CONTACT GEOMETRY OF SECOND ORDER II

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Abstract. This is the continuation of our previous paper "Contact Geometry of Second Order I" , where we have formulated the contact equivalence of systems of second order partial differential equations for a scalar function as the geometry of PD manifolds of second order. In this paper, we will discuss the Two Step Reduction procedure in Contact Geometry of Second Order. In fact we establish the Second Reduction Theorem for PD manifolds $(R; D^1, D^2)$ of second order admitting the first order covariant systems $\tilde{N}$. Utilizing the covariant system $\tilde{N}$, we construct the intermediate object $(W; C^{\cdot}, N)$, called the IG manifold of corank r, as a submanifold of the Involutive Grassmann bundle $I^r(J)$ over the contact manifold (J, C), where $J = R/\mathrm{Ch}\,(D^1)$. We will seek the condition when the equivalence of $(R; D^1, D^2)$ is reducible to that of $(W; C^*, N)$. Moreover, when $\mathrm{Ch}\,(N)$ is non-trivial, the equivalence of $(W; C^*, N)$ is further reducible to that of $(Y; D^*_N, D_N)$, where $Y = W/\mathrm{Ch}\,(N)$. This theorem gives a sufficient condition for the existence of higher dimensional characteristics of $(R; D^1, D^2)$. By analyzing the construction parts of the Two Step Reduction procedure, we will show several examples of Parabolic Geometries, which are, through the Second Reduction Theorem, associated with the geometry of PD manifolds of second order.

1. Introduction

This manuscript is the sequel to Contact Geometry of Second Order I [27], and will discuss the Second Reduction Theorem in **Contact Geometry of Second Order**. In [27], we have formulated the contact equivalence problem of systems of second order partial differential equations for a scalar function as the geometry of PD (partial differential) manifolds $(R; D^1, D^2)$ of second order, where R is a manifold equipped with a pair of differential systems D^1 and D^2 satisfying the appropriate conditions (see §4 [27]). We have also established the **First Reduction Theorem** for PD manifolds $(R; D^1, D^2)$ admitting non-trivial Cauchy characteristic systems $\mathrm{Ch}\,(D^2)$, which reduces the equivalence of $(R; D^1, D^2)$ to the geometry of (X, D), where $X = R/\mathrm{Ch}\,(D^2)$ is the leaf space of the foliation $\mathrm{Ch}\,(D^2)$ on R and D is the differential system on X such that $D^2 = \rho_*^{-1}(D)$ and $\rho : R \to X$ is the projection. Moreover we have exhibitted several examples of **Parabolic Geometries** which are, directly or through the First Reduction Theorem, associated with the geometry of PD manifolds of second order.

In this manuscript, we will establish the **Second Reduction Theorem** for PD manifolds $(R; D^1, D^2)$ admitting the first order covariant systems $\tilde{N}$ such that $D^1 \supset \tilde{N} \supset D^2$. Here $\tilde{N}$ is called covariant if each isomorphism φ of $(R; D^1, D^2)$ preserves $\tilde{N}$, i.e., $\varphi_*(\tilde{N}) = \tilde{N}$. At each point $v \in R$, for every integral element V of $(R; D^1, D^2)$ at v, i.e., V is an integral element of (R, D^2) such that $\mathfrak{s}_{-1}(v) = D^2(v) = V \oplus \mathrm{Ch}\,(D^1)(v)$, $\tilde{N}(v)$ defines the r-dimensional subspace E of V through the symbol algebra identification; $\mathfrak{s}_{-3}(v) = T_v(R)/D^1(v) \cong \mathbb{R}$, $\mathfrak{s}_{-2}(v) = D^1(v)/D^2(v) \cong V^*$, $\mathrm{Ch}\,(D^2)(v) \cong \mathfrak{f} \subset S^2(V^*)$ so that $\tilde{N}(v)/D^2(v) \subset \mathfrak{s}_{-2}(v)$ corresponds to $E^\perp \subset V^*$, where $\mathfrak{s}(v) = \mathfrak{s}_{-3}(v) \oplus \mathfrak{s}_{-2}(v) \oplus \mathfrak{s}_{-1}(v)$ is the symbol algebra of $(R; D^1, D^2)$ at v and r is the codimension of $\tilde{N}$ in D^1 (see §3.1 and §1 [27]). Moreover the first order covariant system

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$\tilde{N}$ defines the map $\eta : R \to I^r(J)$ by

$$\eta(v) = p_*(\tilde{N}(v)) \in I^r(J) \qquad \text{for} \qquad v \in R,$$

where $p : R \to J = R/\mathrm{Ch}\,(D^1)$ is the projection onto the contact manifold (J, C) of dimension $2n+1$ such that $D^1 = p_*^{-1}(C)$. Here $I^r(J)$ is the **Involutive Grassmann Bundle** over (J, C) consisting of involutive (coisotropic) subspaces w of codimension r in the symplectic vector space $(C(u), d\varpi)$, i.e., $w^\perp \subset w$, where ϖ is a local contact form on J around $u \in J$ and $w^\perp = \{X \in C(u) \mid d\varpi(X, Y) = 0 \quad \text{for} \quad \forall Y \in w\}$ (see §2.1). We will consider the image $W = \mathrm{Im}(\eta)$ of this map $\eta : R \to I^r(J)$. Under a mild regularity condition, W is a submanifold of $I^r(J)$ such that W carries two differential systems C^* and N, where C^* is the lift of C and N is the restriction to W of the canonical differential system $\bar{N}$ on $I^r(J)$ defined by the grassmannian construction (see §2.2). $(W; C^*, N)$ is called the IG (involutive grassmann) manifold of corank r associated with $(R; D^1, D^2, \tilde{N})$ (see §3.2). A submanifold W of $I^r(J)$ defines a subvariety $\bar{R}(W)$ of the Lagrange Grassmann bundle $L(J)$ over (J, C) by

$$\bar{R}(W) = \{v \in L(J) \mid v \subset \exists w \in W\}.$$

Technically we construct the Lagrange Grassmann bundle $R(W)$ over $(W; C^*, N)$ by

$$R(W) = \bigcup_{w \in W} R_w, \qquad R_w = \{\hat{v} \subset N(w) \mid d\hat{\varpi}\,|_{\hat{v}} = 0, \quad \hat{v} \text{ is maximal}\},$$

where $C^* = \{\hat{\varpi} = 0\}$ around $w \in W$. Then we have a map $\zeta : R(W) \to L(J)$ defined by $\zeta(\hat{v}) = q_*(\hat{v})$, where $q : W \to J$ is the projection and $\bar{R}(W) = \zeta(R(W))$. Here $\zeta(R_w) \cong L(w/w^\perp)$ is the Lagrange Grassmann manifold of the symplectic vector space $w/w^\perp$ of dimension $2(n-r)$ (see §4.1). Moreover $R(W)$ carries three differential systems D^1_W, D^2_W and N_W, where D^1_W and N_W are lifts of C^* and N respectively and D^2_W is the canonical system defined by the grassmmannian construction (see §4.2). Furthermore we have a map $\kappa_1 : R \to R(W)$ defined by

$$\kappa_1(v) = \eta_*(D^2(v)) \in R_w, \qquad w = \eta(v).$$

κ_1 is actually an immersion by the Realization Lemma and preserves D^1, D^2 and $\tilde{N}$, i.e., $(\kappa_1)_*^{-1}(D^1_W) = D^1$, $(\kappa_1)_*^{-1}(D^2_W) = D^2$ and $(\kappa_1)_*^{-1}(N_W) = \tilde{N}$ (see §5.1). We will seek the condition for $\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ to be a local isomorphism, which gives us the first part of the Second Reduction Theorem. For this purpose, we consider the following covariant systems $\mathfrak{f}(E)$ and $C(E)$ of $(R; D^1, D^2)$: As a regularity condition for $(R; D^1, D^2)$, we assume the constancy of symbol algebras, i.e., the symbol algebra $\mathfrak{s}(v)$ of $(R; D^1, D^2)$ at each $v \in R$ is isomorphic to a fixed subalgebra $\mathfrak{s}$ of $\mathfrak{c}^2(n)$, where $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$ is defined by

$$\mathfrak{s}_{-3} = \mathbb{R}, \quad \mathfrak{s}_{-2} = V^*, \quad \mathfrak{s}_{-1} = V \oplus \mathfrak{f}, \quad \mathfrak{f} \subset S^2(V^*).$$

Under this reguarity condition, $(R; D^1, D^2)$ is called regular of type $\mathfrak{s}$. In this case, the first covariant system $\tilde{N}$ of $(R; D^1, D^2)$ corresponds to the $G_0(\mathfrak{s})$-invariant subspace $E \subset V$, where $G_0(\mathfrak{s}) = \{\sigma \in G(\mathfrak{s}) \mid \sigma(V) = V\}$ and $G(\mathfrak{s})$ is the group of graded Lie algebra automorphisms of $\mathfrak{s}$ (see §3.1). We define subspaces $\mathfrak{f}_E$ and $\mathfrak{c}_E$ of $\mathfrak{f}$ and $\mathfrak{s}_{-1}$ by

$$\mathfrak{f}_E = \mathfrak{f} \cap S^2(E^\perp) \subset \mathfrak{f}, \qquad \mathfrak{c}_E = \hat{E} \oplus \mathfrak{f}_E \subset \mathfrak{s}_{-1} = V \oplus \mathfrak{f},$$

where $\hat{E} = \{v \in E \mid v \circledcirc E \subset \mathfrak{f}^\perp\}$ and $\mathfrak{f}^\perp$ is the annihilator of $\mathfrak{f}$ in $S^2(V) \cong (S^2(V^*))^*$. Then, by the $G_0(\mathfrak{s})$-invariance of E, we can define covariant systems $\mathfrak{f}(E)$ and $C(E)$ by

$$\mathfrak{f}(E)(v) = \phi^{-1}(\mathfrak{f}_E) \subset C(E)(v) = \phi^{-1}(\mathfrak{c}_E) \subset D^2(v) = \mathfrak{s}_{-1}(v) \quad \text{for} \quad v \in R,$$

where ϕ is a graded Lie algebra isomorphism of $\mathfrak{s}(v)$ onto $\mathfrak{s}$. By a symbol algebra calculation, we have $\mathrm{Ch}\,(\tilde{N}) \subset C(E)$ (see §5.2). In fact we will obtain the **first part of Second Reduction**

Theorem for PD manifolds of second order admitting the first order covariant systems as follows.

Theorem 5.1. *Let $(R; D^1, D^2)$ be a PD manifold of second order, which is regular of type $\mathfrak{s}$. Assume that there exists $G_0(\mathfrak{s})$-invariant subspace E of V of dimension r such that $\mathfrak{f}^\perp \subset E \otimes_S V$.*

(1) *In case $r < n-1$, if $\mathfrak{f}(E)$ is completely integrable, then $\mathfrak{f}(E) = Ch(D^1) \cap Ch(\tilde{N})$ and $\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ is a local isomorphism.*

(2) *In case $r < n-1$, if $C(E)$ is completely integrable, then $C(E) = Ch(\tilde{N})$ and $\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ is a local isomorphism.*

(3) *In case $r = n-1$, further assume that* $\operatorname{rank} Ch(D^2) < \dim \hat{E}$, *if $C(E)$ is completely integrable, then $C(E) = Ch(\tilde{N})$ and $\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ is a local isomorphism.*

When $\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ is a local isomorphism, the (local) integration problem of $(R; D^1, D^2)$ is reduced to that of $(W; C^*, N)$ (see §5.1). Moreover, for every integral element V of $(R; D^1, D^2)$ at $v \in R$, $\tilde{N}(v)$ defines the r-dimentional subspace E of V. Then the condition $\mathfrak{f}^\perp \subset E \otimes_S V$ implies that E is a Monge element (Definition 7.9 [20], see also Proposition 7.4, Lemmas 7.6 [20]). In this case, $C(E)$ does not necessarily coincide with the Monge characteristic system corresponding to E in the sence of §7.3 [20] (see §6, Remark 6.2 (1)).

A little generally the submanifold theory of $I^r(J)$ will be formulated as the geometry of IG **manifolds** $(W; C^*, N)$ **of corank** r in §2.2. Moreover we will describe the condition, when $(R(W); D^1_W, D^2_W)$ becomes a PD manifold, in terms of covariant systems of $(W; C^*, N)$ in §4.2.

Furthermore, in the above Theorem, when $\hat{E} \neq \{0\}$, we have the second step reduction procedure for $(W; C^*, N)$, similarly as in the case of the First Reduction Theorem for $(R; D^1, D^2)$ admitting non-trivial Cauchy characteristic system $\mathrm{Ch}\,(D^2)$, as follows (see §5.3): Assume that $C(E) = \mathrm{Ch}\,(\tilde{N})$. When $\hat{E} \neq \{0\}$, since $\tilde{N} = \eta_*^{-1}(N)$, N has non-trivial Cauchy characteristic system $\mathrm{Ch}\,(N)$ on W such that $\operatorname{rank} \mathrm{Ch}\,(N) = \dim \hat{E} = s > 0$. Here assume that W is regular with respect to $\mathrm{Ch}\,(N)$, i.e., the space $Y = W/\mathrm{Ch}\,(N)$ of leaves of this foliation is a manifold such that each fibre of the projection $\beta : W \to Y$ is connected and β is a submersion. We further assume that $C(E) \subset \mathrm{Ch}\,(\tilde{N}^*)$, where $\tilde{N}^*$ is the first order covariant system of $(R; D^1, D^2)$ corresponding to $\hat{E}$. Then $\mathrm{Ch}\,(N) \subset \mathrm{Ch}\,(N^*) \subset N$ on W, where N^* is a covariant system of $(W; C^*, N)$ such that $\eta_*^{-1}(N^*) = \tilde{N}^*$ (see §2.3). Hence there exist differential systems D^*_N and D_N on Y of codimension $s+1$ and $r+1$ respectively such that $N^* = \beta_*^{-1}(D^*_N)$, $N = \beta_*^{-1}(D_N)$, $D_N \supset \mathrm{Ch}\,(D^*_N)$ and $\mathrm{Ch}\,(D_N)$ is trivial. In this situation, from $(Y; D^*_N, D_N)$, we can reconstruct the IG manifold $(W; C^*, N)$, at least locally, as follows. First let us consider the collection $\tilde{W}(Y)$ of hyperplanes w in each tangent space $T_y(Y)$ at $y \in Y$ which contains the fibre $D^*_N(y)$ of D^*_N.

$$\tilde{W}(Y) = \bigcup_{y \in Y} \tilde{W}_y, \quad \tilde{W}_y = \{w \in Gr(T_y(Y), m-1) \mid w \supset D^*_N(y)\} \cong P(T_y(Y)/D^*_N(y)) = \mathbb{P}^s,$$

where $m = \dim Y$ and $s = \dim \hat{E}$. Moreover C^*_Y is the canonical system obtained by the grassmannian construction and N^*_Y, N_Y are the lifts of D^*_N, D_N. Then we have a map κ_2 of W into $\tilde{W}(Y)$ given by

$$\kappa_2(w) = \beta_*(C^*(w)) \subset T_y(Y),$$

for each $w \in W$ and $y = \beta(w)$. In fact $\kappa_2 : (W; C^*, N) \to (\tilde{W}(Y); C^*_Y, N_Y)$ is a local isomorphism (see §5.3). Thus $(W; C^*, N)$ is reconstructed from $(Y; D^*_N, D_N)$, at least locally, as a part of $(\tilde{W}(Y); C^*_Y, N_Y)$.

Now the main theorem of this manuscript (**Two Step Reduction Theorem** for PD manifolds of second order) can be stated as follows.

Theorem 5.2. *Let* $(R; D^1, D^2)$ *and* $(\hat{R}; \hat{D}^1, \hat{D}^2)$ *be* PD *manifolds of second order, which are regular of type* $\mathfrak{s}$. *Assume that there exists a* $G_0(\mathfrak{s})$*-invariant subspace* E *of* V *of dimension* r *satisfying* $\mathfrak{f}^\perp \subset E \otimes_S V$ *and* $\dim \hat{E} = s > 0$. *Moreover assume the following two conditions for the covariant sytems of each* PD *manifold;*

(*i*) $C(E)$ *and* $\hat{C}(E)$ *are completely integrable* (*when* $r = n-1$, *assume further* $\operatorname{rank} Ch(D^2) < s$ *and* $\operatorname{rank} Ch(\hat{D}^2) < s$).

(*ii*) $C(E) \subset Ch(\tilde{N}^*)$ *and* $\hat{C}(E) \subset Ch(\hat{N}^*)$.

Let $(W; C^*, N)$ *and* $(\hat{W}; \hat{C}^*, \hat{N})$ *be the associated* IG *manifolds of corank* r *of* $(R; D^1, D^2)$ *and* $(\hat{R}; \hat{D}^1, \hat{D}^2)$ *respectively. Assume that* W *and* $\hat{W}$ *are regular with respect to* $Ch(N)$ *and* $Ch(\hat{N})$ *respectively. Let* $(Y; D^*_N, D_N)$ *and* $(\hat{Y}, D^*_{\hat{N}}, D_{\hat{N}})$ *be the leaf spaces, where* $Y = W/Ch(N)$ *and* $\hat{Y} = \hat{W}/Ch(\hat{N})$. *Let us fix points* $v_o \in R$ *and* $\hat{v}_o \in \hat{R}$ *and put* $w_o = \eta(v_o)$, $y_o = \beta(w_o)$ *and* $\hat{w}_o = \hat{\eta}(\hat{v}_o)$, $\hat{y}_o = \hat{\beta}(\hat{w}_o)$. *Then a local isomorphism* $\psi : (R; D^1, D^2) \to (\hat{R}; \hat{D}^1, \hat{D}^2)$ *such that* $\psi(v_o) = \hat{v}_o$ *induces a local isomorphism* $\varphi : (W; C^*, N) \to (\hat{W}, \hat{C}^*, \hat{N})$ *such that* $\varphi(w_o) = \hat{w}_o$ *and* $\varphi_*(\kappa_1(v_o)) = \hat{\kappa}_1(\hat{v}_o)$, *and vice versa. Furthermore a local isomorphism* $\varphi : (W; C^*, N) \to (\hat{W}, \hat{C}^*, \hat{N})$ *such that* $\varphi(w_o) = \hat{w}_o$ *induces a local isomorphism* $\phi : (Y; D^*_N, D_N) \to (\hat{Y}; D^*_{\hat{N}}, D_{\hat{N}})$ *such that* $\phi(y_o) = \hat{y}_o$ *and* $\phi_*(\kappa_2(w_o)) = \hat{\kappa}_2(\hat{w}_o)$, *and vice versa.*

Here we remark that, when $\hat{E}$ coincides with E, i.e., when $s = r$, we have $N^* = N$ and $D^*_N = D_N$. Hence, in this case, the condition (ii) is automatically satisfied under the condition (i) and the equivalence of $(R; D^1, D^2)$ is reducible to that of (Y, D_N).

In Theorem 5.2, every (local) solution of $(W; C^*, N)$ is foliated by $\mathrm{Ch}\,(N)$ so that every (local) solution of $(R; D^1, D^2)$ is foliated by s-dimensional integral manifolds of $C(E) = \mathrm{Ch}\,(\tilde{N})$. Thus $(R; D^1, D^2)$, satisfying the conditions in Theorem 5.2, admits s-dimensional characteristics. In fact, in [6], for involutive systems of second order partial differential equations for a scalar function with 3 independent variables, E.Cartan first classified involutive subspaces $\mathfrak{f} \subset S^2(V^*)$ when $\dim V = 3$, and immediately wrote the Structure Equation for each involutive system. Then he argued about the existence of 1 or 2-dimensional characteristics for such systems. As for the existemce of characteristics, Theorem 5.2 covers many of his arguments (see §8.1).

In [23], we have exhibitted typical examples of involutive symbols $\mathfrak{f} = \mathfrak{f}^1(r)$, $\mathfrak{f}^2(r)$ or $\mathfrak{f}^3(r)$ in $S^2(V^*)$, which are the only invariants of the involutive systems of second order of type $\mathfrak{s}$, where $\mathfrak{s}_{-1} = V \oplus \mathfrak{f}$. Namely we have exhibitted that, for a PD manifold $(R; D^1, D^2)$ of second order, which is regular of type $\mathfrak{s}$, where $\mathfrak{f}$ is $\mathfrak{f}^1(r)$, $\mathfrak{f}^2(r)$ or $\mathfrak{f}^3(r)$, R can be transformed to the model linear equation by a contact transformation, similarly as in the case of the system of first order partial differential equation for a scalar function (see Theorem in [23]). We will explain these phenomena for $\mathfrak{f}^2(r)$ and $\mathfrak{f}^1(r)$ in terms of the Second Reduction Theorem in §6 (the explanation for $\mathfrak{f}^3(r)$ has been given in terms of the First Reduction Theorem in §6.1 [27]).

In §7, we will discuss the construction parts of the Two Step Reduction procedure of Theorem 5.2, depending on whether $D^*_N = D_N$ or not. Finally in §8, as examples of Second Reduction Theorem, we first review the arguments in [6] and, in the rest of this section, we will show several examples of Parabolic Geometries, which are, through the Second Reduction Theorem, associated with the geometry of PD manifolds of second order.

