On fundamental solutions for non-local diffusion equations with divergence free drift

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Abstract
We are concerned with non-local diffusion equations in the presence of a divergence free drift term. By using the classical Nash approach we show the existence of fundamental solutions, together with the continuity estimates, under weak regularity assumptions on the kernel of the diffusion term and the velocity of the drift term. As an application, our result gives the alternative proof of the global regularity for the two-dimensional dissipative quasi-geostrophic equations in the critical case.

Keywords: non-local diffusion equations; divergence free drift; fundamental solutions; Nash iteration; 2D dissipative quasi-geostrophic equations.

1 Introduction
In this paper we consider the non-local diffusion equations in the presence of a drift term
\[ \partial_t \theta + A_K(t)\theta + v \cdot \nabla \theta = 0, \quad t > 0, \quad x \in \mathbb{R}^d, \quad (1.1) \]
where \( d \geq 2 \) and \( A_K(t) \) is a linear operator formally defined by
\[ (A_K(t)f)(x) = \text{P.V.} \int_{\mathbb{R}^d} (f(x) - f(y))K(t,x,y) \, dy. \quad (1.2) \]
Here \( K(t,x,y) \) is a positive function satisfying \( K(t,x,y) = K(t,y,x) \) and \( v(t) = (v_1(t), \ldots, v_d(t)) \) is a vector field in \( \mathbb{R}^d \) satisfying the divergence free condition, \( \nabla \cdot v(t) = 0 \). In particular, \( A_K(t) \) will be supposed to possess a diffusion effect like \( (-\Delta)^{\alpha/2} \) for some \( \alpha \in (0, 2) \). Note that in the case \( A_K(t) = (-\Delta)^{\alpha/2} \) with \( \alpha \in (0, 2) \) the kernel \( K \) is given by \( K(t,x,y) = C_{d,\alpha}|x-y|^{-d-\alpha} \) for some positive constant \( C_{d,\alpha} \). The aim of this paper is to prove the existence and the continuity of fundamental solutions for (1.1) under less regularity assumptions on \( K \) and \( v \).

When there is no drift term (i.e., \( v = 0 \)) this problem appears in the theory of Dirichlet forms of jump type. For the diffusion operator \( A_K(t) \) defined by (1.2) the associated Dirichlet form is
\[ \mathcal{E}_K^{(1)}(f,g) = \frac{1}{2} \iint_{\mathbb{R}^{2d}} [f](x,y)K(t,x,y) \, dx \, dy, \quad [f](x,y) = f(x) - f(y), \quad (1.3) \]
and it has been investigated mainly from the probabilistic approach [6, 24, 25, 3, 1, 21, 2]. On the other hand, in recent years the case with a drift term has also attracted much attention, especially in the field of fluid mechanics, mathematical finance, biology, and so on. For example, many works have been done for the two-dimensional dissipative quasi-geostrophic equations (QG), where \( A_K(t) = (-\Delta)^{\alpha/2} \) and the drift
term is a nonlinear term such that \( v \) is given in terms of \( \theta \) via the Riesz transform; [9]. For such nonlinear problems it is crucial to obtain detailed informations of solutions under less regularity conditions on \( v \).

In [4, 19] fundamental solutions were constructed when \( A_K(t) = (-\Delta)^{\alpha/2} \) with \( \alpha \in (1, 2) \) and \( v \) belongs to a suitable Kato class without assuming the divergence free condition. In this case the diffusion term is the leading term and they showed two-sided heat kernel estimates by using perturbation arguments. However, despite of the increasing interest, there seems to be still few works on fundamental solutions for \( \alpha \in (0, 1) \). In such cases the diffusion term formally becomes the leading term and is no longer regarded as a simple perturbation of \( A_K(t) \), which causes difficulties in the study of (1.1). For example, so far little seems to be known about the uniqueness of weak solutions for such cases and this makes even the semigroup property of fundamental solutions nontrivial.

To state our main results we give the precise assumptions on the kernel \( K \) and the velocity \( v \). We assume that there are \( \alpha \in (0, 2) \) and \( C_0 > 0 \) such that

\[
K(t, x, y) = K(t, y, x) \quad \text{for a.e. } (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \tag{1.4}
\]

\[
\operatorname{ess.sup}_{t > 0, x \in \mathbb{R}^d} \int_{|x-y| \leq M} |x-y|^\alpha K(t, x, y) dy \leq C_0 M^{2-\alpha} \quad \text{for each } M \in (0, \infty), \tag{1.5}
\]

\[
\operatorname{ess.inf}_{t > 0, x, y \in \mathbb{R}^d} |x-y|^{d+\alpha} K(t, x, y) \geq C_0^{-1}. \tag{1.6}
\]

Following the conventions in (QG), we will call the case \( \alpha \in (1, 2) \) subcritical, the case \( \alpha = 1 \) critical, and the case \( \alpha \in (0, 1) \) supercritical. We note that if \( K \) satisfies

\[
C^{-1} |x-y|^{-d-\alpha} \leq K(t, x, y) \leq C |x-y|^{-d-\alpha}, \tag{1.7}
\]

then (1.5) and (1.6) are satisfied. The condition (1.6) guarantees the diffusion effect like \((-\Delta)^{\alpha/2}\). Taking this in mind, we assume that \( v \) belongs to a class of functions which is invariant under the scaling

\[
v(t, x) \mapsto r^{1-\alpha/v} v(rt, r^{1/\alpha}x), \quad r > 0. \tag{1.8}
\]

This scaling is natural in the following sense: if \( \theta(t, x) \) is a solution to (1.1) with \( A_K(t) = (-\Delta)^{\alpha/2} \) and \( v = v(t, x) \) then the rescaled function \( \theta(rt, r^{1/\alpha}x) \) satisfies (1.1) with \( A_K(t) = (-\Delta)^{\alpha/2} \) and \( v = r^{1-\alpha/v} v(rt, r^{1/\alpha}x) \), instead of \( v(t, x) \). Heuristically it is essential that \( v \) belongs to a function space which is invariant with respect to (1.8), in order to ensure a smoothing effect by \((-\Delta)^{\alpha/2}\), for example, see [29, 33, 30] and references there in for second order parabolic equations and [5, 22, 32] for the fractional diffusion equations.

To describe the regularity assumption on \( v \) let us introduce the Campanato spaces; see [16].

\[
\mathcal{L}^{p, \lambda}(\mathbb{R}^d) = \{ f \in \mathcal{L}_{loc}^p(\mathbb{R}^d) \mid \| f \|_{\mathcal{L}^{p, \lambda}} = \sup_B \left( R^{-\lambda} \int_B |f(x) - \frac{1}{|B|} \int_B f dx| \right)^{1/\lambda} < \infty \}. \tag{1.9}
\]

Here the supremum is taken over all balls \( B = B_R(y) \) (the ball with radius \( R > 0 \) centered at \( y \in \mathbb{R}^d \)), and \( |B| \) is the volume of the ball \( B \). We will sometimes write \( B_R \) for \( B_R(0) \) for simplicity of notations. The value \( \int_B f dx \) is defined by

\[
\int_B f = \frac{1}{|B|} \int_B f(x) dx. \tag{1.10}
\]

Then it is easy to see that the continuous embedding

\[
\mathcal{L}^{p, \lambda}(\mathbb{R}^d) \hookrightarrow \mathcal{L}^{1, \mu}(\mathbb{R}^d) \quad \text{if } \quad \mu = \frac{\lambda - d}{p} + d \tag{1.11}
\]

holds. Moreover, we have

\[
\dot{L}_w^{p, \mu}(\mathbb{R}^d) \hookrightarrow \mathcal{L}^{p, \lambda}(\mathbb{R}^d) \quad \text{if } \quad 0 < \lambda < d, \tag{1.12}
\]

\[
\mathcal{L}^{p, \lambda}(\mathbb{R}^d) = \text{BMO}(\mathbb{R}^d) \quad \text{if } \quad \lambda = d, \tag{1.13}
\]

\[
\mathcal{L}^{p, \lambda}(\mathbb{R}^d) = \dot{C}^{1, \frac{d}{\lambda}}(\mathbb{R}^d) \quad \text{if } \quad d < \lambda \leq d + p. \tag{1.14}
\]

Here \( L_w^p(\mathbb{R}^d) \) is the weak \( L^p \) space and \( \dot{C}^{\beta}(\mathbb{R}^d) \), \( \beta \in (0, 1] \), is the homogeneous Hölder space of the order \( \beta \), i.e.,

\[
\dot{C}^{\beta}(\mathbb{R}^d) = \{ f \in C(\mathbb{R}^d) \mid \| f \|_{\dot{C}^{\beta}} = \sup_{x, y \in \mathbb{R}^d} \frac{|f(x) - f(y)|}{|x-y|^{\beta}} < \infty \}. \]
Next we introduce the Morrey type spaces of $L^{p, \lambda}$-valued functions. 
\[ L^{p, \lambda}(0, \infty; L^{p, \lambda}(\mathbb{R}^d)) = \{ f \in L^p_{loc}(0, \infty; L^{p, \lambda}(\mathbb{R}^d)) \mid \| f \|_{L^{p, \lambda}(0, \infty; L^{p, \lambda}(\mathbb{R}^d))} = \sup_{t>0} \sup_{0<s<t} (t-s)^{-\lambda} \int_s^t \| f(\tau) \|^p_{L^{p, \lambda}(\mathbb{R}^d)} d\tau \}^{\frac{1}{p}} < \infty. \] (1.15)

For $1 \leq p, q \leq \infty$ let $L^p_{loc}(0, \infty; L^q_{loc}(\mathbb{R}^d))$ be the class of functions defined by 
\[ L^p_q(0, \infty; L^q_{loc}(\mathbb{R}^d)) = \{ f \in L^p_{loc}(0, \infty; \mathbb{R}^d) \mid \| f \|_{L^p_q(0, \infty; L^q_{loc}(\mathbb{R}^d))} < \infty \quad \text{for all} \quad R > 0. \] (1.16)

When $K(t, x, y)$ satisfies (1.5) and (1.6) for some $\alpha \in (0, 2)$ the velocity $v$ is assumed to satisfy the following two conditions:

(C1) there are $\lambda \in [2d/\alpha - d, 2d/\alpha + d]$ and $1 < q \leq \infty$ such that 
\[ \| v \|_{L^{p, \lambda}(0, \infty; \mathbb{R}^d)} \cap L^\infty_{loc}(0, \infty; (L^{p, \lambda}(\mathbb{R}^d))^d), \] (1.17)

where $p_\lambda = 1$ if $\lambda \in [2d/\alpha - d, d]$ and $p_\lambda = \infty$ if $\lambda \in (d, 2d/\alpha + d)$.

(C2) $\nabla \cdot v(t) = 0$ for a.e. $t > 0$ in the sense of distributions.

Remark 1.1 The space $L^{1, \frac{1}{d} + \frac{1}{d} - \frac{d}{\alpha}}(0, \infty; (L^{\frac{d}{\alpha}}(\mathbb{R}^d))^d)$ covers the following cases: $v \in L^{\infty}(0, \infty; (L^{\frac{d}{\alpha}}(\mathbb{R}^d))^d)$ for the subcritical case; $v \in L^{\infty}(0, \infty; (BMO(\mathbb{R}^d))^d)$ for the critical case; $v \in L^{\infty}(0, \infty; (C^{1, \alpha}(\mathbb{R}^d))^d)$ for the supercritical case.

Under the divergence free condition (C2) the drift term becomes skew-symmetric with respect to the usual $L^p(\mathbb{R}^d)$ inner product, and hence, the adjoint equation for (1.1) takes the same form as (1.1). This additional structure is essentially used in constructing fundamental solutions under weak regularity condition (C1). The divergence free condition sometimes plays important roles also in the second order parabolic equations with singular drifts; [29, 33, 30].

For simplicity of notations we will introduce the seminorm
\[ \| v \|_{X_\lambda} = \| v \|_{L^{1, \frac{1}{d} + \frac{1}{d} - \frac{d}{\alpha}}(0, \infty; (L^{\frac{d}{\alpha}}(\mathbb{R}^d))^d)}, \] (1.19)

For $T > 0$ and $x \in \mathbb{R}^d$ we also set
\[ \| v \|_{Y^\gamma_{T, \alpha}} = \| v \|_{L^\gamma(0, T; L^p_{X_\lambda}(B_1(x)))}, \] (1.20)

where $p_\lambda$ is as in (C1). The main result of this paper is the existence of fundamental solutions for (1.1). The precise definition of fundamental solutions will be given in the next section.

Theorem 1.2 Suppose that (1.4) - (1.6) and (C1) - (C2) hold. Then there exists a fundamental solution $P_{K, v}(t, x; s, y)$ for (1.1) satisfying the following properties.

\[ \int_{\mathbb{R}^d} P_{K, v}(t, x; s, y) dx = \int_{\mathbb{R}^d} P_{K, v}(t, x; s, y) dy = 1, \] (1.21)

\[ 0 \leq P_{K, v}(t, x; s, y) \leq C(t-s)^{-\frac{\alpha}{d}}, \] (1.22)

\[ P_{K, v}(t, x; s, y) = \int_{\mathbb{R}^d} P_{K, v}(t-x; t-s, y) P_{K, v}(t, s, y) dx, \quad t > s \geq 0, \] (1.23)

\[ |P_{K, v}(t, x; s, y) - P_{K, v}(t, x; s, y_1)| \leq C^\prime (|x_1 - x_2|^\beta + |y_1 - y_2|^\beta), \] (1.24)

and for $T \geq t_i > s_i \geq 0, i = 1, 2,$

\[ |P_{K, v}(t_1, x; s_1, y) - P_{K, v}(t_2, x; s_2, y)| \leq \frac{C_{T, v}|t_1 - t_2|^\beta + C_{T, v}|s_1 - s_2|^\beta}{(t_1 - s_1, t_2 - s_2)^\beta}, \] (1.25)

Here the positive constant $C$ depends only on $d, \alpha,$ and $C_0,$ the positive constants $c^\prime,$ $c,$ $\beta$ depend only on $d,$ $\alpha$ and $C_0,$ the positive constant $C_T, \gamma$ or $(C_{T, \gamma})$ depends only on $T, d, \alpha, C_0, \lambda, q, \| v \|_{X_\lambda},$ and $\| v \|_{Y^\gamma_{T, \alpha}}$ or $(\| v \|_{Y^\gamma_{T, \alpha}})$, and the positive constants $c^\prime, \beta^\prime$ depend only on $d, \alpha, C_0, \lambda, q, \| v \|_{X_\lambda}.$
Remark 1.3 In the proof of Theorem 1.2 we will also show that

$$\theta(t, x) := \int_{\mathbb{R}^d} P_{K, v}(t, x; s, y) f(y) \, dy \in C([0, \infty); L^p(\mathbb{R}^d)) \quad \text{if} \quad f \in L^p(\mathbb{R}^d), \quad 1 \leq p < \infty,$$

and the energy inequality

$$\|\theta(t)\|_{L^2}^2 + 2 \int_0^t \langle \partial_\tau \theta, \theta(\tau) \rangle \, d\tau \leq \|f\|_{L^2}^2,$$

for $t > s \geq 0$ if $f \in L^2(\mathbb{R}^d)$; see the proof in Section 4.2.

The estimates (1.24) and (1.25) show the Hölder continuity of the fundamental solution, where the Hölder exponents and the constant $C'$ are estimated uniformly in time and space, while the constants $C_{T, v}$ and $C_{T, v}$ can be larger as $|x|$ and $|y|$ increase, if $v$ grows at $|x| \to \infty$. We note that for some class of $(K, v)$ and solutions the Hölder continuity is obtained in [5, 22] for the critical case and also in [11, 32] for the supercritical case. In [5, 22, 11] the case $A_K(t) = (-\Delta)^{\alpha/2}$ and $v \in L^\infty(0, \infty; (L^{2d/\alpha, 2d/\alpha-d}_s)^d)$ was treated under the condition (C2), and [32] dealt with the case (1.7) and $v \in (C^{1-\alpha}((0, \infty) \times \mathbb{R}^d))^d$ but without (C2).

In order to prove Theorem 1.2 it is important to obtain the a priori estimates for fundamental solutions of the approximate equations which are, roughly speaking, of the form $\partial_\tau \theta + \delta(-\Delta)^{\alpha/2} \theta + A_K(t) \theta + \tilde{v} \cdot \nabla \theta = 0$ with $\delta > 0$. Here $\alpha \in (1, 2)$, and $\tilde{K}$ and $\tilde{v}$ are suitable mollifications of $K$ and $v$. It is more or less well known that the unique existence of fundamental solutions holds for such mollified equations due to the fact that the leading term is the extra diffusion term. So our main step is to prove the a priori (equi-)continuity estimates of the fundamental solutions. For the purpose we will use the Nash-type arguments by Komatsu [24, 25] where he studied the non-local diffusion equations without the drift term. As in [28, 24, 25], the arguments consist of four steps; the moment bound, the relative entropy bound, the overlap estimate, and the iteration estimate. However, due to the presence of the non-smooth drift term, it seems to be difficult to obtain the same estimates as in [24, 25]. For example, the moment of the fundamental solution is shifted by the drift, and we have to take this effect into account. To this end we will use time-dependent coordinates along the trajectory determined by a local average of $v$, instead of the usual coordinates $\mathbb{R}^d$. Although similar coordinates were used in [5, 11, 22], we have to choose the appropriate trajectory in each step carefully. We also note that, in fact, the arguments in [28, 24, 25] highly rely on the scaling property of (1.1), while the above approximation does not preserve such property. As a result, for example, it is difficult to get the equi-continuity estimates for solutions to the approximate equations, which causes another technicality in taking the limit and showing the desired estimate rigorously.

Theorem 1.2 has an application for the global regularity of solutions to (QG) in the critical case, as in [23, 5, 22]. Indeed, our result gives the alternative approach to this problem, based on the Nash-type arguments for fundamental solutions. In particular, it should be noted that different from [5] we need not study extension problems to use special property of the fractional Laplacian.

Before concluding the introduction we also state the spatial decay of fundamental solutions.

Theorem 1.4 Let $P_{K, v}(t, x; s, y)$ be the fundamental solution obtained by Theorem 1.2. Then there is $h_0 > 0$ such that for all $R_0 \geq 1$, $t \in (0, h_0]$, and $y \in (0, \min(\alpha, 2-\alpha))$, if $R \geq 1$ is sufficiently large then

$$\sup_{|x| \leq R_0, |y| \leq R_0} \sup_{0 \leq \tau \leq \frac{R}{4}} P_{K, v}(t + \tau, x; s + \tau, y) \leq C t^{-\frac{d-2}{2} + \frac{2}{\alpha}} R^{-\frac{d}{2}},$$

$$\sup_{|y| \leq R_0, |x| \leq R_0} \sup_{0 \leq \tau \leq \frac{R}{4}} P_{K, v}(t + \tau, x; s + \tau, y) \leq C t^{-\frac{d-2}{2} + \frac{2}{\alpha}} R^{-\frac{d}{2}}.$$

Here $C$ depends only on $d, \alpha, C_0, \gamma, \lambda, q, R_0 + h_0$, $\|v\|_{X_{\alpha}},$ and $\|v\|_{Y_{\alpha, \lambda}^{0, h_0, 0}}$.

Remark 1.5 The smallness of $h_0$ is required only when $\lambda = 2d/\alpha + d$. In this case $v$ might have a linear growth at spatial infinity, and the smallness of $h_0$ cannot be removed in general. More detailed pointwise estimates of fundamental solutions are studied in the forthcoming paper [26].

The rest of this paper is organized as follows. In Section 2 we give the precise definition of fundamental solutions and establish some inequalities which are used throughout of this paper. We also introduce the approximate equations for (1.1). Section 3 is the core of this paper, where we establish the a priori estimates for fundamental solutions to the approximate equations. In Section 4 we prove Theorem 1.2 and Theorem 1.4 by taking the limit based on the estimates obtained in Section 3. In Section 5 the uniqueness of weak solutions to (1.1) is discussed in the subcritical and critical cases for a specific class of $K$ and $v$. In Section 6 is devoted to the application of Theorem 1.2 to (QG).
2 Preliminaries

In this section we give a definition of fundamental solutions of (1.1) and prepare several inequalities which will be used in the proofs.

2.1 Definition of fundamental solutions

Let $K(t, x, y)$ be a positive measurable function satisfying $K(t, x, y) = K(t, y, x)$, and let $v(t, x) = (v_1(t, x), \ldots, v_d(t, x))$ be a vector field satisfying $\nabla \cdot v(t) = 0$. Then we formally define the bilinear forms $\mathcal{E}_{\nu_1}(\cdot, \cdot)$ and $\mathcal{B}^{(i)}_{K}(\cdot, \cdot)$ by

$$E_{\nu_1}(f, g) = -\langle f, v(t) \cdot \nabla g \rangle := \int_{\mathbb{R}^d} f(x) v(t, x) \cdot \nabla g(x) \, dx,$$

$$B^{(i)}_{K}(f, g) = \mathcal{E}_{K}^{(i)}(f, g) + \mathcal{E}_{\nu_1}(f, g),$$

where $\mathcal{E}_{K}^{(i)}(\cdot, \cdot)$ is the Dirichlet form associated with $K$ defined by (1.3). Then we formally define weak solutions and fundamental solutions to (1.1) as follows.

Let $T > s \geq 0$. A function $\theta \in L^\infty([s,T]; L^2(\mathbb{R}^d))$ is said to be a weak solution to (1.1) for $t \in [s, T)$ with initial data $\theta_s$ at $t = s$ if $\theta$ satisfies

$$\int_s^T \mathcal{E}_{K}^{(i)}(\theta(t), \theta(t)) \, dt < \infty,$$

and

$$- \int_s^T \theta(t), \partial_t \varphi(t) > dt + \int_s^T B^{(i)}_{K}(\theta(t), \varphi(t)) \, dt = \theta_s, \varphi(s) >, \forall \varphi \in C^0_{0}\left([s,T] \times \mathbb{R}^d\right).$$

Here $C^0_{0}\left([s,T] \times \mathbb{R}^d\right)$ is the set of compactly supported smooth functions on $[s, T) \times \mathbb{R}^d$. Then a measurable function $P_K, v(t, x, y)$ on $\{(t, s, x, y) \mid t > s \geq 0, x, y \in \mathbb{R}^d\}$ is said to be a fundamental solution to (1.1) if for each $T > s \geq 0$ and $f \in L^2(\mathbb{R}^d)$ the function

$$(P_K, v)(t, x, y) := \int_{\mathbb{R}^d} P_K(v, t, x, y) f(y) \, dy,$$

is a weak solution to (1.1) for $t \in [s, T)$ with initial data $f$ at $t = s$.

**Remark 2.1.** When $K$ satisfies the condition (1.6) a weak solution in our definition belongs to $L^2(s, T; \dot{H}^{\alpha/2}(\mathbb{R}^d))$ due to (2.3) and the coercive estimate

$$\text{ess. inf}_{t > s} \mathcal{E}_{K}^{(i)}(f, f) \geq C\|(-\Delta)^{\frac{\alpha}{2}} f\|_{L^2}^2.$$

For $K(t, x, y)$ and $M \in (0, \infty)$ we set

$$J_l(K) = \text{ess. sup}_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^l K(t, x, y) \, dy,$$

$$J_{\gamma_1}(K) = K(t, x, y) \chi_{\{|x - y| \leq M\}}(x, y),$$

$$J_{\gamma_2}(K) = K(t, x, y) \chi_{\{|x - y| > M\}}(x, y).$$

Here $\chi_{A}(x, y)$ is the characteristic function of the set $A \subset \mathbb{R}^d \times \mathbb{R}^d$. When $K$ satisfies (1.5) we have

$$J_{\gamma_1}(K) \leq C M^{\gamma_1 - \alpha}, \text{ if } \gamma_1 > \alpha,$$

$$J_{\gamma_2}(K) \leq C' M^{\gamma_2 - \alpha} \text{, if } \gamma_2 \in [0, \alpha),$$

Indeed, by the monotone convergence theorem we have

$$\int_{\mathbb{R}^d} |x - y|^{\gamma_1} K(t, x, y) \, dy = \sum_{k = 0}^{\infty} \int_{2^{-k-1}M \leq |x - y| \leq 2^{-k}M} |x - y|^{\gamma_1} K(t, x, y) \, dy$$

$$\leq C \sum_{k = 0}^{\infty} (2^{-k}M)^{\gamma_1 - 2} \int_{2^{-k-1}M \leq |x - y| \leq 2^{-k}M} |x - y|^2 K(t, x, y) \, dy$$

$$\leq C C_0 \sum_{k = 0}^{\infty} (2^{-k}M)^{\gamma_1 - 2}(2^k M)^{2-\alpha} \leq C C_0 M^{\gamma_1 - \alpha},$$

if $\gamma_1 > \alpha$. This proves (2.10). The estimate (2.11) is obtained similarly and we omit the details.
For later use it will be convenient to define the “norm” of the kernel $K$:

$$
\|K\|_\gamma = \sum_{k=1}^\infty k^2 2^{(\gamma-2)} J_2(K_{2^k}) + \sum_{k=1}^\infty 2^\frac{2\gamma}{\alpha} J_0(K_{\gamma,2^k}).
$$

From (2.10) and (2.11) it is easy to see that $\|K\|_\gamma < \infty$ if $\gamma < \alpha$ when $K$ satisfies (1.5).

For $t > s \geq 0$ and $r > 0$ we define

\begin{align*}
(i,f)(t,x) &= f(t+s,x), \\
(j,f)(s,x) &= f(t-s,x), \\
(R_t f)(t,x) &= f(rt, r^{\frac{1}{\alpha}} x), \\
(R_t,K)(t,x,y) &= K(rt, r^{\frac{1}{\alpha}} x, r^{\frac{1}{\alpha}} y).
\end{align*}

Note that $(j_{x-x},i)(\tau,x) = (j,f)(\tau,x)$ holds. Let $P_{K,v}(t; x, y)$, $t > s \geq 0$, $x, y \in \mathbb{R}^d$, be the fundamental solution to (1.1). Then the scaling, the translation, and the adjoint properties of (1.1) imply

\begin{align*}
P_{K,v}(t; x, y) &= P_{i,K,i,v}(t-s,x; 0, y) = P_{j_{x-x},j_{x-x},v}(t-s,y; 0, x) = P_{j_{x-x},v}(t-s,y; 0, x), \\
R_t^{\frac{1}{\alpha}} P_{K,v}(r_t, \frac{1}{\alpha} x; r_\alpha x, r_\alpha y) &= P_{r_t,1/\alpha K,1/\alpha v}(t; x, y).
\end{align*}

Here, for example, $P_{j_{x-x}}(\tau; y, r, (0 \leq r < \tau \leq t, y \in \mathbb{R}^d)$ is a fundamental solution to the equation

$$
\partial_\tau \theta(\tau, y) + A_{j_{x-x}}(\tau) \theta(\tau, y) + (\partial_\tau - \partial_y) \nabla \theta(\tau, y) = 0, \quad \tau \in (0, t], \quad y \in \mathbb{R}^d,
$$

and $P_{r_t,1/\alpha K,1/\alpha v}(t; x, y)$ is a fundamental solution to the equation

\begin{equation}
\partial_\tau \theta + A_{r_t,1/\alpha R_{r_t}}(t) \theta + (r_t^{1/\alpha} R_{r_t} \cdot \nabla) \theta = 0, \quad t > 0, \quad x \in \mathbb{R}^d.
\end{equation}

### 2.2 Inequalities from real analysis

**Lemma 2.2** Let $f \in L^{1,\mu}(\mathbb{R}^d)$ for some $\mu \in [0, d+1]$. Let $x_1, x_2 \in \mathbb{R}^d$ and $R_1 \geq R_2 > 0$. Then

\begin{equation}
\left| \int_{B_{R_1}(x_1)} f - \int_{B_{R_2}(x_2)} f \right| \leq \begin{cases} 
C \|f\|_{L^{1,\mu}(R^{d-d}_1)} & \text{if } 0 \leq \mu < d, \\
C \|f\|_{L^{1,\mu}(R^{d-d}_2)} \log \frac{R_1}{R_2} & \text{if } \mu = d, \\
C \|f\|_{L^{1,\mu}(R^{d-d}_1)} \log \frac{R_1}{R_2} & \text{if } d < \mu \leq d+1.
\end{cases}
\end{equation}

Here $C$ depends only on $d$ and $\mu$.

**Proof.** If $x_1 = x_2 = x$ and $2\kappa - 2^{k-1} R_2 \leq R_1 \leq 2^{k} R_2$ for some $k \in \mathbb{N}$, then (2.20) follows from

\begin{equation}
\left| \int_{B_{2^k R_2}(x)} f - \int_{B_{R_1}(x)} f \right| \leq C \|f\|_{L^{1,\mu}(R^{d-d})},
\end{equation}

and

\begin{equation}
\left| \int_{B_{2^{k-1} R_2}(x)} f - \int_{B_{R_2}(x)} f \right| \leq \sum_{l=1}^K \int_{B_{R_2}(x)} f - \int_{B_{R_2}(x)} f \leq C \|f\|_{L^{1,\mu}(R^{d-d})}. \sum_{l=1}^K 2^{(\mu-d)}.
\end{equation}

We turn to the case $R_1 = R_2 = R$. We may assume that $x_2 = 0$. Then (2.20) for the case $\mu \in (d,d+1]$ is trivial since the relation $L^{1,\mu}(\mathbb{R}^d) = C^{\beta}(\mathbb{R}^d)$ with $\beta = \mu - d$. The case $\mu = d$ follows from the estimate for functions in $BMO(\mathbb{R}^d)$; for example, see [17]. As for the case $\mu \in [0,d]$, if $|x_1| \leq R$ then $B_R(x_1) \subset B_{2R}(0)$ and hence

\begin{equation}
\left| \int_{B_{R}(x_1)} f - \int_{B_{R}(0)} f \right| \leq \left| \int_{B_{R}(x_1)} f - \int_{B_{2R}(0)} f \right| + \left| \int_{B_{2R}(0)} f - \int_{B_{R}(0)} f \right| \leq C \|f\|_{L^{1,\mu}(R^{d-d})}.
\end{equation}

If $2^{k-1} R \leq |x_1| \leq 2^k R$ for some $k \in \mathbb{N}$ then $B_{2^k R}(x_1) \subset B_{2^{k+1} R}(0)$, and thus from (2.20) with the case $x_1 = x_2 = x$ proved above,

\begin{equation}
\left| \int_{B_{R}(x_1)} f - \int_{B_{R}(0)} f \right| \leq \left| \int_{B_{R}(x_1)} f - \int_{B_{2^k R}(x_1)} f \right| + \left| \int_{B_{2^k R}(x_1)} f - \int_{B_{2^{k+1} R}(0)} f \right| + \left| \int_{B_{2^{k+1} R}(0)} f - \int_{B_{R}(0)} f \right| \leq C \|f\|_{L^{1,\mu}(R^{d-d})}.
\end{equation}
Collecting these, we get (2.20). This completes the proof.

