On fundamental solutions for non-local diffusion equations with divergence free drift

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Abstract

We are concerned with non-local diffusion equations in the presence of a divergence free drift term. By using the classical Nash approach we show the existence of fundamental solutions, together with the continuity estimates, under weak regularity assumptions on the kernel of the diffusion term and the velocity of the drift term. As an application, our result gives the alternative proof of the global regularity for the two-dimensional dissipative quasi-geostrophic equations in the critical case.

Keywords: non-local diffusion equations; divergence free drift; fundamental solutions; Nash iteration; 2D dissipative quasi-geostrophic equations.

1 Introduction

In this paper we consider the non-local diffusion equations in the presence of a drift term

$$\partial_t\theta + A_K(t)\theta + v\cdot\nabla\theta = 0, \qquad t>0, \quad x\in\mathbb{R}^d, \tag{1.1}$$

where $d\geq 2$ and $A_K(t)$ is a linear operator formally defined by

$$\big(A_K(t)f\big)(x) = P.V.\int_{\mathbb{R}^d}(f(x)-f(y))K(t,x,y)\,\mathrm{d}y. \tag{1.2}$$

Here $K(t,x,y)$ is a positive function satisfying $K(t,x,y)=K(t,y,x)$ and $v(t,x)=(v_1(t,x),\cdots,v_d(t,x))$ is a vector field in $\mathbb{R}^d$ satisfying the divergence free condition, $\nabla\cdot v(t)=0$. In particular, $A_K(t)$ will be supposed to possess a diffusion effect like $(-\Delta)^{\alpha/2}$ for some $\alpha\in(0,2)$. Note that in the case $A_K(t)=(-\Delta)^{\alpha/2}$ with $\alpha\in(0,2)$ the kernel K is given by $K(t,x,y)=C_{d,\alpha}|x-y|^{-d-\alpha}$ for some positive constant $C_{d,\alpha}$. The aim of this paper is to prove the existence and the continuity of fundamental solutions for (1.1) under less regularity assumptions on K and v.

When there is no drift term (i.e., $v=0$) this problem appears in the theory of Dirichlet forms of jump type. For the diffusion operator $A_K(t)$ defined by (1.2) the associated Dirichlet form is

$$\mathcal{E}_K^{(t)}(f,g) = \frac{1}{2}\iint_{\mathbb{R}^{2d}}[f][g](x,y)K(t,x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad [f](x,y)=f(x)-f(y), \tag{1.3}$$

and it has been investigated mainly from the probabilistic approach [6, 24, 25, 3, 1, 21, 2]. On the other hand, in recent years the case with a drift term has also attracted much attention, especially in the field of fluid mechanics, mathematical finance, biology, and so on. For example, many works have been done for the two-dimensional dissipative quasi-geostrophic equations (QG), where $A_K(t)=(-\Delta)^{\alpha/2}$ and the drift term is a nonlinear term such that v is given in terms of θ via the Riesz transform; [9]. For such nonlinear problems it is crucial to obtain detailed informations of solutions under less regularity conditions on v.

In [4, 19] fundamental solutions were constructed when $A_K(t) = (-\Delta)^{\alpha/2}$ with $\alpha \in (1,2)$ and v belongs to a suitable Kato class without assuming the divergence free condition. In this case the diffusion term is the leading term and they showed two-sided heat kernel estimates by using perturbation arguments. However, despite of the increasing interest, there seems to be still few works on fundamental solutions for $\alpha \in (0,1]$. In such cases the drift term formally becomes the leading term and is no longer regarded as a simple perturbation of $A_K(t)$, which causes difficulties in the study of (1.1). For example, so far little seem to be known about the uniqueness of weak solutions for such cases and this makes even the semigroup property of fundamental solutions nontrivial.

To state our main results we give the precise assumptions on the kernel K and the velocity v. We assume that there are $\alpha \in (0,2)$ and $C_0 > 0$ such that

$$K(t,x,y) \;=\; K(t,y,x) \qquad \text{for } \; a.e.\ (t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d, \tag{1.4}$$

$$\operatorname{ess.sup}_{t>0,x\in\mathbb{R}^d} \int_{|x-y|\le M} |x-y|^2 K(t,x,y)\,\mathrm{d}y \;\le\; C_0 M^{2-\alpha} \qquad \text{for each } \; M \in (0,\infty), \tag{1.5}$$

$$\operatorname{ess.inf}_{t>0,x,y\in\mathbb{R}^d} |x-y|^{d+\alpha} K(t,x,y) \;\ge\; C_0^{-1}. \tag{1.6}$$

Following the conventions in (QG), we will call the case $\alpha \in (1,2)$ subcritical, the case $\alpha = 1$ critical, and the case $\alpha \in (0,1)$ supercritical. We note that if K satisfies

$$C^{-1}|x-y|^{-d-\alpha} \le K(t,x,y) \le C|x-y|^{-d-\alpha}, \tag{1.7}$$

then (1.5) and (1.6) are satisfied. The condition (1.6) guarantees the diffusion effect like $(-\Delta)^{\alpha/2}$. Taking this in mind, we assume that v belongs to a class of functions which is invariant under the scaling

$$v(t,x) \longmapsto r^{1-1/\alpha} v(rt, r^{1/\alpha}x), \quad r > 0. \tag{1.8}$$

This scaling is natural in the following sense: if $\theta(t,x)$ is a solution to (1.1) with $A_K(t) = (-\Delta)^{\alpha/2}$ and $v = v(t,x)$ then the rescaled function $\theta(rt, r^{1/\alpha}x)$ satisfies (1.1) with $A_K(t) = (-\Delta)^{\alpha/2}$ and $v = r^{1-1/\alpha}v(rt, r^{1/\alpha}x)$, instead of $v(t,x)$. Heuristically it is essential that v belongs to a function space which is invariant with respect to (1.8), in order to ensure a smoothing effect by $(-\Delta)^{\alpha/2}$; for example, see [29, 33, 30] and references there in for second order parabolic equations and [5, 22, 32] for the fractional diffusion equations.

To describe the regularity assumption on v let us introduce the Campanato spaces; see [16].

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^d) = \{ f \in L^p_{loc}(\mathbb{R}^d) \mid \|f\|_{\mathcal{L}^{p,\lambda}} = \sup_B \big(R^{-\lambda} \int_B |f(x) - \fint_B f|^p \,\mathrm{d}x \big)^{\frac{1}{p}} < \infty \}. \tag{1.9}$$

Here the supremum is taken over all balls $B = B_R(y)$ (the ball with radius $R > 0$ centered at $y \in \mathbb{R}^d$), and $|B|$ is the volume of the ball B. We will sometimes write B_R for $B_R(0)$ for simplicity of notations. The value $\fint_B f$ is defined by

$$\fint_B f = \frac{1}{|B|} \int_B f(x)dx. \tag{1.10}$$

Then it is easy to see that the continuous embedding

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^d) \hookrightarrow \mathcal{L}^{1,\mu}(\mathbb{R}^d) \qquad \text{if} \qquad \mu = \frac{\lambda - d}{p} + d \tag{1.11}$$

holds. Moreover, we have

$$L_{\mathrm{w}}^{\frac{pd}{d-\lambda}}(\mathbb{R}^d) \;\hookrightarrow\; \mathcal{L}^{p,\lambda}(\mathbb{R}^d) \qquad \text{if} \quad 0 < \lambda < d, \tag{1.12}$$

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^d) \;=\; BMO(\mathbb{R}^d) \qquad \text{if} \quad \lambda = d, \tag{1.13}$$

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^d) \;=\; \dot{C}^{\frac{\lambda-d}{p}}(\mathbb{R}^d) \qquad \text{if} \quad d < \lambda \le d+p. \tag{1.14}$$

Here $L^p_{\mathrm{w}}(\mathbb{R}^d)$ is the weak L^p space and $\dot{C}^\beta(\mathbb{R}^d)$, $\beta \in (0,1]$, is the homogeneous Hölder space of the order β, i.e.,

$$\dot{C}^\beta(\mathbb{R}^d) = \{ f \in C(\mathbb{R}^d) \mid \|f\|_{\dot{C}^\beta} = \sup_{x,y\in\mathbb{R}^d} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty \}.$$

Next we introduce the Morrey type spaces of $\mathcal{L}^{p,\lambda}$-valued functions.

$$L^{p,\lambda_1}(0,\infty;\mathcal{L}^{q,\lambda_2}(\mathbb{R}^d)) = \{f\in L^p_{loc}(0,\infty;\mathcal{L}^{q,\lambda_2}(\mathbb{R}^d)) \mid$$
$$\|f\|_{L^{p,\lambda_1}(0,\infty;\mathcal{L}^{q,\lambda_2}(\mathbb{R}^d))} = \sup_{t>0}\sup_{0<s<t}\big((t-s)^{-\lambda_1}\int_s^t \|f(\tau)\|^p_{\mathcal{L}^{q,\lambda_2}}\,\mathrm{d}\tau\big)^{\frac{1}{p}} < \infty\}. \tag{1.15}$$

For $1\le p,q\le\infty$ let $L^q_{loc}(0,\infty;L^p_{loc}(\mathbb{R}^d))$ be the class of functions defined by

$$L^q_{loc}(0,\infty;L^p_{loc}(\mathbb{R}^d)) = \{f\in L^1_{loc}((0,\infty)\times\mathbb{R}^d) \mid \|f\|_{L^q(0,R;L^p(B_R))}<\infty \quad \text{for all} \quad R>0\}. \tag{1.16}$$

When $K(t,x,y)$ satisfies (1.5) and (1.6) for some $\alpha\in(0,2)$ the velocity v is assumed to satisfy the following two conditions:

(C1) there are $\lambda\in[2d/\alpha-d,2d/\alpha+d]$ and $1<q\le\infty$ such that

$$v\in L^{1,\frac12+\frac1\alpha-\frac{\lambda}{2d}}\big(0,\infty;(\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))^d\big) \ \cap\ L^q_{loc}\big(0,\infty;(L^{p_\lambda}_{loc}(\mathbb{R}^d))^d\big), \tag{1.17}$$

where $p_\lambda=1$ if $\lambda\in[2d/\alpha-d,d]$ and $p_\lambda=\infty$ if $\lambda\in(d,2d/\alpha+d]$.

(C2) $\nabla\cdot v(t)=0$ for a.e. $t>0$ in the sense of distributions.

Remark 1.1 The space $L^{1,\frac12+\frac1\alpha-\frac{\lambda}{2d}}(0,\infty;\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))$ is invariant under the scaling (1.8). The condition $v\in L^{1,\frac12+\frac1\alpha-\frac{\lambda}{2d}}(0,\infty;(\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))^d)$ covers the following cases: $v\in L^\infty(0,\infty;(L_{\mathrm{w}}^{\frac{d}{\alpha-1}}(\mathbb{R}^d))^d)$ for the subcritical case; $v\in L^\infty(0,\infty;(BMO(\mathbb{R}^d))^d)$ for the critical case; $v\in L^\infty(0,\infty;(\dot{C}^{1-\alpha}(\mathbb{R}^d))^d)$ for the supercritical case. One of the advantages to use the Campanato spaces (1.9) is that for some exponents (p,λ) they contain functions growing at spatial infinity. In particular, the case $\lambda=2d/\alpha+d$ in **(C1)** allows v to grow at most linearly as $|x|\to\infty$. We also note that the condition $v\in L^{1,\frac12+\frac1\alpha-\frac{\lambda}{2d}}(0,\infty;(\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))^d)$ includes the case

$$|t-t_0|^{\frac{\lambda}{2d}+\frac12-\frac1\alpha}v(t)\in L^\infty(0,\infty;(\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))^d) \qquad \text{for some} \quad t_0\in[0,\infty). \tag{1.18}$$

Under the divergence free condition **(C2)** the drift term becomes skew-symmetric with respect to the usual $L^2(\mathbb{R}^d)$ inner product, and hence, the adjoint equation for (1.1) takes the same form as (1.1). This additional structure is essentially used in constructing fundamental solutions under weak regularity condition **(C1)**. The divergence free condition sometimes plays important roles also in the second order parabolic equations with singular drifts; [29, 33, 30].

For simplicity of notations we will introduce the seminorm

$$\|v\|_{X_\lambda} = \|v\|_{L^{1,\frac12+\frac1\alpha-\frac{\lambda}{2d}}(0,\infty;\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))}. \tag{1.19}$$

For $T>0$ and $x\in\mathbb{R}^d$ we also set

$$\|v\|_{Y^{q,\lambda}_{T,x}} = \|v\|_{L^q(0,T;L^{p_\lambda}(B_1(x)))}, \tag{1.20}$$

where p_λ is as in **(C1)**. The main result of this paper is the existence of fundamental solutions for (1.1). The precise definition of fundamental solutions will be given in the next section.

Theorem 1.2 *Suppose that (1.4) - (1.6) and* **(C1)** - **(C2)** *hold. Then there exists a fundamental solution $P_{K,v}(t,x;s,y)$ for (1.1) satisfying the following properties.*

$$\int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)\,\mathrm{d}x = \int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)\,\mathrm{d}y = 1, \tag{1.21}$$

$$0\le P_{K,v}(t,x;s,y)\le C(t-s)^{-\frac{d}{\alpha}}, \tag{1.22}$$

$$P_{K,v}(t,x;s,y) = \int_{\mathbb{R}^d} P_{K,v}(t,x;\tau,z)P_{K,v}(\tau,z;s,y)\,\mathrm{d}z, \qquad t>\tau>s\ge0, \tag{1.23}$$

$$|P_{K,v}(t,x_1;s,y_1)-P_{K,v}(t,x_2;s,y_2)| \le \frac{C'\big(|x_1-x_2|^\beta+|y_1-y_2|^\beta\big)}{(t-s)^c}, \tag{1.24}$$

and for $T\ge t_i>s_i\ge0$, $i=1,2$,

$$|P_{K,v}(t_1,x;s_1,y)-P_{K,v}(t_2,x;s_2,y)| \le \frac{C_{T,x}|t_1-t_2|^{\beta'}+C_{T,y}|s_1-s_2|^{\beta'}}{(\min\{t_1-s_1,t_2-s_2\})^{c'}}. \tag{1.25}$$

Here the positive constant C depends only on d, α, and C_0, the positive constants C', c, β depend only on d, α, C_0, λ, and $\|v\|_{X_\lambda}$, the positive constant $C_{T,x}$ (or $C_{T,y}$) depends only on T, d, α, C_0, λ, q, $\|v\|_{X_\lambda}$, and $\|v\|_{Y^{q,\lambda}_{T,x}}$ (or $\|v\|_{Y^{q,\lambda}_{T,y}}$), and the positive constants c', β' depend only on d, α, C_0, λ, q, $\|v\|_{X_\lambda}$.

Remark 1.3 In the proof of Theorem 1.2 we will also show that

$$\theta(t,x) := \int_{\mathbb{R}^d} P_{K,v}(t,x;s,y) f(y)\,\mathrm{d}y \in C([s,\infty); L^p(\mathbb{R}^d)) \quad \text{if} \quad f \in L^p(\mathbb{R}^d), \quad 1 \le p < \infty, \tag{1.26}$$

and the energy inequality

$$\|\theta(t)\|_{L^2}^2 + 2\int_s^t \mathcal{E}_K^{(\tau)}\big(\theta(\tau),\theta(\tau)\big)\,\mathrm{d}\tau \le \|f\|_{L^2}^2, \tag{1.27}$$

for $t > s \ge 0$ if $f \in L^2(\mathbb{R}^d)$; see the proof in Section 4.2.

The estimates (1.24) and (1.25) show the Hölder continuity of the fundamental solution, where the Hölder exponents and the constant C' are estimated uniformly in time and space, while the constants $C_{T,x}$ and $C_{T,y}$ can be larger as $|x|$ and $|y|$ increase, if v grows at $|x| \to \infty$. We note that for some class of (K,v) and solutions the Hölder continuity is obtained in [5, 22] for the critical case and also in [11, 32] for the supercritical case. In [5, 22, 11] the case $A_K(t) = (-\Delta)^{\alpha/2}$ and $v \in L^\infty(0,\infty; (\mathcal{L}^{2d/\alpha, 2d/\alpha - d})^d)$ was treated under the condition **(C2)**, and [32] dealt with the case (1.7) and $v \in (C^{1-\alpha}((0,\infty)\times\mathbb{R}^d))^d$ but without **(C2)**.

In order to prove Theorem 1.2 it is important to obtain the a priori estimates for fundamental solutions of the approximate equations which are, roughly speaking, of the form $\partial_t\theta + \delta(-\Delta)^{\tilde\alpha/2}\theta + A_{\tilde K}(t)\theta + \tilde v\cdot\nabla\theta = 0$ with $\delta > 0$. Here $\tilde\alpha \in (1,2)$, and $\tilde K$ and $\tilde v$ are suitable mollifications of K and v. It is more or less well known that the unique existence of fundamental solutions holds for such mollified equations due to the fact that the leading term is the extra diffusion term. So our main step is to prove the a priori (equi-)continuity estimates of the fundamental solutions. For the purpose we will use the Nash-type arguments by Komatsu [24, 25] where he studied the non-local diffusion equations without the drift term. As in [28, 24, 25], the arguments consist of four steps; the moment bound, the relative entropy bound, the overlap estimate, and the iteration estimate. However, due to the presence of the nonsmooth drift term, it seems to be difficult to obtain the same estimates as in [24, 25]. For example, the moment of the fundamental solution is shifted by the drift, and we have to take this effect into account. To this end we will use time-dependent coordinates along the trajectory determined by a local average of v, instead of the usual coordinates $\mathbb{R}^d$. Although similar coordinates were used in [5, 11, 22], we have to choose the appropriate trajectory in each step carefully. We also note that, in fact, the arguments in [28, 24, 25] highly rely on the scaling property of (1.1), while the above approximation does not preserve such property. As a result, for example, it is difficult to get the equi-continuity estimates for solutions to the approximate equations, which causes another technicality in taking the limit and showing the desired estimate rigorously.

Theorem 1.2 has an application for the global regularity of solutions to (QG) in the critical case, as in [23, 5, 22]. Indeed, our result gives the alternative approach to this problem, based on the Nash-type arguments for fundamental solutions. In particular, it should be noted that different from [5] we need not study extension problems to use special property of the fractional Laplacian.

Before concluding the introduction we also state the spatial decay of fundamental solutions.

Theorem 1.4 *Let $P_{K,v}(t,x;s,y)$ be the fundamental solution obtained by Theorem 1.2. Then there is $h_0 > 0$ such that for all $R_0 \ge 1$, $t \in (0,h_0]$, and $\gamma \in (0, \min\{\alpha, 2-\alpha\})$, if $R \ge 1$ is sufficiently large then*

$$\sup_{|x|\le R_0, 0\le\tau\le R_0} \ \sup_{|y|\ge R,\ 0\le s\le \frac{t}{4}} P_{K,v}(t+\tau,x;s+\tau,y) \le C t^{-\frac{d}{\alpha}+\frac{\gamma}{2\alpha}} R^{-\frac{\gamma}{2}}, \tag{1.28}$$

$$\sup_{|y|\le R_0, 0\le\tau\le R_0} \ \sup_{|x|\ge R,\ 0\le s\le \frac{t}{4}} P_{K,v}(t+\tau,x;s+\tau,y) \le C t^{-\frac{d}{\alpha}+\frac{\gamma}{2\alpha}} R^{-\frac{\gamma}{2}}. \tag{1.29}$$

Here C depends only on d, α, C_0, γ, λ, q, $R_0 + h_0$, $\|v\|_{X_\lambda}$, and $\|v\|_{Y^{q,\lambda}_{R_0+h_0,0}}$.

Remark 1.5 The smallness of h_0 is required only when $\lambda = 2d/\alpha + d$. In this case v might have a linear growth at spatial infinity, and the smallness of h_0 cannot be removed in general. More detailed pointwise estimates of fundamental solutions are studied in the forthcoming paper [26].

The rest of this paper is organized as follows. In Section 2 we give the precise definition of fundamental solutions and establish some inequalities which are used throughout of this paper. We also introduce the approximate equations for (1.1). Section 3 is the core of this paper, where we establish the a priori estimates for fundamental solutions to the approximate equations. In Section 4 we prove Theorem 1.2 and Theorem 1.4 by taking the limit based on the estimates obtained in Section 3. In Section 5 the uniqueness of weak solutions to (1.1) is discussed in the subcritical and critical cases for a specific class of K and v. In Section 6 is devoted to the application of Theorem 1.2 to (QG).

2 Preliminaries

In this section we give a definition of fundamental solutions of (1.1) and prepare several inequalities which will be used in the proofs.

2.1 Definition of fundamental solutions

Let $K(t,x,y)$ be a positive measurable function satisfying $K(t,x,y) = K(t,y,x)$, and let $v(t,x) = (v_1(t,x), \cdots, v_d(t,x))$ be a vector field satisfying $\nabla \cdot v(t) = 0$. Then we formally define the bilinear forms $\mathcal{E}_{v(t)}(\cdot,\cdot)$ and $\mathcal{B}_K^{(t)}(\cdot,\cdot)$ by

$$\begin{aligned} \mathcal{E}_{v(t)}(f,g) &= -< f, v(t)\cdot\nabla g > := -\int_{\mathbb{R}^d} f(x)v(t,x)\cdot\nabla g(x)\,\mathrm{d}x, && (2.1)\\ \mathcal{B}_K^{(t)}(f,g) &= \mathcal{E}_K^{(t)}(f,g) + \mathcal{E}_{v(t)}(f,g), && (2.2)\end{aligned}$$

where $\mathcal{E}_K^{(t)}(\cdot,\cdot)$ is the Dirichlet form associated with K defined by (1.3). Then we formally define weak solutions and fundamental solutions to (1.1) as follows.

Let $T > s \geq 0$. A function $\theta \in L^\infty(s,T;L^2(\mathbb{R}^d))$ is said to be a weak solution to (1.1) for $t \in [s,T)$ with initial data θ_s at $t = s$ if θ satisfies

$$\int_s^T \mathcal{E}_K^{(t)}\big(\theta(t),\theta(t)\big)\,\mathrm{d}t < \infty, \tag{2.3}$$

and

$$-\int_s^T < \theta(t), \partial_t\varphi(t) > \,\mathrm{d}t + \int_s^T \mathcal{B}_K^{(t)}(\theta(t),\varphi(t))\,\mathrm{d}t = < \theta_s, \varphi(s) >, \qquad \forall \varphi \in C_0^\infty([s,T)\times\mathbb{R}^d). \tag{2.4}$$

Here $C_0^\infty([s,T)\times\mathbb{R}^d)$ is the set of compactly supported smooth functions on $[s,T)\times\mathbb{R}^d$.
Then a measurable function $P_{K,v}(t,x;s,y)$ on $\{(t,s,x,y) \mid t > s \geq 0,\ x,y \in \mathbb{R}^d\}$ is said to be a fundamental solution to (1.1) if for each $T > s \geq 0$ and $f \in L^2(\mathbb{R}^d)$ the function

$$(P_{K,v}f)(t,s,x) := \int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)f(y)\,\mathrm{d}y, \tag{2.5}$$

is a weak solution to (1.1) for $t \in [s,T)$ with initial data f at $t = s$.

Remark 2.1 When K satisfies the condition (1.6) a weak solution in our definition belongs to $L^2(s,T;\dot{H}^{\alpha/2}(\mathbb{R}^d))$ due to (2.3) and the coercive estimate

$$\text{ess.inf}_{t>0}\mathcal{E}_K^{(t)}(f,f) \geq C\|(-\Delta)^{\frac{\alpha}{4}}f\|_{L^2}^2. \tag{2.6}$$

For $K(t,x,y)$ and $M \in (0,\infty)$ we set

$$\begin{aligned} J_l(K) &= \text{ess.sup}_{t\geq 0, x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|x-y|^l K(t,x,y)\,\mathrm{d}y, \qquad l \geq 0, && (2.7)\\ K_M(t,x,y) &= K(t,x,y)\chi_{\{|x-y|\leq M\}}(x,y), && (2.8)\\ K_{\vee,M}(t,x,y) &= K(t,x,y)\chi_{\{|x-y|\geq M\}}(x,y). && (2.9)\end{aligned}$$

Here $\chi_A(x,y)$ is the characteristic function of the set $A \subset \mathbb{R}^d\times\mathbb{R}^d$. When K satisfies (1.5) we have

$$\begin{aligned} J_{\gamma_1}(K_M) &\leq CM^{\gamma_1-\alpha}, \quad \text{if } \ \gamma_1 > \alpha, && (2.10)\\ J_{\gamma_2}(K_{\vee,M}) &\leq C'M^{\gamma_2-\alpha}, \quad \text{if } \ \gamma_2 \in [0,\alpha), && (2.11)\end{aligned}$$

Indeed, by the monotone convergence theorem we have

$$\begin{aligned} \int_{\mathbb{R}^d}|x-y|^{\gamma_1}K_M(t,x,y)\,\mathrm{d}y &= \sum_{k=0}^\infty \int_{2^{-k-1}M\leq|x-y|\leq 2^{-k}M}|x-y|^{\gamma_1}K(t,x,y)\,\mathrm{d}y\\ &\leq C\sum_{k=0}^\infty (2^{-k}M)^{\gamma_1-2}\int_{2^{-k-1}M\leq|x-y|\leq 2^{-k}M}|x-y|^2K(t,x,y)\,\mathrm{d}y\\ &\leq CC_0\sum_{k=0}^\infty(2^{-k}M)^{\gamma_1-2}(2^kM)^{2-\alpha} \leq CC_0M^{\gamma_1-\alpha},\end{aligned}$$

if $\gamma_1 > \alpha$. This proves (2.10). The estimate (2.11) is obtained similarly and we omit the details.

For later use it will be convenient to define the "norm" of the kernel K:

$$\|K\|_\gamma = \sum_{k=1}^{\infty} k^2 2^{k(\gamma-2)} J_2(K_{2^k}) + \sum_{k=1}^{\infty} 2^{\frac{\gamma k}{2}} J_0(K_{\vee,2^k}). \tag{2.12}$$

From (2.10) and (2.11) it is easy to see that $\|K\|_\gamma < \infty$ if $\gamma < \alpha$ when K satisfies (1.5).

For $t > s \geq 0$ and $r > 0$ we define

$$(i_s f)(t,x) = f(t+s,x), \tag{2.13}$$

$$(j_t f)(s,x) = f(t-s,x), \tag{2.14}$$

$$(R_r f)(t,x) = f(rt, r^{\frac{1}{\alpha}}x), \qquad (R_r K)(t,x,y) = K(rt, r^{\frac{1}{\alpha}}x, r^{\frac{1}{\alpha}}y). \tag{2.15}$$

Note that $(j_{t-s} i_s f)(\tau,x) = (j_t f)(\tau,x)$ holds. Let $P_{K,v}(t,x;s,y)$, $t > s \geq 0$, $x,y \in \mathbb{R}^d$, be the fundamental solution to (1.1). Then the scaling, the translation, and the adjoint properties of (1.1) imply

$$\begin{aligned} P_{K,v}(t,x;s,y) = P_{i_s K, i_s v}(t-s,x;0,y) &= P_{j_{t-s} i_s K, -j_{t-s} i_s v}(t-s,y;0,x) \\ &= P_{j_t K, -j_t v}(t-s,y;0,x), \end{aligned} \tag{2.16}$$

$$r^{\frac{d}{\alpha}} P_{K,v}(rt, r^{\frac{1}{\alpha}}x; rs, r^{\frac{1}{\alpha}}y) = P_{r^{1+\frac{d}{\alpha}} R_r K, r^{1-\frac{1}{\alpha}} R_r v}(t,x;s,y). \tag{2.17}$$

Here, for example, $P_{j_t K, -j_t v}(\tau,y;r,x)$ ($0 \leq r < \tau \leq t$, $y,x \in \mathbb{R}^d$) is a fundamental solution to the equation

$$\partial_\tau \theta(\tau,y) + A_{j_t K}(\tau)\theta(\tau,y) + (-j_t v)\cdot\nabla\theta(\tau,y) = 0, \quad \tau \in (0,t], \quad y \in \mathbb{R}^d, \tag{2.18}$$

and $P_{r^{1+d/\alpha} R_r K, r^{1-1/\alpha} R_r v}(t,x;s,y)$ is a fundamental solution to the equation

$$\partial_t \theta + A_{r^{1+\frac{d}{\alpha}} R_r K}(t)\theta + (r^{1-\frac{1}{\alpha}} R_r v)\cdot\nabla\theta = 0, \quad t > 0, \; x \in \mathbb{R}^d. \tag{2.19}$$

2.2 Inequalities from real analysis

Lemma 2.2 *Let $f \in \mathcal{L}^{1,\mu}(\mathbb{R}^d)$ for some $\mu \in [0, d+1]$. Let $x_1, x_2 \in \mathbb{R}^d$ and $R_1 \geq R_2 > 0$. Then*

$$\left|\fint_{B_{R_1}(x_1)} f - \fint_{B_{R_2}(x_2)} f\right| \leq \begin{cases} C\|f\|_{\mathcal{L}^{1,\mu}} R_2^{\mu-d} & \text{if } 0 \leq \mu < d, \\ C\|f\|_{\mathcal{L}^{1,\mu}} \big(\log(e + \frac{|x_1-x_2|}{R_2}) + \log\frac{R_1}{R_2}\big) & \text{if } \mu = d, \\ C\|f\|_{\mathcal{L}^{1,\mu}} \big(|x_1-x_2|^{\mu-d} + R_1^{\mu-d}\big) & \text{if } d < \mu \leq d+1. \end{cases} \tag{2.20}$$

Here C depends only on d and μ.

Proof. If $x_1 = x_2 = x$ and $2^{k-1}R_2 \leq R_1 \leq 2^k R_2$ for some $k \in \mathbb{N}$, then (2.20) follows from

$$\left|\fint_{B_{2^k R_2}(x)} f - \fint_{B_{R_1}(x)} f\right| \leq C\|f\|_{\mathcal{L}^{1,\mu}} R_1^{\mu-d},$$

and

$$\left|\fint_{B_{2^k R_2}(x)} f - \fint_{B_{R_2}(x)} f\right| \leq \sum_{l=1}^{k} \left|\fint_{B_{2^l R_2}(x)} f - \fint_{B_{2^{l-1} R_2}(x)} f\right| \leq C\|f\|_{\mathcal{L}^{1,\mu}} R_2^{\mu-d} \sum_{l=1}^{k} 2^{(\mu-d)l}.$$

We turn to the case $R_1 = R_2 = R$. We may assume that $x_2 = 0$. Then (2.20) for the case $\mu \in (d, d+1]$ is trivial since the relation $\mathcal{L}^{1,\mu}(\mathbb{R}^d) = \dot{C}^\beta(\mathbb{R}^d)$ with $\beta = \mu - d$. The case $\mu = d$ follows from the estimate for functions in $BMO(\mathbb{R}^d)$; for example, see [17]. As for the case $\mu \in [0,d)$, if $|x_1| \leq R$ then $B_R(x_1) \subset B_{2R}(0)$ and hence

$$\begin{aligned} \left|\fint_{B_R(x_1)} f - \fint_{B_R(0)} f\right| &\leq \left|\fint_{B_R(x_1)} f - \fint_{B_{2R}(0)} f\right| + \left|\fint_{B_{2R}(0)} f - \fint_{B_R(0)} f\right| \\ &\leq C\|f\|_{\mathcal{L}^{1,\mu}} R^{\mu-d}. \end{aligned}$$

If $2^{k-1}R \leq |x_1| \leq 2^k R$ for some $k \in \mathbb{N}$ then $B_{2^k R}(x_1) \subset B_{2^{k+1}R}(0)$, and thus from (2.20) with the case $x_1 = x_2 = x$ proved above,

$$\begin{aligned} &\left|\fint_{B_R(x_1)} f - \fint_{B_R(0)} f\right| \\ \leq\; &\left|\fint_{B_R(x_1)} f - \fint_{B_{2^k R}(x_1)} f\right| + \left|\fint_{B_{2^k R}(x_1)} f - \fint_{B_{2^{k+1}R}(0)} f\right| + \left|\fint_{B_{2^{k+1}R}(0)} f - \fint_{B_{2R}(0)} f\right| \\ \leq\; &C\|f\|_{\mathcal{L}^{1,\mu}} R^{\mu-d}. \end{aligned}$$

Collecting these, we get (2.20). This completes the proof.

Next we consider the trajectory determined by an average of u:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\xi_u(t;x,R) = \fint_{B_R(x+\xi_u(t;x,R))} u(t), \qquad 0 \le t \le t_0, \\ \xi_u(0;x,R) = 0. \end{cases} \tag{2.21}$$

Here $x \in \mathbb{R}^d$ and $R > 0$. The solutions of ODEs of the form (2.21) play important roles in the analysis of (1.1). Set

$$F_R[u](t_0,x) = \sup_{0<t<t_0} \Big| \int_0^t \fint_{B_R(x)} u(s)\,\mathrm{d}s \Big|. \tag{2.22}$$

Proposition 2.3 *Assume that $u \in L^1(0,t_0;\mathcal{L}^{1,\mu}(\mathbb{R}^d))$ for some $\mu \in [0,d+1]$. Let $\xi_u(t;x,R)$ be the solution to (2.21). Then we have:*

(i)

$$\begin{aligned} &|\xi_u(t_0;x,R)| \\ \le &\begin{cases} CR^{\mu-d}\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})} + F_R[u](t_0,x) & \text{if } \ \mu \in [0,d), \\ C\Big(\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})} \log\big(e + R^{-1}\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}\big) + F_R[u](t_0,x)\Big) & \text{if } \ \mu = d, \\ C\Big(\|u\|^{\frac{1}{d+1-\mu}}_{L^1(0,t_0;\mathcal{L}^{1,\mu})} + F_R[u](t_0,x)\Big) & \text{if } \ \mu \in (d,d+1), \\ F_R[u](t_0,x)\exp(C\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}) & \text{if } \ \mu = d+1. \end{cases} \end{aligned} \tag{2.23}$$

Here C depends only on d and μ.

(ii) *For $x_1, x_2 \in \mathbb{R}^d$ and $R_1 \ge R_2 > 0$ it follows that*

$$\begin{aligned} &|\xi_u(t_0;x_1,R_1) - \xi_u(t_0;x_2,R_2)| \\ \le &\begin{cases} C\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})} R_2^{\mu-d} & \text{if } \ \mu \in [0,d), \\ C\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})} \log\Big(e + \dfrac{\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})} + |x_1-x_2|}{R_2} + \dfrac{R_1}{R_2}\Big) & \text{if } \ \mu = d, \\ C\Big(\|u\|^{\frac{1}{d+1-\mu}}_{L^1(0,t_0;\mathcal{L}^{1,\mu})} + \|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}\big(|x_1-x_2|^{\mu-d} + R_1^{\mu-d}\big)\Big) & \text{if } \ \mu \in (d,d+1), \\ C\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}\big(|x_1-x_2| + R_1\big)\exp(C\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}) & \text{if } \ \mu = d+1. \end{cases} \end{aligned} \tag{2.24}$$

Here C depends only on d and μ.