Throughout this manuscript we always assume the differentiability of class C^∞, though the argument goes through in real or complex analytic category with suitable modifications.

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2. Involutive Grassmann Bundle over a Contact Manifold (J, C)

2.1. **Involutive Grassmann Bundle** $I^r(J)$**.** Let (J, C) be a contact manifold of dimension $2n+1$, i.e., J is a manifold of dimension $2n+1$ and C is a subbundle of the tangent bundle $T(J)$ of J of corank 1 such that the Cauchy characteristic system $\mathrm{Ch}\,(C)$ of C is trivial. We will consider the **Involutive Grassmann Bundle** $I^r(J)$ of (J, C) of codimension r $(1 \leqq r < n)$;

$$I^r(J) = \bigcup_{u \in J} I_u \xrightarrow{\pi} J, \quad I_u = \{\text{involutive subspaces } w \text{ of } \ (C(u), d\varpi) \ \text{ of codimension } r\},$$

i.e., w is a subspace of the symplectic vector space $(C(u), d\varpi)$ of codimension r such that $w^\perp \subset w$, where $C = \{\varpi = 0\}$ around u and $w^\perp = \{X \in C(u) \mid d\varpi(X, Y) = 0 \ \text{ for } \ \forall\, Y \in w\}$. Let us fix a reference point w_o of $I^r(J)$ and put $u_o = \pi(w_o)$, where $\pi : I^r(J) \to J$ is the projection. Take a canonical coordinate system $(x_1, \ldots, x_n, z, p_1, \ldots, p_n)$ of (J, C) defined on a neighborhood U' with origin u_o such that $w_o = \{X \in T_{u_o}(J) \mid \varpi(X) = dp_1(X) = \cdots = dp_r(X) = 0\}$, where $\varpi = d\,z - \sum_{i=1}^n p_i d\,x_i$. We can introduce coordinate system $(x_i, x_\alpha, z, p_i, p_\alpha, a_i^\alpha, b_i^\alpha, s_{ij})$ $(1 \leqq i \leqq j \leqq r, r+1 \leqq \alpha \leqq n)$ of $I^r(J)$ on the following neighborhood $\bar{U}$ of w_o;

$$\bar{U} = \{w \in \pi^{-1}(U') \mid dp_{r+1}, \ldots, dp_n, dx_1, \ldots, dx_n \text{ are linearly independent on } \ w\}.$$

By expressing $dp_i \mid_w$ as a linear combination of $dp_\alpha \mid_w$, $dx_i \mid_w$ and $dx_\alpha \mid_w$, we see that w is defined by

$$w = \{X \in T_u(J) \mid \varpi(X) = \varpi_1^*(X) = \cdots = \varpi_r^*(X) = 0\},$$

where $u = \pi(w)$ and

$$\varpi = dz - \sum_{a=1}^{n} p_a dx_a, \quad \varpi_i^* = dp_i - \sum_{\alpha=r+1}^{n} a_i^\alpha dp_\alpha - \sum_{\alpha=r+1}^{n} b_i^\alpha dx_\alpha - \sum_{j=1}^{r}(s_{ij} - \sum_{\alpha=r+1}^{n} a_i^\alpha b_j^\alpha) dx_j$$

for $1 \leqq i \leqq r$. We calculate

$$\begin{aligned}
d\varpi &= \sum_{i=1}^{r} dx_i \wedge dp_i + \sum_{\alpha=r+1}^{n} dx_\alpha \wedge dp_\alpha \\
&= \sum_{i=1}^{r} dx_i \wedge \varpi_i^* + \sum_{\alpha=r+1}^{n} dx_\alpha \wedge dp_\alpha + \sum_{i,\alpha} a_i^\alpha dx_i \wedge dp_\alpha \\
&\qquad\qquad + \sum_{i,\alpha} b_i^\alpha dx_i \wedge dx_\alpha + \sum_{i,j} s_{ij} dx_i \wedge dx_j - \sum_{i,j,\alpha} a_i^\alpha b_j^\alpha dx_i \wedge dx_j, \\
&= \sum_{i=1}^{r} dx_i \wedge \varpi_i^* + \sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \varpi_\alpha^* + \sum_{i,j=1}^{r} s_{ij} dx_i \wedge dx_j \\
&\equiv \sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \varpi_\alpha^* + \sum_{i,j=1}^{r} s_{ij} dx_i \wedge dx_j \pmod{\varpi, \varpi_1^*, \ldots, \varpi_r^*}
\end{aligned}$$

where we put

$$\omega^\alpha = dx_\alpha + \sum_{i=1}^{r} a_i^\alpha dx_i, \quad \varpi_\alpha^* = dp_\alpha - \sum_{j=1}^{r} b_j^\alpha dx_j \quad (r+1 \leqq \alpha \leqq n).$$

Here w is involutive if and only if $d\varpi \mid_w$ is of rank $2(n-r)$, which implies $s_{ij} = s_{ji}$ $(1 \leqq i, j \leqq r)$. Moreover we calculate

$$\begin{aligned}
d\varpi_i^* &= \sum_{\alpha=r+1}^{n} dp_\alpha \wedge da_i^\alpha + \sum_{\alpha=r+1}^{n} dx_\alpha \wedge db_i^\alpha + \sum_{j=1}^{r} dx_j \wedge ds_{ij} - \sum_{j,\alpha} dx_j \wedge (a_i^\alpha db_j^\alpha + b_j^\alpha da_i^\alpha) \\
&= \sum_{\alpha=r+1}^{n} \varpi_\alpha^* \wedge da_i^\alpha + \sum_{\alpha=r+1}^{n} \omega^\alpha \wedge db_i^\alpha + \sum_{j=1}^{r} dx_j \wedge \varpi_{ij}
\end{aligned}$$

where $\varpi_{ij} = ds_{ij} - \sum_{\alpha=r+1}^{n}(a_i^\alpha db_j^\alpha + a_j^\alpha db_i^\alpha)$ $(1 \leqq i, j \leqq r)$. Hence, for these 1-forms, we have

$$
(2.1) \qquad \begin{cases} d\varpi = \sum_{i=1}^{r} dx_i \wedge \varpi_i^* + \sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \varpi_\alpha^* \\ d\varpi_i^* = \sum_{\alpha=r+1}^{n} \varpi_\alpha^* \wedge da_i^\alpha + \sum_{\alpha=r+1}^{n} \omega^\alpha \wedge db_i^\alpha + \sum_{j=1}^{r} dx_j \wedge \varpi_{ij} \\ d\varpi_\alpha^* = \sum_{i=1}^{r} dx_i \wedge db_i^\alpha \\ d\omega^\alpha = \sum_{i=1}^{r} da_i^\alpha \wedge dx_i \\ d\varpi_{ij} = \sum_{\alpha=r+1}^{n} (db_i^\alpha \wedge da_j^\alpha + db_j^\alpha \wedge da_i^\alpha) \end{cases}
$$

We have several differential systems naturally defined on $I^r(J)$ as follows: $\bar{C} = \pi_*^{-1}(C)$ is the lift of C and canonical systems $\bar{N}$, $\bar{N}^\perp$ are defined by

$$
\bar{N}^\perp(w) = \pi_*^{-1}(w^\perp) \subset \bar{N}(w) = \pi_*^{-1}(w) \subset T_w(I^r(J)) \xrightarrow{\pi_*} T_u(J) \quad \text{at each} \quad w \in I^r(J).
$$

By the above calculation, we have, on a neighborhood $\bar{U}$, $\bar{C} = \{\varpi = 0\}$ and

$$
\bar{N} = \{\varpi = \varpi_1^* = \cdots = \varpi_r^* = 0\}, \quad \bar{N}^\perp = \{\varpi = \varpi_i^* = \varpi_\alpha^* = \omega^\alpha = 0 \ (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\}.
$$

For the Cauchy characteristic systems of these systems, we see $\mathrm{Ch}\,(\bar{N}) = \mathrm{Ch}\,(\bar{N}^\perp) = \{0\}$ and

$$
\mathrm{Ch}\,(\bar{C}) = \{\varpi = \varpi_i^* = \varpi_\alpha^* = dx_i = \omega^\alpha = 0 \quad (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\}.
$$

The dual frame of the coframe $\{\varpi, \varpi_i^*, \varpi_\alpha^*, \omega^\alpha, dx_i, db_i^\alpha, da_i^\alpha, \varpi_{ij} (1 \leqq i \leqq j \leqq r, r+1 \leqq \alpha \leqq n)\}$ on $\bar{U}$ consists of the following vector fields;

$$
\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial p_i}, \quad \frac{d^*}{d\,p_\alpha} = \frac{\partial}{\partial p_\alpha} + \sum_{j=1}^{r} a_j^\alpha \frac{\partial}{\partial p_j}, \quad \frac{d^*}{d\,x_\alpha} = \frac{\partial}{\partial x_\alpha} + p_\alpha \frac{\partial}{\partial z} + \sum_{j=1}^{r} b_j^\alpha \frac{\partial}{\partial p_j},
$$

$$
\frac{d^*}{dx_i} = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial z} + \sum_{j=1}^{r}(s_{ij} - \sum_{\alpha=r+1}^{n} a_j^\alpha b_i^\alpha) \frac{\partial}{\partial p_j} + \sum_{\alpha=r+1}^{n} b_i^\alpha \frac{d^*}{dp_\alpha} - \sum_{\alpha=r+1}^{n} a_i^\alpha \frac{d^*}{dx_\alpha},
$$

$$
\frac{d^*}{db_i^\alpha} = \frac{\partial}{\partial b_i^\alpha} + \sum_{j=1}^{r} a_j^\alpha \frac{\partial}{\partial s_{ij}} + a_i^\alpha \frac{\partial}{\partial s_{ii}}, \quad \frac{\partial}{\partial a_i^\alpha}, \quad \frac{\partial}{\partial s_{ij}}.
$$

Hence, by calculating brackets of the above vector fields, or by (2.1), we obtain

$$
\partial \bar{N}^\perp = \bar{C}, \qquad \partial \bar{N} = T(I^r(J)),
$$

where ∂D denotes the derived system of D. Moreover, putting

$$
A(\bar{C}) = \{X \in \mathrm{Ch}\,(\bar{C}) \mid \varpi_{ij}(X) = 0 \quad (1 \leqq i, j \leqq r)\},
$$

we have $\partial(A(\bar{C})) = \mathrm{Ch}\,(\bar{C})$. Here we note that $A(\bar{C})$ corresponds to the standard diffrential system of each fibre (the Involutive Grassmann manifold of codimension r) of the projection $\pi : I^r(J) \to J$, which is an R-space of type $(C_{n-1}, \{\alpha_r\})$ (cf. §4 in [24]).

2.2. IG **manifold** $(W; C^*, N)$ **of corank** r**.** Let W be a submanifold of $I^r(J)$ satisfying the following condition:

$$(W.0) \qquad q : W \to J : \quad \text{submersion},$$

where $q = \pi\mid_W$ and $\pi : I^r(J) \to J$ is the projection. From the two differential systems $\bar{C}$ and $\bar{N}$ on $I^r(J)$, we obtain the differential systems C^* and N on W by restricting $\bar{C}$ and $\bar{N}$ to W. At each point $w \in W$, we define the bilinear map $\gamma_w : N(w) \times N(w) \to \mathfrak{t}_{-3}(w) = T_w(W)/C^*(w)$ as follows; For vectors X and Y in $N(w)$, let $\tilde{X} \in \Gamma(N)$ and $\tilde{Y} \in \Gamma(N)$ be vectorfields such that $X = (\tilde{X})_w$ and $Y = (\tilde{Y})_w$, where $\Gamma(N)$ denotes the space of sections of N. Then γ_w is defined by

$$\gamma_w(X, Y) = \pi_{-3}([\tilde{X}, \tilde{Y}]_w),$$

where $\pi_{-3}; T_w(W) \to \mathfrak{t}_{-3}(w) = T_w(W)/C^*(w)$ is the projection. Let w_o be an arbitrary point of W. We will introduce a coordinate system $(x_i, x_\alpha, z, p_i, p_\alpha, a_i^\alpha, b_i^\alpha, s_{ij})$ $(1 \leqq i \leqq j \leqq r, r+1 \leqq \alpha \leqq n)$ of $I^r(J)$ on a neighborhood $\bar{U}$ of w_o as in §2.1. Then, by the condition $(W.0)$, 1-forms $\{\varpi, \varpi_i^*, \varpi_\alpha^*, \omega^\alpha, dx_i (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\}$ remain linearly independent at each point of $W \cap \bar{U}$, when restricted to the submanifold W. Hence we have

$$C^* = \{\varpi = 0\}, \qquad N = \{\varpi = \varpi_1^* = \cdots = \varpi_r^* = 0\},$$

where we denote the restricted 1-forms on W by the same symbols as those on $\bar{U}$. The contact form ϖ fixes the basis Z of $\mathfrak{t}_{-3}(w)$ by $Z = \pi_{-3}(\hat{Z})$, $\hat{Z} \in T_w(W)$ such that $\varpi(\hat{Z}) = 1$, Then we have

$$\gamma_w(X, Y) = \varpi([\tilde{X}, \tilde{Y}]_w) \cdot Z = -d\varpi(X, Y) \cdot Z,$$

which shows that $\gamma_w(X, Y)$ is well-defined for $X, Y \in N(w)$ (cf. §3.1 [27]). Moreover, from (2.1), we have

$$(2.2) \qquad d\varpi = \sum_{i=1}^{r} dx_i \wedge \varpi_i^* + \sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \varpi_\alpha^* \equiv \sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \varpi_\alpha^* \quad (\text{mod} \quad \varpi, \varpi_1^*, \ldots, \varpi_r^*).$$

Hence we have

$$\begin{aligned} \mathrm{Ch}\,(C^*) &= \{\varpi = \varpi_i^* = \varpi_\alpha^* = \omega^\alpha = dx_i = 0 \quad (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\} \\ &= \{dz = dp_1 = \cdots = dp_n = dx_1 = \cdots = dx_n = 0\} = \mathrm{Ker}\, p_* \end{aligned}$$

and

$$\begin{aligned} \mathrm{Ker}\,\gamma_w &= \{X \in N(w) \mid \gamma_w(X, Y) = 0 \quad \text{for} \quad \forall\, Y \in N(w)\} \\ &= \{\varpi = \varpi_i = \varpi_\alpha^* = \omega^\alpha = 0 \quad (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\} \end{aligned}$$

Thus $N^\perp = \mathrm{Ker}\,\gamma = \bigcup_{w \in W} \mathrm{Ker}\,\gamma_w$ is a subbundle of N which coincides with the restriction to W of $\bar{N}^\perp$ on $I^r(J)$. Summarizing, $(W; C^*, N)$ satisfies the following conditions:

$(W.1)$ *C^* and N are differential systems of codimension 1 and $r+1$ respectively such that $N \subset C^*$.*

$(W.2)$ *$Ch\,(C^*)$ is a subbundle of N of codimension $2n - r$.*

$(W.3)$ $\mathrm{rank}\,\gamma_w = 2(n - r)$ *at each $w \in W$.*

$(W.4)$ $\mathrm{Ch}\,(C^*)(w) \cap \mathrm{Ch}\,(N)(w) = \{0\}$ *at each $w \in W$.*

In fact the last condition $(W.4)$ follows from the Realization Lemma for (W, N, q, J) (see §4.1 [27]). The condition $(W.3)$ is equivalent to the following condition:

$(W.3')$ *$N^\perp = \mathrm{Ker}\,\gamma$ is a subbundle of N of codimension $2(n - r)$.*

Conversely these four conditions, at least locally, characterize submanifolds in $I^r(J)$ satisfying $(W.0)$ as in the following.

We call the triplet $(W; C^*, N)$ of a manifold and two differential systems on it an IG (**Involutive Grassmann) manifold of corank** r if these satisfy the above four conditions (W.1) to (W.4). We have the (local) Realization Theorem for IG manifolds as follows: From conditions (W.1) and (W.2), the codimension of the foliation defined by $\mathrm{Ch}\,(C^*)$ is $2n+1$. Assume that W is regular with respect to $\mathrm{Ch}\,(C^*)$, i.e., the space $J = W/\mathrm{Ch}\,(C^*)$ of leaves of this foliation is a manifold of dimension $2n+1$ such that each fibre of the projection $q : W \to J = W/\mathrm{Ch}\,(C^*)$ is connected and q is a submersion. Then C^* drops down to J. Namely there exists a differential system C on J of codimension 1 such that $C^* = q_*^{-1}(C)$. From $\mathrm{Ch}\,(C) = \{0\}$, (J, C) becomes a contact manifold of dimension $2n+1$. Conditions (W.2) and (W.3) imply that the image of the following map ι is an involutive subspace of codimension r in the symplectic vector space $(C(u), d\varpi)$:

$$\iota(w) = q_*(N(w)) \subset C(u), \qquad u = q(w).$$

Namely ι is a map from W into $I^r(J)$, which is an immersion by the Realization Lemma for (W, N, q, J) and (W.4). Moreover, similarly as in Theorem 4.1 [27], we obtain

Theorem 2.1. *Let $(W; C^*, N)$ and $(\hat{W}; \hat{C}^*, \hat{N})$ be IG manifolds of corank r. Assume that W and $\hat{W}$ are regular with respect to Ch$\,(C^*)$ and Ch$\,(\hat{C}^*)$ respectively. Let (J, C) and $(\hat{J}, \hat{C})$ be the associated contact manifolds. Then an isomorphism $\Phi : (W; C^*, N) \to (\hat{W}; \hat{C}^*, \hat{N})$ induces a contact diffeomorphism $\varphi : (J, C) \to (\hat{J}, \hat{C})$ such that the following commutes;*

$$\begin{array}{ccc} W & \xrightarrow{\ \iota\ } & I^r(J) \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_*} \\ \hat{W} & \xrightarrow{\ \hat{\iota}\ } & I^r(\hat{J}). \end{array}$$

By this theorem, the submanifold theory for $I^r(J)$ is reformulated as the geometry of IG manifolds of corank r .

2.3. **Covariant systems of** $(W; C^*, N)$**.** Let $(W; C^*, N)$ be an IG manifold of corank r. In this subsection, we will consider several covariant systems of $(W; C^*, N)$.

(1) Covariant system $N^\perp$.

$N^\perp$ is defined by $N^\perp(w) = \mathrm{Ker}\,\gamma_w$ at each $w \in W$. Hence, by (W.3), $N^\perp$ is a subbundle of N of codimension $2(n-r)$ and contains $\mathrm{Ch}\,(C^*)$ as a subbundle of codimension r. In fact, utilizing the local Realization Theorem for $(W; C^*, N)$ and canonical coordinate system as in §2.1, we have

$$N = \{\varpi = \varpi_1^* = \cdots = \varpi_r^* = 0\}, \quad N^\perp = \{\varpi = \varpi_i^* = \varpi_\alpha^* = \omega^\alpha = 0 \quad (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\}.$$

Hence we see, from (2.2), $\mathrm{Ch}\,(N)(w) \subset N^\perp(w)$ at each $w \in W$. Moreover we see, from (2.1), $\mathrm{Ch}\,(C^*)(w) \cap \mathrm{Ch}\,(N^\perp)(w) = \{0\}$ at each $w \in W$.

(2) Covariant system $N^* = \partial N^\perp + N$ and the weak symbol algebra $\mathfrak{t}(w)$ of $(W; C^*, N)$.

