but the obtained result becomes weaker than in the case (2.1). So we focus on the class of  $K$  stated as (2.1) in this paper, and the result for the class (2.10) will be stated in Remark 3.4 without proofs.

## 2.2 Logarithmic Sobolev Inequality and Estimates for the Trajectory

We first recall the logarithmic Sobolev inequality with fractional order proved in [14].

**Lemma 2.2** ([14]) *Let  $f$  be a function in  $H^\alpha(\mathbb{R}^d)$  and  $\beta > 0$  be any positive number. Then*

$$\left( \int |f|^2 \log \frac{|f|^2}{\|f\|_{L^2}^2} dx + (d + \log \frac{\alpha \Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2\alpha})} + \frac{d}{2\alpha} \log \beta) \|f\|_{L^2}^2 \right) \leq \frac{\beta}{\pi^\alpha} \|(-\Delta)^{\frac{\alpha}{2}} f\|_{L^2}^2$$

holds.

Next we state a couple of lemmas for the estimate of the drift term. The following lemma is useful to estimate local averages.

**Lemma 2.3** ([21, Lemma 2.2]) *Let  $f \in \mathcal{L}^{1,\mu}(\mathbb{R}^d)$  for some  $\mu \in [0, d+1]$ . Let  $x_1, x_2 \in \mathbb{R}^d$  and  $R_1 \geq R_2 > 0$ . Then*

$$\left| \int_{B_{R_1}(x_1)} f - \int_{B_{R_2}(x_2)} f \right| \leq \begin{cases} C \|f\|_{\mathcal{L}^{1,\mu}} R_2^{\mu-d} & \text{if } 0 \leq \mu < d, \\ C \|f\|_{\mathcal{L}^{1,\mu}} \left( \log(e + \frac{|x_1 - x_2|}{R_2}) + \log \frac{R_1}{R_2} \right) & \text{if } \mu = d, \\ C \|f\|_{\mathcal{L}^{1,\mu}} (|x_1 - x_2|^{\mu-d} + R_1^{\mu-d}) & \text{if } d < \mu \leq d+1. \end{cases} \quad (2.11)$$

Here  $C$  depends only on  $d$  and  $\mu$ .

We now consider the trajectory generated by the local average of the vector field  $u$ :

$$\begin{cases} \frac{d}{dt} \xi_u(t; x, R) = \int_{B_R(x + \xi_u(t; x, R))} u(t), & 0 \leq t \leq t_0, \\ \xi_u(0; x, R) = 0, \end{cases} \quad (2.12)$$

where  $x \in \mathbb{R}^d$  and  $R > 0$ . The next lemma plays a fundamental role for the estimate of the drift term:

**Lemma 2.4** *Let  $\xi_u(t; x, R)$  be the solution to (2.12). Assume that  $u$  satisfies (C) for  $\lambda \in [d, 2d/\alpha + d]$ . Let  $R \geq t_0^{1/\alpha}$ . Then*

$$|\xi_u(t_0; x, R)| \leq C(R \|u\|_{X_\lambda} + \sup_{0 < t < t_0} \left| \int_0^t \int_{B_R(x)} u(\tau) d\tau \right|) \quad \lambda > d, \quad (2.13)$$

$$|\xi_u(t_0; x, R)| \leq C(R \|u\|_{X_\lambda} (1 + \log \|u\|_{X_\lambda}) + \sup_{0 < t < t_0} \left| \int_0^t \int_{B_R(x)} u(\tau) d\tau \right|) \quad \lambda = d. \quad (2.14)$$

Here  $C$  depends only on  $d, \alpha, p$ . Moreover the same estimate (2.13) also holds for the case  $\lambda = 2d/\alpha + d$  provided  $\|u\|_{X_\lambda}$  is sufficiently small.

*Proof.* By the definition of  $\xi_u(t; x, R)$  we have

$$|\xi_u(t; x, R)| \leq \int_0^t \left| \int_{B_R(x + \xi_u(s; x, R))} u(s) - \int_{B_R(x)} u(s) \right| ds + \left| \int_0^t \int_{B_R(x)} u(s) ds \right|.$$

Set  $\mu = \alpha(\lambda - d)/(2d) + d$ , where  $\lambda$  is the number in **(C)**. Applying Lemma 2.3, we have

$$\begin{aligned} & |\xi_u(t; x, R)| \\ \leq & \begin{cases} C \int_0^t \|u(s)\|_{\mathcal{L}^{1,d}} \log(e + \frac{|\xi_u(s; x, R)|}{R}) ds + \sup_{0 < t < t_0} \left| \int_0^t \int_{B_R(x)} u(\tau) d\tau \right| & \text{if } \mu = d, \\ C \int_0^t \|u(s)\|_{\mathcal{L}^{1,\mu}} |\xi_u(s; x, R)|^{\mu-d} ds + \sup_{0 < t < t_0} \left| \int_0^t \int_{B_R(x)} u(\tau) d\tau \right| & \text{if } \mu > d. \end{cases} \end{aligned}$$

From these estimates it is easy to see (2.13) since  $\mathcal{L}^{2d/\alpha, \lambda} \hookrightarrow \mathcal{L}^{1,\mu}$ . Thus we only prove (2.14). Let

$$g = \sup_{0 < t < t_0} |\xi_u(t; x, R)|/R, \quad g_1 = CR^{-1} \int_0^{t_0} \|u(s)\|_{\mathcal{L}^{1,\mu}} ds, \quad g_2 = R^{-1} \sup_{0 < t < t_0} \left| \int_0^t \int_{B_R(x)} u(s) ds \right|,$$

then the above estimate yields

$$\begin{aligned} g & \leq g_1 \left( \log 2g_1 + \log \left( \frac{e+g}{2C_1} \right) \right) + g_2 \leq g_1 (1 + \log g_1) + g_1 \left( e + \frac{e+g}{2g_1} \right) + g_2 \\ & \leq Cg_1 (1 + \log C_1) + g/2 + g_2, \end{aligned}$$

which implies  $g \leq C(g_1(1 + \log g_1) + g_2)$ . Since  $R \geq t_0^{1/\alpha}$ , we have  $g_1 \leq CR^{-1}t_0^{\frac{1}{\alpha}} \|u\|_{X_\lambda} \leq C\|u\|_{X_\lambda}$ . This completes the proof.  $\square$

### 2.3 Approximation of the equation

The existence of fundamental solutions for (2.6) is not trivial under the weak regularity condition **(C)** on  $v$ . In [21] the fundamental solution is constructed by introducing the approximating equation, and we will use this approximation also in this paper. For convenience to the reader, in this section we will briefly describe this approximation procedure.