Proof. (i) By the definition of $\xi_u(t;x,R)$ we have

$$|\xi_u(t;x,R)| \ \le \ \int_0^t \Big| \fint_{B_R(x+\xi_u(s;x,R))} u(s) - \fint_{B_R(x)} u(s)\Big|\,\mathrm{d}s + \Big|\int_0^t \fint_{B_R(x)} u(s)\,\mathrm{d}s\Big|.$$

Then Lemma 2.2 yields

$$\begin{aligned} &|\xi_u(t;x,R)| \\ \le &\begin{cases} CR^{\mu-d}\displaystyle\int_0^t \|u(s)\|_{\mathcal{L}^{1,\mu}}\,\mathrm{d}s + \Big|\int_0^t \fint_{B_R(x)} u(s)\,\mathrm{d}s\Big| & \text{if } \ \mu \in [0,d), \\ C\displaystyle\int_0^t \|u(s)\|_{\mathcal{L}^{1,\mu}} \log\big(e + \frac{|\xi_u(s;x,R)|}{R}\big)\,\mathrm{d}s + \Big|\int_0^t \fint_{B_R(x)} u(s)\,\mathrm{d}s\Big| & \text{if } \ \mu = d, \\ C\displaystyle\int_0^t \|u(s)\|_{\mathcal{L}^{1,\mu}} |\xi_u(s;x,R)|^{\mu-d}\,\mathrm{d}s + \Big|\int_0^t \fint_{B_R(x)} u(s)\,\mathrm{d}s\Big| & \text{if } \ \mu \in (d,d+1]. \end{cases} \end{aligned} \tag{2.25}$$

Then the case $\mu \in [0,d)$ is obvious. When $\mu = d$ the above estimate implies that there is $\tilde{C} > 0$ depending only on C such that

$$\begin{aligned} &|\xi_u(t_0;x,R)| \\ \le \ &\tilde{C}\|u\|_{L^1(0,t_0;\mathcal{L}^{1,d})}\Big(\log(e + R^{-1}\|u\|_{L^1(0,t_0;\mathcal{L}^{1,d})}) + \log(e + R^{-1}\sup_{0<t<t_0}\Big|\int_0^t \fint_{B_R(x)} u(s)\,\mathrm{d}s\Big|)\Big) \\ &+ \sup_{0<t<t_0}\Big|\int_0^t \fint_{B_R(x)} u(s)\,\mathrm{d}s\Big|. \end{aligned}$$

When $\mu \in (d, d+1)$ there is $\tilde{C} > 0$ depending only on C and μ such that

$$|\xi_u(t_0; x, R)| \leq \tilde{C}\Big(\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}^{\frac{1}{d+1-\mu}} + \|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}\big(\sup_{0<t<t_0}|\int_0^t \fint_{B_R(x)} u(s)\,\mathrm{d}s|\big)^{\mu-d}\Big) + \sup_{0<t<t_0}|\int_0^t \fint_{B_R(x)} u(s)\,\mathrm{d}s|.$$

When $\mu = d+1$ the Gronwall inequality yields

$$|\xi_u(t_0; x, R)| \leq e^{C\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}} \sup_{0<t<t_0}|\int_0^t \fint_{B_R(x)} u(s)\,\mathrm{d}s|.$$

Then it is not difficult to get (2.23). This completes the proof of (i).

(ii) We may assume that $2^k R_2 \leq R_1 \leq 2^{k+1} R_2$ for some $k \in \mathbb{N} \cup \{0\}$. Set $\varphi_i(s) = \xi_u(s; x_i, R_i)$. Again from the ODEs, we have

$$\begin{aligned}|\varphi_1(t) - \varphi_2(t)| &\leq \int_0^{t_0} |\fint_{B_{R_1}(x_1+\varphi_1(s))} u(s) - \fint_{B_{R_2}(x_1+\varphi_1(s))} u(s)|\,\mathrm{d}s \\ &\quad + \int_0^t |\fint_{B_{R_2}(x_1+\varphi_1(s))} u(s) - \fint_{B_{R_2}(x_2+\varphi_2(s))} u(s)|\,\mathrm{d}s \\ &=: I_1(t_0) + I_2(t).\end{aligned}$$

Then Lemma 2.2 yields similar estimate as in the right-hand side of (2.25). Hence as in the proof of (i) we get

$$\begin{aligned}&|\varphi_1(t_0) - \varphi_2(t_0)| \\ &\leq \begin{cases} CR_2^{\mu-d}\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})} + I_1(t_0) & \text{if } \mu \in [0,d), \\ C\Big(\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}\log\big(e + R_2^{-1}\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}\big) \\ \qquad + \|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}\log(e + \dfrac{|x_1-x_2|}{R_2}) + I_1(t_0)\Big) & \text{if } \mu = d, \\ C\Big(\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}^{\frac{1}{d+1-\mu}} + \|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}|x_1-x_2|^{\mu-d} + I_1(t_0)\Big) & \text{if } \mu \in (d, d+1), \\ C\Big(\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}|x_1-x_2|^{\mu-d} + I_1(t_0)\Big)\exp(C\|u\|_{L^1(0,t_0;\mathcal{L}^{1,\mu})}) & \text{if } \mu = d+1. \end{cases}\end{aligned} \tag{2.26}$$

Here C depends only on d and μ. The term $I_1(t_0)$ is estimated from Lemma 2.2 and we get the desired estimates. This proves the assertion (ii).

2.3 Approximation of kernel

To prove the existence of fundamental solutions satisfying the desired estimates rigorously, we have to approximate the kernel $K(t,x,y)$ suitably. For this purpose it will be useful to define the class of smooth kernels K. Let C_0 be the number in (1.6). We say $K(t,x,y)$ is a smooth kernel of the order $\alpha' \in (1,2)$ if $K(t,x,y)$ is of the form

$$K(t,x,y) = |x-y|^{-d-\alpha}k(t,x,y) + \delta|x-y|^{-d-\alpha'}, \tag{2.27}$$

where $\delta > 0$, $\alpha' \geq \alpha$, and $k(t,x,y)$ is a function defined on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ such that

$$k(t,x,y) = k(t,y,x), \qquad \sup_{t\in\mathbb{R},x,y\in\mathbb{R}^d} \sum_{|\beta|\leq 1} |\nabla_{t,x,y}^{\beta} k(t,x,y)| < \infty, \qquad \inf_{t\in\mathbb{R},x,y\in\mathbb{R}^d} k(t,x,y) \geq C_0^{-1}. \tag{2.28}$$

If $K(t,x,y)$ is a smooth kernel and v is smooth and bounded, then it is not difficult to prove that there exists a unique and positive fundamental solution $P_{K,v}(t,x;s,y)$ to (1.1). In particular, when $f \in C_0^\infty(\mathbb{R}^d)$ the function $(P_{K,v}f)(t,s,x)$ defined by (2.5) solves (1.1) for $t > s$ in the classical sense rather than the weak sense. Moreover, under the assumption of $\nabla \cdot v(t) = 0$, $P_{K,v}(t,x;s,y)$ satisfies

$$\int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)\,\mathrm{d}y = \int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)\,\mathrm{d}x = 1, \tag{2.29}$$

$$P_{K,v}(t,x;s,y) = \int_{\mathbb{R}^d} P_{K,v}(t,x;\tau,\xi)P_{K,v}(\tau,\xi,s,y)\,\mathrm{d}\xi \qquad t > \tau > s \geq 0, \tag{2.30}$$

$$P_{K,v}(t,x;s,y) \leq C(t-s)^{-\frac{d}{\alpha}}, \tag{2.31}$$

$$\int_s^T \|P_{K,v}f(t,s)\|^2_{\dot H^{\frac{\alpha}{2}}}\,\mathrm{d}t \leq C\|f\|^2_{L^2}, \tag{2.32}$$

$$\|P_{K,v}f(t,s)\|_{L^p} \leq \|f\|_{L^p} \qquad\qquad 1 \leq p \leq \infty. \tag{2.33}$$

Here C depends only on d, α, and C_0; as for (2.31), see [26]. Especially, C is independent of α' and δ. Let $T_N(\sigma)$, $N \gg 1$ be a truncated function such that $T_N(\sigma) = \sigma$ if $\sigma \leq N$ and $T_N(\sigma) = N$ if $\sigma \geq N$. Then we define the approximation of $K(t,x,y)$ satisfying (1.5) and (1.6) by

$$K^{(N,\delta)}(t,x,y) = |x-y|^{-d-\alpha}(\nu_\delta * k_N)(t,x,y) + \delta|x-y|^{-d-1-\frac{\alpha}{2}}, \tag{2.34}$$

where

$$k_N(t,x,y) = T_N(|x-y|^{d+\alpha}K(t,x,y)), \quad t > 0, \qquad k_N(t,x,y) = C_0^{-1}, \quad t \leq 0, \tag{2.35}$$

which satisfies $k_N(t,x,y) \geq C_0^{-1}$ if $N \geq C_0^{-1}$ from the definition and (1.6). Here $\nu_\delta *$ is a mollifier on (t,x,y) variables such as

$$(\nu_\delta * k_N)(t,x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}^{2d}} \nu_{1,\delta}(t-s)\nu_{d,\delta}(x-\xi)\nu_{d,\delta}(y-\eta)k_N(s,\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}s, \tag{2.36}$$

where $\nu_{n,\delta}(z) = \delta^{-n}\nu_n(z/\delta)$ and $\nu_n(z)$ is a smooth non-negative function on $\mathbb{R}^n$ satisfying $\mathrm{supp}\,\nu_n \subset \{z \in \mathbb{R}^n \mid |z| \leq 2\}$, $\int_{\mathbb{R}^n} \nu_n(z)\,\mathrm{d}z = 1$. We note that the above mollification preserves the symmetry and the estimate such as

$$(\nu_\delta * k_N)(t,x,y) = (\nu_\delta * k_N)(t,y,x), \qquad (\nu_\delta * k_N)(t,x,y) \geq C_0^{-1}. \tag{2.37}$$

It is clear that $K^{(N,\delta)}(t,x,y)$ is a smooth kernel of the order $1+\alpha/2$. If $8\delta \leq M$ then we have

$$J_2(K_M^{(N,\delta)}) \leq C(\delta^{2-\alpha}N + M^{2-\alpha} + \delta M^{1-\frac{\alpha}{2}}), \tag{2.38}$$

$$J_0(K_{\vee,M}^{(N,\delta)}) \leq C(M^{-\alpha} + \delta M^{-1-\frac{\alpha}{2}}), \tag{2.39}$$

and if $M \leq 8\delta$ then

$$J_2(K_M^{(N,\delta)}) \leq C(\delta^{2-\alpha}N + \delta M^{1-\frac{\alpha}{2}}), \tag{2.40}$$

$$J_0(K_{\vee,M}^{(N,\delta)}) \leq C(M^{-\alpha} + \delta^{2-\alpha}M^{-2}N + \delta M^{-1-\frac{\alpha}{2}}), \tag{2.41}$$

where C depends only on d, α, and C_0. Indeed, from the definition of $K^{(N,\delta)}$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} |x-y|^2 K_M^{(N,\delta)}(t,x,y)\,\mathrm{d}y \\
\leq\ & \int_{|x-y|\leq M}\int \nu_{2d+1}(\tau,\xi,\eta)\frac{k_N(t-\delta\tau,x-\delta\xi,y-\delta\eta)}{|x-y|^{d+\alpha-2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\tau\,\mathrm{d}y \\
&\quad + \delta\int_{|x-y|\leq M} |x-y|^{1-\frac{\alpha}{2}-d}\,\mathrm{d}y \\
\leq\ & \int_{8\delta\leq|x-y|\leq M}\int \nu_{2d+1}(\tau,\xi,\eta)\frac{k_N(t-\delta\tau,x-\delta\xi,y-\delta\eta)}{|x-y|^{d+\alpha-2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\tau\,\mathrm{d}y \\
&\quad + \int_{|x-y|\leq 8\delta}\int \nu_{2d+1}(\tau,\xi,\eta)\frac{k_N(t-\delta\tau,x-\delta\xi,y-\delta\eta)}{|x-y|^{d+\alpha-2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\tau\,\mathrm{d}y + C\delta M^{1-\frac{\alpha}{2}} \\
=:\ & I_1 + I_2 + C\delta M^{1-\frac{\alpha}{2}}.
\end{aligned}$$

Here we have set $\nu_{2d+1}(\tau,\xi,\eta) = \nu_1(\tau)\nu_d(\xi)\nu_d(\eta)$ for simplicity of notations. As for I_1, since $|x-\delta\xi-(y-\delta\eta)| \geq |x-y| - \delta(|\xi|+|\eta|) \geq |x-y| - 4\delta \geq |x-y|/2$ for $|\xi|+|\eta| \leq 4$, we see

$$I_1 \leq \int \nu_{2d+1}(\tau,\xi,\eta)\int_{|x-\delta\xi-(y-\delta\eta)|\leq CM} \frac{k_N(t-\delta\tau,x-\delta\xi,y-\delta\eta)}{|x-\delta\xi-(y-\delta\eta)|^{d+\alpha-2}}\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\tau \leq CM^{2-\alpha},$$

by the definition of k_N and (1.5). As for I_2, from $k_N \leq N$ it is easy to get $I_2 \leq C\delta^{2-\alpha}N$. This completes the proof of (2.38). The estimates (2.39), (2.40), and (2.41) are proved similarly and the details are omitted here.

2.4 Approximation of velocity field

In order to prove the main theorems we need to approximate v by a smooth and bounded vector field satisfying $\nabla \cdot v(t) = 0$. To this end the next lemma is essential.

Lemma 2.4 *Let $\lambda \in [2d/\alpha - d, 2d/\alpha + d]$ and set $p_\lambda = 1$ if $\lambda \in [2d/\alpha - d, d]$ and $p_\lambda = \infty$ if $\lambda \in (d, 2d/\alpha + d]$. Let $q_1 \in [1, \infty)$, $q_2 \in [1, \infty]$, and $\mu \in [0, 1]$. Let $v \in L^{q_1,\mu}(0, \infty; (\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))^d) \cap L^{q_2}_{loc}(0, \infty; (L^{p_\lambda}_{loc}(\mathbb{R}^d))^d)$ be a given vector field satisfying $\nabla \cdot v(t) = 0$. Then there is a sequence of smooth and bounded vector fields $\{v^{(N)}\}$ satisfying $\nabla \cdot v^{(N)}(t) = 0$ and*

$$\sup_N \|v^{(N)}\|_{\mathcal{L}^{q_1,\mu}(L^{\frac{2d}{\alpha},\lambda})} \leq C_1 \|v\|_{L^{q_1,\mu}(\mathcal{L}^{\frac{2d}{\alpha},\lambda})}, \tag{2.42}$$

$$\limsup_{N\to\infty} \|v^{(N)}\|_{L^{q_2}(0,R_1;L^{p_\lambda}(B_{R_2}(x)))} \leq C_2 \|v\|_{L^{q_2}(0,2R_1;L^{p_\lambda}(B_{2R_2}(x)))} \qquad R_1,\ R_2 > 0,\ x \in \mathbb{R}^d, \tag{2.43}$$

and

$$v^{(N)} \to v \quad \text{in} \quad (L^1_{loc}((0,\infty) \times \mathbb{R}^d))^d. \tag{2.44}$$

Here C_1 depends only on d, α, λ, μ, and q_1, and C_2 depends only on d and q_2.

The proof of Lemma 2.4 will be given in the appendix.

3 A priori estimates for fundamental solutions

In this section we establish the a priori estimates on the regularity of $P_{K,v}(t, x; s, y)$.

3.1 Moment bound

We first consider the case that $K(t, x, y)$ and v are smooth. For $R > 0$ let $\chi_R(x)$, $0 \leq \chi_R \leq 1$, be a smooth cut-off function such that $\chi_R(x) = 1$ if $|x| \leq R$ and $\chi_R(x) = 0$ if $|x| \geq 2R$, and $|\nabla \chi_R(x)| \leq C/R$. For $\gamma \in (0, 2]$ we set

$$\rho_\gamma(x) = |x|^{\frac{\gamma}{2}}, \qquad \rho_{\gamma,R}(x) = \rho_\gamma(x)\chi_R(x). \tag{3.1}$$

Let K be a smooth kernel of the order $\alpha' \in (1, 2)$. Then it is not difficult to see that if $\gamma < \alpha$ then $\|K\|_\gamma < \infty$. For $\lambda \in [2d/\alpha - d, 2d/\alpha + d]$ we set

$$\|u\|_{X_\lambda(s,t)} = \sup_{s \leq s' < t' \leq t} (t' - s')^{\frac{\lambda}{2d} - \frac{1}{\alpha} - \frac{1}{2}} \int_{s'}^{t'} \|u(\tau)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} \, d\tau. \tag{3.2}$$

Note that $\|j_t u\|_{X_\lambda(0,t-s)} = \|u\|_{X_\lambda(s,t)}$ holds, where j_t is defined by (2.14).

Proposition 3.1 *Let $K(\tau, y, x)$ be a smooth kernel of the order $\alpha' \in (1, 2)$. Let $\lambda \in [2d/\alpha - d, 2d/\alpha + d]$ and $\gamma \in (0, \min\{\alpha, 2 - \alpha\})$. Assume that u is a smooth and bounded vector field satisfying $\nabla \cdot u(\tau) = 0$ in $(0, 1) \times \mathbb{R}^d$. Let $\xi_u(\tau; x) \in \mathbb{R}^d$, $0 \leq \tau \leq 1$ be the solution to the ODE (2.21) with $R = 1$. Then for any $\tau \in (0, 1]$,*

$$C_1 \leq m_\gamma(\tau; x) := \int_{\mathbb{R}^d} \rho_\gamma(y - x - \xi_u(\tau; x)) P_{K,u}(\tau, y; 0, x) \, dy \leq C_2(1 + \|K\|_\gamma). \tag{3.3}$$

Here C_1 depends only on d, α, and γ, and C_2 depends only on d, α, γ, λ, and $\|u\|_{X_\lambda(0,1)}$.

Proof. Without loss of generality we may assume $x = 0$. For simplicity of notations we write $P(\tau, y)$ and $\xi(\tau)$ for $P_{K,u}(\tau, y; 0, 0)$ and $\xi_u(\tau; 0)$, respectively. Then set

$$m_{\gamma,R}(\tau) = \int_{\mathbb{R}^d} \rho_{\gamma,R}(y - \xi(\tau)) P(\tau, y) \, dy, \qquad 0 < \tau \leq 1. \tag{3.4}$$

We first observe that

$$P(\tau, \tilde{y}) \leq C\tau^{-\frac{d}{\alpha}} =: e^b \tau^{-\frac{d}{\alpha}}, \tag{3.5}$$

where b depends only on d, α, and C_0. As in [28, 24, 25], we set

$$Q(\tau) = -\int_{\mathbb{R}^d} P(\tau, y) \log P(\tau, y) \, dy, \tag{3.6}$$

$$f(\tau) = Q(\tau) - \frac{d}{\alpha} \log \tau + b, \tag{3.7}$$

$$h(\tau) = \int_1^\tau \mathcal{E}_K^{(r)}(P(r), \log P(r)) \, dr - \frac{d}{\alpha} \log \tau. \tag{3.8}$$

Lemma 3.2 *For any $0 \le \tau_2 < \tau_1 \le 1$, we have*

$$
\begin{aligned}
Q(\tau_1) - Q(\tau_2) &\ge \int_{\tau_2}^{\tau_1} \mathcal{E}_K^{(r)}(P(r), \log P(r))\,\mathrm{d}r, &(3.9)\\
f(\tau) &\ge 0, &(3.10)\\
f(\tau_1) - f(\tau_2) &\ge h(\tau_1) - h(\tau_2). &(3.11)
\end{aligned}
$$

Proof of Lemma 3.2. For every $\epsilon > 0$ and $n \ge 1$ we have

$$
\begin{aligned}
&-\int_{\mathbb{R}^d} \chi_n(y) P(\tau_1, y) \log(P(\tau_1, y) + \epsilon)\,\mathrm{d}y + \int_{\mathbb{R}^d} \chi_n(y) P(\tau_2, y) \log(P(\tau_2, y) + \epsilon)\,\mathrm{d}y\\
=\;& -\int_{\tau_2}^{\tau_1} \int_{\mathbb{R}^d} \partial_r P(r, y) \chi_n(y) \{\log(P(r, y) + \epsilon) + \frac{P(r, y)}{P(r, y) + \epsilon}\}\,\mathrm{d}y\\
=\;& \int_{\tau_2}^{\tau_1} \mathcal{E}_K^{(r)}\big(P(r), \chi_n(\log(P(r) + \epsilon) + \frac{P(r)}{P(r) + \epsilon})\big)\,\mathrm{d}r\\
&- \int_{\tau_2}^{\tau_1} \int_{\mathbb{R}^d} u(r, y) \cdot \nabla \chi_n(y) P(r, y) \log(P(r, y) + \epsilon)\,\mathrm{d}y\,\mathrm{d}r.
\end{aligned}
$$

From the boundedness of u, it is not difficult to see that the last term in the right-hand side of the above equality goes to zero as $n \to \infty$. Thus we have

$$
\begin{aligned}
&-\int_{\mathbb{R}^d} P(\tau_1, y) \log(P(\tau_1, y) + \epsilon)\,\mathrm{d}y + \int_{\mathbb{R}^d} P(\tau_2, y) \log(P(\tau_2, y) + \epsilon)\,\mathrm{d}y\\
=\;& \int_{\tau_2}^{\tau_1} \mathcal{E}_K^{(r)}(P(r), \log(P(r) + \epsilon) + \frac{P(r)}{P(r) + \epsilon})\,\mathrm{d}r\\
\ge\;& \int_{\tau_2}^{\tau_1} \mathcal{E}_K^{(r)}(P(r), \log(P(r) + \epsilon))\,\mathrm{d}r.
\end{aligned}
$$

Letting $\epsilon \to 0$, we get (3.9). The inequality (3.11) follows from (3.9). This completes the proof of Lemma 3.2.

Lemma 3.3 *There is $C > 0$ depending only on d, α, and γ such that*

$$
m_{\gamma,\infty}(\tau) \ge C e^{\frac{\gamma}{2d} Q(\tau)}. \tag{3.12}
$$

Proof of Lemma 3.3. The proof is the same as in [28]. We use the inequality $T \log T + \lambda T \ge -e^{-\lambda - 1}$ which holds for $T,\ \lambda > 0$. By setting $\lambda = a_1 \rho_\gamma(y - \xi(\tau)) + a_2$ with $a_1,\ a_2 \ge 0$, we have

$$
\begin{aligned}
-Q(\tau) + a_1 m_{\gamma,\infty}(\tau) + a_2 &= \int_{\mathbb{R}^d} P\big(\log P + a_1 \rho_\gamma(y - \xi(\tau)) + a_2\big)\,\mathrm{d}y\\
&\ge -\int_{\mathbb{R}^d} e^{-a_1 \rho_\gamma(y - \xi(\tau)) - a_2 - 1}\,\mathrm{d}y\\
&= -C e^{-a_2} \int_0^\infty e^{-a_1 r^{\frac{\gamma}{2}}} r^{d-1}\,\mathrm{d}r = -C e^{-a_2} a_1^{-\frac{2d}{\gamma}}.
\end{aligned}
$$

Hence by taking $a_1 = m_{\gamma,\infty}^{-1}$ and $a_2 = 2d\gamma^{-1} \log m_{\gamma,\infty}$, we obtain

$$
-Q(\tau) + 1 + \frac{2d}{\gamma} \log m_{\gamma,\infty} \ge -c.
$$

This completes the proof of Lemma 3.3.

Proof of Proposition 3.1(continued). From (3.7), (3.10), and Lemma 3.3, it is easy to obtain

$$
m_{\gamma,\infty}(\tau) \ge C_1 \tau^{\frac{\gamma}{2\alpha}}, \qquad \tau \in (0, 1], \tag{3.13}
$$

where C_1 depends only on d, α, C_0, and γ. This proves the lower bound in (3.3). Next we consider the upper bound. Let $R,\ M \ge 1$. From the definition of $m_{2,R}(\tau)$ we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau} m_{2,R}(\tau) &= -\mathcal{E}_{K_M}^{(\tau)}(\rho_{2,R}(\cdot - \xi(\tau)), P(\tau)) - \mathcal{E}_{K_{\vee,M}}^{(\tau)}(\rho_{2,R}(\cdot - \xi(\tau)), P(\tau))\\
&\quad + \int_{\mathbb{R}^d} (u(\tau, y) - \frac{\mathrm{d}}{\mathrm{d}\tau}\xi(\tau)) \cdot \nabla_y \rho_{2,R}(y - \xi(\tau)) P(\tau, y)\,\mathrm{d}y\\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

(i) Estimate of I_1: The inequality $(1-\theta)^2 \le -(1-\theta)\log\theta$ for $\theta \in (0,1]$ yields

$$
\begin{aligned}
I_1 \quad &\le \quad \int_{P(\tau,y)\ge P(\tau,z)} |[\rho_{2,R}(\cdot-\xi(\tau)]|(1-P(\tau,z)/P(\tau,y))P(\tau,y)K_M(\tau,y,z)\,\mathrm{d}z\,\mathrm{d}y \\
&\quad + \int_{P(\tau,y)<P(\tau,z)} |[\rho_{2,R}(\cdot-\xi(\tau)]|(1-P(\tau,z)/P(\tau,y))P(\tau,y)K_M(\tau,y,z)\,\mathrm{d}z\,\mathrm{d}y \\
&\le \quad \Big(\int_{P(\tau,y)\ge P(\tau,z)} |[\rho_{2,R}(\cdot-\xi(\tau)]|^2 P(\tau,y)K_M(\tau,y,z)\,\mathrm{d}z\,\mathrm{d}y\Big)^{\frac{1}{2}} \\
&\qquad \cdot \Big(\int_{P(\tau,y)\ge P(\tau,z)} (1-P(\tau,z)/P(\tau,y))^2 P(\tau,y)K_M(\tau,y,z)\,\mathrm{d}z\,\mathrm{d}y\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_{P(\tau,y)<P(\tau,z)} |[\rho_{2,R}(\cdot-\xi(\tau)]|^2 P(\tau,z)K_M(\tau,y,z)\,\mathrm{d}z\,\mathrm{d}y\Big)^{\frac{1}{2}} \\
&\qquad \cdot \Big(\int_{P(\tau,y)<P(\tau,z)} (1-P(\tau,y)/P(\tau,z))^2 P(\tau,z)K_M(\tau,y,z)\,\mathrm{d}z\,\mathrm{d}y\Big)^{\frac{1}{2}} \\
&\le \quad CJ_2^{\frac{1}{2}}(K_M)\big(\mathcal{E}_K^{(\tau)}(P(\tau),\log P(\tau))\big)^{\frac{1}{2}}.
\end{aligned}
\tag{3.14}
$$

Here we have also used $|[\rho_{2,R}(\cdot-\xi(\tau)](y,z)| \le C|y-z|$.

(ii) Estimate of I_2: Since $\rho_{2,R}$, P, and $K_{\vee,M}$ are nonnegative functions, we have

$$
\begin{aligned}
I_2 \quad &= \quad -\int_{\mathbb{R}^{2d}} [\rho_{2,R}(\cdot-\xi(\tau))](y,z)[P(\tau,\cdot)](y,z)K_{\vee,M}(\tau,y,z)\,\mathrm{d}y\,\mathrm{d}z \\
&\le \quad 2\int_{\mathbb{R}^{2d}} \rho_{2,R}(y-\xi(\tau))P(\tau,z)K_{\vee,M}(\tau,y,z)\,\mathrm{d}y\,\mathrm{d}z \\
&\le \quad CRJ_0(K_{\vee,M}).
\end{aligned}
\tag{3.15}
$$

(iii) Estimate of I_3: We first consider the case $\lambda \in [2d/\alpha - d, d]$. Then, since $|\nabla_y \rho_{2,R}(y)| \le C$, we have

$$
\begin{aligned}
I_3 \quad &\le \quad C\int_{|y-\xi(\tau)|\le 2R} |u(\tau,y) - \fint_{B_1(\xi(\tau))} u(\tau)|P(\tau,y)\,\mathrm{d}y \\
&\le \quad C\int_{|y-\xi(\tau)|\le 2R} |u(\tau,y) - \fint_{B_{2R}(\xi(\tau))} u(\tau)|P(\tau,y)\,\mathrm{d}y + C|\fint_{B_{2R}(\xi(\tau))} u(\tau) - \fint_{B_1(\xi(\tau))} u(\tau)| \\
&\le \quad C\|u(\tau)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\big(\tau^{-\frac{1}{2}}R^{\frac{\alpha\lambda}{2d}} + \log R\big) \\
&\le \quad C\|u(\tau)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\tau^{-\frac{1}{2}}R^{\frac{\alpha\lambda}{2d}}.
\end{aligned}
\tag{3.16}
$$

In the second last line in (3.16), the term $\log R$ appears only when $\lambda = d$. Next we consider the case $\lambda \in (d, 2d/\alpha + d]$. In this case we have $\mathcal{L}^{2d/\alpha,\lambda}(\mathbb{R}^d) = \dot{C}^{\beta_\lambda}(\mathbb{R}^d)$ with equivalent seminorms, where $\beta_\lambda = \alpha\lambda/(2d) - \alpha/2 \in (0,1]$. Hence it follows that

$$
\begin{aligned}
I_3 \quad &\le \quad C\int_{\mathbb{R}^d} |u(\tau,y) - \fint_{B_1(\xi(\tau))} u(\tau)||\nabla\rho_2(y-\xi(\tau))|\chi_R(y-\xi(\tau))P(\tau,y)\,\mathrm{d}y \\
&\qquad + \int_{\mathbb{R}^d} |u(\tau,y) - \fint_{B_1(\xi(\tau))} u(\tau)||y-\xi(\tau)||\nabla\chi_R(y-\xi(\tau))|P(\tau,y)\,\mathrm{d}y \\
&\le \quad C\|u(\tau)\|_{\dot{C}^{\beta_\lambda}} \int_{\mathbb{R}^d} |y-\xi(\tau)|^{\beta_\lambda}\chi_R(y-\xi(\tau))P(\tau,y)\,\mathrm{d}y + C\|u(\tau)\|_{\dot{C}^{\beta_\lambda}} \\
&\qquad + C\|u(\tau)\|_{\dot{C}^{\beta_\lambda}} \int_{R\le|y-\xi(\tau)|\le 2R} |y-\xi(\tau)|^{\beta_\lambda}P(\tau,y)\,\mathrm{d}y \\
&\le \quad C\|u(\tau)\|_{\dot{C}^{\beta_\lambda}}\big(1 + m_{2,R}(\tau) + R^{\beta_\lambda}\int_{R\le|y-\xi(\tau)|\le 2R} P(\tau,y)\,\mathrm{d}y\big).
\end{aligned}
\tag{3.17}
$$

For the case $\lambda \in [2d/\alpha - d, d]$, (3.14) - (3.17) yield

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau^{\frac{1}{2}}m_{2,R}(\tau)) \quad &\le \quad C\tau^{\frac{1}{2}}J_2(K_M)^{\frac{1}{2}}\big(\mathcal{E}_K^{(\tau)}(P(\tau),\log P(\tau))\big)^{\frac{1}{2}} + C\tau^{\frac{1}{2}}RJ_0(K_{\vee,M}) \\
&\qquad + C\|u(\tau)\|_{\mathcal{L}^{p,\frac{2d}{\alpha}}}R^{\frac{\alpha\lambda}{2d}} + \frac{1}{2}\tau^{-\frac{1}{2}}m_{2,R}(\tau).
\end{aligned}
\tag{3.18}
$$

Regarding the case $\lambda \in (d, 2d/\alpha + d]$, one gets

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\big(e^{-C\int_0^\tau \|u(s)\|_{\dot{C}^{\beta_\lambda}}\,\mathrm{d}s} m_{2,R}(\tau)\big) &\le CJ_2(K_R)^{\frac12}\big(\mathcal{E}_K^{(\tau)}(P(\tau), \log P(\tau))\big)^{\frac12} + CRJ_0(K_{\vee,R}) \\ &+ C\|u(\tau)\|_{\dot{C}^{\beta_\lambda}}\big(1 + R^{\beta_\lambda}\int_{R\le|y-\xi(\tau)|\le 2R} P(\tau, y)\,\mathrm{d}y\big). \qquad (3.19)\end{aligned}$$

By the definition of $h(\tau)$ we have

$$\mathcal{E}_K^{(\tau)}(P(\tau), \log P(\tau))^{\frac12} \le \Big(\frac{d}{\alpha\tau} + \frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau}\Big)^{\frac12} \le \tau^{-\frac12}\Big(C + \tau\frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau}\Big).$$

This with (3.10) and (3.11) gives

$$\begin{aligned}\int_0^\tau \mathcal{E}_K^{(s)}(P(s), \log P(s))^{\frac12}\,\mathrm{d}s &\le \int_0^\tau s^{-\frac12}\Big(C + s\frac{\mathrm{d}h(s)}{\mathrm{d}s}\Big)\,\mathrm{d}s \\ &\le C\tau^{\frac12} + \tau^{\frac12}h(\tau) - \frac12\int_0^\tau s^{-\frac12}h(s)\,\mathrm{d}s \\ &= C\tau^{\frac12} + \frac12\int_0^\tau s^{-\frac12}(h(\tau) - h(s))\,\mathrm{d}s \\ &\le C\tau^{\frac12} + \frac12\int_0^\tau s^{-\frac12}(f(\tau) - f(s))\,\mathrm{d}s \le \tau^{\frac12}(C + f(\tau)). \qquad (3.20)\end{aligned}$$

Then for $\lambda \in [2d/\alpha - d, d]$, (3.18) and (3.20) yield

$$\begin{aligned}\tau^{\frac12}m_{2,R}(\tau) \le\ & C\tau J_2^{\frac12}(K_M)(1 + f(\tau)) + C\tau^{\frac32}RJ_0(K_{\vee,M}) + CR^{\frac{\alpha\lambda}{2d}}\int_0^\tau \|u(s)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\,\mathrm{d}s \\ &+ \frac12\int_0^\tau s^{-\frac12}m_{2,R}(s)\,\mathrm{d}s.\end{aligned}$$