N^* is defined by $N^*(w) = \partial N^\perp(w) + N(w)$ at each $w \in W$, where $\partial N^\perp$ denotes the derived system of $N^\perp$. From $d\varpi \equiv 0 \pmod{\varpi, \varpi_1^*, \ldots, \varpi_n^*}$, it follows that $\partial N^\perp \subset C^*$. In the following, we will assume that N^* is a subbundle of $T(W)$, hence of C^*, which contains N. Here $N^\perp$ and C^* are subbundles of $T(W)$ such that $\partial N^\perp \subset C^*$. Hence subbundles $N^\perp$, C^* and $T(W)$ define a filtration on W. Namely, putting $T^{-1} = N^\perp$, $T^{-2} = C^*$, $T^p = T(W)$ for $p \leqq -3$, we have

$$[\mathcal{T}^p . \mathcal{T}^q] \subset \mathcal{T}^{p+q} \qquad \text{for} \quad p, q < 0,$$

where $\mathcal{T}^p = \Gamma(T^p)$. We can form the weak symbol algebra $\mathfrak{t}(w)$ of $(W; C^*, N)$ at $w \in W$ as follows: Put $\mathfrak{t}_{-3}(w) = T_w(W)/C^*(w)$, $\mathfrak{t}_{-2}(w) = C^*(w)/N^\perp(w)$ and $\mathfrak{t}_{-1}(w) = N^\perp(w)$. Then as in the symbol algebra of PD manifolds, we can introduce the Lie brackets in

$$\mathfrak{t}(w) = \mathfrak{t}_{-3}(w) \oplus \mathfrak{t}_{-2}(w) \oplus \mathfrak{t}_{-1}(w),$$

as follows; For $X \in \mathfrak{t}_p(w)$ and $Y \in \mathfrak{t}_q(w)$, let us take $\tilde{X} \in \mathcal{T}^p$ and $\tilde{Y} \in \mathcal{T}^q$ such that $X = \pi_p((\tilde{X})_w)$ and $Y = \pi_q((\tilde{Y})_w)$, where $\pi_p : T^p(w) \to \mathfrak{t}_p(w)$ is the projection. Then the bracket product is defined by

$$[X, Y] = \pi_{p+q}([\tilde{X}, \tilde{Y}]_w) \in \mathfrak{t}_{p+q}(w).$$

Now N defines a subspace $\mathfrak{t}_N(w) = N(w)/N^\perp(w)$ of $\mathfrak{t}_{-2}(w)$. Then γ_w induces the non-degenerate paing $\hat{\gamma}_w : \mathfrak{t}_N(w) \times \mathfrak{t}_N(w) \to \mathfrak{t}_{-3}(w)$. Moreover, in Lie brackets of $\mathfrak{t}(w)$, we have

$$[\mathfrak{t}_{-1}(w), \mathfrak{t}_N(w)] = 0,$$

which is equivalent to $[\Gamma(N^\perp), \Gamma(N)] \subset \Gamma(C^*)$. Namely $\mathfrak{t}_N(w)$ is an abelian ideal of $\mathfrak{t}(w)$. Thus we can form the quotient Lie algebra

$$\tilde{\mathfrak{t}}(w) = \mathfrak{t}(w)/\mathfrak{t}_N(w) = \mathfrak{t}_{-3}(w) \oplus \tilde{\mathfrak{t}}_{-2}(w) \oplus \mathfrak{t}_{-1}(w),$$

where $\tilde{\mathfrak{t}}_{-2}(w) = C^*(w)/N(w)$. Conversely, for $Y \in \mathfrak{t}_{-2}(w)$, $[Y, \mathfrak{t}_{-1}(w)] = 0$ implies $Y \in \mathfrak{t}_N(w)$, which can be checked as follows; Take a vector field $\tilde{Y} \in \Gamma(C^*)$ such that $\pi_{-2}((\tilde{Y})_w) = Y$. Then, from (2.2), we see that $[\tilde{Y}, \Gamma(N^\perp)] \subset \Gamma(C^*)$ iff

$$\tilde{Y} \rfloor d\varpi \equiv -\sum_{i=1}^r \varpi_i^*(\tilde{Y}) dx_i \equiv 0 \pmod{N^\perp} \quad \Longleftrightarrow \quad \varpi_i^*(\tilde{Y}) = 0 \quad \text{for} \quad i = 1, \ldots, r,$$

which implies $\tilde{Y} \in \Gamma(N)$. Thus, in the Lie algebra $\tilde{\mathfrak{t}}(w)$, we have

$(a.1)$ If $\bar{Y} \in \tilde{\mathfrak{t}}_{-2}(w)$ and $[\bar{Y}, \mathfrak{t}_{-1}(w)] = 0$, then $\bar{Y} = 0$.

$(a.2)$ $\mathrm{Ch}\,(C^*)(w) = \{X \in \mathfrak{t}_{-1}(w) \mid [X, \tilde{\mathfrak{t}}_{-2}(w)] = 0\}$.

Here $(a.2)$ can be checked as follows; Take a vector field $\tilde{X} \in \Gamma(N^\perp)$ such that $(\tilde{X})_w = X \in \mathfrak{t}_{-1}(w) = N^\perp(w)$. Then $[X, \tilde{\mathfrak{t}}_{-2}(w)] = 0$ iff $[\tilde{X}, \Gamma(C^*)] \subset \Gamma(C^*)$, which is equivalent to $\tilde{X} \in \Gamma(\mathrm{Ch}\,(C^*))$. By $(a.1)$ and $(a.2)$, the Lie bracket in $\tilde{\mathfrak{t}}(w)$ induces the non-degenerate paring

$$\tilde{\gamma}_w : E(w) \times \tilde{\mathfrak{t}}_{-2}(w) \to \mathfrak{t}_{-3}(w),$$

where $E(w) = \mathfrak{t}_{-1}(w)/\mathrm{Ch}\,(C^*)(w)$. Thus $\tilde{\mathfrak{t}}_{-2}(w) \cong \mathfrak{t}_{-3}(w) \otimes E(w)^*$.

(3) Covariant systems $H(N)$ and $S(N)$.

Here we define, at each $w \in W$,

$$H(N)(w) = \{X \in N^\perp(w) \mid [X, \mathfrak{t}_{-1}(w)] = 0 \text{ in } \tilde{\mathfrak{t}}(w)\} = \{X \in N^\perp(w) \mid [X, \Gamma(N^\perp)] \subset \Gamma(N)\},$$

and

(2.3) $$S(N)(w) = H(N)(w) \cap \mathrm{Ch}\,(C^*)(w) = \{X \in \mathrm{Ch}\,(C^*)(w) \mid [X, \Gamma(N^\perp)] \subset \Gamma(N)\}.$$

Utilizing the canonical coordinate system as in §2.2, we have, for $X \in \mathrm{Ch}\,(C^*)(w)$,

$$X \rfloor d\varpi \equiv 0, \quad X \rfloor d\varpi_i^* \equiv -\sum_{j=1}^r \varpi_{ij}(X) dx_j \pmod{N^\perp}$$

Hence we have

$$S(N)(w) = \{\varpi = \varpi_i^* = \varpi_\alpha^* = \omega^\alpha = dx_i = \varpi_{ij} = 0 \quad (1 \leqq i \leqq j \leqq r, r+1 \leqq \alpha \leqq n)\}.$$

Now, for a vector f of $\mathrm{Ch}\,(C^*)(w)$, we have a map $\kappa : \mathrm{Ch}\,(C^*)(w) \to \mathfrak{t}_{-3}(w) \otimes S^2(E(w)^*)$ by taking brackets in $\tilde{\mathfrak{t}}(w)$

$$\kappa(f)(\bar{X}, \bar{Y}) = [[f, X], Y] = [[f, Y], X] \in \mathfrak{t}_{-3}(w),$$

for $X, Y \in \mathfrak{t}_{-1}(w)$ such that $\bar{X} = \pi_{-1}(X), \bar{Y} = \pi_{-1}(Y)$, where $[f, [X, Y]] = 0$ and $\pi_{-1} : \mathfrak{t}_{-1}(w) \to E(w)$ is the projection. Then κ induces the injective map

$$\hat{\kappa} : \mathrm{Ch}\,(C^*)(w)/S(N)(w) \to \mathfrak{t}_{-3}(w) \otimes S^2(E(w)^*).$$

Moreover, when $S(N)$ forms a subbundle, we have an injective map

$$\gamma_c : S(N)(w) \to \mathfrak{t}_N(w) \otimes E(w)^* \tag{2.4}$$

as follows; For vectors $X \in S(N)(w)$ and $Y \in N^{\perp}(w)$, we take $\tilde{X} \in \Gamma(S(N))$ and $\tilde{Y} \in \Gamma(N^{\perp})$ such that $X = \tilde{X}_w$ and $Y = \tilde{Y}_w$. Then $[\tilde{X}, \tilde{Y}] \in \Gamma(N)$ and $[\tilde{X}, \tilde{Y}] \in \Gamma(\mathrm{Ch}\,(C^*))$ when $\tilde{Y} \in \Gamma(\mathrm{Ch}\,(C^*))$. From $[f\tilde{X}, g\tilde{Y}] = fg[\tilde{X}, \tilde{Y}] + f(\tilde{X}(g))Y - g(\tilde{Y}(f))X$, we see

$$\hat{\gamma}_c(X, Y) = \pi_N([\tilde{X}, \tilde{Y}]_w) \in \mathfrak{t}_N(w) = N(w)/N^{\perp}(w)$$

depends only on X and Y and $\hat{\gamma}_c(X, Y) = 0$ if $Y \in \mathrm{Ch}\,(C^*)(w)$, where $\pi_N : N(w) \to \mathfrak{t}_N(w)$ is the projection. Thus $\hat{\gamma}_c$ induces the map $\gamma_c : S(N)(w) \to \mathfrak{t}_N(w) \otimes E(w)^*$. Assume that $\gamma_c(X) = 0$. This implies that $\hat{\gamma}_c(X, Y) = \pi_N([\tilde{X}, \tilde{Y}]_w) = 0$ for any $Y \in N^{\perp}(w)$. Hence we have $[X, \Gamma(N^{\perp})] \subset \Gamma(N^{\perp})$, i.e., $X \in \mathrm{Ch}\,(N^{\perp})(w)$. Thus $X = 0$ follows from $\mathrm{Ch}\,(C^*)(w) \cap \mathrm{Ch}\,(N^{\perp})(w) = \{0\}$ at each $w \in W$, which shows that γ_c is injective.

Now we prepare the following proposition for the later use in §7.2. We assume the following compatibility condition for $(W; C^*, N)$;

(C^*) There exists, at each $w \in W$, an n-dimensional integral element E of (W, N) such that

$$E \cap \mathrm{Ch}\,(C^*)(w) = \{0\}.$$

Then, $\pi_N(E)$ is an isotropic subspace of the symplectic vector space $(\mathfrak{t}_N(w), \hat{\gamma}_W)$, where $\pi_N : N(w) \to \mathfrak{t}_N(w) = N(w)/N^{\perp}(w)$ is the projection. Hence $E' = E \cap N^{\perp}(w)$ is an r-dimensional subspace such that

$$\mathfrak{t}_{-1}(w) = N^{\perp}(w) = E' \oplus \mathrm{Ch}\,(C^*)(w) \quad \text{and} \quad [E', E'] = 0 \quad \text{in} \quad \tilde{\mathfrak{t}}(w). \tag{2.5}$$

We have (cf. [27] Proposition 4.1)

Proposition 2.1. *Let $(W; C^*, N)$ be an IG manifold of corank r satisfying the condition (C^*) above. Then $\dim C^*(w)/N^*(w) = \dim H(N)(w)/S(N)(w)$ at each $w \in W$.*

Proof. We utilize the above decomposition of $N^{\perp}(w) = E' \oplus \mathrm{Ch}\,(C^*)(w)$. For $X \in H(w)$, we decompose $X = v_X + A_X$, where $v_X \in E'$ and $A_X \in \mathrm{Ch}\,(C^*)(w)$. From $[X, \mathfrak{t}_{-1}(w)] = 0$, we have $[X, v] = [A_X, v] = 0$ for $\forall v \in E'$ and $[X, A] = [v_X, A] = 0$ for $\forall A \in \mathrm{Ch}\,(C^*)(w)$. Hence $A_X \in S(N)(w)$ and we get

$$H(N)(w) = E'' \oplus S(N)(w),$$

where $E'' = E' \cap H(N)(w) = \{v \in E' \mid [v, A] = 0 \text{ for } \forall A \in \mathrm{Ch}\,(C^*)(w)\}$.

On the other hand, $N^*(w)$ defines the subspace $\mathfrak{n}^*(w) = \tilde{\pi}_{-2}(N^*(w)) \subset \tilde{\mathfrak{t}}_{-2}(w) = C^*(w)/N(w)$. In fact $\mathfrak{n}^*(w) = [\mathfrak{t}_{-1}(w), \mathfrak{t}_{-1}(w)]$ in $\tilde{\mathfrak{t}}(w)$. This subspace $\mathfrak{n}^*(w)$ defines the subspace F of $E(w)^*$ via the non-degenerate paring $\tilde{\gamma}_w$. Identifying E' with $E(w) = N^{\perp}(w)/\mathrm{Ch}\,(C^*)(w)$, let $F^{\perp} \subset E'$ be the annihilator of F. Then we get $[v, \mathfrak{n}^*(w)] = 0$ for $v \in F^{\perp}$ and we calculate

$$[v, [\hat{v}, A]] = [\hat{v}, [v, A]] = 0 \qquad \text{for} \quad \forall \hat{v} \in E', \forall A \in \mathrm{Ch}\,(C^*)(w).$$

Hence, by $(a.1)$ and $(a.2)$, we obtain $[v, A] = 0$ for $\forall A \in \mathrm{Ch}\,(C^*)(w)$, which shows $F^{\perp} = E''$. This completes the proof of Proposition. □

3. IG manifolds induced from PD manifolds of second order

3.1. **Covariant Systems associated with a PD manifold of second order.** Let $(R; D^1, D^2)$ be a PD manifold of second order. We will assume the regularity of $(R; D^1, D^2)$ in terms of the symbol algebras as follows; Let $\mathfrak{c}^2(n) = \mathfrak{c}_{-3} \oplus \mathfrak{c}_{-2} \oplus \mathfrak{c}_{-1}$ be the symbol algebra of $(L(J), E)$ given by

$$\mathfrak{c}_{-3} = \mathbb{R}, \quad \mathfrak{c}_{-2} = V^*, \quad \mathfrak{c}_{-1} = V \oplus S^2(V^*),$$

where $\dim V = n$ and $\dim J = 2n+1$ (cf. §2.5 [27]). Let $\mathfrak{f}$ be a fixed subspace of $S^2(V^*)$ and $\mathfrak{s}$ be a subalgebra of $\mathfrak{c}^2(n)$, which is defined by

$$\mathfrak{s}_{-3} = \mathbb{R}, \quad \mathfrak{s}_{-2} = V^*, \quad \mathfrak{s}_{-1} = V \oplus \mathfrak{f}, \qquad \mathfrak{f} \subset S^2(V^*),$$

$(R; D^1, D^2)$ is called regular of type $\mathfrak{s}$ iff the symbol algebra $\mathfrak{s}(v)$ of $(R; D^1, D^2)$ at each $v \in R$ is isomorphic to $\mathfrak{s}$ (cf. [20],[23], §1 [27]). We note here that R satisfies the compatibility condition (C):

$$(C) \qquad p^{(1)} : R^{(1)} \to R \textit{ is onto.}$$

where $R^{(1)}$ is the first prolongation of $(R; D^1, D^2)$ (cf. §4.2 [27]).

Here we first recall, from §3 in [20], the structure of the group $G(\mathfrak{c}^2(n))$ of graded Lie algebra automorphisms of $\mathfrak{c}^2(n)$. Let $\kappa : \mathfrak{c}_{-1} \to V_c = \mathfrak{c}_{-1}/\mathfrak{F}$, $\mathfrak{F} = S^2(V^*) \subset \mathfrak{c}_{-1}$, be the projection. Then $\kappa_0 = \kappa \mid_V$ is a linear isomorphism of V onto V_c. Since $\mathfrak{F} = \{X \in \mathfrak{c}_{-1} \mid [X, \mathfrak{c}_{-2}] = 0\}$, it follows that $\phi(\mathfrak{F}) = \mathfrak{F}$ for $\phi \in G(\mathfrak{c}^2(n))$. Hence ϕ induces the linear isomorphism $\hat{\phi} : V_c \to V_c$ such that $\hat{\phi} \cdot \kappa = \kappa \cdot \phi$. We define the closed normal subgroup $N(\mathfrak{c}^2(n))$ of $G(\mathfrak{c}^2(n))$ by setting

$$N(\mathfrak{c}^2(n)) = \{\phi \in G(\mathfrak{c}^2(n)) \mid \phi \mid_{\mathfrak{c}_{-3}} = id_{\mathfrak{c}_{-3}} \quad \text{and} \quad \hat{\phi} = id_{V_c} \}.$$

We define the homomorphism $\chi : GL(V) \times GL(\mathbb{R}) \to G(\mathfrak{c}^2(n))$, for $a \in GL(V)$ and $b \in GL(\mathbb{R}) = \mathbb{R}^\times$, by putting

$$\chi(a,b) \mid_V = a, \quad \chi(a,b) \mid_{\mathfrak{c}_{-3}} = b \cdot id_{\mathfrak{c}_{-3}}, \quad \chi(a,b) \mid_{V^*} = b \cdot (a^*)^{-1} \quad \text{and} \quad \chi(a,b) \mid_{S^2(V^*)} = b \cdot \otimes^2 (a^*)^{-1},$$

where a^* is the adjoint linear map of a. We put $G_0(\mathfrak{c}^2(n)) = \chi(GL(V) \times GL(\mathbb{R}))$. Moreover, let $S(\mathfrak{c}^2(n))$ be the set of abelian subalgebras $\hat{V}$ of $\mathfrak{c}^2(n)$ such that $\mathfrak{c}_{-1} = \hat{V} \oplus \mathfrak{F}$ (direct sum). Then we have (Proposition 3.7 [20])

(1) $N(\mathfrak{c}^2(n))$ is canonically isomorphic to the vector group $S^3(V^*)$. Furthermore $N(\mathfrak{c}^2(n))$ acts simply transitively on $S(\mathfrak{c}^2(n))$.
(2) $G_0(\mathfrak{c}^2(n)) = \{\phi \in G(\mathfrak{c}^2(n)) \mid \phi(V) = V \}$ and $G(\mathfrak{c}^2(n)) = G_0(\mathfrak{c}^2(n)) \cdot N(\mathfrak{c}^2(n))$ is the semi-direct product.

The action of $N(\mathfrak{c}^2(n))$ can be explicitly described as follows: First we identify

$$S^3(V^*) \cong \{\rho : V \to S^2(V^*) \mid v_1 \rfloor \rho(v_2) = v_2 \rfloor \rho(v_1) \}.$$

Then, for $\rho \in S^3(V^*)$, we define the element $A_\rho \in N(\mathfrak{c}^2(n))$ by

$$A_\rho \mid_{\mathfrak{c}_{-3}} = id_{\mathfrak{c}_{-3}}, \quad A_\rho \mid_{V^*} = id_{V^*}, \quad A_\rho \mid_{\mathfrak{F}} = id_{\mathfrak{F}} \quad \text{and} \quad A_\rho \mid_V = id_V + \rho.$$

Let $G(\mathfrak{s})$ be the group of graded Lie algebra automorphisms of $\mathfrak{s}$. Then $G(\mathfrak{s})$ is a subgroup of $G(\mathfrak{c}^2(n))$. In fact, we have (Corollary 5.8 [20])

$$G(\mathfrak{s}) = \{\sigma \in G(\mathfrak{c}^2(n)) \mid \sigma(\mathfrak{s}) = \mathfrak{s} \} = \{\sigma \in G(\mathfrak{c}^2(n)) \mid \sigma(\mathfrak{s}_{-1}) = \mathfrak{s}_{-1} \subset \mathfrak{c}_{-1} \}.$$

Thus $G(\mathfrak{s})$ is a semi-direct product $G_0(\mathfrak{s}) \cdot N(\mathfrak{s})$, where

$$G_0(\mathfrak{s}) = \{\sigma \in G(\mathfrak{s}) \mid \sigma(V) = V\} = \{\sigma \in G_0(\mathfrak{c}^2(n)) \mid \sigma(\mathfrak{f}) = \mathfrak{f} \subset \mathfrak{F} = S^2(V^*) \},$$

$$N(\mathfrak{s}) = \{\sigma \in N(\mathfrak{c}^2(n)) \mid \sigma(\mathfrak{s}_{-1}) = \mathfrak{s}_{-1} \} = \{A_\rho \in N(\mathfrak{c}^2(n)) \mid \rho(V) \subset \mathfrak{f}\} \cong \mathfrak{f}^{(1)}.$$

Here $\mathfrak{f}^{(1)}$ denotes the prolongation of $\mathfrak{f}$ (see §5 [20] for the detail).

Now, starting from an invariant subspace $E \subset V$ of $G_0(\mathfrak{s})$, we will construct the first order covariant system $\tilde{N} = \tilde{N}(E)$ of $(R; D^1, D^2)$ as in the following; Let $E^\perp \subset V^*$ be the annihilator subspace of E. Then $E^\perp$ is an $G(\mathfrak{s})$-invariant subspace of $\mathfrak{s}_{-2} = V^*$. Let v be any point of R and let $\mathfrak{s}(v)$ be the symbol algebra at $v \in R$. Take a graded Lie algebra isomorphism ϕ of $\mathfrak{s}(v)$ onto $\mathfrak{s}$. Let $\mathfrak{n}(E)(v)$ denote the linear subspace of $\mathfrak{s}_{-2}(v)$ defined by

$$\mathfrak{n}(E)(v) = \phi^{-1}(E^\perp).$$

Then, since $E^\perp$ is $G(\mathfrak{s})$-invariant, it follows that $\mathfrak{n}(E)(v)$ is well-defined. We define the linear subspace $\tilde{N}(E)(v)$ of $D^1(v)$ by setting

$$\tilde{N}(E)(v) = (\pi_{-2})^{-1}(\mathfrak{n}(E)(v)),$$

where $\pi_{-2} : D^1(v) \to \mathfrak{s}_{-2}(v) = D^1(v)/D^2(v)$ is the projection. Then it follows that the assignment $v \mapsto \tilde{N}(E)(v)$ defines a subbundle $\tilde{N} = \tilde{N}(E)$ of D^1, which contains D^2.