We first state the approximation of the kernel  $K$  used in [21]. Especially, this approximation is applicable for the class of the kernel (2.10) which is more general than (2.1). Let  $C_0$  be the number in (2.10). Following [21], we say  $K(t, x, y)$  is a smooth kernel of the order  $\alpha' \in (1, 2)$  if  $K(t, x, y)$  is of the form

$$K(t, x, y) = |x - y|^{-d-\alpha'} k(t, x, y) + \delta |x - y|^{-d-\alpha'}, \quad (2.15)$$

where  $\delta > 0$ ,  $\alpha' \geq \alpha$ , and  $k(t, x, y)$  is a function defined on  $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$  such that

$$k(t, x, y) = k(t, y, x), \quad \sup_{t \in \mathbb{R}, x, y \in \mathbb{R}^d} \sum_{|\beta| \leq 1} |\nabla_{t,x,y}^\beta k(t, x, y)| < \infty, \quad \inf_{t \in \mathbb{R}, x, y \in \mathbb{R}^d} k(t, x, y) \geq C_0^{-1}. \quad (2.16)$$

If  $K(t, x, y)$  is a smooth kernel and  $v$  is smooth and bounded, then it is not difficult to prove that there exists a unique and positive fundamental solution  $P_{K,v}(t, x; s, y)$  to (2.6). In particular, when  $f \in C_0^\infty(\mathbb{R}^d)$  the function  $(P_{K,v}f)(t, s, x)$  defined by (2.8) solves (2.6) for  $t > s$  in the classical sense rather than the weak sense. Moreover, under the assumption of  $\nabla \cdot v(t) = 0$ ,  $P_{K,v}(t, x; s, y)$  satisfies

$$\begin{aligned} & \int_{\mathbb{R}^d} P_{K,v}(t, x; s, y) dy = \int_{\mathbb{R}^d} P_{K,v}(t, x; s, y) dx = 1, \\ & P_{K,v}(t, x; s, y) = \int_{\mathbb{R}^d} P_{K,v}(t, x; \tau, \xi) P_{K,v}(\tau, \xi, s, y) d\xi \quad t > \tau > s \geq 0, \\ & \int_s^T \|P_{K,v}f(t, s)\|_{\dot{H}^{\frac{\alpha}{2}}}^2 dt \leq C \|f\|_{L^2}^2, \end{aligned}$$



Here  $C$  depends only on  $d$ ,  $\alpha$ , and  $C_0$ . Especially,  $C$  is independent of  $\alpha'$  and  $\delta$ . Let  $T_N(\sigma)$ ,  $N \gg 1$  be a truncated function such that  $T_N(\sigma) = \sigma$  if  $\sigma \leq N$  and  $T_N(\sigma) = N$  if  $\sigma \geq N$ . Then we define the approximation of  $K(t, x, y)$  satisfying (2.10) by

$$K^{(N, \delta)}(t, x, y) = |x - y|^{-d-\alpha} (\nu_\delta * k_N)(t, x, y) + \delta |x - y|^{-d-1-\frac{\alpha}{2}},$$

where

$$k_N(t, x, y) = T_N(|x - y|^{d+\alpha} K(t, x, y)), \quad t > 0, \quad k_N(t, x, y) = C_0^{-1}, \quad t \leq 0,$$

which satisfies  $k_N(t, x, y) \geq C_0^{-1}$  if  $N \geq C_0^{-1}$  from the definition and (2.10). Here  $\nu_\delta *$  is a mollifier on  $(t, x, y)$  variables such as

$$(\nu_\delta * k_N)(t, x, y) = \int_{\mathbb{R}} \int_{\mathbb{R}^{2d}} \nu_{1, \delta}(t - s) \nu_{d, \delta}(x - \xi) \nu_{d, \delta}(y - \eta) k_N(s, \xi, \eta) d\xi d\eta ds,$$

where  $\nu_{n, \delta}(z) = \delta^{-n} \nu_n(z/\delta)$  and  $\nu_n(z)$  is a smooth non-negative function on  $\mathbb{R}^n$  satisfying  $\text{supp } \nu_n \subset \{z \in \mathbb{R}^n \mid |z| \leq 2\}$ ,  $\int_{\mathbb{R}^n} \nu_n(z) dz = 1$ . We note that the above mollification preserves the symmetry and the estimate such as

$$(\nu_\delta * k_N)(t, x, y) = (\nu_\delta * k_N)(t, y, x), \quad (\nu_\delta * k_N)(t, x, y) \geq C_0^{-1}.$$

We next recall the lemma for the approximation of  $v$ .

**Lemma 2.5** ([21, Lemma 2.4]) *Let  $\lambda \in [2d/\alpha - d, 2d/\alpha + d]$  and set  $p_\lambda = 1$  if  $\lambda \in [2d/\alpha - d, d]$  and  $p_\lambda = \infty$  if  $\lambda \in (d, 2d/\alpha + d]$ . Let  $q_1 \in [1, \infty)$ ,  $q_2 \in [1, \infty]$ , and  $\mu \in [0, 1]$ . Let  $v \in L^{q_1, \mu}(0, \infty; (\mathcal{L}^{\frac{2d}{\alpha}, \lambda}(\mathbb{R}^d))^d) \cap L_{loc}^{q_2}(0, \infty; (L_{loc}^{p_\lambda}(\mathbb{R}^d))^d)$  be a given vector field satisfying  $\nabla \cdot v(t) = 0$ . Then there is a sequence of smooth and bounded vector fields  $\{v^{(N)}\}$  satisfying  $\nabla \cdot v^{(N)}(t) = 0$  and*

$$\sup_N \|v^{(N)}\|_{L^{q_1, \mu}(0, \infty; \mathcal{L}^{\frac{2d}{\alpha}, \lambda}(\mathbb{R}^d))} \leq C_1 \|v\|_{L^{q_1, \mu}(0, \infty; \mathcal{L}^{\frac{2d}{\alpha}, \lambda}(\mathbb{R}^d))}, \quad (2.17)$$

$$\limsup_{N \rightarrow \infty} \|v^{(N)}\|_{L^{q_2}(0, R_1; L^{p_\lambda}(B_{R_2}(x)))} \leq C_2 \|v\|_{L^{q_2}(0, 2R_1; L^{p_\lambda}(B_{2R_2}(x)))} \quad R_1, R_2 > 0, x \in \mathbb{R}^d, \quad (2.18)$$

and

$$v^{(N)} \rightarrow v \quad \text{in } (L_{loc}^1((0, \infty) \times \mathbb{R}^d))^d. \quad (2.19)$$

Here  $C_1$  depends only on  $d$ ,  $\alpha$ ,  $\lambda$ ,  $\mu$ , and  $q_1$ , and  $C_2$  depends only on  $d$  and  $q_2$ .

### 3 Pointwise upper bounds

In this section we will prove our main result. By using the formulation in the previous section our result is stated as follows.

**Theorem 3.1** *Assume that (2.1) and (C) hold. Then there exists a fundamental solution  $P_{K, v}(t, x; s, y)$  to (2.6) such that for all  $t > s \geq 0$  and  $x, y \in \mathbb{R}^d$ ,*

$$P_{K, v}(t, x; s, y) \leq C_1 (t - s)^{-\frac{d}{\alpha}}, \quad (3.1)$$

$$P_{K, v}(t, x; s, y) \leq C_2 (t - s)^{-\frac{d}{\alpha}} \left( 1 + \frac{(|x - y| - CF[v](t, s, x, y))_+}{(t - s)^{\frac{1}{\alpha}}} \right)^{-d-\alpha}. \quad (3.2)$$

Here  $C_1$  depends only on  $d$ ,  $\alpha$ , and  $C_0$ ,  $C_2$  depends only on  $d$ ,  $\alpha$ ,  $C_0$ , and  $\|v\|_{X_\lambda}$ , and  $C > 1$  is some absolute constant.