Next we consider the trajectory determined by an average of $u$:

$$
\begin{align*}
\begin{cases}
\frac{d}{dt} \xi_u(t; x, R) &= \int_{B_R(x+\xi_u(t; x, R))} u(t), \quad 0 \leq t \leq t_0, \\
\xi_u(0; x, R) &= 0,
\end{cases}
\end{align*}
$$

(2.21)

Here $x \in \mathbb{R}^d$ and $R > 0$. The solutions of ODEs of the form (2.21) play important roles in the analysis of (1.1). Set

$$
F_R[u](t_0, x) = \sup_{0 < t < t_0} \left| \int_{B_R(x)} u(s) \, ds \right|.
$$

(2.22)

Proposition 2.3 Assume that $u \in \mathcal{L}^1(0, t_0; \mathcal{L}^{1, \mu}(\mathbb{R}^d))$ for some $\mu \in [0, d+1]$. Let $\xi_u(t; x, R)$ be the solution to (2.21). Then we have:

(i) By the definition of $\xi_u(t; x, R)$

$$
\begin{align*}
|\xi_u(t_0; x, R)| &\leq C R^{n-d} \|u\|_{L^1(0,t_0,L^{1,\mu})} + F_R[u](t_0, x)
\end{align*}
$$

(2.23)

Here $C$ depends only on $d$ and $\mu$.

(ii) For $x_1, x_2 \in \mathbb{R}^d$ and $R_1 \geq R_2 > 0$ it follows that

$$
\begin{align*}
|\xi_u(t_0; x_1, R_1) - \xi_u(t_0; x_2, R_2)| &\leq C\|u\|_{L^1(0,t_0,L^{1,\mu})} R_2^{n-d} + C \left( \|u\|_{L^1(0,t_0,L^{1,\mu})} \right) \left( |x_1 - x_2| + \frac{R_1}{R_2} \right)
\end{align*}
$$

(2.24)

Here $C$ depends only on $d$ and $\mu$.

Proof. (i) By the definition of $\xi_u(t; x, R)$ we have

$$
|\xi_u(t; x, R)| \leq \int_{B_R(x+\xi_u(t; x, R))} u(s) - \int_{B_R(x)} u(s) \, ds + \int_{0}^{t} \int_{B_R(x)} u(s) \, ds.
$$

Then Lemma 2.2 yields

$$
\begin{align*}
|\xi_u(t; x, R)| &\leq C R^{n-d} \int_{0}^{t} \|u(s)\|_{L^{1,\mu}} \, ds + \int_{0}^{t} \int_{B_R(x)} u(s) \, ds
\end{align*}
$$

(2.25)

Then the case $\mu \in [0, d)$ is obvious. When $\mu = d$ the above estimate implies that there is $\bar{C} > 0$ depending only on $C$ such that

$$
|\xi_u(t_0; x, R)| \leq \bar{C}\|u\|_{L^1(0,t_0,L^{1,\mu})} \left( \log(e + R^{-1}\|u\|_{L^1(0,t_0,L^{1,\mu})}) + \log(e + R^{-1} \sup_{0 < t < t_0} \left| \int_{B_R(x)} u(s) \, ds \right|) \right)
$$

$$
+ \sup_{0 < t < t_0} \left| \int_{B_R(x)} u(s) \, ds \right|.
$$
When $\mu \in (d, d+1)$ there is $\tilde{C} > 0$ depending only on $C$ and $\mu$ such that
\[
|\xi_u(t_0; x, R)| \leq \tilde{C}\left(\|u\|_{L^1(0,t_0;L^1,\mu)} + \|u\|_{L^1(0,t_0;L^1,\mu)}\left(\sup_{0 < t < t_0} \int_0^t \int_{B_R(x)} u(s) \, ds\right)\right) + \sup_{0 < t < t_0} \int_0^t \int_{B_R(x)} u(s) \, ds.
\]
When $\mu = d + 1$ the Gronwall inequality yields
\[
|\xi_u(t_0; x, R)| \leq e^{C\|u\|_{L^1(0,t_0;L^1,\mu)}} \sup_{0 < t < t_0} \int_0^t \int_{B_R(x)} u(s) \, ds.
\]
Then it is not difficult to get (2.23). This completes the proof of (i). (ii) We may assume that $2^k R_2 \leq R_1 \leq 2^{k+1} R_2$ for some $k \in \mathbb{N} \cup \{0\}$. Set $\varphi_j(s) = \xi_u(s; x_j, R_j)$. Again from the ODEs, we have
\[
|\varphi_1(t) - \varphi_2(t)| \leq \int_0^t |\int_{B_{R_1}(x_1 + \varphi_1(s))} u(s) - \int_{B_{R_2}(x_1 + \varphi_1(s))} u(s)| \, ds
\]
\[
+ \int_0^t |\int_{B_{R_2}(x_1 + \varphi_1(s))} u(s) - \int_{B_{R_2}(x_2 + \varphi_2(s))} u(s)| \, ds
\]
\[
=: I_1(t_0) + I_2(t).
\]
Then Lemma 2.2 yields similar estimate as in the right-hand side of (2.25). Hence as in the proof of (i) we get
\[
|\varphi_1(t_0) - \varphi_2(t_0)| \leq C R_2^{\alpha - d} \|u\|_{L^1(0,t_0;L^1,\mu)} + I_1(t_0)
\]
\[
+ C\left(\|u\|_{L^1(0,t_0;L^1,\mu)} \log \left(e + R_2^{-1} \|u\|_{L^1(0,t_0;L^1,\mu)}\right)
\]
\[
+ \|u\|_{L^1(0,t_0;L^1,\mu)} \log(e + \frac{|x_1 - x_2|}{R_2}) + I_1(t_0)\right) \quad \text{if } \mu = d,
\]
\[
C\left(\|u\|_{L^1(0,t_0;L^1,\mu)} |x_1 - x_2|^{d - \alpha} + I_1(t_0)\right) \quad \text{if } \mu \in (d, d+1),
\]
\[
C\left(\|u\|_{L^1(0,t_0;L^1,\mu)} |x_1 - x_2|^{d - \alpha} + I_1(t_0)\right) \quad \text{if } \mu = d + 1.
\]
Here $C$ depends only on $d$ and $\mu$. The term $I_1(t_0)$ is estimated from Lemma 2.2 and we get the desired estimates. This proves the assertion (ii).

### 2.3 Approximation of kernel

To prove the existence of fundamental solutions satisfying the desired estimates rigorously, we have to approximate the kernel $K(t, x, y)$ suitably. For this purpose it will be useful to define the class of smooth kernels $K$. Let $C_0$ be the number in (1.6). We say $K(t, x, y)$ is a smooth kernel of the order $\alpha' \in (1, 2)$ if $K(t, x, y)$ is of the form
\[
K(t, x, y) = |x - y|^{-d - \alpha} k(t, x, y) + \delta|x - y|^{-d - \alpha'},
\]
where $\delta > 0$, $\alpha' \geq \alpha$, and $k(t, x, y)$ is a function defined on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ such that
\[
k(t, x, y) = k(t, y, x), \quad \sup_{t \in \mathbb{R}, x, y \in \mathbb{R}^d} \sum_{|\beta| \leq 1} |\nabla^\beta x y k(t, x, y)| < \infty, \quad \inf_{t \in \mathbb{R}, x, y \in \mathbb{R}^d} k(t, x, y) \geq C_0^{-1}.
\]
sense. Moreover, under the assumption of $\nabla \cdot v(t) = 0$, $P_{K,\nu}(t, x; s, y)$ satisfies
\[
\int_{\mathbb{R}^d} P_{K,\nu}(t, x; s, y) \, dy = \int_{\mathbb{R}^d} P_{K,\nu}(t, x; s, y) \, dx = 1,
\]
(2.29)
\[
P_{K,\nu}(t, x; s, y) = \int_{\mathbb{R}^d} P_{K,\nu}(t, x; \tau, \xi) P_{K,\nu}(\tau, \xi, s, y) \, d\xi \quad t > \tau > s \geq 0,
\]
(2.30)
\[
P_{K,\nu}(t, x; s, y) \leq C(t-s)^{-\frac{d}{2}},
\]
(2.31)
\[
\int_{s}^{T} \|P_{K,\nu}(t, s)\|_{L^{2}}^2 \, dt \leq C\|f\|_{L^{2}}^2,
\]
(2.32)
\[
\|P_{K,\nu}(t, s)\|_{L^{p}} \leq \|f\|_{L^{p}} 1 \leq p \leq \infty.
\]
(2.33)
Here $C$ depends only on $d$, $\alpha$, and $C_{0}$ as for (2.31), see [26]. Especially, $C$ is independent of $\alpha'$ and $\delta$. Let $T_{\Sigma}(\sigma)$, $N \gg 1$ be a truncated function such that $T_{\Sigma}(\sigma) = \sigma$ if $\sigma \leq N$ and $T_{\Sigma}(\sigma) = N$ if $\sigma \geq N$. Then we define the approximation of $K(t, x, y)$ satisfying (1.5) and (1.6) by
\[
K^{(N, \delta)}(t, x, y) = |x - y|^{-d - \alpha}(\nu_{\delta} * k_{N})(t, x, y) + \delta |x - y|^{-d - 1 - \frac{d}{2}},
\]
(2.34)
where
\[
k_{N}(t, x, y) = T_{\Sigma}(|x - y|^{d + \alpha} K(t, x, y)), \quad t > 0, \quad k_{N}(t, x, y) = C_{0}^{-1}, \quad t \leq 0,
\]
(2.35)
which satisfies $k_{N}(t, x, y) \geq C_{0}^{-1}$ if $N \geq C_{0}^{-1}$ from the definition and (1.6). Here $\nu_{\delta}$ is a mollifier on $(t, x, y)$ variables such as
\[
(\nu_{\delta} * k_{N})(t, x, y) = \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \nu_{\delta}(t-s) \nu_{\delta}(x-\xi) \nu_{\delta}(y-\eta) k_{N}(s, \xi, \eta) \, d\xi \, d\eta \, ds,
\]
(2.36)
where $\nu_{\delta}(z) = \delta^{-n} \nu_{\delta}(z/\delta)$ and $\nu_{\delta}(z)$ is a smooth non-negative function on $\mathbb{R}^{n}$ satisfying $\text{supp} \, \nu_{\delta} \subset \{z \in \mathbb{R}^{n} \mid |z| \leq 2\}$, $\int_{\mathbb{R}^{n}} \nu_{\delta}(z) \, dz = 1$. We note that the above mollification preserves the symmetry and the estimate such as
\[
(\nu_{\delta} * k_{N})(t, x, y) = (\nu_{\delta} * k_{N})(t, y, x), \quad (\nu_{\delta} * k_{N})(t, x, y) \geq C_{0}^{-1}.
\]
(2.37)
It is clear that $K^{(N, \delta)}(t, x, y)$ is a smooth kernel of the order $1 + \alpha/2$. If $8\delta \leq M$ then we have
\[
J_{2}(K^{(N, \delta)}_{M}) \leq C(\delta^{-\alpha} N + M^{2-\alpha} + \delta M^{1-\frac{d}{2}}),
\]
(2.38)
\[
J_{0}(K^{(N, \delta)}_{M}) \leq C(M^{-\alpha} + \delta M^{1-\frac{d}{2}}),
\]
(2.39)
and if $M \leq 8\delta$ then
\[
J_{2}(K^{(N, \delta)}_{M}) \leq C(\delta^{-\alpha} N + \delta M^{1-\frac{d}{2}}),
\]
(2.40)
\[
J_{0}(K^{(N, \delta)}_{M}) \leq C(M^{-\alpha} + \delta^{2-\alpha} M^{-2} N + \delta M^{1-\frac{d}{2}}),
\]
(2.41)
where $C$ depends only on $d$, $\alpha$, and $C_{0}$. Indeed, from the definition of $K^{(N, \delta)}$ we get
\[
\int_{\mathbb{R}^{d}} |x - y|^{2} K^{(N, \delta)}_{M}(t, x, y) \, dy \
\leq \int_{|x - y| \leq M} \nu_{2\delta+1}(\tau, \xi, \eta) \frac{k_{N}(t - \tau, x - \delta \xi, y - \delta \eta)}{|x - y|^{d + \alpha - 2}} \, d\xi \, d\eta \, d\tau \, dy \
+ \delta \int_{|x - y| \leq M} |x - y|^{-d - \alpha - 2} \, dy \
\leq \int_{|x - y| \leq M} \nu_{2\delta+1}(\tau, \xi, \eta) \frac{k_{N}(t - \tau, x - \delta \xi, y - \delta \eta)}{|x - y|^{d + \alpha - 2}} \, d\xi \, d\eta \, d\tau \, dy \
+ \int_{|x - y| \leq \delta \delta} \nu_{2\delta+1}(\tau, \xi, \eta) \frac{k_{N}(t - \tau, x - \delta \xi, y - \delta \eta)}{|x - y|^{d + \alpha - 2}} \, d\xi \, d\eta \, d\tau \, dy + C\delta M^{1-\frac{d}{2}} \\
= I_{1} + I_{2} + C\delta M^{1-\frac{d}{2}}.
\]
Here we have set $\nu_{2\delta+1}(\tau, \xi, \eta) = \nu_{1}(\tau) \nu_{\delta}(\xi) \nu_{\delta}(\eta)$ for simplicity of notations. As for $I_{1}$, since $|x - \delta \xi - (y - \delta \eta)| \geq |x - y| - \delta(|\xi| + |\eta|) \geq |x - y| - 4\delta \geq |x - y|/2$ for $|\xi| + |\eta| \leq 4$, we see
\[
I_{1} \leq \int_{|x - \delta \xi - (y - \delta \eta)| \leq CM_{\delta}} \frac{k_{N}(t - \tau, x - \delta \xi, y - \delta \eta)}{|x - \delta \xi - (y - \delta \eta)|^{d + \alpha - 2}} \, dy \, d\xi \, d\eta \, d\tau \leq C M^{2-\alpha},
\]
by the definition of $k_{N}$ and (1.5). As for $I_{2}$, from $k_{N} \leq N$ it is easy to get $I_{2} \leq C\delta^{2-\alpha} N$. This completes the proof of (2.38). The estimates (2.39), (2.40), and (2.41) are proved similarly and the details are omitted here.
2.4 Approximation of velocity field

In order to prove the main theorems we need to approximate $v$ by a smooth and bounded vector field satisfying $\nabla \cdot v(t) = 0$. To this end the next lemma is essential.

**Lemma 2.4** Let $\lambda \in [2d/\alpha - d, 2d/\alpha + d]$ and set $p_1 = 1$ if $\lambda \in [2d/\alpha - d, d]$ and $p_3 = \infty$ if $\lambda \in (d, 2d/\alpha + d]$.

Let $q_1 \in [1, \infty)$, $q_2 \in [1, \infty]$, and $\mu \in [0, 1]$. Let $v \in L^{q_1, \mu}(0, \infty; (L^{\frac{2d}{\alpha}}(\mathbb{R}^d))^d) \cap L^{q_2}_{\text{loc}}(0, \infty; (L^{p_3}_{\text{loc}}(\mathbb{R}^d))^d)$ be a given vector field satisfying $\nabla \cdot v(t) = 0$. Then there is a sequence of smooth and bounded vector fields $\{v^{(N)}\}$ satisfying $\nabla \cdot v^{(N)}(t) = 0$ and

\[ \sup_{N} \|v^{(N)}\|_{L^{q_1, \mu}(\mathbb{R}^d)} \leq C_1 \|v\|_{L^{q_1, \mu}(\mathbb{R}^d)}, \] (2.42)

\[ \lim_{N \to \infty} \sup_{N} \|v^{(N)}\|_{L^{q_2}_{\text{loc}}(0, R_1; L^{p_3}_{\text{loc}}(B_{R_2}(x)))} \leq C_2 \|v\|_{L^{q_2}_{\text{loc}}(0, R_1; L^{p_3}_{\text{loc}}(B_{R_2}(x)))}, \] (2.43)

and

\[ v^{(N)} \to v \quad \text{in} \quad (L^{1}_{\text{loc}}((0, \infty) \times \mathbb{R}^d))^d. \] (2.44)

Here $C_1$ depends only on $d$, $\alpha$, $\lambda$, and $q_1$, and $C_2$ depends only on $d$ and $q_2$.

The proof of Lemma 2.4 will be given in the appendix.

3 A priori estimates for fundamental solutions

In this section we establish the a priori estimates on the regularity of $P_{K,v}(t, x; s, y)$.

3.1 Moment bound

We first consider the case that $K(t, x, y)$ and $v$ are smooth. For $R > 0$ let $\chi_R(x)$, $0 \leq \chi_R \leq 1$, be a smooth cut-off function such that $\chi_R(x) = 1$ if $|x| \leq R$ and $\chi_R(x) = 0$ if $|x| \geq 2R$, and $|\nabla \chi_R(x)| \leq C/R$. For $\gamma \in (0, 2]$ we set

\[ \rho_{\gamma}(x) = |x|^{2 - \gamma}, \quad \rho_{\gamma, R}(x) = \rho_{\gamma}(x) \chi_R(x). \] (3.1)

Let $K$ be a smooth kernel of the order $\alpha' \in (1, 2)$. Then it is not difficult to see that if $\gamma < \alpha$ then $\|K\|_{\gamma} < \infty$. For $\lambda \in [2d/\alpha - d, 2d/\alpha + d]$ we set

\[ \|u\|_{X_{\lambda}(s, t)} = \sup_{s \leq u \leq t} \left( t - \frac{1}{2} \int_{s}^{t} \|u(\tau)\|_{\frac{2d}{\alpha + \lambda}} d\tau \right). \] (3.2)

Note that $\|j \nu\|_{X_{\lambda}(0, s - 1)} = \|\nu\|_{X_{\lambda}(s, t)}$ holds, where $j_\nu$ is defined by (2.14).

**Proposition 3.1** Let $K(t, x, y)$ be a smooth kernel of the order $\alpha' \in (1, 2)$. Let $\lambda \in [2d/\alpha - d, 2d/\alpha + d]$ and $\gamma \in (0, \min\{(2, 2 - \alpha')\})$. Assume that $u$ is a smooth and bounded vector field satisfying $\nabla \cdot u(t) = 0$ in $(0, 1) \times \mathbb{R}^d$. Let $\xi(t, x) \in \mathbb{R}^d$, $0 \leq \tau \leq 1$ be the solution to the ODE (2.21) with $R = 1$. Then for any $\tau \in (0, 1]$,

\[ C_1 \leq m_{\gamma}(\tau; x) := \int_{\mathbb{R}^d} \rho_{\gamma}(y - x - \xi(\tau; x)) P_{K,v}(\tau, y, 0, x) dy \leq C_2 (1 + \|K\|_{\gamma}). \] (3.3)

Here $C_1$ depends only on $d$, $\alpha$, and $\gamma$, and $C_2$ depends only on $d$, $\alpha$, $\gamma$, $\lambda$, and $\|u\|_{X_{\lambda}(0, 1)}$.

**Proof**. Without loss of generality we may assume $x = 0$. For simplicity of notations we write $P(\tau, y)$ and $\xi(\tau)$ for $P_{K,v}(\tau, y; 0, 0)$ and $\xi(\tau; 0)$, respectively. Then set

\[ m_{\gamma}(\tau; y) = \int_{\mathbb{R}^d} \rho_{\gamma}(y - \xi(\tau)) P(\tau, y) dy, \quad 0 < \tau \leq 1. \] (3.4)

We first observe that

\[ P(\tau, \bar{y}) \leq C \tau^{\frac{d}{2}} = e^{b \tau^{\frac{d}{2}}}, \] (3.5)

where $b$ depends only on $d$, $\alpha$, and $C_0$. As in [28, 24, 25], we set

\[ Q(\tau) = -\int_{\mathbb{R}^d} P(\tau, y) \log P(\tau, y) dy, \] (3.6)

\[ f(\tau) = Q(\tau) - \frac{d}{\alpha} \log \tau + b, \] (3.7)

\[ h(\tau) = \int_{0}^{\tau} \mathbb{E}_{K}^{(r)}(P(r), \log P(r)) dr - \frac{d}{\alpha} \log \tau. \] (3.8)
Lemma 3.2 For any $0 \leq \tau_2 < \tau_1 \leq 1$, we have

$$Q(\tau_1) - Q(\tau_2) \geq \int_{\tau_2}^{\tau_1} \mathcal{E}_K^{(r)}(P(r), \log P(r)) \, dr,$$

(3.9)

$$f(\tau) \geq 0,$$

(3.10)

$$f(\tau_1) - f(\tau_2) \geq h(\tau_1) - h(\tau_2).$$

(3.11)

Proof of Lemma 3.2. For every $\epsilon > 0$ and $n \geq 1$ we have

$$- \int_{\mathbb{R}^d} \chi_n(y) P(\tau_1, y) \log(P(\tau_1, y) + \epsilon) \, dy + \int_{\mathbb{R}^d} \chi_n(y) P(\tau_2, y) \log(P(\tau_2, y) + \epsilon) \, dy \geq 0.$$

Hence by taking $\epsilon > 0$ and $n \geq 1$ we have

$$- \int_{\mathbb{R}^d} \chi_n(y) P(\tau_1, y) \log(P(\tau_1, y) + \epsilon) \, dy + \int_{\mathbb{R}^d} \chi_n(y) P(\tau_2, y) \log(P(\tau_2, y) + \epsilon) \, dy \geq 0.$$

Letting $\epsilon \to 0$, we get (3.9). The inequality (3.11) follows from (3.9). This completes the proof of Lemma 3.2.

Lemma 3.3 There is $C > 0$ depending only on $d$, $\alpha$, and $\gamma$ such that

$$m_{\gamma, \infty}(\tau) \geq Ce^{\mathcal{E}_K^{(r)}}.$$

(3.12)

Proof of Lemma 3.3. The proof is the same as in [28]. We use the inequality $T \log T + \lambda T \geq -e^{-\lambda} - 1$ which holds for $T, \lambda > 0$. By setting $\lambda = a_1 \rho_{\gamma}(y - \xi(\tau)) + a_2$ with $a_1$, $a_2 \geq 0$, we have

$$-Q(\tau) + a_1 m_{\gamma, \infty}(\tau) + a_2 = \int_{\mathbb{R}^d} P(\log P + a_1 \rho_{\gamma}(y - \xi(\tau)) + a_2) \, dy \geq - \int_{\mathbb{R}^d} e^{-a_1 \rho_{\gamma}(y - \xi(\tau)) - a_2} \, dy = -Ce^{-a_2} \int_{\mathbb{R}^d} e^{-a_1} \, r \, dr = -Ce^{-a_2} a_1 \frac{2d}{\gamma}.$$

Hence by taking $a_1 = m_{\gamma, \infty}^{-1}$ and $a_2 = 2d \gamma^{-1} \log m_{\gamma, \infty}$, we obtain

$$-Q(\tau) + 1 + \frac{2d}{\gamma} \log m_{\gamma, \infty} \geq -c.$$

This completes the proof of Lemma 3.3.

Proof of Proposition 3.1(continued). From (3.7), (3.10), and Lemma 3.3, it is easy to obtain

$$m_{\gamma, \infty}(\tau) \geq C_1 \tau^{\frac{2d}{\gamma}}, \quad \tau \in (0, 1],$$

(3.13)

where $C_1$ depends only on $d$, $\alpha$, $C_0$, and $\gamma$. This proves the lower bound in (3.3). Next we consider the upper bound. Let $R$, $M \geq 1$. From the definition of $m_{2, R}(\tau)$ we have

$$\frac{d}{d\tau} m_{2, R}(\tau) = -\mathcal{E}_{K, M}^{(r)}(\rho_{2, R}(\tau - \xi(\tau)), P(\tau)) - \mathcal{E}_{K, \gamma, \alpha}^{(r)}(\rho_{2, R}(\tau - \xi(\tau)), P(\tau)) + \int_{\mathbb{R}^d} (u(\tau, y) - \frac{d}{d\tau} \xi(\tau)) \cdot \nabla y \rho_{2, R}(\tau - \xi(\tau)) P(\tau, y) \, dy = I_1 + I_2 + I_3.$$
(i) Estimate of $I_1$: The inequality $\frac{(1 - \theta)^2}{\Delta t} \leq \frac{1}{(1 - \theta)^2} \log \theta$ for $\theta \in (0, 1)$ yields
\[
I_1 \leq \int_{\Omega} |\rho_2,\phi(y(\tau) + \xi(\tau))| \cdot (1 - P(\tau, \theta)) \int_{\Omega} P(\tau, \theta) K_M(\tau, y, z) \, dw + dy
\]
\[
\leq \left( \int_{\Omega} |\rho_2,\phi(y(\tau) + \xi(\tau))|^2 P(\tau, \theta) K_M(\tau, y, z) \, dw + dy \right)^{\frac{1}{2}}
\]
\[
\leq C J_2^2 (K_M)(\xi(\tau)(P(\tau), \log P(\tau)))^{\frac{1}{2}}.
\] (3.14)

Here we have also used $|\rho_2,\phi(y(\tau) + \xi(\tau))| \leq C |y - z|.$

(ii) Estimate of $I_2$: Since $\rho_2,\phi, P,$ and $K_{v,\alpha}$ are nonnegative functions, we have
\[
I_2 = \int_{\Omega} |\rho_2,\phi(y(\tau) + \xi(\tau))| P(\tau, \theta) K_{v,\alpha}(\tau, y, z) \, dw + dy
\]
\[
\leq 2 \int_{\Omega} |\rho_2,\phi(y(\tau) + \xi(\tau))| P(\tau, \theta) K_{v,\alpha}(\tau, y, z) \, dw + dy
\]
\[
\leq C R J_0(K_{v,\alpha}).
\] (3.15)

(iii) Estimate of $I_3$: We first consider the case $\lambda \in [2d/\alpha - d, d]$. Then, since $|\nabla y,\rho_2,\phi(y)| \leq C$, we have
\[
I_3 \leq C \int_{|y(\tau) - \xi(\tau)| \leq 2R} |u(\tau, y) - \int_{B_{\lambda}(\xi(\tau))} u(\tau)| P(\tau, y) \, dy
\]
\[
\leq C \int_{|y(\tau) - \xi(\tau)| \leq 2R} |u(\tau)| P(\tau, y) \, dy + C \int_{B_{2R}(\xi(\tau))} |u(\tau)| - \int_{B_{\lambda}(\xi(\tau))} u(\tau)|
\]
\[
\leq C ||u(\tau)||_{L^\infty} (1 + 2 \log R)
\] (3.16)

In the second last line in (3.16), the term $\log R$ appears only when $\lambda = d$. Next we consider the case $\lambda \in (d, 2d/\alpha + d]$. In this case we have $L^{2,\beta,\lambda}(\Omega) = C_{\beta,\lambda}(\Omega)$ with equivalent seminorms, where $\beta = \alpha \lambda / (2d) - \alpha / 2 \in (0, 1)$. Hence it follows that
\[
I_3 \leq C \int_{\Omega} |u(\tau, y) - \int_{B_{1}(\xi(\tau))} u(\tau)\|\nabla \rho_2,\phi(y - \xi(\tau))| \chi_{\lambda}(y - \xi(\tau)) P(\tau, y) \, dy
\]
\[
\leq C \int_{\Omega} |u(\tau)| \chi_{\lambda}(y - \xi(\tau)) P(\tau, y) \, dy + C ||u(\tau)||_{C^{\beta,\lambda}}
\]
\[
\leq C ||u(\tau)||_{C^{\beta,\lambda}} \int_{\Omega} |y - \xi(\tau)| \chi_{\lambda}(y - \xi(\tau)) P(\tau, y) \, dy
\] (3.17)

For the case $\lambda \in [2d/\alpha - d, d]$, (3.14) - (3.17) yield
\[
\frac{d}{d\tau} (\tau^\frac{1}{2} m_2,\phi(\tau)) \leq C \tau^\frac{1}{2} J_2(K_M) \left( \frac{\xi(\tau)}{K_M} (P, \log P(\tau)) \right)^{\frac{1}{2}} + C \tau^\frac{1}{2} R J_0(K_{v,\alpha})
\]
\[
+ C ||u(\tau)||_{L^\infty} R^{\frac{1}{2}} + \frac{1}{2} \tau^\frac{-1}{2} m_2,\phi(\tau). 
\] (3.18)
Regarding the case \( \lambda \in (d, 2d/\alpha + d) \), one gets
\[
\frac{d}{d\tau} (e^{-C \int_0^\tau \|u(s)\|_{C, \alpha} ds} m_{2,R}(\tau)) \leq C J_2(K_R)^{\frac{2}{3}} (E_K^{(r)}(P(\tau), \log P(\tau))^\frac{2}{3} + CR J_0(K_{\nu, R})
+ C\|u(\tau)\|_{C, \alpha} (1 + R^{\alpha_3} \int_{R \leq |y - \xi(\tau)| \leq 2R} P(\tau, y) dy). \tag{3.19}
\]