Moreover we will define the covariant systems $\tilde{N}^\perp = \tilde{N}^\perp(E)$ and $\tilde{N}^* = \tilde{N}^*(E)$ of $(R; D^1, D^2)$ as follows; Take a graded Lie algebra isomorphism ϕ of $\mathfrak{s}(v)$ onto $\mathfrak{s}$. Let $\tilde{N}^\perp(E)(v)$ denote the linear subspace of $D^2(v) = \mathfrak{s}_{-1}(v)$ defined by

$$\tilde{N}^\perp(E)(v) = \phi^{-1}(E \oplus \mathfrak{f}) \subset D^2(v),$$

Then, since $E \oplus \mathfrak{f}$ is $G(\mathfrak{s})$-invariant, it follows that $\tilde{N}^\perp(E)(v)$ is well-defined. Thus the assignment $v \to \tilde{N}^\perp(E)(v)$ defines a subbundle $\tilde{N}^\perp = \tilde{N}^\perp(E)$ of D^2.

Then $\tilde{N}^* = \tilde{N}^*(E)$ is defined by $\tilde{N}^*(E)(v) = \partial\tilde{N}^\perp(E)(v) + \tilde{N}(E)(v)$ at each $v \in R$, where $\partial\tilde{N}^\perp(E)$ denotes the derived system of $\tilde{N}^\perp(E)$. From $\tilde{N}^\perp(E) \subset D^2 \subset \tilde{N}(E)$ and $\partial D^2 \subset D^1$, it follows that $\partial\tilde{N}^\perp(E) \subset D^1$ and $D^2 \subset \tilde{N}^*(E) \subset D^1$. In terms of the symbol algebra, we calculate $[E \oplus \mathfrak{f}, E \oplus \mathfrak{f}] = E \rfloor \mathfrak{f}$. Hence $\tilde{N}^*(E)$ corresponds to the subspace $E \rfloor \mathfrak{f} + E^\perp$ of $\mathfrak{s}_{-2} = V^*$. We put

$$\hat{E} = E \cap (E \rfloor \mathfrak{f})^\perp = \{v \in E \mid v \circledcirc E \subset \mathfrak{f}^\perp\}.$$

Then $\tilde{N}^*(E)$ coincides with the first order covariant system corresponding to the $G_0(\mathfrak{s})$-invariant subspace $\hat{E}$. Namely take a graded Lie algebra isomorphism ϕ of $\mathfrak{s}(v)$ onto $\mathfrak{s}$. Let $\mathfrak{n}^*(E)(v)$ denote the linear subspace of $\mathfrak{s}_{-2}(v)$ defined by

$$\mathfrak{n}^*(E)(v) = \phi^{-1}((\hat{E})^\perp).$$

Then we have

$$\tilde{N}^*(E)(v) = (\pi_{-2})^{-1}(\mathfrak{n}^*(E)(v)).$$

In particular $\tilde{N}^* = \tilde{N}^*(E)$ is a subbundle under our regularity condition for $(R; D^1, D^2)$.

3.2. *IG* **manifold** $(W; C^*, N)$ **associated with** $(R; D^1, D^2, \tilde{N}(E))$**.** Let $(R; D^1, D^2)$ be a *PD* manifold of second order, which is regular of type $\mathfrak{s}$. Moreover we assume that R is regular with respect to $\mathrm{Ch}\,(D^1)$, i.e., the space $J = R/\mathrm{Ch}\,(D^1)$ of leaves of this foliation is a manifold of dimension $2n+1$ such that each fibre of the projection $p : R \to J$ is connected and p is a submersion. Then we have a differential system C on J of codimension 1 such that $D^1 = p_*^{-1}(C)$ and (J, C) becomes a contact manifold.

Now assume that there exists a $G_0(\mathfrak{s})$-invariant subspace E of V of dimension r. Then we have the first order covariant system $\tilde{N} = \tilde{N}(E)$ of $(R; D^1, D^2)$. $\tilde{N}$ contains D^2 and is a subbundle of D^1 of codimension r. For a point $v \in R$, we see that $p_*(D^1(v)) = C(u)$, $u = p(v)$ and $\iota(v) = p_*(D^2(v))$ is a legendrian subspace of (J, C), where $\iota : R \to L(J)$ is the canonical immersion (cf. §4.1 [27]). Thus $w = p_*(\tilde{N}(v)) \subset C(u)$ is an involutive subspace of $(C(u), d\varpi)$

of codimension r such that $w^\perp \subset \iota(v) \subset w$. Let $I^r(J)$ be the Involutive Grassmann bundle of (J, C) of codimension r. Utilizing $\tilde{N}$, we will consider the map $\eta : R \to I^r(J)$ defined by

$$\eta(v) = p_*(\tilde{N}(v)) \in I^r(J) \qquad \text{for} \quad v \in R.$$

By Realization Lemma for $(R, \tilde{N}, p, J)$, we have $\operatorname{Ker}\eta_* = \operatorname{Ch}(D^1) \cap \operatorname{Ch}(\tilde{N})$. Thus, if $\operatorname{Ch}(D^1) \cap \operatorname{Ch}(\tilde{N})$ is a subbundle, η is a map of constant rank so that the image $W = \operatorname{Im}(\eta)$ of η is, at least locally, a submanifold of $I^r(J)$ such that $q : W \to J$ is a submersion, where $q = \pi \mid_W$ and $\pi : I^r(J) \to J$ is the projection.

In the rest of this subsection, we assume that $\operatorname{Ch}(D^1) \cap \operatorname{Ch}(\tilde{N})$ is a subbundle so that $W = \operatorname{Im}(\eta)$ is a submanifold of $I^r(J)$ and will consider the relations of several covariant systems on R and W. As a submanifold of $I^r(J)$ satisfying the condition $(W.0)$, W carries two differential systems C^* and N such that $C^* = q_*^{-1}(C)$. $(W; C^*, N)$ is an IG manifold of corank r. $(W; C^*, N)$ is called the IG manifold of corank r associated with $(R; D^1, D^2, \tilde{N}(E))$. Then, by the definition of the canonical systems $\bar{C}$, $\bar{N}$ of $I^r(J)$ and the map η, we have

$$\eta_*^{-1}(C^*) = D^1 \qquad \text{and} \qquad \eta_*^{-1}(N) = \tilde{N}.$$

Moreover, putting $\eta_*^{-1}(S(N)(w)) = S(\tilde{N})(v) \subset \operatorname{Ch}(D^1)(v)$ for $w = \eta(v)$, we have

Lemma 3.1. (1) $\eta_*^{-1}(N^\perp) = \tilde{N}^\perp$.
(2) $\eta_*^{-1}(N^*) = \tilde{N}^*$.
(3) *$Ch(\tilde{N})(v) \subset \tilde{N}^\perp(v) \subset D^2(v)$ for $v \in R$.*
(4) *For a graded Lie algebra isomorphism ϕ of $\mathfrak{s}(v)$ onto $\mathfrak{s}$,*
$\phi(S(\tilde{N})(v)) = \mathfrak{f} \cap (E^\perp \otimes_S V^*) \subset S^2(V^*)$.

Proof. (1) Put $\hat{N} = \eta_*^{-1}(N^\perp)$. For a point $v \in R$, from $p = q \cdot \eta$, we have $p_*(\hat{N}(v)) = q_*(N^\perp(w)) = w^\perp \subset \iota(v)$, where $w = \eta(v)$, which implies $\hat{N} \subset D^2$. From $\operatorname{Ker} q_* = \operatorname{Ch}(C^*) \subset N^\perp$ and $\operatorname{Ker} p_* = \operatorname{Ch}(D^1) = \eta_*^{-1}(\operatorname{Ch}(C^*))$, we have $\operatorname{Ch}(D^1) \subset \hat{N} \subset D^2$. On the other hand, we have$\tilde{N}^\perp(v) = \{X \in D^2(v) = \mathfrak{s}_{-1}(v) \mid [X, \mathfrak{n}(E)] = 0\}$ in terms of the symbol algebra $\mathfrak{s}(v)$. Moreover, from $[\Gamma(N^\perp), \Gamma(N)] \subset \Gamma(C^*)$, we have $[\Gamma(\hat{N}), \Gamma(\tilde{N})] \subset \Gamma(D^1)$. This implies $\hat{N} \subset \tilde{N}^\perp$. Then, comparing the ranks of both sides, we obtain $\hat{N} = \tilde{N}^\perp$.

(2) follows immediately from (1) and (3) follows from $\operatorname{Ch}(N)(w) \subset N^\perp(w)$ at each $w \in W$ (see §2.3 (1)).

(4) From §2.3 (3), we have $S(N)(w) = \{X \in \operatorname{Ch}(C^*)(w) \mid [X, \Gamma(N^\perp)] \subset \Gamma(N)\}$. Thus, in terms of the symbol algebra, we get

$$\phi(S(\tilde{N})(v)) = \{f \in \mathfrak{f} \mid [f, E] \subset E^\perp\} = \{f \in \mathfrak{f} \mid [[f, E], E] = 0\} = \mathfrak{f} \cap (E^\perp \otimes_S V^*) \subset S^2(V^*).$$

□

In particular, by Lemma 3.1 (4), $S(N)$ becomes a subbundle of $\operatorname{Ch}(C^*)$ under our regularity condition for $(R; D^1, D^2, \tilde{N}(E))$.

4. Lagrange Grassmann Bundle $R(W)$ over an IG manifold $(W; C^*, N)$

4.1. **Lagrange Grassmann Bundle $R(W)$.** Let $(W; C^*, N)$ be an IG manifold of corank r. We wil construct the Lagrange Grassmann bundle $R(W)$ over $(W; C^*, N)$ and will examine the conditions when $R(W)$ becomes a PD manifold of second order.

We assume that W is regular with respect to $\operatorname{Ch}(C^*)$, i.e., the space $J = W/\operatorname{Ch}(C^*)$ of leaves of this foliation $\operatorname{Ch}(C^*)$ is a manifold of dimension $2n+1$ such that (J, C) is a contact manifold, where $C^* = q_*^{-1}(C)$ and $q : W \to J$ is a submersion. For a point $w \in W$, we will consider maximal isotropic subspaces of $(N(w), \gamma_w)$. Namely, we consider the Lagrange Grassmann bundle $R(W)$ over $(W; C^*, N)$:

$$R(W) = \bigcup_{w \in W} R_w, \qquad R_w = \{\hat{v} \subset N(w) \mid \gamma_w \mid_{\hat{v}} = 0, \quad \hat{v} \text{ is maximal}\}.$$

$R(W)$ is a submanifold of the Grassmann bundle $J(W, n+t)$ over W, where $t = \operatorname{rank} \operatorname{Ch}(C^*)$. Moreover $v = q_*(\hat{v})$ is a legendrian subspace of (J, C) such that $v \subset \iota(w) = q_*(N(w))$, where $\iota : W \to I^r(J)$ is the canonical immersion (see Theorem 2.1). Thus we define a map $\zeta : R(W) \to L(J)$ by $\zeta(\hat{v}) = q_*(\hat{v})$. Then we have

$$\zeta(R_w) = \{v \in L(J) \mid v \subset \bar{w} \subset C(u)\} \cong L(\bar{w}/\bar{w}^\perp) \cong U(n-r)/O(n-r),$$

where $u = q(w)$, $\bar{w} = \iota(w)$ and $L(\bar{w}/\bar{w}^\perp)$ denotes the Lagrange Grassmann manifold of the symplectic vector space $\bar{w}/\bar{w}^\perp$ of dimension $2(n-r)$. Hence $R(W)$ is a manifold of dimension $k + \dfrac{1}{2}(n-r)(n-r+1)$, where $k = \dim W$ and $\zeta(R(W))$ is the collection of legendrian subspaces v such that $v \subset \bar{w}$ for $\bar{w} \in \iota(W) \subset I^r(J)$.

Now we will describe the map $\zeta : R(W) \to L(J)$ in suitable coordinates. Let us fix a reference point $\hat{v}_o \in R(W)$ and put $w_o = \tau(\hat{v}_o)$, where $\tau : R(W) \to W$ is the projection. Moreover put $u_o = q(w_o) \in J$. $v_o = \zeta(\hat{v}_o)$ is a legendrian subspace of $(C(u_o), d\varpi)$ such that $v_o \subset \iota(w_o) = q_*(N(w_o)) \in I^r(J)$. Take a canonical coordinate system $(x_1, \ldots, x_n, z, p_1, \ldots, p_n)$ of (J, C) defined on a neiborhood U' with origin u_o such that

$$v_o = \{X \in T_{u_o}(J) \mid \varpi(X) = dp_1(X) = \cdots = dp_n(X) = 0\} \subset$$
$$\iota(w_o) = \{X \in T_{u_o}(J) \mid \varpi(X) = dp_1(X) = \cdots = dp_r(X) = 0\},$$

where $\varpi = dz - \sum_{i=1}^n p_i dx_i$. Then we can introduce coordinate system $(x_i, x_\alpha, z, p_i, p_\alpha, a_i^\alpha, b_i^\alpha, s_{ij})$ $(1 \leqq i \leqq j \leqq r, r+1 \leqq \alpha \leqq n)$ of $I^r(J)$ aroud $\iota(w_o)$ with origin $\iota(w_o)$ as in §2.1. Since $q : W \to J$ is a submersion, let us take a coordinate system $(x_1, \ldots, x_n, z, p_1, \ldots, p_n, \lambda_1, \ldots, \lambda_t)$ of W on a neiborhood $U^* \subset q^{-1}(U')$ of w_o. Here $(x_1, \ldots, x_n, z, p_1, \ldots, p_n)$ are pullbacks to U^* of coordinate functions on U'. Then, on U^*, we have, from §2.2 and §2.3, $C^* = \{\varpi = 0\}$,

$$N = \{\varpi = \varpi_1^* = \cdots = \varpi_r^* = 0\}, \quad N^\perp = \{\varpi = \varpi_i^* = \varpi_\alpha^* = \omega^\alpha = 0 \quad (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\},$$

where

$$\varpi_i^* = dp_i - \sum_{\alpha=r+1}^n a_i^\alpha dp_\alpha - \sum_{\alpha=r+1}^n b_i^\alpha dx_\alpha - \sum_{j=1}^r (s_{ij} - \sum_{\alpha=r+1}^n a_i^\alpha b_j^\alpha) dx_j \quad (1 \leqq i \leqq r),$$

$$\varpi_\alpha^* = dp_\alpha - \sum_{j=1}^r b_j^\alpha dx_j \quad \text{and} \quad \omega^\alpha = dx_\alpha + \sum_{i=1}^r a_i^\alpha dx_i \quad (r+1 \leqq \alpha \leqq n).$$

Here we note that $a_i^\alpha = \iota^*(a_i^\alpha)$, $b_i^\alpha = \iota^*(b_i^\alpha)$ and $s_{ij} = \iota^*(s_{ij})$ are functions of $(x_1, \ldots, x_n, z, p_1, \ldots, p_n, \lambda_1, \ldots, \lambda_t)$ on U^*. Moreover we have, from §2.2 and §2.3,

$$\operatorname{Ch}(C^*) = \{\varpi = \varpi_i^* = \varpi_\alpha^* = \omega^\alpha = dx_i = 0 \quad (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\},$$

and

$$\text{(4.1)} \quad S(N)(w) = \{\varpi = \varpi_i^* = \varpi_\alpha^* = \omega^\alpha = dx_i = \varpi_{ij} = 0 \quad (1 \leqq i \leqq j \leqq r, r+1 \leqq \alpha \leqq n)\}.$$

where $\varpi_{ij} = ds_{ij} - \sum_{\alpha=r+1}^n (a_i^\alpha db_j^\alpha + a_j^\alpha db_i^\alpha)$.

On U^*, 1-forms $\{\varpi, \varpi_i^*, dp_\alpha, dx_i, dx_\alpha, d\lambda_a \quad (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n, 1 \leqq a \leqq t)\}$ form a coframe. By our choice of the coordinate system, we have

$$\hat{v}_o = q_*^{-1}(v_o) = \{X \in T_{w_o}(W) \mid \varpi(X) = \varpi_i^*(X) = dp_\alpha(X) = 0 \quad (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\}.$$

Hence we will work on the following neiborhood $\hat{U}$ of $\hat{v}_o$ in $J(W, n+t)$:

$$\hat{U} = \{\hat{v} \in \hat{\tau}^{-1}(U^*) \mid dx_1, \ldots, dx_n, d\lambda_1, \ldots, d\lambda_t \quad \text{are linearly independent on} \quad \hat{v}\},$$

where $\hat{\tau} : J(W, n+t) \to W$ is the projection. For a point $\hat{v} \in R(W) \cap \hat{U}$, from $\hat{v} \subset N(w)$, we have

$$\varpi \mid_{\hat{v}} = \varpi_i^* \mid_{\hat{v}} = 0 \quad (i = 1, \ldots, r).$$

Thus, by expressing $dp_\alpha \mid_{\hat{v}}$ as a linear combination of $dx_i \mid_{\hat{v}}, dx_\alpha \mid_{\hat{v}}$ and $d\lambda_a \mid_{\hat{v}}$, we see that $\hat{v}$ is defined by

$$\hat{v} = \{X \in T_w(W) \mid \varpi(X) = \varpi_i^*(X) = \pi_\alpha^*(\hat{v})(X) = 0\},$$

where

$$\pi_\alpha^*(\hat{v}) = dp_\alpha - \sum_{i=1}^{r} p_{\alpha i}^*(\hat{v}) dx_i - \sum_{\beta=r+1}^{n} p_{\alpha\beta}^*(\hat{v}) dx_\beta - \sum_{a=1}^{t} p_\alpha^a(\hat{v}) d\lambda_a.$$

Here $(x_i, x_\alpha, z, p_i, p_\alpha, \lambda_a, p_{\alpha i}^*, p_{\alpha\beta}^*, p_\alpha^a)(1 \leqq i \leqq r, r+1 \leqq \alpha, \beta \leqq n, 1 \leqq a \leqq t)$ constitute a coordinate system on $\hat{U}$ of the submanifold $J(W, n+t; N) = \{\hat{v} \in J(W, n+t) \mid \hat{v} \subset N(w)\}$ of $J(W, n+t)$. On this coordinate system, $\gamma_w \mid_{\hat{v}} = 0$ is equivalent to $d\varpi \mid_{\hat{v}} = 0$, i.e.,

$$d\varpi \equiv 0 \qquad (\text{mod} \quad \varpi, \varpi_1^*, \ldots, \varpi_r^*, \pi_{r+1}^*(\hat{v}), \ldots, \pi_n^*(\hat{v})).$$

Hence, from $\pi_\alpha^* = \varpi_\alpha^* - \sum_{\beta=r+1}^{n} p_{\alpha\beta}^* \omega^\beta - \sum_{i=1}^{r} (p_{\alpha i}^* - b_i^\alpha - \sum_{\beta=r+1}^{n} p_{\alpha\beta}^* a_i^\beta) dx_i - \sum_{a=1}^{t} p_\alpha^a d\lambda_a$, we calculate

$$\begin{aligned} d\varpi &\equiv \sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \varpi_\alpha^* && (\text{mod} \quad \varpi, \varpi_1^*, \ldots, \varpi_r^*) \\ &\equiv \sum_{\alpha=r+1}^{n} \{ \sum_{\beta=r+1}^{n} p_{\alpha\beta}^* \omega^\alpha \wedge \omega^\beta + \sum_{i=1}^{r} (p_{\alpha i}^* - b_i^\alpha - \sum_{\beta=r+1}^{n} p_{\alpha\beta}^* a_i^\beta) \omega^\alpha \wedge dx_i + \sum_{a=1}^{t} p_\alpha^a \omega^\alpha \wedge d\lambda_a \} \\ & && (\text{mod} \quad \varpi, \varpi_1^*, \ldots, \varpi_r^*, \pi_{r+1}^*, \ldots, \pi_n^*). \end{aligned}$$

Thus we see that $R(W)$ is defined in this coordinate system by

$$p_{\alpha\beta}^* = p_{\beta\alpha}^*, \quad p_{\alpha i}^* = b_i^\alpha + \sum_{\beta=r+1}^{n} p_{\alpha\beta}^* a_i^\beta, \quad p_\alpha^a = 0 \quad (r+1 \leqq \alpha, \beta \leqq n, 1 \leqq i \leqq r, 1 \leqq a \leqq t).$$

In this way, we obtain a coordinate system $(x_1, \ldots, x_n, z, p_1, \ldots, p_n, \lambda_1, \ldots, \lambda_t, p_{\alpha\beta}^*)$ $(r+1 \leqq \alpha \leqq \beta \leqq n)$ of $R(W)$ on $\hat{U} \cap R(W)$. On the other hand, we have a canonical coordinate system $(x_1, \ldots, x_n, z, p_1, \ldots, p_n, p_{kl})$ $(1 \leqq k \leqq l \leqq n)$, $p_{kl} = p_{lk}$, of $(L(J), E)$, which is subordinate to $U'; (x_1, \ldots, x_n, z, p_1, \ldots, p_n)$, defined on a neiborhood U with origin $v_o = \zeta(\hat{v}_o)$ such that

$$E = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\},$$

where $\varpi_k = dp_k - \sum_{l=1}^{n} p_{kl} dx_l$ $(1 \leqq k \leqq n)$, $U = \{v \in \pi^{-1}(U') \mid dx_1 \wedge \cdots \wedge dx_n \mid_v \neq 0\}$ and $\pi : L(J) \to J$ is the projection. By the definition of the canonical system E on $L(J)$, we have $E(v) = \pi_*^{-1}(v)$. Hence, from $\pi \cdot \zeta = q \cdot \tau$ and $v = q_*(\hat{v})$, we have

$$\zeta_*^{-1}(E(v)) = (\pi \cdot \zeta)_*^{-1}(v) = (q \cdot \tau)_*^{-1}(v) = \tau_*^{-1}(\hat{v}). \tag{4.2}$$