In the condition (C) the endpoint case  $\lambda = 2d/\alpha + d$  is excluded. In this case, we have the following auxiliary result:

**Theorem 3.2** Assume that (2.1) holds. Then there exists small constant  $\delta_0 > 0$  such that if

$$(\mathbf{C}'') \ v \in L^1(0, \infty; (\text{Lip}(\mathbb{R}^d))^d) \cap L_{loc}^q(0, \infty; (L_{loc}^\infty(\mathbb{R}^d))^d), \quad \|v\|_{X_{\frac{2d}{\alpha}+d}^+} := \|v\|_{L^1(0, \infty; \text{Lip}(\mathbb{R}^d))} < \delta_0$$

for some  $1 < q < \infty$ , then there exists a fundamental solution  $P_{K,v}(t, x; s, y)$  to (2.6) such that for all  $t > s \geq 0$  and  $x, y \in \mathbb{R}^d$ ,

$$P_{K,v}(t, x; s, y) \leq C_1(t-s)^{-\frac{d}{\alpha}}, \quad (3.3)$$

$$P_{K,v}(t, x; s, y) \leq C_2(t-s)^{-\frac{d}{\alpha}} \left( 1 + \frac{(|x-y| - CF[v](t, s, x, y))_+}{(t-s)^{\frac{1}{\alpha}}} \right)^{-d-\alpha}. \quad (3.4)$$

Here  $C_1$  depends only on  $d$  and  $C_0$ ,  $C_2$  depends only on  $d$ ,  $C_0$ , and  $\|v\|_{X_{d+2d/\alpha}}$ , and  $C > 1$  is some absolute constant.

**Remark 3.3** In fact, the smallness condition in  $(\mathbf{C}'')$  is not needed for the existence of fundamental solution; see [21]. Furthermore, it is easy to show that (3.4) is valid whenever  $|t-s| \ll 1$  without the smallness condition. Indeed, by the local smallness of the  $X_{d+2d/\alpha}$ , the proof is reduced to that of Theorem 3.2. We will prove Theorem 3.2 in the end of this section.

**Remark 3.4** As stated in Remark 2.1, we can deal with the kernels of the class (2.10) with minor modifications of the proofs. But in this case we can only obtain the weaker pointwise estimate:

$$\begin{aligned} P_{K,v}(t, x; s, y) &\leq C_2(t-s)^{-\frac{d}{\alpha}} \left( 1 + \frac{(|x-y| - CF[v](t, s, x, y))_+}{(t-s)^{\frac{1}{\alpha}}} \right)^{-d-\alpha} \\ &\quad + C_3(t-s)^{-\frac{d}{\alpha}} \left( 1 + \frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}} \right)^{-\alpha}, \end{aligned} \quad (3.5)$$

where  $C_3$  depends only on  $d$ ,  $\alpha$ , and  $C_0$ . That is, the additional term, which decays with the power  $-\alpha$  as  $|x-y| \rightarrow \infty$ , is required. This term appears when we estimate the Duhamel term of the right-hand side of (3.33) under the condition of (2.10). Since the modifications are not difficult we will skip the details.

For the proof of Theorem 3.1, it is difficult to estimate the solution of (2.6) directly as explained in the previous section. So we will derive uniform estimates of the fundamental solutions to the approximating equation. Let  $K^{(N,\delta)}(t, x, y)$  with  $N \gg 1$  and  $0 < \delta \ll 1$  be the approximation of  $K(t, x, y)$  in Section 2.3 and consider the equation:

$$\partial_t \theta + A_{K^{(N,\delta)}}(t) \theta + v^{(\delta)} \cdot \nabla \theta = 0, \quad t > 0, \quad x \in \mathbb{R}^d, \quad (3.6)$$

where  $v^{(\delta)}(t, x)$  is a smooth and bounded solenoidal vector field and  $A_{K^{(N,\delta)}}(t)$  is a linear operator defined in (2.4) with  $K = K^{(N,\delta)}$ . Then for each  $M \in (0, \infty]$  the function  $K_M^{(N,\delta)}$ , the bilinear forms  $\mathcal{E}_{K_M^{(N,\delta)}}^{(t)}$ ,  $\mathcal{B}_{K_M^{(N,\delta)}}^{(t)}$ , and the fundamental solution  $P_{K_M^{(N,\delta)}}(t, x; s, y)$  are defined in the similar ways as in Section 2.1. From the definition of  $K_M^{(N,\delta)}$  there exists a unique fundamental solution  $P_{K_M^{(N,\delta)}}(t, x; s, y)$  to (2.6) with  $K$  replaced by  $K_M^{(N,\delta)}$ . In particular,  $P_{K_M^{(N,\delta)}}(t, x; s, y)$  gives the classical solutions rather than weak solutions.

Let  $\text{Lip}_0(\mathbb{R}^d)$  be the class of compactly-supported Lipschitz functions. For  $\Psi \in \text{Lip}([0, \infty) \times \mathbb{R}^d)$  with  $\Psi(t, \cdot) \in \text{Lip}_0(\mathbb{R}^d)$ , we set

$$\Gamma_M^{(N,\delta)}(\Psi)(t, x) = e^{-2\Psi(t,x)} \Gamma_M^{(N,\delta)}(e^\Psi, e^\Psi)(t, x), \quad (3.7)$$

$$\Lambda_M^{(N,\delta)}(\Psi) = \max\{\|\Gamma_M^{(N,\delta)}(\Psi)\|_{L_{t,x}^\infty}, \|\Gamma_M^{(N,\delta)}(-\Psi)\|_{L_{t,x}^\infty}\}, \quad (3.8)$$

where  $\Gamma_M^{(N,\delta)}(f, g)$  is the function defined by

$$\Gamma_M^{(N,\delta)}(f, g)(t, x) = \int_{\mathbb{R}^d} [f][g](x, y) K_M^{(N,\delta)}(t, x, y) dy. \quad (3.9)$$

For simplicity of notations we will write  $\tilde{K}_M, \tilde{\Lambda}_M, \tilde{\Gamma}_M, \tilde{v}$  for  $K_M^{(N,\delta)}, \Lambda_M^{(N,\delta)}, \Gamma_M^{(N,\delta)}, v^{(\delta)}$ .

We now recall a coercive-type estimate for the Dirichlet form  $\mathcal{E}_{\tilde{K}}^{(t)}$ .

**Lemma 3.5 ([9, Theorem 3.9])** *Let  $\Psi \in \text{Lip}([0, \infty) \times \mathbb{R}^d)$  with  $\Psi(t, \cdot) \in \text{Lip}_0(\mathbb{R}^d)$ . For  $r \in [1, \infty)$  it follows that*

$$\mathcal{E}_{\tilde{K}_M}^{(t)}(e^\Psi f^{r-1}, e^{-\Psi} f) \geq \frac{1}{r} \mathcal{E}_{\tilde{K}_M}^{(t)}(f^{\frac{r}{2}}, f^{\frac{r}{2}}) - Cr \tilde{\Lambda}_M(\Psi) \|f\|_{L^r}^r, \quad \forall f \in C_0^\infty(\mathbb{R}^d), f \geq 0. \quad (3.10)$$

Here  $C$  is a numerical constant (which is also independent of  $M$ ).

In fact, [9] considered the case when the kernel  $K$  and  $\Psi$  are independent of  $t$ . The dependence on  $t$  however does not change any arguments to obtain (3.10). So we omit the details here.

On the other hand, the divergence free condition for  $\tilde{v}$  with the integral by parts immediately yields the following identity for the Dirichlet form  $\mathcal{E}_{\tilde{v}(t)}$ .