Thus from the Gronwall-type inequality we obtain

$$\begin{aligned}\tau^{\frac12}m_{2,R}(\tau) \le\ & C\tau J_2^{\frac12}(K_M)(1 + f(\tau)) + C\tau^{\frac32}RJ_0(K_{\vee,M}) + CR^{\frac{\alpha\lambda}{2d}}\int_0^\tau \|u(s)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\,\mathrm{d}s \\ &+ C\tau^{\frac12}\int_0^\tau s^{-\frac32}\Big(sJ_2^{\frac12}(K_M)(1 + f(s)) + s^{\frac32}RJ_0(K_{\vee,M}) + CR^{\frac{\alpha\lambda}{2d}}\int_0^s \|u(r)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\,\mathrm{d}r\Big)\,\mathrm{d}s \\ \le\ & CJ_2^{\frac12}(K_M)\Big(\tau(1 + f(\tau)) + \tau^{\frac12}\int_0^\tau s^{-\frac12}(1 + f(s))\,\mathrm{d}s\Big) + C\tau^{\frac32}RJ_0(K_{\vee,M}) \\ &+ CR^{\frac{\alpha\lambda}{2d}}\tau^{\frac12 + \frac1\alpha - \frac{\lambda}{2d}}\|u\|_{X_\lambda(0,1)}. \qquad (3.21)\end{aligned}$$

For $\lambda \in (d, 2d/\alpha + d]$ (3.19) and (3.20) give

$$\begin{aligned}m_{2,R}(\tau) \le\ & Ce^{C\int_0^\tau \|u(s)\|_{\dot{C}^{\beta_\lambda}}\,\mathrm{d}s}\Big(J_2(K_M)^{\frac12}\tau^{\frac12}(1 + f(\tau)) + RJ_0(K_{\vee,M})\tau \\ &+ \int_0^\tau \|u(s)\|_{\dot{C}^{\beta_\lambda}}\big(1 + R^{\beta_\lambda}\int_{R\le|y-\xi(s)|\le 2R} P(s, y)\,\mathrm{d}y\big)\,\mathrm{d}s\Big). \qquad (3.22)\end{aligned}$$

We choose $\gamma \in (0, 2)$ so that

$$\sum_{k=0}^\infty \big(k^2 2^{k(\gamma-2)}J_2(K_{2^k}) + 2^{\frac{\gamma k}{2}}J_0(K_{\vee,2^k}) + 2^{k\{\frac\gamma2 - 1 + \frac{\alpha\lambda}{2d}\}}\big) < \infty, \qquad \text{if } \lambda \in [2d/\alpha - d, d], \qquad (3.23)$$

and

$$\sum_{k=0}^\infty \big(k^2 2^{k(\gamma-2)}J_2(K_{2^k}) + 2^{\frac{\gamma k}{2}}J_0(K_{\vee,2^k})\big) < \infty, \qquad \text{if } \lambda \in (d, 2d/\alpha + d]. \qquad (3.24)$$

Note that $\gamma \in (0, \min\{\alpha, 2 - \alpha\})$ always satisfies (3.23) and (3.24). Then for $\lambda \in [2d/\alpha - d, d]$, (3.7) and

Lemma 3.3 imply

$$
\begin{aligned}
& m_{\gamma,\infty}(\tau) \\
= \ & \int_{|y-\xi(\tau)|\le 1} \rho_\gamma(y-\xi(\tau))P(\tau,y)\,\mathrm{d}y + \sum_{k=1}^{\infty}\int_{2^{k-1}\le|y-\xi(\tau)|\le 2^k} \rho_\gamma(y-\xi(\tau))P(\tau,y)\,\mathrm{d}y \\
\le \ & \int_{|y-\xi(\tau)|\le 1} \rho_\gamma(y-\xi(\tau))P(\tau,y)\,\mathrm{d}y + \sum_{k=1}^{\infty} 2^{(k-1)(\frac{\gamma}{2}-1)} m_{2,2^k}(\tau) \\
\le \ & 1 + C\Big(\tau^{\frac12}(1+f(\tau)) + \int_0^\tau s^{-\frac12}(1+f(s))\,\mathrm{d}s\Big)\sum_{k=0}^{\infty} 2^{k(\frac{\gamma}{2}-1)} J_2^{\frac12}(K_{2^k}) + C\tau\sum_{k=0}^{\infty} 2^{\frac{\gamma k}{2}} J_0(K_{\vee,2^k}) \\
& + C\tau^{\frac1\alpha-\frac{\lambda}{2d}}\|u\|_{X_\lambda(0,1)}\sum_{k=1}^{\infty} 2^{k\{\frac{\gamma}{2}-1+\frac{\alpha\lambda}{2d}\}} \\
\le \ & 1 + C\Big(\tau^{\frac12}\big(1-\frac{d}{\alpha}\log\tau + \frac{2d}{\gamma}\log m_{\gamma,\infty}(\tau)\big) + \frac{2d}{\gamma}\int_0^\tau s^{-\frac12}\log m_{\gamma,\infty}(s)\,\mathrm{d}s\Big)\sum_{k=1}^{\infty} 2^{k(\frac{\gamma}{2}-1)} J_2^{\frac12}(K_{2^k}) \\
& + C\tau\sum_{k=1}^{\infty} 2^{\frac{\gamma k}{2}} J_0(K_{\vee,2^k}) + C\tau^{\frac1\alpha-\frac{\lambda}{2d}}\|u\|_{X_\lambda(0,1)}\sum_{k=1}^{\infty} 2^{k\{\frac{\gamma}{2}-1+\frac{\alpha\lambda}{2d}\}}.
\end{aligned}
\tag{3.25}
$$

Similarly, for $\lambda\in(d,2d/\alpha+d]$, we have

$$
\begin{aligned}
m_{\gamma,\infty}(\tau) \ \le \ & 1 + Ce^{C\int_0^\tau \|u(s)\|_{\dot{C}^{\beta_\lambda}}\,\mathrm{d}s}\Big\{\tau^{\frac12}(1+f(\tau))\sum_{k=1}^{\infty} 2^{k(\frac{\gamma}{2}-1)} J_2^{\frac12}(K_{2^k}) + \tau\sum_{k=1}^{\infty} 2^{\frac{\gamma k}{2}} J_0(K_{\vee,2^k}) \\
& + \int_0^\tau \|u(s)\|_{\dot{C}^{\beta_\lambda}}\big(\sum_{k=1}^{\infty} 2^{k(\frac{\gamma}{2}-1)} + \sum_{k=1}^{\infty} 2^{k(\frac{\gamma}{2}-1+\beta_\lambda)}\int_{2^k\le|y-\xi(\tau)|\le 2^{k+1}} P(s,y)\,\mathrm{d}y\big)\,\mathrm{d}s\Big\} \\
\le \ & 1 + Ce^{C\int_0^\tau \|u(s)\|_{\dot{C}^{\beta_p}}\,\mathrm{d}s}\Big\{\tau^{\frac12}\big(1-\frac{d}{\alpha}\log\tau + \frac{2d}{\gamma}\log m_{\gamma,\infty}(\tau)\big)\sum_{k=1}^{\infty} 2^{k(\frac{\gamma}{2}-1)} J_2^{\frac12}(K_{2^k}) \\
& + \tau\sum_{k=1}^{\infty} 2^{\frac{\gamma k}{2}} J_0(K_{\vee,2^k}) + \int_0^\tau \|u(s)\|_{\dot{C}^{\beta_\lambda}}\big(1+m_{\gamma,\infty}(s)\big)\,\mathrm{d}s\Big\}.
\end{aligned}
\tag{3.26}
$$

Here the constants C in (3.25) and (3.26) depend only on d, α, C_0, γ, and λ. Thus (3.25) - (3.26) and the definition of $\|K\|_\gamma$ in (2.12) imply

$$
m_{\gamma,\infty}(\tau) \le C\Big(1 + \big(\sum_{k=1}^{\infty} 2^{k(\frac{\gamma}{2}-1)} J_2^{\frac12}(K_{2^k})\big)^2 + \sum_{k=1}^{\infty} 2^{\frac{\gamma k}{2}} J_0(K_{\vee,2^k})\Big) \le C(1+\|K\|_\gamma),
\tag{3.27}
$$

where C depends only on d, α, C_0, γ, λ, and $\|u\|_{X_\lambda(0,1)}$. This completes the proof.

Proposition 3.4 *Let $K(\tau,y,x)$ be a kernel satisfying (1.5) and (1.6), and let $K^{(N,\delta)}(\tau,y,x)$ is the approximation of K given by Section 2.3. Fix $t>s\ge 0$. Let $v(\tau,y)$ be a smooth and bounded vector field satisfying $\nabla\cdot v(\tau)=0$ in $(s,t)\times\mathbb{R}^d$. Let $\gamma\in(0,\min\{\alpha,2-\alpha\})$. Then for any $x\in\mathbb{R}^d$ there is $\xi_v^*=\xi_v^*(x,t,s)\in\mathbb{R}^d$ such that*

$$
\begin{aligned}
C_1(t-s)^{\frac{\gamma}{2\alpha}} \ \le \ & \int_{\mathbb{R}^d}\rho_\gamma(y-x-\xi_v^*)P_{K^{(N,\delta)},v}(t,y;s,x)\,\mathrm{d}y \\
\le \ & C_2(t-s)^{\frac{\gamma}{2\alpha}}\big(1+\delta^{2-\alpha}(t-s)^{1-\frac{2}{\alpha}}N + \delta(t-s)^{\frac12-\frac1\alpha}\big).
\end{aligned}
\tag{3.28}
$$

Here C_1 depends only on d, α, C_0, and γ, and C_2 depends only on d, α, C_0, γ, λ, and $\|v\|_{X_\lambda(s,t)}$. Moreover, it holds that

$$
\begin{aligned}
& |\xi_v^*(x,t,s)| \\
\le & \begin{cases}
C(t-s)^{\frac1\alpha}\|v\|_{X_\lambda(s,t)} + F_{(t-s)^{\frac1\alpha}}[v(\cdot+s)](t-s,x) & \text{if } \ \lambda\in[\frac{2d}{\alpha}-d,d), \\
C\Big((t-s)^{\frac1\alpha}\|v\|_{X_\lambda(s,t)}\log\big(e+\|v\|_{X_\lambda(s,t)}\big) + F_{(t-s)^{\frac1\alpha}}[v(\cdot+s)](t-s,x)\Big) & \text{if } \ \lambda=d, \\
C\Big((t-s)^{\frac1\alpha}\|v\|_{X_\lambda(s,t)}^{\frac{2d}{2d-\alpha(\lambda-d)}} + F_{(t-s)^{\frac1\alpha}}[v(\cdot+s)](t-s,x)\Big) & \text{if } \ \lambda\in(d,\frac{2d}{\alpha}+d), \\
F_{(t-s)^{\frac1\alpha}}[v(\cdot+s)](t-s,x)\exp(C\|v\|_{X_\lambda(s,t)}) & \text{if } \ \lambda=\frac{2d}{\alpha}+d.
\end{cases}
\end{aligned}
\tag{3.29}
$$

Here $F_R[u](t,x)$ is the function defined by (2.22), and C depends only on d, α, and λ.

Proof. If we put

$$\tilde{K}(\tau,\tilde{y},\tilde{x})=(t-s)^{1+\frac{d}{\alpha}}(R_{t-s}i_sK^{(N,\delta)})(\tau,\tilde{y},\tilde{x}),\qquad \tilde{v}(\tau,\tilde{y})=(t-s)^{1-\frac{1}{\alpha}}(R_{t-s}i_sv)(\tau,\tilde{y}),\tag{3.30}$$

then the scaling property implies

$$P_{K^{(N,\delta)},v}(t,y;s,x)=(t-s)^{-\frac{d}{\alpha}}P_{\tilde{K},\tilde{v}}(1,(t-s)^{-\frac{1}{\alpha}}y;0,(t-s)^{-\frac{1}{\alpha}}x)\tag{3.31}$$

By the definition of $K^{(N,\delta)}$ we have

$$\tilde{K}(\tau,\tilde{y},\tilde{x})=|\tilde{y}-\tilde{x}|^{-d-\alpha}(R_{t-s}i_s(\nu_\delta*k_N))(\tau,\tilde{y},\tilde{x})+\delta(t-s)^{\frac{1}{2}-\frac{1}{\alpha}}|\tilde{y}-\tilde{x}|^{-d-1-\frac{\alpha}{2}}.\tag{3.32}$$

Since $(R_{t-s}i_s(\nu_\delta*k_N))(\tau,\tilde{y},\tilde{x})\geq C_0^{-1}$, the function $\tilde{K}(\tau,\tilde{y},\tilde{x})$, $0\leq\tau\leq 1$, is a smooth kernel of the order $1+\alpha/2$ but with $\delta(t-s)^{1/2-1/\alpha}$ instead of δ in (2.27). Moreover, the equality $\|(t-s)^{1-\frac{1}{\alpha}}(R_{t-s}i_sv)(\tau)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}=(t-s)^{\frac{\lambda}{2d}+\frac{1}{2}-\frac{1}{\alpha}}\|v((t-s)\tau+s)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}$ implies $\|\tilde{v}\|_{X_\lambda(0,1)}=\|v\|_{X_\lambda(s,t)}$.

For any $\tilde{x}\in\mathbb{R}^d$ let $\xi_{\tilde{v}}(\tau;\tilde{x})\in\mathbb{R}^d$, $\tau\in[0,1]$, be the solution to the ODE (2.21) with $u=\tilde{v}$. Then by Proposition 3.1 we have for $\tau\in(0,1]$,

$$C_1\leq\int_{\mathbb{R}^d}\rho_\gamma(\tilde{y}-\tilde{x}-\xi_{\tilde{v}}(\tau;\tilde{x}))P_{\tilde{K},\tilde{v}}(\tau,\tilde{y};0,\tilde{x})\,\mathrm{d}\tilde{y}\leq C_2(1+\|\tilde{K}\|_\gamma).$$

From (2.38) - (2.41) we see

$$J_2(\tilde{K}_{2^k})\leq C\big(2^{k(2-\alpha)}+\delta^{2-\alpha}(t-s)^{1-\frac{2}{\alpha}}N+\delta(t-s)^{\frac{1}{2}-\frac{1}{\alpha}}2^{k(1-\frac{\alpha}{2})}\big),\tag{3.33}$$
$$J_0(\tilde{K}_{\vee,2^k})\leq C\big(2^{-\alpha k}+2^{-2k}\delta^{2-\alpha}(t-s)^{1-\frac{2}{\alpha}}N+\delta(t-s)^{\frac{1}{2}-\frac{1}{\alpha}}2^{-k(1+\frac{\alpha}{2})}\big).\tag{3.34}$$

This gives the estimate

$$\|\tilde{K}\|_\gamma\leq C(1+\delta^{2-\alpha}(t-s)^{1-\frac{2}{\alpha}}N+\delta(t-s)^{\frac{1}{2}-\frac{1}{\alpha}}).\tag{3.35}$$

Hence we have

$$C_1\leq\int_{\mathbb{R}^d}\rho_\gamma(\tilde{y}-\tilde{x}-\xi_{\tilde{v}}(\tau;\tilde{x}))P_{\tilde{K},\tilde{v}}(\tau,\tilde{y};0,\tilde{x})\,\mathrm{d}\tilde{y}\leq CC_2\big(1+\delta^{2-\alpha}(t-s)^{1-\frac{2}{\alpha}}N+\delta(t-s)^{\frac{1}{2}-\frac{1}{\alpha}}\big).\tag{3.36}$$

This implies the desired estimate by taking $\tilde{x}=(t-s)^{-1/\alpha}x$ and by setting $\xi_v^*(x,t,s)=(t-s)^{1/\alpha}\xi_{\tilde{v}}(1;\tilde{x})$. By the definition of $\|\cdot\|_{X_\lambda(s,t)}$ and the embedding $\mathcal{L}^{2d/\alpha,\lambda}(\mathbb{R}^d)\hookrightarrow\mathcal{L}^{1,\mu}(\mathbb{R}^d)$ with $\mu=\alpha\lambda/(2d)+d-\alpha/2$, we have $\|\tilde{v}\|_{L^1(0,1;\mathcal{L}^{1,\mu})}\leq C\|\tilde{v}\|_{X_\lambda(0,1)}=C\|v\|_{X_\lambda(s,t)}$ for this μ. Furthermore, we have the equality

$$F_1[\tilde{v}](1,\tilde{x})=(t-s)^{-\frac{1}{\alpha}}F_{(t-s)^{\frac{1}{\alpha}}}[v(\cdot+s)](t-s,x).$$

Hence (2.23) yields (3.29) by the definition $\xi_v^*(x,t,s)=(t-s)^{1/\alpha}\xi_{\tilde{v}}(1;\tilde{x})$. This completes the proof.

The adjoint relation $P_{K^{(N,\delta)},v}(t,y;s,x)=P_{j_tK^{(N,\delta)},-j_tv}(t-s,x;0,y)$ (see (2.14) for the definition of j_t) and Proposition 3.4 yield the moment bound of the integral with respect to x as follows. Note that $\|j_tv\|_{X_\lambda(0,t-s)}=\|v\|_{X_\lambda(s,t)}$ by the definitions of j_t and $\|\cdot\|_{X_\lambda(s,t)}$.

Proposition 3.5 *Assume that the conditions in Proposition 3.4 hold. Then for any $y\in\mathbb{R}^d$ there is $\eta_v^*=\eta_v^*(y,t,s)\in\mathbb{R}^d$ such that*

$$\begin{aligned}C_1(t-s)^{\frac{\gamma}{2\alpha}}&\leq\int_{\mathbb{R}^d}\rho_\gamma(x-y-\eta_v^*)P_{K^{(N,\delta)},v}(t,y;s,x)\,\mathrm{d}x\\&\leq C_2(t-s)^{\frac{\gamma}{2\alpha}}\big(1+\delta^{2-\alpha}(t-s)^{1-\frac{2}{\alpha}}N+\delta(t-s)^{\frac{1}{2}-\frac{1}{\alpha}}\big).\end{aligned}\tag{3.37}$$

Here C_1 depends only on d, α, C_0, and γ, and C_2 depends only on d, α, C_0, γ, λ, and $\|v\|_{X_\lambda(s,t)}$. Moreover, $\eta_v^*(y,t,s)$ satisfies the same estimate as $\xi_v^*(x,t,s)$ in Proposition 3.4 with v replaced by j_tv.

3.2 Relative entropy bound

In this section we establish the estimate on the relative entropy of fundamental solutions. We set

$$E(x)=H(|x|)=C(1+|x|)^{-d-\alpha}/\log(e+|x|),\qquad\int_{\mathbb{R}^d}E(x)\,\mathrm{d}x=1.\tag{3.38}$$

Proposition 3.6 *Let $\epsilon \in (0,1]$ and $\gamma \in (0, \min\{\alpha, 2-\alpha\})$. Let $K(\tau,y,x)$ be a smooth kernel of the order $\alpha' \in (1,2)$. Assume that u is a smooth and bounded vector field satisfying $\nabla \cdot u(\tau) = 0$ in $(0,1)\times\mathbb{R}^d$. For $x \in \mathbb{R}^d$ let $\xi_u(\tau;x)$ and $m_\gamma(\tau;x)$ be as in Proposition 3.1. Then there are $C_1,\ C_2 > 0$, $l_\lambda \in [0,1)$ such that*

$$\begin{aligned} -C_1 &\le -\int_{\mathbb{R}^d} E(y-x-\xi_u(1;x)) \log\big(P_{K,u}(1,y;0,x)+\epsilon\big)\,\mathrm{d}y \\ &\le C_2 |\log\epsilon|^{l_\lambda} \left(1 + \|K\|_\gamma + \frac{1}{H\big(1+\{\sup_{\tau\in[\frac12,1]} m_\gamma(\tau;x)\}^{\frac{2}{\gamma}}\big)}\right). \end{aligned} \tag{3.39}$$

Here C_1 depends only on d, α and C_0, and C_2 depends only on d, α, C_0, λ, γ, and $\|u\|_{X_\lambda(0,1)}$. The number l_λ depends only on d, α, and λ, and especially, we can take $l_\lambda = 0$ if $\lambda \in (d, 2d/\alpha + d]$.

Proof. As in the proof of Proposition 3.1, we may take $x = 0$. Then we consider

$$g_\epsilon(\tau) = -\int_{\mathbb{R}^d} E(\eta - \tau^{-\frac{1}{\alpha}}\xi_u(\tau;0)) \log\big(\tau^{\frac{d}{\alpha}} P_{K,u}(\tau, \tau^{\frac{1}{\alpha}}\eta; 0,0) + \epsilon\big)\,\mathrm{d}\eta, \quad \frac12 \le \tau \le 1. \tag{3.40}$$

Since $\displaystyle\int_{\mathbb{R}^d} \tau^{\frac{d}{\alpha}} P_{K,u}(\tau,\tau^{\frac{1}{\alpha}}\eta;0,0)\,\mathrm{d}\eta = 1$ it is easy to see that $g_\epsilon(\tau)$ is bounded from below uniformly in τ and ϵ by the Jensen inequality. For simplicity of notations, we set

$$\begin{aligned} P(\tau,\eta) &= \tau^{\frac{d}{\alpha}} P_{K,u}(\tau,\tau^{\frac{1}{\alpha}}\eta;0,0), \\ \xi(\tau) &= \tau^{-\frac{1}{\alpha}}\xi_u(\tau;0), \\ E(\tau,\eta) &= E(\eta - \xi(\tau))). \end{aligned}$$

Since

$$\partial_\tau P(\tau,\eta) = \frac{d}{\alpha\tau} P(\tau,\eta) + \frac{1}{\alpha\tau}\eta\cdot\nabla_\eta P(\tau,\eta) + \tau^{\frac{d}{\alpha}}(\partial_\tau P_{K,u})(\tau,\tau^{\frac{1}{\alpha}}\eta;0,0),$$

we have

$$\begin{aligned} \tau\frac{\mathrm{d}}{\mathrm{d}\tau} g_\epsilon(\tau) &= -\frac{d}{\alpha}\int_{\mathbb{R}^d} E(\tau,\eta)\frac{P(\tau,\eta)}{P(\tau,\eta)+\epsilon}\,\mathrm{d}\eta + \frac{d}{\alpha} g_\epsilon(\tau) \\ &\quad + \frac{1}{\alpha}\int_{\mathbb{R}^d} (\eta - \xi(\tau))\cdot(\nabla E)(\eta-\xi(\tau))\log(P(\tau,\eta)+\epsilon)\,\mathrm{d}\eta \\ &\quad + \frac{\tau^{1+\frac{d}{\alpha}}}{2}\int_{\mathbb{R}^d} [P(\tau,\cdot)][\frac{E(\tau,\cdot)}{P(\tau,\cdot)+\epsilon}](\eta,z)(R_\tau K)(1,\eta,z)\,\mathrm{d}\eta\,\mathrm{d}z \\ &\quad - \tau^{1-\frac{1}{\alpha}}\int_{\mathbb{R}^d} (u(\tau,\tau^{\frac{1}{\alpha}}\eta) - \frac{\mathrm{d}}{\mathrm{d}\tau}\xi_u(\tau;0), \nabla_\eta) E(\tau,\eta)\log\big(P(\tau,\eta)+\epsilon\big)\,\mathrm{d}\eta \\ &\le \frac{d}{\alpha} g_\epsilon(\tau) + \frac{1}{\alpha} I_1 + \frac{\tau^{1+\frac{d}{\alpha}}}{2} I_2 + \tau^{1-\frac{1}{\alpha}} I_3. \end{aligned}$$

(i) Estimate of I_1: From the definition of E and $\log(P(\tau,\eta)+\epsilon) \le C$ where C depends only on d, α, and C_0, it is not difficult to see

$$I_1 \le C g_\epsilon(\tau) + C, \tag{3.41}$$

where C depends only on d, α, and C_0
(ii) Estimate of I_2: Although I_2 is estimated in the same manner as in [24], we give the proof for convenience to the reader. Set

$$\begin{aligned} 2\zeta(\eta,z) &= [\log E(\tau,\cdot)](\eta,z), \\ 2\omega(\eta,z) &= [\log(P(\tau,\cdot)+\epsilon)](\eta,z). \end{aligned}$$

We use the inequality

$$\sinh(\theta-\omega)\sinh\omega \le (\theta-\omega)\omega\frac{\sinh\theta}{\theta}, \tag{3.42}$$

which was proved in [24, Lemma 3.1]). Then we have

$$\begin{aligned} [P(\tau,\cdot)](\eta,z)[\frac{E(\tau,\cdot)}{P(\tau,\cdot)+\epsilon}](\eta,z) &= 2(E(\tau,\eta)+E(\tau,z))\sinh(\zeta-\omega)\sinh(\omega)/\cosh(\zeta) \\ &\le (E(\tau,\eta)+E(\tau,z))\frac{\tanh\ \zeta(\eta,z)}{\zeta(\eta,z)}(\zeta^2(\eta,z)-\omega^2(\eta,z)). \end{aligned}$$

Hence we get

$$I_2 \le \int_{\mathbb{R}^d\times\mathbb{R}^d} (E(\tau,\eta)+E(\tau,z))\frac{\tanh\ \zeta(\eta,z)}{\zeta(\eta,z)}(\zeta^2(\eta,z)-\omega^2(\eta,z))(R_\tau K)(1,\eta,z)\,\mathrm{d}\eta\,\mathrm{d}z.$$

Let $\tau\in[1/2,1]$. The estimate $|\zeta(\eta,z)|\le C\log(1+|\eta-z|)\le C|\eta-z|$ yields

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d} (E(\tau,\eta)+E(\tau,z))\frac{\tanh\ \zeta(\eta,z)}{\zeta(\eta,z)}\zeta^2(\eta,z)(R_\tau K)(1,\eta,z)\,\mathrm{d}\eta\,\mathrm{d}z\\
\le\ & C\inf_{M\in(0,\infty)}\big(J_2[(R_\tau K)_M]+J_\gamma[(R_\tau K)_{\vee,M}]\big)\\
\le\ & C\inf_{M\in(0,\infty)}\big(J_2[K_M]+J_\gamma[K_{\vee,M}]\big)\le C\|K\|_\gamma<\infty.
\end{aligned}$$

By using the inequalities

$$\frac{\tanh\zeta(\eta,z)}{\zeta(\eta,z)}\ge C(\log(e+|\eta-z|))^{-1},\quad (R_\tau K)(1,\eta,z)\ge C_0\tau^{-1-\frac{d}{\alpha}}|\eta-z|^{-d-\alpha},$$

we have

$$\begin{aligned}
I_2 \le\ & C+C\|K\|_\gamma\\
&-\int_{\mathbb{R}^d\times\mathbb{R}^d}(E(\tau,\eta)+E(\tau,z))\omega^2(\eta,z)(\log(e+|\eta-z|))^{-1}(R_\tau K)(1,\eta,z)\,\mathrm{d}\eta\,\mathrm{d}z\\
\le\ & C+C\|K\|_\gamma-CC_0^{-1}\int_{|\eta-\xi(\tau)|>|z-\xi(\tau)|}E(z-\xi(\tau))\omega^2(\eta,z)E(\eta-\xi(\tau))\,\mathrm{d}\eta\,\mathrm{d}z\\
\le\ & C+C\|K\|_\gamma-C\int_{\mathbb{R}^d\times\mathbb{R}^d}E(\tau,z)\omega^2(\eta,z)E(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}z\\
\le\ & C+C\|K\|_\gamma-C\int_{\mathbb{R}^d}E(\tau,\eta)\big(\int_{\mathbb{R}^d}2\omega(\eta,z)E(\tau,z)\,\mathrm{d}z\big)^2\,\mathrm{d}\eta\\
=\ & C+C\|K\|_\gamma-C\int_{\mathbb{R}^d}E(\tau,\eta)\big(\log(P(\tau,\eta)+\epsilon)+g_\epsilon(\tau)\big)^2\,\mathrm{d}\eta,
\end{aligned}\tag{3.43}$$

for $\tau\in[1/2,1]$, where $C>0$ depending only on d, α, and C_0. Let $l_\epsilon(a,b)$, $a>0$, $b\in\mathbb{R}$ be the function defined by

$$l_\epsilon(a,b)=\frac{1}{a}(\log(a+\epsilon)+b)^2.\tag{3.44}$$

Fix $\kappa_1>0$ which is to be determined later. Since $l_\epsilon(\cdot,b)$ is decreasing on $[\kappa_1^{3-b},\infty)$, if $g_\epsilon(\tau)\ge 3-\log\kappa_1$ then we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}E(\tau,\eta)\big(\log(P(\tau,\eta)+\epsilon)+g_\epsilon(\tau)\big)^2\,\mathrm{d}\eta\\
=\ & \int_{\mathbb{R}^d}E(\tau,\eta)P(\tau,\eta)l_\epsilon(P(\tau,\eta),g_\epsilon(\tau))\,\mathrm{d}\eta\\
\ge\ & l_\epsilon(\max_\eta P(\tau,\eta),g_\epsilon(\tau))\int_{P(\tau,\eta)\ge\kappa_1}E(\eta-\xi(\tau))P(\tau,\eta)\,\mathrm{d}\eta\\
\ge\ & H(R)l_\epsilon(\max_\eta P(\tau,\eta),g_\epsilon(\tau))\int_{P(\tau,\eta)\ge\kappa_1,\ |\eta-\xi(\tau)|\le R}P(\tau,\eta)\,\mathrm{d}\eta\\
\ge\ & H(R)l_\epsilon(\max_\eta P(\tau,\eta),g_\epsilon(\tau))\big(\int_{|\eta-\xi(\tau)|\le R}P(\tau,\eta)\,\mathrm{d}\eta-\kappa_1\int_{|\eta-\xi(\tau)|\le R}\mathrm{d}\eta\big).
\end{aligned}$$

Thus from Proposition 3.1 with a fixed $\gamma\in(0,\min\{\alpha,2-\alpha\})$ we get

$$\begin{aligned}
\int_{|\eta-\xi(\tau)|\le R}P(\tau,\eta)\,\mathrm{d}\eta =\ & 1-\int_{|\eta-\xi(\tau)|\ge R}P(\tau,\eta)\,\mathrm{d}\eta\\
\ge\ & 1-\rho_\gamma(R)^{-1}\int_{|\eta-\xi(\tau)|\ge R}\rho_\gamma(\eta-\xi(\tau))P(\tau,\eta)\,\mathrm{d}\eta\\
\ge\ & 1-\rho_\gamma(R)^{-1}\int_{\mathbb{R}^d}\rho_\gamma\Big(\frac{\eta-\xi(\tau)}{\tau^{\frac{1}{\alpha}}}\Big)P_{K,u}(\tau,\eta;0,0)\,\mathrm{d}\eta\\
=\ & 1-\rho_\gamma(R)^{-1}\tau^{-\frac{\gamma}{2\alpha}}m_\gamma(\tau;0)\ge\frac{3}{4},
\end{aligned}$$

if $\tau \in [1/2, 1]$ and $R \geq 1$ is sufficiently large so that

$$2^{2+\frac{\gamma}{2\alpha}} \sup_{\tau\in[\frac{1}{2},1]} m_\gamma(\tau;0) \leq R^{\frac{\gamma}{2}}. \tag{3.45}$$

By setting $4\kappa_1 = |B_R(0)|^{-1}$ and by using the bound

$$P(\tau,\eta) = \tau^{\frac{d}{\alpha}} P_{K,u}(\tau, \tau^{\frac{1}{\alpha}}\eta; 0, 0) \leq C',$$

where C' depends only on d, α, and C_0, we finally get

$$\begin{aligned} \int_{\mathbb{R}^d} E(\tau,\eta)\big(\log(P(\tau,\eta)+\epsilon) + g_\epsilon(\tau)\big)^2 \,\mathrm{d}\eta &\geq \frac{1}{2} H(R) l_\epsilon(C', g_\epsilon(\tau)) \\ &= CH(R)\big(\log(C'+\epsilon) + g_\epsilon(\tau)\big)^2 \\ &\geq H(R)\big(cg_\epsilon^2(\tau) - C\big), \end{aligned}$$

with c depending only on d, α, and C_0, which holds for $\tau \in [1/2, 1]$ as long as

$$g_\epsilon(\tau) \geq 3 - \log \kappa_1 = C + d \log R. \tag{3.46}$$

Here the constant C in (3.46) depends only on d and γ. Collecting these, under the conditions of (3.45) and (3.46) we get

$$\begin{aligned} I_2 \leq\ & C + C\|K\|_\gamma - C\kappa_2 \int_{\mathbb{R}^d} E(\tau,\eta)\big(\log(P(\tau,\eta)+\epsilon) + g_\epsilon(\tau)\big)^2 \,\mathrm{d}\eta \\ & - c(1-\kappa_2) H(R) g_\epsilon^2(\tau). \end{aligned} \tag{3.47}$$

for every $\kappa_2 \in [0, 1)$ and $\tau \in [1/2, 1]$, where the above constants C and c depend only on d, α, γ, and C_0.
(iii) Estimate of I_3: Let $\lambda \in [2d/\alpha - d, d]$. As in the proof of Proposition 3.4, we set

$$\begin{aligned} D_0 &= \{z \in \mathbb{R}^d \mid |z - \xi_u(\tau;0)| < 1\}, \\ D_k &= \{z \in \mathbb{R}^d \mid 2^{k-1} \leq |z - \xi_u(\tau;0)| < 2^k\}, \qquad k \in \mathbb{N}. \end{aligned}$$

Since $|\nabla_\eta E(\eta)| \leq C(1+|\eta|)^{-1} E(\eta) \leq C(1+|\eta|)^{-(d+\alpha+2)/2} E(\eta)^{1/2}$ we have

$$\begin{aligned} & |I_3| \\ \leq\ & \Big(\int_{\mathbb{R}^d} |u(\tau, \tau^{\frac{1}{\alpha}}\eta) - \fint_{B_1(\xi_u(\tau))} u(\tau)|^{\frac{2d}{\alpha}} (1+|\eta - \xi(\tau)|)^{-\frac{2d}{\alpha}} E(\tau,\eta)\,\mathrm{d}\eta\Big)^{\frac{\alpha}{2d}} \\ & \times \Big(\int_{\mathbb{R}^d} E(\tau,\eta)\big|\log(P(\tau,\eta)+\epsilon)\big|^{\frac{2d}{2d-\alpha}} \,\mathrm{d}\eta\Big)^{\frac{2d-\alpha}{2d}} \\ \leq\ & C\tau^{-\frac{d}{2\alpha}} \bigg(\sum_{k=0}^{\infty} \int_{D_k} |u(\tau,z) - \fint_{B_1(\xi_u(\tau))} u(\tau)|^{\frac{2d}{\alpha}} \Big(1 + \frac{|z - \xi_u(\tau;0)|}{\tau^{\frac{1}{\alpha}}}\Big)^{-\frac{2d}{\alpha}} E\Big(\frac{z - \xi_u(\tau)}{\tau^{\frac{1}{\alpha}}}\Big)\,\mathrm{d}z\bigg)^{\frac{\alpha}{2d}} \\ & \times \Big(\int_{\mathbb{R}^d} E(\tau,\eta)\big|\log(P(\tau,\eta)+\epsilon)\big|^{\frac{2d}{2d-\alpha}} \,\mathrm{d}\eta\Big)^{\frac{2d-\alpha}{2d}}. \end{aligned}$$