By the definition of \( h(\tau) \) we have
\[
E_K^{(r)}(P(\tau), \log P(\tau))^\frac{1}{2} \leq (\frac{d}{\alpha} + \frac{dh(\tau)}{d\tau})^\frac{1}{2} \leq \tau^{-\frac{1}{2}} (C + \tau \frac{dh(\tau)}{d\tau}).
\]
This with (3.10) and (3.11) gives
\[
\int_0^\tau E_K^{(r)}(P(s), \log P(s))^\frac{1}{2} ds \leq \int_0^\tau s^{-\frac{1}{2}} (C + s \frac{dh(s)}{ds}) ds
\leq C\tau^\frac{1}{2} + \frac{1}{2} \int_0^\tau s^{-\frac{1}{2}} h(s) ds
= C\tau^\frac{1}{2} + \frac{1}{2} \int_0^\tau s^{-\frac{1}{2}} (f(\tau) - f(s)) ds \leq \tau^\frac{1}{2} (C + f(\tau)). \tag{3.20}
\]
Then for \( \lambda \in [2d/\alpha - d, d] \), (3.18) and (3.20) yield
\[
\tau^\frac{1}{2} m_{2,R}(\tau) \leq C\tau^\frac{1}{2} J_2^\frac{1}{2} (K_M)(1 + f(\tau)) + CR^\frac{\alpha_3}{2} \int_0^\tau \|u(s)\|_{L^\infty, \lambda} ds
+ \frac{1}{2} \int_0^\tau s^{-\frac{1}{2}} m_{2,R}(s) ds.
\]
Thus from the Gronwall-type inequality we obtain
\[
\tau^\frac{1}{2} m_{2,R}(\tau) \leq C\tau^\frac{1}{2} J_2^\frac{1}{2} (K_M)(1 + f(\tau)) + CR^\frac{\alpha_3}{2} \int_0^\tau \|u(s)\|_{L^\infty, \lambda} ds
+ C\tau^\frac{1}{2} \int_0^\tau s^{-\frac{1}{2}} (1 + f(s)) ds + C\tau^\frac{1}{2} R J_0(K_{\nu, M})
+ CR^\frac{\alpha_3}{2} \tau^\frac{1}{2} + \frac{1}{2} \int_0^\tau \|u\|_{X_\lambda(0, 1)} ds. \tag{3.21}
\]
For \( \lambda \in (d, 2d/\alpha + d) \) (3.19) and (3.20) give
\[
m_{2,R}(\tau) \leq C e^{C J_2^\frac{1}{2} \|u(\tau)\|_{C, \alpha} ds} \left( J_2(K_M)^{\frac{2}{3}} \tau^\frac{1}{2} (1 + f(\tau)) + R J_0(K_{\nu, M}) \tau
+ \int_0^\tau u(s) \|u\|_{C, \alpha} (1 + R^{\alpha_3} \int_{R \leq |y - \xi(s)| \leq 2R} P(s, y) dy) ds. \right) \tag{3.22}
\]
We choose \( \gamma \in (0, 2) \) so that
\[
\sum_{k=0}^\infty \left( k^{\alpha_2} \gamma^{-\alpha_2} J_2(K_{\nu, 2k}) + 2^k J_0(K_{\nu, 2k}) + 2^{k\frac{1}{2}-1 + \frac{\alpha_3}{2}} \right) < \infty, \quad \text{if} \quad \lambda \in [2d/\alpha - d, d], \tag{3.23}
\]
and
\[
\sum_{k=0}^\infty \left( k^{\alpha_2} \gamma^{-\alpha_2} J_2(K_{\nu, 2k}) + 2^k J_0(K_{\nu, 2k}) \right) < \infty, \quad \text{if} \quad \lambda \in (d, 2d/\alpha + d). \tag{3.24}
\]
Note that \( \gamma \in (0, \min\{\alpha, 2 - \alpha\}) \) always satisfies (3.23) and (3.24). Then for \( \lambda \in [2d/\alpha - d, d], \) (3.7) and
Lemma 3.3 imply
\[
m_{\gamma,\infty}(\tau) = \int_{|y - \xi(\tau)| \leq 1} \rho_s(y - \xi(\tau))P(\tau, y) \, dy + \sum_{k=1}^{\infty} \int_{2^{k-1}|y - \xi(\tau)| \leq 2^k} \rho_s(y - \xi(\tau))P(\tau, y) \, dy \\
\leq \int_{|y - \xi(\tau)| \leq 1} \rho_s(y - \xi(\tau))P(\tau, y) \, dy + \sum_{k=1}^{2^{(k-1)}2^{1-1}m_{\gamma,\infty}(\tau)} \left(1 + C \left(\tau \cdot f(\tau) + \int_0^\tau s^{-\frac{1}{\alpha}}(1 + f(s)) \, ds\right) \sum_{k=0}^{\infty} 2^k(\frac{2}{\gamma} - 1)J^2_2(K_{2k}) + C_T \sum_{k=0}^{\infty} 2^k \lambda J_0(K_{\gamma,\infty}) + C_T \left(\tau \cdot f(\tau) + \int_0^\tau s^{-\frac{1}{\alpha}} \log \, dy \right) \sum_{k=1}^{\infty} 2^k(\frac{2}{\gamma} - 1)J^2_2(K_{2k}) \right)
\]
Similarly, for \( \lambda \in (d, 2d/\alpha + d) \), we have
\[
m_{\gamma,\infty}(\tau) \leq 1 + C e^{C |s|} \left(\sum_{k=1}^{\infty} 2^k(\frac{2}{\gamma} - 1)J^2_2(K_{2k}) + C_T \sum_{k=0}^{\infty} 2^k \lambda J_0(K_{\gamma,\infty}) \right) + \int_0^\tau \left(1 + d \log \tau + \frac{2d}{\gamma} \int_0^\tau s^{-\frac{1}{\alpha}} \log m_{\gamma,\infty}(s) \, ds\right) \sum_{k=1}^{\infty} 2^k(\frac{2}{\gamma} - 1)J^2_2(K_{2k}) \cdot \int_0^\tau \left(1 + m_{\gamma,\infty}(s) \right) \sum_{k=1}^{\infty} 2^k(\frac{2}{\gamma} - 1)J^2_2(K_{2k}) \right)
\]
Here the constants \( C \) in (3.25) and (3.26) depend only on \( d, \alpha, C_0, \gamma, \) and \( \lambda \). Thus (3.25) - (3.26) and the definition of \( \|K\|_\gamma \) in (2.12) imply
\[
m_{\gamma,\infty}(\tau) \leq C \left(1 + \int \rho_s(y - x - \xi_0)P_{K(\gamma,x)}(t, y; s, x) \, dy \right) \leq C_2(t - s)^{\frac{d}{\alpha}} \left(1 + \delta^{2-\alpha}(t - s)^{\frac{d}{\alpha}} N + \delta(t - s)^{\frac{d}{\alpha}} \right).
\]
where \( C \) depends only on \( d, \alpha, C_0, \) and \( \gamma, \) and \( C_2 \) depends only on \( d, \alpha, \gamma, \) and \( \|v\|_{X_\lambda(s, t)} \). Moreover, it holds that
\[
|\xi_0^\ast(x, t, s)| \leq \begin{cases} 
C(t - s)^{\frac{d}{\alpha}} \|v\|_{X_\lambda(s, t)} + F(t - s)^{\frac{d}{\alpha}} |v^\ast(t - s, x)| \quad \text{if } \lambda \in \left[\frac{2d}{\alpha} - d, d\right], \\
C(t - s)^{\frac{d}{\alpha}} \|v\|_{X_\lambda(s, t)} \log (c + \|v\|_{X_\lambda(s, t)}) + F(t - s)^{\frac{d}{\alpha}} |v^\ast(t - s, x)| \quad \text{if } \lambda = d, \\
C(t - s)^{\frac{d}{\alpha}} \|v\|_{X_\lambda(s, t)} \exp(C|v|_{X_\lambda(s, t)}) \quad \text{if } \lambda \in \left(d, \frac{2d}{\alpha} + d\right), \\
F(t - s)^{\frac{d}{\alpha}} |v^\ast(t - s, x)| \exp(C|v|_{X_\lambda(s, t)}) \quad \text{if } \lambda = \frac{2d}{\alpha} + d.
\end{cases}
\]
Here \( F_{\gamma}[u](t, x) \) is the function defined by (2.22), and \( C \) depends only on \( d, \alpha, \) and \( \lambda \).
Proof. If we put
\[ \tilde{K}(\tau, \tilde{y}, \tilde{x}) = (t-s)^{1+\frac{d}{2}} (R_{t-s} \ast_x K(N,\delta))(\tau, \tilde{y}, \tilde{x}), \quad \tilde{v}(\tau, \tilde{y}) = (t-s)^{1+\frac{\alpha}{2}} (R_{t-s} \ast_y v)(\tau, \tilde{y}), \quad (3.30) \]
then the scaling property implies
\[ P_{K(N,\delta),\nu}(t, y, s, x) = (t-s)^{-\frac{d}{2}} P_{\tilde{K},\nu}(1, (t-s)^{-\frac{1}{2}} y; 0, (t-s)^{-\frac{1}{2}} x) \quad (3.31) \]
By the definition of \( K(N,\delta) \) we have
\[ \tilde{K}(\tau, \tilde{y}, \tilde{x}) = |\tilde{y} - \tilde{x}|^{-d-\alpha} (R_{t-s} \ast_x \nu \ast k_N)(\tau, \tilde{y}, \tilde{x}) + \delta(t-s)^{\frac{1}{2}-\frac{\alpha}{2}} |\tilde{y} - \tilde{x}|^{-d-1-\frac{\alpha}{2}}. \quad (3.32) \]
Since \( (R_{t-s} \ast_x \nu \ast k_N)(\tau, \tilde{y}, \tilde{x}) \geq C_0^{-1} \), the function \( \tilde{K}(\tau, \tilde{y}, \tilde{x}) \), \( 0 \leq \tau \leq 1 \), is a smooth kernel of the order \( 1+\alpha/2 \) but with \( \delta(t-s)^{1/2-\alpha/2} \) instead of \( \delta \) in (2.27). Moreover, the equality \( \| (t-s)^{1/2-\alpha/2} (R_{t-s} \ast_y v)(\tau) \|_{L^{\infty,N}} = \| (t-s)^{1/2-\alpha/2} \|_{L^{\infty,N}} \) implies \( \| \tilde{v} \|_{X_\lambda(0,1)} = \| v \|_{X_\lambda(s,t)} \).
For any \( \tilde{x} \in \mathbb{R}^d \) let \( \tilde{\xi}_\lambda(\tau; \tilde{x}) \in \mathbb{R}^d \), \( \tau \in [0, 1] \), be the solution to the ODE (2.21) with \( u = \tilde{v} \). Then by Proposition 3.1 we have for \( \tau \in (0, 1] \),
\[ C_1 \leq \int_{\mathbb{R}^d} \rho_\alpha(\tilde{y} - \tilde{x} - \xi_\lambda(\tau; \tilde{x})) P_{\tilde{K},\tilde{v}}(\tau, \tilde{y}; 0, \tilde{x}) \, d\tilde{y} \leq C_2 (1 + \| \tilde{K} \|_\tau). \]
From (3.38) - (2.41) we see
\[ J_2(\tilde{K}_{2k}) \leq C(2^{k(2-\alpha)} + \delta(1-\alpha)(t-s)^{1-\frac{\alpha}{2}} N + \delta(t-s)^{\frac{1}{2} - \frac{\alpha}{2} 2^{k(1-\frac{\alpha}{2})}}), \]
\[ J_0(\tilde{K}_{2k}) \leq C(2^{-\alpha k} + 2^{-\alpha} \delta^2 \alpha (t-s)^{1-\frac{\alpha}{2}} N + \delta(t-s)^{\frac{1}{2} - \frac{\alpha}{2} 2^{-\alpha} \alpha (t-s)^{1/2}}). \]
This gives the estimate
\[ \| \tilde{K} \|_\gamma \leq C(1 + \delta^{2-\alpha}(t-s)^{1-\frac{\alpha}{2}} N + \delta(t-s)^{\frac{1}{2} - \frac{\alpha}{2}}), \]
Hence we have
\[ C_1 \leq \int_{\mathbb{R}^d} \rho_\alpha(\tilde{y} - \tilde{x} - \xi_\lambda(\tau; \tilde{x})) P_{\tilde{K},\tilde{v}}(\tau, \tilde{y}; 0, \tilde{x}) \, d\tilde{y} \leq C C_2 (1 + \delta^{2-\alpha}(t-s)^{1-\frac{\alpha}{2}} N + \delta(t-s)^{\frac{1}{2} - \frac{\alpha}{2}}). \]
This implies the desired estimate by taking \( \tilde{x} = (t-s)^{-1/\alpha} x \) and by setting \( \xi_\lambda(\tau; s, t) = (t-s)^{1/\alpha} \xi_\lambda(1; \tilde{x}) \).
By the definition of \( \| \|_{X_\lambda(s,t)} \) and the embedding \( L^{2d/\alpha,\lambda}(\mathbb{R}^d) \hookrightarrow L^{1,\mu}(\mathbb{R}^d) \) with \( \mu = \alpha \lambda/(2d) + d - \alpha/2 \), we have \( \| \tilde{v} \|_{L^{1,\mu}} \leq C \| v \|_{X_\lambda(s,t)} = C \| v \|_{X_\lambda(s,t)} \) for this \( \mu \). Furthermore, we have the equality
\[ F_1(\tilde{v})(1, \tilde{x}) = (t-s)^{-\frac{\alpha}{2}} \frac{1}{(t-s)^{1/2}} [v(\cdot + s)](t-s, x). \]
Hence (2.23) yields (3.29) by the definition \( \xi_\lambda(\tau; s, t) = (t-s)^{1/\alpha} \xi_\lambda(1; \tilde{x}) \). This completes the proof.

The adjoint relation \( P_{K(N,\delta),\nu}(t, y; s, x) = P_{K(N,\delta),\nu}(t-s, y; 0, x) \) (see (2.14) for the definition of \( j_t \)) and Proposition 3.4 yield the moment bound of the integral with respect to \( x \) as follows. Note that \( \| j_t v \|_{X_\lambda(0,1)} = \| v \|_{X_\lambda(s,t)} \) by the definitions of \( j_t \) and \( \| \|_{X_\lambda(s,t)} \).

Proposition 3.5 Assume that the conditions in Proposition 3.4 hold. Then for any \( y \in \mathbb{R}^d \) there is \( \eta_\nu^* = \eta_\nu^*(y, t, s) \in \mathbb{R}^d \) such that
\[ C_1(t-s)^{\frac{d}{2}} \leq \int_{\mathbb{R}^d} \rho_\alpha(x - y - \eta_\nu^*) P_{K(N,\delta),\nu}(t, y; s, x) \, dx \leq C_2(t-s)^{\frac{d}{2}} (1 + \delta^{2-\alpha}(t-s)^{1-\frac{\alpha}{2}} N + \delta(t-s)^{\frac{1}{2} - \frac{\alpha}{2}}). \quad (3.37) \]
Here \( C_1 \) depends only on \( d, \alpha, C_0, \) and \( \gamma \), and \( C_2 \) depends only on \( d, \alpha, C_0, \gamma, \lambda, \) and \( \| v \|_{X_\lambda(s,t)} \). Moreover, \( \eta_\nu^*(y, t, s) \) satisfies the same estimate as \( \xi_\lambda(\tau; s, t) \) in Proposition 3.4 with \( v \) replaced by \( j_t v \).

3.2 Relative entropy bound
In this section we establish the estimate on the relative entropy of fundamental solutions. We set
\[ E(x) = H(|x|) = C(1 + |x|)^{-d-\alpha}/\log(e + |x|), \quad \int_{\mathbb{R}^d} E(x) \, dx = 1. \quad (3.38) \]
Proposition 3.6 Let $\epsilon \in (0, 1]$ and $\gamma \in (0, \min\{\alpha, 2 - \alpha\})$. Let $K(\tau, y, x)$ be a smooth kernel of the order $\alpha' \in (1, 2)$. Assume that $u$ is a smooth and bounded vector field satisfying $\nabla \cdot u(\tau) = 0$ in $(0, 1) \times \mathbb{R}^d$. For $x \in \mathbb{R}^d$ let $u_0(\tau; x)$ and $m_\alpha(\tau; x)$ be as in Proposition 3.1. Then there are $C_1, C_2 > 0$, $l_x \in [0, 1)$ such that
\[
-C_1 \leq - \int_{\mathbb{R}^d} E(y - x - \xi_u(1; x)) \log \left( P_{K,u}(1, y; 0, x) + \epsilon \right) \, dy \\
\leq C_2 |\log \epsilon|^{l_x} \left( 1 + ||K||_+ + \frac{1}{H(1 + \sup_{\tau \in [\frac{1}{2}, 1]} m_\alpha(\tau; x))} \right).
\]
(3.39)

Here $C_1$ depends only on $d$, $\alpha$ and $C_0$, and $C_2$ depends only on $d$, $\alpha$, $C_0$, $\lambda$, $\gamma$, and $||u||_{X_\lambda(0,1)}$. The number $l_x$ depends only on $d$, $\alpha$, and $\lambda$, and especially, we can take $l_x = 0$ if $\lambda \in (d, 2d/\alpha + d]$.

Proof. As in the proof of Proposition 3.1, we may take $x = 0$. Then we consider
\[
g_\tau(\tau) = - \int_{\mathbb{R}^d} E(\eta - \tau, \xi_u(\tau; 0)) \log \left( \tau \frac{d}{d \tau} P_{K,u}(\tau, \tau; \eta, 0; 0) + \epsilon \right) \, d\eta, \quad \frac{1}{2} \leq \tau \leq 1.
\]
(3.40)

Since $\int_{\mathbb{R}^d} \tau \frac{d}{d \tau} P_{K,u}(\tau, \tau; \eta, 0; 0) \, d\eta = 1$ it is easy to see that $g_\tau(\tau)$ is bounded from below uniformly in $\tau$ and $\epsilon$ by the Jensen inequality. For simplicity of notations, we set
\[
P(\tau, \eta) = \tau \frac{d}{d \tau} P_{K,u}(\tau, \tau; \eta, 0; 0), \\
\xi(\tau) = \tau^{-\frac{1}{2}} \xi_u(\tau; 0), \\
E(\tau, \eta) = E(\eta - \xi(\tau)).
\]

Since
\[
\partial_\tau P(\tau, \eta) = \frac{d}{d \tau} P(\tau, \eta) + \frac{1}{\alpha} \eta \cdot \nabla_\eta P(\tau, \eta) + \tau \frac{d}{d \tau} (\partial_\tau P_{K,u})(\tau, \tau; \eta, 0; 0),
\]
we have
\[
\tau \frac{d}{d \tau} g_\tau(\tau) = - \frac{d}{\alpha} \int_{\mathbb{R}^d} E(\tau, \eta) P(\tau, \eta) (\tau \frac{d}{d \tau} P(\tau, \eta) + \epsilon) \, d\eta + \frac{1}{\alpha} \int_{\mathbb{R}^d} (\eta - \xi(\tau)) \cdot (\nabla E)(\eta - \xi(\tau)) \log(P(\tau, \eta) + \epsilon) \, d\eta \\
+ \frac{\tau^{1+\frac{\alpha}{2}}}{2} \int_{\mathbb{R}^d} [P(\tau, \cdot) \frac{E(\tau, \cdot)}{P(\tau, \cdot) + \epsilon}](\eta, z) (R_u K)(1, \eta, z) \, d\eta \, dz \\
- \tau^{1+\frac{\alpha}{2}} \int_{\mathbb{R}^d} (u(\tau, \tau; \eta, 0) - \frac{1}{d \tau} \xi_u(\tau; 0, \nabla_\eta) E(\tau, \eta) \log(P(\tau, \eta) + \epsilon) \, d\eta \\
\leq \frac{d}{\alpha} g_\tau(\tau) + \frac{1}{\alpha} I_1 + \frac{\tau^{1+\frac{\alpha}{2}}}{2} I_2 + \tau^{1+\frac{\alpha}{2}} I_3.
\]

(i) Estimate of $I_1$: From the definition of $E$ and $\log(P(\tau, \eta) + \epsilon) \leq C$ where $C$ depends only on $d$, $\alpha$, and $C_0$, it is not difficult to see
\[
I_1 \leq C g_\tau(\tau) + C, \quad \text{(3.41)}
\]
where $C$ depends only on $d$, $\alpha$, and $C_0$.

(ii) Estimate of $I_2$: Although $I_2$ is estimated in the same manner as in [24], we give the proof for convenience to the reader. Set
\[
2\zeta(\eta, z) = [\log E(\tau, \cdot)](\eta, z), \\
2\omega(\eta, z) = [\log(P(\tau, \cdot) + \epsilon)](\eta, z).
\]

We use the inequality
\[
\sinh(\theta - \omega) \sinh \omega \leq (\theta - \omega) \omega \frac{\sinh \theta}{\theta},
\]
(3.42)
which was proved in [24, Lemma 3.1]). Then we have
\[
[P(\tau, \cdot)](\eta, z) \frac{E(\tau, \cdot)}{P(\tau, \cdot) + \epsilon}(\eta, z) = 2(E(\tau, \eta) + E(\tau, z)) \sinh(\zeta - \omega) \sinh(\omega) / \cosh(\zeta) \\
\leq (E(\tau, \eta) + E(\tau, z)) \frac{\tanh(\zeta(\eta, z))}{\zeta(\eta, z)} \frac{\zeta^2(\eta, z) - \omega^2(\eta, z))}.
\]

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Hence we get

\[ I_2 \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} (E(\tau, \eta) + E(\tau, z) \tan \frac{\zeta(\eta, z)}{\zeta(\eta, z)} (\zeta^2(\eta, z) - \omega^2(\eta, z))(R_\tau K)(1, \eta, z) \, d\eta \, dz. \]

Let \( \tau \in [1/2, 1] \). The estimate \( |\zeta(\eta, z)| \leq C \log(1 + |\eta - z|) \leq C|\eta - z| \) yields

\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} (E(\tau, \eta) + E(\tau, z)) \tan \frac{\zeta(\eta, z)}{\zeta(\eta, z)} \zeta^2(\eta, z) (R_\tau K)(1, \eta, z) \, d\eta \, dz \]
\[ \leq C \inf_{M \in (0, \infty)} (J_2[(R_\tau K)_M] + J_1[(R_\tau K)_M]) \]
\[ \leq C \inf_{M \in (0, \infty)} (J_2[K_M] + J_1[K_M]) \leq C\|K\|_\gamma < \infty. \]

By using the inequalities

\[ \frac{\tan \frac{\zeta(\eta, z)}{\zeta(\eta, z)}}{\zeta(\eta, z)} \geq C(\log(e + |\eta - z|))^{-1}, \quad (R_\tau K)(1, \eta, z) \geq C_0 \tau^{-d} |\eta - z|^{-d - \alpha}, \]

we have

\[ I_2 \leq C + C\|K\|_\gamma \]
\[ - \int_{\mathbb{R}^d \times \mathbb{R}^d} (E(\tau, \eta) + E(\tau, z)) \omega^2(\eta, z)(\log(e + |\eta - z|))^{-1}(R_\tau K)(1, \eta, z) \, d\eta \, dz \]
\[ \leq C + C\|K\|_\gamma - C\int_{\mathbb{R}^d} E(\tau, z) \omega^2(\eta, z)E(\tau, \eta) \, d\eta \, dz \]
\[ \leq C + C\|K\|_\gamma - C\int_{\mathbb{R}^d} E(\tau, \eta) \left( \int_{\mathbb{R}^d} 2\omega(\eta, z)E(\tau, z) \, dz \right)^2 \, d\eta \]
\[ = C + C\|K\|_\gamma - C\int_{\mathbb{R}^d} E(\tau, \eta) \left( \log(P(\tau, \eta) + \epsilon) + g_\epsilon(\tau) \right)^2 \, d\eta, \quad (3.43) \]

for \( \tau \in [1/2, 1] \), where \( C > 0 \) depending only on \( d, \alpha \), and \( C_0 \). Let \( l_\epsilon(a, b), a > 0, b \in \mathbb{R} \) be the function defined by

\[ l_\epsilon(a, b) = \frac{1}{a} (\log(a + \epsilon) + b)^2. \quad (3.44) \]

Fix \( \kappa_1 > 0 \) which is to be determined later. Since \( l_\epsilon(., b) \) is decreasing on \([\kappa_1^2 - b, \infty)\), if \( g_\epsilon(\tau) \geq 3 - \log \kappa_1 \) then we have

\[ \int_{\mathbb{R}^d} E(\tau, \eta) \left( \log(P(\tau, \eta) + \epsilon) + g_\epsilon(\tau) \right)^2 \, d\eta \]
\[ = \int_{\mathbb{R}^d} E(\tau, \eta)P(\tau, \eta)l_\epsilon(P(\tau, \eta), g_\epsilon(\tau)) \, d\eta \]
\[ \geq l_\epsilon(\max \frac{\eta}{\eta}, \rho_\epsilon(\tau)) \int_{\mathbb{R}^d \times \mathbb{R}^d} E(\eta - \xi(\tau))P(\tau, \eta) \, d\eta \]
\[ \geq H(R)l_\epsilon(\max \frac{\eta}{\eta}, g_\epsilon(\tau)) \int_{\mathbb{R}^d \times \mathbb{R}^d} P(\tau, \eta) \, d\eta \]
\[ \geq H(R)l_\epsilon(\max \frac{\eta}{\eta}, g_\epsilon(\tau)) \left( \int_{\mathbb{R}^d} P(\tau, \eta) \, d\eta - \kappa_1 \int_{|\eta - \xi(\tau)| \leq R} P(\tau, \eta) \, d\eta \right). \]

Thus from Proposition 3.1 with a fixed \( \gamma \in (0, \min\{\alpha, 2 - \alpha\}) \) we get

\[ \int_{|\eta - \xi(\tau)| \leq R} P(\tau, \eta) \, d\eta = 1 - \int_{|\eta - \xi(\tau)| \geq R} P(\tau, \eta) \, d\eta \]
\[ \geq 1 - \rho_\epsilon(R)^{-1} \int_{|\eta - \xi(\tau)| \geq R} \rho_\epsilon(\eta - \xi(\tau))P(\tau, \eta) \, d\eta \]
\[ \geq 1 - \rho_\epsilon(R)^{-1} \int_{\mathbb{R}^d} \rho_\epsilon(\frac{\eta - \xi(\tau)}{\tau})P_{K, \alpha}(\tau, \eta; 0, 0) \, d\eta \]
\[ = 1 - \rho_\epsilon(R)^{-1} \tau^{-\alpha} m_\epsilon(\tau; 0) \geq \frac{3}{4}, \]

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if $\tau \in [1/2, 1]$ and $R \geq 1$ is sufficiently large so that
\begin{equation}
2^{2 + \frac{d}{\alpha}} \sup_{\tau \in [1/2, 1]} m_\gamma(\tau; 0) \leq R^2. \tag{3.45}
\end{equation}

By setting $4\kappa_1 = |B_R(0)|^{-1}$ and by using the bound
\[ P(\tau, \eta) = \tau \log \tau \eta; 0, 0 \leq C', \]
where $C'$ depends only on $d, \alpha,$ and $C_0$, we finally get
\[ \int_{\mathbb{R}^d} E(\tau, \eta)(\log(P(\tau, \eta) + \epsilon) + g_\epsilon(\tau))^2 \, d\eta \geq \frac{1}{2} H(R) I_1(C', g_\epsilon(\tau)) = CH(R)(\log(C' + \epsilon) + g_\epsilon(\tau))^2 \geq H(R)(\epsilon g_\epsilon^2(\tau) - C), \]
with $c$ depending only on $d, \alpha,$ and $C_0$, which holds for $\tau \in [1/2, 1]$ as long as
\begin{equation}
g_\epsilon(\tau) \geq 3 - \log \kappa_1 = C + d \log R. \tag{3.46}
\end{equation}

Here the constant $C$ in (3.46) depends only on $d$ and $\gamma$. Collecting these, under the conditions of (3.45) and (3.46) we get
\[ I_2 \leq C + C\|K\|_{\gamma} - \kappa_2 \int_{\mathbb{R}^d} E(\tau, \eta)(\log(P(\tau, \eta) + \epsilon) + g_\epsilon(\tau))^2 \, d\eta - \epsilon(1 - \kappa_2) \log(R) \epsilon^2. \tag{3.47}
\]
for every $\kappa_2 \in [0, 1)$ and $\tau \in [1/2, 1]$, where the above constants $C$ and $c$ depend only on $d, \alpha, \gamma,$ and $C_0$.

(iii) Estimate of $I_3$: Let $\lambda \in [2d/\alpha - d, d]$. As in the proof of Proposition 3.4, we set
\[ D_0 = \{ z \in \mathbb{R}^d \mid |z - \xi_\alpha(\tau; 0)| < 1 \}, \]
\[ D_k = \{ z \in \mathbb{R}^d \mid 2^k - 1 \leq |z - \xi_\alpha(\tau; 0)| < 2^k \}, \quad k \in \mathbb{N}. \]
Since $|\nabla_\eta E(\eta)| \leq C(1 + |\eta|)^{-1}E(\eta) \leq C(1 + |\eta|)^{-(d+\alpha+2)/2}E(\eta)^{1/2}$ we have
\[ |I_3| \leq \left( \int_{\mathbb{R}^d} |u(\tau, \tau^{\frac{1}{\alpha}} \eta) - \int_{B_1(\xi_\alpha(\tau))} u(\tau)| \right. \left. \tau^{\frac{2d}{\alpha}} (1 + |\eta - \xi_\alpha(\tau)|)^{-\frac{2d}{\alpha}} E(\tau, \eta) \right) \frac{d\tau}{\eta} \]
\[ \times \left( \int_{\mathbb{R}^d} E(\tau, \eta) \left| \log (P(\tau, \eta) + \epsilon) \right| \tau^{\frac{2d}{\alpha}} \right) \frac{d\eta}{\tau} \leq C \tau^{-\frac{d}{\alpha}} \left( \sum_{k=0}^{\infty} \int_{D_k} |u(\tau, z) - \int_{B_1(\xi_\alpha(\tau))} u(\tau)| \right. \left. \tau^{\frac{2d}{\alpha}} (1 + \frac{|z - \xi_\alpha(\tau; 0)|}{\tau})^{-\frac{2d}{\alpha}} E(\frac{z - \xi_\alpha(\tau; 0)}{\tau}) \right) \frac{dz}{\tau} \]
\[ \times \left( \int_{\mathbb{R}^d} E(\tau, \eta) \left| \log (P(\tau, \eta) + \epsilon) \right| \tau^{\frac{2d}{\alpha}} \right) \frac{d\eta}{\tau}. \]

From (2.20) we get
\[ \int_{D_k} |u(\tau, z) - \int_{B_1(\xi_\alpha(\tau))} u(\tau)| \tau^{\frac{2d}{\alpha}} (1 + \frac{|z - \xi_\alpha(\tau; 0)|}{\tau})^{-\frac{2d}{\alpha}} E(\frac{z - \xi_\alpha(\tau; 0)}{\tau}) \right) \frac{dz}{\tau} \]
\[ \leq C 2^{-\frac{d}{2} (\frac{d}{\alpha} + d/2)} \int_{B_1(\xi_\alpha(\tau))} |u(\tau| \tau^{\frac{2d}{\alpha}} \right) \frac{dz}{\tau} \]
\[ + C 2^{-\frac{d}{2} (\frac{d}{\alpha} + d/2)} \int_{B_1(\xi_\alpha(\tau))} |u(\tau)| \tau^{\frac{2d}{\alpha}} \frac{dz}{\tau} \]
\[ \leq C \left( 2^{\lambda - \frac{d}{2} (\frac{d}{\alpha} + d/2)} + 2^{\frac{d}{2} k} \right) \|u(\tau)\|_{L^{\frac{2d}{\alpha}}}. \]
for $\tau \in [1/2, 1]$. Then from the interpolation inequality and the estimate
\[ \int_{\mathbb{R}^d} E(\tau, \eta) \left| \log (P(\tau, \eta) + \epsilon) \right| \frac{d\eta}{\tau} \leq C \log \tau + g_\epsilon(\tau) \leq C + g_\epsilon(\tau) \quad \text{for } \tau \in [1/2, 1], \]
we have
\[ |I_3| \leq C\|u(\tau)\|_{L^{2d/\alpha,\lambda}} \left( \int_{\mathbb{R}^d} E(\tau, \eta) \left| \log(P(\tau, \eta) + \epsilon) \right|^{\frac{2d}{2d-\alpha}} d\eta \right)^{\frac{2d-\alpha}{2d}} \]
\[ \leq C\|u(\tau)\|_{L^{2d/\alpha,\lambda}} (C + g_\lambda(\tau)) \left( \int_{\mathbb{R}^d} E(\tau, \eta) \left| \log(P(\tau, \eta) + \epsilon) \right|^2 d\eta \right)^{\frac{1}{2}} \]  
(3.48)
if \( \tau \in [1/2, 1] \). In particular, from \( |\log(P(\tau, \eta) + \epsilon)| \leq C|\log \epsilon| \) we have
\[ |I_3| \leq C|\log \epsilon|\|u(\tau)\|_{L^{2d/\alpha,\lambda}}. \]
(3.49)