Thus, on $R(W) \cap \hat{U}$, we obtain

$$\{\zeta^* \varpi = \zeta^* \varpi_1 = \cdots = \zeta^* \varpi_n = 0\} = \{\varpi = \varpi_1^* = \cdots = \varpi_r^* = \pi_{r+1}^* = \cdots = \pi_n^* = 0\}.$$

We have

$$\zeta^* \varpi = \varpi, \quad \zeta^* \varpi_k = dp_k - \sum_{l=1}^{n} \zeta^* p_{kl} \, dx_l \quad (1 \leqq k \leqq n)$$

and

$$\pi^*_\alpha = \varpi^*_\alpha - \sum_{\beta=r+1}^{n} p^*_{\alpha\beta}\omega^\beta = dp_\alpha - \sum_{i=1}^{r} p^*_{\alpha i} dx_i - \sum_{\beta=r+1}^{n} p^*_{\alpha\beta} dx_\beta \quad (r+1 \leqq \alpha \leqq n).$$

Moreover we calculate

$$\begin{aligned}
\varpi^*_i &= dp_i - \sum_{\alpha=r+1}^{n} a^\alpha_i dp_\alpha - \sum_{\alpha=r+1}^{n} b^\alpha_i dx_\alpha - \sum_{j=1}^{r}(s_{ij} - \sum_{\alpha=r+1}^{n} a^\alpha_i b^\alpha_j) dx_j \\
&\equiv dp_i - \sum_{\alpha=r+1}^{n} a^\alpha_i (\sum_{j=1}^{r} p^*_{\alpha j} dx_j + \sum_{\beta=r+1}^{n} p^*_{\alpha\beta} dx_\beta) - \sum_{\alpha=r+1}^{n} b^\alpha_i dx_\alpha - \sum_{j=1}^{r}(s_{ij} - \sum_{\alpha=r+1}^{n} a^\alpha_i b^\alpha_j) dx_j \\
&\equiv dp_i - \sum_{j=1}^{r} p^*_{ij} dx_j - \sum_{\alpha=r+1}^{n} p^*_{\alpha i} dx_\alpha \quad (\mathrm{mod} \quad \pi^*_{r+1}, \ldots, \pi^*_n).
\end{aligned}$$

where $p^*_{ij} = s_{ij} + \sum_{\alpha,\beta=r+1}^{n} p^*_{\alpha\beta} a^\alpha_i a^\beta_j$. Hence we obtain

$$\zeta^* p_{\alpha\beta} = p^*_{\alpha\beta}, \quad \zeta^* p_{\alpha i} = b^\alpha_i + \sum_{\beta=r+1}^{n} p^*_{\alpha\beta} a^\beta_i, \quad \zeta^* p_{ij} = s_{ij} + \sum_{\alpha=r+1}^{n}\sum_{\beta=r+1}^{n} p^*_{\alpha\beta} a^\alpha_i a^\beta_j, \tag{4.3}$$

which describe the map $\zeta : R(W) \to L(J)$ in the above coordinate systems.

4.2. **Differential Systems on** $R(W)$**.** We have several differential systems naturally defined on $R(W)$. First, $D^1_W = \tau_*^{-1}(C^*)$, $N_W = \tau_*^{-1}(N)$ and $N^\perp_W = \tau_*^{-1}(N^\perp)$ are the lifts of C^*, N and $N^\perp$ respectively and the canonical system D^2_W is defined by

$$D^2_W(\hat{v}) = \tau_*^{-1}(\hat{v}) \subset T_{\hat{v}}(R(W)) \xrightarrow{\tau_*} T_w(W) \qquad \text{at each} \quad \hat{v} \in R(W).$$

Then, on $R(W) \cap \hat{U}$, we have

$D^1_W = \{\varpi = 0\}$, $\quad \mathrm{Ch}\,(D^1_W) = \{\varpi = \varpi^*_i = \varpi^*_\alpha = \omega^\alpha = dx_i = 0 \quad (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\}$,
$N_W = \{\varpi = \varpi^*_i = 0 \ (1 \leqq i \leqq r)\}$, $\ N^\perp_W = \{\varpi = \varpi^*_i = \varpi^*_\alpha = \omega^\alpha = 0 \ (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\}$,

$$D^2_W = \{\varpi = \varpi^*_i = \varpi^*_\alpha - \sum_{\beta=r+1}^{n} p^*_{\alpha\beta}\omega^\beta = 0 \quad (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\}.$$

For these systems, we have

Lemma 4.1. (1) $\;\zeta_*^{-1}(E) = D^2_W$ *and* $\zeta_*^{-1}(\partial E) = D^1_W$.
(2) D^1_W *and* D^2_W *are differential systems of codimension* 1 *and* $n+1$ *respectively.*
(3) $\partial D^2_W \subset D^1_W$
(4) *Ch* (D^1_W) *is a subbundle of* D^2_W *of codimension* n*.*
(5) N_W *is a subbundle of* D^1_W *of codimension* r*, which contains* D^2_W *such that* *Ch* $(N_W)(\hat{v}) \subset D^2_W(\hat{v})$ *at each* $\hat{v} \in R(W)$*.*
(6) *Ch* $(D^1_W) \cap$ *Ch* $(N_W) = \mathrm{Ker}\,\tau_*$.
(7) *Ch* $(D^1_W)(\hat{v}) \cap$ *Ch* $(D^2_W)(\hat{v}) = \mathrm{Ker}\,\zeta_*(\hat{v})$ *at each* $\hat{v} \in R(W)$*.*
(8) $\mathrm{Ker}\,\tau_* \cap \mathrm{Ker}\,\zeta_* = \{0\}$.

Proof. (1) follows from (4.2) and $\zeta^*\varpi = \varpi$. (2) and (4) are obvious from the above and (3) follows from $d\varpi \equiv 0$ (mod $\varpi, \varpi^*_1, \ldots, \varpi^*_r, \pi^*_{r+1}, \ldots, \pi^*_n$), where $\pi^*_\alpha = \varpi^*_\alpha - \sum_{\beta=r+1}^{n} p^*_{\alpha\beta}\omega^\beta$ $(r+1 \leqq \alpha \leqq n)$. The first half of (5) is obvious from the above and $\mathrm{Ch}\,(N_W)(\hat{v}) \subset D^2_W(\hat{v})$ follows from $N^\perp_W \subset D^2_W$ and $\mathrm{Ch}\,(N)(w) \subset N^\perp(w)$, where $w = \tau(\hat{v})$. (6) follows from $\mathrm{Ch}\,(C^*) \cap \mathrm{Ch}\,(N) = \{0\}$. By $\mathrm{Ker}(q \cdot \tau)_* = \tau_*^{-1}(\mathrm{Ch}\,(C^*)) = \mathrm{Ch}\,(D^1_W)$ and $\zeta(\hat{v}) = q_*(\hat{v}) = q_*(\tau_*(D^2_W(\hat{v})))$,

(7) follows from the Realization Lemma for $(R(W), D^2_W, q \cdot \tau, J)$. Finally (8) follows from $\operatorname{Ker} \zeta_* \subset \{dp^*_{\alpha\beta} = 0 \ (r+1 \leqq \alpha \leqq \beta \leqq n)\}$ and

$$\operatorname{Ker} \tau_* = \{dz = dp_1 = \cdots = dp_n = dx_1 = \cdots = dx_n = d\lambda_1 = \cdots = d\lambda_t = 0\}.$$

□

Thus, by (2),(3),(4) and (7) of Lemma 4.1, we see that $(R(W); D^1_W, D^2_W)$ is a PD manifold of second order iff $\zeta : R(W) \to L(J)$ is an immersion. We will describe this condition in terms of invariants of $(W; C^*, N)$. For a point $\hat{v} \in R(W)$, put

$$A(\hat{v}) = \tau_*(\mathrm{Ch}\,(D^1_W)(\hat{v}) \cap \mathrm{Ch}\,(D^2_W)(\hat{v})) \subset \mathrm{Ch}\,(C^*)(w) = \tau_*(\mathrm{Ch}\,(D^1_W)(\hat{v})), \quad w = \tau(\hat{v}).$$

By Lemma 4.1 (8), $A(\hat{v}) = \{0\}$ iff $\operatorname{Ker} \zeta_*(\hat{v}) = \{0\}$, i.e., ζ is an immersion around $\hat{v}$. On the neiborhood $R(W) \cap \hat{U}$, we have

$$\begin{aligned} d\pi^*_\alpha &= d\varpi^*_\alpha - \sum_{\beta=r+1}^{n} p^*_{\alpha\beta} d\omega^\beta + \sum_{\beta=r+1}^{n} \omega^\beta \wedge dp^*_{\alpha\beta} \\ &= \sum_{i=1}^{r} dx_i \wedge (db^\alpha_i + \sum_{\beta=r+1}^{n} p^*_{\alpha\beta} da^\beta_i) + \sum_{\beta=r+1}^{n} \omega^\beta \wedge dp^*_{\alpha\beta}. \end{aligned}$$

Hence, from (2.1), we get

$$(4.4) \qquad \begin{cases} d\varpi \equiv 0 \\ d\varpi^*_i \equiv \displaystyle\sum_{j=1}^{r} dx_j \wedge \varpi_{ij} + \sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \pi_{\alpha i} \quad (\mathrm{mod}\ \varpi, \varpi^*_1, \ldots, \varpi^*_r, \pi^*_{r+1}, \ldots, \pi^*_n) \\ d\pi_\alpha \equiv \displaystyle\sum_{i=1}^{r} dx_i \wedge \pi_{\alpha i} + \sum_{\beta=r+1}^{n} \omega^\beta \wedge dp^*_{\alpha\beta}. \end{cases}$$

for $1 \leqq i \leqq r$, $r+1 \leqq \alpha \leqq n$, where $\pi_{\alpha i} = db^\alpha_i + \sum_{\beta=r+1}^{n} p^*_{\alpha\beta} da^\beta_i$. From

$$\mathrm{Ch}\,(D^1_W) = \{\varpi = \varpi^*_i = \varpi^*_\alpha = \omega^\alpha = dx_i = 0 \quad (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\},$$

we obtain

$$\begin{aligned} &\mathrm{Ch}\,(D^1_W)(\hat{v}) \cap \mathrm{Ch}\,(D^2_W)(\hat{v}) = \\ &\{X \in \mathrm{Ch}\,(D^1_W)(\hat{v}) \mid \varpi_{ij}(X) = \pi_{\alpha i}(X) = dp^*_{\alpha\beta}(X) = 0 \ (1 \leqq i \leqq j \leqq r, r+1 \leqq \alpha \leqq \beta \leqq n)\}. \end{aligned}$$

Thus, by (4.1), we get

$$(4.5) \qquad A(\hat{v}) = \{X \in S(N)(w) \mid (db^\alpha_i + \sum_{\beta=r+1}^{n} p^*_{\alpha\beta} da^\beta_i)(X) = 0 \ (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\},$$

on $R(W) \cap \hat{U}$ and $\hat{v} = \{\varpi = \varpi^*_i = \varpi^*_\alpha - \sum_{\beta=r+1}^{n} p^*_{\alpha\beta}\omega^\beta = 0 \ (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\}$. We will describe this subspace of $S(N)(w)$ in terms of invariants of $(W; C^*, N)$.

For a point $\hat{v} \in R(W)$, $\hat{v}$ is a maximal isotropic subspace in $N(w)$, $w = \tau(\hat{v})$, which contains $N^\perp(w)$. We put $E^\perp(\hat{v}) = N(w)/\hat{v}$ ($\cong N_W(\hat{v})/D^2_W(\hat{v})$). Then we get the projection $\pi_{\hat{v}} : \mathfrak{t}_N(w) = N(w)/N^\perp(w) \to E^\perp(\hat{v}) = N(w)/\hat{v}$, which induces, from (2.3), the following map:

$$\gamma_c(\hat{v}) : S(N)(w) \to E^\perp(\hat{v}) \otimes E(w)^*$$

by $\gamma_c(\hat{v}) = (\pi_{\hat{v}} \otimes id_{E(w)^*}) \cdot \gamma_c$. We can describe $A(\hat{v})$ as follows;

Lemma 4.2. (1) $A(\hat{v}) = \operatorname{Ker}\gamma_c(\hat{v})$

(2) $A(\hat{v}) = \{X \in S(N)(w) \mid X\rfloor d\varpi \equiv X\rfloor d\varpi_1^* \equiv \cdots \equiv X\rfloor d\varpi_r^* \equiv 0 \pmod{(\hat{v})^{\perp}}\}$, *where* $N = \{\varpi = \varpi_1^* = \cdots = \varpi_r^* = 0\}$.

Proof. By (4.4), for a vector $X \in S(N)(w)$, we have

$$\begin{cases} X\rfloor d\varpi \equiv 0, \quad X\rfloor d\varpi_i^* \equiv 0 \pmod{\varpi, \varpi_i^*, \varpi_\alpha^*, \omega^\alpha\ (1 \leqq r \leqq r, r+1 \leqq \alpha \leqq n)} \\ X\rfloor d\pi_\alpha^* \equiv -\displaystyle\sum_{i=1}^{r}(db_i^\alpha + \sum_{\beta=r+1}^{n} p_{\alpha\beta}^* da_i^\beta)(X)\ dx_i. \end{cases}$$

Hence we have

$$A(\hat{v}) = \{X \in S(N)(w) \mid [X, \Gamma(N^\perp)] \subset \hat{v}\}.$$

On the other hand, for $X \in S(N)(w)$ and $Y \in N^\perp(w)$, we take $\tilde{X} \in \Gamma(S(N))$ and $\tilde{Y} \in \Gamma(N^\perp)$ such that $X = \tilde{X}_w$ and $Y = \tilde{Y}_w$. Then, from (2.3), $\gamma_c : S(N)(w) \to \mathfrak{t}_N(w) \otimes E^*(w)$ is defined by $\hat{\gamma}_c(X, Y) = \pi_N([\tilde{X}, \tilde{Y}]_w) \in \mathfrak{t}_N(w) = N(w)/N^\perp(w)$. Thus $\gamma_c(\hat{v})$ is defined by

$$\hat{\gamma}_c(\hat{v})(X, Y) = \hat{\pi}_{\hat{v}}([\tilde{X}, \tilde{Y}]_w) \in E^\perp(\hat{v}) = N(w)/\hat{v},$$

where $\hat{\pi}_{\hat{v}} : N(w) \to E^\perp(\hat{v})$ is the projection. Hence $[\tilde{X}, \tilde{Y}]_w \in \hat{v}$ for $\forall Y \in N^\perp(w)$ iff $\hat{\gamma}_c(\hat{v})(X, Y) = 0$ for $\forall Y \in N^\perp(w)$, i.e., $\gamma_c(\hat{v})(X) = 0$, which implies $A(\hat{v}) = \operatorname{Ker}\gamma_c(\hat{v})$.

Moreover, by (4.4), for a vector $X \in S(N)(w)$, we have

$$\begin{cases} X\rfloor d\varpi \equiv 0 \qquad\qquad (\operatorname{mod} \varpi, \varpi_1^*, \ldots, \varpi_r^*, \pi_{r+1}^*, \ldots, \pi_n^*) \\ X\rfloor d\varpi_i^* \equiv -\displaystyle\sum_{\alpha=r+1}^{n} \pi_{\alpha i}(X)\omega^\alpha \quad (1 \leqq i \leqq r) \end{cases}$$

Here $\pi_\alpha^* = \varpi_\alpha^* - \sum_{\beta=r+1}^{n} p_{\alpha\beta}^*\omega^\beta$ and $\pi_{\alpha i} = db_i^\alpha + \sum_{\beta=r+1}^{n} p_{\alpha\beta}^* da_i^\beta$ $(1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)$. Hence, from (4.5), we get

$$\begin{aligned} A(\hat{v}) &= \{X \in S(N)(w) \mid X\rfloor d\varpi \equiv X\rfloor d\varpi_1^* \equiv \cdots \equiv X\rfloor d\varpi_r^* \equiv 0 \pmod{(\hat{v})^\perp}\} \\ &= \{X \in S(N)(w) \mid [X, Y] \in N(w) \quad \text{for} \quad \forall Y \in \hat{v}\} \end{aligned}$$

□

Finally we will indicate the relation between the symbol algebra $\mathfrak{s}(\hat{v})$ of $(R(W); D_W^1, D_W^2)$ at $\hat{v} \in R(W)$ and the invariants of $(W; C^*, N)$ at $w = \tau(\hat{v})$, when $\operatorname{Ker}\zeta_*$ is trivial. In the rest of this subsection, we assume that $\zeta : R(W) \to L(J)$ is an immersion. Hence $(R(W); D_W^1, D_W^2)$ is a PD manifold of second order. Moreover we assume that $R(W)$ satisfies the following condition (C):

$$(C) \qquad \hat{p}^{(1)} : R(W)^{(1)} \to R(W) \text{ is onto.}$$

where $R(W)^{(1)}$ is the first prolongation of $(R(W); D_W^1, D_W^2)$, i.e., there exists an n-dimensional integral element $V = V(\hat{v})$ of $(R(W), D_W^2)$ at each $\hat{v} \in R(W)$ such that $D_W^2(\hat{v}) = V \oplus \mathfrak{f}(\hat{v})$, where $\mathfrak{f}(\hat{v}) = \operatorname{Ker}(q \cdot \tau)_*(\hat{v}) = \operatorname{Ch}(D_W^1)(\hat{v}) = \tau_*^{-1}(\operatorname{Ch}(C^*)(w))$ and $w = \tau(\hat{v})$.

Let $\mathfrak{s}(\hat{v}) = \mathfrak{s}_{-3}(\hat{v}) \oplus \mathfrak{s}_{-2}(\hat{v}) \oplus \mathfrak{s}_{-1}(\hat{v})$ be the symbol algebra of $(R(W); D_W^1, D_W^2)$ at $\hat{v} \in R(W)$, where $\mathfrak{s}_{-3}(\hat{v}) = T_{\hat{v}}(R(W))/D_W^1(\hat{v})$, $\mathfrak{s}_{-2}(\hat{v}) = D_W^1(\hat{v})/D_W^2(\hat{v})$ and $\mathfrak{s}_{-1}(\hat{v}) = D_W^2(\hat{v})$. Fixing a basis of $\mathfrak{s}_{-3}(\hat{v})$, we have (see §3.1 [27])

$$\mathfrak{s}_{-3}(\hat{v}) \cong \mathbb{R}, \quad \mathfrak{s}_{-2}(\hat{v}) \cong V^*, \quad \mathfrak{s}_{-1}(\hat{v}) = V \oplus \mathfrak{f}(\hat{v}) \quad \text{and} \quad \mathfrak{f}(\hat{v}) \subset S^2(V^*).$$

From $N_W^\perp(\hat{v}) \supset \operatorname{Ch}(D_W^1)(\hat{v}) = \mathfrak{f}(\hat{v})$, we obtain the subspace $E(\hat{v})$ of $V(\hat{v})$ of dimension r by $E(\hat{v}) = V(\hat{v}) \cap N_W^\perp(\hat{v})$. Then $N_W(\hat{v})/D_W^2(\hat{v}) \subset \mathfrak{s}_{-2}(\hat{v}) = D_W^1(\hat{v})/D_W^2(\hat{v})$ corresponds to

$E(\hat{v})^\perp \subset V(\hat{v})^*$ in the identification $\mathfrak{s}_{-2}(\hat{v}) \cong V(\hat{v})^*$. Moreover $\tau_* : T_{\hat{v}}(R(W)) \to T_w(W)$ induces the identifications:

$$\mathfrak{s}_{-3}(\hat{v}) \cong \mathfrak{t}_{-3}(w) \cong \mathbb{R}, \quad E(\hat{v}) \cong E(w) = N^\perp(w)/\mathrm{Ch}\,(C^*)(w), \quad N_W(\hat{v})/D^2_W(\hat{v}) \cong N(w)/\hat{v}.$$

In particular, $\tau_* : \mathfrak{f}(\hat{v}) \to \mathrm{Ch}\,(C^*)(w)$ induces the identification:

$$\mathfrak{f}(\hat{v})/S(N_W)(\hat{v}) \cong \mathrm{Ch}\,(C^*)(w)/S(N)(w),$$

where $S(N_W)(\hat{v}) = \tau_*^{-1}(S(N)(w))$. By Lemma 3.1 (4), we have

$$S(N_W)(\hat{v}) = \mathfrak{f}(\hat{v}) \cap (E(\hat{v})^\perp \otimes_S V(\hat{v})^*) \subset S^2(V(\hat{v})^*).$$

Let us take a complimentary subspace H of V such that $V = E \oplus H$. Then $V^* = E^\perp \oplus H^\perp$ and $H^\perp$ is naturally identified with $V^*/E^\perp$, hence with E^*. Then we have

$$S^2(V^*) = E^\perp \otimes_S V^* \oplus S^2(H^\perp) \quad \text{and} \quad E^\perp \otimes_S V^* = S^2(E^\perp) \oplus E^\perp \otimes_S H^\perp.$$

Now the inclusion map $\iota : E \to V$ induces $\iota^* : V^* \to E^*$ and $\hat{\iota} : S^2(V^*) \to S^2(E^*)$ such that $\mathrm{Ker}\,\iota^* = E^\perp$ and $\mathrm{Ker}\,\hat{\iota} = E^\perp \otimes_S V^*$.