**Lemma 3.6** *Let  $\psi \in \text{Lip}_0(\mathbb{R}^d)$ . For  $r \in [1, \infty)$  it follows that*

$$\mathcal{E}_{\tilde{v}(t)}(e^\psi f^{r-1}, e^{-\psi} f) = \int_{\mathbb{R}^d} f^r(x) \tilde{v}(t, x) \cdot \nabla \psi(x) dx. \quad (3.11)$$

Fix  $L \geq 0, R > 0, t_0 > 0$ , and  $x_0, y_0 \in \mathbb{R}^d$ . Let  $\psi$  be the function defined by

$$\psi(x) = L(R - |x - x_0|)_+. \quad (3.12)$$

Set  $\tilde{\xi}(t; x_0, R) \in \mathbb{R}^d$  be the solution to (2.12) with  $R > 0$  and  $u(t, x) = \tilde{v}(t_0 - t, x)$ ,  $0 \leq t \leq t_0$ . If we put  $\xi(t; x_0) = \tilde{\xi}(t_0 - t; x_0, R)$  then  $\xi(t; x_0)$  solves the ODE

$$\begin{cases} \frac{d}{dt} \xi(t; x_0) = - \int_{B_R(x_0 + \xi(t; x_0))} \tilde{v}(t), & 0 \leq t \leq t_0, \\ \xi(t_0; x_0) = 0. \end{cases} \quad (3.13)$$

We also set

$$\Psi(t, x) = \psi(x - \xi(t; x_0)), \quad 0 \leq t \leq t_0. \quad (3.14)$$

Then it is easy to see

$$\text{Lip}(\Psi(t)) \leq L, \quad \text{supp } \Psi(t) = B_R(x_0 + \xi(t; x_0)). \quad (3.15)$$

Moreover we also have the following estimate for  $\tilde{\Lambda}_M(\Psi)$ :

**Lemma 3.7** *Let  $\Psi$  be the function defined by (3.14). Then*

$$\tilde{\Lambda}_M(\Psi) = \Lambda_M^{(N,\delta)}(\Psi) \leq CL^2 e^{2LM} (\delta^{2-\alpha} N + \delta M^{1-\frac{\alpha}{2}} + M^{2-\alpha}). \quad (3.16)$$

Here  $C$  depends only on  $d, \alpha$ , and  $C_0$ .

*Proof.* From  $(e^t - 1)^2 \leq t^2 e^{2t}$  and (3.15) we have

$$\begin{aligned} e^{-2\Psi(t,x)} \int_{\mathbb{R}^d} [e^{\Psi(t,\cdot)}]^2(x, y) \tilde{K}_M(t, x, y) dy &= \int_{\mathbb{R}^d} (e^{\Psi(t,x) - \Psi(t,y)} - 1)^2 \tilde{K}_M(t, x, y) dy \\ &\leq L^2 e^{2LM} \int_{\mathbb{R}^d} |x - y|^2 \tilde{K}_M(t, x, y) dy. \end{aligned}$$

Then (3.16) follows from the definition of  $\tilde{K}_M = K_M^{(N,\delta)}$  and the direct calculation of  $\int_{\mathbb{R}^d} |x - y|^2 \tilde{K}_M(t, x, y) dy$ . This completes the proof.  $\square$

We next state a weighted estimate for the fundamental solution  $P_{\tilde{K}_M}(t, x; s, y)$  which is the core of the proof of Theorem 3.1. Without loss of generality we may take  $s = 0$ .

**Proposition 3.8** *Assume that (2.1) and (C) (or (C'')) for the case  $\lambda = 2d/\alpha + d$  hold. Let  $\Psi$  be the function defined by (3.14).*

1. If  $\lambda \in (2d/\alpha - d, d]$ ,

$$P_{\tilde{K}_M, \tilde{v}}(t, x; 0, y) \leq Ct^{-\frac{d}{\alpha}} \exp\left(-\Psi(t, x) + \Psi(0, y) + C(\tilde{\Lambda}_M(\Psi)t + M^{-\alpha}t + \|\tilde{v}\|_{X_\lambda}^2 L^2 R^{\frac{\alpha\lambda}{2d}} t^{\frac{2}{\alpha} - \frac{\lambda}{d}})\right) \quad (3.17)$$

holds for all  $t > 0$ ,  $x, y \in \mathbb{R}^d$ .

2. If  $\lambda \in (d, 2d/\alpha + d]$ ,

$$P_{\tilde{K}_M, \tilde{v}}(t, x; 0, y) \leq Ct^{-\frac{d}{\alpha}} \exp\left(-\Psi(t, x) + \Psi(0, y) + C(\tilde{\Lambda}_M(\Psi)t + M^{-\alpha}t + \|\tilde{v}\|_{X_\lambda} L R^{\frac{\alpha\lambda}{2d} - \frac{\alpha}{2}} t^{\frac{1}{\alpha} - \frac{\lambda}{2d} + \frac{1}{2}})\right) \quad (3.18)$$

holds for all  $t > 0$ ,  $x, y \in \mathbb{R}^d$ .

Here the positive constant  $C$  depends only on  $d, \alpha$ , and  $C_0$ .

*Proof.* Set

$$\theta_M(t, x) = e^{\Psi(t, x)} \int_{\mathbb{R}^d} P_{\tilde{K}_M, \tilde{v}}(t, x; 0, y) e^{-\Psi(0, y)} f(y) dy, \quad f \in C_0^\infty(\mathbb{R}^d), \quad f \geq 0, \quad (3.19)$$

and let  $r : [0, t_0) \rightarrow [1, \infty)$  be a continuously differentiable function to be specified later. By direct calculation, we have

$$\frac{d}{dt} \log \|\theta_M(t)\|_{L^{r(t)}} = \frac{r'}{r^2} \|\theta_M\|_{L^r}^{-r} \int |\theta_M|^r \log \frac{|\theta_M|^r}{\|\theta_M\|_{L^r}^r} dx + \|\theta_M\|_{L^r}^{-r} \int \theta_M^{r-1} \partial_t \theta_M dx.$$

Then we have from Lemma 3.5, Lemma 3.6, and (2.1),

$$\begin{aligned} & \int \theta_M^{r-1} \partial_t \theta_M dx \\ &= \int_{\mathbb{R}^d} \theta_M^r \partial_t \Psi dx + \langle e^\Psi \theta_M^{r-1}, \partial_t (e^{-\Psi} \theta_M) \rangle \\ &= -\mathcal{E}_{\tilde{K}_M}(e^\Psi \theta_M^{r-1}, e^{-\Psi} \theta_M) - \mathcal{E}_{\tilde{v}(t)}(e^\Psi \theta_M^{r-1}, e^{-\Psi} \theta_M) + \int_{\mathbb{R}^d} \theta_M^r \partial_t \Psi dx \\ &\leq -\frac{2}{r} \mathcal{E}_{\tilde{K}_M}(\theta_M^{\frac{r}{2}}, \theta_M^{\frac{r}{2}}) + Cr \tilde{\Lambda}_M(\Psi) \|\theta_M\|_{L^r}^r + \int_{\mathbb{R}^d} \theta_M^r (\partial_t \Psi - \tilde{v} \cdot \nabla \Psi) dx \\ &\leq -\frac{2c_0}{r} \|(-\Delta)^{\frac{\alpha}{4}} \theta_M^{\frac{r}{2}}\|_{L^2}^2 + C\left(\frac{M^{-\alpha}}{r} + r \tilde{\Lambda}_M(\Psi)\right) \|\theta_M\|_{L^r}^r + \int_{\mathbb{R}^d} \theta_M^r (\partial_t \Psi - \tilde{v} \cdot \nabla \Psi) dx. \quad (3.20) \end{aligned}$$

Here  $c_0$  is a constant depending only on  $d, \alpha$ , and  $C_0$ .