Next we consider the case \( \lambda \in (d, 2d/\alpha + d) \). In this case we have \( L^{2d/\alpha,\lambda}(\mathbb{R}^d) = \dot{C}^{\beta_\lambda}(\mathbb{R}^d) \) with \( \beta_\lambda = \alpha(\lambda - d)/(2d) \), and hence,
\[ I_3 \leq C \int_{\mathbb{R}^d} |u(\tau, \tau^{\beta_\lambda})| - \int \frac{B_1(\xi, \tau)}{\xi_{\alpha, \tau}} u(\tau)(1 + \lambda - \xi(\tau))^{-1} E(\tau, \eta) \left| \log(P(\tau, \eta) + \epsilon) \right| d\eta \]
\[ = C\tau^{-\beta_\lambda} \int_{\mathbb{R}^d} |u(\tau, \tau^{\beta_\lambda})| - \int B_1(\xi, \tau) u(\tau)(1 + \frac{\lambda - \xi(\tau)}{\tau^{\beta_\lambda}})^{-1} E(\tau, \eta) \left| \log(P(\tau, \eta) + \epsilon) \right| d\eta \]
\[ \leq C\tau^{-\beta_\lambda} |u(\tau)|_{L^{2d/\alpha,\lambda}} \int_{\mathbb{R}^d} (1 + \frac{\lambda - \xi(\tau)}{\tau^{\beta_\lambda}})^{-1+\beta_\lambda} \left| \log(P(\tau, \eta) + \epsilon) \right| d\eta \]
\[ \leq C\|u(\tau)\|_{L^{2d/\alpha,\lambda}} \lambda \int_{\mathbb{R}^d} E(\tau, \eta) \left| \log(P(\tau, \eta) + \epsilon) \right| d\eta \]
\[ \leq C\|u(\tau)\|_{L^{2d/\alpha,\lambda}} \int_{\mathbb{R}^d} E(\tau, \eta) \left| \log(P(\tau, \eta) + \epsilon) \right| d\eta \]
(3.50)

From (3.41), (3.47), and (3.49), we observe for \( \lambda \in [2d/\alpha - d, d] \) that
\[ \frac{d}{d\tau} g_\lambda(\tau) \leq C + C\|K\|_{\gamma} + C\|u(\tau)\|_{L^{2d/\alpha,\lambda}} \left( \frac{1}{\log(H(R)g_\lambda(\tau))} \right) \]
if \( \tau \in [1/2, 1] \), (3.45), and (3.46). As for the case \( \lambda \in (d, 2d/\alpha + d) \), from (3.41), (3.47), and (3.50) we have
\[ \frac{d}{d\tau} g_\lambda(\tau) \leq C + C\|K\|_{\gamma} + C\|u(\tau)\|_{L^{2d/\alpha,\lambda}} (1 + g_\lambda(\tau)) \left( \frac{1}{\log(H(R)g_\lambda(\tau))} \right) \]
(3.51)
(3.52)
Here the constants \( C \) and \( c \) in (3.51) - (3.52) depend only on \( d, \alpha, C_0, \lambda, \) and \( \gamma \). These differential inequalities imply the bound of \( g_\lambda(1) \). We give a proof only for the case \( \lambda \in [2d/\alpha - d, d] \), since the other cases are proved in the same manner. Est. (3.51) implies that
\[ L_\lambda(\tau; \tau_0) = g_\lambda(\tau) - C\int_{\tau_0}^{\tau} \left( 1 + \|K\|_{\gamma} + \left| \log \epsilon \right| \|u(s)\|_{L^{2d/\alpha,\lambda}} \right) ds \int_{\mathbb{R}^d} \left| \frac{1}{\log(H(R)g_\lambda(\tau))} \right| \]
is non-increasing for \( \tau \in [\tau_0, 1] \), where \( \tau_0 \in [1/2, 1] \) is fixed. Set
\[ M = \left( C'(1 + \|K\|_{\gamma} + \left| \log \epsilon \right| \|u\|_{X_\lambda(0,1)}) \right) \frac{2}{cH(R)} \], \( l = \frac{3ad + 2d - \alpha \lambda}{2(3ad + 2d - \alpha \lambda)} \in \left[ \frac{1}{2}, 1 \right) \),
(3.53)
where \( C' \) is a large constant depending only on \( d, \alpha, C_0, \lambda, \) and \( \gamma \). Let us take \( \tau_0 = 1 - \left| \log \epsilon \right|^{-2(\alpha-1)} \). Recalling the general bound \( \sup_{0 < \tau \leq 1} |g_\lambda(\tau)| \leq C|\log \epsilon| \), we claim that there is \( \tau_1 \in [\tau_0, 1] \) such that
\[ g_\lambda(\tau_1) \leq M|\log \epsilon|^l \]
(3.54)
Indeed, otherwise it follows from \( L_\lambda(1; \tau_0) \leq L_\lambda(\tau_1; \tau_0) = g_\lambda(\tau_0) \) that
\[ g_\lambda(1) \leq C\left( |\log \epsilon| + (1 - \tau_0)\|K\|_{\gamma} + \left| \log \epsilon \right| \|1 - \tau_0\|^{\frac{3}{2} - \frac{d}{2} + \frac{\alpha \lambda}{2d}} \|u\|_{X_\lambda(0,1)} \right) - cH(R)M(1 - \tau_0)\left| \log \epsilon \right|^{2l} \]
\[ \leq (C - C') \left( 1 + \|K\|_{\gamma} + \left| \log \epsilon \right| \|u\|_{X_\lambda(0,1)} \right) \left| \log \epsilon \right| \leq 0, \]
if \( C' \) is taken so that \( C' \geq C \), which is a contradiction. Hence there is \( \tau_1 \in [\tau_0, 1] \) such that (3.54) holds, and then \( L_\lambda(1; \tau_1) \leq L_\lambda(\tau_1; \tau_1) = g_\lambda(\tau_1) \) obtain
\[ g_\lambda(1) \leq M|\log \epsilon|^l + C\left( |\log \epsilon|^{-2(\alpha-1)}\|K\|_{\gamma} + \left| \log \epsilon \right|^{-2(\alpha-1)}\left( 1 + \lambda - \frac{d}{2} + \frac{\alpha \lambda}{2d} \right) \|u\|_{X_\lambda(0,1)} \right) \]
\[ \leq (M + \|u\|_{X_\lambda(0,1)} + \|K\|_{\gamma})|\log \epsilon|^l \]
(3.55)
This proves the case \( \lambda \in [2d/\alpha - d, d] \). For the case \( \lambda \in (d, 2d/\alpha + d] \) we have the bound of \( g_\lambda(1) \) uniformly in \( \epsilon \in (0, 1] \), thanks to (3.52). The details are omitted. The proof of Proposition 3.6 is now complete.
Remark 3.7 By using (3.41), (3.47), and (3.48), it is possible to take \( l_\lambda = 0 \) in Proposition 3.6 even when \( \lambda \in [2d/\alpha - d, d] \), if \( u \) belongs to \( L^{p(1-\frac{1}{\alpha})}((x, \frac{1}{2})^d \cap \{0, \infty; (L^{2d/p}(R^d))^{d'}\}) \) with \( p = 2(2d - \alpha)/(4d - 3\alpha) \), which is slightly stronger than (C1).

As in Proposition 3.4, we will derive the relative entropy bound of \( P_{K(N,\lambda),\alpha}(t; x, s; y) \) by using the scaling property.

Proposition 3.8 Let \( K(t, y, x) \) be a kernel satisfying (1.5) and (1.6), and let \( K(N,\lambda)(t, y, x) \) be the approximation of \( K \) given by Section 2.3. Fix \( t > s \geq 0 \). Let \( v(t, y) \) be a smooth and bounded vector field satisfying \( \nabla \cdot v(\tau) = 0 \) in \((s, t) \times R^d \). Assume that \( N \gg 1, 0 < \delta \leq N^{-1/(2-\alpha)} \) and

\[
 t - s \geq \mu_{N,\delta} := \delta^a N^{-\frac{1}{2\alpha}}. \tag{3.56}
\]

Fix any \( x, y \in R^d \) and let \( \xi_\alpha^*(x,t,s), \eta_\alpha^* = \eta_\alpha^*(y,t,s) \in R^d \) be the vectors in Propositions 3.4, 3.5. Let \( l_\lambda \in (0,1) \) be the number in Proposition 3.6. Then

\[
 C_1 \leq - \int_{R^d} E(\bar{y} - \bar{x} - \xi_\alpha(t; \bar{x})) \log (P_{K,\alpha} \xi_\alpha(t; \bar{y}, 0, \bar{x}) + \epsilon) \, d\bar{y} \leq C_2 \log \|v\|_{X(\lambda,0,1)}, \tag{3.57}
\]

\[
 C_1 \leq - \int_{R^d} E(\bar{y} - \bar{x} - \xi_\alpha(t; \bar{x})) \log (P_{K,\alpha} \xi_\alpha(t; \bar{y}, 0, \bar{x}) + \epsilon) \, d\bar{y} \leq C_2 \log \|v\|_{X(\lambda,0,1)}. \tag{3.58}
\]

Here \( C_1 \) depends only on \( d, \alpha, \) and \( C_0 \), and \( C_2 \) depends only on \( d, \alpha, C_0, \lambda, \gamma, \) and \( \|v\|_{X(\lambda,0,1)} \).

Proof. It suffices to show (3.57). The proof is similar to Proposition 3.4. Noticing the scaling relation (3.31) and (3.30), we apply Proposition 3.6 to \( P_{K,\alpha}(t, y; 0, \bar{z}) \). Then we have

\[
 C_1 \leq - \int_{R^d} E(\bar{y} - \bar{x} - \xi_\alpha(t; \bar{x})) \log (P_{K,\alpha} \xi_\alpha(t; \bar{y}, 0, \bar{x}) + \epsilon) \, d\bar{y} \leq C_2 \log \|v\|_{X(\lambda,0,1)}(1 + \|v\|_{X(\lambda,0,1)} + \|K\|_\gamma + \frac{1}{H(1 + \sup_{\varepsilon \in (1/2,1)] m_\alpha(\varepsilon))} \}, \tag{3.59}
\]

where \( C_1 \) depends only on \( d, \alpha, \) and \( C_0 \), and \( C_2 \) depends only on \( d, \alpha, C_0, \lambda, \gamma, \) and \( \|v\|_{X(\lambda,0,1)} \).

By (3.36) and \( 2\alpha/(2-\alpha) \leq C \), if \( \varepsilon \in (0,1) \) and \( N \geq 1 \) satisfy \( \delta^\alpha N^{\alpha/(2-\alpha)} \leq t - s \), then \( \sup_{\varepsilon \in (1/2,1)] m_\alpha(\varepsilon) \leq C, \) where \( C \) depends only on \( d, \alpha, C_0, \lambda, \gamma, \) and \( \|v\|_{X(\lambda,0,1)} \). Moreover, (3.35) implies \( \|K\|_\gamma \leq C \), where \( C \) depends only on \( d, \alpha, C_0, \gamma. \) Finally, we observe the relation \( \|v\|_{X(\lambda,0,1)} = \|v\|_{X(\lambda,0,1)} \). This completes the proof.

We have the following estimates for \( \xi_\alpha^*(x,t,s) \) and \( \eta_\alpha^*(x,t,s) \), which will be used later.

Proposition 3.9 Let \( T \geq t > s \geq 0 \) and \( x, x_1, x_2 \in R^d \). Let \( \xi_\alpha^*(x,t,s), \eta_\alpha^*(x,t,s) \) be the vectors in Proposition 3.4, 3.5. Then

\[
 |\xi_\alpha^*(x,t,s)| + |\eta_\alpha^*(x,t,s)| \leq \begin{cases} 
 C_1 |T|^{-\frac{1}{\alpha}} + (1 - \frac{1}{\alpha}) |t - s|^{-\frac{1}{\alpha}} \|v\|_{Y^{1,\lambda}} & \text{if } \lambda \in [\frac{2d}{\alpha} - d, d), \\
 C_1 (t - s)^{-\frac{1}{\alpha}} + C_2 (t - s)^{-\frac{1}{\alpha}} \|v\|_{Y^{1,\lambda}} & \text{if } \lambda = d,
 \end{cases}
\]

and

\[
 |\xi_\alpha^*(x_1,t,s) - \xi_\alpha^*(x_2,t,s)| + |\eta_\alpha^*(x_1,t,s) - \eta_\alpha^*(x_2,t,s)| \leq \begin{cases} 
 C_3 \|v\|_{X(\lambda,0,1)}(t - s)^{-\frac{1}{\alpha}} & \text{if } \lambda \in [\frac{2d}{\alpha} - d, d), \\
 C_3 \|v\|_{X(\lambda,0,1)}(t - s)^{-\frac{1}{\alpha}} \log (e + \frac{|x_1 - x_2|}{(t - s)^{\frac{1}{\alpha}}}) & \text{if } \lambda = d,
 \end{cases}
\]

Here \( C_1 \) and \( C_3 \) depend only on \( d, \alpha, \lambda, \) and \( \|v\|_{X(\lambda,0,1)} \), and \( C_2 \) depends only on \( d \) and \( q \).
Proof. In order to estimate \( \xi_0^*(x,t,s) \), by Proposition 3.4 it suffices to give the estimate for \( F_{(t-s)^{1/\alpha}}[v^{(t+s)}(t-s)] \). Let \( \lambda \in (2d/\alpha - d,d) \). For every \( r \in (0,t-s) \) we have
\[
\left| \int_0^r \int_{B_r(t-s)^{1/\alpha}} v(\tau + s) d\tau \right| \leq \int_0^r \int_{B_r(t-s)^{1/\alpha}} v(\tau + s) - \int_{B_r(z)} v(\tau + s) d\tau + \int_r^t \int_{B_r(z)} v(\tau + s) d\tau = I_1 + I_2.
\]
Since it is easy to get \( I_2 \leq C(t-s)^{1-1/\alpha} \|v\|_{L_{\rho,\lambda}} \), it suffices to consider \( I_1 \), but Proposition 2.3 with \( \mu = \alpha \lambda/(2d) - \alpha/2 + d \) implies
\[
I_1 \leq C(1 + (t-s)\frac{\mu - d}{\alpha}) \int_0^{t-s} \|v(\tau + s)\|_{L_{1,\rho}} d\tau \leq C(1 + (t-s)^{1-1/2}) (t-s)^{1/2 - \frac{1}{\alpha} + \frac{\alpha}{2} - \frac{d}{\alpha}} \|v\|_{L_{\rho,\lambda}}.
\]
The case \( \lambda \in [d,2d/\alpha + d] \) is obtained similarly and the details are omitted here. In order to estimate \( |\xi_0^*(x_1,t,s) - \xi_0^*(x_2,t,s)| \) we recall that \( \xi_0^*(x,z,s) \) is given by \( \xi_0^*(z,t,s) = (t-s)^{-1/\alpha} \xi_0^*(1,\tilde{z}) \), where \( \tilde{z} = (t-s)^{-1/\alpha} z \) and \( \xi_0(\tau,\tilde{z}) \) is the solution to (2.21) with \( u = \tau = (t-s)^{-1/\alpha} R_{t-s}v, x = \tilde{x}, \) and \( R = 1 \). Hence, from Proposition 2.3 and \( \|v\|_{L_{1,1;\rho,\lambda}} \leq C\|v\|_{L_{\rho,\lambda}} \) for \( \mu = \alpha \lambda/(2d) - \alpha/2 + d \) it is easy to get the desired estimates. The estimates for \( \xi_0^*(x,t,s) \) are obtained similarly. This completes the proof.

Remark 3.10 By the triangle inequality \( |\xi_0^*(x,t,s)| \leq |\xi_0^*(x,t,s) - \xi_0^*(0,t,s)| + |\xi_0^*(0,t,s)| \), applying Proposition 3.9, we also have the estimates
\[
|\xi_0^*(x,t,s)| \leq \begin{cases}
C_1(T^{1/2-\frac{d}{2\alpha}} + 1) (t-s)^{1/2} + C_2(t-s)^{1-\frac{d}{2\alpha}} \|v\|_{L_{\rho,\lambda}} & \text{if } \lambda \in (\frac{2d}{\alpha} - d,d), \\
C_1(t-s)^{1/2} \|v\|_{L_{\rho,\lambda}} & \text{if } \lambda = d,
\end{cases}
\]
\[
C_1((t-s)^{1/2} \|v\|_{L_{\rho,\lambda}} + \|v\|_{L_{\rho,\lambda}}) & \text{if } \lambda \in (d,\frac{2d}{\alpha} + d].
\]
Here \( C_1 \) depends only on \( d, \alpha, \lambda, \) and \( \|v\|_{L_{\rho,\lambda}} \), and \( C_2 \) depends only on \( d \) and \( q \).

3.3 Overlap estimates

In this section we will show the overlap estimates based on Proposition 3.8.

Proposition 3.11 Let \( K(\tau,y,x) \) be a kernel satisfying (1.5) and (1.6), and let \( K^{(N,0)}(N,0) \) be the approximation of \( K \) given by Section 2.3. Assume that \( N \gg 1 \) and \( \delta \leq N^{-1/(2-\alpha)} \). Then there exists a positive decreasing function \( \phi = \phi(r) \) such that
\[
\int_{\mathbb{R}^d} P_{K^{(N,0)}},(t,y;s,x_1) \wedge P_{K^{(N,0)}},(t,y;s,x_2) dy \geq \phi \left( \frac{|x_1 - x_2|}{(t-s)^\gamma} \right),
\]
for \( x_1, x_2 \in \mathbb{R}^d, t > s \geq 0 \) satisfying \( t-s \geq \mu_{N,\delta} \) and \( |x_2 - x_2| \leq 2(t-s)^{1/\alpha} \). Here \( \mu_{N,\delta} \) is the number in Proposition 3.8, and \( \phi \) is taken depending only on \( d, \alpha, C_0, \lambda, \) and \( \|v\|_{L_{\rho,\lambda}} \).

Proof. Let \( \xi_0^*(x_i) = \xi_0^*(x_i,t,s) \) be the vector in Proposition 3.8. We follow the arguments in [28]. The inequality \( a_1 b_1 + a_2 b_2 \leq \max a_i \max b_j + \min a_i \min b_j \) and Proposition 3.8 yields that for \( t-s \geq \mu_{N,\delta} \),
\[
(t-s)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \min E \left( \frac{|y - x_i - \xi_0^*(x_i)|}{(t-s)^{1/2}} \right) \min \log \left( \frac{t-s}{(t-s)^{1/2}} \right) P_{K^{(N,0)}},(t,y;s,x_i) + \epsilon dy \geq (t-s)^{-\frac{d}{2}} \sum_i \int_{\mathbb{R}^d} E \left( \frac{|y - x_i - \xi_0^*(x_i)|}{(t-s)^{1/2}} \right) \log \left( \frac{t-s}{(t-s)^{1/2}} \right) P_{K^{(N,0)}},(t,y;s,x_i) + \epsilon dy
\]
\[
-(t-s)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \max E \left( \frac{|y - x_i - \xi_0^*(x_i)|}{(t-s)^{1/2}} \right) \max \log \left( \frac{t-s}{(t-s)^{1/2}} \right) P_{K^{(N,0)}},(t,y;s,x_i) + \epsilon dy \geq -2C_2|\log| + 2C.
\]
where $C_2$ and $l_3 \in [0, 1)$ are the constants in Proposition 3.8 and $C$ is the constant such that $P_{K(N,\delta),\nu}(t,y;s,x) \leq C(t-s)^{-d/\alpha}$. On the other hand, we have

\[
\text{L.H.S of (3.61)} \leq \log \epsilon \cdot w \left( \frac{|x_1 + \xi^*_t(x_1) - x_2 - \xi^*_t(x_2)|}{(t-s)^{\frac{d}{\alpha}}} \right)
\]

\[
+ (t-s)^{-\frac{d}{\alpha}} \min_{y \in \mathbb{R}^d} E \left( \frac{|y-x_1 - \xi^*_t(x_1)|}{(t-s)^{\frac{d}{\alpha}}} \right) \min \log \left( 1 + \frac{(t-s)^{\frac{d}{\alpha}} P_{K(N,\delta),\nu}(t,y;s,x)}{\epsilon} \right) dy
\]

\[
\leq \log \epsilon \cdot w \left( \frac{|x_1 + \xi^*_t(x_1) - x_2 - \xi^*_t(x_2)|}{(t-s)^{\frac{d}{\alpha}}} \right) + \frac{C}{\epsilon} \int_{\mathbb{R}^d} \min_{y \in \mathbb{R}^d} P_{K(N,\delta),\nu}(t,y;s,x_1) dy.
\]

(3.62)

Here $w(\sigma)$ is the function on $\mathbb{R}_+$ defined by

\[
w(|z_1 - z_2|) = \int_{\mathbb{R}} \min E(2 \sigma) \, dz.
\]

(3.63)

Note that, by the radial symmetry and the positivity of $E$, $w$ is positive decreasing. Estimates (3.61) and (3.62) yield

\[
\int_{\mathbb{R}^d} \min_{y \in \mathbb{R}^d} P_{K(N,\delta),\nu}(t,y;s,x_1) dy \geq \frac{\epsilon}{C} \left( -\log \epsilon \cdot w \left( \frac{|x_1 + \xi^*_t(x_1) - x_2 - \xi^*_t(x_2)|}{(t-s)^{\frac{d}{\alpha}}} \right) - 2C_2 \log \epsilon^{(\alpha/2)} - 2C \right).
\]

(3.64)

By Proposition 3.9 it is not difficult to see that for each case of $\lambda \in [2d/\alpha - d, 2d/\alpha + d]$, $|\xi^*_t(x_1, t,s) - \xi^*_t(x_2, t,s)| \leq C((t-s)^{\frac{d}{\alpha}} + |x_1 - x_2|)$,

(3.65)

where $C$ depends only on $d, \alpha, \lambda$, and $\|\cdot\|_{X_\lambda}$. Then, since $w$ is positive decreasing, from (3.64) and $l_3 \in [0,1)$ we have

\[
\int_{\mathbb{R}^d} \min_{y \in \mathbb{R}^d} P_{K(N,\delta),\nu}(t,y;s,x_1) dy \geq \phi \left( \frac{|x_1 - x_2|}{(t-s)^{\frac{d}{\alpha}}} \right)
\]

(3.66)

for a positive decreasing function $\phi$, by taking $\epsilon > 0$ sufficiently small. This completes the proof.

### 3.4 Continuity estimates by the Nash iteration

In this section we establish continuity estimates for fundamental solutions by using the Nash iteration argument.

**Proposition 3.12** Let $K(t,x,y)$ be a function in Section 2.1 and let $K^{(N,\beta)}(t,x,y)$ be the approximation of $K$ in Section 2.3. Fix $N \geq 1$ and let $\tilde{\delta}^{2-N} \leq 1$. Let $\mu_{N,\beta}$ be the number in Proposition 3.8. Then

\[
|P_{K(N,\delta),\nu}(\tilde{t},x_1;\tilde{s},y) - P_{K(N,\delta),\nu}(\bar{t},x_2;\bar{s},y)| \leq C(\bar{t} - \tilde{s})^{-\frac{d}{\alpha}} \left( \frac{|x_1 - x_2| + \mu_{N,\beta}^{\frac{1}{\alpha}}}{(\bar{t} - \tilde{s})^{\frac{d}{\alpha}}} \right)^{\beta},
\]

(3.67)

for all $\bar{t} > \tilde{s} \geq 0$ and $x_1, x_2, y \in \mathbb{R}^d$ satisfying

\[
\frac{|x_1 - x_2| + \mu_{N,\beta}^{\frac{1}{\alpha}}}{(\bar{t} - \tilde{s})^{\frac{d}{\alpha}}} \leq C'.
\]

(3.68)

Here $C$, $\beta$, and $C'$ depend only on $d, \alpha, C_0, \lambda$, and $\|\cdot\|_{X_\lambda}$.

**Proof.** Without loss of generality we may assume that $\tilde{s} = 0$. Fix $\bar{t} > 0$. Let $\phi$ be a positive decreasing function in Proposition 3.11. Then we set

\[
\omega = 1 - \frac{\phi(2)}{4} \in (0,1), \quad x_0 = \frac{x_1 + x_2}{2},
\]

\[
\tilde{v}(\tau,x) = -\int_0^\tau v(t,x) \, dt, \quad \tilde{K}(\tau,x,y) = K^{(N,\delta)}(\tilde{t} - \tau,x,y), \quad 0 \leq \tau < \bar{t},
\]

\[
z_n(\tau) = \int_0^\tau \int_{B_{r_n}(x_0)} \tilde{v}(s,\cdot + z_n(s)) \, ds, \quad r_n > 0, \quad n \in \mathbb{N} \cup \{0\},
\]

\[
\tilde{z}_n(\tau) = z_n(\tau) - z_{n-1}(\tau), \quad n \in \mathbb{N},
\]

\[
\tilde{v}_n(\tau,x) = \tilde{v}(\tau,x + z_n(\tau)) - \int_{B_{r_n}(x_0)} \tilde{v}(\tau,\cdot + z_n(\tau)), \quad 0 \leq \tau < \bar{t},
\]

\[
\tilde{K}_n(\tau,x,y) = \tilde{K}(\tau,x + z_n(\tau),y + z_n(\tau)).
\]

Here $\{r_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $r_n \leq r_{n+1}$ for all $n$, which will be determined later. Note that $z_n(\tau)$ is a solution to (2.21) with $u = \tilde{v}, x = x_0, R = r_n$. The following lemma is important.
Lemma 3.13 Let $\zeta_n(\tau)$ be the vector defined as above, and let $\xi^*_{v_n}(x, \tau, s)$ be the vector in Proposition 3.4. Then there is a positive constant $C$ depending only on $d$, $\alpha$, $\lambda$, and $\|v\|_{X_\lambda}$ such that

$$
|\zeta_n(\tau)| \leq \begin{cases} 
C \left( \frac{d}{\alpha} + \frac{1}{\alpha} \right) r_n^{\frac{d}{\alpha} - 1}, & \lambda \in \left[ \frac{2d}{\alpha} - d, d \right], \\
C \left( 1 + \log(e + \frac{\tau}{r_n}) \right) \log \frac{r_n}{r_n - 1}, & \lambda = d, \\
C (\tau^{\frac{d}{\alpha}} + r_n), & \lambda \in (d, d + \frac{2d}{\alpha}], 
\end{cases}
$$

(3.69)

and

$$
|\xi^*_{v_n}(x, \tau, s)| \leq C \left( |\tau - s|^{\frac{d}{\alpha}} + |x - x_0| \right),
$$

(3.70)

and

$$
|\xi^*_{v_n}(x, \tau, s) - \xi^*_{v_n}(x_0, \tau, s)| \leq C \left( |\tau - s|^{\frac{d}{\alpha}} + |x - x_0| \right).
$$

(3.71)

Proof of Lemma 3.13. The estimate (3.69) follows from (2.24) and $\|v\|_{L^1(0, \tau, C^{1,\omega})} \leq C \frac{1}{2} + \frac{d}{2} + d$. The estimate (3.70) follows from (3.29) and Lemma 2.2. The estimate (3.71) is already proved in (3.65). This completes the proof of Lemma 3.13.

In the sequel we will freely use the relation

$$
P_{K^{(N,d)}}(l, x_1; -l, \tau, y + z_n(\tau))
= P_{K^{(N,d)}}(\tau, y + z_n(\tau); 0, x_1)
= \int_{\mathbb{R}^d} P_{K^{(N,d)}}(\tau, y + z_n(\tau); s, \xi) P_{K^{(N,d)}}(s, \xi; 0, x_1) d\xi
= \int_{\mathbb{R}^d} P_{K^{(N,d)}}(\tau, y; s, \xi - \zeta_n(s)) P_{K^{(N,d)}}(s, \xi + z_n(\tau); 0, x_1) d\xi.
$$

(3.72)

Set

$$
P_{+n}(\tau, y) = \left[ P_{K^{(N,d)}}(\tau, y + z_n(\tau); 0, x_1) - P_{K^{(N,d)}}(\tau, y + z_n(\tau); 0, x_2) \right] \vee 0,
\quad P_{-n}(\tau, y) = \left[ P_{K^{(N,d)}}(\tau, y + z_n(\tau); 0, x_2) - P_{K^{(N,d)}}(\tau, y + z_n(\tau); 0, x_1) \right] \vee 0.
$$

Then we have

$$
P_{+n}(\tau, y) + P_{-n}(\tau, y) = [P_{K^{(N,d)}}(\tau, y + z_n(\tau); 0, x_1) - P_{K^{(N,d)}}(\tau, y + z_n(\tau); 0, x_2)],
\quad P_{+n}(\tau, y) - P_{-n}(\tau, y) = [P_{K^{(N,d)}}(\tau, y + z_n(\tau); 0, x_1) - P_{K^{(N,d)}}(\tau, y + z_n(\tau); 0, x_2)].
$$

We define $F(\tau)$ by

$$
F(\tau) := \int_{\mathbb{R}^d} P_{+n}(\tau, y) dy = \int_{\mathbb{R}^d} P_{-n}(\tau, y) dy = \frac{1}{2} \int_{\mathbb{R}^d} |P_{K^{(N,d)}}(\tau, y + z_n(\tau); 0, x_1) - P_{K^{(N,d)}}(\tau, y + z_n(\tau); 0, x_2)| dy
= \frac{1}{2} \int_{\mathbb{R}^d} |P_{K^{(N,d)}}(\tau, y; 0, x_1) - P_{K^{(N,d)}}(\tau, y; 0, x_2)| dy
= 1 - \int_{\mathbb{R}^d} P_{K^{(N,d)}}(\tau, y; l, x_1; l - \tau, y) & P_{K^{(N,d)}}(\tau, y; x_2; l - \tau, y) dy.
$$

In particular, $F(\tau)$ does not depend either on $n \in \mathbb{N}$ or on the choice of $\{r_n\}_{n=0}^\infty$, which is the key in the following arguments. Set

$$
W_n(\tau, \xi, \eta) = P_{+n}(\tau, \xi) P_{-n}(\tau, \eta) / F(\tau).
$$

(3.73)

For any $l \geq \tau > s \geq 0$ we observe from (3.72) that

$$
P_{K^{(N,d)}}(\tau, y + z_n(\tau); 0, x_1) - P_{K^{(N,d)}}(\tau, y + z_n(\tau); 0, x_2)
= \int_{\mathbb{R}^d} \left[ P_{K^{(N,d)}}(\tau, y; \xi - \zeta_n(s)) - P_{K^{(N,d)}}(\tau, y; \xi - \zeta_n(s)) \right] W_{n-1}(s, \xi, \eta) d\xi dy.
$$

(3.74)
This implies
\[ F(\tau) \leq \int_{\mathbb{R}^{2d}} \frac{1}{2} |P_{K_{n},\delta}^{\prime}(\tau, y; s, \xi - \zeta_{n}(s)) - P_{K_{n},\delta}^{\prime}(\tau, y; \eta - \zeta_{n}(s))| W_{n-1}(s, \xi, \eta) \, d\xi \, d\eta. \]  
(3.75)

Since \( \int_{\mathbb{R}^{d}} |P_{K_{n},\delta}^{\prime}(\tau, y; s, \xi - \zeta_{n}(s)) - P_{K_{n},\delta}^{\prime}(\tau, y; \eta - \zeta_{n}(s))| \, dy \leq 2 \), we always have
\[ F(\tau) \leq F(s), \quad \bar{\ell} \geq \tau \geq s \geq 0. \]  
(3.76)