Furthermore, under the identification: $\mathfrak{s}_{-3}(\hat{v}) \cong \mathfrak{t}_{-3}(w) \cong \mathbb{R}$ by fixing a basis of $\mathfrak{s}_{-3}(\hat{v})$, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathfrak{f}(\hat{v}) & \xrightarrow{\ \epsilon\ } & S^2(V(\hat{v})^*) \\
{\scriptstyle \tau_*}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\iota}(\hat{v})} \\
\mathrm{Ch}\,(C^*)(w) & \xrightarrow{\ \kappa\ } & S^2(E(w)^*).
\end{array}$$

where $\epsilon : \mathfrak{f}(\hat{v}) \to S^2(V(\hat{v})^*)$ is given by $\epsilon(f)(v_1, v_2) = [[f, v_1], v_2]$ in $\mathfrak{s}(\hat{v})$ for $f \in \mathfrak{f}(\hat{v})$ and $v_1, v_2 \in V(\hat{v})$, $\tilde{\iota}(\hat{v}) = (\hat{\tau}^*)^{-1} \cdot \hat{\iota}(\hat{v})$, $\hat{\iota}(\hat{v}) : S^2(V(\hat{v})^*) \to S^2(E(\hat{v})^*)$ and $\hat{\tau}^* : S^2(E(w)^*) \to S^2(E(\hat{v})^*)$ is induced by the linear isomorphism $\hat{\tau} : E(\hat{v}) \to E(w)$. Hence $\kappa(\mathrm{Ch}\,(C^*)(w)) = \hat{\kappa}(\mathrm{Ch}\,(C^*)(w)/S(N)(w)) \subset S^2(E(w)^*)$ is sent, by $\hat{\tau}^*$, to the image $\hat{\iota}(\hat{v})(\mathfrak{f}(\hat{v})) \subset S^2(E(\hat{v})^*)$. Thus $\hat{\kappa}(\mathrm{Ch}\,(C^*)(w)/S(N)(w))$ describes the image $\hat{\iota}(\hat{v})(\mathfrak{f}(\hat{v})) \cong \mathfrak{f}(\hat{v})/S(N_W)(\hat{v})$ of $\mathfrak{f}(\hat{v})$ by the projection $\hat{\iota}(\hat{v}) : S^2(V(\hat{v})^*) \to S^2(E(\hat{v})^*)$.

On the other hand, we have the following commutative diagram:

$$\begin{array}{ccc}
S(N_W)(\hat{v}) & \xrightarrow{\ \hat{\epsilon}\ } & E(\hat{v})^\perp \otimes E(\hat{v})^* \\
{\scriptstyle \tau_*}\big\downarrow & & \big\downarrow{\scriptstyle id\otimes(\hat{\tau}^*)^{-1}} \\
S(N)(w) & \xrightarrow{\ \gamma_c(\hat{v})\ } & E(\hat{v})^\perp \otimes E(w)^*.
\end{array}$$

where $\hat{\epsilon} : S(N_W)(\hat{v}) \to E(\hat{v})^\perp \otimes E(\hat{v})^*$ is given by $\hat{\epsilon}(f)(v) = [f, v] \in E(\hat{v})^\perp \subset V(\hat{v})^* = \mathfrak{s}_{-2}(\hat{v})$ in $\mathfrak{s}(\hat{v})$ for $f \in S(N_W)(\hat{v}) = \mathfrak{f}(\hat{v}) \cap (E(\hat{v})^\perp \otimes_S V(\hat{v})^*) \subset S^2(V(\hat{v})^*)$ and $v \in E(\hat{v}) \subset \mathfrak{s}_{-1}(\hat{v})$ (see the proof of Lemma 3.1 (4)). Here we note that $\mathrm{Ker}\,\hat{\epsilon} = S^2(E(\hat{v})^\perp) = \mathrm{Ker}\,\tau_*$. Thus $\gamma_c(\hat{v})(S(N)(w))$ describes the structure of $S(N_W)(\hat{v}) = \mathfrak{f}(\hat{v}) \cap \mathrm{Ker}\,\hat{\iota}(\hat{v})$.

5. Second Reduction Theorem

5.1. **Equivalence of** $(R; D^1, D^2)$ **and** $(W; C^*.N)$**.** As in §3.1, let $(R; D^1, D^2)$ be a PD manifold of second order, which is regular of type $\mathfrak{s}$, where $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$ is a subalgebra of $\mathfrak{c}^2(n)$, which is defined by

$$\mathfrak{s}_{-3} = \mathbb{R}, \quad \mathfrak{s}_{-2} = V^*, \quad \mathfrak{s}_{-1} = V \oplus \mathfrak{f}, \qquad \mathfrak{f} \subset S^2(V^*).$$

We assume that R is regular with respect to $\mathrm{Ch}\,(D^1)$ and let $p : R \to J = R/\mathrm{Ch}\,(D^1)$ be the projection.

Now we assume that there exists a $G_0(\mathfrak{s})$-invariant subspace E of V of dimension r. Then, as in §3.1, we have the first order covariant system $\tilde{N}(E)$ and covariant systems $\tilde{N}^{\perp}(E)$ and $\tilde{N}^*(E)$. Moreover, as in §3.2, let $(W; C^*, N)$ be the IG manifold of corank r associated with $(R; D^1, D^2, \tilde{N}(E))$. Namely we consider the map $\eta : R \to I^r(J)$ defined by

$$\eta(v) = p_*(\tilde{N}(v)) \in I^r(J) \qquad \text{for} \quad v \in R,$$

where $I^r(J)$ is the Involutive Grassmann bundle of (J, C) of codimension r. We assume that $\operatorname{Ker} \eta_* = \operatorname{Ch}(D^1) \cap \operatorname{Ch}(\tilde{N})$ is a subbundle such that $W = Im(\eta)$ is a submanifold of $I^r(J)$. Let $R(W)$ be the Lagrange Grassmann bundle over $(W; C^*, N)$. Then we have the map $\kappa_1 : R \to R(W)$ defined by

$$\kappa_1(v) = \eta_*(D^2(v)) \in R_w, \quad w = \eta(v).$$

In fact $\hat{v} = \eta_*(D^2(v))$ is a subspace of $N(w) \subset T_w(W)$ of dimension $n+t$ such that $\gamma \mid_{\hat{v}} = 0$, which follows from $D^2(v) \subset \tilde{N}(v)$, $\operatorname{Ch}(D^1)$ is a subbundle of D^2 of codimension n and $\partial D^2 \subset D^1$. Moreover, by Realization Lemma for (R, D^2, η, W), we have $\operatorname{Ker}(\kappa_1)_* = \operatorname{Ch}(D^2) \cap \operatorname{Ker} \eta_* \subset \operatorname{Ch}(D^1) \cap \operatorname{Ch}(D^2) = \{0\}$, which implies $\operatorname{Ker}(\kappa_1)_*$ is trivial, i.e., κ_1 is an immersion such that $\eta = \tau \cdot \kappa_1$, where $\tau : R(W) \to W$ is the projection. Thus, by the definitions of D^1_W and D^2_W on $R(W)$, we obtain

Proposition 5.1. *$\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ is a local isomorphism if and only if*

$$\text{rank}\, Ch\,(D^1) \cap Ch\,(\tilde{N}) = \frac{1}{2}(n-r)(n-r+1).$$

When $\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ is a local isomorphism, by the construction of $(R(W); D^1_W, D^2_W)$ from $(W; C^*, N)$, we see that the local equivalence of $(R; D^1, D^2)$ is reducible to that of $(W; C^*, N)$ as in the following: Let $(R; D^1, D^2)$ and $(\hat{R}; \hat{D}^1, \hat{D}^2)$ be PD manifolds of second order, which are regular of type $\mathfrak{s}$, and let $(W; C^*, N)$ and $(\hat{W}; \hat{C}^*, \hat{N})$ be the associated IG manifolds of $(R; D^1, D^2, \tilde{N}(E))$ and $(\hat{R}; \hat{D}^1, \hat{D}^2, \hat{N}(E))$ respectively. Moreover let $(R(W); D^1_W, D^2_W)$ and $(R(\hat{W}); D^1_{\hat{W}}, D^2_{\hat{W}})$ be the Lagrange Grassmann bundle over $(W; C^*, N)$ and $(\hat{W}; \hat{C}^*, \hat{N})$ respectively. For points $v_o \in R$ and $\hat{v}_o \in \hat{R}$, put $w_o = \eta(v_o)$ and $\hat{w}_o = \hat{\eta}(\hat{v}_o)$. We assume that $\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ and $\hat{\kappa}_1; (\hat{R}; \hat{D}^1, \hat{D}^2) \to (R(\hat{W}); D^1_{\hat{W}}, D^2_{\hat{W}})$ are local isomorphisms around v_o and $\hat{v}_o$ respectively. Then a local isomorphism $\psi : (R; D^1, D^2) \to (\hat{R}, \hat{D}^1, \hat{D}^2)$ such that $\psi(v_o) = \hat{v}_o$ induces a local isomorphism $\varphi : (W; C^*, N) \to (\hat{W}; \hat{C}^*, \hat{N})$ such that $\varphi(w_o) = \hat{w}_o$ and $\varphi_*(\kappa_1(v_o)) = \hat{\kappa}_1(\hat{v}_o)$, and vice versa.

Moreover, when $\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ is a local isomorphism, we observe here the correspondence of local integral manifolds of $(R; D^1, D^2)$ and $(W; C^*, N)$ as in the following: First we observe, by the uniqueness of the map in the Realization Lemma for (R, D^2, p, J), that the map $\tilde{\kappa} = \zeta \cdot \kappa_1 : R \to L(J)$ coincides with the canonical immersion $\iota : R \to L(J)$, from $\pi \cdot \tilde{\kappa} = q \cdot \tau \cdot \kappa_1 = q \cdot \eta = p$ and $\tilde{\kappa}_*^{-1}(E) = (\kappa_1)_*^{-1}(D^2_W) = D^2$, where $\pi : L(J) \to J$ is the projection.

Let Λ be an integral manifold of $(R; D^1, D^2)$ passing through v_o, i.e., Λ is an n-dimensional integral manifold of (R, D^2) such that $T_v(\Lambda) \cap \operatorname{Ch}(D^1)(v) = \{0\}$ for $v \in \Lambda$. Then we have $T_v(\Lambda) \cap \operatorname{Ker} \eta_*(v) = \{o\}$. Hence, from $\tilde{N} \supset D^2$, $\Lambda' = \eta(\Lambda)$ is, at least locally, an n-dimensional integral manifold of $(W; C^*, N)$ passing through $w_o = \eta(v_o)$ such that $\kappa_1(v) = T_w(\Lambda') \oplus \operatorname{Ch}(C^*)(w)$ for $v \in \Lambda$ and $w = \eta(v)$. Conversely let Λ' be an n-dimensional integral manifold of $(W; C^*, N)$ passing through $w_o = \eta(v_o)$ such that $T_{w_o}(\Lambda') \oplus \operatorname{Ch}(C^*)(w_o) = \kappa_1(v_o)$, i.e., Λ' is an n-dimensional integral manifold of (W, N) such that $T_w(\Lambda') \cap \operatorname{Ch}(C^*)(w) = \{0\}$ for $w \in \Lambda'$ and $T_{w_o}(\Lambda') \oplus \operatorname{Ch}(C^*)(w_o) = \kappa_1(v_o)$. Then, from $N \subset C^*$, $\Lambda^o = q(\Lambda')$ is, at least locally, a legendrian submanifold of (J, C). Hence we have the lift $\sigma(\Lambda^o) \subset L(J)$ of Λ^o

by $\sigma(u) = T_u(\Lambda^o) \in L(J)$ for $u \in \Lambda^o$. Moreover we have a map $\lambda : \Lambda' \to R(W)$ defined by $\lambda(w) = T_w(\Lambda') \oplus \mathrm{Ch}\,(C^*)(w) \in R(W)$. Then, from $\tau \cdot \lambda = id_{\Lambda'}$ and by the definition of $\zeta : R(W) \to L(J)$, we see that λ is an immersion such that $\lambda(w_o) = \kappa_1(v_o)$ and $\zeta \cdot \lambda(w) = \sigma \cdot q(w)$ for $w \in \Lambda'$. Hence $\Lambda = (\kappa_1)^{-1}(\lambda(\Lambda'))$ is an integral manifold of $(R; D^1, D^2)$ passing through $v_o \in R$ such that $\eta(\Lambda) = \tau \cdot \kappa_1(\Lambda) = \Lambda'$ and $\iota(\Lambda) = \zeta \cdot \kappa_1(\Lambda) = \zeta \cdot \lambda(\Lambda') = \sigma \cdot q(\Lambda') = \sigma(\Lambda^o)$ around $v_o \in R$.

5.2. **Covariant systems $\mathfrak{f}(E)$ and $C(E)$.** We will first consider the condition in Proposition 5.1 in terms of the symbol algebra $\mathfrak{s}$ of $(R; D^1, D^2)$. Here $(R; D^1, D^2)$ is regular of type $\mathfrak{s}$. Namely the symbol algebra $\mathfrak{s}(v) = \mathfrak{s}_{-3}(v) \oplus \mathfrak{s}_{-2}(v) \oplus \mathfrak{s}_{-1}(v)$ at each $v \in R$ is isomorphic to $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$. We define subspaces $\mathfrak{f}_E$ and $\mathfrak{c}_E$ of $\mathfrak{f}$ and $\mathfrak{s}_{-1}$ by

$$\mathfrak{f}_E = \mathfrak{f} \cap S^2(E^\perp) \subset \mathfrak{f}, \qquad \mathfrak{c}_E = \hat{E} \oplus \mathfrak{f}_E \subset \mathfrak{s}_{-1} = V \oplus \mathfrak{f},$$

where $\hat{E} = \{v \in E \mid v \circledcirc E \subset \mathfrak{f}^\perp\}$. Then we have

Lemma 5.1. *$\mathfrak{f}_E$ and $\mathfrak{c}_E$ are $G(\mathfrak{s})$-invariant.*

Proof. Since E is $G_o(\mathfrak{s})$-invariant, $\hat{E}$ and $E^\perp$ are $G_0(\mathfrak{s})$-invariant subspaces of V and V^* respectively, which also implies $S^2(E^\perp)$ is a $G_0(\mathfrak{s})$-invariant subspace of $S^2(V^*)$. Hence $\mathfrak{f}_E$ is a $G(\mathfrak{s})$-invariant subspace of $\mathfrak{f}$. Since $\hat{E}$ is $G_o(\mathfrak{s})$-invariant, to show that $\mathfrak{c}_E$ is $G(\mathfrak{s})$-invariant, it suffices to check that $\rho(\hat{E}) \subset \mathfrak{f}_E$ for each $\rho \in \mathfrak{f}^{(1)}$, where $\mathfrak{f}^{(1)} = \mathfrak{f} \otimes V^* \cap S^3(V^*)$ is the first prolongation of $\mathfrak{f}$ and $\rho : V \to \mathfrak{f} \subset S^2(V^*)$ satisfies $v_1 \rfloor \rho(v_2) = v_2 \rfloor \rho(v_1)$ (see §5.2 [20] for the detail). From $(S^2(E^\perp))^\perp = E \otimes_S V \subset S^2(V)$, we have $(\mathfrak{f}_E)^\perp = \mathfrak{f}^\perp + E \otimes_S V$. Hence, from $\hat{E} \otimes_S E \subset \mathfrak{f}^\perp$, $\rho(\hat{E}) \subset \mathfrak{f}_E$ follows from

$$\mathfrak{f}^\perp + E \otimes_S V \subset (\hat{E} \rfloor \mathfrak{f}^{(1)})^\perp = \{a \in S^2(V) \mid \hat{E} \circledcirc a \subset (\mathfrak{f}^{(1)})^\perp\}.$$

□

Hence we can define the covariant systems $\mathfrak{f}(E)$ and $C(E)$ of $(R; D^1, D^2)$ as follows; Take a graded Lie algebra isomorphism ϕ of $\mathfrak{s}(v)$ onto $\mathfrak{s}$. We put

$$\mathfrak{f}(E)(v) = \phi^{-1}(\mathfrak{f}_E) \subset C(E)(v) = \phi^{-1}(\mathfrak{c}_E) \subset D^2(v) = \mathfrak{s}_{-1}(v).$$

Then, by the above lemma, it follows that $\mathfrak{f}(E)(v)$ and $C(E)(v)$ are well-defined and we obtain subbundles $\mathfrak{f}(E)$ and $C(E)$ of D^2.

Moreover, starting from the $G(\mathfrak{s})$-invariant subspace $\hat{E}$, we put

$$\mathfrak{f}_{\hat{E}} = \mathfrak{f} \cap S^2((\hat{E})^\perp) \subset \mathfrak{f}, \qquad \mathfrak{c}_{\hat{E}} = \hat{E} \oplus \mathfrak{f}_{\hat{E}} \subset \mathfrak{s}_{-1} = V \oplus \mathfrak{f}.$$

$\mathfrak{f}_{\hat{E}}$ and $\mathfrak{c}_{\hat{E}}$ are also $G(\mathfrak{s})$-invariant and we can define the covariant systems $\mathfrak{f}(\hat{E})$ and $C(\hat{E})$ of $(R; D^1.D^2)$ as follows;

$$\mathfrak{f}(\hat{E})(v) = \phi^{-1}(\mathfrak{f}_{\hat{E}}) \subset C(\hat{E})(v) = \phi^{-1}(\mathfrak{c}_{\hat{E}}) \subset D^2(v) = \mathfrak{s}_{-1}(v),$$

for a graded Lie algebra isomorphism ϕ of $\mathfrak{s}(v)$ onto $\mathfrak{s}$. Here we note $\hat{(\hat{E})} = \hat{E}$.

For these systems, we have

Lemma 5.2. (1) *$\mathfrak{f}(E) = Ch\,(D^1) \cap Ch\,(\tilde{N})$, if $C(E) = Ch\,(\tilde{N})$.*
(2) *$Ch\,(D^1)(v) \cap Ch\,(\tilde{N})(v) \subset \mathfrak{f}(E)(v)$ and $Ch\,(\tilde{N})(v) \subset C(E)(v)$ at each $v \in R$.*
(3) *$Ch\,(D^1)(v) \cap Ch\,(\tilde{N}^*)(v) \subset \mathfrak{f}(\hat{E})(v)$ and $Ch\,(\tilde{N}^*)(v) \subset C(\hat{E})(v)$ at each $v \in R$.*
(4) rank *$Ch\,(D^1) \cap Ch\,(\tilde{N}) = \frac{1}{2}(n-r)(n-r+1)$ iff $\mathfrak{f}^\perp \subset E \otimes_S V$ and $Ch\,(D^1) \cap Ch\,(\tilde{N}) = \mathfrak{f}(E)$.*
(5) *If $\mathfrak{f}^\perp \subset E \otimes_S V$, then $Ch\,(D^2)(v) \subset C(E)(v)$ at each $v \in R$.*

Proof. (1) By definition, $\mathfrak{f}(E) = C(E) \cap \mathrm{Ch}\,(D^1)$.

(2) By Lemma 3.1, $\mathrm{Ch}\,(\tilde{N})(v) \subset D^2(v) = \mathfrak{s}_{-1}(v)$. Let ϕ be a graded Lie algebra isomorphism of $\mathfrak{s}(v)$ onto $\mathfrak{s}$. For a vector $X \in \mathrm{Ch}\,(\tilde{N})(v)$, we have $[X, \Gamma(\tilde{N})] \subset \Gamma(\tilde{N})$. Moreover $\tilde{N}(v)$ corresponds to the subspace $E^\perp \oplus \mathfrak{s}_{-1}$ of $\mathfrak{s}$ under ϕ (see §3.1). Thus $\phi(X) \in \mathfrak{s}_{-1}$ satisfies $[\phi(X), \mathfrak{s}_{-1}] \subset E^\perp$ and $[\phi(X), E^\perp] = 0$. Then, for $\phi(X) = v + a$, $v \in V$, $a \in \mathfrak{f}$, we have $v \rfloor \mathfrak{f} \subset E^\perp, V \rfloor a \subset E^\perp$ and $\langle v, E^\perp \rangle = 0$. Hence we have $v \in E$, $v \circledcirc E \subset \mathfrak{f}^\perp$ and $X \rfloor a = 0$ for $\forall X \in E$, which implies $v \in \hat{E}$ and $a \in \mathfrak{f}_E$.