From (2.20) we get

$$\begin{aligned} & \int_{D_k} |u(\tau,z) - \fint_{B_1(\xi_u(\tau))} u(\tau)|^{\frac{2d}{\alpha}} \Big(1 + \frac{|z - \xi_u(\tau;0)|}{\tau^{\frac{1}{\alpha}}}\Big)^{-\frac{2d}{\alpha}} E\Big(\frac{z - \xi_u(\tau;0)}{\tau^{\frac{1}{\alpha}}}\Big)\,\mathrm{d}z \\ \leq\ & C 2^{-(\frac{2d}{\alpha} + d + \frac{\alpha}{2})k} \int_{B_{2^k}(\xi_u(\tau;0))} |u(\tau,z) - \fint_{B_{2^k}(\xi_u(\tau))} u(\tau)|^{\frac{2d}{\alpha}} \,\mathrm{d}z \\ & + C 2^{-\frac{2dk}{\alpha}} \Big|\fint_{B_{2^k}(\xi_u(\tau))} u(\tau) - \fint_{B_1(\xi_u(\tau))} u(\tau)\Big|^{\frac{2d}{\alpha}} \\ \leq\ & C\Big(2^{(\lambda - \frac{2d}{\alpha} - d - \frac{\alpha}{2})k} + 2^{-\frac{2dk}{\alpha}} k\Big) \|u(\tau)\|^{\frac{2d}{\alpha}}_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}. \end{aligned}$$

for $\tau \in [1/2, 1]$. Then from the interpolation inequality and the estimate

$$\int_{\mathbb{R}^d} E(\tau,\eta)\big|\log(P(\tau,\eta)+\epsilon)\big| \,\mathrm{d}\eta \leq -C\log\tau + g_\epsilon(\tau) \leq C + g_\epsilon(\tau) \qquad \text{for } \tau \in [\tfrac{1}{2}, 1],$$

we have

$$\begin{aligned}
|I_3| &\le C\|u(\tau)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} \Big(\int_{\mathbb{R}^d} E(\tau,\eta) \big|\log(P(\tau,\eta)+\epsilon)\big|^{\frac{2d}{2d-\alpha}} \,\mathrm{d}\eta \Big)^{\frac{2d-\alpha}{2d}} \\
&\le C\|u(\tau)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} \big(C+g_\epsilon(\tau)\big)^{\frac{2(d-\alpha)}{2d-\alpha}} \Big(\int_{\mathbb{R}^d} E(\tau,\eta) \big|\log(P(\tau,\eta)+\epsilon)\big|^{2} \,\mathrm{d}\eta \Big)^{\frac{\alpha}{2(2d-\alpha)}} \qquad (3.48)
\end{aligned}$$

if $\tau \in [1/2,1]$. In particular, from $|\log(P(\tau,\eta)+\epsilon)| \le C|\log\epsilon|$ we have

$$|I_3| \le C|\log\epsilon| \|u(\tau)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}. \qquad (3.49)$$

Next we consider the case $\lambda \in (d, 2d/\alpha + d]$. In this case we have $\mathcal{L}^{2d/\alpha,\lambda}(\mathbb{R}^d) = \dot{C}^{\beta_\lambda}(\mathbb{R}^d)$ with $\beta_\lambda = \alpha(\lambda-d)/(2d)$, and hence,

$$\begin{aligned}
I_3 &\le C\int_{\mathbb{R}^d} |u(\tau,\tau^{\frac{1}{\alpha}}\eta) - \fint_{B_1(\xi_u(\tau))} u(\tau)|(1+|\eta-\xi(\tau)|)^{-1} E(\tau,\eta)\big|\log(P(\tau,\eta)+\epsilon)\big| \,\mathrm{d}\eta \\
&= C\tau^{-\frac{d}{\alpha}} \int_{\mathbb{R}^d} |u(\tau,z) - \fint_{B_1(\xi_u(\tau))} u(\tau)|(1+\frac{|z-\xi_u(\tau;0)|}{\tau^{\frac{1}{\alpha}}})^{-1} E(\tau,\frac{z}{\tau^{\frac{1}{\alpha}}})|\log(P(\tau,\frac{z}{\tau^{\frac{1}{\alpha}}})+\epsilon)| \,\mathrm{d}z \\
&\le C\tau^{-\frac{d}{\alpha}} \|u(\tau)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} \int_{\mathbb{R}^d} (1+\frac{|z-\xi_u(\tau;0)|}{\tau^{\frac{1}{\alpha}}})^{-1+\beta_\lambda} E(\tau,\frac{z}{\tau^{\frac{1}{\alpha}}})|\log(P(\tau,\frac{z}{\tau^{\frac{1}{\alpha}}})+\epsilon)| \,\mathrm{d}z \\
&\le C\|u(\tau)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} \int_{\mathbb{R}^d} E(\tau,\eta)|\log(P(\tau,\eta)+\epsilon)| \,\mathrm{d}z \\
&\le C\|u(\tau)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} (1+g_\epsilon(\tau)). \qquad (3.50)
\end{aligned}$$

From (3.41), (3.47), and (3.49), we observe for $\lambda \in [2d/\alpha - d, d]$ that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_\epsilon(\tau) \quad\le\quad C + C\|K\|_\gamma + C|\log\epsilon| \; \|u(\tau)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} - cH(R) g_\epsilon^2(\tau), \qquad (3.51)$$

if $\tau \in [1/2,1]$, (3.45), and (3.46). As for the case $\lambda \in (d, 2d/\alpha + d]$, from (3.41), (3.47), and (3.50) we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_\epsilon(\tau) \quad\le\quad C + C\|K\|_\gamma + C\|u(\tau)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} (1+g_\epsilon(\tau)) - cH(R) g_\epsilon^2(\tau), \qquad (3.52)$$

Here the constants C and c in (3.51) - (3.52) depend only on d, α, C_0, λ, and γ. These differential inequalities imply the bound of $g_\epsilon(1)$. We give a proof only for the case $\lambda \in [2d/\alpha - d, d]$, since the other cases are proved in the same manner. Est. (3.51) implies that

$$L_\epsilon(\tau;\tau_0) = g_\epsilon(\tau) - C\int_{\tau_0}^{\tau} (1+\|K\|_\gamma + |\log\epsilon| \; \|u(s)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}) \,\mathrm{d}s + cH(R)\int_{\tau_0}^{\tau} g_\epsilon(s)^2 \,\mathrm{d}s$$

is non-increasing for $\tau \in [\tau_0, 1]$, where $\tau_0 \in [1/2, 1)$ is fixed. Set

$$M = \Big(\frac{C'(1+\|K\|_\gamma + \|u\|_{X_\lambda(0,1)})}{cH(R)} \Big)^{\frac{1}{2}}, \qquad l = \frac{3\alpha d + 2d - \alpha\lambda}{2(2\alpha d + 2d - \alpha\lambda)} \in (\frac{1}{2}, 1), \qquad (3.53)$$

where C' is a large constant depending only on d, α, C_0, λ, and γ. Let us take $\tau_0 = 1 - |\log\epsilon|^{-(2l-1)}$. Recalling the general bound $\sup_{0<\tau\le 1} |g_\epsilon(\tau)| \le C|\log\epsilon|$, we claim that there is $\tau_1 \in [\tau_0, 1]$ such that

$$g_\epsilon(\tau_1) \le M|\log\epsilon|^l. \qquad (3.54)$$

Indeed, otherwise it follows from $L_\epsilon(1;\tau_0) \le L_\epsilon(\tau_0;\tau_0) = g_\epsilon(\tau_0)$ that

$$\begin{aligned}
g_\epsilon(1) &\le C\{|\log\epsilon| + (1-\tau_0)\|K\|_\gamma + |\log\epsilon|(1-\tau_0)^{\frac{1}{2}+\frac{1}{\alpha}-\frac{\lambda}{2d}} \|u\|_{X_\lambda(0,1)}\} - cH(R)M^2(1-\tau_0)|\log\epsilon|^{2l} \\
&\le (C - C')\big(1+\|K\|_\gamma + \|u\|_{X_\lambda(0,1)}\big)|\log\epsilon| \le 0,
\end{aligned}$$

if C' is taken so that $C' \ge C$, which is a contradiction. Hence there is $\tau_1 \in [\tau_0, 1]$ such that (3.54) holds, and then $L_\epsilon(1;\tau_1) \le L_\epsilon(\tau_1;\tau_1) = g_\epsilon(\tau_1)$ obtain

$$\begin{aligned}
g_\epsilon(1) &\le M|\log\epsilon|^l + C\{|\log\epsilon|^{-(2l-1)}\|K\|_\gamma + |\log\epsilon|^{1-(2l-1)\left(\frac{1}{2}+\frac{1}{\alpha}-\frac{\lambda}{2d}\right)} \|u\|_{X_\lambda(0,1)}\} \\
&\le \big(M + \|u\|_{X_\lambda(0,1)} + \|K\|_\gamma\big)|\log\epsilon|^l. \qquad (3.55)
\end{aligned}$$

This proves the case $\lambda \in [2d/\alpha - d, d]$. For the case $\lambda \in (d, 2d/\alpha + d]$ we have the bound of $g_\epsilon(1)$ uniformly in $\epsilon \in (0,1]$, thanks to (3.52). The details are omitted. The proof of Proposition 3.6 is now complete.

Remark 3.7 By using (3.41), (3.47), and (3.48), it is possible to take $l_\lambda = 0$ in Proposition 3.6 even when $\lambda \in [2d/\alpha - d, d]$, if u belongs to $L^{p,1-p(\frac{\lambda}{2d}+\frac{1}{2}-\frac{1}{\alpha})}\big(0,\infty;(\mathcal{L}^{\frac{2d}{\alpha},\lambda}(\mathbb{R}^d))^d\big)$ with $p = 2(2d-\alpha)/(4d-3\alpha)$, which is slightly stronger than **(C1)**.

As in Proposition 3.4, we will derive the relative entropy bound of $P_{K^{(N,\delta)},v}(t,x;s,y)$ by using the scaling property.

Proposition 3.8 *Let $K(\tau,y,x)$ be a kernel satisfying (1.5) and (1.6), and let $K^{(N,\delta)}(\tau,y,x)$ be the approximation of K given by Section 2.3. Fix $t > s \geq 0$. Let $v(\tau,y)$ be a smooth and bounded vector field satisfying $\nabla \cdot v(\tau) = 0$ in $(s,t)\times\mathbb{R}^d$. Assume that $N \gg 1$, $0 < \delta \leq N^{-1/(2-\alpha)}$ and*

$$t - s \geq \mu_{N,\delta} := \delta^\alpha N^{\frac{\alpha}{2-\alpha}}. \tag{3.56}$$

Fix any $x, y \in \mathbb{R}^d$ and let $\xi_v^* = \xi_v^*(x,t,s)$, $\eta_v^* = \eta_v^*(y,t,s) \in \mathbb{R}^d$ be the vectors in Propositions 3.4, 3.5. Let $l_\lambda \in [0,1)$ be the number in Proposition 3.6. Then

$$C_1 \leq -(t-s)^{-\frac{d}{\alpha}} \int_{\mathbb{R}^d} E\Big(\frac{y-x-\xi_v^*}{(t-s)^{\frac{1}{\alpha}}}\Big) \log\big((t-s)^{\frac{d}{\alpha}} P_{K^{(N,\delta)},v}(t,y;s,x)+\epsilon\big)\,\mathrm{d}y \leq C_2|\log\epsilon|^{l_\lambda}, \tag{3.57}$$

$$C_1 \leq -(t-s)^{-\frac{d}{\alpha}} \int_{\mathbb{R}^d} E\Big(\frac{x-y-\eta_v^*}{(t-s)^{\frac{1}{\alpha}}}\Big) \log\big((t-s)^{\frac{d}{\alpha}} P_{K^{(N,\delta)},v}(t,y;s,x)+\epsilon\big)\,\mathrm{d}x \leq C_2|\log\epsilon|^{l_\lambda}. \tag{3.58}$$

Here C_1 depends only on d, α, and C_0, and C_2 depends only on d, α, C_0, λ, and $\|v\|_{X_\lambda(s,t)}$.

Proof. It suffices to show (3.57). The proof is similar to Proposition 3.4. Noticing the scaling relation (3.31) and (3.30), we apply Proposition 3.6 to $P_{\tilde{K},\tilde{v}}(\tau,\tilde{y};0,\tilde{x})$. Then we have

$$\begin{aligned} C_1 &\leq -\int_{\mathbb{R}^d} E(\tilde{y}-\tilde{x}-\xi_{\tilde{v}}(1;\tilde{x}))\log\big(P_{\tilde{K},\tilde{v}}(1,\tilde{y};0,\tilde{x})+\epsilon\big)\,\mathrm{d}\tilde{y} \\ &\leq C_2|\log\epsilon|^{l_\lambda}\Bigg(1+\|\tilde{v}\|_{X_\lambda(0,1)}+\|\tilde{K}\|_\gamma+\frac{1}{H\big(1+\{\sup_{\tau\in[\frac{1}{2},1]}\tilde{m}_\gamma(\tau)\}^{\frac{2}{\gamma}}\big)}\Bigg), \end{aligned} \tag{3.59}$$

where C_1 depends only on d, α, and C_0, and C_2 depends only on d, α, C_0, λ, γ, and $\|\tilde{v}\|_{X_\lambda(0,1)}$. Here

$$\tilde{m}_\gamma(\tau) = \int_{\mathbb{R}^d} \rho_\gamma(\tilde{y}-\tilde{x}-\xi_{\tilde{v}}(\tau;\tilde{x}))P_{\tilde{K},\tilde{v}}(\tau,\tilde{y};0,\tilde{x})\,\mathrm{d}\tilde{y}.$$

By (3.36) and $2\alpha/(2-\alpha) < \alpha$, if $\delta \in (0,1)$ and $N \geq 1$ satisfy $\delta^\alpha N^{\alpha/(2-\alpha)} \leq t-s$, then $\sup_{\tau\in[1/2,1]} m_\gamma(\tau) \leq C$, where C depends only on d, α, C_0, λ, γ, and $\|\tilde{v}\|_{X_\lambda}$. Moreover, (3.35) implies $\|\tilde{K}\|_\gamma \leq C$, where C depends only on d, α, C_0, and γ. Finally, we observe the relation $\|\tilde{v}\|_{X_\lambda(0,1)} = \|v\|_{X_\lambda(s,t)}$. This completes the proof.

We have the following estimates for $\xi_v^*(x,t,s)$ and $\eta_v^*(x,t,s)$, which will be used later.

Proposition 3.9 *Let $T \geq t > s \geq 0$ and $x, x_1, x_2 \in \mathbb{R}^d$. Let $\xi_v^*(x,t,s), \eta_v^*(x,t,s)$ be the vectors in Proposition 3.4, 3.5. Then*

$$\begin{aligned} &|\xi_v^*(x,t,s)| + |\eta_v^*(x,t,s)| \\ &\leq \begin{cases} C_1(T^{\frac{1}{2}(1-\frac{\lambda}{d})}+1)(t-s)^{\frac{1}{\alpha}} + C_2(t-s)^{1-\frac{1}{q}}\|v\|_{Y_{T,x}^{q,\lambda}} & \text{if } \lambda \in [\frac{2d}{\alpha}-d, d), \\ C_1(t-s)^{\frac{1}{\alpha}}(1+|\log(t-s)|) + C_2(t-s)^{1-\frac{1}{q}}\|v\|_{Y_{T,x}^{q,\lambda}} & \text{if } \lambda = d, \\ C_1(t-s)^{\frac{1}{\alpha}} + C_2(t-s)^{1-\frac{1}{q}}\|v\|_{Y_{T,x}^{q,\lambda}} & \text{if } \lambda \in (d, \frac{2d}{\alpha}+d], \end{cases} \end{aligned}$$

and

$$\begin{aligned} &|\xi_v^*(x_1,t,s) - \xi_v^*(x_2,t,s)| + |\eta_v^*(x_1,t,s) - \eta_v^*(x_2,t,s)| \\ &\leq \begin{cases} C_3\|v\|_{X_\lambda(s,t)}(t-s)^{\frac{1}{\alpha}} & \text{if } \lambda \in [\frac{2d}{\alpha}-d, d), \\ C_3\|v\|_{X_\lambda(s,t)}(t-s)^{\frac{1}{\alpha}}\log\big(e+\frac{|x_1-x_2|}{(t-s)^{\frac{1}{\alpha}}}\big) & \text{if } \lambda = d, \\ C_3\|v\|_{X_\lambda(s,t)}(t-s)^{\frac{1}{\alpha}}\big(1+(\frac{|x_1-x_2|}{(t-s)^{\frac{1}{\alpha}}})^{\frac{\alpha\lambda}{2d}-\frac{\alpha}{2}}\big) & \text{if } \lambda \in (d, \frac{2d}{\alpha}+d]. \end{cases} \end{aligned}$$

Here C_1 and C_3 depend only on d, α, λ, and $\|v\|_{X_\lambda}$, and C_2 depends only on d and q.

Proof. In order to estimate $\xi_v^*(x,t,s)$, by Proposition 3.4 it suffices to give the estimate for $F_{(t-s)^{1/\alpha}}[v(\cdot+s)](t-s,x)$. Let $\lambda\in(2d/\alpha-d,d)$. For every $r\in(0,t-s]$ we have

$$\begin{aligned}|\int_0^r \fint_{B_{(t-s)^{\frac{1}{\alpha}}}(x)} v(\tau+s)\,\mathrm{d}\tau| &\le \int_0^r |\fint_{B_{(t-s)^{\frac{1}{\alpha}}}(x)} v(\tau+s) - \fint_{B_1(x)} v(\tau+s)|\,\mathrm{d}\tau + \int_0^r |\fint_{B_1(x)} v(\tau+s)|\,\mathrm{d}\tau \\ &= I_1+I_2.\end{aligned}$$

Since it is easy to get $I_2\le C(t-s)^{1-1/q}\|v\|_{Y^{q,\lambda}_{T,x}}$, it suffices to consider I_1, but Proposition 2.3 with $\mu=\alpha\lambda/(2d)-\alpha/2+d$ implies

$$\begin{aligned}I_1\le C\big(1+(t-s)^{\frac{\mu-d}{\alpha}}\big)\int_0^{t-s}\|v(\tau+s)\|_{\mathcal{L}^{1,\mu}}\,\mathrm{d}\tau &\le C\big(1+(t-s)^{\frac{\lambda}{2d}-\frac12}\big)(t-s)^{\frac12+\frac1\alpha-\frac{\lambda}{2d}}\|v\|_{X_\lambda(s,t)} \\ &\le C\big(T^{\frac12(1-\frac{\lambda}{d})}+1\big)(t-s)^{\frac1\alpha}\|v\|_{X_\lambda(s,t)}.\end{aligned}$$

The case $\lambda\in[d,2d/\alpha+d]$ is obtained similarly and the details are omitted here. In order to estimate $|\xi_v^*(x_1,t,s)-\xi_v^*(x_2,t,s)|$ we recall that $\xi_v^*(z,t,s)$ is given by $\xi_v^*(z,t,s)=(t-s)^{1/\alpha}\xi_{\tilde v}(1;\tilde z)$, where $\tilde z=(t-s)^{-1/\alpha}z$ and $\xi_{\tilde v}(\tau;\tilde z)$ is the solution to (2.21) with $u=\tilde v=(t-s)^{1-1/\alpha}R_{t-s}i_s v$, $x=\tilde z$, and $R=1$. Hence, from Proposition 2.3 and $\|\tilde v\|_{L^1(0,1;\mathcal{L}^{1,\mu})}\le C\|v\|_{X_\lambda(s,t)}$ for $\mu=\alpha\lambda/(2d)-\alpha/2+d$ it is easy to get the desired estimates. The estimates for $\eta_v^*(x,t,s)$ are obtained similarly. This completes the proof.

Remark 3.10 By the triangle inequality $|\xi_v^*(x,t,s)|\le|\xi_v^*(x,t,s)-\xi_v^*(0,t,s)|+|\xi_v^*(0,t,s)|$, applying Proposition 3.9, we also have the estimates

$$\begin{aligned}&|\xi_v^*(x,t,s)|+|\eta_v^*(x,t,s)| \\ \le&\begin{cases} C_1(T^{\frac12(1-\frac{\lambda}{d})}+1)(t-s)^{\frac1\alpha}+C_2(t-s)^{1-\frac1q}\|v\|_{Y^{q,\lambda}_{T,0}} & \text{if } \lambda\in[\frac{2d}{\alpha}-d,d), \\ C_1(t-s)^{\frac1\alpha}\big(|\log(t-s)|+\log(e+|x|)\big)+C_2(t-s)^{1-\frac1q}\|v\|_{Y^{q,\lambda}_{T,0}} & \text{if } \lambda=d, \\ C_1\big((t-s)^{\frac1\alpha}+\|v\|_{X_\lambda(s,t)}(t-s)^{\frac1\alpha+\frac12-\frac{\lambda}{2d}}|x|^{\frac{\alpha\lambda}{2d}-\frac\alpha2}\big) & \\ \qquad +C_2(t-s)^{1-\frac1q}\|v\|_{Y^{q,\lambda}_{T,0}} & \text{if } \lambda\in(d,\frac{2d}{\alpha}+d].\end{cases}\end{aligned}$$

Here C_1 depends only on d, α, λ, and $\|v\|_{X_\lambda}$, and C_2 depends only on d and q.

3.3 Overlap estimates

In this section we will show the overlap estimates based on Proposition 3.8.

Proposition 3.11 *Let $K(\tau,y,x)$ be a kernel satisfying (1.5) and (1.6), and let $K^{(N,\delta)}$ be the approximation of K given by Section 2.3. Assume that $N\gg1$ and $\delta\le N^{-1/(2-\alpha)}$. Then there exists a positive decreasing function $\phi=\phi(r)$ such that*

$$\int_{\mathbb{R}^d} P_{K^{(N,\delta)},v}(t,y;s,x_1)\wedge P_{K^{(N,\delta)},v}(t,y;s,x_2)\,\mathrm{d}y\ge\phi\Big(\frac{|x_1-x_2|}{(t-s)^{\frac1\alpha}}\Big), \tag{3.60}$$

for $x_1,\ x_2\in\mathbb{R}^d$, $t>s\ge0$ satisfying $t-s\ge\mu_{N,\delta}$ and $|x_1-x_2|\le2(t-s)^{1/\alpha}$. Here $\mu_{N,\delta}$ is the number in Proposition 3.8, and ϕ is taken depending only on d, α, C_0, λ, and $\|v\|_{X_\lambda}$.

Proof. Let $\xi_v^*(x_i)=\xi_v^*(x_i,t,s)$ be the vector in Proposition 3.8. We follow the arguments in [28]. The inequality $a_1b_1+a_2b_2\le\max_i a_i\max_j b_j+\min_i a_i\min_j b_j$ and Proposition 3.8 yields that for $t-s\ge\mu_{N,\delta}$,

$$\begin{aligned}&(t-s)^{-\frac{d}{\alpha}}\int_{\mathbb{R}^d}\min_i E\Big(\frac{y-x_i-\xi_v^*(x_i)}{(t-s)^{\frac1\alpha}}\Big)\min_i\log\big((t-s)^{\frac{d}{\alpha}}P_{K^{(N,\delta)},v}(t,y;s,x_i)+\epsilon\big)\,\mathrm{d}y \\ \ge\ &(t-s)^{-\frac{d}{\alpha}}\sum_i\int_{\mathbb{R}^d}E\Big(\frac{y-x_i-\xi_v^*(x_i)}{(t-s)^{\frac1\alpha}}\Big)\log\big((t-s)^{\frac{d}{\alpha}}P_{K^{(N,\delta)},v}(t,y;s,x_i)+\epsilon\big)\,\mathrm{d}y \\ &-(t-s)^{-\frac{d}{\alpha}}\int_{\mathbb{R}^d}\max_i E\Big(\frac{y-x_i-\xi_v^*(x_i)}{(t-s)^{\frac1\alpha}}\Big)\max_i\log\big((t-s)^{\frac{d}{\alpha}}P_{K^{(N,\delta)},v}(t,y;s,x_i)+\epsilon\big)\,\mathrm{d}y \\ \ge\ &-2C_2|\log\epsilon|^{l_\lambda}-2C,\end{aligned}\tag{3.61}$$

where C_2 and $l_\lambda \in [0,1)$ are the constants in Proposition 3.8 and C is the constant such that $P_{K^{(N,\delta)},v}(t,y;s,x) \le C(t-s)^{-d/\alpha}$. On the other hand, we have

$$
\begin{aligned}
& \text{L.H.S of (3.61)} \\
\le \;& \log\epsilon \cdot w\big(\frac{|x_1+\xi_v^*(x_1)-x_2-\xi_v^*(x_2)|}{(t-s)^{\frac{1}{\alpha}}}\big) \\
+ \;& (t-s)^{-\frac{d}{\alpha}} \int_{\mathbb{R}^d} \min_i E\big(\frac{y-x_i-\xi_v^*(x_i)}{(t-s)^{\frac{1}{\alpha}}}\big) \min_i \log\big(1+\frac{(t-s)^{\frac{d}{\alpha}} P_{K^{(N,\delta)},v}(t,y;s,x_i)}{\epsilon}\big)\,\mathrm{d}y \\
\le \;& \log\epsilon \cdot w\big(\frac{|x_1+\xi_v^*(x_1)-x_2-\xi_v^*(x_2)|}{(t-s)^{\frac{1}{\alpha}}}\big) + \frac{C}{\epsilon}\int_{\mathbb{R}^d} \min_i P_{K^{(N,\delta)},v}(t,y;s,x_i)\big)\,\mathrm{d}y. \qquad (3.62)
\end{aligned}
$$

Here $w(\sigma)$ is the function on $\mathbb{R}_+$ defined by

$$
w(|z_1-z_2|) = \int_{\mathbb{R}^d} \min_i E(z-z_i)\,\mathrm{d}z. \tag{3.63}
$$

Note that, by the radial symmetry and the positivity of E, w is positive decreasing. Estimates (3.61) and (3.62) yield

$$
\int_{\mathbb{R}^d} \min_i P_{K^{(N,\delta)},v}(t,y;s,x_i)\big)\,\mathrm{d}y \ge \frac{\epsilon}{C}\big(-\log\epsilon\cdot w\big(\frac{|x_1+\xi_v^*(x_1)-x_2-\xi_v^*(x_2)|}{(t-s)^{\frac{1}{\alpha}}}\big) - 2C_2|\log\epsilon|^{l_\lambda} - 2C\big). \tag{3.64}
$$

By Proposition 3.9 it is not difficult to see that for each case of $\lambda \in [2d/\alpha - d, 2d/\alpha + d]$,

$$
|\xi_v^*(x_1,t,s) - \xi_v^*(x_2,t,s)| \le C\big((t-s)^{\frac{1}{\alpha}} + |x_1-x_2|\big), \tag{3.65}
$$

where C depends only on d, α, λ, and $\|v\|_{X_\lambda}$. Then, since w is positive decreasing, from (3.64) and $l_\lambda \in [0,1)$ we have

$$
\int_{\mathbb{R}^d} \min_i P_{K^{(N,\delta)},v}(t,y;s,x_i)\big)\,\mathrm{d}y \ge \phi\big(\frac{|x_1-x_2|}{(t-s)^{\frac{1}{\alpha}}}\big) \tag{3.66}
$$

for a positive deceasing function ϕ, by taking $\epsilon > 0$ sufficiently small. This completes the proof.

3.4 Continuity estimates by the Nash iteration

In this section we establish continuity estimates for fundamental solutions by using the Nash iteration argument.

Proposition 3.12 *Let $K(t,x,y)$ be a function in Section 2.1 and let $K^{(N,\delta)}(t,x,y)$ be the approximation of K in Section 2.3. Fix $N \gg 1$ and let $\delta^{2-\alpha}N \le 1$. Let $\mu_{N,\delta}$ be the number in Proposition 3.8. Then*

$$
|P_{K^{(N,\delta)},v}(\bar t,x_1;\bar s,y) - P_{K^{(N,\delta)},v}(\bar t,x_2;\bar s,y)| \le C(\bar t-\bar s)^{-\frac{d}{\alpha}}\left(\frac{|x_1-x_2| + \mu_{N,\delta}^{\frac{1}{\alpha}}}{(\bar t-\bar s)^{\frac{1}{\alpha}}}\right)^{\beta}, \tag{3.67}
$$

for all $\bar t > \bar s \ge 0$ and $x_1,\ x_2,\ y \in \mathbb{R}^d$ satisfying

$$
\frac{|x_1-x_2| + \mu_{N,\delta}^{\frac{1}{\alpha}}}{(\bar t-\bar s)^{\frac{1}{\alpha}}} \le C'. \tag{3.68}
$$

Here C, β, and C' depend only on d, α, C_0, λ, and $\|v\|_{X_\lambda}$.

Proof. Without loss of generality we may assume that $\bar s = 0$. Fix $\bar t > 0$. Let ϕ be a positive decreasing function in Proposition 3.11. Then we set

$$
\begin{aligned}
& \omega = 1 - \frac{\phi(2)}{4} \in (0,1), \qquad x_0 = \frac{x_1+x_2}{2}, \\
& \hat v(\tau,x) = -j_{\bar t} v(\tau,x) = -v(\bar t-\tau,x), \qquad \hat K(\tau,x,y) = K^{(N,\delta)}(\bar t-\tau,x,y), \qquad 0 \le \tau < \bar t, \\
& z_n(\tau) = \int_0^\tau ⨍_{B_{r_n}(x_0)} \hat v(s,\cdot+z_n(s))\,\mathrm{d}s, \qquad r_n > 0, \qquad n \in \mathbb{N}\cup\{0\}, \\
& \zeta_n(\tau) = z_n(\tau) - z_{n-1}(\tau), \qquad n \in \mathbb{N}, \\
& \hat v_n(\tau,x) = \hat v(\tau,x+z_n(\tau)) - ⨍_{B_{r_n}(x_0)} \hat v(\tau,\cdot+z_n(\tau)), \qquad 0 \le \tau < \bar t, \\
& \hat K_n(\tau,x,y) = \hat K(\tau,x+z_n(\tau),y+z_n(\tau)).
\end{aligned}
$$

Here $\{r_n\}_{n=0}^\infty$ is a sequence of positive numbers such that $r_n \le r_{n+1}$ for all n, which will be determined later. Note that $z_n(\tau)$ is a solution to (2.21) with $u = \hat v$, $x = x_0$, $R = r_n$. The following lemma is important.

Lemma 3.13 *Let $\zeta_n(\tau)$ be the vector defined as above, and let $\xi^*_{\hat{v}_n}(x,\tau,s)$ be the vector in Proposition 3.4. Then there is a positive constant C depending only on d, α, λ, and $\|v\|_{X_\lambda}$ such that*

$$|\zeta_n(\tau)| \le \begin{cases} C\tau^{\frac{1}{2}+\frac{1}{\alpha}-\frac{\lambda}{2d}} r_{n-1}^{\frac{\alpha\lambda}{2d}-\frac{\alpha}{2}}, & \lambda \in [\frac{2d}{\alpha}-d, d), \\ C\tau^{\frac{1}{\alpha}}\big(1+\log(e+\frac{\tau^{\frac{1}{\alpha}}}{r_{n-1}})+\log\frac{r_n}{r_{n-1}}\big), & \lambda = d, \\ C(\tau^{\frac{1}{\alpha}}+r_n), & \lambda \in (d, \frac{2d}{\alpha}+d], \end{cases} \tag{3.69}$$

$$\begin{aligned} &|\xi^*_{\hat{v}_n}(x,\tau,s)| \\ \le\ & \begin{cases} C\big((\tau-s)^{\frac{1}{\alpha}}+(\tau-s)^{\frac{1}{2}+\frac{1}{\alpha}-\frac{\lambda}{2d}} r_n^{\frac{\alpha\lambda}{2d}-\frac{\alpha}{2}}\big), & \text{if } \lambda \in [\frac{2d}{\alpha}-d, d), \\ C(\tau-s)^{\frac{1}{\alpha}}\big(\log(e+\frac{|x-x_0|}{(\tau-s)^{\frac{1}{\alpha}}})+|\log\frac{r_n}{(\tau-s)^{\frac{1}{\alpha}}}|\big), & \text{if } \lambda = d, \\ C\big((\tau-s)^{\frac{1}{\alpha}}+|x-x_0|+r_n\big), & \text{if } \lambda \in (d, \frac{2d}{\alpha}+d], \end{cases} \end{aligned} \tag{3.70}$$

and

$$|\xi^*_{\hat{v}_n}(x,\tau,s)-\xi^*_{\hat{v}_n}(x_0,\tau,s)| \le C\big((\tau-s)^{\frac{1}{\alpha}}+|x-x_0|\big). \tag{3.71}$$

Proof of Lemma 3.13. The estimate (3.69) follows from (2.24) and $\|v\|_{L^1(0,\tau;\mathcal{L}^{1,\mu})} \le C\tau^{\frac{1}{2}+\frac{1}{\alpha}-\frac{\lambda}{2d}}\|v\|_{X_\lambda}$ if $\mu = \alpha\lambda/(2d)-\alpha/2+d$. The estimate (3.70) follows from (3.29) and Lemma 2.2. The estimate (3.71) is already proved in (3.65). This completes the proof of Lemma 3.13.