Let \( t_{m} \leq \bar{\ell}/2 \) be the time such that \( F(t_{m}) = \omega^{m} \) if it exists. Then \( F(t_{0}) = 1 \) which implies that \( t_{0} = 0 \) since \( P_{K_{n},\delta}^{\prime}(\tau, y; s, \xi - \zeta_{n}(s)) \) is strictly positive. Again we note that each \( t_{m} \) does not depend on the choice of \( \{r_{n}\}_{n=0}^{\infty} \). We set \( t_{m}^{(0)} = t_{m} + (m - 1)\mu_{N,\delta} \) if \( m \geq 1 \) and \( t_{0}^{(0)} = 0 \). Let \( \gamma \in (0, \min\{\alpha, 2 - \alpha\}) \) and set
\[ M_{\pm, n}(s) = \int_{\mathbb{R}^{d}} \rho_{\gamma}(y - x_{0}) P_{\pm, n}(s, y) \, dy, \]
\[ M_{n}(s) = M_{+}, n(s) \vee M_{-}, n(s), \]
\[ M_{n} = M_{n}(t_{m}^{(0)}), \quad \text{if} \quad t_{m}^{(0)} \leq \bar{\ell}/2. \]

In particular, we have \( M_{0} = M_{0}(0) = 2^{-\gamma/2}|x_{1} - x_{2}|^{\gamma/2} \). We also set
\[ P_{\pm, n}^{\prime}(s, \xi, \eta) = P_{\pm, n}(s, \xi) \quad \text{if} \quad \rho_{\gamma}(\xi - x_{0}) \leq 2 F(t_{m}^{(0)})^{-1} M_{n}, \quad P_{\pm, n}^{\prime}(s, \xi, \eta) = 0 \quad \text{otherwise}, \]
\[ W_{n}^{\prime}(s, \xi, \eta) = F(t_{m}^{(0)})^{-1} P_{\pm, n}^{\prime}(s, \xi) P_{\pm, n}(s, \eta). \]

Note that \( W_{n}(t_{m}^{(0)}, \xi, \eta) \geq W_{n}^{\prime}(t_{m}^{(0)}, \xi, \eta) \) holds. Hence when \( t_{m}^{(0)} < \tau \leq \bar{\ell}/2 \) we have from (3.75),
\[ F(\tau) \leq \int_{\mathbb{R}^{2d}} \left( W_{n}(t_{m}^{(0)}, \xi, \eta) - W_{n}^{\prime}(t_{m}^{(0)}, \xi, \eta) \right) \, d\xi \, d\eta \]
\[ + \frac{1}{2} \int_{\mathbb{R}^{2d}} |P_{K_{n+1},\delta}^{\prime} - P_{K_{n},\delta}^{\prime}| W_{n}^{\prime}(t_{m}^{(0)}, \xi, \eta) \, d\xi \, d\eta \]

Since \( W_{n}^{\prime}(t_{m}^{(0)}, \xi, \eta) > 0 \) only when \( |\xi - x_{0}| \leq (2 F(t_{m}^{(0)})^{-1} M_{n})^{2/\gamma} \) and \( |\eta - x_{0}| \leq (2 F(t_{m}^{(0)})^{-1} M_{n})^{2/\gamma} \), we have
\[ |\xi - \eta| \leq 2(2 F(t_{m}^{(0)})^{-1} M_{n})^{2/\gamma} \] in this case. Now we take
\[ \tau = \tau_{n} = t_{m}^{(0)} + \max\{(2 F(t_{m}^{(0)})^{-1} M_{n})^{2/\gamma}, \mu_{N,\delta}\}. \]  
(3.77)

Then \( |\xi - \zeta_{n}(s) - (\eta - \zeta_{n}(s))| = |\xi - \eta| \leq 2(\tau_{n} - t_{m}^{(0)})^{1/\alpha} \) and \( \tau_{n} - t_{m}^{(0)} \geq \mu_{N,\delta} \). Thus, by Proposition 3.11, if \( \tau_{n} \leq \bar{\ell}/2 \) then
\[ F(\tau_{n}) \leq \int_{\mathbb{R}^{2d}} \left( W_{n}(t_{m}^{(0)}, \xi, \eta) - W_{n}^{\prime}(t_{m}^{(0)}, \xi, \eta) \right) \, d\xi \, d\eta \]
\[ + \int_{\mathbb{R}^{2d}} W_{n}(t_{m}^{(0)}, \xi, \eta) \left( 1 - \phi_{2}(\frac{|\xi - \eta|}{(\tau_{n} - t_{m}^{(0)})^{1/\alpha}}) \right) \, d\xi \, d\eta \]
\[ \leq \int_{\mathbb{R}^{2d}} \left( W_{n}(t_{m}^{(0)}, \xi, \eta) - W_{n}^{\prime}(t_{m}^{(0)}, \xi, \eta) \right) \, d\xi \, d\eta + \int_{\mathbb{R}^{2d}} W_{n}(t_{m}^{(0)}, \xi, \eta) \, d\xi \, d\eta \]
\[ = \int_{\mathbb{R}^{2d}} W_{n}(t_{m}^{(0)}, \xi, \eta) \, d\xi \, d\eta \]

Hence from the estimate
\[ \int_{\mathbb{R}^{d}} P_{\pm, n}^{\prime}(t_{m}^{(0)}, \xi) \, d\xi = F(t_{m}^{(0)}) - \int_{|\xi - x_{0}| \geq (2 F(t_{m}^{(0)}) M_{n})^{1/\alpha}} P_{\pm, n}(t_{m}^{(0)}, \xi) \, d\xi \]
\[ \geq F(t_{m}^{(0)}) - (2 M_{n})^{-1} F(t_{m}^{(0)}) \int_{\mathbb{R}^{d}} \rho_{\gamma}(\xi - x_{0}) P_{\pm, n}(t_{m}^{(0)}, \xi) \, d\xi \]
\[ \geq \frac{1}{2} F(t_{m}^{(0)}). \]

we have
\[ F(\tau_{n}) \leq F(t_{m}^{(0)}) - \phi(2) F(t_{m}^{(0)})^{-1} \left( \frac{F(t_{m}^{(0)})}{2} \right)^{2} = F(t_{m}^{(0)})(1 - \frac{1}{4} \phi(2)) \leq F(t_{0}) \omega = \omega^{n+1}. \]

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That is, \( t_{n+1} \leq \tau_n \), which gives the estimate
\[
t_{n+1}^{(\delta)} = t_{n+1} + n\mu_{N,\delta} \leq \tau_n + n\mu_{N,\delta} \leq (T_n^{(\delta)} + (n + 1)\mu_{N,\delta} + (2F(t_n^{(\delta)})^{-1}M_n)^\frac{2n}{\omega_n}.
\]
(3.78)

Next we estimate \( M_n \). The equality (3.74) leads to the inequality
\[
P_{\pm,n+1}(\tau, y) \leq \int_{\mathbb{R}^d} P_{K_{n+1},\epsilon_{n+1}^{-1}}(\tau, y; s, x - \zeta_{n+1}(s))P_{\pm,n}(s, x) \, dx,
\]
and hence, we have
\[
M_{\pm,n+1}(\tau) \leq \int_{\mathbb{R}^d} \rho_\tau(y - x_0) P_{K_{n+1},\epsilon_{n+1}^{-1}}(\tau, y; s, x - \zeta_{n+1}(s))P_{\pm,n}(s, x) \, dx \, dy
\]
\[
\leq \int_{\mathbb{R}^d} \left\{ \rho_\tau(x - x_0) + \rho_\tau(\zeta_{n+1}^*(x - \zeta_{n+1}(s), \tau, s) - \zeta_{n+1}^*(x_0 - \zeta_{n+1}(s), \tau, s)) + \rho_\tau(y - \zeta_{n+1}^*(x - \zeta_{n+1}(s), \tau, s) - x + \zeta_{n+1}(s)) \right. \\
+ \rho_\tau(\zeta_{n+1}^*(x_0 - \zeta_{n+1}(s), \tau, s) - \zeta_{n+1}(s)) \left. \right\} \times P_{\pm,n}(s, x)P_{K_{n+1},\epsilon_{n+1}^{-1}}(\tau, y; s, x - \zeta_{n+1}(s)) \, dx \, dy
\]
\[
\leq M_{\pm,n}(s) + \int_{\mathbb{R}^d} [\zeta_{n+1}^*(x - \zeta_{n+1}(s), \tau, s) - \zeta_{n+1}^*(x_0 - \zeta_{n+1}(s), \tau, s)]^\frac{2}{\omega} P_{\pm,n}(s, x) \, dx

+ F(s) \sup_{x} \int_{\mathbb{R}^d} \rho_\tau(y - \zeta_{n+1}^*(x, \tau, s) - x + \zeta_{n+1}(s))P_{K_{n+1},\epsilon_{n+1}^{-1}}(\tau, y; s, x) \, dy

+ F(s)[\zeta_{n+1}^*(x_0 - \zeta_{n+1}(s), \tau, s) - \zeta_{n+1}(s)]^\frac{2}{\omega}.
\]
(3.79)

From Proposition 3.4 and Lemma 3.13, if \( \tau - s \geq \mu_{N,\delta} \) then we have the estimate of \( M_{\pm,n+1}(\tau) \) such as
\[
M_{\pm,n+1}(\tau) \leq CM_{\pm,n}(s) + \left( C(\tau - s)\frac{\omega}{\omega_n} + [\zeta_{n+1}^*(x_0 - \zeta_{n+1}(s), \tau, s)]^\frac{2}{\omega} + 2[\zeta_{n+1}(s)]^\frac{2}{\omega} \right) F(s).
\]
(3.80)

From \( t_{n+1}^{(\delta)} - t_n^{(\delta)} \geq \mu_{N,\delta} \) and (3.80) we have
\[
M_n + 1 \leq CM_n + F(t_n^{(\delta)}) \left( C(t_{n+1}^{(\delta)} - t_n^{(\delta)})\frac{\omega}{\omega_n} + [\zeta_{n+1}^*(x_0 - \zeta_{n+1}(t_n^{(\delta)}), t_{n+1}^{(\delta)}, t_n^{(\delta)})]^{\frac{2}{\omega}} + 2[\zeta_{n+1}(t_n^{(\delta)})]^{\frac{2}{\omega}} \right).
\]
(3.81)

Let us take \( r_0 = r_1 = (t_n^{(\delta)})^{1/\alpha} \) and \( r_n = (t_n^{(\delta)})^{1/\alpha} \) for \( n \geq 2 \). Then \( \zeta(t_0) = \zeta(0) = 0 \) and Lemma 3.13 yields
\[
[\zeta_{n+1}(t_n^{(\delta)})] \leq C((t_{n+1}^{(\delta)} - t_n^{(\delta)})^{1/\omega} + (t_n^{(\delta)})^{1/\omega}),
\]
(3.82)
and also
\[
[\zeta_{n+1}^*(x_0 - \zeta_{n+1}(t_n^{(\delta)}), t_{n+1}^{(\delta)}, t_n^{(\delta)})] \leq C((t_{n+1}^{(\delta)} - t_n^{(\delta)})^{1/\omega} + (t_n^{(\delta)})^{1/\omega}).
\]
(3.83)

Hence we finally get
\[
M_{n+1} \leq C_1 M_n + C_2 F(t_n^{(\delta)}) ((t_{n+1}^{(\delta)} - t_n^{(\delta)})^{1/\omega} + (t_n^{(\delta)})^{1/\omega}),
\]
(3.84)
as long as \( t_{n+1}^{(\delta)} = t_{n+1} + n\mu_{N,\delta} \leq \bar{t}/2 \). From (3.78), (3.84), and \( F(t_n^{(\delta)}) \leq F(t_\bar{\tau}) = \omega_n \), we have
\[
M_{n+1} \leq C_1 M_n + C_2 F(t_n^{(\delta)}) \sum_{k=0}^{n} (t_{k+1}^{(\delta)} - t_k^{(\delta)}) \frac{\omega_n}{\omega_n}
\]
\[
\leq C_1 M_n + C_2 F(t_n^{(\delta)}) \sum_{k=0}^{n} \left( (k + 1)\mu_{N,\delta} + (2F(t_n^{(\delta)})^{-1}M_n)^{\frac{2n}{\omega_n}} \right)^\frac{\omega_n}{\omega_n}
\]
\[
\leq C_1 M_n + 2C_2 \sum_{k=0}^{n} M_k + C_2 \mu_{N,\delta}^\frac{2n}{\omega_n} \omega_n
\]
\[
\leq C_1 M_n + 2C_2 \sum_{k=0}^{n} M_k + C_3 \mu_{N,\delta}^\frac{2n}{\omega_n}.
\]
(3.85)

Here we used the inequality \((n + 1)^{1+\frac{2n}{\omega_n}} \omega_n \leq C\) for all \( n \geq 0 \). Then we have
\[
M_n \leq C_4 (M_0 + \mu_{N,\delta}^\frac{2n}{\omega_n}), \quad C_4 = \max\{3C_1, 1 + 6C_2, 3C_3\}.
\]
(3.86)
Indeed, (3.86) is valid for $n = 0$. Assuming that (3.86) holds up to $n$, we have from (3.85),

$$M_{n+1} \leq C_1 C_4^n (M_0 + \mu_{N,d}^{\gamma/\alpha}) + 2C_2 (M_0 + \mu_{N,d}^{\gamma/\alpha}) \sum_{k=0}^n C_4^k + C_3 \mu_{N,d}^{\gamma/\alpha}\$$

This proves (3.86) for $n + 1$ due to the choice of $C_4$. Then by (3.78) and (3.86), if $\tau_n \leq \bar{t}/2$, we have

$$t_{n+1}^{(k)} = \sum_{k=0}^n t_{k+1}^{(k)} + t_{k}^{(k)} \leq \sum_{k=0}^n (k + 1) \mu_{N,d} + (2F(t_k^{(k)})^{-1} M_k)^{2\alpha} \leq (n + 1)^2 \mu_{N,d} + \left(\frac{2}{F(\bar{t}/2)}\right)^{2\alpha} \sum_{k=0}^n M_{k}^{2\alpha} \leq C_2^{n+1} F(\bar{t}/2)^{-2\alpha} \left(\frac{x_1 - x_2}{\bar{t}}\right)^2 + \mu_{N,d}^{\gamma/\alpha} \right)^{\frac{2\alpha}{\gamma}} \leq C_2^{n+1} \left(\frac{\sigma}{F(\bar{t}/2)}\right)^{\frac{2\alpha}{t}} \bar{t},$$

(3.87)

where

$$\sigma = \left(\frac{|x_1 - x_2|}{\bar{t}}\right)^2 + \left(\frac{\mu_{N,d}}{\bar{t}}\right)^{\frac{\gamma}{\alpha}}.$$  

(3.88)

Here we used the estimates $F(\bar{t}/2) \leq F(t_k^{(k)})$ and $M_0 = 2^{-\gamma/2} |x_1 - x_2|^{\gamma/2}$. If $F(\bar{t}/2) \geq \sigma^{1/2}$ then (3.87) implies that

$$t_n^{(k)} \leq C_2^n \sigma^{\frac{k}{\alpha}} \bar{t}.$$  

(3.89)

Hence we can take $n = n(\sigma)$ as

$$-\frac{\alpha}{2\gamma \log C_5} \log \sigma \leq n(\sigma) < -\frac{\alpha}{2\gamma \log C_5} + 1,$$

(3.90)

which satisfies both $n(\sigma) \to \infty$ as $\sigma \to 0$ and $C_2^n \sigma^{\frac{k}{\alpha}} \leq C_5 \sigma^{\frac{k}{\alpha}/(2\gamma)} \to 0$ as $\sigma \to 0$. In particular, if $\sigma \leq (2C_5)^{-2\gamma/\alpha}$ then $t_n^{(k)} \leq \bar{t}/2$ and so

$$F(\bar{t}/2) \leq F(t_{n(\sigma)}^{(k)}) \leq F(t_{n(\sigma)}) = \omega^{n(\sigma)} = e^{n(\sigma) \log \omega} \leq \sigma^{-\frac{\alpha \log \omega}{2\gamma \log C_5}}.$$ 

Thus we have $F(\bar{t}/2) \leq \sigma^\beta$ with $\beta = \min\{1/2, -\alpha \log \omega/(2\gamma \log C_5)\}$. This estimate yields

$$|P_{K,\lambda}(\xi, x, 0, y) - P_{K,\lambda}(\xi, x, 0, y)| = |P_{K,\lambda}(\xi, 0, x, y) - P_{K,\lambda}(\xi, x, y; 0, x_2)| \leq \int_{\mathbb{R}^d} P_{K,\lambda}(\xi, y, \bar{t}/2, \xi) |P_{K,\lambda}(\xi, 0, x_1) - P_{K,\lambda}(\xi, 0, x_2)| d\xi \leq C \left(\frac{\sigma}{F(\bar{t}/2)}\right)^{\frac{2\alpha}{\gamma}}$$

if

$$\frac{\left(\frac{|x_1 - x_2|}{\bar{t}}\right)^2 + \left(\frac{\mu_{N,d}}{\bar{t}}\right)^{\frac{\gamma}{\alpha}}}{(2C_5)^{-2\gamma/\alpha}}.$$  

(3.91)

Here $C$, $\beta$, and $C_5$ depend only on $d$, $\alpha$, $C_0$, $\lambda$, and $\|v\|_{L_\lambda}$. This completes the proof.

**Corollary 3.14** Under the assumptions of Proposition 3.12 it follows that

$$|P_{K,\lambda}(\xi, x, 0, y) - P_{K,\lambda}(\xi, x, 0, y)| \leq \frac{C}{(t-s)^{\beta}} \left(\frac{|x_1 - x_2| + |y_1 - y_2| + \mu_{N,d}^{\gamma/\alpha}}{(t-s)^{\beta}}\right)^{\frac{1}{\gamma}},$$  

(3.92)

for all $x_i, y_i \in \mathbb{R}^d$ and $t > s \geq 0$ such that $t - s \geq C \mu_{N,d}$. Here $C$, $\beta$, and $C'$ depend only on $d$, $\alpha$, $C_0$, $\lambda$, and $\|v\|_{L_\lambda}$.
Proof. It suffices to consider the case \( y_1 = y_2 = y \), for the continuity estimate with respect to \( y \) follows from the adjoint relation. Let \( C' \) be the constant in Proposition 3.12. Assume that \( \mu_{N,\delta} \leq (2^{-1}C')^{\alpha} (t-s) \). Then if \( |x_1 - x_2| \leq 2^{-1} C'(t-s)^{1/\alpha} \) the estimate \((3.92)\) follows from Proposition 3.12. If \( |x_1 - x_2| \geq 2^{-1} C'(t-s)^{1/\alpha} \) then

\[
|P_{K(t,s),v}(t,x_1; s,y) - P_{K(t,s),v}(t,x_2; s,y)| \leq C(t-s)^{-\frac{d}{2}} = C(t-s)^{-\frac{d}{2}} \frac{(t-s)^{\frac{d}{2}} |x_1 - x_2|}{|x_1 - x_2|^\beta} \leq C(t-s)^{-\frac{d}{2}} \left( \frac{|x_1 - x_2|}{(t-s)^{\frac{d}{2}}} \right)^\beta.
\]

This completes the proof.

**Corollary 3.15** Under the assumptions of Proposition 3.12 it follows that

\[
|P_{K(t,s),v}(t_1,x; s_1,y) - P_{K(t,s),v}(t_2,x; s_2,y)| \leq C \left( \min \{ t_1 - s_1, t_2 - s_2 \} \right)^{\frac{d}{2}} \left( C_{\nu,v} |t_1 - t_2|^{\nu'} + C_{\nu,v} |s_1 - s_2|^{\nu'} + \mu_{N,\delta} \right),
\]

for all \( x, y \in \mathbb{R}^d \) and \( T \geq t_i > s_i \geq 0 \) such that \( \min \{ t_1 - t_2, s_1 - s_2, t_1 - s_1, t_2 - s_2 \} \geq C' \mu_{N,\delta} \). Here \( C, c, \) and \( C' \) depend only on \( d, \alpha, C_0, \lambda, \) and \( ||v||_{X_{\gamma}} \). The constant \( \beta' > 0 \) depends only on \( d, \alpha, C_0, \lambda, q, \) and \( ||v||_{X_{\gamma}} \). The constant \( C_{\nu,v} \) (or \( C_{\nu,v} \)) depends only on \( T, d, \alpha, \lambda, q, ||v||_{X_{\gamma}}, \) and \( ||v||_{V_{\gamma}^{T,\lambda}} \).

Proof. It suffices to consider the case \( s_1 = s_2 = s \). Let \( R > 0 \). By the semigroup property of fundamental solutions, we have

\[
|P_{K(t,s),v}(t_1,x; s_1,y) - P_{K(t,s),v}(t_2,x; s_2,y)|
\]

\[
\leq \int_{|x-z| \leq R} P_{K(t,s),v}(t_1,x,t_2,z) ||P_{K(t,s),v}(t_2,z; s_2,y) - P_{K(t,s),v}(t_2,x; s_2,y)|| dz
\]

\[
= \int_{|x-z| \leq R} P_{K(t,s),v}(t_1,x,t_2,z) ||P_{K(t,s),v}(t_2,z; s_2,y) - P_{K(t,s),v}(t_2,x; s_2,y)|| dz
\]

\[
+ \int_{|x-z| \geq R} P_{K(t,s),v}(t_1,x,t_2,z) ||P_{K(t,s),v}(t_2,z; s_2,y) - P_{K(t,s),v}(t_2,x; s_2,y)|| dz
\]

\[
= I_1 + I_2.
\]

Applying Corollary 3.14 to \( I_1 \) yields \( I_1 \leq C(t_2-s)^{-(d+\beta)/\alpha}(R^{d/\alpha} + \mu_{N,\delta}^{\frac{d}{2}}) \) if \( t_2 - s \geq C' \mu_{N,\delta} \). As for \( I_2 \), we use the moment bound in Proposition 3.5 and then there is \( \eta^* = \eta^*(x,t_1,t_2) \) such that

\[
I_2 \leq \frac{C}{(t_2 - s)^{\frac{d}{2}} R^2} \int_{|x-z| \geq R} P_{K(t,s),v}(t_1,x,t_2,z) \rho_{\nu}(z-x) dz
\]

\[
\leq \frac{C}{(t_2 - s)^{\frac{d}{2}} R^2} \int_{|x-z| \geq R} P_{K(t,s),v}(t_1,x,t_2,z) (\rho_{\nu}(z-x - \eta^*_v) + ||\eta^*_v||^{\frac{d}{2}}) dz
\]

\[
\leq \frac{C}{(t_2 - s)^{\frac{d}{2}} R^2} \{ (t_1 - t_2)^{\frac{d}{2}} (1 + (t_1 - t_2)^{1-\frac{d}{2}} \delta^{2-\alpha} N + \delta (t_1 - t_2)^{1-\frac{d}{2}}) + ||\eta^*_v||^{\frac{d}{2}} \}
\]

\[
\leq \frac{C}{(t_2 - s)^{\frac{d}{2}} R^2} \{ (t_1 - t_2)^{\frac{d}{2}} + ||\eta^*(x,t_1,t_2)||^{\frac{d}{2}} \},
\]

if \( t_1 - t_2 \geq C' \mu_{N,\delta} \). By Proposition 3.9 the vector \( \eta^*_v(x,t_1,t_2) \) satisfies the estimate

\[
||\eta^*_v(x,t_1,t_2)|| \leq C \left( (t_1 - t_2)^{\frac{d}{2}} (1 + \log(1 + |y_1 - y_2|)) + \frac{1}{(t_1 - t_2)^{\frac{d}{2}}} \right),
\]

where \( C \) depends only on \( T, d, \alpha, \lambda, ||v||_{X_{\gamma}}, \) and \( ||v||_{V_{\gamma}^{T,\lambda}} \). Collecting \((3.94)-(3.96)\) and taking \( R \) of the form \( (t_1 - t_2)^{\nu} \) for suitable \( \nu \) depending only on \( \alpha \) and \( q \), we get \((3.93)\). This completes the proof.

### 4 Existence of fundamental solutions

In this section we complete the proof of Theorems 1.2 and 1.4. We first take the limit \( \delta \to 0 \) of \( P_{K(t,s),v}(t,x; s,y) \) and next we consider the limit \( N \to \infty \).
4.1 Limit $\delta \to 0$

Set $D = \{(t,s) \in \mathbb{R}^2 \mid t > s \geq 0 \}$, $D_m = \{(t,s) \in D \mid t > s + 1/m \}$ with $m \geq 1$, $\Omega = D \times \mathbb{R}^d \times \mathbb{R}^d$, and $\Omega_m = D_m \times \mathbb{R}^d \times \mathbb{R}^d$. Fix $N \gg 1$. First we assume that $v$ is smooth and bounded. As a function on $\Omega$, by (2.31) $P_{K(N,\delta),v}(t,x,s,y)$ satisfies the following estimates:

$$P_{K(N,\delta),v}(t,x,s,y) \leq C(t-s)^{-\frac{\gamma}{2}},$$

(4.1)

where $C$ depends only on $d$, $\alpha$, and $C_0$. So there is a sequence $\{\delta_n\}_{n=1}^{\infty}$ with $\delta_n \to 0$ as $n \to \infty$ such that $P_{K(N,\delta_n),v}(t,x,s,y) \to P_{K(N),v}(t,x,s,y)$ as $n \to \infty$ in $L^\infty(\Omega_m)$ for each $m \geq 1$ for some nonnegative function $P_{K(N),v}(t,x,s,y) \in L^\infty(\Omega)$, that is,

$$\lim_{n \to \infty} \int_{\Omega_m} P_{K(N,\delta_n),v}(t,x,s,y) \varphi(t,x,s,y) \, dx \, dy \, ds = \int_{\Omega} P_{K(N),v}(t,x,s,y) \varphi(t,x,s,y) \, dx \, dy \, ds,$$

(4.2)

for every $\varphi \in L^1(\Omega)$ supported in $\Omega_m$ for some $m \geq 1$. Taking $\varphi(t,s,x,y)$ of the form $\varphi(t,s,x,y)$ in (4.2), by the duality argument we see that $P_{K(N),v}(t,x,s,y) \in L^\infty(D;L^p(\mathbb{R}^d))$ for all $p \in [1, \infty]$ and satisfies

$$\|P_{K(N),v}\|_{L^\infty(D;L^p)} \leq \|f\|_{L^p}.$$

(4.3)

By taking $\varphi(t,s,x,y)$ of the form $(t-s)^{\alpha/\gamma} \varphi(t,s,x,y)$ in (4.2) we also have

$$P_{K(N),v}(t,x,s,y) \leq C(t-s)^{-\frac{\gamma}{2}},$$

(4.4)

where $C$ depends only on $d$, $\alpha$, and $C_0$.

(1) Moment bound. Next we prove the moment bound

$$C_1(t-s)^{\frac{\gamma}{2}} \leq \int_{\mathbb{R}^d} \rho_{\gamma}(y-x - \xi_{(t,s)}(x,t,s)) P_{K(N),v}(t,x,s,y) \, dy \, dx \, dt \, ds,$$

(4.5)

where $\xi_{(t,s)}(x,t,s)$ is the vector given by Proposition 3.4, $\gamma \in [0, \min\{\alpha, 2 - \alpha\}]$, and $C_1$, $C_2$ depend only on $d$, $\alpha$, $C_0$, $\gamma$, and $\|\|_{\chi_{\lambda}}$. To prove the upper bound, we take $\varphi(t,s,x,y)$ of the form $(t-s)^{-\gamma/(2\alpha)} \rho_{\gamma}(y-x - \xi_{(t,s)}(x,t,s)) \varphi(t,s,x,y)$ in (4.2). Then we have from Proposition 3.4,

$$\int_D \int_{\mathbb{R}^d} P_{K(N),v}(t,x,s,y) \varphi(t,s,x,y) \rho_{\gamma}(y-x - \xi_{(t,s)}(x,t,s)) \, dy \, dx \, dt \, ds = \int_D \int_{\mathbb{R}^d} P_{K(N,\delta_n),v}(t,x,s,y) \varphi(t,s,x,y) \rho_{\gamma}(y-x - \xi_{(t,s)}(x,t,s)) \, dy \, dx \, dt \, ds$$

$$\leq C_2 \int_{\mathbb{R}^d} \int_D \int_{\mathbb{R}^d} \rho_{\gamma}(y-x) \varphi(t,s,x,y) \rho_{\gamma}(y-x - \xi_{(t,s)}(x,t,s)) \varphi(t,s,x,y) \, dy \, dx \, dt \, ds$$

This proves the upper bound of (4.5). Regarding the lower bound, for all $\varphi \in C_0^\infty(D \times \mathbb{R}^d)$ with $\varphi \geq 0$, $\gamma' \in (\gamma, \min\{\alpha, 2 - \alpha\})$, and $R \geq 1$, we have

$$\int_D \int_{\mathbb{R}^d} \varphi(t,s,x) \rho_{\gamma}(y-x - \xi_{(t,s)}(x,t,s)) P_{K(N),v}(t,x,s,y) \, dy \, dx \, dt \, ds$$

$$\leq C_2 \int_D \int_{\mathbb{R}^d} \rho_{\gamma}(y-x - \xi_{(t,s)}(x,t,s)) P_{K(N,\delta_n),v}(t,x,s,y) \, dy \, dx \, dt \, ds$$

$$\geq C_2 \int_D \int_{\mathbb{R}^d} \varphi(t,s,x) (t-s)^{-\frac{\gamma}{2}} \, dx \, dt \, ds$$

$$- CR^{-\frac{\gamma'}{2}} \int_D \int_{\mathbb{R}^d} \varphi(t,s,x)(t-s)^{\frac{\gamma'}{2}} \, dx \, dt \, ds$$

$$- \limsup_{n \to \infty} \int_D \int_{\mathbb{R}^d} \varphi(t,s,x) \rho_{\gamma}(y-x - \xi_{(t,s)}(x,t,s)) P_{K(N,\delta_n),v}(t,x,s,y) \, dy \, dx \, dt \, ds$$

$$- CR^{-\frac{\gamma'}{2}} \int_D \int_{\mathbb{R}^d} \varphi(t,s,x)(t-s)^{\frac{\gamma'}{2}} \, dx \, dt \, ds$$

$$\geq C_2 \int_D \int_{\mathbb{R}^d} \varphi(t,s,x)(t-s)^{\frac{\gamma'}{2}} \, dx \, dt \, ds - CR^{-\frac{\gamma'}{2}} \int_D \int_{\mathbb{R}^d} \varphi(t,s,x)(t-s)^{\frac{\gamma'}{2}} \, dx \, dt \, ds.$$
By taking $R \to \infty$ we get
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(t, s, x) \int_{\mathbb{R}^d} \rho_\gamma(y - x - \xi^*_\gamma(x, t, s)) P_{K(N), \nu}(t, x; s, y) \, dy \, dx \, dt \, ds \\
\geq \ C_1 \int_{\mathbb{D}} \int_{\mathbb{R}^d} \psi(t, s, x)(t-s) \, dx \, dt \, ds.
\]
Since $\psi \in C_0^\infty(D \times \mathbb{R}^d)$ is an arbitrary nonnegative function, we get the desired lower bound. This completes the proof of (4.5). The same argument leads to
\[
C_1(t-s) \frac{\gamma}{\gamma} \leq \int_{\mathbb{R}^d} \rho_\gamma(y - \xi^*_\gamma(y, t, s)) P_{K(N), \nu}(t, x; s, y) \, dx \leq C_2(t-s) \frac{\gamma}{\gamma}.
\] (4.6)
By Remark 3.10 the vectors $\xi^*_\gamma(x, t, s)$ and $\eta^*_\gamma(y, t, s)$ in (4.5) and (4.6) satisfy for all $R > 0$,
\[
\sup_{|x| \leq R, 0 \leq s \leq t \leq R} |\xi^*_\gamma(x, t, s)| + \sup_{|y| \leq R, 0 \leq s \leq t \leq R} |\eta^*_\gamma(y, t, s)| \leq CR < \infty.
\] (4.7)
(ii) Mass conservation. Next we show
\[
\int_{\mathbb{R}^d} P_{K(N), \nu}(t, x; s, y) \, dy = \int_{\mathbb{R}^d} P_{K(N), \nu}(t, x; s, y) \, dx = 1.
\] (4.8)
Since $P_{K(N), \nu}(t, x; s, y)$ is nonnegative we already know that
\[
\int_{\mathbb{R}^d} P_{K(N), \nu}(t, x; s, y) \, dy \leq 1, \quad \int_{\mathbb{R}^d} P_{K(N), \nu}(t, x; s, y) \, dx \leq 1
\]
for a.e. by (4.3). So it suffices to prove the lower bound. For any $\phi(t, s, x) \in C_0^\infty(D \times \mathbb{R}^d)$ with $\phi \geq 0$, we have from the moment bounds for $P_{K(N), \nu}(t, x; s, y)$ and $P_{K(N, \delta_n), \nu}(t, x; s, y)$,
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(t, s, x) \int_{\mathbb{R}^d} P_{K(N), \nu}(t, x; s, y) \, dy \, dx \, dt \, ds
\]
\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(t, s, x) \int_{|y - x - \xi^*_\gamma(x, t, s)| \leq R} P_{K(N), \nu}(t, x; s, y) \, dy \, dx \, dt \, ds
\]
\[
+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(t, s, x) \int_{|y - x - \xi^*_\gamma(x, t, s)| \geq R} P_{K(N), \nu}(t, x; s, y) \, dy \, dx \, dt \, ds
\]
\[
\geq \lim_{n \to \infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(t, s, x) \int_{|y - x - \xi^*_\gamma(x, t, s)| \leq R} P_{K(N, \delta_n), \nu}(t, x; s, y) \, dy \, dx \, dt \, ds
\]
\[
\quad - C_2 R^{-\frac{\gamma}{\gamma}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(t, s, x) \, dx \, dt \, ds - CR^{-\frac{\gamma}{\gamma}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(t, s, x) \, dx \, dt \, ds.
\]
Here $\gamma \in (0, \min(\alpha/2, \alpha - \alpha))$. Taking $R \to \infty$ we get
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(t, s, x) \int_{\mathbb{R}^d} P_{K(N), \nu}(t, x; s, y) \, dy \, dx \, dt \, ds \geq \int_{\mathbb{R}^d} \phi(t, s, x) \, dx \, dt \, ds,
\]
for all $\phi \in C_0(D \times \mathbb{R}^d)$ with $\phi \geq 0$, which proves $\int_{\mathbb{R}^d} P_{K(N), \nu}(t, x; s, y) \, dy \geq 1$ for a.e. $(t, s, x)$. The same argument is applied for $\int_{\mathbb{R}^d} P_{K(N), \nu}(t, x; s, y) \, dx$. This proves (4.8).