We now divide the proof by the value of  $\lambda$  in the assumption (C) and first consider the case  $\lambda \leq d$ . By using (3.13)-(3.15) and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \theta_M^r (\partial_t \Psi - \tilde{v} \cdot \nabla \Psi) dx &= \int_{B_R(x_0 + \xi(t; x_0))} \theta_M^r \left( \int_{B_R(x_0 + \xi(t; x_0))} \tilde{v} - \tilde{v} \right) \cdot \nabla \Psi dx \\ &\leq L \|\theta_M^{\frac{r}{2}}\|_{L^{\frac{4d}{2d-\alpha}}}^2 \left( \int_{B_R(x_0 + \xi(t; x_0))} |\tilde{v} - \int_{B_R(x_0 + \xi(t; x_0))} \tilde{v}|^{\frac{2d}{\alpha}} dx \right)^{\frac{\alpha}{2d}} \\ &\leq CLR^{\frac{\alpha\lambda}{2d}} \|(-\Delta)^{\frac{\alpha}{4}} \theta_M^{\frac{r}{2}}\|_{L^2} \|\theta_M\|_{L^r}^{\frac{r}{2}} \|\tilde{v}\|_{\mathcal{L}^{\frac{2d}{\alpha}, \lambda}} \\ &\leq \frac{c_0}{r} \|(-\Delta)^{\frac{\alpha}{4}} \theta_M^{\frac{r}{2}}\|_{L^2}^2 + Cr L^2 R^{\frac{\alpha\lambda}{d}} \|\tilde{v}\|_{\mathcal{L}^{\frac{2d}{\alpha}, \lambda}}^2 \|\theta_M\|_{L^r}^r. \quad (3.21) \end{aligned}$$

Plugging this in (3.20) we have

$$\int \theta_M^{r-1} \partial_t \theta_M dx \leq -\frac{c_0}{r} \|(-\Delta)^{\frac{\alpha}{4}} \theta_M^{\frac{r}{2}}\|_{L^2}^2 + C\left(\frac{M^{-\alpha}}{r} + r\tilde{\Lambda}_M(\Psi)\|\theta_M\|_{L^r}^r + rL^2R^{\frac{\alpha\lambda}{d}}\|\tilde{v}\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}^2\right)\|\theta_M\|_{L^r}^r.$$

Then we apply Lemma 2.2 with  $\beta = \frac{c_0\pi^{\frac{\alpha}{2}}r}{r'}$  to get

$$\begin{aligned} & \frac{d}{dt} \log\|\theta_M(t)\|_{L^{r(t)}} \\ & \leq -\frac{r'}{r^2}\left(d + \frac{\alpha\Gamma(\frac{d}{2})}{2\Gamma(\frac{d}{\alpha})}\right) + \frac{d}{\alpha}\left(\log\frac{\pi^{\frac{\alpha}{2}}}{c_0} + \log\frac{r}{r'}\right) + Cr\left(r^{-2}M^{-\alpha} + \tilde{\Lambda}_M(\Psi) + L^2R^{\frac{\alpha\lambda}{d}}\|\tilde{v}\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}^2\right). \end{aligned}$$

Set  $s(t) = 1/r(t)$ . Then we have

$$\frac{d}{dt} \log\|\theta_M(t)\|_{L^{\frac{1}{s}}} \leq s'(C_{d,\alpha} + \frac{d}{\alpha}\log(-\frac{s}{s'})) + \frac{C}{s}(M^{-\alpha} + \tilde{\Lambda}_M(\Psi) + L^2R^{\frac{\alpha\lambda}{d}}\|\tilde{v}\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}^2).$$

Integrating from 0 to  $t_0$ , we get

$$\begin{aligned} \log\|\theta_M(t_0)\|_{L^{\frac{1}{s(t_0)}}} - \log\|\theta_M(0)\|_{L^{\frac{1}{s(0)}}} & \leq \int_0^{t_0} s'(C_{d,\alpha} + \frac{d}{\alpha}\log s) dt - \frac{d}{\alpha} \int_0^{t_0} s' \log(-s') dt \\ & \quad + \int_0^{t_0} \frac{C}{s} (M^{-\alpha} + \tilde{\Lambda}_M(\Psi) + L^2R^{\frac{\alpha\lambda}{d}}\|\tilde{v}\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}^2) dt \end{aligned}$$

Choosing  $s(t) = (1 - t/t_0)^q$  so that  $s(t_0) = 0$ ,  $s(0) = 1$  with  $q \in (0, 2/\alpha - \lambda/d)$ , we have

$$\int_0^{t_0} s'(C_{d,\alpha} + \frac{d}{\alpha}\log s) dt = [C_{d,\alpha}s(t) - \frac{\alpha}{d}s(t)(\log s(t) - 1)]_{t=0}^{t_0} = -C_{d,\alpha}.$$

Moreover the other integrals are estimated as follows:

$$-\int_0^{t_0} s' \log(-s') dt \leq -\log t_0 + C, \quad \int_0^{t_0} \frac{dt}{s} = Ct_0,$$

$$\begin{aligned} \int_0^{t_0} \|\tilde{v}\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}^2 \frac{dt}{s} & = \int_0^{t_0} \left(-\int_t^{t_0} \|\tilde{v}\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}^2 d\tau\right)' \frac{dt}{s} \\ & = \int_0^{t_0} \|\tilde{v}\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}^2 d\tau - \int_0^{t_0} s' s^{-2} \int_t^{t_0} \|\tilde{v}\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}^2 d\tau dt \\ & \leq C\|\tilde{v}\|_{X_\lambda}^2 t_0^{\frac{2}{\alpha} - \frac{\lambda}{d}}. \end{aligned}$$

Summing up these estimates and replacing  $t_0$  by  $t$ , we obtain

$$\log\|\theta_M(t)\|_{L^\infty} - \log\|\theta_M(0)\|_{L^1} \leq -\frac{d}{\alpha}\log t + C\left(1 + \tilde{\Lambda}_M(\Psi)t + M^{-\alpha}t + \|\tilde{v}\|_{X_\lambda}^2 L^2 R^{\frac{\alpha\lambda}{d}} t^{\frac{2}{\alpha} - \frac{\lambda}{d}}\right), \quad (3.22)$$

which proves the desired estimate.

We next consider the case  $d < \lambda \leq 2d/\alpha + d$ . By using the characterization  $\mathcal{L}^{2d/\alpha,\lambda} = \dot{C}^{\alpha\lambda/(2d)-\alpha/2}$ , the last term in (3.20) can be estimated as the follows

$$\begin{aligned} \int_{\mathbb{R}^d} \theta_M^r (\partial_t \Psi - \tilde{v} \cdot \nabla \Psi) dx & = \int_{B_R(x_0+\xi(t;x_0))} \theta_M^r \left( \int_{B_R(x_0+\xi(t;x_0))} \theta_M^r \left( \int_{B_R(x_0+\xi(t;x_0))} \tilde{v} - \tilde{v} \right) \cdot \nabla \Psi dx \right. \\ & \leq R^{\frac{\alpha\lambda}{2d} - \frac{\alpha}{2}} \sup_{x,y \in \mathbb{R}^d} \frac{|\tilde{v}(x) - \tilde{v}(y)|}{|x-y|^{\frac{\alpha\lambda}{2d} - \frac{\alpha}{2}}} L \|\theta_M\|_{L^r}^r \\ & \leq \|\tilde{v}\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} R^{\frac{\alpha\lambda}{2d} - \frac{\alpha}{2}} L \|\theta_M\|_{L^r}^r. \end{aligned} \quad (3.23)$$

Thus arguing as the preceding case we get

$$\log\|\theta_M(t)\|_{L^\infty} - \log\|\theta_M(0)\|_{L^1} \leq -\frac{d}{\alpha}\log t + C(1 + \tilde{\Lambda}_M(\Psi)t + M^{-\alpha}t + \|\tilde{v}\|_{X_\lambda} LR^{\frac{\alpha\lambda}{2d} - \frac{\alpha}{2}} t^{\frac{1}{2} + \frac{1}{\alpha} - \frac{\lambda}{2d}}). \quad (3.24)$$

This completes the proof.  $\square$

Since  $C$  in Proposition 3.8 does not depend on  $\|\tilde{v}\|_{X_\lambda}$ , by taking  $L = 0$  and letting  $M \rightarrow \infty$ , we obtain (3.1) in Theorem 3.1 as follows.