In the sequel we will freely use the relation

$$\begin{aligned} & P_{K^{(N,\delta)},v}(\bar{t},x_i;\bar{t}-\tau,y+z_n(\tau)) \\ =\ & P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);0,x_i) \\ =\ & \int_{\mathbb{R}^d} P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);s,\xi)P_{\hat{K},\hat{v}}(s,\xi;0,x_i)\,\mathrm{d}\xi \\ =\ & \int_{\mathbb{R}^d} P_{\hat{K}_n,\hat{v}_n}(\tau,y;s,\xi-\zeta_n(s))P_{\hat{K},\hat{v}}(s,\xi+z_{n-1}(s);0,x_i)\,\mathrm{d}\xi. \end{aligned} \tag{3.72}$$

Set

$$\begin{aligned} P_{+,n}(\tau,y) &= [P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);0,x_1)-P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);0,x_2)]\vee 0, \\ P_{-,n}(\tau,y) &= [P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);0,x_2)-P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);0,x_1)]\vee 0. \end{aligned}$$

Then we have

$$\begin{aligned} P_{+,n}(\tau,y)+P_{-,n}(\tau,y) &= |P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);0,x_1)-P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);0,x_2)|, \\ P_{+,n}(\tau,y)-P_{-,n}(\tau,y) &= P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);0,x_1)-P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);0,x_2). \end{aligned}$$

We define $F(\tau)$ by

$$\begin{aligned} F(\tau) &:= \int_{\mathbb{R}^d} P_{+,n}(\tau,y)dy = \int_{\mathbb{R}^d} P_{-,n}(\tau,y)\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathbb{R}^d} |P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);0,x_1)-P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);0,x_2)|\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathbb{R}^d} |P_{\hat{K},\hat{v}}(\tau,y;0,x_1)-P_{\hat{K},\hat{v}}(\tau,y;0,x_2)|\,\mathrm{d}y \\ &= 1-\int_{\mathbb{R}^d} P_{K^{(N,\delta)},v}(\bar{t},x_1;\bar{t}-\tau,y)\wedge P_{K^{(N,\delta)},v}(\bar{t},x_2;\bar{t}-\tau,y)\,\mathrm{d}y. \end{aligned}$$

In particular, $F(\tau)$ does not depend either on $n\in\mathbb{N}$ or on the choice of $\{r_n\}_{n=0}^\infty$, which is the key in the following arguments. Set

$$W_n(\tau,\xi,\eta) = P_{+,n}(\tau,\xi)P_{-,n}(\tau,\eta)/F(\tau). \tag{3.73}$$

For any $\bar{t}\ge\tau>s\ge 0$ we observe from (3.72) that

$$\begin{aligned} & P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);0,x_1)-P_{\hat{K},\hat{v}}(\tau,y+z_n(\tau);0,x_2) \\ =\ & \int_{\mathbb{R}^{2d}} \big(P_{\hat{K}_n,\hat{v}_n}(\tau,y;s,\xi-\zeta_n(s))-P_{\hat{K}_n,\hat{v}_n}(\tau,y;s,\eta-\zeta_n(s))\big)W_{n-1}(s,\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned} \tag{3.74}$$

This implies

$$F(\tau) \le \int_{\mathbb{R}^{3d}} \frac{1}{2}\big|P_{\hat{K}_n,\hat{v}_n}(\tau,y;s,\xi-\zeta_n(s)) - P_{\hat{K}_n,\hat{v}_n}(\tau,y;s,\eta-\zeta_n(s))\big|W_{n-1}(s,\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}y. \tag{3.75}$$

Since $\int_{\mathbb{R}^d} \big|P_{\hat{K}_n,\hat{v}_n}(\tau,y;s,\xi-\zeta_n(s)) - P_{\hat{K}_n,\hat{v}_n}(\tau,y;s,\eta-\zeta_n(s))\big|\,\mathrm{d}y \le 2$, we always have

$$F(\tau) \le F(s), \qquad \bar{t} \ge \tau \ge s \ge 0. \tag{3.76}$$

Let $t_m \le \bar{t}/2$ be the time such that $F(t_m) = \omega^m$ if it exists. Then $F(t_0) = 1$ which implies that $t_0 = 0$ since $P_{\hat{K}_n,\hat{v}_n}(\tau,y;s,\xi-\zeta_n(s))$ is strictly positive. Again we note that each t_m does not depend on the choice of $\{r_n\}_{n=0}^{\infty}$. We set $t_m^{(\delta)} = t_m + (m-1)\mu_{N,\delta}$ if $m \ge 1$ and $t_0^{(\delta)} = 0$. Let $\gamma \in (0, \min\{\alpha, 2-\alpha\})$ and set

$$\begin{aligned}
M_{\pm,n}(s) &= \int_{\mathbb{R}^d} \rho_\gamma(y-x_0)P_{\pm,n}(s,y)\,\mathrm{d}y,\\
M_n(s) &= M_{+,n}(s) \vee M_{-,n}(s),\\
M_n &= M_n(t_n^{(\delta)}), \qquad \text{if } \ t_n^{(\delta)} \le \bar{t}/2.
\end{aligned}$$

In particular, we have $M_0 = M_0(0) = 2^{-\gamma/2}|x_1-x_2|^{\gamma/2}$. We also set

$$\begin{aligned}
P'_{\pm,n}(s,\xi) &= P_{\pm,n}(s,\xi) \quad \text{if } \ \rho_\gamma(\xi-x_0) \le 2F(t_n^{(\delta)})^{-1}M_n, \qquad P'_{\pm,n}(s,\xi) = 0 \quad \text{otherwise},\\
W'_n(s,\xi,\eta) &= F(t_n^{(\delta)})^{-1}P'_{+,n}(s,\xi)P'_{-,n}(s,\eta).
\end{aligned}$$

Note that $W_n(t_n^{(\delta)},\xi,\eta) \ge W'_n(t_n^{(\delta)},\xi,\eta)$ holds. Hence when $t_n^{(\delta)} < \tau \le \bar{t}/2$ we have from (3.75),

$$\begin{aligned}
F(\tau) &\le \int_{\mathbb{R}^{2d}} \big(W_n(t_n^{(\delta)},\xi,\eta) - W'_n(t_n^{(\delta)},\xi,\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&+\frac{1}{2}\int_{\mathbb{R}^{3d}} \big|P_{\hat{K}_{n+1},\hat{v}_{n+1}}(\tau,y;t_n^{(\delta)},\xi-\zeta_n(t_n^{(\delta)})) - P_{\hat{K}_{n+1},\hat{v}_{n+1}}(\tau,y;t_n^{(\delta)},\eta-\zeta_n(t_n^{(\delta)}))\big|W'_n(t_n^{(\delta)},\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}y
\end{aligned}$$

Since $W'_n(t_n^{(\delta)},\xi,\eta) > 0$ only when $|\xi-x_0| \le (2F(t_n^{(\delta)})^{-1}M_n)^{2/\gamma}$ and $|\eta-x_0| \le (2F(t_n^{(\delta)})^{-1}M_n)^{2/\gamma}$, we have $|\xi-\eta| \le 2(2F(t_n^{(\delta)})^{-1}M_n)^{2/\gamma}$ in this case. Now we take

$$\tau = \tau_n = t_n^{(\delta)} + \max\{(2F(t_n^{(\delta)})^{-1}M_n)^{2\alpha/\gamma}, \mu_{N,\delta}\}. \tag{3.77}$$

Then $|\xi-\zeta_n(s)-(\eta-\zeta_n(s))| = |\xi-\eta| \le 2(\tau_n - t_n^{(\delta)})^{1/\alpha}$ and $\tau_n - t_n^{(\delta)} \ge \mu_{N,\delta}$. Thus, by Proposition 3.11, if $\tau_n \le \bar{t}/2$ then

$$\begin{aligned}
F(\tau_n) &\le \int_{\mathbb{R}^{2d}} \big(W_n(t_n^{(\delta)},\xi,\eta) - W'_n(t_n^{\delta},\xi,\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&\qquad + \int_{\mathbb{R}^{2d}} W'_n(t_n^{(\delta)},\xi,\eta)\big(1-\phi(\frac{|\xi-\eta|}{(\tau_n-t_n^{(\delta)})^{\frac{1}{\alpha}}})\big)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&\le \int_{\mathbb{R}^{2d}} \big(W_n(t_n^{(\delta)},\xi,\eta) - W'_n(t_n^{(\delta)},\xi,\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta + (1-\phi(2))\int_{\mathbb{R}^{2d}} W'_n(t_n^{(\delta)},\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&= \int_{\mathbb{R}^{2d}} W_n(t_n^{(\delta)},\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta - \phi(2)\int_{\mathbb{R}^{2d}} W'_n(t_n^{(\delta)},\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned}$$

Hence from the estimate

$$\begin{aligned}
\int_{\mathbb{R}^d} P'_{\pm,n}(t_n^{(\delta)},\xi)\,\mathrm{d}\xi &= F(t_n^{(\delta)}) - \int_{|\xi-x_0|\ge(2F(t_n^{(\delta)})M_n)^{\frac{2}{\gamma}}} P_{\pm,n}(t_n^{(\delta)},\xi)\,\mathrm{d}\xi\\
&\ge F(t_n^{(\delta)}) - (2M_n)^{-1}F(t_n^{(\delta)})\int_{\mathbb{R}^d} \rho_\gamma(\xi-x_0)P_{\pm,n}(t_n^{(\delta)},\xi)\,\mathrm{d}\xi\\
&\ge \frac{1}{2}F(t_n^{(\delta)}),
\end{aligned}$$

we have

$$F(\tau_n) \le F(t_n^{(\delta)}) - \phi(2)F(t_n^{(\delta)})^{-1}(\frac{F(t_n^{(\delta)})}{2})^2 = F(t_n^{(\delta)})(1-\frac{1}{4}\phi(2)) \le F(t_n)\omega = \omega^{n+1}.$$

That is, $t_{n+1} \le \tau_n$, which gives the estimate

$$\begin{aligned} t_{n+1}^{(\delta)} = t_{n+1} + n\mu_{N,\delta} &\le \tau_n + n\mu_{N,\delta} \\ &\le t_n^{(\delta)} + (n+1)\mu_{N,\delta} + (2F(t_n^{(\delta)})^{-1}M_n)^{\frac{2\alpha}{\gamma}}. \end{aligned} \tag{3.78}$$

Next we estimate M_n. The equality (3.74) leads to the inequality

$$P_{\pm,n+1}(\tau, y) \le \int_{\mathbb{R}^d} P_{\hat{K}_{n+1},\hat{v}_{n+1}}(\tau, y; s, x - \zeta_{n+1}(s)) P_{\pm,n}(s, x)\,\mathrm{d}x,$$

hence, we have

$$\begin{aligned} M_{\pm,n+1}(\tau) &\le \int_{\mathbb{R}^{2d}} \rho_\gamma(y - x_0) P_{\hat{K}_{n+1},\hat{v}_{n+1}}(\tau, y; s, x - \zeta_{n+1}(s)) P_{\pm,n}(s, x)\,\mathrm{d}x\,\mathrm{d}y \\ &\le \int_{\mathbb{R}^{2d}} \Big\{ \rho_\gamma(x - x_0) + \rho_\gamma\big(\xi^*_{\hat{v}_{n+1}}(x - \zeta_{n+1}(s), \tau, s) - \xi^*_{\hat{v}_{n+1}}(x_0 - \zeta_{n+1}(s), \tau, s)\big) \\ &\qquad + \rho_\gamma\big(y - \xi^*_{\hat{v}_{n+1}}(x - \zeta_{n+1}(s), \tau, s) - x + \zeta_{n+1}(s)\big) \\ &\qquad\qquad + \rho_\gamma\big(\xi^*_{\hat{v}_{n+1}}(x_0 - \zeta_{n+1}(s), \tau, s) - \zeta_{n+1}(s)\big) \Big\} \\ &\quad \times P_{\pm,n}(s, x) P_{\hat{K}_{n+1},\hat{v}_{n+1}}(\tau, y; s, x - \zeta_{n+1}(s))\,\mathrm{d}x\,\mathrm{d}y \\ &\le M_{\pm,n}(s) + \int_{\mathbb{R}^d} |\xi^*_{\hat{v}_{n+1}}(x - \zeta_{n+1}(s), \tau, s) - \xi^*_{\hat{v}_{n+1}}(x_0 - \zeta_{n+1}(s), \tau, s)|^{\frac{\gamma}{2}} P_{\pm,n}(s, x)\,\mathrm{d}x \\ &\quad + F(s) \sup_x \int_{\mathbb{R}^d} \rho_\gamma(y - \xi^*_{\hat{v}_{n+1}}(x, \tau, s) - x + \zeta_{n+1}(s)) P_{\hat{K}_{n+1},\hat{v}_{n+1}}(\tau, y; s, x)\,\mathrm{d}y \\ &\quad + F(s) |\xi^*_{\hat{v}_{n+1}}(x_0 - \zeta_{n+1}(s), \tau, s) - \zeta_{n+1}(s)|^{\frac{\gamma}{2}}. \end{aligned} \tag{3.79}$$

From Proposition 3.4 and Lemma 3.13, if $\tau - s \ge \mu_{N,\delta}$ then we have the estimate of $M_{\pm,n+1}(\tau)$ such as

$$M_{\pm,n+1}(\tau) \le CM_{\pm,n}(s) + \left(C(\tau - s)^{\frac{\gamma}{2\alpha}} + |\xi^*_{\hat{v}_{n+1}}(x_0 - \zeta_{n+1}(s), \tau, s)|^{\frac{\gamma}{2}} + 2|\zeta_{n+1}(s)|^{\frac{\gamma}{2}} \right) F(s). \tag{3.80}$$

From $t_{n+1}^{(\delta)} - t_n^{(\delta)} \ge \mu_{N,\delta}$ and (3.80) we have

$$M_{n+1} \le CM_n + F(t_n^{(\delta)}) \left(C(t_{n+1}^{(\delta)} - t_n^{(\delta)})^{\frac{\gamma}{2\alpha}} + |\xi^*_{\hat{v}_{n+1}}(x_0 - \zeta_{n+1}(t_n^{(\delta)}), t_{n+1}^{(\delta)}, t_n^{(\delta)})|^{\frac{\gamma}{2}} + 2|\zeta_{n+1}(t_n^{(\delta)})|^{\frac{\gamma}{2}} \right). \tag{3.81}$$

Let us take $r_0 = r_1 = (t_1^{(\delta)})^{1/\alpha}$ and $r_n = (t_n^{(\delta)})^{1/\alpha}$ for $n \ge 2$. Then $\zeta_1(t_0) = \zeta_1(0) = 0$ and Lemma 3.13 yields

$$|\zeta_{n+1}(t_n^{(\delta)})| \le C\big((t_{n+1}^{(\delta)} - t_n^{(\delta)})^{\frac{1}{\alpha}} + (t_n^{(\delta)})^{\frac{1}{\alpha}}\big), \tag{3.82}$$

and also

$$|\xi^*_{\hat{v}_{n+1}}(x_0 - \zeta_{n+1}(t_n^{(\delta)}), t_{n+1}^{(\delta)}, t_n^{(\delta)})| \le C\big((t_{n+1}^{(\delta)} - t_n^{(\delta)})^{\frac{1}{\alpha}} + (t_n^{(\delta)})^{\frac{1}{\alpha}}\big). \tag{3.83}$$

Hence we finally get

$$M_{n+1} \le C_1 M_n + C_2 F(t_n^{(\delta)})\big((t_{n+1}^{(\delta)} - t_n^{(\delta)})^{\frac{\gamma}{2\alpha}} + (t_n^{(\delta)})^{\frac{\gamma}{2\alpha}}\big), \tag{3.84}$$

as long as $t_{n+1}^{(\delta)} = t_{n+1} + n\mu_{N,\delta} \le \bar{t}/2$. From (3.78), (3.84), and $F(t_n^{(\delta)}) \le F(t_n) = \omega^n$, we have

$$\begin{aligned} M_{n+1} &\le C_1 M_n + C_2 F(t_n^{(\delta)}) \sum_{k=0}^n (t_{k+1}^{(\delta)} - t_k^{(\delta)})^{\frac{\gamma}{2\alpha}} \\ &\le C_1 M_n + C_2 F(t_n^{(\delta)}) \sum_{k=0}^n \left((k+1)\mu_{N,\delta} + \big(2F(t_k^{(\delta)})^{-1} M_k\big)^{\frac{2\alpha}{\gamma}} \right)^{\frac{\gamma}{2\alpha}} \\ &\le C_1 M_n + 2C_2 \sum_{k=0}^n M_k + C_2 \mu_{N,\delta}^{\frac{\gamma}{2\alpha}} (n+1)^{1+\frac{\gamma}{2\alpha}} \omega^n \\ &\le C_1 M_n + 2C_2 \sum_{k=0}^n M_k + C_3 \mu_{N,\delta}^{\frac{\gamma}{2\alpha}}. \end{aligned} \tag{3.85}$$

Here we used the inequality $(n+1)^{1+\frac{\gamma}{2\alpha}} \omega^n \le C$ for all $n \ge 0$. Then we have

$$M_n \le C_4^n (M_0 + \mu_{N,\delta}^{\frac{\gamma}{2\alpha}}), \qquad C_4 = \max\{3C_1, 1 + 6C_2, 3C_3\}. \tag{3.86}$$

Indeed, (3.86) is valid for $n=0$. Assuming that (3.86) holds up to n, we have from (3.85),

$$\begin{aligned}
M_{n+1} &\leq C_1 C_4^n (M_0 + \mu_{N,\delta}^{\frac{\gamma}{2\alpha}}) + 2C_2 (M_0 + \mu_{N,\delta}^{\frac{\gamma}{2\alpha}}) \sum_{k=0}^{n} C_4^k + C_3 \mu_{N,\delta}^{\frac{\gamma}{2\alpha}} \\
&\leq \big(C_1 C_4^n + \frac{2C_2 C_4^{n+1}}{C_4 - 1}\big) M_0 + \big(C_1 C_4^n + \frac{2C_2 C_4^{n+1}}{C_4 - 1} + C_3\big) \mu_{N,\delta}^{\frac{\gamma}{2\alpha}}.
\end{aligned}$$

This proves (3.86) for $n+1$ due to the choice of C_4. Then by (3.78) and (3.86), if $\tau_n \leq \bar{t}/2$, we have

$$\begin{aligned}
t_{n+1}^{(\delta)} = \sum_{k=0}^{n} t_{k+1}^{(\delta)} - t_k^{(\delta)} &\leq \sum_{k=0}^{n} (k+1)\mu_{N,\delta} + \big(2F(t_k^{(\delta)})^{-1} M_k\big)^{\frac{2\alpha}{\gamma}} \\
&\leq (n+1)^2 \mu_{N,\delta} + \big(\frac{2}{F(\bar{t}/2)}\big)^{\frac{2\alpha}{\gamma}} \sum_{k=0}^{n} M_k^{\frac{2\alpha}{\gamma}} \\
&\leq (n+1)^2 \mu_{N,\delta} + \big(\frac{2}{F(\bar{t}/2)}\big)^{\frac{2\alpha}{\gamma}} (M_0 + \mu_{N,\delta}^{\frac{\gamma}{2\alpha}})^{\frac{2\alpha}{\gamma}} \sum_{k=0}^{n} C_4^{\frac{2k\alpha}{\gamma}} \\
&\leq C_5^{n+1} F(\bar{t}/2)^{-\frac{2\alpha}{\gamma}} (|x_1 - x_2|^{\frac{\gamma}{2}} + \mu_{N,\delta}^{\frac{\gamma}{2\alpha}})^{\frac{2\alpha}{\gamma}} \\
&= C_5^{n+1} \bigg(\frac{\sigma}{F(\bar{t}/2)}\bigg)^{\frac{2\alpha}{\gamma}} \bar{t},
\end{aligned} \tag{3.87}$$

where

$$\sigma = \big(\frac{|x_1 - x_2|}{\bar{t}^{\frac{1}{\alpha}}}\big)^{\frac{\gamma}{2}} + \big(\frac{\mu_{N,\delta}}{\bar{t}}\big)^{\frac{\gamma}{2\alpha}}. \tag{3.88}$$

Here we used the estimates $F(\bar{t}/2) \leq F(t_k^{(\delta)})$ and $M_0 = 2^{-\gamma/2}|x_1 - x_2|^{\gamma/2}$. If $F(\bar{t}/2) \geq \sigma^{1/2}$ then (3.87) implies that

$$t_n^{(\delta)} \leq C_5^n \sigma^{\frac{\alpha}{\gamma}} \bar{t}. \tag{3.89}$$

Hence we can take $n = n(\sigma)$ as

$$-\frac{\alpha}{2\gamma \log C_5} \log \sigma \leq n(\sigma) < -\frac{\alpha}{2\gamma \log C_5} + 1, \tag{3.90}$$

which satisfies both $n(\sigma) \to \infty$ as $\sigma \to 0$ and $C_5^{n(\sigma)} \sigma^{\frac{\alpha}{\gamma}} \leq C_5 \sigma^{\alpha/(2\gamma)} \to 0$ as $\sigma \to 0$. In particular, if $\sigma \leq (2C_5)^{-2\gamma/\alpha}$ then $t_{n(\sigma)}^{(\delta)} \leq \bar{t}/2$ and so

$$F(\frac{\bar{t}}{2}) \leq F(t_{n(\sigma)}^{(\delta)}) \leq F(t_{n(\sigma)}) = \omega^{n(\sigma)} = e^{n(\sigma)\log\omega} \leq \sigma^{-\frac{\alpha \log \omega}{2\gamma \log C_5}}.$$

Thus we have $F(\bar{t}/2) \leq \sigma^{\beta}$ with $\beta = \min\{1/2, -\alpha \log\omega/(2\gamma \log C_5)\}$. This estimate yields

$$\begin{aligned}
&|P_{K^{(N,\delta)},v^{(\delta)}}(\bar{t}, x_1; 0, y) - P_{K^{(N,\delta)},v^{(\delta)}}(\bar{t}, x_2; 0, y)| = |P_{\hat{K},\hat{v}}(\bar{t}, y; 0, x_1) - P_{\hat{K},\hat{v}}(\bar{t}, y; 0, x_2)| \\
\leq\ & \int_{\mathbb{R}^d} P_{\hat{K},\hat{v}}(\bar{t}, y; \frac{\bar{t}}{2}, \xi) |P_{\hat{K},\hat{v}}(\frac{\bar{t}}{2}, \xi; 0, x_1) - P_{\hat{K},\hat{v}}(\frac{\bar{t}}{2}, \xi; 0, x_2)| \, \mathrm{d}\xi \\
\leq\ & C\bar{t}^{-\frac{d}{\alpha}} F(\frac{\bar{t}}{2}) \leq C\bar{t}^{-\frac{d}{\alpha}} \bigg(\big(\frac{|x_1 - x_2|}{\bar{t}^{\frac{1}{\alpha}}}\big)^{\frac{\gamma}{2}} + \big(\frac{\mu_{N,\delta}}{\bar{t}}\big)^{\frac{\gamma}{2\alpha}} \bigg)^{\beta},
\end{aligned}$$

if

$$\big(\frac{|x_1 - x_2|}{\bar{t}^{\frac{1}{\alpha}}}\big)^{\frac{\gamma}{2}} + \big(\frac{\mu_{N,\delta}}{\bar{t}}\big)^{\frac{\gamma}{2\alpha}} \leq (2C_5)^{-2\gamma/\alpha}. \tag{3.91}$$

Here C, β, and C_5 depend only on d, α, C_0, λ, and $\|v\|_{X_\lambda}$. This completes the proof.

Corollary 3.14 *Under the assumptions of Proposition 3.12 it follows that*

$$|P_{K^{(N,\delta)},v}(t, x_1; s, y_1) - P_{K^{(N,\delta)},v}(t, x_2; s, y_2)| \leq \frac{C}{(t-s)^{\frac{d}{\alpha}}} \bigg(\frac{|x_1 - x_2| + |y_1 - y_2| + \mu_{N,\delta}^{\frac{1}{\alpha}}}{(t-s)^{\frac{1}{\alpha}}} \bigg)^{\beta}, \tag{3.92}$$

for all $x_i,\ y_i \in \mathbb{R}^d$ and $t > s \geq 0$ such that $t - s \geq C'\mu_{N,\delta}$. Here C, β, and C' depend only on d, α, C_0, λ, and $\|v\|_{X_\lambda}$.

Proof. It suffices to consider the case $y_1 = y_2 = y$, for the continuity estimate with respect to y follows from the adjoint relation. Let C' be the constant in Proposition 3.12. Assume that $\mu_{N,\delta} \le (2^{-1}C')^\alpha (t-s)$. Then if $|x_1 - x_2| \le 2^{-1}C'(t-s)^{1/\alpha}$ the estimate (3.92) follows from Proposition 3.12. If $|x_1 - x_2| \ge 2^{-1}C'(t-s)^{1/\alpha}$ then

$$\begin{aligned} |P_{K^{(N,\delta)},v}(t,x_1;s,y) - P_{K^{(N,\delta)},v}(t,x_2;s,y)| &\le C(t-s)^{-\frac{d}{\alpha}} = C(t-s)^{-\frac{d}{\alpha}}\Big(\frac{(t-s)^{\frac{1}{\alpha}}}{|x_1-x_2|}\frac{|x_1-x_2|}{(t-s)^{\frac{1}{\alpha}}}\Big)^\beta \\ &\le C(t-s)^{-\frac{d}{\alpha}}\Big(\frac{|x_1-x_2|}{(t-s)^{\frac{1}{\alpha}}}\Big)^\beta. \end{aligned}$$

This completes the proof.

Corollary 3.15 *Under the assumptions of Proposition 3.12 it follows that*

$$\begin{aligned} &|P_{K^{(N,\delta)},v}(t_1,x;s_1,y) - P_{K^{(N,\delta)},v}(t_2,x;s_2,y)| \\ \le\ & \frac{C}{(\min\{t_1-s_1,t_2-s_2\})^{\frac{d+c}{\alpha}}}\{C_{T,x}|t_1-t_2|^{\beta'} + C_{T,y}|s_1-s_2|^{\beta'} + \mu_{N,\delta}^{\frac{c}{\alpha}}\}, \end{aligned} \tag{3.93}$$

for all $x,\ y \in \mathbb{R}^d$ and $T \ge t_i > s_i \ge 0$ such that $\min\{t_1-t_2, s_1-s_2, t_1-s_1, t_2-s_2\} \ge C'\mu_{N,\delta}$. Here C, c, and C' depend only on d, α, C_0, λ, and $\|v\|_{X_\lambda}$. The constant $\beta' > 0$ depends only on d, α, C_0, λ, q, and $\|v\|_{X_\lambda}$. The constant $C_{T,x}$ (or $C_{T,y}$) depends only on T, d, α, λ, q, $\|v\|_{X_\lambda}$, and $\|v\|_{Y^{q,\lambda}_{T,x}}$ (or $\|v\|_{Y^{q,\lambda}_{T,y}}$).

Proof. It suffices to consider the case $s_1 = s_2 = s$. Let $R > 0$. By the semigroup property of fundamental solutions, we have

$$\begin{aligned} &|P_{K^{(N,\delta)},v}(t_1,x;s,y) - P_{K^{(N,\delta)},v}(t_2,x;s,y)| \\ \le\ & \int_{\mathbb{R}^d} P_{K^{(N,\delta)},v}(t_1,x;t_2,z)|P_{K^{(N,\delta)},v}(t_2,z;s,y) - P_{K^{(N,\delta)},v}(t_2,x;s,y)|\,\mathrm{d}z \\ =\ & \int_{|x-z|\le R} P_{K^{(N,\delta)},v}(t_1,x;t_2,z)|P_{K^{(N,\delta)},v}(t_2,z;s,y) - P_{K^{(N,\delta)},v}(t_2,x;s,y)|\,\mathrm{d}z \\ &+ \int_{|x-z|\ge R} P_{K^{(N,\delta)},v}(t_1,x;t_2,z)|P_{K^{(N,\delta)},v}(t_2,z;s,y) - P_{K^{(N,\delta)},v}(t_2,x;s,y)|\,\mathrm{d}z \\ =\ & I_1 + I_2. \end{aligned} \tag{3.94}$$

Applying Corollary 3.14 to I_1 yields $I_1 \le C(t_2-s)^{-(d+\beta)/\alpha}(R^\beta + \mu_{N,\delta}^{\frac{\beta}{\alpha}})$ if $t_2 - s \ge C'\mu_{N,\delta}$. As for I_2, we use the moment bound in Proposition 3.5 and then there is $\eta_v^* = \eta_v^*(x,t_1,t_2)$ such that

$$\begin{aligned} I_2 &\le \frac{C}{(t_2-s)^{\frac{d}{\alpha}}R^{\frac{\gamma}{2}}}\int_{|x-z|\ge R} P_{K^{(N,\delta)},v}(t_1,x;t_2,z)\rho_\gamma(z-x)\,\mathrm{d}z \\ &\le \frac{C}{(t_2-s)^{\frac{d}{\alpha}}R^{\frac{\gamma}{2}}}\int_{\mathbb{R}^d} P_{K^{(N,\delta)},v}(t_1,x;t_2,z)\big(\rho_\gamma(z-x-\eta_v^*) + |\eta_v^*|^{\frac{\gamma}{2}}\big)\,\mathrm{d}z \\ &\le \frac{C}{(t_2-s)^{\frac{d}{\alpha}}R^{\frac{\gamma}{2}}}\big\{(t_1-t_2)^{\frac{\gamma}{2\alpha}}\big(1 + (t_1-t_2)^{1-\frac{2}{\alpha}}\delta^{2-\alpha}N + \delta(t_1-t_2)^{\frac{1}{2}-\frac{1}{\alpha}}\big) + |\eta_v^*|^{\frac{\gamma}{2}}\big\} \\ &\le \frac{C}{(t_2-s)^{\frac{d}{\alpha}}R^{\frac{\gamma}{2}}}\big\{(t_1-t_2)^{\frac{\gamma}{2\alpha}} + |\eta_v^*(x,t_1,t_2)|^{\frac{\gamma}{2}}\big\}, \end{aligned} \tag{3.95}$$

if $t_1 - t_2 \ge C'\mu_{N,\delta}$. By Proposition 3.9 the vector $\eta_v^*(x,t_1,t_2)$ satisfies the estimate

$$|\eta_v^*(x,t_1,t_2)| \le C\big((t_1-t_2)^{\frac{1}{\alpha}}(1 + |\log(t_1-t_2)|) + (t_1-t_2)^{1-\frac{1}{q}}\big), \tag{3.96}$$

where C depends only on T, d, α, λ, $\|v\|_{X_\lambda}$, and $\|v\|_{Y^{q,\lambda}_{T,x}}$. Collecting (3.94)-(3.96) and taking R of the form $(t_1-t_2)^\nu$ for suitable ν depending only on α and q, we get (3.93). This completes the proof.

4 Existence of fundamental solutions

In this section we complete the proof of Theorems 1.2 and 1.4. We first take the limit $\delta \to 0$ of $P_{K^{(N,\delta)},v^{(N)}}(t,x;s,y)$ and next we consider the limit $N \to \infty$.

4.1 Limit $\delta \to 0$

Set $D = \{(t,s) \in \mathbb{R}^2 \mid t > s \geq 0\}$, $D_m = \{(t,s) \in D \mid t > s + 1/m\}$ with $m \geq 1$, $\Omega = D \times \mathbb{R}^d \times \mathbb{R}^d$, and $\Omega_m = D_m \times \mathbb{R}^d \times \mathbb{R}^d$. Fix $N \gg 1$. First we assume that v is smooth and bounded. As a function on Ω, by (2.31) $P_{K^{(N,\delta)},v}(t,x;s,y)$ satisfies the following estimates:

$$P_{K^{(N,\delta)},v}(t,x;s,y) \leq C(t-s)^{-\frac{d}{\alpha}}, \tag{4.1}$$

where C depends only on d, α, and C_0. So there is a sequence $\{\delta_n\}_{n=1}^\infty$ with $\delta_n \to 0$ as $n \to \infty$ such that $P_{K^{(N,\delta_n)},v}(t,x;s,y) \rightharpoonup^* P_{K^{(N)},v}(t,x;s,y)$ as $n \to \infty$ in $L^\infty(\Omega_m)$ for each $m \geq 1$ for some nonnegative function $P_{K^{(N)},v}(t,x;s,y) \in L^\infty_{loc}(\Omega)$, that is,

$$\lim_{n\to\infty} \int_\Omega P_{K^{(N,\delta_n)},v}(t,x;s,y)\varphi(t,s,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}s = \int_\Omega P_{K^{(N)},v}(t,x;s,y)\varphi(t,s,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}s, \tag{4.2}$$

for every $\varphi \in L^1(\Omega)$ supported in Ω_m for some $m \geq 1$. Taking $\varphi(t,s,x,y)$ of the form $\varphi(t,s,x)f(y)$ in (4.2), by the duality argument we see that $(P_{K^{(N)},v}f)(t,s,x) = \int_{\mathbb{R}^d} P_{K^{(N)},v}(t,x;s,y)f(y)\,\mathrm{d}y$ belongs to $L^\infty(D; L^p(\mathbb{R}^d))$ for all $p \in [1,\infty]$ and satisfies

$$\|P_{K^{(N)},v}f\|_{L^\infty(D;L^p)} \leq \|f\|_{L^p}. \tag{4.3}$$

By taking $\varphi(t,s,x,y)$ of the form $(t-s)^{d/\alpha}\varphi(t,s,x,y)$ in (4.2) we also have

$$P_{K^{(N)},v}(t,x;s,y) \leq C(t-s)^{-\frac{d}{\alpha}}, \tag{4.4}$$

where C depends only on d, α, and C_0.