(3) can be shown similarly as in (2) by taking $\hat{E}$ in place of E.

(4) By $\mathfrak{f}_E = \mathfrak{f} \cap S^2(E^\perp)$, $\dim \mathfrak{f}_E \leqq \frac{1}{2}(n-r)(n-r+1)$ and the equality holds iff $\mathfrak{f}_E = S^2(E^\perp)$, i.e., iff $\mathfrak{f} \supset S^2(E^\perp)$. Thus (4) follows immediately from (2).

(5) For a vector $X \in \mathrm{Ch}\,(D^2)(v)$, we have $[X, \Gamma(D^2)] \subset \Gamma(D^2)$. Hence we have $[\phi(X), \mathfrak{s}_{-1}] = 0$ for an isomorphism $\phi : \mathfrak{s}(v) \to \mathfrak{s}$. Thus, for $\phi(X) = v + a$, $v \in V$, $a \in \mathfrak{f}$, we have $v \rfloor \mathfrak{f} = 0$ and $V \rfloor a = 0$. Hence, from $\mathfrak{f} \supset S^2(E^\perp)$, we have $v \in E$, $v \circledcirc E \subset \mathfrak{f}^\perp$ and $a = 0$, which implies $\phi(X) \in \hat{E}$. This completes the proof of (5). $\square$

Now we assume that there exists $G_0(\mathfrak{s})$-invariant subspace E of V of dimension r such that $\mathfrak{f}^\perp \subset E \otimes_S V$. Let s be the dimension of $\hat{E} = \{v \in E \mid v \circledcirc E \subset \mathfrak{f}^\perp\}$. First we describe the structure equation of the graded Lie algebra $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$, where

$$\mathfrak{s}_{-3} = \mathbb{R}, \quad \mathfrak{s}_{-2} = V^*, \quad \mathfrak{s}_{-1} = V \oplus \mathfrak{f}, \qquad \mathfrak{f} \subset S^2(V^*).$$

We have $\hat{E} \otimes_S E \subset \mathfrak{f}^\perp \subset E \otimes_S V$. Thus we have a basis $\{e_1, \ldots, e_n\}$ of V such that

$$\hat{E} = \langle \{e_1, \ldots, e_s\} \rangle \subset E = \langle \{e_1, \ldots, e_r\} \rangle,$$

$$\langle \{e_a \circledcirc e_k (1 \leqq a \leqq s, 1 \leqq k \leqq r)\} \rangle \subset \mathfrak{f}^\perp \subset \langle \{e_i \circledcirc e_\ell (1 \leqq i \leqq r, 1 \leqq \ell \leqq n)\} \rangle.$$

Namely we have

$$\begin{aligned} S^2(E^\perp) = \langle \{e^*_\alpha \circledcirc e^*_\beta (r+1 \leqq \alpha \leqq \beta \leqq n)\} \rangle \subset \mathfrak{f} \subset \\ S^2(E^\perp) \cup \langle \{e^*_k \circledcirc e^*_\alpha (1 \leqq k \leqq r, r+1 \leqq \alpha \leqq n)\} \rangle \cup \langle \{e^*_i \circledcirc e^*_j (s+1 \leqq i \leqq j \leqq r)\} \rangle, \end{aligned}$$

where $\{e^*_1, \ldots, e^*_n\}$ are the dual basis of $\{e_1, \ldots, e_n\}$ in V^*. Let us take the complimentary subspace H of V such that $V = E \oplus H$, where $H = \langle \{e_{r+1}, \ldots, e_n\} \rangle$, and take a complimentary subspace T of $E \otimes_S V$ such that $E \otimes_S V = \mathfrak{f}^\perp \oplus T$. Then, from $S^2(V) = E \otimes_S V \oplus S^2(H)$, $\dim T = \dim \mathfrak{f} - \dim S^2(E^\perp) (= t)$. Let $\{\tilde{\pi}_\lambda (1 \leqq \lambda \leqq t)\}$ be a basis of T and put $\tilde{\pi}_{kl} = e_k \circledcirc e_l$ $(1 \leqq k \leqq r, 1 \leqq l \leqq n)$ and $\tilde{\varpi}_{\alpha\beta} = e_\alpha \circledcirc e_\beta$ $(1 \leqq \alpha, \beta \leqq n)$. Under the identification: $S^2(V) \cong (S^2(V^*))^*$, we restrict these covectors to the subspace $\mathfrak{f} \subset S^2(V^*)$ and put

$$\hat{\pi}_\lambda = \tilde{\pi}_\lambda \mid_\mathfrak{f}, \quad \hat{\pi}_{kl} = \tilde{\pi}_{kl} \mid_\mathfrak{f} \quad \text{and} \quad \hat{\varpi}_{\alpha\beta} = \tilde{\varpi}_{\alpha\beta} \mid_\mathfrak{f}.$$

Then we have $\hat{\pi}_{ak} = 0$ $(1 \leqq a \leqq s, 1 \leqq k \leqq r)$ and $\{\hat{\pi}_\lambda (1 \leqq \lambda \leqq t)\}$ forms a basis of $\langle \{\hat{\pi}_{k,\alpha}\ (1 \leqq k \leqq r, r+1 \leqq \alpha \leqq n), \hat{\pi}_{ij}\ (s+1 \leqq i, j \leqq r)\} \rangle$. Moreover $\{\hat{\pi}_\lambda (1 \leqq \lambda \leqq t), \hat{\varpi}_{\alpha\beta} (1 \leqq \alpha \leqq \beta \leqq n)\}$ forms a basis of $\mathfrak{f}^*$. Then, firstly fixing a basis of $\mathfrak{s}_{-3}$, we have a basis of $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$, $\mathfrak{s}_{-3} \cong \mathbb{R}$, $\mathfrak{s}_{-2} \cong V^*$ and $\mathfrak{s}_{-1} = V \oplus \mathfrak{f}$, by fixing the basis $\{e_1, \ldots, e_n\}$ of V as above. Thus we have covectors

$$\begin{aligned} \{\hat{\varpi}, \hat{\varpi}_1, \ldots, \hat{\varpi}_n, \hat{\omega}^1, \ldots, \hat{\omega}^n, \hat{\varpi}_{\alpha\beta}\ (r+1 \leqq \alpha, \beta \leqq n), \\ \hat{\pi}_{k,\alpha}\ (1 \leqq k \leqq r, r+1 \leqq \alpha \leqq n), \hat{\pi}_{ij}\ (s+1 \leqq i, j \leqq r)\} \end{aligned}$$

in $\mathfrak{s}^*$ such that $\hat{\varpi}_{\alpha\beta} = \hat{\varpi}_{\beta\alpha}$, $\hat{\pi}_{ij} = \hat{\pi}_{ji}$, and that

$$\begin{cases} d\hat{\varpi} = \hat{\omega}^1 \wedge \hat{\varpi}_1 + \cdots + \hat{\omega}^n \wedge \hat{\varpi}_n, \\ d\hat{\varpi}_a = \displaystyle\sum_{\alpha=r+1}^{n} \hat{\omega}^\alpha \wedge \hat{\pi}_{a\alpha} \quad (1 \leqq a \leqq s), \\ d\hat{\varpi}_i = \displaystyle\sum_{j=s+1}^{r} \hat{\omega}^j \wedge \hat{\pi}_{ij} + \sum_{\alpha=r+1}^{n} \hat{\omega}^\alpha \wedge \hat{\pi}_{i\alpha} \quad (s+1 \leqq i \leqq r), \\ d\hat{\varpi}_\alpha = \displaystyle\sum_{a=1}^{s} \hat{\omega}^a \wedge \hat{\pi}_{a\alpha} + \sum_{i=s+1}^{r} \hat{\omega}^i \wedge \hat{\pi}_{i\alpha} + \sum_{\beta=r+1}^{n} \hat{\omega}^\beta \wedge \hat{\varpi}_{\alpha\beta} \quad (r+1 \leqq \alpha \leqq n), \end{cases}$$

where $\{\hat{\varpi}, \hat{\varpi}_1, \ldots, \hat{\varpi}_n, \hat{\omega}^1, \ldots, \hat{\omega}^n, \hat{\varpi}_{\alpha\beta}\ (r+1 \leqq \alpha \leqq \beta \leqq n), \hat{\pi}_\lambda\ (1 \leqq \lambda \leqq t)\}$ forms a basis of $\mathfrak{s}^*$ such that $\{\hat{\pi}_\lambda (1 \leqq \lambda \leqq t)\}$ is a basis of $\langle\{\hat{\pi}_{k,\alpha}\ (1 \leqq k \leqq r, r+1 \leqq \alpha \leqq n), \hat{\pi}_{ij}\ (s+1 \leqq i,j \leqq r)\}\rangle$. Moreover, putting $\hat{\Omega}_i = \sum_{j=s+1}^{r} \hat{\omega}^j \wedge \hat{\pi}_{ij}$, we see that $\{\hat{\Omega}_{s+1}, \ldots, \hat{\Omega}_r\}$ are linearly independent as follows; If $\sum_{i=s+1}^{r} a_i \hat{\Omega}_i = 0$, we have $\sum_{i=s+1}^{r} a_i \hat{\pi}_{ij} = 0$ for $s+1 \leqq j \leqq r$. From $\hat{\pi}_{ia} = 0$ $(s+1 \leqq i \leqq r, 1 \leqq a \leqq s)$, we have $\sum_{i=s+1}^{r} a_i \hat{\pi}_{ik} = 0$ for $1 \leqq k \leqq r$, which implies $(\sum_{i=s+1}^{r} a_i e_i) \circledcirc e_k \in \mathfrak{f}^\perp$ for $1 \leqq k \leqq r$. Then by the definition of $\hat{E}$, we get $\sum_{i=s+1}^{r} a_i e_i \in \hat{E}$, which shows $a_{s+1} = \cdots = a_r = 0$, by the choice of our basis.

Now let $(R; D^1, D^2)$ be a PD manifold of second order, which is regular of type $\mathfrak{s}$. Let us fix a point $v \in R$. Then, as in §1 of [23], there exists a coframe $\{\varpi, \varpi_1, \ldots, \varpi_n, \omega^1, \ldots, \omega^n, \varpi_{\alpha\beta}\ (r+1 \leqq \alpha \leqq \beta \leqq n), \pi_\lambda\ (1 \leqq \lambda \leqq t)\}$ $(t = \dim \mathfrak{f} - \dim S^2(E^\perp))$ and 1-forms $\{\pi_{k,\alpha}\ (1 \leqq k \leqq r, r+1 \leqq \alpha \leqq n), \pi_{ij}\ (s+1 \leqq i,j \leqq r)\}$ defined around $v \in R$ such that

$$D^1 = \{\varpi = 0\}, \quad D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\}, \quad \tilde{N} = \tilde{N}(E) = \{\varpi = \varpi_1 = \cdots = \varpi_r = 0\},$$

$$\tilde{N}^* = \tilde{N}(\hat{E}) = \{\varpi = \varpi_1 = \cdots = \varpi_s = 0\}, \quad \tilde{N}^\perp = \{\varpi = \varpi_1 = \cdots = \varpi_n = \omega^{r+1} = \cdots = \omega^n = 0\},$$

and the following structure equations hold;

(A) $$d\varpi \equiv \omega^1 \wedge \varpi_1 + \cdots + \omega^n \wedge \varpi_n \pmod{\varpi},$$

(B.1) $$\begin{cases} d\varpi_a \equiv \displaystyle\sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \pi_{a\alpha}\ (1 \leqq a \leqq s), \\ d\varpi_i \equiv \displaystyle\sum_{j=s+1}^{r} \omega^j \wedge \pi_{ij} + \sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \pi_{i\alpha}\ (s+1 \leqq i \leqq r), \\ d\varpi_\alpha \equiv \displaystyle\sum_{a=1}^{s} \omega^a \wedge \pi_{a\alpha} + \sum_{i=s+1}^{r} \omega^i \wedge \pi_{i\alpha} + \sum_{\beta=r+1}^{n} \omega^\beta \wedge \varpi_{\alpha\beta}\ (r+1 \leqq \alpha \leqq n), \end{cases}$$

(mod $\varpi, \varpi_1, \ldots, \varpi_n$), where $\varpi_{\alpha\beta} = \varpi_{\beta\alpha}$, $\pi_{ij} = \pi_{ji}$, and $\{\pi_\lambda (1 \leqq \lambda \leqq t)\}$ is a basis of $\langle\{\pi_{k,\alpha}\ (1 \leqq k \leqq r, r+1 \leqq \alpha \leqq n), \pi_{ij}\ (s+1 \leqq i,j \leqq r)\}\rangle$ (mod $\varpi, \varpi_1, \ldots, \varpi_n$). Moreover $\{\Omega_{s+1}, \ldots, \Omega_r\}$ are linearly independent (mod $\varpi, \varpi_1 \ldots, \varpi_n$), where $\Omega_i = \sum_{j=s+1}^{r} \omega^j \wedge \pi_{ij}$. Furthermore, we have

$$C(E) = \{\varpi = \varpi_1 = \cdots = \varpi_n = \omega^{s+1} = \cdots = \omega^n = \pi_1 = \cdots = \pi_t = 0\},$$

and

$$\mathfrak{f}(E) = \{\varpi = \varpi_1 = \cdots = \varpi_n = \omega^1 = \cdots = \omega^n = \pi_1 = \cdots = \pi_t = 0\}.$$

By utilizing the structure equations (A) and (B.1), we will investigate the properties of $C(E)$ and $\mathfrak{f}(E)$. In the rest of this section, we will adopt the Einstein's convention for indices. The index ranges are as follows; $1 \leqq a, b \leqq s,\ s+1 \leqq i, j, k \leqq r,\ r+1 \leqq \alpha, \beta \leqq n$ and $1 \leqq \lambda \leqq t$. From (B.1), we have

$$\text{(B.2)} \qquad \begin{cases} d\varpi_a \equiv & \omega^\alpha \wedge \pi_{a\alpha} + A_a^{\alpha\beta}\varpi_\alpha \wedge \varpi_\beta + B_a^{\alpha\lambda}\varpi_\alpha \wedge \pi_\lambda + C_a^{\alpha\beta\gamma}\varpi_\alpha \wedge \varpi_{\beta\gamma} \\ & \quad + D_{ab}^\alpha \varpi_\alpha \wedge \omega^b + E_{aj}^\alpha \varpi_\alpha \wedge \omega^j + F_{a\beta}^\alpha \varpi_\alpha \wedge \omega^\beta, \\ d\varpi_i \equiv \omega^j \wedge \pi_{ij} + & \omega^\alpha \wedge \pi_{i\alpha} + A_i^{\alpha\beta}\varpi_\alpha \wedge \varpi_\beta + B_i^{\alpha\lambda}\varpi_\alpha \wedge \pi_\lambda + C_i^{\alpha\beta\gamma}\varpi_\alpha \wedge \varpi_{\beta\gamma} \\ & \quad + D_{ib}^\alpha \varpi_\alpha \wedge \omega^b + E_{ij}^\alpha \varpi_\alpha \wedge \omega^j + F_{i\beta}^\alpha \varpi_\alpha \wedge \omega^\beta, \end{cases}$$

(mod $\varpi, \varpi_1, \ldots, \varpi_r$). From (A), we have $d\varpi \equiv 0$ (mod $\varpi, \varpi_1, \ldots, \varpi_r, \omega^{r+1}, \ldots, \omega^n$). Hence we have

$$d\omega^\alpha \wedge \varpi_\alpha \equiv \omega^a \wedge d\varpi_a + \omega^i \wedge d\varpi_i \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_r, \omega^{r+1}, \ldots, \omega^n).$$

Then, from (B.2), we get

$$\text{(5.1)} \quad d\omega^\alpha + \omega^a \wedge (B_a^{\alpha\lambda}\pi_\lambda + C_a^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ab}^\alpha \omega^b + E_{aj}^\alpha \omega^j) \\ + \omega^i \wedge (B_i^{\alpha\lambda}\pi_\lambda + C_i^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ib}^\alpha \omega^b + E_{ij}^\alpha \omega^j) \equiv 0 \ (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_n, \omega^{r+1}, \ldots, \omega^n).$$

Putting $\mathfrak{M}_1 = \langle\{\varpi, \varpi_1, \ldots, \varpi_n, \pi_1, \ldots, \pi_t\}\rangle = \langle\{\varpi, \varpi_1, \ldots, \varpi_n, \pi_{k,\alpha}\ (1 \leqq k \leqq r, r+1 \leqq \alpha \leqq n), \pi_{ij}\ (s+1 \leqq i, j \leqq r)\}\rangle$, we have

$$d\varpi \equiv d\varpi_a \equiv d\varpi_i \equiv 0, \quad d\varpi_\alpha \equiv \omega^\delta \wedge \varpi_{\alpha\delta} \quad (\mathrm{mod} \quad \mathfrak{M}_1).$$

Then, taking exterior derivatives of both sides of (B.2) and calculating mod $\mathfrak{M}_1$, we get

$$\omega^\alpha \wedge d\pi_{a\alpha} \equiv \omega^\delta \wedge \varpi_{\alpha\delta} \wedge (C_a^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ab}^\alpha \omega^b + E_{aj}^\alpha \omega^j + F_{a\beta}^\alpha \omega^\beta) \quad (\mathrm{mod} \quad \mathfrak{M}_1),$$

and

$$\omega^j \wedge d\pi_{ij} + \omega^\alpha \wedge d\pi_{i\alpha} \equiv \omega^\delta \wedge \varpi_{\alpha\delta} \wedge (C_i^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ib}^\alpha \omega^b + E_{ij}^\alpha \omega^j + F_{i\beta}^\alpha \omega^\beta) \quad (\mathrm{mod} \quad \mathfrak{M}_1).$$

Thus we obtain

$$\text{(5.2)} \qquad d\pi_{a\alpha} \equiv C_a^{\delta\beta\gamma}\varpi_{\alpha\delta} \wedge \varpi_{\beta\gamma} + D_{ab}^\delta \varpi_{\alpha\delta} \wedge \omega^b + E_{aj}^\delta \varpi_{\alpha\delta} \wedge \omega^j$$

(mod $\varpi, \varpi_1, \ldots, \varpi_n, \omega^{r+1}, \ldots, \omega^n, \pi_1, \ldots, \pi_t$), and

$$\text{(5.3)} \qquad \begin{cases} d\pi_{i\alpha} \equiv C_i^{\delta\beta\gamma}\varpi_{\alpha\delta} \wedge \varpi_{\beta\gamma} + D_{ib}^\delta \varpi_{\alpha\delta} \wedge \omega^b, \\ d\pi_{ij} \equiv 0 \qquad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_n, \omega^{s+1}, \ldots, \omega^n, \pi_1, \ldots, \pi_t). \end{cases}$$

Moreover, from (B.2), we have, for $s+1 \leqq p, q \leqq n$,

$$\text{(B.3)} \quad d\varpi_a \equiv \omega^\alpha \wedge \pi_{a\alpha} + A_a^{pq}\varpi_p \wedge \varpi_q + B_a^{j\lambda}\varpi_j \wedge \pi_\lambda + B_a^{\alpha\lambda}\varpi_\alpha \wedge \pi_\lambda \\ + C_a^{j\beta\gamma}\varpi_j \wedge \varpi_{\beta\gamma} + C_a^{\alpha\beta\gamma}\varpi_\alpha \wedge \varpi_{\beta\gamma} + D_{ab}^j \varpi_j \wedge \omega^b + D_{ab}^\alpha \varpi_\alpha \wedge \omega^b \\ + E_{ak}^j \varpi_j \wedge \omega^k + E_{aj}^\alpha \varpi_\alpha \wedge \omega^j + F_{a\beta}^j \varpi_j \wedge \omega^\beta + F_{a\beta}^\alpha \varpi_\alpha \wedge \omega^\beta \ (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_s).$$

From (A), we have $d\varpi \equiv 0$ (mod $\varpi, \varpi_1, \ldots, \varpi_s, \omega^{s+1}, \ldots, \omega^n$). Hence we have

$$d\omega^i \wedge \varpi_i + d\omega^\alpha \wedge \varpi_\alpha \equiv \omega^a \wedge d\varpi_a \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_s, \omega^{s+1}, \ldots, \omega^n).$$

Then, from (B.3), we get

$$\text{(5.4)} \qquad d\omega^i + \omega^a \wedge (B_a^{i\lambda}\pi_\lambda + C_a^{i\beta\gamma}\varpi_{\beta\gamma} + D_{ab}^i \omega^b) \equiv 0 \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_n, \omega^{s+1}, \ldots, \omega^n).$$

Here we have

Lemma 5.3. (1) *$\mathfrak{f}(E) = Ch(D^1) \cap Ch(\tilde{N})$ iff $C_a^{\alpha\beta\gamma} = C_i^{\alpha\beta\gamma} = 0$.*
(2) *$C(E) = Ch(\tilde{N})$ iff $C_a^{\alpha\beta\gamma} = C_i^{\alpha\beta\gamma} = D_{ab}^\alpha = D_{ib}^\alpha = 0$.*
(3) *$C(E) \subset Ch(\tilde{N}^*)$ iff $C_a^{\alpha\beta\gamma} = C_a^{i\beta\gamma} = D_{ab}^\alpha = D_{ab}^i = 0$.*

Proof. (1) From $Ch(D^1) \cap \mathrm{Ch}\,(\tilde{N}) \subset \mathfrak{f}(E) \subset \mathrm{Ch}\,(D^1)$, we see that $\mathfrak{f}(E) = \mathrm{Ch}\,(D^1) \cap \mathrm{Ch}\,(\tilde{N})$ iff $\mathfrak{f}(E) \subset \mathrm{Ch}\,(\tilde{N})$. For a vector $X \in \mathfrak{f}(E)(v)$, $v \in R$, by (A) and (B.2), we have

$$\begin{cases} X \rfloor d\varpi \equiv 0 \qquad (\mathrm{mod}\ \varpi), \\ X \rfloor d\varpi_a \equiv -C_a^{\alpha\beta\gamma}\varpi_{\beta\gamma}(X)\varpi_\alpha \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_r), \\ X \rfloor d\varpi_i \equiv -C_i^{\alpha\beta\gamma}\varpi_{\beta\gamma}(X)\varpi_\alpha \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_r). \end{cases}$$

Thus $X \in \mathrm{Ch}\,(\tilde{N})(v)$ iff $C_a^{\alpha\beta\gamma}\varpi_{\beta\gamma}(X) = C_i^{\alpha\beta\gamma}\varpi_{\beta\gamma}(X) = 0$, which completes the proof of (1).