(iii) Continuity. Next we prove the Hölder continuity of $P_{K(N), \nu}(t, x; s, y)$. To this end we set $\psi_\mu(t, s, h, k) = (t - s)^{-(d+\beta)/\mu}(|h| + |k| + \mu)^\beta$, $0 < \mu \ll 1$, where $\beta$ is the constant in Corollary 3.14. Then from (4.2) we have for every $\phi \in C_0^\infty(\Omega)$,
\[
\left| \int_{\Omega} P_{K(N), \nu}(t, x; s, y) \int_{\Omega} P_{K(N, \delta_n), \nu}(t, x + k; s, y + k) \frac{\phi(t, s, x, y)}{\psi_\mu(t, s, h, k)} \, dx \, dy \, dt \, ds \right|
\]
\[
= \left| \lim_{n \to \infty} \int_{\Omega} P_{K(N, \delta_n), \nu}(t, x; s, y) \int_{\Omega} P_{K(N, \delta_n), \nu}(t, x + k; s, y + k) \frac{\phi(t, s, x, y)}{\psi_\mu(t, s, h, k)} \, dx \, dy \, dt \, ds \right|
\]
\[
\leq C \lim_{n \to \infty} \int_{\Omega} \frac{\psi_\mu(t, s, h, k)}{\psi_\mu(t, s, h, k)} |\phi(t, s, x, y)| \, dx \, dy \, dt \, ds
\]
\[
= C \|\phi\|_{L^1(\Omega)}.
\]
by Proposition 3.14. This implies
\[ |P_{K^{(N)},h}(t, x; s, y) - P_{K^{(N)},h}(t, x + h; s, y + h)| \leq C \psi_h(t, s, h, k) = C(t - s)^{-\nu/\alpha}(|h| + |k|)^\beta, \] (4.9)
where \( C \) and \( \beta \) depend only on \( d, \alpha, C_0, \) and \( \|v\|_{X_\alpha} \). Thus \( P_{K^{(N)},h}(t, x; s, y) \) may be regarded as a Hölder continuous function with respect to the spatial variables. As for the time regularity, we set \( \tilde{\psi}_h(t, s, \tau) = (t - s)^{-\nu/\alpha} |\tau|^{\beta'} + \mu^{\nu/\alpha}, \) \( 0 < \mu < 1, \tau \in [0, 1) \), where \( c \) and \( \beta' \) are the constants in Corollary 3.15. Then by the same arguments as in (4.9) we obtain from Corollary 3.15,
\[ |P_{K^{(N)},h}(t, x; s, y) - P_{K^{(N)},h}(t + \tau(t - s), x; s, y)| \leq C_{\tau} \tau^{\beta'} (t - s)^{\nu}. \] (4.10)
This proves the Hölder continuity with respect to \( t \), since \( \tau \in [0, 1) \) is arbitrary. The continuity with respect to \( s \) is shown similarly. Collecting these above, we obtain the Hölder continuity with respect to \( (t, x, s, y) \) for \( P_{K^{(N)},h}(t, x; s, y) \). Moreover, the estimates (4.3), (4.5), and (4.10) imply
\[ P_{K^{(N)},h} f(t, s) \in C([s, \infty); L^p), \quad \text{if} \quad f \in L^p(\mathbb{R}^d), 1 \leq p < \infty. \] (4.11)
For (4.11) we will check the continuity at \( t = s \) only. By the interpolation and the density arguments based on (4.3) it suffices to consider the case \( p = 1 \). Let \( f \in C_0^\infty(\mathbb{R}^d) \) and let \( \sup \int f \subset B_{R_1}(0) \) for some \( R_1 > 0 \). Take any \( R_2 \geq \{ R_1, C R_1 \} \), where \( C R_2 \) is the constant defined in (4.7). Then from \( |f(x) - f(y)| \leq C|x - y|^{\gamma/2}, \) for \( \gamma > 0 \) \( \|f\|_{L^1} \leq C_{\gamma} \), we have
\[ \|P_{K^{(N)},h} f(t, s) - f\|_{L^1} \leq C_R \left[ \sup_{|x| \leq 2R_2} \rho_1(y - x - \xi_1(x, y, s))P_{K^{(N)},h}(t, x; s, y) dy + \sup_{|x| \leq 2R_2} |\xi_1(x, t, s)|^{\gamma} \right] \]
\[ + R_2^{-\gamma} \left[ \sup_{|x| \leq 2R_2} \rho_1(y - x - \xi_1(y, t, s))P_{K^{(N)},h}(x, t; y, s) dy \right] \]
\[ + R_2^{-\gamma} \left( |\xi_1(x, y, s)|^{\gamma} + C R_2^{-\gamma} (t - s) \right). \]

Hence, from the estimate of \( \xi_1(x, t, s) \) in Proposition 3.9 we have \( \lim_{t \to s} \|P_{K^{(N)},h} f(t, s) - f\|_{L^1} = 0 \) if \( f \in C_0^\infty(\mathbb{R}^d) \). The density arguments and (4.3) give \( \lim_{t \to s} \|P_{K^{(N)},h} f(t, s) - f\|_{L^1} = 0 \). This completes the proof of (4.11). Similarly, we also have
\[ P_{K^{(N)},h} f \in C(\overline{D}; L^p), \quad \text{if} \quad f \in L^p(\mathbb{R}^d), 1 \leq p < \infty. \] (4.12)

The details are omitted here.

(iv) Energy inequality. We next show that the energy inequality
\[ \|P_{K^{(N)},h} f(t, s)\|_{L^2}^2 + 2 \int s \frac{E_K^{(N)}}{C_{K^{(N)}}} (P_{K^{(N)},h} f) (\tau, s), (P_{K^{(N)},h} f) (\tau, s) d\tau \leq \|f\|_{L^2}^2, \] (4.13)
holds for all \( f \in L^2(\mathbb{R}^d) \) and all \( t > s \geq 0 \). Let \( \{ f_t \}_{t \geq 0} \subset L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) be a countable and dense subset of \( L^2(\mathbb{R}^d) \). Then by taking a subsequence of \( \{ \delta_n \} \) if necessary, due to the diagonal arguments, we may assume that \( \{ P_{K^{(N)},h} f_t (\cdot) \}_{t \geq 0} \} \) \( \Rightarrow \) \( (P_{K^{(N)},h} f_t (\cdot)) \) as \( n \to \infty \) in \( L^\infty(s, \infty; L^2(\mathbb{R}^d)) \) for each \( t \in \mathbb{N} \) and \( s \in \mathbb{Q} \). Fix \( \epsilon > 0, \lambda \in \mathbb{N}, s \in \mathbb{Q} \), and set
\[ G_{\delta_n, f_t}(t, x, y; s) = \left[ (P_{K^{(N),h}} f_t)(t, x; y) \right] (\nu_{\delta_n} \ast k_N(t, x, y)) \left( \frac{|x - y|}{\lambda(x - y)} \right)^{\frac{1}{2}}, \]
\[ G_{\delta_n, f_t}(t, x, y; s) = \left[ (P_{K^{(N),h}} f_t)(t, x; y) \right] \left( \frac{|x - y|}{\lambda(x - y)} \right)^{\frac{1}{2}}, \]
where \( \{ f_t \}_{t \geq 0} \) = \( f(x) - f(y) \). Then it is not difficult to see that \( G_{\delta_n, f_t}(t, x, y; s) \) is bounded in \( L^\infty(s, \infty; L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)) \) uniformly in \( n \). Since \( (P_{K^{(N),h}} f_t)(\cdot, s) \Rightarrow (P_{K^{(N),h}} f_t)(\cdot, s) \) in \( L^\infty(s, \infty; L^2(\mathbb{R}^d)) \) and \( \nu_{\delta_n} * \)
\[ k_Y(t, x, y) \to k_Y(t, x, y) \] strongly in \( L^p_{\text{loc}}((s, \infty) \times \mathbb{R}^d \times \mathbb{R}^d) \) for \( 1 \leq p < \infty \), we have \( G_{\delta_n}(f(t, x, y, s) \to G_Y(f(t, x, y, s)) \) in \( L^2_{\text{loc}}((s, \infty) ; L^2(\mathbb{R}^d)^2 \times L^2(\mathbb{R}^d)^2) \). By the lower semicontinuity of the seminorms in \( L^2_{\text{loc}}((s, \infty) ; L^2(\mathbb{R}^d)^2 \times L^2(\mathbb{R}^d)^2) \) implies that for any \( \phi(t) \in C_0(s, \infty) \) with \( \varphi \geq 0 \),

\[
\int_s^\infty \| G_x(f(t)) \|^2_{L^2(\omega,t,\mathbb{R}^d)} \phi(t) \, dt \leq \int_s^\infty \liminf_{n \to \infty} \| G_{\delta_n}(f(t)) \|^2_{L^2(\omega,t,\mathbb{R}^d)} \phi(t) \, dt
\]

\[
\leq \liminf_{n \to \infty} \int_s^\infty \| G_{\delta_n}(f(t)) \|^2_{L^2(\omega,t,\mathbb{R}^d)} \phi(t) \, dt
\]

\[
\leq \liminf_{n \to \infty} \int_s^\infty \left( \| f(t) \|^2_{L^2} - \| (P_{K(N,s_n)}f(t, s)) \|^2_{L^2} \right) \phi(t) \, dt
\]

\[
\leq \int_s^\infty \left( \| f(t) \|^2_{L^2} - \| (P_{K(N)}f(t, s)) \|^2_{L^2} \right) \phi(t) \, dt.
\]

Here we have used the energy inequality for \( (P_{K(N,s_n)}f(t, s)) \) and the lower semicontinuity in \( L^2_{\text{loc}}((s, \infty) ; L^2(\mathbb{R}^d)^2) \). This yields

\[
\| (P_{K(N)}f(t, s)) \|^2_{L^2} + \| G_x(f(t)) \|^2_{L^2(\omega,t,\mathbb{R}^d)} \leq \| f(t) \|^2_{L^2},
\]

for a.e. \( t \in (s, \infty) \) and \( s \in \mathbb{Q} \). But the fact \( P_{K(N)}f(t) \in C(\overline{\mathbb{D}}, L^2(\mathbb{R}^d)) \) implies that (4.14) holds for all \( t > s \geq 0 \). Then by the density arguments using (4.3) we conclude that

\[
\| (P_{K(N)}f(t, s)) \|^2_{L^2} + \| G_x(f(t)) \|^2_{L^2(\omega,t,\mathbb{R}^d)} \leq \| f(t) \|^2_{L^2},
\]

for all \( t > s \geq 0 \) and \( f \in L^2(\mathbb{R}^d) \). By tending \( \epsilon \) to zero in (4.15) and by using the monotone convergence theorem, we obtain (4.13).

(v) **Semigroup property.** Next we check the semigroup property

\[
(P_{K(N)} f)(t, x) = \int_\mathbb{R}^d P_{K(N)}(t, x; \tau, z) P_{K(N)}(\tau, z; y) \, dz,
\]

for \( t > \tau > s \geq 0, x, y \in \mathbb{R}^d \). It suffices to consider the case \( s > 0 \); then the case \( s = 0 \) follows from the continuity with respect to \( s \). Assume that \( t > s + 1/m, m \in \mathbb{N} \). Let \( \phi \in C_0^\mathbb{N}(\Omega_m) \) and let \( \epsilon > 0 \). We take \( \tau > 0 \) so that \( t - \tau > \tau > s + 1/(2m) \) for all \( t, s \in \cup_{x,y} \text{supp} \, \phi(t, x, y) \). Then there is \( R > 0 \) such that

\[
\int_\Omega \int_{|z| > R} P_{K(N),\nu}^1(t, x; \tau, z) P_{K(N),\nu}^0(\tau, z; s, y) \phi(t, s, x, y) \, dz \, dy \, dt \, ds < \epsilon.
\]

For sufficiently large \( l \in \mathbb{N} \) we set

\[
P_{K(N),\nu}^{(l)}(t, x; s, y) = \int_\Omega \nu_1, \nu_2 \left( t - t' \right) \nu_1, \nu_2 \left( s - s' \right) \nu_1, \nu_2 \left( x - x' \right) \nu_1, \nu_2 \left( y - y' \right) P_{K(N),\nu}^0(t', x', s', y') \, dx' \, dy' \, dt' \, ds',
\]

and

\[
P_{K(N),\nu}^{(l)}(t, x; s, y) = \int_\Omega \nu_1, \nu_2 \left( t - t' \right) \nu_1, \nu_2 \left( s - s' \right) \nu_1, \nu_2 \left( x - x' \right) \nu_1, \nu_2 \left( y - y' \right) P_{K(N),\nu}^0(t', x', s', y') \, dx' \, dy' \, dt' \, ds'.
\]

From the uniform continuity of \( P_{K(N),\nu}^0(t, x; s, y) \) on \( \overline{\Omega_m} \) we have for \( l \gg 1 \),

\[
\sup_{(x, t, s, y) \in \mathbb{R}^d \times \text{supp} \, \varphi} \left( |P_{K(N),\nu}^0(t, x; \tau, z) - P_{K(N),\nu}^{(l)}(t, x; \tau, z)| + |P_{K(N),\nu}^0(\tau, z; s, y) - P_{K(N),\nu}^{(l)}(\tau, z; s, y)| \right) < \epsilon.
\]

Furthermore, Corollaries 3.14 and 3.15 imply

\[
\sup_{(x, t, s, y) \in \mathbb{R}^d \times \text{supp} \, \varphi} \left( |P_{K(N),\nu}^0(t, x; \tau, z) - P_{K(N),\nu}^{(l)}(t, x; \tau, z)| + |P_{K(N),\nu}^0(\tau, z; s, y) - P_{K(N),\nu}^{(l)}(\tau, z; s, y)| \right) < \epsilon,
\]

for all sufficiently small \( \delta > 0 \). On the other hand, for \( l \) satisfying (4.20) we have

\[
\lim_{n \to \infty} P_{K(N,s_n),\nu}^{(l)}(t, x; \tau, z) = P_{K(N),\nu}^{(l)}(t, x; \tau, z), \quad \lim_{n \to \infty} P_{K(N,s_n),\nu}^0(\tau, z; s, y) = P_{K(N),\nu}^0(\tau, z; s, y),
\]

for every \((z, t, s, y) \in \mathbb{R}^d \times \text{supp} \, \varphi, \) and we also have the uniform bound

\[
\sup_{n \in \mathbb{N}} \sup_{(x, t, s, y) \in \mathbb{R}^d \times \text{supp} \, \varphi} \left( P_{K(N,s_n),\nu}^{(l)}(t, x; \tau, z) + P_{K(N,s_n),\nu}^{(l)}(\tau, z; s, y) \right) \leq C_m.
\]

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So the Lebesgue convergence theorem yields for each $R > 0$,

\[
\int_0^R \int_{|z| \leq R} \varphi(t, s, x, y) P^{(4.24)}_{K(N), \nu}(t, \tau, z) \nu_{\tau, z} \left( \varphi(t, s, x, y) \right) \, dz \, dy \, dt \, ds \\
= \lim_{n \to \infty} \int_0^R \int_{|z| \leq R} \varphi(t, s, x, y) P^{(4.24)}_{K(N, \delta_n), \nu}(t, \tau, z) \nu_{\tau, z} \left( \varphi(t, s, x, y) \right) \, dz \, dy \, dt \, ds. \quad (4.24)
\]

Let $0 < \epsilon < 1$. From the moment bound of $P_K(N, \delta_n, \nu)(t, x, s, y)$ in Proposition 3.4 we can take $R > 0$ large enough but independent of $n \gg 1$ so that

\[
\int_{|z| \geq R} P_{K(N, \delta_n), \nu}(t, \tau, z) P_{K(N, \delta_n), \nu}(t, s, \tau, y) \, dz \leq C\epsilon.
\]

Then from (4.21) we have

\[
\int_{|z| \leq R} P^{(4.24)}_{K(N, \delta_n), \nu}(t, \tau, z) \nu_{\tau, z} \left( \varphi(t, s, x, y) \right) \, dz \\
= \int_{|z| \leq R} P_{K(N, \delta_n), \nu}(t, \tau, z) P_{K(N, \delta_n), \nu}(t, s, \tau, y) \, dz + O(\epsilon) \\
= P_{K(N, \delta_n), \nu}(t, x, s, y) - \int_{|z| \geq R} P_{K(N, \delta_n), \nu}(t, \tau, z) P_{K(N, \delta_n), \nu}(t, s, \tau, y) \, dz + O(\epsilon) \\
= P_{K(N, \delta_n), \nu}(t, x, s, y) + O(\epsilon).
\]

Here $O(\epsilon)$ satisfies $|O(\epsilon)| \leq C\epsilon$ with $C$ independent of $n$ and $(t, s, x, y) \in \supp \varphi$. Hence from (4.17), (4.20), and (4.24), it is not difficult to see

\[
\int_{\Omega} \int_{\mathbb{R}^d} P_{K(N), \nu}(t, \tau, z) P_{K(N), \nu}(t, s, y) \varphi(t, s, x, y) \, dz \, dy \, dt \, ds \\
= \int_{\Omega} \int_{\mathbb{R}^d} P_{K(N), \nu}(t, x, s, y) \varphi(t, s, x, y) \, dz \, dy \, dt \, ds + O(\epsilon). \quad (4.25)
\]

Since $\epsilon > 0$ and $\varphi \in C_0(\Omega_m)$ is arbitrary, we get (4.16) for $t - 1/(2m) > \tau > s + 1/(2m)$. Then since $m \in \mathbb{N}$ is arbitrary, we have (4.16) for $t > \tau > s > 0$. This completes the proof of (4.16).

(vi) **End of the proof.** Finally we show that $P_{K(N), \nu}(t, x, s, y)$ is a fundamental solution to (1.1) with

\[
K(t, x, y) = K^{(N)}(t, x, y) = |x - y|^{-d - \alpha} k_N(t, x, y).
\]

For $f \in C_0^\infty(\mathbb{R}^d)$, the function $(P_{K(N), \nu} f)(t, s, z)$ satisfies

\[
\int_D < (P_{K(N), \nu} f)(t, s), \partial_t \varphi(t, s) > \, dt \, ds \\
= \lim_{n \to \infty} \int_D < (P_{K(N, \delta_n), \nu} f)(t, s), \partial_t \varphi(t, s) > \, dt \, ds \\
= \lim_{n \to \infty} \int_D \mathcal{E}_{K(N, \delta_n)}^{(4.24)}(\nu_{\tau, z} \left( \varphi(t, s) \right)) \, dt \, ds,
\]

(4.26)

for all $\varphi \in C_0^\infty(D \times \mathbb{R}^d)$. For any $\epsilon > 0$ we decompose the first term in the right hand side as

\[
\int_D \mathcal{E}_{K(N, \delta_n)}^{(4.24)}(\nu_{\tau, z} \left( \varphi(t, s) \right)) \, dt \, ds \\
= \int_D \int_{|x - z| \leq \delta_n} \frac{\nu_{\tau, z} \left( \varphi(t, s) \right)}{|x - z|^{d + \alpha}} \, dx \, dz \, dt \, ds \\
+ \int_D \int_{|x - z| \leq \delta_n} \frac{\nu_{\tau, z} \left( \varphi(t, s) \right)}{|x - z|^{d + \alpha}} \, dx \, dz \, dt \, ds \\
+ \delta_n \int_D \int_{|x - z| \leq \delta_n} \frac{\nu_{\tau, z} \left( \varphi(t, s) \right)}{|x - z|^{d + \alpha}} \, dx \, dz \, dt \, ds \\
= I_1 + II_1 + III_1.
\]
Let \( R, T > 0 \) be such that \( \text{supp} \, \varphi \subset \{(t, s, x) \in D \times \mathbb{R}^d \mid 0 \leq s < t < T, \, |x| \leq R \} \). Then we have

\[
|I| \leq N \int_D \|P_{K(N, \delta_n, \nu)} f(t, s)\|_{H^{2+\frac{d}{2}}} \left( \int_{|x-z| \leq \varepsilon} \frac{|\varphi(t, s)|^2(x, z)}{|x-z|^{d+\alpha}} \, dx \, dz \right)^{\frac{1}{2}} \, dt \, ds \\
\leq N\|f\|_{L^2} \int_0^T \left( \int_{|x-z| \leq \varepsilon} \frac{|\varphi(t, s)|^2(x, z)}{|x-z|^{d+\alpha}} \, dx \, dz \right)^{\frac{1}{2}} \, dt \\
\leq NR^2T^{\frac{d}{2}} \|f\|_{L^2} \varepsilon^{\frac{2-d}{2}}.
\]

For \( II \), we have

\[
II = \int_D \int_{|x-\xi| \geq \varepsilon} \frac{|P_{K(N, \delta_n, \nu)} f(t, s)|}{|x-\xi|^{d+\alpha}} (\nu_{\delta_n} + k_N)(t, x, \xi) \, dx \, dt \, d\xi \\
= \int_D \int_{|x-\xi| \geq \varepsilon} \frac{|P_{K(N, \delta_n, \nu)} f(t, s)|}{|x-\xi|^{d+\alpha}} (\nu_{\delta_n} + k_N)(t, x, \xi) \, dx \, dt \, d\xi \\
+ \int_D \int_{|x-\xi| \geq \varepsilon} \frac{|P_{K(N, \delta_n, \nu)} f(t, s)|}{|x-\xi|^{d+\alpha}} k_N(t, x, \xi) \, dx \, d\xi \, dt \\
= II_{1, 1} + II_{1, 2},
\]

and

\[
|II_{1, 1}| \leq 2N^2 \|f\|_{L^2} T \int_0^T \int_{|x-\xi| \geq \varepsilon} \frac{|\varphi(t, s)|^2(x, z)}{|x-z|^{d+\alpha}} (\nu_{\delta_n} + k_N)(t, x, z) \, dx \, dz \, dt \, d\xi \, ds \\
\to 0 \quad \text{as} \quad n \to \infty,
\]

by the Lebesgue convergence theorem, since \((\nu_{\delta_n} + k_N)(t, x, \xi)\) converges to \(k_N(t, x, \xi)\) for a.e. \((t, x, \xi)\). As for \(II_{1, 2}\), we claim that

\[
II_{1, 2} \to \int_0^\infty \int_{|x-\xi| \geq \varepsilon} \frac{|P_{K(N, \delta_n, \nu)} f(t, s)|}{|x-\xi|^{d+\alpha}} k_N(t, x, \xi) \, dx \, d\xi \, ds,
\]

as \( n \to \infty \). Indeed, first we observe that

\[
\int_D \int_{|x-\xi| \geq \varepsilon} \frac{|P_{K(N, \delta_n, \nu)} f(t, s, x)|}{|x-z|^{d+\alpha}} k_N(t, x, z) \, dx \, dz \, dt \\
= \int_D \int_{\mathbb{R}^{2d}} P_{K(N, \delta_n, \nu)} f(t, s, y) \varphi(t, s, y) \int_{|x-\xi| \geq \varepsilon} \frac{k_N(t, x, z)}{|x-z|^{d+\alpha}} \, dy \, dz \, dt \\
= \int_D \int_{\mathbb{R}^{2d}} P_{K(N, \delta_n, \nu)} f(t, s, y) \varphi(t, s, y) \, dz \int_{|x-\xi| \geq \varepsilon} \frac{k_N(t, x, z)}{|x-z|^{d+\alpha}} \, dy \, dt \\
\to \int_D \int_{\mathbb{R}^{2d}} P_{K(N, \delta_n, \nu)} f(t, s, y) \varphi(t, s, y) \, dz \int_{|x-\xi| \geq \varepsilon} \frac{k_N(t, x, z)}{|x-z|^{d+\alpha}} \, dy \, dt,
\]

as \( n \to \infty \), since \(\varphi(t, s, x)\int_{|x-\xi| \geq \varepsilon} \frac{k_N(t, x, z)}{|x-z|^{d+\alpha}} \, dz \, dy \in L^1(\Omega_m)\) for some \(m \geq 1\). Next we see

\[
\int_D \int_{|x-\xi| \geq \varepsilon} \frac{|P_{K(N, \delta_n, \nu)} f(t, s, z)|}{|x-\xi|^{d+\alpha}} k_N(t, x, z) \, dx \, dz \, dt \\
= \int_D \int_{\mathbb{R}^{2d}} P_{K(N, \delta_n, \nu)} f(t, s, y) \varphi(t, s, y) \int_{|x-\xi| \geq \varepsilon} \frac{\varphi(t, s, z)k_N(t, x, z)}{|x-z|^{d+\alpha}} \, dy \, dz \, dt \\
= \int_D \int_{\mathbb{R}^{2d}} P_{K(N, \delta_n, \nu)} f(t, s, y) \varphi(t, s, y) \, dz \int_{|x-\xi| \geq \varepsilon} \frac{\varphi(t, s, z)k_N(t, x, z)}{|x-z|^{d+\alpha}} \, dy \, dt \\
\to \int_D \int_{\mathbb{R}^{2d}} P_{K(N, \delta_n, \nu)} f(t, s, y) \varphi(t, s, y) \, dz \int_{|x-\xi| \geq \varepsilon} \frac{\varphi(t, s, z)k_N(t, x, z)}{|x-z|^{d+\alpha}} \, dy \, dt,
\]

as \( n \to \infty \), for \(f(y)\int_{|x-\xi| \geq \varepsilon} \frac{\varphi(t, s, z)k_N(t, x, z)}{|x-z|^{d+\alpha}} \, dz \, dy \in L^1(\Omega_m)\) for some \(m \geq 1\). Collecting these two, we get

(4.27). Next we consider the term \(III\). We rewrite it as

\[
III = C\delta_n \int_D \int_{\mathbb{R}^{d}} (P_{K(N, \delta_n, \nu)} f)(t, s, x) ((-\Delta)^{1+\frac{d}{2}} \varphi)(t, s, x) \, dx \, dt \, ds,
\]

and hence,

\[
|III| \leq C\delta_n \|f\|_{L^2} \int_0^T \int_s^T \|\varphi(t, s)\|_{H^{1+\frac{d}{2}}} \, dt \, ds \to 0, \quad \text{as} \quad n \to \infty.
\]

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On the other hand, it is not difficult to see
\[
\lim_{n \to \infty} \int_D \mathcal{E}_{v(t)}((P_{K(v,N),v}f)(t,s), \phi(t,s)) \, dt \, ds = \int_D \mathcal{E}_{v(t)}((P_{K(N),v}f)(t,s), \phi(t,s)) \, dt \, ds.
\]
Collecting these above, by taking \( n \to \infty \) first then \( \epsilon \to 0 \) in (4.26) we obtain
\[
\int_D < (P_{K(N),v}f)(t,s), \partial_t \phi(t,s) > \, dt \, ds
= \int_D \mathcal{E}^{(t)}_{K(N)}((P_{K(N),v}f)(t,s), \phi(t,s)) + \mathcal{E}_{v(t)}((P_{K(N),v}f)(t,s), \phi(t,s)) \, dt \, ds.
\] (4.28)
We take \( \phi(t,s) \) of the form \( \phi(t,x)\phi_m(t,s)\psi(s) \) in (4.28), where \( \phi \in C^\infty_0([0, \infty) \times \mathbb{R}^d), \psi \in C^\infty_0([0, \infty)), \) and \( \phi_m \in C^\infty_0(D), \phi \geq 0, \) is such that
\[
\phi_m(t,s) = \begin{cases} 1 & \text{for } t \geq s + 2/m, \\ 0 & \text{for } t \leq s + 1/m, \end{cases}
\] (4.29)
for \( m \geq 1. \) Then setting \( \varphi_m(t,s,x) = \phi(t,x)\phi_m(t,s) \), we have
\[
0 = \int_0^\infty \psi(s) \left( \int_0^\infty \{ < (P_{K(N),v}f)(t,s), \partial_t \varphi_m(t,s) > - \mathcal{E}^{(t)}_{K(N)}((P_{K(N),v}f)(t,s), \varphi_m(t,s)) \right.
- \mathcal{E}_{v(t)}((P_{K(N),v}f)(t,s), \varphi_m(t,s)) \} \, dt \right) \, ds,
\]
which implies for a.e. \( s \in [0, \infty), \)
\[
\int_0^\infty (P_{K(N),v}f)(t,s), \partial_t \varphi_m(t,s) > dt
= \int_0^\infty \left( \mathcal{E}^{(t)}_{K(N)}((P_{K(N),v}f)(t,s), \varphi_m(t,s)) + \mathcal{E}_{v(t)}((P_{K(N),v}f)(t,s), \varphi_m(t,s)) \right) \, dt.
\] (4.30)
But since each term in (4.30) is continuous with respect to \( s, \) (4.30) is valid for all \( s \in [0, \infty). \) Then by standard arguments we can take the limit \( m \to \infty \) to get
\[
\int_0^\infty (P_{K(N),v}f)(t,s), \partial_t \varphi(t) > dt + < f, \varphi(s) >
= \int_0^\infty \left( \mathcal{E}^{(t)}_{K(N)}((P_{K(N),v}f)(t,s), \varphi(t)) + \mathcal{E}_{v(t)}((P_{K(N),v}f)(t,s), \varphi(t)) \right) \, dt.
\] (4.31)
For a general initial data \( f \in L^2(\mathbb{R}^d) \) it suffices to approximate \( f \) by \( \{ f_n \}_{n=1}^\infty \subset C^\infty_0(\mathbb{R}^d). \) Then \( (P_{K(N),v}f)(t,s) \) is shown to satisfy (4.31) by using (4.3) and (4.13). Hence \( P_{K(N),v}(t,x,s,y) \) is a fundamental solution to (1.1) with \( K(t,x,y) = K(N)(t,x,y). \)