(i) Moment bound. Next we prove the moment bound

$$C_1(t-s)^{\frac{\gamma}{2\alpha}} \leq \int_{\mathbb{R}^d} \rho_\gamma(y-x-\xi_v^*(x,t,s))P_{K^{(N)},v}(t,x;s,y)\,\mathrm{d}y \leq C_2(t-s)^{\frac{\gamma}{2\alpha}}, \tag{4.5}$$

where $\xi_v^*(x,t,s)$ is the vector given by Proposition 3.4, $\gamma \in [0, \min\{\alpha, 2-\alpha\})$, and C_1, C_2 depend only on d, α, C_0, γ, and $\|v\|_{X_\lambda}$. To prove the upper bound, we take $\varphi(t,s,x,y)$ of the form $(t-s)^{-\gamma/(2\alpha)}\rho_\gamma(y-x-\xi_v^*(x,t,s))\varphi(t,s,x,y)$ in (4.2). Then we have from Proposition 3.4,

$$\begin{aligned}
&\int_D\int_{\mathbb{R}^{2d}} P_{K^{(N)},v}(t,x;s,y)(t-s)^{-\frac{\gamma}{2\alpha}}\rho_\gamma(y-x-\xi_v^*(x,t,s))\varphi(t,s,x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
= &\lim_{n\to\infty}\int_D\int_{\mathbb{R}^{2d}} P_{K^{(N,\delta_n)},v}(t,x;s,y)(t-s)^{-\frac{\gamma}{2\alpha}}\rho_\gamma(y-x-\xi_v^*(x,t,s))\varphi(t,s,x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
\leq\ &C_2\lim_{n\to\infty}\int_D \|\varphi(t,s)\|_{L^1(\mathbb{R}^d_x;L^\infty(\mathbb{R}^d_y))}\big(1+(1+(t-s)^{1-\frac{2}{\alpha}})\delta_n^{2-\alpha}N+\delta_n(t-s)^{\frac12-\frac1\alpha}\big)\,\mathrm{d}t\,\mathrm{d}s\\
\leq\ &C_2\|\varphi\|_{L^1(D\times\mathbb{R}^d_x;L^\infty(\mathbb{R}^d_y))}.
\end{aligned}$$

This proves the upper bound of (4.5). Regarding the lower bound, for all $\varphi \in C_0^\infty(D\times\mathbb{R}^d)$ with $\varphi \geq 0$, $\gamma' \in (\gamma, \min\{\alpha, 2-\alpha\})$, and $R \geq 1$, we have

$$\begin{aligned}
&\int_D\int_{\mathbb{R}^d}\varphi(t,s,x)\int_{\mathbb{R}^d}\rho_\gamma(y-x-\xi_v^*(x,t,s))P_{K^{(N)},v}(t,x;s,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
= &\int_D\int_{\mathbb{R}^d}\varphi(t,s,x)\int_{|y-x-\xi_v^*(x,t,s)|\leq R}\rho_\gamma(y-x-\xi_v^*(x,t,s))P_{K^{(N)},v}(t,x;s,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
&+\int_D\int_{\mathbb{R}^d}\varphi(t,s,x)\int_{|y-x-\xi_v^*(x,t,s)|\geq R}\rho_\gamma(y-x-\xi_v^*(x,t,s))P_{K^{(N)},v}(t,x;s,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
\geq &\lim_{n\to\infty}\int_D\int_{\mathbb{R}^d}\varphi(t,s,x)\int_{|y-x-\xi_v^*(x,t,s)|\leq R}\rho_\gamma(y-x-\xi_v^*(x,t,s))P_{K^{(N,\delta_n)},v}(t,x;s,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
&-CR^{\frac{\gamma-\gamma'}{2}}\int_D\int_{\mathbb{R}^d}\varphi(t,s,x)(t-s)^{\frac{\gamma'}{2\alpha}}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
\geq &C_2\int_D\int_{\mathbb{R}^d}\varphi(t,s,x)(t-s)^{\frac{\gamma}{2\alpha}}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
&-\limsup_{n\to\infty}\int_D\int_{\mathbb{R}^d}\varphi(t,s,x)\int_{|y-x-\xi_v^*(x,t,s)|\geq R}\rho_\gamma(y-x-\xi_v^*(x,t,s))P_{K^{(N,\delta_n)},v}(t,x;s,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
&-CR^{\frac{\gamma-\gamma'}{2}}\int_D\int_{\mathbb{R}^d}\varphi(t,s,x)(t-s)^{\frac{\gamma'}{2\alpha}}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
\geq &C_2\int_D\int_{\mathbb{R}^d}\varphi(t,s,x)(t-s)^{\frac{\gamma}{2\alpha}}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s - CR^{\frac{\gamma-\gamma'}{2}}\int_D\int_{\mathbb{R}^d}\varphi(t,s,x)(t-s)^{\frac{\gamma'}{2\alpha}}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

By taking $R \to \infty$ we get

$$\begin{aligned} & \int_D \int_{\mathbb{R}^d} \varphi(t,s,x) \int_{\mathbb{R}^d} \rho_\gamma(y-x-\xi_v^*(x,t,s)) P_{K^{(N)},v}(t,x;s,y)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s \\ \geq \quad & C_2 \int_D \int_{\mathbb{R}^d} \varphi(t,s,x)(t-s)^{\frac{\gamma}{2\alpha}}\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s. \end{aligned}$$

Since $\varphi \in C_0^\infty(D \times \mathbb{R}^d)$ is an arbitrary nonnegative function, we get the desired lower bound. This completes the proof of (4.5). The same argument leads to

$$C_1(t-s)^{\frac{\gamma}{2\alpha}} \leq \int_{\mathbb{R}^d} \rho_\gamma(x-y-\eta_v^*(y,t,s)) P_{K^{(N)},v}(t,x;s,y)\, \mathrm{d}x \leq C_2(t-s)^{\frac{\gamma}{2\alpha}}. \tag{4.6}$$

By Remark 3.10 the vectors $\xi_v^*(x,t,s)$ and $\eta_v^*(x,t,s)$ in (4.5) and (4.6) satisfy for all $R > 0$,

$$\sup_{|x| \leq R, 0 \leq s < t \leq R} |\xi_v^*(x,t,s)| + \sup_{|y| \leq R, 0 \leq s < t \leq R} |\eta_v^*(y,t,s)| \leq C_R < \infty. \tag{4.7}$$

(ii) Mass conservation. Next we show

$$\int_{\mathbb{R}^d} P_{K^{(N)},v}(t,x;s,y)\, \mathrm{d}y = \int_{\mathbb{R}^d} P_{K^{(N)},v}(t,x;s,y)\, \mathrm{d}x = 1. \tag{4.8}$$

Since $P_{K^{(N)},v}(t,x;s,y)$ is nonnegative we already know that

$$\int_{\mathbb{R}^d} P_{K^{(N)},v}(t,x;s,y)\, \mathrm{d}y \leq 1, \ \int_{\mathbb{R}^d} P_{K^{(N)},v}(t,x;s,y)\, \mathrm{d}x \leq 1$$

for a.e. by (4.3). So it suffices to prove the lower bound. For any $\varphi(t,s,x) \in C_0^\infty(D \times \mathbb{R}^d)$ with $\varphi \geq 0$, we have from the moment bounds for $P_{K^{(N)},v}(t,x;s,y)$ and $P_{K^{(N,\delta_n)},v}(t,x;s,y)$,

$$\begin{aligned} & \int_D \int_{\mathbb{R}^d} \varphi(t,s,x) \int_{\mathbb{R}^d} P_{K^{(N)},v}(t,x;s,y)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s \\ = \quad & \int_D \int_{\mathbb{R}^d} \varphi(t,s,x) \int_{|y-x-\xi_v^*(x,t,s)| \leq R} P_{K^{(N)},v}(t,x;s,y)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s \\ & + \int_D \int_{\mathbb{R}^d} \varphi(t,s,x) \int_{|y-x-\xi_v^*(x,t,s)| \geq R} P_{K^{(N)},v}(t,x;s,y)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s \\ \geq \quad & \lim_{n \to \infty} \int_D \int_{\mathbb{R}^d} \varphi(t,s,x) \int_{|y-x-\xi_v^*(x,t,s)| \leq R} P_{K^{(N,\delta_n)},v}(t,x;s,y)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s \\ & - C_2 R^{-\frac{\gamma}{2}} \int_D \int_{\mathbb{R}^d} \varphi(t,s,x)\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s \\ \geq \quad & \int_D \int_{\mathbb{R}^d} \varphi(t,s,x)\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s - C R^{-\frac{\gamma}{2}} \int_D \int_{\mathbb{R}^d} \varphi(t,s,x)\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s. \end{aligned}$$

Here $\gamma \in (0, \min\{\alpha, 2-\alpha\})$. Taking $R \to \infty$ we get

$$\int_D \int_{\mathbb{R}^d} \varphi(t,s,x) \int_{\mathbb{R}^d} P_{K^{(N)},v}(t,x;s,y)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s \geq \int_D \int_{\mathbb{R}^d} \varphi(t,s,x)\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s,$$

for all $\varphi \in C_0(D \times \mathbb{R}^d)$ with $\varphi \geq 0$, which proves $\int_{\mathbb{R}^d} P_{K^{(N)},v}(t,x;s,y)\, \mathrm{d}y \geq 1$ for a.e. (t,s,x). The same argument is applied for $\int_{\mathbb{R}^d} P_{K^{(N)},v}(t,x;s,y)\, \mathrm{d}x$. This proves (4.8).

(iii) Continuity. Next we prove the Hölder continuity of $P_{K^{(N)},v}(t,x;s,y)$. To this end we set $\psi_\mu(t,s,h,k) = (t-s)^{-(d+\beta)/\alpha}(|h|+|k|+\mu)^\beta$, $0 < \mu \ll 1$, where β is the constant in Corollary 3.14. Then from (4.2) we have for every $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} & \left| \int_\Omega \frac{P_{K^{(N)},v}(t,x;s,y) - P_{K^{(N)},v}(t,x+h;s,y+k)}{\psi_0(t,s,h,k)} \varphi(t,s,x,y)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t\, \mathrm{d}s \right| \\ = \quad & \left| \lim_{n \to \infty} \int_\Omega \frac{P_{K^{(N,\delta_n)},v}(t,x;s,y) - P_{K^{(N,\delta_n)},v}(t,x+h;s,y+k)}{\psi_0(t,s,h,k)} \varphi(t,s,x,y)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t\, \mathrm{d}s \right| \\ \leq \quad & C \limsup_{n \to \infty} \int_\Omega \frac{\psi_{\mu_{N,\delta_n}}(t,s,h,k)}{\psi_0(t,s,h,k)} |\varphi(t,s,x,y)|\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t\, \mathrm{d}s| \\ = \quad & C\|\varphi\|_{L^1(\Omega)}, \end{aligned}$$

by Proposition 3.14. This implies

$$|P_{K^{(N)},v}(t,x;s,y) - P_{K^{(N)},v}(t,x+h;s,y+k)| \le C\psi_0(t,s,h,k) = C(t-s)^{-(d+\beta)/\alpha}(|h|+|k|)^{\beta}, \qquad (4.9)$$

where C and β depend only on d, α, C_0, and $\|v\|_{X_\lambda}$. Thus $P_{K^{(N)},v}(t,x;s,y)$ may be regarded as a Hölder continuous function with respect to the spatial variables. As for the time regularity, we set $\tilde{\psi}_\mu(t,s,\tau) = (t-s)^{-(d+c)/\alpha}(\tau^{\beta'}(t-s)^{\beta'} + \mu^{c/\alpha})$, $0<\mu\ll 1$, $\tau\in[0,1)$, where c and β' are the constants in Corollary 3.15. Then by the same arguments as in (4.9) we obtain from Corollary 3.15,

$$|P_{K^{(N)},v}(t,x;s,y) - P_{K^{(N)},v}(t+\tau(t-s),x;s,y)| \le C\frac{C_{T,x}\tau^{\beta'}(t-s)^{\beta'}}{(t-s)^{(d+c)/\alpha}}. \qquad (4.10)$$

This proves the Hölder continuity with respect to t, since $\tau\in[0,1)$ is arbitrary. The continuity with respect to s is shown similarly. Collecting these above, we obtain the Hölder continuity with respect to (t,s,x,y) for $P_{K^{(N)},v}(t,x;s,y)$. Moreover, the estimates (4.3), (4.5), and (4.10) imply

$$P_{K^{(N)},v}f(\cdot,s)\in C([s,\infty);L^p), \quad \text{if} \quad f\in L^p(\mathbb{R}^d),\ 1\le p<\infty. \qquad (4.11)$$

For (4.11) we will check the continuity at $t=s$ only. By the interpolation and the density arguments based on (4.3) it suffices to consider the case $p=1$. Let $f\in C_0^\infty(\mathbb{R}^d)$ and let supp $f\subset B_{R_1}(0)$ for some $R_1>0$. Take any $R_2\ge\{R_1,C_{R_1}\}$, where C_R is the constant defined in (4.7). Then from $|f(x)-f(y)|\le C|x-y|^{\gamma/2}$,

$$\begin{aligned}
&\|(P_{K^{(N)},v}f)(t,s)-f\|_{L^1}\\
\le\ & \int_{|x|\le 2R_2}\int_{\mathbb{R}^d} P_{K^{(N)},v}(t,x;s,y)|f(y)-f(x)|\,\mathrm{d}y\,\mathrm{d}x\\
&+\int_{|x|\ge 2R_2}\int_{\mathbb{R}^d} P_{K^{(N)},v}(t,x;s,y)|f(y)-f(x)|\,\mathrm{d}y\,\mathrm{d}x\\
\le\ & CR_2^d\Big(\sup_{x\in\mathbb{R}^d}\int_{R^d}\rho_\gamma(y-x-\xi_v^*(x,y,s))P_{K^{(N)},v}(t,x;s,y)\,\mathrm{d}y + \sup_{|x|\le 2R_2}|\xi_v^*(x,t,s)|^{\frac{\gamma}{2}}\Big)\\
&+\int_{|x|\ge 2R_2}\int_{|y|\le R_1} P_{K^{(N)},v}(t,x;s,y)|f(y)|\,\mathrm{d}y\,\mathrm{d}x\\
\le\ & CR_2^d\big((t-s)^{\frac{\gamma}{2\alpha}}+\sup_{|x|\le 2R_2}|\xi_v^*(x,t,s)|^{\frac{\gamma}{2}}\big)\\
&+R_2^{-\frac{\gamma}{2}}\int_{|y|\le R_1}\int_{|x-y-\eta_v^*(y,t,s)|\ge R_2}\rho_\gamma(x-y-\eta_v^*(y,t,s))P_{K^{(N)},v}(t,x;s,y)\,\mathrm{d}x|f(y)|\,\mathrm{d}y\\
\le\ & CR_2^d\big((t-s)^{\frac{\gamma}{2\alpha}}+\sup_{|x|\le 2R_2}|\xi_v^*(x,t,s)|^{\frac{\gamma}{2}}\big)+CR_2^{-\frac{\gamma}{2}}(t-s)^{\frac{\gamma}{2\alpha}}.
\end{aligned}$$

Hence, from the estimate of $\xi_v^*(x,t,s)$ in Proposition 3.9 we have $\lim_{t\to s}\|(P_{K^{(N)},v}f)(t,s)-f\|_{L^1}=0$ if $f\in C_0^\infty(\mathbb{R}^d)$. The density arguments and (4.3) give $\lim_{t\to s}\|(P_{K^{(N)},v}f)(t,s)-f\|_{L^1}=0$ for all $f\in L^1(\mathbb{R}^d)$. This completes the proof of (4.11). Similarly, we also have

$$P_{K^{(N)},v}f\in C(\overline{D};L^p), \quad \text{if} \quad f\in L^p(\mathbb{R}^d),\ 1\le p<\infty. \qquad (4.12)$$

The details are omitted here.
(iv) Energy inequality. We next show that the energy inequality

$$\|(P_{K^{(N)},v}f)(t,s)\|_{L^2}^2 + 2\int_s^t \mathcal{E}_{K^{(N)}}^{(\tau)}\big((P_{K^{(N)},v}f)(\tau,s),(P_{K^{(N)},v}f)(\tau,s)\big)\,\mathrm{d}\tau \le \|f\|_{L^2}^2, \qquad (4.13)$$

holds for all $f\in L^2(\mathbb{R}^d)$ and all $t>s\ge 0$. Let $\{f_l\}_{l=1}^\infty\subset L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ be a countable and dense subset of $L^2(\mathbb{R}^d)$. Then by taking a subsequence of $\{\delta_n\}$ if necessary, due to the diagonal arguments, we may assume that $(P_{K^{(N,\delta_n)},v}f_l)(\cdot,s)\rightharpoonup^*(P_{K^{(N)},v}f_l)(\cdot,s)$ as $n\to\infty$ in $L_t^\infty(s,\infty;L^2(\mathbb{R}^d))$ for each $l\in\mathbb{N}$ and $s\in\mathbb{Q}$. Fix $\epsilon>0$, $l\in\mathbb{N}$, $s\in\mathbb{Q}$, and set

$$\begin{aligned}
G_{\delta_n,\epsilon}(f_l)(t,x,y;s) &= [(P_{K^{(N,\delta_n)},v}f_l)(t,s)](x,y)\{\nu_{\delta_n}*k_N(t,x,y)\}^{\frac12}|x-y|^{-\frac{d+\alpha}{2}}\chi_{\{|x-y|\ge\epsilon\}}^{\frac12}(x,y),\\
G_\epsilon(f_l)(t,x,y;s) &= [(P_{K^{(N)},v}f_l)(t,s)](x,y)k_N^{\frac12}(t,x,y)|x-y|^{-\frac{d+\alpha}{2}}\chi_{\{|x-y|\ge\epsilon\}}^{\frac12}(x,y),
\end{aligned}$$

where $[f](x,y)=f(x)-f(y)$. Then it is not difficult to see that $G_{\delta_n,\epsilon}(f_l)(t,x,y;s)$ is bounded in $L^\infty(s,\infty;L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d))$ uniformly in n. Since $(P_{K^{(N,\delta_n)},v}f_l)(\cdot,s)\rightharpoonup^*(P_{K^{(N)},v}f_l)(\cdot,s)$ in $L^\infty(s,\infty;L^2(\mathbb{R}^d))$ and $\nu_{\delta_n}*$

$k_N(t,x,y) \to k_N(t,x,y)$ strongly in $L^p_{loc}((s,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$ for $1 \le p < \infty$, we have $G_{\delta_n,\epsilon}(f_l)(t,x,y;s) \rightharpoonup G_\epsilon(f_l)(t,x,y;s)$ in $L^2_{loc}(s,\infty;L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d))$. By the lower semicontinuity of the seminorms in $L^2_{loc}(s,\infty;L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d))$ implies that for any $\varphi(t)\in C_0(s,\infty)$ with $\varphi \ge 0$,

$$\begin{aligned}\int_s^\infty \|G_\epsilon(f_l)\|^2_{L^2(s,t;L^2\times L^2)}\varphi(t)\,\mathrm{d}t \quad &\le \quad \int_s^\infty \liminf_{n\to\infty}\|G_{\delta_n,\epsilon}(f_l)\|^2_{L^2(s,t;L^2\times L^2)}\varphi(t)\,\mathrm{d}t\\ &\le \quad \liminf_{n\to\infty}\int_s^\infty \|G_{\delta_n,0}(f_l)\|^2_{L^2(s,t;L^2\times L^2)}\varphi(t)\,\mathrm{d}t\\ &\le \quad \liminf_{n\to\infty}\int_s^\infty \big(\|f_l\|^2_{L^2}-\|(P_{K^{(N,\delta_n)},v}f_l)(t,s)\|^2_{L^2}\big)\varphi(t)\,\mathrm{d}t\\ &\le \quad \int_s^\infty \big(\|f_l\|^2_{L^2}-\|(P_{K^{(N)},v}f_l)(t,s)\|^2_{L^2}\big)\varphi(t)\,\mathrm{d}t.\end{aligned}$$

Here we have used the energy inequality for $(P_{K^{(N,\delta_n)},v}f_l)(t,s)$ and the lower semicontinuity in $L^2_{loc}(s,\infty;L^2(\mathbb{R}^d))$. This yields

$$\|(P_{K^{(N)},v}f_l)(t,s)\|^2_{L^2}+\|G_\epsilon(f_l)\|^2_{L^2(s,t;L^2\times L^2)} \le \|f_l\|^2_{L^2}, \tag{4.14}$$

for a.e. $t\in(s,\infty)$ and $s\in\mathbb{Q}$. But the fact $P_{K^{(N)},v}f_l \in C(\overline{D};L^2(\mathbb{R}^d))$ implies that (4.14) holds for all $t > s \ge 0$. Then by the density arguments using (4.3) we conclude that

$$\|(P_{K^{(N)},v}f)(t,s)\|^2_{L^2}+\|G_\epsilon(f)\|^2_{L^2(s,t;L^2\times L^2)} \le \|f\|^2_{L^2}, \tag{4.15}$$

for all $t > s \ge 0$ and $f \in L^2(\mathbb{R}^d)$. By tending ϵ to 0 in (4.15) and by using the monotone convergence theorem, we obtain (4.13).

(v) Semigroup property. Next we check the semigroup property

$$P_{K^{(N)},v}(t,x;s,y)=\int_{\mathbb{R}^d}P_{K^{(N)},v}(t,x;\tau,z)P_{K^{(N)},v}(\tau,z;s,y)\,\mathrm{d}z, \tag{4.16}$$

for $t > \tau > s \ge 0$, $x,\ y\in\mathbb{R}^d$. It suffices to consider the case $s>0$; then the case $s=0$ follows from the continuity with respect to s. Assume that $t > s+1/m$, $m\in\mathbb{N}$. Let $\varphi\in C_0^\infty(\Omega_m)$ and let $\epsilon>0$. We take $\tau>0$ so that $t-1/(2m) > \tau > s+1/(2m)$ for all $(t,s)\in\cup_{x,y}\mathrm{supp}\ \varphi(\cdot,\cdot,x,y)$. Then there is $R>0$ such that

$$\int_\Omega\int_{|z|\ge R}P_{K^{(N)},v}(t,x;\tau,z)P_{K^{(N)},v}(\tau,z;s,y)\varphi(t,s,x,y)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}s<\epsilon. \tag{4.17}$$

For sufficiently large $l\in\mathbb{N}$ we set

$$P^{(l)}_{K^{(N)},v}(t,x;s,y)=\int_\Omega \nu_{1,\frac1l}(t-t')\nu_{1,\frac1l}(s-s')\nu_{d,\frac1l}(x-x')\nu_{d,\frac1l}(y-y')P_{K^{(N)},v}(t',x';s',y')\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}t'\,\mathrm{d}s', \tag{4.18}$$

and

$$P^{(l)}_{K^{(N,\delta)},v}(t,x;s,y)=\int_\Omega \nu_{1,\frac1l}(t-t')\nu_{1,\frac1l}(s-s')\nu_{d,\frac1l}(x-x')\nu_{d,\frac1l}(y-y')P_{K^{(N,\delta)},v}(t',x';s',y')\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}t'\,\mathrm{d}s'. \tag{4.19}$$

From the uniform continuity of $P_{K^{(N)},v}(t,x;s,y)$ on $\overline{\Omega_m}$ we have for $l\gg 1$,

$$\sup_{(z,t,s,x,y)\in\mathbb{R}^d\times\mathrm{supp}\ \varphi}\Big(|P_{K^{(N)},v}(t,x;\tau,z)-P^{(l)}_{K^{(N)},v}(t,x;\tau,z)|+|P_{K^{(N)},v}(\tau,z;s,y)-P^{(l)}_{K^{(N)},v}(\tau,z;s,y)|\Big)<\epsilon. \tag{4.20}$$

Furthermore, Corollaries 3.14 and 3.15 imply

$$\begin{aligned}\sup_{(z,t,s,x,y)\in\mathbb{R}^d\times\mathrm{supp}\ \varphi}\Big(&|P_{K^{(N,\delta)},v}(t,x;\tau,z)-P^{(l)}_{K^{(N,\delta)},v}(t,x;\tau,z)|\\ &+|P_{K^{(N,\delta)},v}(\tau,z;s,y)-P^{(l)}_{K^{(N,\delta)},v}(\tau,z;s,y)|\Big)<\epsilon,\end{aligned} \tag{4.21}$$

for all sufficiently small $\delta>0$. On the other hand, for l satisfying (4.20) we have

$$\lim_{n\to\infty}P^{(l)}_{K^{(N,\delta_n)},v}(t,x;\tau,z)=P^{(l)}_{K^{(N)},v}(t,x;\tau,z),\qquad \lim_{n\to\infty}P^{(l)}_{K^{(N,\delta_n)},v}(\tau,z;s,y)=P^{(l)}_{K^{(N)},v}(\tau,z;s,y), \tag{4.22}$$

for every $(z,t,s,x,y)\in\mathbb{R}^d\times\mathrm{supp}\ \varphi$, and we also have the uniform bound

$$\sup_{n\in\mathbb{N}}\sup_{(z,t,s,x,y)\in\mathbb{R}^d\times\mathrm{supp}\ \varphi}\Big(P^{(l)}_{K^{(N,\delta_n)},v}(t,x;\tau,z)+P^{(l)}_{K^{(N,\delta_n)},v}(\tau,z;s,y)\Big)\le C_m. \tag{4.23}$$

So the Lebesgue convergence theorem yields for each $R > 0$,

$$
\begin{aligned}
&\int_\Omega \int_{|z|\le R} \varphi(t,s,x,y) P^{(l)}_{K^{(N)},v}(t,x;\tau,z) P^{(l)}_{K^{(N)},v}(\tau,z;s,y)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}s \\
= \ &\lim_{n\to\infty} \int_\Omega \int_{|z|\le R} \varphi(t,s,x,y) P^{(l)}_{K^{(N,\delta_n)},v}(t,x;\tau,z) P^{(l)}_{K^{(N,\delta_n)},v}(\tau,z;s,y)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}s. \qquad (4.24)
\end{aligned}
$$

Let $0 < \epsilon \ll 1$. From the moment bound of $P_{K^{(N,\delta_n)},v}(t,x;s,y)$ in Proposition 3.4 we can take $R > 0$ large enough but independent of $n \gg 1$ so that

$$
\int_{|z|\ge R} P_{K^{(N,\delta_n)},v}(t,x;\tau,z) P_{K^{(N,\delta_n)},v}(\tau,z;s,y)\,\mathrm{d}z \le C\epsilon.
$$

Then from (4.21) we have

$$
\begin{aligned}
&\int_{|z|\le R} P^{(l)}_{K^{(N,\delta_n)},v}(t,x;\tau,z) P^{(l)}_{K^{(N,\delta_n)},v}(\tau,z;s,y)\,\mathrm{d}z \\
= \ &\int_{|z|\le R} P_{K^{(N,\delta_n)},v}(t,x;\tau,z) P_{K^{(N,\delta_n)},v}(\tau,z;s,y)\,\mathrm{d}z + O(\epsilon) \\
= \ &P_{K^{(N,\delta_n)},v}(t,x;s,y) - \int_{|z|\ge R} P_{K^{(N,\delta_n)},v}(t,x;\tau,z) P_{K^{(N,\delta_n)},v}(\tau,z;s,y)\,\mathrm{d}z + O(\epsilon) \\
= \ &P_{K^{(N,\delta_n)},v}(t,x;s,y) + O(\epsilon).
\end{aligned}
$$

Here $O(\epsilon)$ satisfies $|O(\epsilon)| \le C\epsilon$ with C independent of n and $(t,s,x,y) \in \operatorname{supp} \varphi$. Hence from (4.17), (4.20), and (4.24), it is not difficult to see

$$
\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^d} P_{K^{(N)},v}(t,x;\tau,z) P_{K^{(N)},v}(\tau,z;s,y) \varphi(t,s,x,y)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}s \\
= \ &\int_\Omega P_{K^{(N)},v}(t,x;s,y) \varphi(t,s,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}s + O(\epsilon). \qquad (4.25)
\end{aligned}
$$

Since $\epsilon > 0$ and $\varphi \in C_0(\Omega_m)$ is arbitrary, we get (4.16) for $t - 1/(2m) > \tau > s + 1/(2m)$. Then since $m \in \mathbb{N}$ is arbitrary, we have (4.16) for $t > \tau > s > 0$. This completes the proof of (4.16).

(vi) End of the proof. Finally we show that $P_{K^{(N)},v}(t,x;s,y)$ is a fundamental solution to (1.1) with

$$
K(t,x,y) = K^{(N)}(t,x,y) = |x-y|^{-d-\alpha} k_N(t,x,y).
$$

For $f \in C_0^\infty(\mathbb{R}^d)$, the function $(P_{K^{(N)},v} f)(t,s,x)$ satisfies

$$
\begin{aligned}
&\int_D < (P_{K^{(N)},v} f)(t,s), \partial_t \varphi(t,s) > \,\mathrm{d}t\,\mathrm{d}s \\
= \ &\lim_{n\to\infty} \int_D < (P_{K^{(N,\delta_n)},v} f)(t,s), \partial_t \varphi(t,s) > \,\mathrm{d}t\,\mathrm{d}s \\
= \ &\lim_{n\to\infty} \int_D \mathcal{E}^{(t)}_{K^{(N,\delta_n)}}((P_{K^{(N,\delta_n)},v} f)(t,s), \varphi(t,s)) + \mathcal{E}_{v(t)}((P_{K^{(N,\delta_n)},v} f)(t,s), \varphi(t,s))\,\mathrm{d}t\,\mathrm{d}s, \qquad (4.26)
\end{aligned}
$$

for all $\varphi \in C_0^\infty(D \times \mathbb{R}^d)$. For any $\epsilon > 0$ we decompose the first term in the right hand side as

$$
\begin{aligned}
&\int_D \mathcal{E}^{(t)}_{K^{(N,\delta_n)}}((P_{K^{(N,\delta_n)},v} f)(t,s), \varphi(t,s))\,\mathrm{d}t\,\mathrm{d}s \\
= \ &\int_D \int_{|x-z|\le\epsilon} \frac{[P_{K^{(N,\delta_n)},v} f(t,s)](x,z) [\varphi(t,s)](x,z)}{|x-z|^{d+\alpha}} (\nu_{\delta_n} * k_N)(t,x,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\,\mathrm{d}s \\
&+ \int_D \int_{|x-z|\ge\epsilon} \frac{[P_{K^{(N,\delta_n)},v} f(t,s)](x,z) [\varphi(t,s)](x,z)}{|x-z|^{d+\alpha}} (\nu_{\delta_n} * k_N)(t,x,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\,\mathrm{d}s \\
&+ \delta_n \int_D \int_{\mathbb{R}^{2d}} \frac{[P_{K^{(N,\delta_n)},v} f(t,s)](x,z) [\varphi(t,s)](x,z)}{|x-z|^{d+1+\frac{\alpha}{2}}}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\,\mathrm{d}s \\
= \ &I_\epsilon + II_\epsilon + III.
\end{aligned}
$$

Let $R,\ T>0$ be such that $\operatorname{supp}\varphi\subset\{(t,s,x)\in D\times\mathbb{R}^d\mid 0\le s<t<T,\ |x|\le R\}$. Then we have

$$\begin{aligned}
|I_\epsilon| &\le N\int_D \|P_{K^{(N,\delta_n)},v}f(t,s)\|_{\dot H^{\frac{\alpha}{2}}}\Big(\int_{|x-z|\le\epsilon}\frac{[\varphi(t,s)]^2(x,z)}{|x-z|^{d+\alpha}}\,\mathrm{d}x\,\mathrm{d}z\Big)^{\frac12}\,\mathrm{d}t\,\mathrm{d}s\\
&\le N\|f\|_{L^2}\int_0^\infty\Big(\int_s^\infty\int_{|x-z|\le\epsilon}\frac{[\varphi(t,s)]^2(x,z)}{|x-z|^{d+\alpha}}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\Big)^{\frac12}\,\mathrm{d}s\\
&\le NR^{\frac d2}T^{\frac32}\|f\|_{L^2}\epsilon^{\frac{2-\alpha}{2}}.
\end{aligned}$$

For II_ϵ we have

$$\begin{aligned}
II_\epsilon &= \int_D\int_{|x-\xi|\ge\epsilon}\frac{[P_{K^{(N,\delta_n)},v}f(t,s)][\varphi(t,s)](x,\xi)}{|x-\xi|^{d+\alpha}}(\nu_{\delta_n}*k_N)(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}s\\
&= \int_D\int_{|x-\xi|\ge\epsilon}\frac{[P_{K^{(N,\delta_n)},v}f(t,s)][\varphi(t,s)](x,\xi)}{|x-\xi|^{d+\alpha}}(\nu_{\delta_n}*k_N-k_N)(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}s\\
&\quad+\int_D\int_{|x-\xi|\ge\epsilon}\frac{[P_{K^{(N,\delta_n)},v}f(t,s)][\varphi(t,s)](x,\xi)}{|x-\xi|^{d+\alpha}}k_N(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}s\\
&= II_{\epsilon,1}+II_{\epsilon,2},
\end{aligned}$$

and

$$\begin{aligned}
|II_{\epsilon,1}| &\le 2N^{\frac12}\|f\|_{L^2}\int_0^T\Big(\int_s^T\int_{|x-z|\ge\epsilon}\frac{[\varphi(t,s)]^2(x,z)}{|x-z|^{d+\alpha}}|\nu_{\delta_n}*k_N-k_N|(t,x,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\Big)^{\frac12}\,\mathrm{d}s\\
&\to 0 \quad\text{as}\quad n\to\infty,
\end{aligned}$$

by the Lebesgue convergence theorem, since $(\nu_{\delta_n}*k_N)(t,x,\xi)$ converges to $k_N(t,x,\xi)$ for a.e. (t,x,ξ). As for $II_{\epsilon,2}$ we claim that

$$II_{\epsilon,2}\to\int_0^\infty\int_s^\infty\int_{|x-\xi|\ge\epsilon}\frac{[P_{K^{(N)},v}f(t,s)][\varphi(t,s)](x,\xi)}{|x-\xi|^{d+\alpha}}k_N(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}s, \tag{4.27}$$

as $n\to\infty$. Indeed, first we observe that

$$\begin{aligned}
&\int_D\int_{|x-\xi|\ge\epsilon}\frac{P_{K^{(N,\delta_n)},v}f(t,s,x)\varphi(t,s,x)}{|x-z|^{d+\alpha}}k_N(t,x,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\,\mathrm{d}s\\
=&\int_D\int_{\mathbb{R}^{2d}}P_{K^{(N,\delta_n)},v}(t,x;s,y)f(y)\varphi(t,s,x)\int_{|x-z|\ge\epsilon}\frac{k_N(t,x,z)}{|x-z|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
\to&\int_D\int_{\mathbb{R}^{2d}}P_{K^{(N)},v}(t,x;s,y)f(y)\varphi(t,s,x)\int_{|x-z|\ge\epsilon}\frac{k_N(t,x,z)}{|x-z|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s,
\end{aligned}$$

as $n\to\infty$, since $\varphi(t,s,x)\int_{|x-z|\ge\epsilon}\frac{k_N(t,x,z)}{|x-z|^{d+\alpha}}\,\mathrm{d}zf(y)\in L^1(\Omega_m)$ for some $m\ge1$. Next we see

$$\begin{aligned}
&\int_D\int_{|x-\xi|\ge\epsilon}\frac{P_{K^{(N,\delta_n)},v}f(t,s,x)\varphi(t,s,z)}{|x-\xi|^{d+\alpha}}k_N(t,x,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\,\mathrm{d}s\\
=&\int_D\int_{\mathbb{R}^{2d}}P_{K^{(N,\delta_n)},v}(t,x;s,y)f(y)\int_{|x-z|\ge\epsilon}\frac{\varphi(t,s,z)k_N(t,x,z)}{|x-\xi|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
\to&\int_D\int_{\mathbb{R}^{2d}}P_{K^{(N)},v}(t,x;s,y)f(y)\int_{|x-z|\ge\epsilon}\frac{\varphi(t,s,z)k_N(t,x,z)}{|x-z|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s,
\end{aligned}$$

as $n\to\infty$, for $f(y)\int_{|x-z|\ge\epsilon}\frac{\varphi(t,s,z)k_N(t,x,z)}{|x-\xi|^{d+\alpha}}\,\mathrm{d}\xi\in L^1(\Omega_m)$ for some $m\ge1$. Collecting these two, we get (4.27). Next we consider the term III. We rewrite it as

$$III=C\delta_n\int_D\int_{\mathbb{R}^d}(P_{K^{(N,\delta_n)},v}f)(t,s,x)\big((-\Delta_x)^{1+\frac{\alpha}{2}}\varphi\big)(t,s,x)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s,$$

and hence,

$$|III| \le C\delta_n\|f\|_{L^2}\int_0^T\int_s^T\|\varphi(t,s)\|_{\dot H^{1+\frac{\alpha}{2}}}\,\mathrm{d}t\,\mathrm{d}s\to0,\quad\text{as}\quad n\to\infty.$$

On the other hand, it is not difficult to see

$$\lim_{n\to\infty}\int_D \mathcal{E}_{v(t)}((P_{K^{(N,\delta_n)},v}f)(t,s),\varphi(t,s))\,\mathrm{d}t\,\mathrm{d}s=\int_D \mathcal{E}_{v(t)}((P_{K^{(N)},v}f)(t,s),\varphi(t,s))\,\mathrm{d}t\,\mathrm{d}s.$$

Collecting these above, by taking $n\to\infty$ first then $\epsilon\to 0$ in (4.26) we obtain

$$\begin{aligned}&\int_D <(P_{K^{(N)},v}f)(t,s),\partial_t\varphi(t,s)>\,\mathrm{d}t\,\mathrm{d}s\\ =&\int_D \mathcal{E}^{(t)}_{K^{(N)}}((P_{K^{(N)},v}f)(t,s),\varphi(t,s))+\mathcal{E}_{v(t)}((P_{K^{(N)},v}f)(t,s),\varphi(t,s))\,\mathrm{d}t\,\mathrm{d}s.\end{aligned}\tag{4.28}$$

We take $\varphi(t,s,x)$ of the form $\varphi(t,x)\phi_m(t,s)\psi(s)$ in (4.28), where $\varphi\in C_0^\infty([0,\infty)\times\mathbb{R}^d)$, $\psi\in C_0^\infty([0,\infty))$, and $\phi_m\in C_0^\infty(D)$, $\phi\geq 0$, is such that

$$\phi_m(t,s)=\begin{cases}1 & \text{for } t\geq s+2/m,\\ 0 & \text{for } t\leq s+1/m,\end{cases}\tag{4.29}$$

for $m\geq 1$. Then setting $\varphi_m(t,s,x)=\varphi(t,x)\phi_m(t,s)$, we have

$$\begin{aligned}0 =& \int_0^\infty \psi(s)\Big(\int_0^\infty\{<(P_{K^{(N)},v}f)(t,s),\partial_t\varphi_m(t,s)>-\mathcal{E}^{(t)}_{K^{(N)}}((P_{K^{(N)},v}f)(t,s),\varphi_m(t,s))\\ &-\mathcal{E}_{v(t)}((P_{K^{(N)},v}f)(t,s),\varphi_m(t,s))\}\,\mathrm{d}t\Big)\,\mathrm{d}s,\end{aligned}$$

which implies for a.e. $s\in[0,\infty)$,

$$\begin{aligned}&\int_s^\infty <(P_{K^{(N)},v}f)(t,s),\partial_t\varphi_m(t,s)>\,\mathrm{d}t\\ =&\int_s^\infty\Big(\mathcal{E}^{(t)}_{K^{(N)}}((P_{K^{(N)},v}f)(t,s),\varphi_m(t,s))+\mathcal{E}_{v(t)}((P_{K^{(N)},v}f)(t,s),\varphi_m(t,s))\Big)\,\mathrm{d}t.\end{aligned}\tag{4.30}$$

But since each term in (4.30) is continuous with respect to s, (4.30) is valid for all $s\in[0,\infty)$. Then by standard arguments we can take the limit $m\to\infty$ to get

$$\begin{aligned}&\int_s^\infty <(P_{K^{(N)},v}f)(t,s),\partial_t\varphi(t)>\,\mathrm{d}t+<f,\varphi(s)>\\ =&\int_s^\infty\Big(\mathcal{E}^{(t)}_{K^{(N)}}((P_{K^{(N)},v}f)(t,s),\varphi(t))+\mathcal{E}_{v(t)}((P_{K^{(N)},v}f)(t,s),\varphi(t))\Big)\,\mathrm{d}t.\end{aligned}\tag{4.31}$$

For a general initial data $f\in L^2(\mathbb{R}^d)$ it suffices to approximate f by $\{f_n\}_{n=1}^\infty\subset C_0^\infty(\mathbb{R}^d)$. Then $(P_{K^{(N)},v}f)(t,s)$ is shown to satisfy (4.31) by using (4.3) and (4.13). Hence $P_{K^{(N)},v}(t,x;s,y)$ is a fundamental solution to (1.1) with $K(t,x,y)=K^{(N)}(t,x,y)$.