**Corollary 3.9** *For all  $t > 0$ ,  $x, y \in \mathbb{R}^d$  it follows that*

$$P_{\tilde{K}, \tilde{v}}(t, x; 0, y) \leq Ct^{-\frac{d}{\alpha}}. \quad (3.25)$$

Here  $C$  depends only on  $d, \alpha$ , and  $C_0$ .

**Proposition 3.10** *Assume that (2.1) and (C) hold. Assume that  $0 < \delta \ll 1$  and  $\delta^{1-\alpha/2} \leq (t-s)^{1/\alpha} N^{-1/2}$ . Then for all  $x, y \in \mathbb{R}^d$  it follows that*

$$P_{\tilde{K}, \tilde{v}}(t, x; s, y) \leq C(t-s)^{-\frac{d}{\alpha}} \left(1 + \frac{(|x-y| - 2F[\tilde{v}](t, s, x, y))_+}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha}, \quad (3.26)$$

where  $C$  depends only on  $d, \alpha, C_0$ , and  $\|\tilde{v}\|_{X_\lambda}$ .

*Proof.* We give the proof only for the case  $\lambda \in (d, 2d/\alpha + d)$ ; the other case is shown similarly. Without loss of generality, we may assume  $s = 0$ . Fix  $x_0, y_0 \in \mathbb{R}^d$ ,  $t_0 > 0$ , and let  $\delta^{1-\alpha/2} \leq t_0^{1/\alpha} N^{-1/2}$ . Let us take  $R = |x_0 - y_0|$ ,  $M = \eta R$  in Proposition 3.8 with  $\eta \in (0, 1)$  to be determined later. First we consider the case  $M \leq C_* t_0^{\frac{1}{\alpha}}$ , where  $C_* \geq 1$  will be specified later. In this case we have from Corollary 3.9,

$$P_{\tilde{K}, \tilde{v}}(t_0, x_0; 0, y_0) \leq Ct_0^{-\frac{d}{\alpha}} \leq Ct_0^{-\frac{d}{\alpha}} (\eta + C_*^{-1} t_0^{-\frac{1}{\alpha}} M)^{-d-\alpha} \leq CC_*^{d+\alpha} \eta^{-d-\alpha} t_0^{-\frac{d}{\alpha}} (1 + t_0^{-\frac{1}{\alpha}} R)^{-d-\alpha}. \quad (3.27)$$

Next we consider the case  $M \geq C_* t_0^{\frac{1}{\alpha}}$ . We may assume that  $R \geq 2F[\tilde{v}](t_0, 0, x_0, y_0)$ , otherwise (3.26) always holds by Corollary 3.9. Take  $L = M^{-1} \log(M^\alpha/t_0)$ . Then Lemma 3.7 and the smallness of  $\delta$  imply

$$\tilde{\Lambda}_M(\Psi)t_0 + M^{-\alpha}t_0 \leq C, \quad (3.28)$$

where  $C$  depends only on  $d, \alpha, \gamma$ , and  $C_0$ . Hence Applying Proposition 3.8 and then taking  $M = \eta R$  with  $\eta \in (0, 1)$  and  $\Psi(0, y_0) = 0$ , we have

$$P_{\tilde{K}_M, \tilde{v}}(t_0, x_0; 0, y_0) \leq Ct_0^{-\frac{d}{\alpha}} \exp(-\Psi(t_0, x_0) + C\|\tilde{v}\|_{X_\lambda} LR^{\frac{\alpha\lambda}{2d} - \frac{\alpha}{2}} t_0^{\frac{1}{\alpha} - \frac{\lambda}{2d} + \frac{1}{2}}).$$

Taking  $C_*$  sufficient large depending on  $d, \alpha$  and  $\lambda$ , we can estimate

$$LR^{\frac{\alpha\lambda}{2d} - \frac{\alpha}{2}} t_0^{\frac{1}{\alpha} - \frac{\lambda}{2d} + \frac{1}{2}} = \eta^{-\frac{\alpha\lambda}{2d} + \frac{\alpha}{2}} \left(\frac{t_0}{M^\alpha}\right)^{\frac{1}{\alpha} - \frac{\lambda}{2d} + \frac{1}{2}} \log\left(\frac{M^\alpha}{t_0}\right) \leq C\eta^{-\frac{\alpha\lambda}{2d} + \frac{\alpha}{2}}.$$

for  $M \geq C_* t_0^{1/\alpha}$ . Thus, by the definition of  $\Psi$ , we get

$$P_{\tilde{K}_M, \tilde{v}}(t_0, x_0; 0, y_0) \leq Ct_0^{-\frac{d}{\alpha}} \exp(-L(R - |\xi(t_0; x_0)|)_+). \quad (3.29)$$

Now we will prove

$$-(R - |\xi(t_0; x_0)|)_+ \leq -\frac{R}{4} \quad (3.30)$$

when  $R \geq C_* t_0^{\frac{1}{\alpha}}$  and  $R \geq 2F[\tilde{v}](t_0, 0, x_0, y_0)$ . From the definition of  $x(t_0; x_0)$ , we have

$$\begin{aligned} -(R - |\xi(t_0; x_0)|)_+ &\leq -(R - |\xi(t_0; x_0)|) \\ &= -R + \left| \int_0^{t_0} \int_{B_R(x_0 + \xi(s; x_0))} \tilde{v}(s) \, ds \right| \\ &\leq -(R - \left| \int_0^{t_0} \int_{B_R(x_0)} \tilde{v}(s) \, ds \right|) - \left| \int_0^{t_0} \left( \int_{B_R(x_0)} \tilde{v}(s) - \int_{B_R(x_0 + \xi(s; x_0))} \tilde{v}(s) \right) \, ds \right| \\ &= I + II. \end{aligned}$$

Since  $R > 2F[\tilde{v}](t_0, 0, x_0, y_0)$ , the first term is estimated as follows:

$$I = \left| \int_0^{t_0} \int_{B_R(x_0)} \tilde{v}(s) \, ds \right| - R \leq F[\tilde{v}](t_0, 0, x_0, y_0) - R \leq -\frac{R}{2}.$$

On the other hand, For  $\mu = \frac{\alpha}{2d}(\lambda - d) + d$ , Lemma 2.3, Lemma 2.4 and (1.4) yield

$$\begin{aligned} II &\leq C \int_0^{t_0} \|\tilde{v}(s)\|_{\mathcal{L}^{1,\mu}} (|\xi(s, x_0)|^{\mu-d} + R^{\mu-d}) \, ds \\ &\leq C \int_0^{t_0} \|\tilde{v}(s)\|_{\mathcal{L}^{1,\mu}} \, ds \{ (R \|\tilde{v}\|_{X_\lambda} + F[\tilde{v}](t_0, 0, x_0, y_0))^{\mu-d} + R^{\mu-d} \} \\ &\leq C t_0^{\frac{1}{\alpha} - \frac{\lambda-d}{2d}} \|\tilde{v}\|_{X_\lambda} \{ (R(\|\tilde{v}\|_{X_\lambda} + 1))^{\frac{\alpha(\lambda-d)}{2d}} + F[\tilde{v}](t_0, 0, x_0, y_0)^{\frac{\alpha(\lambda-d)}{2d}} \} \\ &\leq C t_0^{\frac{1}{\alpha} - \frac{\lambda-d}{2d}} \|\tilde{v}\|_{X_\lambda} (1 + \|\tilde{v}\|_{X_\lambda})^{\frac{\alpha(\lambda-d)}{2d}} R^{\frac{\alpha(\lambda-d)}{2d}} + C' \|\tilde{v}\|_{X_\lambda}^{\frac{2d}{2d-\alpha(\lambda-d)}} t_0^{\frac{1}{\alpha}} + \frac{1}{4} F[\tilde{v}](t_0, 0, x_0, y_0). \end{aligned}$$