4.2 Limit \( N \to \infty \)

Let \( \{ v(N) \} \) be the approximation of \( v \) obtained by Lemma 2.4. In this section we take the limit \( N \to \infty \) of \( P_{K(v,N),v,N}(t,x,s,y) \) and complete the proof of Theorems 1.2 and 1.4. From the continuity estimates (4.9)-(4.10), by taking a subsequence if necessary, \( P_{K(v,N),v,N}(t,x,s,y) \) converges to a function \( P_{K,v}(t,x,s,y) \) uniformly on each compact subset of \( \Omega = D \times \mathbb{R}^d \times \mathbb{R}^d. \) From (4.3) and (4.4) we easily see that \( (P_{K,v}f)(t,s,x) = \int_{\mathbb{R}^d} P_{K,v}(t,x,s,y)f(y) \, dy \) satisfies
\[
\| P_{K,v}f \|_{L^\infty(D,L^p)} \leq \| f \|_{L^p}, \quad 1 \leq p \leq \infty,
\] (4.32)
\[
P_{K,v}(t,x,s,y) \leq C(t-s)^{-\frac{d}{2}},
\] (4.33)
and by the interpolation, we also have
\[
\| (P_{K,v}f)(t,s) \|_{L^q} \leq C(t-s)^{-(\frac{d+1}{2}-\frac{1}{p})}\| f \|_{L^p}, \quad 1 \leq q \leq p \leq \infty.
\] (4.34)
Note that \( C \) in (4.33) and (4.34) is taken depending only on \( d, \alpha, \) and \( C_0. \) Next we show the moment bound, i.e., for any \( x \in \mathbb{R}^d, t \geq s \geq 0 \) there is \( \xi(t,x,t,s) \in \mathbb{R}^d \) such that
\[
C_1(t-s)^{-\frac{d}{2}} \leq \int_{\mathbb{R}^d} \rho_\gamma(y-x-\xi(t,x,t,s)) P_{K,v}(t,x,s,y) \, dy \leq C_2(t-s)^{-\frac{d}{2}},
\] (4.35)
where $\gamma \in (0, \min\{\alpha, 2 - \alpha\})$. Here $C_1$ and $C_2$ are the constants in (4.5) depending only on $d$, $\alpha$, $C_0$, $\gamma$, and $\|v\|_{X_\lambda}$. Indeed, by Proposition 3.9 and Lemma 2.4 the vector $\xi(t,x,s) \in \mathbb{R}^d$ in (4.5) satisfies
\[
\sup_{s \leq r, 0 \leq t \leq R_0} \|\xi(t,x,s)\|_0 \leq C_0 R_0
\] for each $R_0 > 0$, where $C_0 R_0$ is independent of $N$. Hence for any $(x,t,s) \in \mathbb{R}^d$ there is $\xi(t,x,s) \in \mathbb{R}^d$ such that $\xi(t,x,s) \to \xi(t,x,s)$ as $N \to \infty$ for a subsequence $\{N^\prime\} \subset \{N\}$. Then the upper bound of (4.35) follows from (4.5) and the Fatou lemma. The lower bound is obtained from the same (or simpler) arguments as in the proof of (4.5), so we omit the details here. From (4.6) we also have for any $(y,t,s)$ there is $\eta^\ast(y,t,s) \in \mathbb{R}^d$ satisfying
\[
C_1(t-s)^{\frac{3}{35}} \leq \int_{\mathbb{R}^d} \rho_\gamma(x-y-\eta^\ast(y,t,s)) P_{K,v}(t,x,s,y) \, dx \leq C_2(t-s)^{\frac{3}{35}},
\] (4.36)
where $\gamma \in (0, \min\{\alpha, 2 - \alpha\})$. From the above proof and by Lemma 2.4 and Remark 3.10 we observe that $\xi(t,x,s)$ and $\eta^\ast(t,x,s)$ in (4.35) and (4.36) satisfy
\[
\left|\xi^\ast(t,x,s) + |\eta^\ast(t,x,s)|\right| \leq \begin{cases} 
C_1 (T^{\frac{1}{2}}(1 - \frac{2}{\alpha}) + 1)(t-s)^{\frac{1}{2}} + C_2 (t-s)^{1 - \frac{1}{\alpha} \frac{1}{2}} \|v\|_{Y^\ast_{\infty,0}} & \text{if } \lambda \in \left[\frac{2d}{\alpha} - d, d\right), \\
C_1 (t-s)^{\frac{1}{2}} \left(\|\log(t-s)\| + \log(e + |x|)\right) + C_2 (t-s)^{1 - \frac{1}{\alpha - d} \frac{1}{2}} \|v\|_{Y^\ast_{\infty,0}} & \text{if } \lambda = d, \\
C_1 (t-s)^{\frac{1}{2}} + \|v\|_{X_{\lambda}(t,s)}(t-s)^{\frac{1}{2} + \frac{1}{\alpha} \frac{1}{2}} + C_2 (t-s)^{1 - \frac{1}{\alpha} \frac{1}{2}} \|v\|_{Y^\ast_{\infty,0}} & \text{if } \lambda \in (d, \frac{2d}{\alpha} + d].
\end{cases}
\] (4.37)
Here $T \geq t > s \geq 0$, and $C_1$ depends only on $d$, $\alpha$, $\lambda$, and $\|v\|_{X_{\lambda}}$, and $C_2$ depends only on $d$ and $\alpha$. The similar argument as in the proof of (4.8) leads to
\[
\int_{\mathbb{R}^d} P_{K,v}(t,x; s, y) \, dy = \int_{\mathbb{R}^d} P_{K,v}(t,x; s, y) \, dx = 1,
\] (4.38)
and the details are omitted. The semigroup property
\[
P_{K,v}(t,x; s, y) = \int_{\mathbb{R}^d} P_{K,v}(t, \tau; z) P_{K,v}(\tau, z; s, y) \, dz \quad \text{for all } t > \tau > s \geq 0, \quad x, y \in \mathbb{R}^d,
\] (4.39)
is obtained by dividing the integral
\[
\int_{\mathbb{R}^d} P_{K,v}(t, \tau; z) P_{K,v}(\tau, z; s, y) \, dz = \int_{|z| \leq R} P_{K,v}(t, \tau; z) P_{K,v}(\tau, z; s, y) \, dz
\]
\[
+ \int_{|z| \geq R} P_{K,v}(t, \tau; z) P_{K,v}(\tau, z; s, y) \, dz,
\]
and then by using the moment bounds (4.5) - (4.6) and (4.35) - (4.36), the convergence of $P_{K,v}(t, x, s)$ to $P_{K,v}(t, x, s, y)$, and the semigroup property (4.16). Moreover, by arguing as in (4.11) we can show
\[
(P_{K,v}(\tau, f) (t, s) \in C(\overline{\mathbb{R}}^d; L^p) \quad \text{if } f \in L^p(\mathbb{R}^d), \quad 1 \leq p < \infty.
\] (4.40)
The details are left to the reader and omitted here.

Next we show that $P_{K,v}(t, x, s, y)$ is a fundamental solution to (1.1). For all $f \in C_0^\infty(\mathbb{R}^d)$ we have from (4.31),
\[
\int_s^\infty < (P_{K,v}(\varphi) f)(t,s), \partial_t \varphi(t) > \, dt + < f, \varphi(s) > = \int_s^\infty \left( \xi^\ast(t) \left( (P_{K,v}(\varphi) f)(t,s), \varphi(t) \right) + \varphi^\ast(t) \left( (P_{K,v}(\varphi) f)(t,s), \varphi(t) \right) \right) \, dt,
\] (4.41)
for all $\varphi \in C_0^\infty([s, \infty) \times \mathbb{R}^d)$. From the fact that $P_{K,v}(\varphi) f(t, x, s, y)$ converges to $P_{K,v}(t, x, s, y)$ uniformly in each compact subset of $\Omega$ and the uniform bound (4.3), it is easy to see that
\[
\int_s^\infty < (P_{K,v}(\varphi) f)(t,s), \partial_t \varphi(t) > \, dt \rightarrow \int_s^\infty < (P_{K,v} f)(t,s), \partial_t \varphi(t) > \, dt, \quad N \to \infty.
\]
Similarly, since $v^N \to v$ in $L^1_{\operatorname{loc}}((0, \infty) \times \mathbb{R}^d)$ we have
\[
\int_s^\infty \xi^\ast(t) \left( (P_{K,v}(\varphi) f)(t,s), \varphi(t) \right) \, dt \rightarrow \int_s^\infty \xi^\ast(t) \left( (P_{K,v} f)(t,s), \varphi(t) \right) \, dt, \quad N \to \infty.
\]
Next we consider the first term of the right-hand side of (4.41). For any \( \epsilon > 0 \), we have

\[
\int_s^\infty \mathcal{E}_{K}^{(t)}(\{(P_{K,v}(t,s)f)(t,s), \varphi(t)\}) \, dt \\
= \int_s^\infty \int_{|x-y| \leq s} \frac{|(P_{K,v}(t,s)f)(t,s)| \varphi(t) |x-y| \, k_{K}^{(N)}(t,x,y) \, dx \, dy}{|x-y|^{d+\alpha}} \, dt \\
+ \int_s^\infty \int_{|x-y| \geq s} |(P_{K,v}(t,s)f)(t,s)| \varphi(t) |x-y| \, k_{K}^{(N)}(t,x,y) \, dx \, dy \, dt \\
= I_1 + I_2.
\]

Let \( \text{supp} \varphi \subseteq [0,T) \times B_{\epsilon}(0) \). From (4.13), \( |x-y|^{-d-\alpha} k_{K}^{(N)}(t,x,y) \leq K(t,x,y) \), and (1.5), we have

\[
|I_1| \leq \left( \int_s^T \mathcal{E}_{K}^{(t)}((P_{K,v}(t,s)f)(t,s), (P_{K,v}(t,s)f)(t,s)) \, dt \right)^{\frac{1}{2}} \left( \int_s^T \mathcal{E}_{K}^{(t)}(\varphi(t), \varphi(t)) \, dt \right)^{\frac{1}{2}} \\
\leq \|f\|_{L^2} \left( \int_s^T \int_{|x-y| \leq s}, |x-y|^{d+\alpha} k_{K}^{(N)}(t,x,y) \, dx \, dy \, dt \right)^{\frac{1}{2}} \\
\leq CT \|f\|_{L^2} \left( \sup_{t>0, \epsilon \in \mathbb{R}^d \setminus \mathbb{R}^d} \int \frac{|x-y|^2 K_\epsilon(t,x,y) \, dy}{|x-y|^{d+\alpha}} \right)^{\frac{1}{2}} \\
\leq CT \|f\|_{L^2} \epsilon^{-\alpha} \|f\|_{L^2}.
\]

On the other hand, since

\[
\frac{[(P_{K,v}(t,s)f)(t,s)] \varphi(t)|x-y| \, k_{K}^{(N)}(t,x,y)}{|x-y|^{d+\alpha}} \rightarrow \frac{[P_{K,v}f](t,s)|\varphi(t)|(x,y)K(t,x,y)}{|x-y|^{d+\alpha}}
\]

for a.e. \( (t,x) \in (s, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \), and

\[
|[(P_{K,v}(t,s)f)(t,s)] |\varphi(t)| |x-y| k_{K}^{(N)}(t,x,y)| \leq 2\|f\|_{L^\infty} \left( |\varphi(t,x)| + |\varphi(t,y)| \right) K(t,x,y),
\]

where the right-hand side is integrable over \( \{(t,x,y) \in (s, \infty) \times \mathbb{R}^d \times \mathbb{R}^d | |x-y| \geq \epsilon \} \) due to (2.11), by the Lebesgue convergence theorem we have

\[
I_2 \rightarrow \int_s^\infty \int_{|x-y| \geq \epsilon} [(P_{K,v}(t,s)f)(t,s)] |\varphi(t)|(x,y)K(t,x,y) \, dx \, dy \, dt.
\]

The Fatou lemma and (4.13) imply

\[
\int_s^T \mathcal{E}_{K}^{(t)}((P_{K,v}f)(t,s), (P_{K,v}f)(t,s)) \, dt \\
\leq \int_s^T \int_{\mathbb{R}^d} \lim_{N \to \infty} \frac{[(P_{K,v}(t,s)f)(t,s)]^2 |x-y| k_{K}^{(N)}(t,x,y)}{|x-y|^{d+\alpha}} \, dx \, dy \, dt \\
\leq \liminf_{N \to \infty} \int_s^T \int_{\mathbb{R}^d} \frac{[(P_{K,v}(t,s)f)(t,s)]^2 |x-y| k_{K}^{(N)}(t,x,y)}{|x-y|^{d+\alpha}} \, dx \, dy \, dt \\
\leq \liminf_{N \to \infty} \|f\|^2_{L^2} - \|(P_{K,v}(t,s)f)(T, s)\|^2_{L^2} \\
\leq \|f\|^2_{L^2} < \infty.
\]

In particular, the integral of the left-hand side of (4.42) converges absolutely. By taking \( \epsilon \to 0 \) after the limit \( N \to \infty \), we finally obtain

\[
\int_s^\infty \mathcal{E}_{K}^{(t)}((P_{K,v}(t,s)f)(t,s), \varphi(t)) \, dt \to \int_s^\infty \mathcal{E}_{K}^{(t)}((P_{K,v}f)(t,s), \varphi(t)) \, dt,
\]

as \( N \to \infty \). Collecting these above, we conclude that \( (P_{K,v}f)(t,s,x), f \in C^\infty_0(\mathbb{R}^d) \), satisfies

\[
\int_s^\infty <(P_{K,v}f)(t,s), \partial_s \varphi(t)> \, dt + <f, \varphi(s)> \\
= \int_s^\infty \left( \mathcal{E}_{K_v}^{(t)}((P_{K,v}f)(t,s), \varphi(t)) + \mathcal{E}_{\varphi(t)}((P_{K,v}f)(t,s), \varphi(t)) \right) \, dt,
\]

(4.43)
for all $\varphi \in C_0^\infty ([s, \infty) \times \mathbb{R}^d)$. By the density arguments using (4.32) and (4.42), the equation (4.43) holds also for all $f \in L^2 (\mathbb{R}^d)$. This proves that $P_{K,v}(t, s, y)$ is a fundamental solution to (1.1). We note that the following energy inequality also holds from (4.42):

$$
\| (P_{K,v} f)(t, s) \|^2_{L^2} + 2 \int_s^t \mathcal{E}^{(2)}_{K,v}( (P_{K,v} f)(\tau, s), (P_{K,v} f)(\tau, s) ) \, d\tau \leq 2 \| f \|^2_{L^2}, \quad \text{for all } t > s \geq 0. \tag{4.44}
$$

Now Theorem 1.2 has been proved. Finally we prove Theorem 1.4 by using the moment bound and the semigroup property. Let $\eta^* (y) = \eta^* (y, t/2 + \alpha + s, \gamma + s)$ and $\zeta^* (x) = \zeta^* (x, t + \alpha, t/2 + \gamma)$ be the vectors in (4.36) and (4.35), respectively. From the semigroup property we have for $|x| + \gamma \leq R_0$ and $s \in [0, t/4]$

$$
P_{K,v}(t + \alpha, t/2 + \gamma, s + \gamma, y) = \int_{|z| \leq \eta^*} P_{K,v}(t + \alpha, t/2 + \gamma, z) P_{K,v}(t/2 + \gamma, z; s + \gamma, y) \, dz
$$

$$
= \int_{|z - y - \eta^*(y)| \leq \frac{\eta}{2}} P_{K,v}(t + \alpha, t/2 + \gamma, z) P_{K,v}(t/2 + \gamma, z; s + \gamma, y) \, dz
$$

$$
+ \int_{|z - y - \eta^*(y)| \geq \frac{\eta}{2}} P_{K,v}(t + \alpha, t/2 + \gamma, z) P_{K,v}(t/2 + \gamma, z; s + \gamma, y) \, dz
$$

$$
= I_1 + I_2. \tag{4.45}
$$

By (4.37) there is $h_0 > 0$ such that when $|y| \geq R \gg R_0 \geq 1$ and $h_0 \geq t > s \geq 0$ then $|y + \eta^* (y)| \geq |y| - |\eta^* (y)| \geq R/2$. The number $h_0$ has to be taken small enough only when $\lambda = 2d/\alpha + d$; in this case (4.37) implies

$$
|\eta^* (y)| = |\eta^* (y, t/2 + \alpha + s, \gamma + s)| \leq C(C' + \| v \|_{L^1 (\alpha + t/2 + \gamma, t\gamma + R_0) R}) \quad \text{for } t + \gamma \leq R_0 + h_0,
$$

where $C$ depends only on $d, \alpha, \lambda$, and $\| v \|_{X_\lambda}$, and $C'$ depends only on $R_0 + h_0, d, \alpha, \lambda, q, \| v \|_{X_\lambda}$, and $\| v \|_{Y_{\alpha, \lambda}^{R_0 + h_0, 0}}$. Note that if $\sup_{0 \leq s \leq R_0} \| v \|_{L^1 (\alpha + t/2 + \gamma, t\gamma + R_0)} \ll 1$ if $0 \leq s < t \leq h_0 \ll 1$. From the moment bound (4.35) we have

$$
I_1 \leq \int_{|z| \geq \frac{\eta}{2}} P_{K,v}(t + \alpha, t/2 + \gamma, z) P_{K,v}(t/2 + \gamma, z; s + \gamma, y) \, dz
$$

$$
\leq C(t + \frac{\alpha}{2} - s) \frac{\eta}{2} \int_{|z| \geq \frac{\eta}{2}} P_{K,v}(t + \alpha, t/2 + \gamma, z) \, dz
$$

$$
\leq Ct\frac{\alpha}{2} \int_{|z - x - \zeta^*(x)| \geq \frac{\eta}{2}} P_{K,v}(t + \alpha, t/2 + \gamma, z) \, dz
$$

$$
\leq Ct\frac{\alpha}{2} R_*^2 \int_{|z - x - \zeta^*(x)| \geq \frac{\eta}{2}} \rho_+(z - x - \zeta^*(x)) P_{K,v}(t + \alpha, t/2 + \gamma, z) \, dz
$$

$$
\leq Ct\frac{\alpha}{2} \frac{\eta^2}{2} R_*^2. \tag{4.46}
$$

Here $R$ is taken large enough depending on $R_0, h_0, \| v \|_{X_\lambda}$, and $\| v \|_{Y_{\alpha, \lambda}^{R_0 + h_0, 0}}$, but not on $s \in [0, t/4]$. The term $I_2$ in (4.45) is similarly estimated by the moment bound (4.36) as

$$
I_2 \leq Ct\frac{\alpha}{2} R_*^2 \int_{|z - y - \eta^*(y)| \leq \frac{\eta}{2}} \rho_+(z - y - \zeta^*(y)) P_{K,v}(z, t/2 + \alpha + s + \gamma, y) \, dz
$$

$$
\leq Ct\frac{\alpha}{2} \frac{\eta^2}{2} R_*^2, \tag{4.47}
$$

if $s \in [0, t/4]$. The estimates (4.46) and (4.47) imply (1.28). The estimate (1.29) is proved in the same way. This completes the proof.

5 Uniqueness of weak solutions for subcritical and critical cases

In this section we study the uniqueness of weak solutions to (1.1) in the case $\alpha \in [1, 2)$. The argument below is just based on the energy inequality, and it might be difficult to adapt it for the supercritical case. For simplicity we consider the special case of (C1) only; see Theorem 5.5.
5.1 Uniqueness for \(L^2(\mathbb{R}^d)\) initial data

We first prove the energy equality of weak solutions when the initial time \(s\) is positive.

**Theorem 5.1** Let \(\alpha \in \{1, 2\}\) and \(T > s \geq 0\). Assume that \(v \in L^\infty(s, T; (L^\infty(\mathbb{R}^d))^d)\) and \(\nabla \cdot v(t) = 0\). Let \(\theta \in L^\infty(\mathbb{R}^d)\) be a weak solution to (1.1) for \(t \in [s, T)\) with the initial data \(f \in L^2(\mathbb{R}^d)\) at the initial time \(t = s\). Assume in addition that there are \(C, c > 0\) satisfying

\[
esup_{|h| < \epsilon} K(t, x + h, y + h) \leq C K(t, x, y) \quad \text{for a.e. } t \in (s, T), \quad |x - y| < c.
\]

Then

\[
\|\theta(t)\|_{L^2}^2 + 2 \int_s^T \mathcal{E}^{(t)}_K(\theta(t), \theta(t)) \, dt = \|f\|_{L^2}^2,
\]

for a.e. \(t \in (s, T)\).

**Remark 5.2** For example, (5.1) is satisfied in the following two cases.

(i) \(C^{-1}|x - y|^{-d-\alpha} \leq K(t, x, y) \leq C|x - y|^{-d-\alpha}\),

(ii) \(K(t, x, y)\) is of the form \(K(t, x, y) = \mathcal{C}(t, x, y)\) is positive.

In the appendix we give an example of \(K\) which satisfies (ii) but not (i).

**Remark 5.3** The energy equality (5.2) actually holds for all \(t \in (s, T)\) by the uniqueness of weak solutions obtained from Theorem 5.1 (see Corollary 5.4) and the continuity (4.40).

**Proof.** By setting \(\theta(t) = f\) for \(\tau \leq s\) we first extend \(\theta(t)\) to the time \(\tau \in (-\infty, T)\). Fix \(t_0 \in (s, T)\) and for \(0 < \epsilon \ll 1, \epsilon + T - t_0 + 4\epsilon\) we set

\[
\theta_{m,\epsilon'}(t, x) = \int_{-\infty}^T \nu_{1,\epsilon}(t - \tau) \int_{\mathbb{R}^d} \nu_{d,\epsilon'}(x - y) \theta(\tau, y) \, dy \, d\tau,
\]

\[
\theta_{m,\epsilon'}(t, x) = (1 - \phi_m(t, t_0)) \theta_{m,\epsilon'}(t, x).
\]

Here \(\phi_m\) is the function defined by (4.29) and \(m \gg 1\). Note that if \(0 < \epsilon \ll 1\) and \(m \gg 1\) then the function \((1 - \phi_m(t, t_0)) \nu_{1,\epsilon}(t - \tau) \nu_{d,\epsilon'}(x - y)\) is taken as a test function with variables \((\tau, y)\) in (2.4). Thus we get

\[
\partial_t \theta_{m,\epsilon'}(t, x) = -(1 - \phi_m(t, t_0)) \int_s^T \nu_{1,\epsilon}(t - \tau) \mathcal{E}^{(t)}(\theta(t), \nu_{d,\epsilon'}(x - \cdot)) \, d\tau
\]

\[
- \int_{s}^{T} \nu_{1,\epsilon}(t - \tau) \mathcal{E}^{(t)}(\theta(t), \nu_{d,\epsilon'}(x - \cdot)) \, d\tau - \partial_t \theta_{m,\epsilon'}(t, x).
\]

This equality yields

\[
\int_{s}^{T} \nu_{1,\epsilon}(t - \tau) \mathcal{E}^{(t)}(\theta(t), \nu_{d,\epsilon'}(x - \cdot)) \, d\tau = \int_{s}^{T} \nu_{1,\epsilon}(t - \tau) \mathcal{E}^{(t)}(\theta(t), \nu_{d,\epsilon'}(x - \cdot)) \, d\tau.
\]

Hence by integrating over \(\mathbb{R}^d\) and then taking the limit \(\epsilon \to 0\) and \(m \to \infty\), we have for a.e. \(t_0 \in (s, T)\),

\[
- \|\nu_{d,\epsilon'} * \theta(t_0)\|_{L^2}^2 + \|\nu_{d,\epsilon'} * f(s)\|_{L^2}^2 = 2 \int_{s}^{T} \mathcal{E}^{(t)}(\theta(t), \nu_{d,\epsilon'} * \theta(t)) \, d\tau + 2 \int_{s}^{T} \mathcal{E}^{(t)}(\theta(t), \nu_{d,\epsilon'} * \theta(t)) \, d\tau
\]

\[
= 2 \int_{s}^{T} \mathcal{E}^{(t)}(\theta(t), \theta(t)) \, d\tau + 2 \int_{s}^{T} \mathcal{E}^{(t)}(\theta(t), \nu_{d,\epsilon'} * \theta(t)) \, d\tau + 2 \int_{s}^{T} \mathcal{E}^{(t)}(\theta(t), \nu_{d,\epsilon'} * \theta(t)) \, d\tau.
\]

Here we have used \(\mathcal{E}^{(t)}(\nu_{d,\epsilon'} * \theta(t), \nu_{d,\epsilon'} * \theta(t)) = 0\) due to \(\nabla \cdot v(t) = 0\). Clearly, the left-hand side of (5.3) converges to \(\|\theta(t_0)\|_{L^2}^2 - \|f(s)\|_{L^2}^2\) as \(\epsilon' \to 0\). As for the second term of the right-hand side of (5.3), let us prove that

\[
\lim_{\epsilon' \to 0} \int_{s}^{T} \mathcal{E}^{(t)}(\nu_{d,\epsilon'} * \theta(t), \nu_{d,\epsilon'} * \theta(t)) \, d\tau = 0.
\]
We divide the integral as
\[
\int_s^T \mathcal{E}_K^{(1)}(\nu_{d,\epsilon'} \ast \theta(\tau) - \theta(\tau), \nu_{d,\epsilon'} \ast \theta(\tau) - \theta(\tau)) \, d\tau \\
= \int_s^T \int_{|x-y| \leq \delta} [\nu_{d,\epsilon'} \ast \theta(\tau) - \theta(\tau)]^2(x, y) K(\tau, x, y) \, dx \, dy \, d\tau \\
+ \int_s^T \int_{|x-y| \geq \delta} [\nu_{d,\epsilon'} \ast \theta(\tau) - \theta(\tau)]^2(x, y) K(\tau, x, y) \, dx \, dy \, d\tau.
\]
(5.5)

It is easy to see that for a fixed \( \delta > 0 \) the second term of the right-hand side of (5.5) goes to zero as \( \epsilon' \to 0 \) by using (2.11). As for the first term of the right-hand side of (5.5), since
\[
[\nu_{d,\epsilon'} \ast \theta(\tau) - \theta(\tau)]^2(x, y) \leq C \int_{\mathbb{R}^d} [\theta(\tau, \cdot - \epsilon' z) - \theta(\tau, \cdot)]^2(x, y) \nu_d(z) \, dz
\]
\[
\leq C \int_{\mathbb{R}^d} [\theta(\tau, \cdot - \epsilon' z)]^2(x, y) \nu_d(z) \, dz + C' [\theta(\tau)]^2(x, y),
\]
we have from (5.1),
\[
\int_s^T \int_{|x-y| \leq \delta} [\nu_{d,\epsilon'} \ast \theta(\tau) - \theta(\tau)]^2(x, y) K(\tau, x, y) \, dx \, dy \, d\tau
\]
\[
\leq C \int_s^T \int_{|x-y| \leq \delta} \int_{\mathbb{R}^d} [\theta(\tau)]^2(x, y) \nu_d(z) K(\tau, x + \epsilon' z, y + \epsilon' z) \, dz \, dx \, dy \, d\tau
\]
\[
+ C \int_s^T \int_{|x-y| \leq \delta} [\theta(\tau)]^2(x, y) K(\tau, x, y) \, dx \, dy \, d\tau
\]
\[
\leq C \int_s^T \int_{|x-y| \leq \delta} [\theta(\tau)]^2(x, y) K(\tau, x, y) \, dx \, dy \, d\tau.
\]
(5.6)

The right-hand side of (5.6) tends to 0 as \( \delta \to 0 \) due to the condition \( \int_s^T \mathcal{E}_K^{(1)}(\theta(\tau), \theta(\tau)) \, d\tau < \infty \). Hence we get (5.4), which implies the second term in the right-hand side of (5.3) goes to zero as \( \epsilon' \to 0 \). As for the third term of the right-hand side of (5.3), we first observe that
\[
\|\theta(\tau) - \nu_{d,\epsilon'} \ast \theta(\tau)\|^2_{L^2} \leq \int_{\mathbb{R}^d} \nu_d(y) \int_{\mathbb{R}^d} |\theta(\tau, x) - \theta(\tau, x - \epsilon' y)|^2 \, dx \, dy
\]
\[
= \int_{\mathbb{R}^d} \nu_d(y) \int_{\mathbb{R}^d} |\theta(\tau, \xi) - e^{-i\epsilon' y \cdot \xi} \hat{\theta}(\tau, \xi)|^2 \, d\xi \, dy
\]
\[
\leq C(\epsilon' M)^2 \int_{|\xi| \leq 2} \nu_d(y) \int_{|\xi| \leq M} \left| \frac{1 - e^{-i\epsilon' y \cdot \xi}}{i\epsilon' y \cdot \xi} \right|^2 |\hat{\theta}(\tau, \xi)|^2 \, d\xi \, dy
\]
\[
+ C\epsilon' \alpha \int_{|\xi| \leq 2} \nu_d(y) \int_{|\xi| \leq M} \left| \frac{1 - e^{-i\epsilon' y \cdot \xi}}{i\epsilon' y \cdot \xi} \right| |\hat{\theta}(\tau, \xi)|^2 \, d\xi \, dy
\]
\[
\leq C \epsilon' M^2 \|\theta(\tau)\|^2_{L^2} + C\epsilon' \alpha \int_{|\xi| \geq M} |\xi|^\alpha |\hat{\theta}(\tau, \xi)|^2 \, d\xi,
\]
for all \( M > 0 \). Hence we have
\[
\int_s^T \|\mathcal{E}_K^{(1)}(\theta(\tau) - \nu_{d,\epsilon'} \ast \theta(\tau), \nu_{d,\epsilon'} \ast \theta(\tau))\| \, d\tau
\]
\[
\leq \|v\|_{L^\infty_t} \int_s^T \|\theta(\tau) - \nu_{d,\epsilon'} \ast \theta(\tau)\|_{L^2} \|\nabla \nu_{d,\epsilon'} \ast \theta(\tau)\|_{L^2} \, d\tau
\]
\[
\leq C\|v\|_{L^\infty_t} \left( \int_s^T \|\theta(\tau) - \nu_{d,\epsilon'} \ast \theta(\tau)\|_{L^2}^2 \, d\tau \right)^{1/2} \left( \int_s^T \int_{\mathbb{R}^d} |\xi|^2 |\nu_d(\epsilon' \xi)|^2 |\hat{\theta}(\tau, \xi)|^2 \, d\xi \, d\tau \right)^{1/2}
\]
\[
\leq C\|v\|_{L^\infty_t} \left( \epsilon' M \left( \int_s^T \|\theta(\tau)\|_{L^2}^2 \, d\tau \right)^{1/2} + \epsilon' \alpha \left( \int_s^T \int_{|\xi| \geq M} |\xi|^\alpha |\hat{\theta}(\tau, \xi)|^2 \, d\xi \, d\tau \right)^{1/2} \right)
\]
\[
\times \epsilon' \frac{2^\alpha - 1}{2^\alpha} \left( \int_s^T \|\theta(\tau)\|_{H^\frac{\alpha}{2}}^2 \, d\tau \right)^{1/2}
\]
\[
\leq C\|v\|_{L^\infty_t} \|\theta\|_{L^\infty_t} \left( \epsilon' M T^{1/2} \|\theta\|_{L^\infty_t} + \epsilon' \alpha \left( \int_s^T \int_{|\xi| \geq M} |\xi|^\alpha |\hat{\theta}(\tau, \xi)|^2 \, d\xi \, d\tau \right)^{1/2} \right).
\]
Noting \( \alpha \in (1, 2) \), by taking \( \epsilon' \to 0 \) first and then by taking \( M \to \infty \), we conclude that the third term of the right-hand side of (5.3) goes to zero as \( \epsilon' \to 0 \). This completes the proof of (5.2).