4.2 Limit $N\to\infty$

Let $\{v^{(N)}\}$ be the approximation of v obtained by Lemma 2.4. In this section we take the limit $N\to\infty$ of $P_{K^{(N)},v^{(N)}}(t,x;s,y)$ and complete the proof of Theorems 1.2 and 1.4. From the continuity estimates (4.9)-(4.10), by taking a subsequence if necessary, $P_{K^{(N)},v^{(N)}}(t,x;s,y)$ converges to a function $P_{K,v}(t,x;s,y)$ uniformly on each compact subset of $\Omega=D\times\mathbb{R}^d\times\mathbb{R}^d$. From (4.3) and (4.4) we easily see that $(P_{K,v}f)(t,s,x)=\int_{\mathbb{R}^d}P_{K,v}(t,x;s,y)f(y)\,\mathrm{d}y$ satisfies

$$\begin{aligned}\|P_{K,v}f\|_{L^\infty(D;L^p)} &\leq \|f\|_{L^p},\quad 1\leq p\leq\infty, &(4.32)\\ P_{K,v}(t,x;s,y) &\leq C(t-s)^{-\frac{d}{\alpha}}, &(4.33)\end{aligned}$$

and by the interpolation, we also have

$$\|(P_{K,v}f)(t,s)\|_{L^p}\leq C(t-s)^{-\frac{\alpha}{d}(\frac{1}{q}-\frac{1}{p})}\|f\|_{L^q},\quad 1\leq q\leq p\leq\infty.\tag{4.34}$$

Note that C in (4.33) and (4.34) is taken depending only on d, α, and C_0. Next we show the moment bound, i.e., for any $x\in\mathbb{R}^d$, $t>s\geq 0$ there is $\xi^*(x,t,s)\in\mathbb{R}^d$ such that

$$C_1(t-s)^{\frac{\gamma}{2\alpha}}\leq\int_{\mathbb{R}^d}\rho_\gamma(y-x-\xi^*(x,t,s))P_{K,v}(t,x;s,y)\,\mathrm{d}y\leq C_2(t-s)^{\frac{\gamma}{2\alpha}},\tag{4.35}$$

where $\gamma \in (0, \min\{\alpha, 2-\alpha\})$. Here C_1 and C_2 are the constants in (4.5) depending only on d, α, C_0, γ, and $\|v\|_{X_\lambda}$. Indeed, by Proposition 3.9 and Lemma 2.4 the vector $\xi^*_{v^{(N)}}(x,t,s) \in \mathbb{R}^d$ in (4.5) satisfies $\sup_{|x|\le R_0, 0\le s<t\le R_0} |\xi^*_{v^{(N)}}(x,t,s)| \le C_{R_0}$ for each $R_0 > 0$, where C_{R_0} is independent of N. Hence for any (x,t,s) there is $\xi^*(x,t,s) \in \mathbb{R}^d$ such that $\xi^*_{v^{(N')}}(x,t,s) \to \xi^*(x,t,s)$ as $N' \to \infty$ for a subsequence $\{N'\} \subset \{N\}$. Then the upper bound of (4.35) follows from (4.5) and the Fatou lemma. The lower bound is obtained from the same (or simpler) arguments as in the proof of (4.5), so we omit the details here. From (4.6) we also have for any (y,t,s) there is $\eta^*(y,t,s) \in \mathbb{R}^d$ satisfying

$$C_1(t-s)^{\frac{\gamma}{2\alpha}} \le \int_{\mathbb{R}^d} \rho_\gamma(x-y-\eta^*(y,t,s)) P_{K,v}(t,x;s,y)\,\mathrm{d}x \le C_2(t-s)^{\frac{\gamma}{2\alpha}}, \tag{4.36}$$

where $\gamma \in (0, \min\{\alpha, 2-\alpha\})$. From the above proof and by Lemma 2.4 and Remark 3.10 we observe that $\xi^*(x,t,s)$ and $\eta^*(x,t,s)$ in (4.35) and (4.36) satisfy

$$\begin{aligned} &|\xi^*(x,t,s)| + |\eta^*(x,t,s)| \\ &\le \begin{cases} C_1(T^{\frac{1}{2}(1-\frac{\lambda}{d})}+1)(t-s)^{\frac{1}{\alpha}} + C_2(t-s)^{1-\frac{1}{q}}\|v\|_{Y^{q,\lambda}_{T,0}} & \text{if } \ \lambda \in [\frac{2d}{\alpha}-d, d), \\ C_1(t-s)^{\frac{1}{\alpha}}\big(|\log(t-s)| + \log(e+|x|)\big) + C_2(t-s)^{1-\frac{1}{q}}\|v\|_{Y^{q,\lambda}_{T,0}} & \text{if } \ \lambda = d, \\ C_1\big((t-s)^{\frac{1}{\alpha}} + \|v\|_{X_\lambda(t,s)}(t-s)^{\frac{1}{\alpha}+\frac{1}{2}-\frac{\lambda}{2d}}|x|^{\frac{\alpha\lambda}{2d}-\frac{\alpha}{2}}\big) & \\ \qquad\qquad + C_2(t-s)^{1-\frac{1}{q}}\|v\|_{Y^{q,\lambda}_{T,0}} & \text{if } \ \lambda \in (d, \frac{2d}{\alpha}+d]. \end{cases} \end{aligned} \tag{4.37}$$

Here $T \ge t > s \ge 0$, and C_1 depends only on d, α, λ, and $\|v\|_{X_\lambda}$, and C_2 depends only on d and α. The similar argument as in the proof of (4.8) leads to

$$\int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)\,\mathrm{d}y = \int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)\,\mathrm{d}x = 1, \tag{4.38}$$

and the details are omitted. The semigroup property

$$P_{K,v}(t,x;s,y) = \int_{\mathbb{R}^d} P_{K,v}(t,x;\tau,z) P_{K,v}(\tau,z;s,y)\,\mathrm{d}z \qquad \text{for all } \ t > \tau > s \ge 0, \ x,y \in \mathbb{R}^d, \tag{4.39}$$

is obtained by dividing the integral

$$\begin{aligned} \int_{\mathbb{R}^d} P_{K,v}(t,x;\tau,z) P_{K,v}(\tau,z;s,y)\,\mathrm{d}z \ &= \ \int_{|z|\le R} P_{K,v}(t,x;\tau,z) P_{K,v}(\tau,z;s,y)\,\mathrm{d}z \\ &\quad + \int_{|z|\ge R} P_{K,v}(t,x;\tau,z) P_{K,v}(\tau,z;s,y)\,\mathrm{d}z, \end{aligned}$$

and then by using the moment bounds (4.5) - (4.6) and (4.35) - (4.36), the convergence of $P_{K^{(N)},v^{(N)}}(t,x,s,y)$ to $P_{K,v}(t,x;s,y)$, and the semigroup property (4.16). Moreover, by arguing as in (4.11) we can show

$$(P_{K,v}f)(\cdot,s) \in C(\overline{D}; L^p) \quad \text{ if } \ f \in L^p(\mathbb{R}^d), \ \ 1 \le p < \infty. \tag{4.40}$$

The details are left to the reader and omitted here.

Next we show that $P_{K,v}(t,x;s,y)$ is a fundamental solution to (1.1). For all $f \in C_0^\infty(\mathbb{R}^d)$ we have from (4.31),

$$\begin{aligned} &\int_s^\infty < (P_{K^{(N)},v^{(N)}}f)(t,s), \partial_t\varphi(t) > \,\mathrm{d}t + < f, \varphi(s) > \\ &= \int_s^\infty \Big(\mathcal{E}^{(t)}_{K^{(N)}}((P_{K^{(N)},v^{(N)}}f)(t,s),\varphi(t)) + \mathcal{E}_{v^{(N)}(t)}((P_{K^{(N)},v^{(N)}}f)(t,s),\varphi(t))\Big)\,\mathrm{d}t. \end{aligned} \tag{4.41}$$

for all $\varphi \in C_0^\infty([s,\infty)\times\mathbb{R}^d)$. From the fact that $P_{K^{(N)},v^{(N)}}(t,x;s,y)$ converges to $P_{K,v}(t,x;s,y)$ uniformly in each compact subset of Ω and the uniform bound (4.3), it is easy to see that

$$\int_s^\infty < (P_{K^{(N)},v^{(N)}}f)(t,s), \partial_t\varphi(t) > \,\mathrm{d}t \to \int_s^\infty < (P_{K,v}f)(t,s), \partial_t\varphi(t) > \,\mathrm{d}t, \qquad N \to \infty.$$

Similarly, since $v^{(N)} \to v$ in $L^1_{loc}((0,\infty)\times\mathbb{R}^d)$ we have

$$\int_s^\infty \mathcal{E}_{v^{(N)}(t)}((P_{K^{(N)},v^{(N)}}f)(t,s),\varphi(t))\,\mathrm{d}t \to \int_s^\infty \mathcal{E}_{v(t)}((P_{K,v}f)(t,s),\varphi(t))\,\mathrm{d}t, \qquad N \to \infty.$$

Next we consider the first term of the right-hand side of (4.41). For any $\epsilon > 0$, we have

$$\begin{aligned}
&\int_s^\infty \mathcal{E}^{(t)}_{K^{(N)}}\big((P_{K^{(N)},v^{(N)}}f)(t,s),\varphi(t)\big)\,\mathrm{d}t \\
=\;& \int_s^\infty \int_{|x-y|\le\epsilon} \frac{[(P_{K^{(N)},v^{(N)}}f)(t,s)][\varphi(t)](x,y)}{|x-y|^{d+\alpha}} k^{(N)}(t,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&+\int_s^\infty \int_{|x-y|\ge\epsilon} \frac{[(P_{K^{(N)},v^{(N)}}f)(t,s)][\varphi(t)](x,y)}{|x-y|^{d+\alpha}} k^{(N)}(t,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
=\;& I_\epsilon + II_\epsilon.
\end{aligned}$$

Let supp $\varphi \subset [0,T)\times B_R(0)$. From (4.13), $|x-y|^{-d-\alpha}k^{(N)}(t,x,y) \le K(t,x,y)$, and (1.5), we have

$$\begin{aligned}
|I_\epsilon| &\le \Big(\int_s^T \mathcal{E}^{(t)}_{K^{(N)}}((P_{K^{(N)},v^{(N)}}f)(t,s),(P_{K^{(N)},v^{(N)}}f)(t,s))\,\mathrm{d}t\Big)^{\frac12}\Big(\int_s^T \mathcal{E}^{(t)}_{K^{(N)}_\epsilon}(\varphi(t),\varphi(t))\,\mathrm{d}t\Big)^{\frac12} \\
&\le \|f\|_{L^2}\Big(\int_s^T\int_{|x-y|\le\epsilon,\ |x|+|y|\le 2R} \frac{[\varphi(t)]^2(x,y)}{|x-y|^{d+\alpha}}k^{(N)}(t,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\Big)^{\frac12} \\
&\le CT^{\frac12}R^{\frac d2}\|f\|_{L^2}\Big(\sup_{t>0,x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|x-y|^2K_\epsilon(t,x,y)\,\mathrm{d}y\Big)^{\frac12} \\
&\le CT^{\frac12}R^{\frac d2}\epsilon^{2-\alpha}\|f\|_{L^2}.
\end{aligned}$$

On the other hand, since

$$\frac{[(P_{K^{(N)},v^{(N)}}f)(t,s)][\varphi(t)](x,y)}{|x-y|^{d+\alpha}}k^{(N)}(t,x,y) \longrightarrow [(P_{K,v}f)(t,s)][\varphi(t)](x,y)K(t,x,y),$$

for a.e. $(t,x,y)\in(s,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$, and

$$\Big|\frac{[(P_{K^{(N)},v^{(N)}}f)(t,s)][\varphi(t)](x,y)}{|x-y|^{d+\alpha}}k^{(N)}(t,x,y)\Big| \le 2\|f\|_{L^\infty}(|\varphi(t,x)|+|\varphi(t,y)|)K(t,x,y),$$

where the right-hand side is integrable over $\{(t,x,y)\in[s,\infty)\times\mathbb{R}^d\times\mathbb{R}^d \mid |x-y|\ge\epsilon\}$ due to (2.11), by the Lebesgue convergence theorem we have

$$II_\epsilon \longrightarrow \int_s^\infty\int_{|x-y|\ge\epsilon}[(P_{K,v}f)(t,s)][\varphi(t)](x,y)K(t,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

The Fatou lemma and (4.13) imply

$$\begin{aligned}
&\int_s^T \mathcal{E}^{(t)}_K((P_{K,v}f)(t,s),(P_{K,v}f)(t,s))\,\mathrm{d}t \\
=\;& \int_s^T\int_{\mathbb{R}^{2d}}\lim_{N\to\infty}\frac{[(P_{K^{(N)},v^{(N)}}f)(t,s)]^2(x,y)}{|x-y|^{d+\alpha}}k^{(N)}(t,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
\le\;& \liminf_{N\to\infty}\int_s^T\int_{\mathbb{R}^{2d}}\frac{[(P_{K^{(N)},v^{(N)}}f)(t,s)]^2(x,y)}{|x-y|^{d+\alpha}}k^{(N)}(t,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
\le\;& \liminf_{N\to\infty}(\|f\|^2_{L^2}-\|(P_{K^{(N)},v^{(N)}}f)(T,s)\|^2_{L^2}) \\
\le\;& \|f\|^2_{L^2}-\|(P_{K,v}f)(T,s)\|^2_{L^2}<\infty. \qquad (4.42)
\end{aligned}$$

In particular, the integral of the left-hand side of (4.42) converges absolutely. By taking $\epsilon\to0$ after the limit $N\to\infty$, we finally obtain

$$\int_s^\infty \mathcal{E}^{(t)}_{K^{(N)}}\big((P_{K^{(N)},v^{(N)}}f)(t,s),\varphi(t)\big)\,\mathrm{d}t \longrightarrow \int_s^\infty \mathcal{E}^{(t)}_K\big((P_{K,v}f)(t,s),\varphi(t)\big)\,\mathrm{d}t,$$

as $N\to\infty$. Collecting these above, we conclude that $(P_{K,v}f)(t,s,x)$, $f\in C_0^\infty(\mathbb{R}^d)$, satisfies

$$\begin{aligned}
&\int_s^\infty < (P_{K,v}f)(t,s),\partial_t\varphi(t) > \,\mathrm{d}t + < f,\varphi(s) > \\
=\;& \int_s^\infty\Big(\mathcal{E}^{(t)}_K\big((P_{K,v}f)(t,s),\varphi(t)\big)+\mathcal{E}_{v(t)}\big((P_{K,v}f)(t,s),\varphi(t)\big)\Big)\,\mathrm{d}t, \qquad (4.43)
\end{aligned}$$

for all $\varphi \in C_0^\infty([s,\infty)\times\mathbb{R}^d)$. By the density arguments using (4.32) and (4.42), the equation (4.43) holds also for all $f \in L^2(\mathbb{R}^d)$. This proves that $P_{K,v}(t,x;s,y)$ is a fundamental solution to (1.1). We note that the following energy inequality also holds from (4.42):

$$\|(P_{K,v}f)(t,s)\|_{L^2}^2 + 2\int_s^t \mathcal{E}_K^{(\tau)}\big((P_{K,v}f)(\tau,s),(P_{K,v}f)(\tau,s)\big)\,\mathrm{d}\tau \le \|f\|_{L^2}^2, \quad \text{for all } \ t > s \ge 0. \tag{4.44}$$

Now Theorem 1.2 has been proved. Finally we prove Theorem 1.4 by using the moment bound and the semigroup property. Let $\eta^*(y) = \eta^*(y, t/2+\tau, s+\tau)$ and $\xi^*(x) = \xi^*(x, t+\tau, t/2+\tau)$ be the vectors in (4.36) and (4.35), respectively. From the semigroup property we have for $|x|+\tau \le R_0$ and $s \in [0,t/4]$,

$$\begin{aligned} P_{K,v}(t+\tau,x;s+\tau,y) &= \int_{\mathbb{R}^d} P_{K,v}(t+\tau,x;\frac{t}{2}+\tau,z)P_{K,v}(\frac{t}{2}+\tau,z;s+\tau,y)\,\mathrm{d}z \\ &= \int_{|z-y-\eta^*(y)|\le\frac{R}{4}} P_{K,v}(t+\tau,x;\frac{t}{2}+\tau,z)P_{K,v}(\frac{t}{2}+\tau,z;s+\tau,y)\,\mathrm{d}z \\ &\quad + \int_{|z-y-\eta^*(y)|\ge\frac{R}{4}} P_{K,v}(t+\tau,x;\frac{t}{2}+\tau,z)P_{K,v}(\frac{t}{2}+\tau,z;s+\tau,y)\,\mathrm{d}z \\ &= I_1 + I_2. \end{aligned} \tag{4.45}$$

By (4.37) there is $h_0 > 0$ such that when $|y| \ge R \gg R_0 \ge 1$ and $h_0 \ge t > s \ge 0$ then $|y+\eta^*(y)| \ge |y| - |\eta^*(y)| \ge R/2$. The number h_0 has to be taken small enough only when $\lambda = 2d/\alpha + d$; in this case (4.37) implies

$$|\eta^*(y)| = |\eta^*(y,t/2+\tau,s+\tau)| \le C\big(C' + \|v\|_{L^1(s+\tau,t/2+\tau;\mathrm{Lip})}R\big) \qquad \text{for } \ t+\tau \le R_0 + h_0,$$

where C depends only on d, α, λ, and $\|v\|_{X_\lambda}$, and C' depends only on R_0+h_0, d, α, λ, q, $\|v\|_{X_\lambda}$, and $\|v\|_{Y^{q,\lambda}_{R_0+h_0,0}}$. Note that $\sup_{0\le\tau\le R_0}\|v\|_{L^1(s+\tau,t/2+\tau;\mathrm{Lip})} \ll 1$ if $0 \le s < t \le h_0 \ll 1$. From the moment bound (4.35) we have

$$\begin{aligned} I_1 &\le \int_{|z|\ge\frac{R}{4}} P_{K,v}(t+\tau,x;\frac{t}{2}+\tau,z)P_{K,v}(\frac{t}{2}+\tau,z;s+\tau,y)\,\mathrm{d}z \\ &\le C(\frac{t}{2}-s)^{-\frac{d}{\alpha}} \int_{|z|\ge\frac{R}{4}} P_{K,v}(t+\tau,x;\frac{t}{2}+\tau,z)\,\mathrm{d}z \\ &\le Ct^{-\frac{d}{\alpha}} \int_{|z-x-\xi^*(x)|\ge\frac{R}{8}} P_{K,v}(t+\tau,x;\frac{t}{2}+\tau,z)\,\mathrm{d}z \\ &\le Ct^{-\frac{d}{\alpha}}R^{-\frac{\gamma}{2}} \int_{|z-x-\xi^*(x)|\ge\frac{R}{8}} \rho_\gamma(z-x-\xi^*(x))P_{K,v}(t+\tau,x;\frac{t}{2}+\tau,z)\,\mathrm{d}z \\ &\le Ct^{-\frac{d}{\alpha}+\frac{\gamma}{2\alpha}}R^{-\frac{\gamma}{2}}. \end{aligned} \tag{4.46}$$

Here R is taken large enough depending on R_0, h_0, $\|v\|_{X_\lambda}$, and $\|v\|_{Y^{q,\lambda}_{R_0+h_0,0}}$, but not on $s \in [0,t/4]$. The term I_2 in (4.45) is similarly estimated by the moment bound (4.36) as

$$\begin{aligned} I_2 &\le Ct^{-\frac{d}{\alpha}}R^{-\frac{\gamma}{2}} \int_{|z-y-\eta^*(y)|\ge\frac{R}{4}} \rho_\gamma(z-y-\xi^*(y))P_{K,v}(z,\frac{t}{2}+\tau;s+\tau,y)\,\mathrm{d}z \\ &\le Ct^{-\frac{d}{\alpha}+\frac{\gamma}{2\alpha}}R^{-\frac{\gamma}{2}}, \end{aligned} \tag{4.47}$$

if $s \in [0,t/4]$. The estimates (4.46) and (4.47) imply (1.28). The estimate (1.29) is proved in the same way. This completes the proof.

5 Uniqueness of weak solutions for subcritical and critical cases

In this section we study the uniqueness of weak solutions to (1.1) in the case $\alpha \in [1,2)$. The argument below is just based on the energy inequality, and it might be difficult to adapt it for the supercritical case. For simplicity we consider the special case of **(C1)** only; see Theorem 5.5.

5.1 Uniqueness for $L^2(\mathbb{R}^d)$ initial data

We first prove the energy equality of weak solutions when the initial time s is positive.

Theorem 5.1 *Let $\alpha \in [1,2)$ and $T > s \geq 0$. Assume that $v \in L^\infty(s,T;(L^\infty(\mathbb{R}^d))^d)$ and $\nabla \cdot v(t) = 0$. Let $\theta \in L^\infty(s,T;L^2(\mathbb{R}^d))$ be a weak solution to (1.1) for $t \in [s,T)$ with the initial data $f \in L^2(\mathbb{R}^d)$ at the initial time $t = s$. Assume in addition that there are $C, c > 0$ satisfying*

$$\text{ess.sup}_{|h|<c} K(t,x+h,y+h) \leq CK(t,x,y) \quad \text{for a.e. } \ t \in (s,T), \ |x-y| < c. \tag{5.1}$$

Then

$$\|\theta(t)\|_{L^2}^2 + 2\int_s^t \mathcal{E}_K^{(\tau)}\big(\theta(\tau),\theta(\tau)\big)\,\mathrm{d}\tau = \|f\|_{L^2}^2, \tag{5.2}$$

for a.e. $t \in (s,T)$.

Remark 5.2 For example, (5.1) is satisfied in the following two cases.
(i) $C^{-1}|x-y|^{-d-\alpha} \leq K(t,x,y) \leq C|x-y|^{-d-\alpha}$,
(ii) $K(t,x,y)$ is of the form $K(t,x-y)$.
In the appendix we give an example of K which satisfies (ii) but not (i).

Remark 5.3 The energy equality (5.2) actually holds for all $t \in (s,T)$ by the uniqueness of weak solutions obtained from Theorem 5.1 (see Corollary 5.4) and the continuity (4.40).

Proof. By setting $\theta(\tau) = f$ for $\tau \leq s$ we first extend $\theta(\tau)$ to the time $\tau \in (-\infty,T)$. Fix $t_0 \in (s,T)$ and for $0 < \epsilon \ll 1$ and $t < T - t_0 + 4\epsilon$ we set

$$\begin{aligned}
\theta_{\epsilon,\epsilon'}(t,x) &= \int_{-\infty}^T \nu_{1,\epsilon}(t-\tau)\int_{\mathbb{R}^d}\nu_{d,\epsilon'}(x-y)\theta(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau,\\
\theta_{m,\epsilon,\epsilon'}(t,x) &= (1-\phi_m(t,t_0))\theta_{\epsilon,\epsilon'}(t,x).
\end{aligned}$$

Here ϕ_m is the function defined by (4.29) and $m \gg 1$. Note that if $0 < \epsilon \ll 1$ and $m \gg 1$ then the function $(1-\phi_m(t,t_0))\nu_{1,\epsilon}(t-\tau)\nu_{d,\epsilon'}(x-y)$ is taken as a test function with variables (τ, y) in (2.4). Thus we get

$$\begin{aligned}
\partial_t\theta_{m,\epsilon,\epsilon'}(t,x) &= -(1-\phi_m(t,t_0))\int_s^T \nu_{1,\epsilon}(t-\tau)(\mathcal{E}_K^{(t)}\big(\theta(\tau),\nu_{d,\epsilon'}(x-\cdot)\big)\,\mathrm{d}\tau\\
&\quad -(1-\phi_m(t,t_0))\int_s^T \nu_{1,\epsilon}(t-\tau)\mathcal{E}_{v(t)}\big(\theta(\tau),\nu_{d,\epsilon'}(x-\cdot))\big)\,\mathrm{d}\tau - \partial_t\phi_m(t,t_0)\theta_{\epsilon,\epsilon'}(t,x).
\end{aligned}$$

This equality yields

$$\begin{aligned}
&\int_s^\infty \theta_{\epsilon,\epsilon'}^2(t,x)\partial_t(1-\phi_m(t,t_0))^2\,\mathrm{d}t + \theta_{m,\epsilon,\epsilon'}^2(s,x)\\
&= 2\int_s^\infty (1-\phi_m(t,t_0))\theta_{m,\epsilon,\epsilon'}(t,x)\int_s^\infty \nu_{1,\epsilon}(t-\tau)\mathcal{E}_K^{(t)}\big(\theta(\tau),\nu_{d,\epsilon'}(x-\cdot)\big)\,\mathrm{d}\tau\,\mathrm{d}t\\
&\quad +2\int_s^\infty (1-\phi_m(t,t_0))\theta_{m,\epsilon,\epsilon'}(t,x)\int_s^\infty \nu_{1,\epsilon}(t-\tau)\mathcal{E}_{v(t)}\big(\theta(\tau),\nu_{d,\epsilon'}(x-\cdot))\big)\,\mathrm{d}\tau\,\mathrm{d}t.
\end{aligned}$$

Hence by integrating over $\mathbb{R}^d$ and then taking the limit $\epsilon \to 0$ and $m \to \infty$, we have for a.e. $t_0 \in (s,T)$,

$$\begin{aligned}
-\|\nu_{d,\epsilon'} * \theta(t_0)\|_{L^2}^2 + \|\nu_{d,\epsilon'} * f(s)\|_{L^2}^2 &= 2\int_s^{t_0} \mathcal{E}_K^{(t)}\big(\theta(\tau),\nu_{d,\epsilon'} * \theta(\tau)\big)\,\mathrm{d}\tau + 2\int_s^{t_0} \mathcal{E}_{v(t)}\big(\theta(\tau),\nu_{d,\epsilon'} * \theta(\tau))\big)\,\mathrm{d}\tau\\
&= 2\int_s^{t_0} \mathcal{E}_K^{(t)}\big(\theta(\tau),\theta(\tau)\big)\,\mathrm{d}\tau + 2\int_s^{t_0} \mathcal{E}_K^{(t)}\big(\theta(\tau),\nu_{d,\epsilon'} * \theta(\tau) - \theta(\tau)\big)\,\mathrm{d}\tau\\
&\quad +2\int_s^{t_0} \mathcal{E}_{v(t)}\big(\theta(\tau) - \nu_{d,\epsilon'} * \theta(\tau),\nu_{d,\epsilon'} * \theta(\tau))\big)\,\mathrm{d}\tau.
\end{aligned}\tag{5.3}$$

Here we have used $\mathcal{E}_{v(t)}\big(\nu_{d,\epsilon'} * \theta(\tau),\nu_{d,\epsilon'} * \theta(\tau))\big) = 0$ due to $\nabla \cdot v(t) = 0$. Clearly, the left-hand side of (5.3) converges to $\|\theta(t_0)\|_{L^2}^2 - \|f(s)\|_{L^2}^2$ as $\epsilon' \to 0$. As for the second term of the right-hand side of (5.3), let us prove that

$$\lim_{\epsilon' \to 0}\int_s^T \mathcal{E}_K^{(t)}\big(\nu_{d,\epsilon'} * \theta(\tau) - \theta(\tau),\nu_{d,\epsilon'} * \theta(\tau) - \theta(\tau)\big)\,\mathrm{d}\tau = 0. \tag{5.4}$$

We divide the integral as

$$\begin{aligned}
&\int_s^T \mathcal{E}_K^{(t)}\big(\nu_{d,\epsilon'}*\theta(\tau)-\theta(\tau),\nu_{d,\epsilon'}*\theta(\tau)-\theta(\tau)\big)\,\mathrm{d}\tau\\
=&\int_s^T\int_{|x-y|\le\delta}[\nu_{d,\epsilon'}*\theta(\tau)-\theta(\tau)]^2(x,y)K(\tau,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\\
&+\int_s^T\int_{|x-y|\ge\delta}[\nu_{d,\epsilon'}*\theta(\tau)-\theta(\tau)]^2(x,y)K(\tau,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau. \qquad (5.5)
\end{aligned}$$

It is easy to see that for a fixed $\delta>0$ the second tern of the right-hand side of (5.5) goes to zero as $\epsilon'\to 0$ by using (2.11). As for the first term of the right-hand side of (5.5), since

$$\begin{aligned}
[\nu_{d,\epsilon'}*\theta(\tau)-\theta(\tau)]^2(x,y) &\le C\int_{\mathbb{R}^d}[\theta(\tau,\cdot-\epsilon'z)-\theta(\tau,\cdot)]^2(x,y)\nu_d(z)\,\mathrm{d}z\\
&\le C\int_{\mathbb{R}^d}[\theta(\tau,\cdot-\epsilon'z)]^2(x,y)\nu_d(z)\,\mathrm{d}z+C[\theta(\tau)]^2(x,y),
\end{aligned}$$

we have from (5.1),

$$\begin{aligned}
&\int_s^T\int_{|x-y|\le\delta}[\nu_{d,\epsilon'}*\theta(\tau)-\theta(\tau)]^2(x,y)K(\tau,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\\
\le\ & C\int_s^T\int_{|x-y|\le\delta}\int_{\mathbb{R}^d}[\theta(\tau)]^2(x,y)\nu_d(z)K(\tau,x+\epsilon'z,y+\epsilon'z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\\
&+C\int_s^T\int_{|x-y|\le\delta}[\theta(\tau)]^2(x,y)K(\tau,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\\
\le\ & C\int_s^T\int_{|x-y|\le\delta}[\theta(\tau)]^2(x,y)K(\tau,x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau. \qquad (5.6)
\end{aligned}$$

The right-hand side of (5.6) tends to 0 as $\delta\to 0$ due to the condition $\int_s^T \mathcal{E}_K^{(\tau)}\big(\theta(\tau),\theta(\tau)\big)\,\mathrm{d}\tau<\infty$. Hence we get (5.4), which implies the second term in the right-hand side of (5.3) goes to zero as $\epsilon'\to 0$. As for the third term of the right-hand side of (5.3), we first observe that

$$\begin{aligned}
\|\theta(\tau)-\nu_{d,\epsilon'}*\theta(\tau)\|_{L^2}^2 &\le \int_{\mathbb{R}^d}\nu_d(y)\int_{\mathbb{R}^d}|\theta(\tau,x)-\theta(\tau,x-\epsilon'y)|^2\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_{\mathbb{R}^d}\nu_d(y)\int_{\mathbb{R}^d}|\hat\theta(\tau,\xi)-e^{-i\epsilon'y\cdot\xi}\hat\theta(\tau,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}y\\
&\le C(\epsilon'M)^2\int_{|y|\le 2}\nu_d(y)\int_{|\xi|\le M}\Big|\frac{1-e^{-i\epsilon'y\cdot\xi}}{i\epsilon'y\cdot\xi}\Big|^2|\hat\theta(\tau,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}y\\
&\quad +C\epsilon'^{\alpha}\int_{|y|\le 2}\nu_d(y)\int_{|\xi|\ge M}\Big|\frac{1-e^{-i\epsilon'y\cdot\xi}}{i\epsilon'y\cdot\xi}\Big|^{\alpha}|\xi|^{\alpha}|\hat\theta(\tau,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}y\\
&\le C\epsilon'^2M^2\|\theta(\tau)\|_{L^2}^2+C\epsilon'^{\alpha}\int_{|\xi|\ge M}|\xi|^{\alpha}|\hat\theta(\tau,\xi)|^2\,\mathrm{d}\xi,
\end{aligned}$$

for all $M>0$. Hence we have

$$\begin{aligned}
&\int_s^{t_0}|\mathcal{E}_{v(t)}\big(\theta(\tau)-\nu_{d,\epsilon'}*\theta(\tau),\nu_{d,\epsilon'}*\theta(\tau))\big)|\,\mathrm{d}\tau\\
\le\ & \|v\|_{L^\infty_{t,x}}\int_s^T\|\theta(\tau)-\nu_{d,\epsilon'}*\theta(\tau)\|_{L^2}\|\nabla\nu_{d,\epsilon'}*\theta(\tau)\|_{L^2}\,\mathrm{d}\tau\\
\le\ & C\|v\|_{L^\infty_{t,x}}\Big(\int_s^T\|\theta(\tau)-\nu_{d,\epsilon'}*\theta(\tau)\|_{L^2}^2\,\mathrm{d}\tau\Big)^{\frac12}\Big(\int_s^T\int_{\mathbb{R}^d}|\xi|^2|\hat\nu_d(\epsilon'\xi)|^2|\hat\theta(\tau,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}\tau\Big)^{\frac12}\\
\le\ & C\|v\|_{L^\infty_{t,x}}\bigg(\epsilon'M\Big(\int_s^T\|\theta(\tau)\|_{L^2}^2\,\mathrm{d}\tau\Big)^{\frac12}+\epsilon'^{\frac{\alpha}{2}}\Big(\int_s^T\int_{|\xi|\ge M}|\xi|^{\alpha}|\hat\theta(\tau,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}\tau\Big)^{\frac12}\bigg)\\
&\times\epsilon'^{\frac{\alpha}{2}-1}\Big(\int_s^T\|\theta(\tau)\|_{\dot H^{\frac{\alpha}{2}}}^2\,\mathrm{d}\tau\Big)^{\frac12}\\
\le\ & C\|v\|_{L^\infty_{t,x}}\|\theta\|_{L^2\dot H^{\frac{\alpha}{2}}}\bigg(\epsilon'^{\frac{\alpha}{2}}MT^{\frac12}\|\theta\|_{L^\infty L^2}+\epsilon'^{\alpha-1}\Big(\int_s^T\int_{|\xi|\ge M}|\xi|^{\alpha}|\hat\theta(\tau,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}\tau\Big)^{\frac12}\bigg).
\end{aligned}$$

Noting $\alpha \in [1,2)$, by taking $\epsilon' \to 0$ first and then by taking $M \to \infty$, we conclude that the third term of the right-hand side of (5.3) goes to zero as $\epsilon' \to 0$. This completes the proof of (5.2).