If we take  $C_* \geq \tilde{C} \max(\|\tilde{v}\|_{X_\lambda}^{\frac{2d+\alpha(\lambda-d)}{2d-\alpha(\lambda-d)}}, \|\tilde{v}\|_{X_\lambda}^{\frac{2d}{2d-\alpha(\lambda-d)}})$  with some absolute constant  $\tilde{C} > 0$ , the right hand side is bounded by  $R/4$ . Here we have also used the condition  $\lambda \in (d, 2d/\alpha + d)$  in the first term and  $R > 2F[\tilde{v}](t_0, 0, x_0, y_0)$  in the third term.

Thus, summing up these estimates, we get the desired estimate (3.30). Taking  $\eta = \frac{\alpha}{4(d+\alpha)}$ , we get

$$\begin{aligned} P_{\tilde{K}_M, \tilde{v}}(t_0, x_0; 0, y_0) &\leq C t_0^{-\frac{d}{\alpha}} \exp(-L(R - |\xi(t_0; x_0)|)_+) \leq C t_0^{-\frac{d}{\alpha}} \exp\left(-\frac{LR}{4}\right) \\ &= C t_0^{-\frac{d}{\alpha}} \exp\left(-\frac{1}{4\eta} \log \frac{M^\alpha}{t_0}\right) \\ &= C t_0 R^{-d-\alpha} \end{aligned}$$

Here  $C$  depends only on  $d, \alpha, C_0$ , and  $\|\tilde{v}\|_{X_\lambda}$ . Let  $A_{\tilde{K}_M^c}(t)$  be the self-adjoint operator in  $L^2(\mathbb{R}^d)$  associated with the symmetric Dirichlet form  $\mathcal{E}_{\tilde{K}_M^c}^{(t)}$ . Then (3.6) is written as

$$\partial_t \theta + A_{\tilde{K}_M}(t) \theta + \tilde{v} \cdot \nabla \theta = -A_{\tilde{K}_M^c}(t) \theta, \quad t > 0. \quad (3.31)$$

We define the evolution operator  $P_{\tilde{K}_M, \tilde{v}}(t, s)$  as

$$(P_{\tilde{K}_M, \tilde{v}}(t, s)f)(x) = \int_{\mathbb{R}^d} P_{\tilde{K}_M, \tilde{v}}(t, x; s, y) f(y) \, dy. \quad (3.32)$$

Then by the Duhamel principle, (3.31) leads to the formula

$$\begin{aligned} &P_{\tilde{K}_M, \tilde{v}}(t_0, x_0; 0, y_0) \\ &= P_{\tilde{K}_M, \tilde{v}}(t_0, x_0; 0, y_0) - \int_0^{t_0} (P_{\tilde{K}_M, \tilde{v}}(t_0, s) A_{\tilde{K}_M^c}(s) P_{\tilde{K}_M, \tilde{v}}(s, \cdot; 0, y_0))(x_0) \, ds. \quad (3.33) \end{aligned}$$

From the definitions of  $A_{\tilde{K}_{M^c}}(s)$  and  $\tilde{K}_{M^c}(t, x, y)$  we have

$$\begin{aligned}
& - \int_0^{t_0} (P_{\tilde{K}_{M,\tilde{v}}}(t_0, s) A_{\tilde{K}_{M^c}}(s) P_{\tilde{K},\tilde{v}}(s, \cdot; 0, y_0))(x_0) ds \\
= & - \int_0^{t_0} \int_{\mathbb{R}^{2d}} [P_{\tilde{K}_{M,\tilde{v}}}(t_0, x_0; s, \cdot)] [P_{\tilde{K},\tilde{v}}(s, \cdot; 0, y_0)](z, \tilde{z}) \tilde{K}_{M^c}(s, z, \tilde{z}) dz d\tilde{z} ds \\
\leq & 2 \int_0^{t_0} \int_{\mathbb{R}^{2d}} P_{\tilde{K}_{M,\tilde{v}}}(t_0, x_0; s, z) P_{\tilde{K},\tilde{v}}(s, \tilde{z}; 0, y_0) \tilde{K}_{M^c}(s, z, \tilde{z}) dz d\tilde{z} ds \\
\leq & 2t_0 (C_0 M^{-d-\alpha} + \delta M^{-d-1-\frac{\alpha}{2}}), \tag{3.34}
\end{aligned}$$

where we have used the definition of  $\tilde{K}_M(t, x, y)$  and the properties

$$\int_{\mathbb{R}^d} P_{\tilde{K}_{M,\tilde{v}}}(t, x; s, z) dz = \int_{\mathbb{R}^d} P_{\tilde{K}_{M,\tilde{v}}}(t, z; s, y) dz = 1.$$

Collecting (3.27), (3.33), and (3.34), we obtain the desired estimates in Proposition 3.10.  $\square$

*Proof of Theorem 3.1.* Set  $D = \{(t, s) \in \mathbb{R}^2 \mid t > s \geq 0\}$  and  $\Omega = D \times \mathbb{R}^d \times \mathbb{R}^d$ . Then fundamental solutions are regarded as functions in  $\Omega$ . In [21] a fundamental solution  $P_{K,v}$  to (2.6) is constructed as a limit of  $P_{K^{(N,\delta)},v^{(N)}}$  at  $\delta \rightarrow 0$  and  $N \rightarrow \infty$  in weak-\* topology of  $L_{loc}^\infty(\Omega)$ . Since  $v^{(N)}$  converges to  $v$  strongly in  $(L_{loc}^1(0, \infty) \times \mathbb{R}^d)^d$  by Lemma 2.5, it is standard that the limit function  $P_{K,v}$  satisfies the estimates (3.1)-(3.2) thanks to Corollary 3.9 and Proposition 3.10. The details are omitted here. This completes the proof.  $\square$

*Proof of Theorem 3.2.* As in Proposition 3.10, it suffices to consider the fundamental solution  $P_{\tilde{K},\tilde{v}}(t_0, x_0; 0, y_0)$  when  $R = |x_0 - y_0|$  satisfies  $R \geq Ct_0^{1/\alpha}$  and  $R \geq 2F[\tilde{v}](t_0, 0; x_0, y_0)$ . By using Proposition 3.8 and the definition of  $\Psi$ , we have

$$\begin{aligned}
P_{\tilde{K}_{M,\tilde{v}}}(t_0, x_0; 0, y_0) & \leq Ct_0^{-\frac{d}{\alpha}} \exp(-\Psi(t_0, x_0) + C\|\tilde{v}\|_{X_{\frac{2d}{\alpha}+d}} LR) \\
& \leq Ct_0^{-\frac{d}{\alpha}} \exp(-L(R - |\xi(t_0; x_0)|) + C\|\tilde{v}\|_{X_{\frac{2d}{\alpha}+d}} LR)
\end{aligned}$$