As an immediate consequence of Theorem 5.1, we have the following uniqueness result.

**Corollary 5.4** Let the assumptions in Theorem 5.1 hold with \( f = 0 \). Then \( \theta(t, x) = 0 \) for all \( (t, x) \in (0, T) \times \mathbb{R}^d \).

Next we prove the uniqueness of weak solutions under the following condition on \( v \) instead of (C1):

\( v(t) \in L^\infty(\mathbb{R}^d, \mathbb{R}) \) for each \( \epsilon > 0 \), and (C1) with \( \lambda = d \) holds, i.e.,

\[
v \in L^{1, \frac{1}{2} + \frac{d}{2} + \epsilon'(0, \infty; (BMO(\mathbb{R}^d))^d)} \cap L^1_{t,x}(0, T; (L^1_{t,x}(\mathbb{R}^d))^d) \quad \text{for some } 1 < q \leq \infty.
\]

**Theorem 5.5** Let \( \alpha \in (1, 2) \). Assume that (C1') and (C2) hold. Let \( \theta \in L^\infty(0, T; L^2(\mathbb{R}^d)) \) be a weak solution to (1.1) for \( t \in (0, T) \) with the initial data \( f \in L^2(\mathbb{R}^d) \) at the initial time \( t = 0 \). Assume in addition that (5.1) holds for a.e. \( s \in (0, T) \). Then \( \theta(t, x) = (P_{K,v,t})(t, 0, x) \) for all \( (t, x) \in (0, T) \times \mathbb{R}^d \).

**Proof.** It suffices to consider the case \( t \in (0, h_0) \) by the uniqueness result in Corollary 5.4, where \( h_0 \) is the number in Theorem 1.4. Let us take the test function of the form \( (1 - \phi_m(t, t_0)) \varphi(x) \) in (2.4) with \( s = 0 \), where \( \phi_m(t, t_0) \) is as in (4.29) with \( t_0 \in (0, T) \) and \( \varphi \in C_0^\infty(\mathbb{R}^d) \). Then by tending \( m \to \infty \) we have for a.e. \( t_0 \in (0, T) \),

\[
- \langle \theta(t_0), \varphi \rangle + < f, \varphi > = \int_{t_0}^t \left( E_K^{(t)}(\theta(t), \varphi) + E_{v(t)}^{(t)}(\theta(t), \varphi) \right) \, dt. \tag{5.7}
\]

The equality (5.7) implies that there is \( \{t_n\}_{n \in \mathbb{N}} \subset (0, T) \), \( t_n \to 0 \) as \( n \to \infty \), such that \( \theta(t_n) \) converges to \( f \) in the sense of distributions. Then from the density argument \( \theta(t_n) \) converges to \( f \) weakly in \( L^2(\mathbb{R}^d) \). The uniqueness result in Corollary 5.4 enables us to write

\[
\theta(t, x) = \int_{\mathbb{R}^d} P_{K,v,t}(t, x; t_n, y) \theta(t_n, y) \, dy
\]

\[
= \int_{\mathbb{R}^d} P_{K,v,t}(t, x; 0, 0) f(y) \, dy + \int_{\mathbb{R}^d} \left( P_{K,v,t}(t, x; t_n, y) - P_{K,v,t}(t, x; 0, y) \right) \theta(t_n, y) \, dy
\]

\[
+ \int_{\mathbb{R}^d} P_{K,v,t}(t, x; 0, 0) \left( \theta(t_n, y) - f(y) \right) \, dy
\]

\[
= \int_{\mathbb{R}^d} P_{K,v,t}(t, x; 0, 0) f(y) \, dy + I_1 + I_2. \tag{5.8}
\]

From the weak convergence of \( \theta(t_n) \) to \( f \) in \( L^2(\mathbb{R}^d) \), \( I_2 \) goes to 0 as \( n \to \infty \). As for \( I_1 \), we have

\[
|I_1| \leq \|P_{K,v,t}(t, x; t_n, \cdot) - P_{K,v,t}(t, x; 0, \cdot)\|_{L^2} \|	heta(t_n)\|_{L^2}
\]

\[
\leq \sup_{s \in (0, T)} \|\theta(s)\|_{L^2} \|P_{K,v,t}(t, x; t_n, \cdot) - P_{K,v,t}(t, x; 0, \cdot)\|_{L^2}
\]

\[
\leq C t^{-\frac{d}{2}} \sup_{s \in (0, T)} \|\theta(s)\|_{L^2} \|P_{K,v,t}(t, x; t_n, \cdot) - P_{K,v,t}(t, x; 0, \cdot)\|_{L^1}^{\frac{1}{2}}.
\]

Then for \( R > 1 \),

\[
\|P_{K,v,t}(t, x; t_n, \cdot) - P_{K,v,t}(t, x; 0, \cdot)\|_{L^1}
\]

\[
\leq \int_{|y| \leq R} |P_{K,v,t}(t, x; t_n, y) - P_{K,v,t}(t, x; 0, y)| \, dy + 2 \sup_{s \leq \frac{t}{2}} \int_{|y| \geq R} P_{K,v,t}(t, s; y) \, dy = I_{2,1} + I_{2,2}.
\]

For each \( R > 1 \) by the continuity of the fundamental solution \( I_{2,1} \) converges to 0 as \( n \to 0 \). Then by taking \( R \to \infty \) the term \( I_{2,2} \) goes to zero by (1.28). This completes the proof.

### 5.2 Uniqueness for finite measure initial data

The advantage of the approach using fundamental solutions will be that once they are constructed, there is a flexibility on the choice of the class of initial data. For example, we can easily extend Theorem 5.5 to the case when the initial data is a finite Radon measure on \( \mathbb{R}^d \). The class of finite Radon measures on \( \mathbb{R}^d \) is denoted by \( M(\mathbb{R}^d) \), and the total variation of a finite Radon measure on \( \mathbb{R}^d \) is defined by

\[
\|\mu\|_M = \sup \left\{ \int_{\mathbb{R}^d} \varphi \, d\mu \mid \varphi \in C_c(\mathbb{R}^d), \|\varphi\|_{L^\infty} \leq 1 \right\}.
\]
For $\mu \in \mathcal{M}(\mathbb{R}^d)$ we say that $\theta \in L^\infty(0,T;L^1(\mathbb{R}^d))$ is a weak solution to (1.1) for $t \in [0,T]$ with the initial data $\mu$ at the initial time $t = 0$ if $\theta$ belongs to $L^\infty(s,T;L^2(\mathbb{R}^d))$ for each $s \in (0,T)$ and is a weak solution to (1.1) for $t \in [s,T)$ with the initial data $\theta(s)$ at the initial time $t = s$, and $\theta(t) \rightarrow \mu$ as $t \rightarrow 0$ in the sense of measures.

For $\mu \in \mathcal{M}(\mathbb{R}^d)$ we set
\begin{equation}
(P_{K,v}\mu)(t,0) = \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)\mu(dy).
\end{equation}

The right hand side of (5.10) is well-defined, since $P_{K,v}(t,x;0,y)$ is continuous and integrable with respect to $y$.

**Theorem 5.6** Let $\alpha \in [1,2]$. Assume that (C1') and (C2) hold. Let $\mu$ be a finite Radon measure on $\mathbb{R}^d$. Let $\theta \in L^\infty(0,T;L^1(\mathbb{R}^d))$ be a weak solution to (1.1) for $t \in [0,T)$ with the initial data $\mu$ at the initial time $t = 0$. Assume in addition that (5.1) holds for a.e. $s \in (0,T)$. Then $\theta(t,x) = (P_{K,v}\mu)(t,0,x)$ for all $(t,x) \in (0,T) \times \mathbb{R}^d$.

**Proof.** The strategy is the same as in the proof of Theorem 5.5. It suffices to show $\theta(t,x) = (P_{K,v}\mu)(t,0,x)$ for all $0 < t \leq h_0$ and $x \in \mathbb{R}^d$. The uniqueness result in Corollary 5.4 yields that for a.e. $s \in (0,t)$ and $R \gg 1$,
\begin{equation}
\theta(t,x) = \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)\theta(s,y)\,dy
= \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)\mu(dy) + \int_{\mathbb{R}^d} (P_{K,v}(t,x;0,y) - P_{K,v}(t,x;0,0))\chi_R(y)\theta(s,y)\,dy
+ \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)(1 - \chi_R(y))\theta(s,y)\,dy - \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)(1 - \chi_R(y))\mu(dy)
+ \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)\chi_R(y)(\theta(s,y)\,dy - \mu(dy))
= \int_{\mathbb{R}^d} P_{K,v}(t,x;0,y)\mu(dy) + I_{1.1} + I_{1.2} + I_{2.1} + I_{2.2}.
\end{equation}

Since $P_{K,v}(t,x;0,y)$ is continuous,
\begin{equation}
|I_{1.1}| \leq C\|\theta\|_{L^\infty(0,T,L^1(\mathbb{R}^d))} \sup_{|y| \leq 2R} |P_{K,v}(t,x;0,y) - P_{K,v}(t,x;0,0)| \rightarrow 0
\end{equation}
as $s \rightarrow 0$ for each $R \gg 1$. Moreover, the convergence of $\theta(s)$ to $\mu$ at $s \rightarrow 0$ in the sense of measures implies $I_{2.1} \rightarrow 0$ as $s \rightarrow 0$ for each $R \gg 1$. Hence it suffices to show that we can make $|I_{1.2} + I_{2.2}|$ arbitrary small by taking $R$ large enough, uniformly in $s$ with $0 < s \leq t/4$. But this assertion follows from (1.28), for it gives the estimate
\begin{equation}
|I_{1.2} + I_{2.2}| \leq (\|\theta\|_{L^\infty(0,T,L^1(\mathbb{R}^d))} + \|\mu\|_{\mathcal{M}(\mathbb{R}^d)}) \sup_{|y| \geq R, \tau \in [0,T]} |P_{K,v}(t,x;\tau,y)|
\leq C(\|\theta\|_{L^\infty(0,T,L^1(\mathbb{R}^d))} + \|\mu\|_{\mathcal{M}(\mathbb{R}^d)}) t^{-\frac{d+2}{4}} \sim R^{-\frac{d+2}{4}},
\end{equation}
for $0 < s \leq \frac{t}{4}$. This completes the proof.

### 6 Applications to two-dimensional critical dissipative quasi-geostrophic equations

In this section we discuss the two-dimensional critical dissipative quasi-geostrophic equations
\begin{equation}
\left\{
\begin{array}{l}
\partial_t \theta + (-\Delta)^{\frac{1}{2}} \theta + v \cdot \nabla \theta = 0, \quad 0 < t < T, \quad x \in \mathbb{R}^2, \\
v = (-R_2 \partial_x \theta, R_1 \theta), \quad 0 < t < T, \quad x \in \mathbb{R}^2, \\
\theta|_{t=0} = \theta_0, \quad x \in \mathbb{R}^2.
\end{array}
\right.
\end{equation}

Here $R_t = \partial_t(-\Delta)^{1/2}$ is the Riesz transform. There are already many works on (6.1); for example, see [9, 8, 7, 12, 27, 18, 20, 23, 34, 10, 11, 13, 14, 5, 22, 31]. In particular, the global regularity of solutions were established by [23, 5, 22]. In [23] the a priori Hölder continuity for smooth periodic solutions was proved by constructing the modulus of continuity which is preserved by (6.1). In [5, 22], which are more related with our approach, the linear problem (1.1) is discussed for $A_K(t) = (-\Delta)^{1/2}$ and for a given $v$ satisfying $\nabla \cdot v(t) = 0$. The key step of the proof in [5] is to establish the local Hölder continuity of $\theta$.
when \( v \in L^\infty(0, \infty; (BMO(\mathbb{R}^d))^d) \cap L^2_{\text{loc}}(0, \infty; (L^2_{\text{loc}}(\mathbb{R}^d))^d) \) and \( \theta \) satisfies the level set energy inequality. They used the harmonic extension of the equation to \((0, \infty) \times \mathbb{R}^{d+1}\) and regarded \((-\Delta)^{1/2}\) as the normal derivative on the boundary. This enables to derive localized level set energy inequalities and to apply the De Giorgi argument. In [22] the a priori Hölder continuity for smooth periodic solutions was obtained in terms of the norm of \( v \) in \( L^\infty(0, \infty; (BMO(T^d))^d) \), where they studied the adjoint problem and used a predual characterization of the Hölder spaces. We note that the moment bound for solutions as in Section 3.1 is important also in their argument. Theorem 1.2 gives the alternative proof for the global regularity of solutions to (6.1).

**Theorem 6.1** Let \( \theta \in L^\infty(0, T; L^2(\mathbb{R}^2)) \) be a weak solution to (6.1). Assume that \( v \in L^\infty(t_1, t_2; (L^\infty(\mathbb{R}^2))^2) \) for some \( 0 \leq t_1 < t_2 \leq T \). Then there is \( \beta \in (0, 1) \) such that \( \theta \in C^\beta([t_0, t_2] \times \mathbb{R}^2) \) for all \( t_0 \in (t_1, t_2) \). Here the exponent \( \beta \) is taken depending only on \( \|v\|_{L^\infty(t_1, t_2; BMO)} \), and \( \|\theta\|_{C^\beta([t_0, t_2] \times \mathbb{R}^2)} \) is estimated depending only on \( t_0 \in (t_1, t_2) \) and \( \|v\|_{L^\infty(t_1, t_2; BMO \cap L^2)} \).

**Remark 6.2** In the proof of Theorem 6.1 the assumption of \( v \in L^\infty(t_1, t_2; (L^\infty(\mathbb{R}^2))^2) \), rather than \( v \) belonging to \( L^\infty(t_1, t_2; (BMO(\mathbb{R}^2))^2) \), is used only to ensure the uniqueness of weak solutions; see Theorem 5.5.

Note that the norm \( \|v\|_{L^\infty(t_0, t_2; BMO \cap L^2)} \) itself can be estimated independently with respect to the norm of \( v(t) \) in \( L^\infty(\mathbb{R}^2) \). For (6.1) it is well known that under some regularity conditions on initial data smooth solutions uniquely exist at least locally in time, together with the global a priori bound of \( v \) in \( L^\infty(0, \infty; (BMO(\mathbb{R}^2))^2) \); for example, see [8]. Hence Theorem 6.1 gives the global regularity of solutions to (6.1) at least for smooth initial data.

**Proof.** Without loss of generality, we may assume that \( t_1 = 0 \) and \( t_2 = T \). Then there is a fundamental solution \( P_v(t, x; s, y) \) defined on \( \{(t, s, x, y) \mid 0 < s < t \leq T, x, y \in \mathbb{R}^2\} \) which satisfies

\[
|P_v(t, x; s, y) - P_v(t', x'; s, y)| \leq C(\min\{t - s, t' - s\})^{-\epsilon}(|x - x'| + |t - t'|)^{\beta'}
\]

for all \( 0 \leq s < t, t' \leq T, x, x', y \in \mathbb{R}^2 \). Here positive constants \( c \) and \( \beta' \) depend only on \( \|v\|_{L^\infty(0, T; BMO)} \) and \( C \) depends only on \( \|v\|_{L^\infty(0, T; BMO \cap L^2)} \). By the uniqueness of weak solutions obtained by Theorem 5.5 for a given \( v \) we have the representation \( \theta(t, x) = \int_{\mathbb{R}^2} P_v(t, x; 0, y) \theta_0(y) \, dy \). Hence \( \theta \) is Hölder continuous uniformly on each \( [t_0, T] \times \mathbb{R}^2 \), \( t_0 \in (0, T) \), due to the Hölder continuity and the semigroup property of \( P_v(t, x; s, y) \) obtained by Theorem 1.2. This completes the proof.

### 7 Appendix

#### 7.1 Example of \( K \)

Here we give an example of \( K \) which satisfies (1.5) and (1.6), but not (1.7). Let \( H \in L^1(\mathbb{R}^d) \) be such that \( H(z) \geq 0, H(z) = H(-z) \), and \( \supp H \subset B_2(0) \setminus B_1(0) \). Set

\[
H_k(z) = H(k^d z), \quad L(z) = \sum_{k=0}^{\infty} H_k(z).
\]

Then \( \supp H_k \cap \supp H_l = \emptyset \) if \( k \neq l \) and \( \int_{\mathbb{R}^d} L(z) \, dz = C \sum_{k=0}^{\infty} 3^{-kd} < \infty \). Moreover, for each \( M \in (0, \infty) \) it follows that

\[
\int_{|z| \leq M} L(z) \, dz \leq CM^d.
\]

Indeed, if \( 3^{-k_0 - 1} \leq M < 3^{-k_0} \) for some \( k_0 \in \mathbb{N} \cup \{0\} \) then \( \int_{|z| \leq M} H_k(z) \, dz = 0 \) if \( k \leq k_0 \), and hence,

\[
\int_{|z| \leq M} L(z) \, dz = \sum_{k=k_0+1}^{\infty} \int_{|z| \leq M} H_k(z) \, dz \leq C \sum_{k=k_0+1}^{\infty} 3^{-kd} = C3^{-k_0 d} \leq CM^d.
\]

If \( M \geq 1 \) then we have \( \int_{|z| \leq M} L(z) \, dz \leq \int_{\mathbb{R}^d} L(z) \, dz \leq C \leq CM^d \). This proves (7.2). Now we set

\[
K(t, x, y) = g(t)|x - y|^{-d-\alpha}(1 + L(x - y)),
\]
for some $\alpha \in (0, 2)$ and $g \in L^\infty(0, \infty)$ such that $C^{-1} \leq g(t) \leq C$ with a constant $C > 0$. Then clearly $K$ satisfies (1.6), but not (1.7) in general, for $H$ (and so $L$) is just in $L^1(\mathbb{R}^d)$. Furthermore, we have from (7.2),

$$\text{ess.sup}_{t > 0, x \in \mathbb{R}^d} \int_{|x-y| \leq M} |x-y|^d K(t, x, y) dy \leq CM^d \quad \text{for each } M \in (0, \infty).$$  

(7.4)

Just as in the proof of (2.10), we can derive (1.5) from (7.4) for some $C_0 > 0$.

### 7.2 Proof of Lemma 2.4

In order to prove Lemma 2.4 we first prove the following

**Lemma 7.1** Let $v \in (L^{2d/\alpha, \lambda}(\mathbb{R}^d))^d$, $\lambda \in [2d/\alpha - d, 2d/\alpha + d]$ be a vector field satisfying $\nabla \cdot v = 0$. Set $p_\alpha = 1$ if $\lambda \in [2d/\alpha - d, d]$ and $p_\alpha = \infty$ if $\lambda \in (d, 2d/\alpha + d]$. Let $R \geq 1$. Then there is a sequence of smooth and bounded vector fields $\{v_N\}$ satisfying $\nabla \cdot v_N = 0$ and

$$\|v_N\|_{L^{2d/\alpha, \lambda}} \leq C_1 \|v\|_{L^{2d/\alpha, \lambda}},$$  

(7.5)

$$\|v_N\|_{L^p(B_R(0))} \leq C_2 \|v\|_{L^p(B_R(0))} + c_R N(\|v\|_{L^p(B_R(0))}),$$  

(7.6)

$$\lim_{N \to \infty} v_N \rightarrow v \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^d).$$  

(7.7)

Here $C_1$ depends only on $d$, $\alpha$, and $\lambda$, $C_2$ depends only on $d$, and $c_R N_{\text{ps}}$ depends only on $d$, $\alpha$, $\lambda$, $R$, $N$, and satisfies $\lim_{N \to \infty} c_R N = 0$ for each $R \geq 1$.

**Proof of Lemma 7.1.** Let $\chi_N \in C^\infty_0(\mathbb{R}^d)$, $0 \leq \chi_N \leq 1$, be a cut-off function such that $\chi_N(x) = 1$ if $|x| \leq N$ and $\chi_N(x) = 0$ if $|x| \geq 2N$. Set

$$w_N = v - \int_{B_{2N}(0)} v,$$  

(7.8)

$$u_N = \chi_N w_N + \int_{B_{2N}(0)} v - \left(\chi_N - \int_{B_1(0)} \chi_N\right),$$  

(7.9)

$$v_N = \nu_{d, \frac{1}{\alpha}} \ast u_N,$$  

(7.10)

where $\chi_N \in (L^1(\mathbb{R}^d))^d$ is a solution to

$$\text{div } \chi_N = w_N \cdot \nabla \chi_N, \quad \text{supp } \chi_N \subset B_{2N}(0),$$  

(7.11)

which satisfies

$$\|\nabla \chi_N\|_{L^q(B_{2N}(0))} \leq C \|w_N \cdot \nabla \chi_N\|_{L^q(B_{2N}(0))}, \quad 1 < q < \infty.$$  

(7.12)

It is easy to see that $\|\chi_N\|_{L^p(B_{2N}(0))} \leq C \|u_N\|_{L^p(B_{2N}(0))}$ and $\|\chi_N\|_{L^p(B_{2N}(0))} \leq C \|\chi_N\|_{L^p(B_{2N}(0))}$ if $N \gg 1$. Since (7.7) holds if $\lim_{N \to \infty} u_N \rightarrow v$ in $L^1_{\text{loc}}(\mathbb{R}^d)$, it suffices to prove (7.5)-(7.7) for $u_N$ with $N \gg 1$, instead of $v_N$. However, since it is not difficult to prove similar estimates as (7.5)-(7.7) for $w_N \chi_N + \int_{B_{2N}(0)} v$, it suffices to show

$$\|\chi_N\|_{L^{p_\alpha}(B_1(0))} \leq C_1 \|v\|_{L^{p_\alpha}(B_1(0))},$$  

(7.13)

$$\|\chi_N - \int_{B_1(0)} \chi_N\|_{L^{p\lambda}(B_{2N}(0))} \leq c_R N \|v\|_{L^{p\lambda}(B_{2N}(0))}. $$  

(7.14)

(i) The case $\lambda \in [2d/\alpha - d, d]$: First we note that the following inequality holds:

$$\|\nabla \chi_N\|_{L^{p_\alpha}} \leq C \|\nabla \chi_N\|_{L^{p_\alpha}}, \quad \frac{d}{p'} = 1 + \frac{\alpha}{2} - \frac{\alpha \lambda}{2d}. $$  

(7.15)

Then since $p^*$ defined in (7.15) satisfies $p^* \leq 2d/\alpha$ in the present case, we have from (7.12),

$$\|\chi_N\|_{L^{p_\alpha}} \leq C \|\nabla \chi_N\|_{L^{p^*}} \leq C N^{1 - \frac{\alpha}{2d}} \|\nabla \chi_N\|_{L^{p_\alpha}} \leq C N^{1 - \frac{\alpha}{2d}} \|w_N \cdot \nabla \chi_N\|_{L^{p_\alpha}} \leq C \|v\|_{L^{p_\alpha}}.$$  

(7.16)

Thus we get (7.13). As for (7.14), from the Hölder inequality and the Poincaré inequality we have

$$\|\chi_N - \int_{B_1(0)} \chi_N\|_{L^1(B_{2N}(0))} \leq C_R \|\nabla \chi_N\|_{L^{p_\alpha}(B_{2N}(0))} \leq C_R \|\nabla \chi_N\|_{L^{p_\alpha}(B_{2N}(0))} \leq C_R N^{-1 + \frac{\alpha}{2d}} \|v\|_{L^{p_\alpha}}.$$

(7.17)

Here $C_R$ is independent of $N$ and $v$. This completes the proof for the case $\lambda \in [2d/\alpha - d, d]$. 

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where \( \beta \in (d, d/\alpha + d) \): The estimate (7.13) follows from (7.16). As for (7.14), we have for \( \beta \in (0, 1) \) and \( d/p_\beta = 1 - \beta \),

\[
\| \zeta_N - \int_{B_1(0)} \zeta_N \|_{L^\infty(B_R(0))} \leq C R^\beta \| \zeta_N \|_{C^\beta} \leq C R^\beta \| \nabla \zeta_N \|_{L^{p_\beta}} \\
\leq C R^\beta N^{1 + \frac{d}{p_\beta}} + \frac{2d}{p} \| v \|_{L^\frac{2d}{d-\lambda} \lambda}.
\]

By taking \( \beta \) near 1, we get (7.14).

(iii) The case \( \lambda = 2d/\alpha + d \): In this case, instead of (7.9), we set \( u_N \) as

\[
u_N = \int_{B_1(0)} \zeta_N - \int_{B_1(0)} \{ x \cdot \nabla \zeta_N \}.
\] (7.18)

Note that \( \int_{B_1(0)} \{ x \cdot \nabla \zeta_N \} = \int_{B_1(0)} \{ w_N \cdot \nabla \chi_N \} = 0 \), and hence \( \nu_N = 0 \).

From the representation formula of \( \zeta_N \) given in the proof of [15, Lemma III-3-1], we have

\[
\| \nabla^2 \zeta_N \|_{L^p} \leq C (\| \nabla (w_N \cdot \nabla \chi_N) \|_{L^p} + N^{-1} \| w_N \cdot \nabla \chi_N \|_{L^p}).
\] (7.19)

By the Gagliardo-Nirenberg inequality

\[
\| \nabla \zeta_N \|_{L^\infty} \leq C \| \nabla^2 \zeta_N \|_{L^p} \| \nabla \zeta_N \|_{L^{-\sigma}} \quad \sigma \left( \frac{1}{p} - \frac{1}{\sigma} \right) + (1 - \sigma) \left( \frac{1}{p} - \frac{1}{\sigma} \right) = 0, \quad p \in (d, \infty], \quad r \in [1, \infty),
\]

we have from (7.12) and (7.19),

\[
\| \nabla \zeta_N \|_{L^\infty} \leq C (\| \nabla (w_N \cdot \nabla \chi_N) \|_{L^p} + N^{-1} \| w_N \cdot \nabla \chi_N \|_{L^p}) \| \nabla \zeta_N \|_{L^{-\sigma}}^\sigma \| \nabla \zeta_N \|_{L^{-\sigma}}^{1 - \sigma} \leq C (N^{-1} \frac{d}{p} \| v \|_{L^\frac{2d}{d-\lambda} \lambda})^{\frac{d}{2d-\lambda}} (N \frac{d}{p} \| v \|_{L^\frac{2d}{d-\lambda} \lambda})^{1 - \sigma} \]

which gives

\[
\| \zeta_N(x) - \int_{B_1(0)} \zeta_N - \int_{B_1(0)} \{ x \cdot \nabla \zeta_N \} \|_{L^\frac{2d}{d-\lambda} \lambda} \leq C \| v \|_{L^\frac{2d}{d-\lambda} \lambda}.
\] (7.20)

Next we observe that

\[
\zeta_N(x) - \int_{B_1(0)} \zeta_N - \int_{B_1(0)} \{ x \cdot \nabla \zeta_N \} = \frac{1}{|B_1(0)|} \int_{B_1(0)} (x - y) \cdot \int_0^1 \{ (\nabla \zeta_N)(\sigma x + (1 - \sigma)y) - \nabla \zeta_N(y) \} d\sigma dy,
\]

and hence, for \( R \geq 1, \beta \in (0, 1) \), and \( d/p_\beta = 1 - \beta \),

\[
\| \zeta_N(x) - \int_{B_1(0)} \zeta_N - \int_{B_1(0)} \{ x \cdot \nabla \zeta_N \} \|_{L^\infty(B_R(0))} \leq C R^{1 + \beta} \| \nabla \zeta_N \|_{C^\beta} \leq C R^{1 + \beta} \| \nabla \zeta_N \|_{L^{p_\beta}} \leq C R^{1 + \beta} \| \nabla \zeta_N \|_{L^{p_\beta}} + N^{-1} \| w_N \cdot \nabla \chi_N \|_{L^{p_\beta}} \]

This gives (7.14) for \( \zeta_N(x) - \int_{B_1(0)} \zeta_N - \int_{B_1(0)} \{ x \cdot \nabla \zeta_N \} \) instead of \( \zeta_N(x) - \int_{B_1(0)} \zeta_N \). The proof of Lemma 7.1 is now complete.

Proof of Lemma 2.4. We first extend \( v(t) \) by 0 for \( t < 0 \) and set

\[
v^{(N)}(t) = \int_R v_1, \frac{1}{N} (t - s) v_N(s) \, ds,
\] (7.21)

where \( v_N(t) \) is the approximation of \( v(t) \) obtained by Lemma 7.1. Then from \( \| v^{(N)}(t) \|_{L^\frac{2d}{d-\lambda} \lambda} \leq C \int_R v_1, \frac{1}{N} (t - s) \| v_N(s) \|_{L^\frac{2d}{d-\lambda} \lambda} \, ds \) and \( \| v^{(N)}(t) \|_{L^{p_\lambda}(B_R(0))} \leq C \int_R v_1, \frac{1}{N} (t - s) \| v_N(s) \|_{L^{p_\lambda}(B_R(0))} \, ds \), it is easy to get (2.42) and (2.43). Moreover, we have

\[
\| v^{(N)}(t) - v(t) \|_{L^1(B_R(0))} \leq \int_R v_1, \frac{1}{N} (t - s) \| v_N(s) - v(s) \|_{L^1(B_R(0))} \, ds + \int_R v_1, \frac{1}{N} (t - s) \| v(s) - v(t) \|_{L^1(B_R(0))} \, ds,
\]
which yields
\[
\|v^{(N)} - v\|_{L^1(0,R: L^1(B_R(0)))} \leq C \int_0^{2R} \|v_N(s) - v(s)\|_{L^1(B_R(0))} \, ds + C \int_{|t| \leq 2} \int_0^R \|v(t - \frac{\tau}{N}) - v(t)\|_{L^1(B_R(0))} \, dt \, d\tau.
\]
Hence $v^{(N)} \to v$ in $L^1_{loc}((0,\infty) \times \mathbb{R}^d)$. This completes the proof.

References