As an immediate consequence of Theorem 5.1, we have the following uniqueness result.

Corollary 5.4 *Let the assumptions in Theorem 5.1 hold with $f = 0$. Then $\theta(t,x) = 0$ for all $(t,x) \in (s,T) \times \mathbb{R}^d$.*

Next we prove the uniqueness of weak solutions under the following condition on v instead of **(C1)**:
(C1') $v(t) \in L^\infty(\epsilon, \epsilon^{-1}; (L^\infty(\mathbb{R}^d))^d)$ for each $\epsilon > 0$, and **(C1)** with $\lambda = d$ holds, i.e.,

$$v \in \mathcal{L}^{1,\frac{1}{2}+\frac{1}{\alpha}-\frac{\lambda}{2d}}(0,\infty;(BMO(\mathbb{R}^d))^d) \cap L^q_{loc}(0,T;(L^1_{loc}(\mathbb{R}^d))^d) \text{ for some } 1 < q \le \infty.$$

Theorem 5.5 *Let $\alpha \in [1,2)$. Assume that* **(C1')** *and* **(C2)** *hold. Let $\theta \in L^\infty(0,T;L^2(\mathbb{R}^d))$ be a weak solution to (1.1) for $t \in [0,T)$ with the initial data $f \in L^2(\mathbb{R}^d)$ at the initial time $t = 0$. Assume in addition that (5.1) holds for a.e. $s \in (0,T)$. Then $\theta(t,x) = (P_{K,v}f)(t,0,x)$ for all $(t,x) \in (0,T) \times \mathbb{R}^d$.*

Proof. It suffices to consider the case $t \in (0,h_0]$ by the uniqueness result in Corollary 5.4, where h_0 is the number in Theorem 1.4. Let us take the test function of the form $(1-\phi_m(t,t_0))\varphi(x)$ in (2.4) with $s = 0$, where $\phi_m(t,t_0)$ is as in (4.29) with $t_0 \in (0,T)$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$. Then by tending $m \to \infty$ we have for a.e. $t_0 \in (0,T)$,

$$-<\theta(t_0),\varphi> + <f,\varphi> = \int_0^{t_0} \Big(\mathcal{E}_K^{(t)}\big((\theta(t),\varphi) + \mathcal{E}_{v(t)}(\theta(t),\varphi)\Big)\,\mathrm{d}t. \tag{5.7}$$

The equality (5.7) implies that there is $\{t_n\}_{n\in\mathbb{N}} \subset (0,T)$, $t_n \to 0$ as $n \to \infty$, such that $\theta(t_n)$ converges to f in the sense of distributions. Then from the density argument $\theta(t_n)$ converges to f weakly in $L^2(\mathbb{R}^d)$. The uniqueness result in Corollary 5.4 enables us to write

$$\begin{aligned}
\theta(t,x) &= \int_{\mathbb{R}^d} P_{K,v}(t,x;t_n,y)\theta(t_n,y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)f(y)\,\mathrm{d}y + \int_{\mathbb{R}^d} \big(P_{K,v}(t,x;t_n,y) - P_{K,v}(t,x;0,y)\big)\theta(t_n,y)\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)\big(\theta(t_n,y) - f(y)\big)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)f(y)\,\mathrm{d}y + I_1 + I_2.
\end{aligned} \tag{5.8}$$

From the weak convergence of $\theta(t_n)$ to f in $L^2(\mathbb{R}^d)$, I_2 goes to 0 as $n \to \infty$. As for I_1, we have

$$\begin{aligned}
|I_2| &\le \|P_{K,v}(t,x;t_n,\cdot) - P_{K,v}(t,x;0,\cdot)\|_{L^2}\|\theta(t_n)\|_{L^2} \\
&\le \sup_{s\in(0,T)} \|\theta(s)\|_{L^2}\|P_{K,v}(t,x;t_n,\cdot) - P_{K,v}(t,x;0,\cdot)\|_{L^2} \\
&\le Ct^{-\frac{d}{2\alpha}} \sup_{s\in(0,T)} \|\theta(s)\|_{L^2}\|P_{K,v}(t,x;t_n,\cdot) - P_{K,v}(t,x;0,\cdot)\|_{L^1}^{\frac{1}{2}}.
\end{aligned}$$

Then for $R \gg 1$,

$$\begin{aligned}
&\|P_{K,v}(t,x;t_n,\cdot) - P_{K,v}(t,x;0,\cdot)\|_{L^1} \\
\le\ & \int_{|y|\le R} |P_{K,v}(t,x;t_n,y) - P_{K,v}(t,x;0,y)|\,\mathrm{d}y + 2\sup_{s\in[0,\frac{t}{4}]}\int_{|y|\ge R} P_{K,v}(t,x;s,y)\,\mathrm{d}y = I_{2,1} + I_{2,2}.
\end{aligned}$$

For each $R \gg 1$ by the continuity of the fundamental solution $I_{2,1}$ converges to 0 as $n \to 0$. Then by taking $R \to \infty$ the term $I_{2,2}$ goes to zero by (1.28). This completes the proof.

5.2 Uniqueness for finite measure initial data

The advantage of the approach using fundamental solutions will be that once they are constructed, there is a flexibility on the choice of the class of initial data. For example, we can easily extend Theorem 5.5 to the case when the initial data is a finite Radon measure on $\mathbb{R}^d$. The class of finite Radon measures on $\mathbb{R}^d$ is denoted by $\mathcal{M}(\mathbb{R}^d)$, and the total variation of a finite Radon measure on $\mathbb{R}^d$ is defined by

$$\|\mu\|_{\mathcal{M}} = \sup\Big\{\int_{\mathbb{R}^d} \varphi\,\mathrm{d}\mu \mid \varphi \in C_0(\mathbb{R}^d), \|\varphi\|_{L^\infty} \le 1\Big\}. \tag{5.9}$$

For $\mu \in \mathcal{M}(\mathbb{R}^d)$ we say that $\theta \in L^\infty(0,T;L^1(\mathbb{R}^d))$ is a weak solution to (1.1) for $t \in [0,T)$ with the initial data μ at the initial time $t=0$ if θ belongs to $L^\infty(s,T;L^2(\mathbb{R}^d))$ for each $s \in (0,T)$ and is a weak solution to (1.1) for $t \in [s,T)$ with the initial data $\theta(s)$ at the initial time $t=s$, and $\theta(t) \rightharpoonup \mu$ as $t \to 0$ in the sense of measures.

For $\mu \in \mathcal{M}(\mathbb{R}^d)$ we set

$$(P_{K,v}\mu)(t,0,x) = \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)\mu(\,\mathrm{d}y). \tag{5.10}$$

The right hand side of (5.10) is well-defined, since $P_{K,v}(t,x;0,y)$ is continuous and integrable with respect to y.

Theorem 5.6 *Let $\alpha \in [1,2)$. Assume that* **(C1')** *and* **(C2)** *hold. Let μ be a finite Radon measure on $\mathbb{R}^d$. Let $\theta \in L^\infty(0,T;L^1(\mathbb{R}^d))$ be a weak solution to (1.1) for $t \in [0,T)$ with the initial data μ at the initial time $t=0$. Assume in addition that (5.1) holds for a.e. $s \in (0,T)$. Then $\theta(t,x) = (P_{K,v}\mu)(t,0,x)$ for all $(t,x) \in (0,T) \times \mathbb{R}^d$.*

Proof. The strategy is the same as in the proof of Theorem 5.5. It suffices to show $\theta(t,x) = (P_{K,v}\mu)(t,0,x)$ for all $0 < t \le h_0$ and $x \in \mathbb{R}^d$. The uniqueness result in Corollary 5.4 yields that for a.e. $s \in (0,t)$ and $R \gg 1$,

$$\begin{aligned}
\theta(t,x) &= \int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)\theta(s,y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)\mu(\,\mathrm{d}y) + \int_{\mathbb{R}^d} \big(P_{K,v}(t,x;s,y) - P_{K,v}(t,x;0,y)\big)\chi_R(y)\theta(s,y)\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^d} P_{K,v}(t,x;s,y)(1-\chi_R(y))\theta(s,y)\,\mathrm{d}y - \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)(1-\chi_R(y))\mu(\,\mathrm{d}y) \\
&\quad + \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)\chi_R(y)\big(\theta(s,y)\,\mathrm{d}y - \mu(\,\mathrm{d}y)\big) \\
&= \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)\mu(\,\mathrm{d}y) + I_{1,1} + I_{1,2} + I_{2,1} + I_{2,2}.
\end{aligned} \tag{5.11}$$

Since $P_{K,v}(t,x;s,y)$ is continuous,

$$|I_{1,1}| \le C\|\theta\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} \sup_{|y|\le 2R} |P_{K,v}(t,x;s,y) - P_{K,v}(t,x;0,y)| \to 0$$

as $s \to 0$ for each $R \gg 1$. Moreover, the convergence of $\theta(s)$ to μ at $s \to 0$ in the sense of measures implies $I_{2,1} \to 0$ as $s \to 0$ for each $R \gg 1$. Hence it suffices to show that we can make $|I_{1,2} + I_{2,2}|$ arbitrary small by taking R large enough, uniformly in s with $0 < s \le t/4$. But this assertion follows from (1.28), for it gives the estimate

$$\begin{aligned}
|I_{1,2} + I_{2,2}| &\le \big(\|\theta\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} + \|\mu\|_{M(\mathbb{R}^d)}\big) \sup_{|y|\ge R,\ \tau\in[0,\frac{t}{4}]} |P_{K,v}(t,x;\tau,y)| \\
&\le C\big(\|\theta\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} + \|\mu\|_{M(\mathbb{R}^d)}\big) t^{-\frac{d}{\alpha}-\frac{\gamma}{2\alpha}} R^{-\frac{\gamma}{2}},
\end{aligned}$$

for $0 < s \le \frac{t}{4}$. This completes the proof.

6 Applications to two-dimensional critical dissipative quasi-geostrophic equations

In this section we discuss the two-dimensional critical dissipative quasi-geostrophic equations

$$\left\{ \begin{array}{l}
\partial_t\theta + (-\Delta)^{\frac{1}{2}}\theta + v\cdot\nabla\theta = 0, \quad 0 < t < T, \quad x \in \mathbb{R}^2, \\
v = (-R_2\theta, R_1\theta), \quad 0 < t < T, \quad x \in \mathbb{R}^2, \\
\theta|_{t=0} = \theta_0, \quad x \in \mathbb{R}^2.
\end{array} \right. \tag{6.1}$$

Here $R_i = \partial_i(-\Delta)^{-1/2}$ is the Riesz transform. There are already many works on (6.1); for example, see [9, 8, 7, 12, 27, 18, 20, 23, 34, 10, 11, 13, 14, 5, 22, 31]. In particular, the global regularity of solutions were established by [23, 5, 22]. In [23] the a priori Hölder continuity for smooth periodic solutions was proved by constructing the modulus of continuity which is preserved by (6.1). In [5, 22], which are more related with our approach, the linear problem (1.1) is discussed for $A_K(t) = (-\Delta)^{1/2}$ and for a given v satisfying $\nabla\cdot v(t) = 0$. The key step of the proof in [5] is to establish the local Hölder continuity of θ

when $v \in L^\infty_{loc}(0,\infty;(BMO(\mathbb{R}^d))^d) \cap L^2_{loc}(0,\infty;(L^2_{loc}(\mathbb{R}^d))^d)$ and θ satisfies the level set energy inequality. They used the harmonic extension of the equation to $(0,\infty)\times\mathbb{R}^{d+1}_+$ and regarded $(-\Delta)^{1/2}$ as the normal derivative on the boundary. This enables to derive localized level set energy inequalities and to apply the De Giorgi argument. In [22] the a priori Hölder continuity for smooth periodic solutions was obtained in terms of the norm of v in $L^\infty(0,\infty;(BMO(\mathbb{T}^d))^d)$, where they studied the adjoint problem and used a predual characterization of the Hölder spaces. We note that the moment bound for solutions as in Section 3.1 is important also in their argument. Theorem 1.2 gives the alternative proof for the global regularity of solutions to (6.1).

Theorem 6.1 *Let $\theta \in L^\infty(0,T;L^2(\mathbb{R}^2))$ be a weak solution to (6.1). Assume that $v \in L^\infty(t_1,t_2;(L^\infty(\mathbb{R}^2))^2)$ for some $0 \le t_1 < t_2 \le T$. Then there is $\beta \in (0,1)$ such that $\theta \in \dot{C}^\beta([t_0,t_2]\times\mathbb{R}^2)$ for all $t_0 \in (t_1,t_2)$. Here the exponent β is taken depending only on $\|v\|_{L^\infty(t_1,t_2;BMO)}$, and $\|\theta\|_{\dot{C}^\beta([t_0,t_2]\times\mathbb{R}^2)}$ is estimated depending only on $t_0 \in (t_1,t_2)$ and $\|v\|_{L^\infty(t_1,t_2;BMO\cap L^2)}$.*

Remark 6.2 In the proof of Theorem 6.1 the assumption of $v \in L^\infty(t_1,t_2;(L^\infty(\mathbb{R}^2))^2)$, rather than v belonging to $L^\infty(t_1,t_2;(BMO(\mathbb{R}^2))^2)$, is used only to ensure the uniqueness of weak solutions; see Theorem 5.5. Note that the norm $\|\theta\|_{\dot{C}^\beta([t_0,t_2]\times\mathbb{R}^2)}$ itself can be estimated independently with respect to the norm of $v(t)$ in $L^\infty(\mathbb{R}^2)$. For (6.1) it is well known that under some regularity conditions on initial data smooth solutions uniquely exist at least locally in time, together with the global a priori bound of v in $L^\infty(0,\infty;(BMO(\mathbb{R}^2))^2)$; for example, see [8]. Hence Theorem 6.1 gives the global regularity of solutions to (6.1) at least for smooth initial data.

Proof. Without loss of generality, we may assume that $t_1 = 0$ and $t_2 = T$. Then there is a fundamental solution $P_v(t,x;s,y)$ defined on $\{(t,s,x,y) \mid 0 \le s < t \le T,\ x,y \in \mathbb{R}^2\}$ which satisfies

$$|P_v(t,x;s,y) - P_v(t',x';s,y)| \le C(\min\{t-s,t'-s\})^{-c}\big(|x-x'|+|t-t'|\big)^{\beta'} \tag{6.2}$$

for all $0 \le s < t,t' \le T$, $x,x',y \in \mathbb{R}^2$. Here positive constants c and β' depend only on $\|v\|_{L^\infty(0,T;BMO)}$ and C depends only on $\|v\|_{L^\infty(0,T;BMO\cap L^2)}$. By the uniqueness of weak solutions obtained by Theorem 5.5 for a given v we have the representation $\theta(t,x) = \displaystyle\int_{\mathbb{R}^2} P_v(t,x;0,y)\theta_0(y)\,\mathrm{d}y$. Hence θ is Hölder continuous uniformly on each $[t_0,T]\times\mathbb{R}^2$, $t_0 \in (0,T)$, due to the Hölder continuity and the semigroup property of $P_v(t,x;s,y)$ obtained by Theorem 1.2. This completes the proof.

7 Appendix

7.1 Example of K

Here we give an example of K which satisfies (1.5) and (1.6), but not (1.7). Let $H \in L^1(\mathbb{R}^d)$ be such that $H(z) \ge 0$, $H(z) = H(-z)$, and $\operatorname{supp} H \subset B_2(0)\backslash B_1(0)$. Set

$$H_k(z) = H(3^k z), \quad L(z) = \sum_{k=0}^\infty H_k(z). \tag{7.1}$$

Then $\operatorname{supp} H_k \cap \operatorname{supp} H_l = \emptyset$ if $k \ne l$ and $\displaystyle\int_{\mathbb{R}^d} L(z)\,\mathrm{d}z = C\sum_{k=0}^\infty 3^{-kd} < \infty$. Moreover, for each $M \in (0,\infty)$ it follows that

$$\int_{|z|\le M} L(z)\,\mathrm{d}z \le CM^d. \tag{7.2}$$

Indeed, if $3^{-k_0-1} \le M < 3^{-k_0}$ for some $k_0 \in \mathbb{N}\cup\{0\}$ then $\displaystyle\int_{|z|\le M} H_k(z)\,\mathrm{d}z = 0$ if $k \le k_0$, and hence,

$$\int_{|z|\le M} L(z)\,\mathrm{d}z = \sum_{k=k_0+1}^\infty \int_{|z|\le M} H_k(z)\,\mathrm{d}z \le C\sum_{k=k_0+1}^\infty 3^{-kd} = C3^{-k_0 d} \le CM^d.$$

If $M \ge 1$ then we have $\displaystyle\int_{|z|\le M} L(z)\,\mathrm{d}z \le \int_{\mathbb{R}^d} L(z)\,\mathrm{d}z \le C \le CM^d$. This proves (7.2). Now we set

$$K(t,x,y) = g(t)|x-y|^{-d-\alpha}\big(1 + L(x-y)\big), \tag{7.3}$$

for some $\alpha \in (0,2)$ and $g \in L^\infty(0,\infty)$ such that $C^{-1} \le g(t) \le C$ with a constant $C > 0$. Then clearly K satisfies (1.6), but not (1.7) in general, for H (and so L) is just in $L^1(\mathbb{R}^d)$. Furthermore, we have from (7.2),

$$\text{ess.sup}_{t>0, x\in\mathbb{R}^d} \int_{|x-y|\le M} |x-y|^{d+\alpha} K(t,x,y)\,\mathrm{d}y \le CM^d \qquad \text{for each} \quad M \in (0,\infty). \tag{7.4}$$

Just as in the proof of (2.10), we can derive (1.5) from (7.4) for some $C_0 > 0$.

7.2 Proof of Lemma 2.4

In order to prove Lemma 2.4 we first prove the following

Lemma 7.1 *Let $v \in (\mathcal{L}^{2d/\alpha,\lambda}(\mathbb{R}^d))^d$, $\lambda \in [2d/\alpha - d, 2d/\alpha + d]$ be a vector field satisfying $\nabla \cdot v = 0$. Set $p_\lambda = 1$ if $\lambda \in [2d/\alpha - d, d]$ and $p_\lambda = \infty$ if $\lambda \in (d, 2d/\alpha + d]$. Let $R \ge 1$. Then there is a sequence of smooth and bounded vector fields $\{v_N\}$ satisfying $\nabla \cdot v_N = 0$ and*

$$\begin{aligned} \|v_N\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} &\le C_1 \|v\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}, &(7.5)\\ \|v_N\|_{L^{p_\lambda}(B_R(0))} &\le C_2\|v\|_{L^{p_\lambda}(B_{2R}(0))} + c_{R,N}(\|v\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} + \|v\|_{L^{p_\lambda}(B_{2R}(0))}), &(7.6)\\ \lim_{N\to\infty} v_N &= v \qquad \text{in} \ \ L^1_{loc}(\mathbb{R}^d). &(7.7)\end{aligned}$$

Here C_1 depends only on d, α, and λ, C_2 depends only on d, and $c_{R,N}$ depends only on d, α, λ, R, N, and satisfies $\lim_{N\to\infty} c_{R,N} = 0$ for each $R \ge 1$.

Proof of Lemma 7.1. Let $\chi_N \in C_0^\infty(\mathbb{R}^d)$, $0 \le \chi_N \le 1$, be a cut-off function such that $\chi_N(x) = 1$ if $|x| \le N$ and $\chi_N(x) = 0$ if $|x| \ge 2N$. Set

$$\begin{aligned} w_N &= v - \fint_{B_{2N}(0)} v, &(7.8)\\ u_N &= w_N\chi_N + \fint_{B_{2N}(0)} v - \big(\zeta_N - \fint_{B_1(0)} \zeta_N\big), &(7.9)\\ v_N &= \nu_{d,\frac{1}{N}} * u_N, &(7.10)\end{aligned}$$

where $\zeta_N \in (L^1(\mathbb{R}^d))^d$ is a solution to

$$\text{div}\ \zeta_N = w_N \cdot \nabla\chi_N, \qquad \text{supp}\ \zeta_N \subset B_{2N}(0), \tag{7.11}$$

which satisfies

$$\|\nabla\zeta_N\|_{L^q(B_{2N}(0))} \le C\|w_N \cdot \nabla\chi_N\|_{L^q(B_{2N}(0))}, \quad 1 < q < \infty. \tag{7.12}$$

It is easy to see that $\|v_N\|_{\mathcal{L}^{2d/\alpha,\lambda}} \le C\|u_N\|_{\mathcal{L}^{2d/\alpha.\lambda}}$ and $\|v_N\|_{L^p(B_R(0))} \le C\|u_N\|_{L^p(B_{2R}(0))}$ if $N \gg 1$. Since (7.7) holds if $\lim_{N\to\infty} u_N \to v$ in $L^1_{loc}(\mathbb{R}^d)$, it suffices to prove (7.5)-(7.7) for u_N with $N \gg 1$, instead of v_N. However, since it is not difficult to prove similar estimates as (7.5)-(7.7) for $w_N\chi_N + \fint_{B_{2N}(0)} v$, it suffices to show

$$\begin{aligned} \|\zeta_N\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} &\le C_1\|v\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}, &(7.13)\\ \|\zeta_N - \fint_{B_1(0)} \zeta_N\|_{L^{p_\lambda}(B_R(0))} &\le c_{R,N}\|v\|_{L^{p_\lambda}(B_{2R}(0))}, &(7.14)\end{aligned}$$

(i) The case $\lambda \in [2d/\alpha - d, d]$: First we note that the following inequality holds:

$$\|\zeta_N\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} \le C\|\nabla\zeta_N\|_{L^{p^*}}, \quad \frac{d}{p^*} = 1 + \frac{\alpha}{2} - \frac{\alpha\lambda}{2d}. \tag{7.15}$$

Then since p^* defined in (7.15) satisfies $p^* \le 2d/\alpha$ in the present case, we have from (7.12),

$$\|\zeta_N\|_{\mathcal{L}^{\frac{2d}{\alpha}}} \le C\|\nabla\zeta_N\|_{L^{p^*}} \le CN^{1-\frac{\alpha\lambda}{2d}}\|\nabla\zeta_N\|_{L^{\frac{2d}{\alpha}}} \le CN^{1-\frac{\alpha\lambda}{2d}}\|w_N\cdot\nabla\chi_N\|_{L^{\frac{2d}{\alpha}}} \le C\|v\|_{\mathcal{L}^{\frac{2d}{\alpha}}}. \tag{7.16}$$

Thus we get (7.13). As for (7.14), from the Hölder inequality and the Poincaré inequality we have

$$\|\zeta_N - \fint_{B_1(0)} \zeta_N\|_{L^1(B_R(0))} \le C_R\|\nabla\zeta_N\|_{L^{\frac{2d}{\alpha}}(B_{2R}(0))} \le C_R\|\nabla\zeta_N\|_{L^{\frac{2d}{\alpha}}(B_{2R}(0))} \le C_R N^{-1+\frac{\alpha\lambda}{2d}}\|v\|_{\mathcal{L}^{\frac{2d}{\alpha}}}. \tag{7.17}$$

Here C_R is independent of N and v. This completes the proof for the case $\lambda \in [2d/\alpha - d, d]$.

(ii) The case $\lambda \in (d, 2d/\alpha + d)$: The estimate (7.13) follows from (7.16). As for (7.14), we have for $\beta \in (0,1)$ and $d/p_\beta = 1-\beta$,

$$\begin{aligned}\|\zeta_N - \fint_{B_1(0)} \zeta_N\|_{L^\infty(B_R(0))} \le CR^\beta \|\zeta_N\|_{\dot{C}^\beta} \le CR^\beta \|\nabla \zeta_N\|_{L^{p_\beta}} &\le CR^\beta \|w_N \cdot \nabla \chi_N\|_{L^{p_\beta}} \\ &\le CR^\beta N^{-1+\frac{d}{p_\beta}+\frac{\alpha\lambda}{2d}-\frac{\alpha}{2}} \|v\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}.\end{aligned}$$

By taking β near 1, we get (7.14).

(iii) The case $\lambda = 2d/\alpha + d$: In this case, instead of (7.9), we set u_N as

$$u_N = w_N\chi_N + \fint_{B_{2N}(0)} v - \big(\zeta_N - \fint_{B_1(0)} \zeta_N - \fint_{B_1(0)} \{(x-\cdot)\cdot \nabla\zeta_N\}\big). \tag{7.18}$$

Note that $\operatorname{div} \fint_{B_1(0)} \{(x-\cdot)\cdot\nabla\zeta_N\} = \fint_{B_1(0)} \operatorname{div}\, \zeta_N = \fint_{B_1(0)} \{w_N \cdot \nabla\chi_N\} = 0$, and hence $\operatorname{div}\, u_N = 0$. From the representation formula of ζ_N given in the proof of [15, Lemma III-3-1], we have

$$\|\nabla^2 \zeta_N\|_{L^q} \le C\big(\|\nabla(w_N\cdot\nabla\chi_N)\|_{L^q} + N^{-1}\|w_N\cdot\nabla\chi_N\|_{L^q}\big). \tag{7.19}$$

By the Gagliardo-Nirenberg inequality

$$\|\nabla\zeta_N\|_{L^\infty} \le C\|\nabla^2\zeta_N\|_{L^p}^\sigma \|\nabla\zeta_N\|_{L^r}^{1-\sigma} \qquad \sigma(\frac{1}{p}-\frac{1}{d}) + (1-\sigma)\frac{1}{r} = 0, \quad p\in(d,\infty],\ r\in[1,\infty],$$

we have from (7.12) and (7.19),

$$\begin{aligned}\|\nabla\zeta_N\|_{L^\infty} &\le C\big(\|\nabla(w_N\cdot\nabla\chi_N)\|_{L^p} + N^{-1}\|w_N\cdot\nabla\chi_N\|_{L^p}\big)^\sigma \|w_N\cdot\nabla\chi_N\|_{L^r}^{1-\sigma} \\ &\le C(N^{-1+\frac{d}{p}}\|v\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}})^\sigma (N^{\frac{d}{r}}\|v\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}})^{(1-\sigma)} \\ &= C\|v\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}},\end{aligned}$$

which gives

$$\|\zeta_N(x) - \fint_{B_1(0)} \zeta_N - \fint_{B_1(0)} \{(x-\cdot)\cdot\nabla\zeta_N\}\|_{\mathcal{L}_x^{\frac{2d}{\alpha},\lambda}} \le C\|v\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}, \tag{7.20}$$

Next we observe that

$$\zeta_N(x) - \fint_{B_1(0)} \zeta_N - \fint_{B_1(0)} \{(x-\cdot)\cdot\nabla\zeta_N\} = \frac{1}{|B_1(0)|}\int_{B_1(0)} (x-y)\cdot \int_0^1 \{(\nabla\zeta_N)(\sigma x + (1-\sigma)y) - \nabla\zeta_N(y)\}\, \mathrm{d}\sigma\, \mathrm{d}y,$$

and hence, for $R \ge 1$, $\beta\in(0,1)$, and $d/p_\beta = 1-\beta$,

$$\begin{aligned}\|\zeta_N(x) - \fint_{B_1(0)} \zeta_N - \fint_{B_1(0)} \{(x-\cdot)\cdot\nabla\zeta_N\}\|_{L^\infty(B_R(0))} &\le CR^{1+\beta}\|\nabla\zeta_N\|_{\dot{C}^\beta} \\ &\le CR^{1+\beta}\|\nabla^2\zeta_N\|_{L^{p_\beta}} \\ &\le CR^{1+\beta}\big(\|\nabla(w_N\cdot\nabla\chi_N)\|_{L^{p_\beta}} + N^{-1}\|w_N\cdot\nabla\chi_N\|_{L^{p_\beta}}\big) \\ &\le CR^{1+\beta}N^{-1+\frac{d}{p_\beta}}\|v\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} = CR^{1+\beta}N^{-\beta}\|v\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}.\end{aligned}$$

This gives (7.14) for $\zeta_N(x) - \fint_{B_1(0)} \zeta_N - \fint_{B_1(0)} \{(x-\cdot)\cdot\nabla\zeta_N\}$ instead of $\zeta_N(x) - \fint_{B_1(0)} \zeta_N$. The proof of Lemma 7.1 is now complete.

Proof of Lemma 2.4. We first extend $v(t)$ by 0 for $t<0$ and set

$$v^{(N)}(t) = \int_{\mathbb{R}} \nu_{1,\frac{1}{N}}(t-s) v_N(s)\, \mathrm{d}s, \tag{7.21}$$

where $v_N(t)$ is the approximation of $v(t)$ obtained by Lemma 7.1. Then from $\|v^{(N)}(t)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}} \le C\int_{\mathbb{R}} \nu_{1,1/N}(t-s)\|v_N(s)\|_{\mathcal{L}^{\frac{2d}{\alpha},\lambda}}\,\mathrm{d}s$ and $\|v^{(N)}(t)\|_{L^{p_\lambda}(B_R(0))} \le C\int_{\mathbb{R}} \nu_{1,1/N}(t-s)\|v_N(s)\|_{L^{p_\lambda}(B_R(0))}\,\mathrm{d}s$, it is easy to get (2.42) and (2.43). Moreover, we have

$$\|v^{(N)}(t) - v(t)\|_{L^1(B_R(0))} \le \int_{\mathbb{R}} \nu_{1,\frac{1}{N}}(t-s)\|v_N(s) - v(s)\|_{L^1(B_R(0))}\,\mathrm{d}s + \int_{\mathbb{R}} \nu_{1,\frac{1}{N}}(t-s)\|v(s)-v(t)\|_{L^1(B_R(0))}\,\mathrm{d}s,$$

which yields

$$\|v^{(N)} - v\|_{L^1(0,R;L^1(B_R(0)))} \le C \int_0^{2R} \|v_N(s) - v(s)\|_{L^1(B_R(0))} \, \mathrm{d}s + C \int_{|\tau|\le 2} \int_0^R \|v(t - \frac{\tau}{N}) - v(t)\|_{L^1(B_R(0))} \, \mathrm{d}t \, \mathrm{d}\tau.$$

Hence $v^{(N)} \to v$ in $L^1_{loc}((0,\infty) \times \mathbb{R}^d)$. This completes the proof.

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