If  $\|\tilde{v}\|_{X_{2d/\alpha+d}}$  is sufficiently small, we can apply Lemma 2.4 to get

$$\begin{aligned}
P_{\tilde{K}_{M,\tilde{v}}}(t_0, x_0; 0, y_0) & \leq Ct_0^{-\frac{d}{\alpha}} \exp(-LR + L(R\|\tilde{v}\|_{X_{\frac{2d}{\alpha}+d}} + F[\tilde{v}](t_0, 0, x_0, y_0)) + C\|\tilde{v}\|_{X_{\frac{2d}{\alpha}+d}} LR) \\
& \leq Ct_0^{-\frac{d}{\alpha}} \exp\left(-\frac{LR}{4}\right)
\end{aligned}$$

Taking  $L = M^{-1} \log(M^\alpha/t_0)$ ,  $M = \eta R$  and  $\eta = \alpha/\{4(d+\alpha)\}$ , we have

$$P_{\tilde{K},\tilde{v}}(t_0, x_0; 0, y_0) \leq Ct_0 R^{-d-\alpha}.$$

The other arguments in the proof is same as those of Proposition 3.10 and Theorem 3.1. So we omit the detail.  $\square$

*Proof of Corollary 1.4.* When  $|x - y| \leq (t - s)^{1/\alpha}$  we have from (1.8),

$$P_{\alpha,v}(t, x; s, y) \leq C(t-s)^{-\frac{d}{\alpha}} \left(1 + \frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}}\right)^{d+\alpha} \left(1 + \frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha} \leq C(t-s)^{-\frac{d}{\alpha}} \left(1 + \frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha}. \tag{3.35}$$

This estimate holds all cases of  $\lambda$  if  $|x - y| \leq (t - s)^{1/\alpha}$ , so from now on we always consider the case  $|x - y| \geq (t - s)^{1/\alpha}$ . To prove (1.13) we observe that if  $\lambda \in [2d/\alpha - d, d)$  and



$|x - y| \leq 1$  then Lemma 2.3 yields

$$\begin{aligned}
F[v](t, s, x, y) &\leq \int_s^t \left| \int_{B_{|x-y|}(x)} v(\tau) - \int_{B_1(x)} v(\tau) \right| d\tau + \int_s^t \left| \int_{B_1(x)} v(\tau) \right| d\tau \\
&\leq C \int_s^t \|v(\tau)\|_{\mathcal{L}^{\frac{2d}{\alpha}, \lambda}} d\tau |x - y|^{\frac{2d}{\alpha}(\lambda-d)} + C \int_s^t \|v(\tau)\|_{L^1_{uloc}} d\tau \\
&\leq C(t-s)^{\frac{1}{\alpha} + \frac{1}{2} - \frac{\lambda}{2d}} |x - y|^{\frac{2d}{\alpha}(\lambda-d)} + C(t-s)^{\frac{1}{\alpha}} \\
&\leq C(t-s)^{\frac{1}{\alpha}} + \frac{|x - y|}{2}.
\end{aligned} \tag{3.36}$$

Hence if  $|x - y| \geq C'(t-s)^{1/\alpha}$  and  $C' > 0$  is large enough then

$$(|x - y| - F[v](t, s, x, y))_+ \geq \frac{|x - y|}{2}, \tag{3.37}$$

which implies (1.13) by (1.9) in this case. When  $|x - y| \leq C'(t-s)^{1/\alpha}$  we get (1.13) by the same argument in (3.35). If  $|x - y| \geq 1$  then **(C')** implies that

$$F[v](t, s, x, y) \leq C \int_s^t \|v(\tau)\|_{L^1_{uloc}} d\tau \leq C \|v\|_{L^{1, \frac{1}{\alpha}}(0, \infty; L^1_{uloc})} (t-s)^{\frac{1}{\alpha}}.$$

Hence, if  $|x - y| \geq C''(t-s)^{1/\alpha}$  with large  $C'' > 0$  then  $(|x - y| - F[v](t, s, x, y))_+ \geq |x - y|/2$ , i.e., we get from (1.9) that  $P_{\alpha, v}(t, x; s, y) \leq C(t-s)^{-d/\alpha} (1 + |x - y|(t-s)^{-1/\alpha})^{-d-\alpha}$ , as desired. When  $|x - y| \leq C''(t-s)^{1/\alpha}$  we have (1.13) by the same argument as (3.35). This completes the proof of (1.13).

Next we consider (1.14). In view of the proof of (1.13), it suffices to give the proof only for the case  $(t-s)^{1/\alpha} \leq |x - y| \leq 1$ . In this case we have from Lemma 2.3,

$$\begin{aligned}
F[v](t, s, x, y) &\leq \int_s^t \left| \int_{B_{|x-y|}(x)} v(\tau) - \int_{B_1(x)} v(\tau) \right| d\tau + \int_s^t \left| \int_{B_1(x)} v(\tau) \right| d\tau \\
&\leq C(1 - \log|x - y|) \int_s^t \|v(\tau)\|_{\mathcal{L}^{1, d}} d\tau + C \int_s^t \|v(\tau)\|_{L^1_{uloc}} d\tau \\
&\leq C(1 + |\log(t-s)|)(t-s)^{\frac{1}{\alpha}}.
\end{aligned} \tag{3.38}$$

Hence, if  $|x - y| \geq C'(1 + |\log(t-s)|)(t-s)^{1/\alpha}$  for  $C' > 0$  large enough, then  $(|x - y| - CF[v](t, s, x, y))_+ \geq |x - y|/2$ . which leads from (1.9) to  $P_{\alpha, v}(t, x; s, y) \leq C(t-s)^{-\frac{d}{\alpha}} (1 + |x - y|(t-s)^{-1/\alpha})^{-d-\alpha}$ . On the other hand, if  $|x - y| \leq C'(1 + |\log(t-s)|)(t-s)^{1/\alpha}$  then we have from (1.8),

$$\begin{aligned}
P_{\alpha, v}(t, x, s, y) &\leq C(t-s)^{-\frac{d}{\alpha}} \left(1 + \frac{|x - y|}{(t-s)^{\frac{1}{\alpha}}}\right)^{d+\alpha} \left(1 + \frac{|x - y|}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha} \\
&\leq C(t-s)^{-\frac{d}{\alpha}} (1 + |\log(t-s)|)^{d+\alpha} \left(1 + \frac{|x - y|}{(t-s)^{\frac{1}{\alpha}}}\right)^{-d-\alpha}.
\end{aligned}$$

This completes the proof of (1.14).

Est.(1.15) is proved similarly. Indeed, the only difference is that when  $\lambda \in (d, 2d/\alpha + d)$  the estimate (3.38) is replaced by

$$F[v](t, s, x, y) \leq C(t-s)^{\frac{1}{\alpha} + \frac{1}{2} - \frac{\lambda}{2d}}, \tag{3.39}$$

for  $(t-s)^{1/\alpha} \leq |x - y| \leq 1$ . Hence by considering two cases  $|x - y| \geq C'(t-s)^{\frac{1}{\alpha} + \frac{1}{2} - \frac{\lambda}{2d}}$  and  $|x - y| \leq C'(t-s)^{\frac{1}{\alpha} + \frac{1}{2} - \frac{\lambda}{2d}}$  with sufficiently large  $C' > 0$ , we get the desired result when  $(t-s)^{1/\alpha} \leq |x - y| \leq 1$ . The other cases  $|x - y| \leq (t-s)^{1/\alpha}$  and  $|x - y| \geq 1$  are proved also in the same way as in the proof of (1.13) and (1.14), so we skip the details. The proof of Corollary 1.4 is now complete.  $\square$

## References

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