(2) From $\mathrm{Ch}\,(\tilde{N}) \subset C(E)$, $C(E) = \mathrm{Ch}\,(\tilde{N})$ iff $C(E) \subset \mathrm{Ch}\,(\tilde{N})$. For a vector $X \in C(E)(v)$, $v \in R$, by (A) and (B.2), we have

$$\begin{cases} X \rfloor d\varpi \equiv \omega^a(X)\varpi_a \quad (\mathrm{mod}\ \varpi) \quad \equiv 0 \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_r), \\ X \rfloor d\varpi_a \equiv -(C_a^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ab}^\alpha\omega^b)(X)\varpi_\alpha \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_r), \\ X \rfloor d\varpi_i \equiv -(C_i^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ib}^\alpha\omega^b)(X)\varpi_\alpha \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_r). \end{cases}$$

Thus $X \in \mathrm{Ch}\,(\tilde{N})(v)$ iff $(C_a^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ab}^\alpha\omega^b)(X) = (C_i^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ib}^\alpha\omega^b)(X) = 0$, which completes the proof of (2).

(3) For a vector $X \in C(E)(v)$, $v \in R$, by (A) and (B.3), we have

$$\begin{cases} X \rfloor d\varpi \equiv \omega^a(X)\varpi_a \quad (\mathrm{mod}\ \varpi) \quad \equiv 0 \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_s), \\ X \rfloor d\varpi_a \equiv -(C_a^{i\beta\gamma}\varpi_{\beta\gamma} + D_{ab}^i\omega^b)(X)\varpi_i - (C_a^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ab}^\alpha\omega^b)(X)\varpi_\alpha \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_s) \end{cases}$$

Thus $X \in \mathrm{Ch}\,(\tilde{N}^*)(v)$ iff $(C_a^{i\beta\gamma}\varpi_{\beta\gamma} + D_{ab}^i\omega^b)(X) = (C_a^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ab}^\alpha\omega^b)(X) = 0$, which completes the proof of (3). □

Thus we get

Lemma 5.4. (1) *If $r < n-1$ and $\mathfrak{f}(E)$ is completely integrable, then $\mathfrak{f}(E) = Ch\,(D^1) \cap Ch\,(\tilde{N})$.*

(2) *If $C(E)$ is completely integrable, then $C_a^{\alpha\beta\gamma} = C_a^{i\beta\gamma} = D_{ab}^\alpha = D_{ib}^\alpha = 0$ and $D_{ab}^i = D_{ba}^i$. Moreover $C_i^{\alpha\beta\gamma} = 0$, when $r < n-1$. In particular, if $r < n-1$ and $C(E)$ is completely integrable, then $C(E) = Ch\,(\tilde{N})$.*

Proof. (1) If $\mathfrak{f}(E)$ is completely integrable, we have

$$d\omega^i \equiv d\omega^\alpha \equiv d\pi_\lambda \equiv 0 \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_n, \omega^1, \ldots, \omega^n, \pi_1, \ldots, \pi_t).$$

Then, if $r < n-1$, $C_a^{\alpha\beta\gamma} = 0$ follows from (5.2) and $C_i^{\alpha\beta\gamma} = 0$ follows from (5.3), which proves (1) by (1) in Lemma 5.3.

(2) If $C(E)$ is completely integrable, we have

$$d\omega^i \equiv d\omega^\alpha \equiv d\pi_\lambda \equiv 0 \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_n, \omega^{s+1}, \ldots, \omega^n, \pi_1, \ldots, \pi_t).$$

Then $C_a^{\alpha\beta\gamma} = 0$ follows from (5.1), $D_{ab}^\alpha = 0$ follows from (5.2), $D_{ib}^\alpha = 0$ follows from (5.3), $C_a^{i\beta\gamma} = 0$ and $D_{ab}^i = D_{ba}^i$ follows from (5.4) respectively. Moreover, when $r < n-1$, $C_i^{\alpha\beta\gamma} = 0$ follows from (5.3). Then the last assertion follows from (2) in Lemma 5.3. □

Moreover we have

Lemma 5.5. *In case $r = n-1$, if $C(E)$ is completely integrable and* $\mathrm{rank}\, Ch\,(D^2) < \dim \hat{E}$, *then $C(E) = Ch\,(\tilde{N})$.*

Proof. We assume that $r = n-1$ and $C(E)$ is completely integrable. Then, by (2) in Lemma 5.4, we have $C_a^{nnn} = C_a^{inn} = D_{ab}^n = D_{ib}^n = 0$ and $D_{ab}^i = D_{ba}^i$. Thus, by (2) in Lemma 5.3, it suffices to show $C_i^{nnn} = 0$ when $\operatorname{rank} \operatorname{Ch}(D^2) < \dim \hat{E}$. Putting $\mathfrak{M}_2 = \langle \{\varpi, \varpi_1, \ldots, \varpi_n, \omega^1, \ldots, \omega^s, \omega^n\} \rangle$, we have

$$d\varpi \equiv d\varpi_a \equiv 0, \quad d\varpi_i \equiv \omega^j \wedge \pi_{ij}, \quad d\varpi_n \equiv \omega^i \wedge \pi_{in}, \quad (\mathrm{mod} \quad \mathfrak{M}_2).$$

Then, taking exterior derivatives of both sides of (B.3) and calculating mod $\mathfrak{M}_2$, we get

$$d\omega^n \wedge \pi_{an} + d\varpi_j \wedge (B_a^{j\lambda} \pi_\lambda + E_{ak}^j \omega^k) + d\varpi_n \wedge (B_a^{n\lambda} \pi_\lambda + E_{aj}^n \omega^j) \equiv 0 \quad (\mathrm{mod} \quad \mathfrak{M}_2).$$

From (5.1), we have $d\omega^n \equiv (B_i^{n\lambda} \pi_\lambda + C_i^{nnn} \varpi_{nn} + E_{ij}^n \omega^j) \wedge \omega^i$ (mod $\mathfrak{M}_2$). Hence we get

$$\begin{aligned}(B_i^{n\lambda} \pi_\lambda + C_i^{nnn} \varpi_{nn} + E_{ij}^n \omega^j) \wedge \omega^i \wedge \pi_{an} + \omega^j \wedge \pi_{ij} \wedge (B_a^{j\lambda} \pi_\lambda + E_{ak}^j \omega^k) \\ + \omega^i \wedge \pi_{in} \wedge (B_a^{n\lambda} \pi_\lambda + E_{aj}^n \omega^j) \equiv 0 \quad (\mathrm{mod} \quad \mathfrak{M}_2).\end{aligned}$$

Then, since $\pi_{an}, \pi_{ij}, \pi_{in}$ are linear combinations of $\{\pi_\lambda (1 \leqq \lambda \leqq t)\}$ (mod $\varpi, \varpi_1, \ldots, \varpi_n$), we obtain $C_i^{nnn} \varpi_{nn} \wedge \omega^i \wedge \pi_{an} = 0$. By the assumption $\operatorname{rank} \operatorname{Ch}(D^2) < \dim \hat{E}$, we have $\pi_{an} \neq 0$ for some a. Hence we get $C_i^{nnn} \varpi_{nn} \wedge \omega^i = 0$, which implies $C_i^{nnn} = 0$. This completes the proof. □

Finally we add the following.

Lemma 5.6. *If $C(E) = Ch(\tilde{N})$ and $\langle \{\pi_{ij} (s+1 \leqq i, j \leqq r+1)\} \rangle \cap \langle \{\pi_{a\alpha} (1 \leqq a \leqq s, r+1 \leqq \alpha \leqq n)\} \rangle = \{0\}$, then $C(E) \subset Ch(\tilde{N}^*)$.*

Proof. From $C(E) = \operatorname{Ch}(\tilde{N})$ and by (2) in Lemma 5.3, we have $C_a^{\alpha\beta\gamma} = C_i^{\alpha\beta\gamma} = D_{ab}^\alpha = D_{ib}^\alpha = 0$. Moreover, by (2) in Lemma 5.4, we have $C_a^{i\beta\gamma} = 0$. Hence, by (3) in Lemma 5.3, it suffices to show $D_{ab}^i = 0$, under the above condition. Putting $\mathfrak{M}_3 = \langle \{\varpi, \varpi_1, \ldots, \varpi_n, \omega^{r+1}, \ldots, \omega^n\} \rangle$, we have

$$d\varpi \equiv d\varpi_a \equiv 0, \quad d\varpi_i \equiv \omega^j \wedge \pi_{ij}, \quad d\varpi_\alpha \equiv \omega^b \wedge \pi_{b\alpha} + \omega^i \wedge \pi_{i\alpha}, \quad (\mathrm{mod} \quad \mathfrak{M}_3).$$

Then, taking exterior derivatives of both sides of (B.3) and calculating mod $\mathfrak{M}_3$, we get

$$d\omega^\alpha \wedge \pi_{a\alpha} + d\varpi_j \wedge (B_a^{j\lambda} \pi_\lambda + D_{ab}^j \omega^b + E_{ak}^j \omega^k) + d\varpi_\alpha \wedge (B_a^{\alpha\lambda} \pi_\lambda + E_{aj}^\alpha \omega^j) \equiv 0 \quad (\mathrm{mod} \quad \mathfrak{M}_3).$$

From (5.1), we have $d\omega^\alpha \equiv (B_b^{\alpha\lambda} \pi_\lambda + E_{bj}^\alpha \omega^j) \wedge \omega^b + (B_i^{\alpha\lambda} \pi_\lambda + E_{ij}^\alpha \omega^j) \wedge \omega^i$ (mod $\mathfrak{M}_3$). Hence we get

$$\begin{aligned}(B_b^{\alpha\lambda} \pi_\lambda + E_{bj}^\alpha \omega^j) \wedge \omega^b \wedge \pi_{a\alpha} + (B_i^{\alpha\lambda} \pi_\lambda + E_{ij}^\alpha \omega^j) \wedge \omega^i \wedge \pi_{a\alpha} + \omega^k \wedge \pi_{jk} \wedge (B_a^{j\lambda} \pi_\lambda + D_{ab}^j \omega^b + E_{ai}^j \omega^i) \\ + (\omega^b \wedge \pi_{b\alpha} + \omega^i \wedge \pi_{i\alpha}) \wedge (B_a^{\alpha\lambda} \pi_\lambda + E_{aj}^\alpha \omega^j) \equiv 0 \quad (\mathrm{mod} \quad \mathfrak{M}_3).\end{aligned}$$

Then, since $\pi_{a\alpha}, \pi_{ij}, \pi_{i\alpha}$ are linear combinations of $\{\pi_\lambda (1 \leqq \lambda \leqq t)\}$ (mod $\varpi, \varpi_1, \ldots, \varpi_n$), we have

$$\omega^b \wedge \omega^j \wedge (E_{bj}^\alpha \pi_{a\alpha} + E_{aj}^\alpha \pi_{b\alpha} - D_{ab}^i \pi_{ij}) \equiv 0 \quad (\mathrm{mod} \quad \mathfrak{M}_3).$$

Then, by the assumption $\langle \{\pi_{ij} (s+1 \leqq i, j \leqq r+1)\} \rangle \cap \langle \{\pi_{a\alpha} (1 \leqq a \leqq s, r+1 \leqq \alpha \leqq n)\} \rangle = \{0\}$, we get $D_{ab}^i \omega^b \wedge \omega^j \wedge \pi_{ij} \equiv 0$ (mod $\mathfrak{M}_3$). Thus we obtain

$$D_{ab}^i \omega^j \wedge \pi_{ij} = D_{ab}^i \Omega_i \equiv 0 \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_n, \omega^a, \omega^\alpha (1 \leqq a \leqq s, r+1 \leqq \alpha \leqq n)).$$

Since $\{\Omega_{s+1}, \ldots, \Omega_r\}$ are linearly independent (mod $\varpi, \varpi_1, \ldots, \varpi_n$), we obtain $D_{ab}^i = 0$, which completes the proof. □

Summarizing the discussion above, we obtain the first part of Second Reduction Theorem for PD manifolds of second order, which are regular of type $\mathfrak{s}$.

Theorem 5.1. *Let $(R; D^1, D^2)$ be a PD manifold of second order, which is regular of type $\mathfrak{s}$. Assume that there exists $G_0(\mathfrak{s})$-invariant subspace E of V of dimension r such that $\mathfrak{f}^\perp \subset E \otimes_S V$.*

(1) *In case $r < n-1$, if $\mathfrak{f}(E)$ is completely integrable, then $\mathfrak{f}(E) = Ch(D^1) \cap Ch(\tilde{N})$ and $\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ is a local isomorphism.*

(2) *In case $r < n-1$, if $C(E)$ is completely integrable, then $C(E) = Ch(\tilde{N})$ and $\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ is a local isomorphism.*

(3) *In case $r = n-1$, further assume that* $\operatorname{rank} Ch(D^2) < \dim \hat{E}$, *if $C(E)$ is completely integrable, then $C(E) = Ch(\tilde{N})$ and $\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ is a local isomorphism.*

Thus in these cases, the equivalence of PD manifolds of second order $(R; D^1, D^2)$, which are regular of type $\mathfrak{s}$, is reducible to that of the associated IG manifolds of corank r $(W; C^*, N)$, as in §5.1.

5.3. **Two Step Reduction.** Let $(R; D^1, D^2)$ be a PD manifold of second order, which is regular of type $\mathfrak{s}$. We assume that there exists $G_0(\mathfrak{s})$-invariant subspace E of V of dimension r such that $\mathfrak{f}^\perp \subset E \otimes_S V$. When $\hat{E} \neq \{0\}$, we can discuss the further reduction procedure as in the following.

We assume that $C(E) = \mathrm{Ch}(\tilde{N})$. If $\hat{E} \neq \{0\}$, since $\tilde{N} = \eta_*^{-1}(N)$, N has non-trivial Cauchy characteristic system $\mathrm{Ch}(N)$ on W such that $\operatorname{rank} \mathrm{Ch}(N) = \dim \hat{E}$. Now we assume that W is regular with respect to $\mathrm{Ch}(N)$, i.e., the space $Y = W/\mathrm{Ch}(N)$ of leaves of this foliation is a manifold such that each fibre of the projection $\beta : W \to Y$ is connected and β is a submersion. We further assume that $C(E) \subset \mathrm{Ch}(\tilde{N}^*)$. Then $\mathrm{Ch}(N) \subset \mathrm{Ch}(N^*)$ on W. Moreover, by Lemma 5.2 (3), $\mathrm{Ch}(N^*) \subset N$ on W. Hence there exist differential systems D_N^* and D_N on Y of codimension $s+1$ and $r+1$ respectively such that $N^* = \beta_*^{-1}(D_N^*)$, $N = \beta_*^{-1}(D_N)$, $D_N \supset \mathrm{Ch}(D_N^*)$ and $\mathrm{Ch}(D_N)$ is trivial. In this situation, from $(Y; D_N^*, D_N)$, we can reconstruct the IG manifold $(W; C^*, N)$, at least locally, as follows. First let us consider the collection $\tilde{W}(Y)$ of hyperplanes w in each tangent space $T_y(Y)$ at $y \in Y$ which contains the fibre $D_N^*(y)$ of D_N^*.

$$\tilde{W}(Y) = \bigcup_{y \in Y} \tilde{W}_y \subset J(Y, m-1),$$

$$\tilde{W}_y = \{w \in Gr(T_y(Y), m-1) \mid w \supset D_N^*(y)\} \cong P(T_y(Y)/D_N^*(y)) = \mathbb{P}^s,$$

where $m = \dim Y$ and $s = \dim \hat{E}$. Moreover C_Y^* is the canonical system obtained by the Grassmannian construction and N_Y^*, N_Y are the lifts of D_N^*, D_N. Precisely we have

$$C_Y^*(w) = \mu_*^{-1}(w) \supset N_Y^*(w) = \mu_*^{-1}(D_N^*(y)) \supset N_Y(w) = \mu_*^{-1}(D_N(y)),$$

for each $w \in \tilde{W}(Y)$ and $y = \mu(w)$, where $\mu : \tilde{W}(Y) \to Y$ is the projection. Then we have a map κ_2 of W into $\tilde{W}(Y)$ given by

$$\kappa_2(w) = \beta_*(C^*(w)) \subset T_y(Y),$$

for each $w \in W$ and $y = \beta(w)$. By Realization Lemma for (W, C^*, β, Y), κ_2 is a map of constant rank such that

$$\mathrm{Ker}(\kappa_2)_* = \mathrm{Ch}(C^*) \cap \mathrm{Ker}\, \beta_* = \mathrm{Ch}(C^*) \cap \mathrm{Ch}(N) = \{0\}.$$

Thus κ_2 is an immersion and, by a dimension count, in fact, a local diffeomorphism of W into $\tilde{W}(Y)$ such that

$$(\kappa_2)_*(C^*) = C_Y^*, \quad (\kappa_2)_*(N^*) = N_Y^*, \quad \text{and} \quad (\kappa_2)_*(N) = N_Y.$$

Namely $\kappa_2 : (W; C^*, N) \to (\tilde{W}(Y); C_Y^*, N_Y)$ is a local isomorphism of IG manifold of corank r. Thus $(W; C^*, N)$ is reconstructed from $(Y; D_N^*, D_N)$, at least locally, as a part of $(\tilde{W}(Y); C_Y^*, N_Y)$.

By the construction of $(\tilde{W}(Y); C^*_Y, N_Y)$, an isomorphism of $(Y; D^*_N, D_N)$ naturally lifts to an isomorphism of $(\tilde{W}(Y); C^*_Y, N_Y)$.

Summarizing the above discussion, we obtain the following Second Reduction Theorem (Two Step Reduction Theorem) for PD manifolds of second order, which are regular of type $\mathfrak{s}$ such that there exists a $G_0(\mathfrak{s})$-invariant subspace E of V of dimension r satisfying $\mathfrak{f}^\perp \subset E \otimes_S V$ and $\dim \hat{E} = s > 0$.

Theorem 5.2. *Let $(R; D^1, D^2)$ and $(\hat{R}; \hat{D}^1, \hat{D}^2)$ be PD manifolds of second order, which are regular of type $\mathfrak{s}$. Assume that there exists a $G_0(\mathfrak{s})$-invariant subspace E of V of dimension r satisfying $\mathfrak{f}^\perp \subset E \otimes_S V$ and $\dim \hat{E} = s > 0$. Moreover assume the following two conditions for the covariant sytems of each PD manifold;*

(i) $C(E)$ and $\hat{C}(E)$ are completely integrable (when $r = n-1$, assume further $\operatorname{rank} Ch\,(D^2) < s$ *and* $\operatorname{rank} Ch\,(\hat{D}^2) < s$ *).*

(ii) $C(E) \subset Ch\,(\tilde{N}^*)$ and $\hat{C}(E) \subset Ch\,(\hat{N}^*)$.

Let $(W; C^*, N)$ and $(\hat{W}; \hat{C}^*, \hat{N})$ be the associated IG manifolds of corank r of $(R; D^1, D^2)$ and $(\hat{R}; \hat{D}^1, \hat{D}^2)$ respectively. Assume that W and $\hat{W}$ are regular with respect to $Ch\,(N)$ and $Ch\,(\hat{N})$ respectively. Let $(Y; D^*_N, D_N)$ and $(\hat{Y}, D^*_{\hat{N}}, D_{\hat{N}})$ be the leaf spaces, where $Y = W/Ch\,(N)$ and $\hat{Y} = \hat{W}/Ch\,(\hat{N})$. Let us fix points $v_o \in R$ and $\hat{v}_o \in \hat{R}$ and put $w_o = \eta(v_o)$, $y_o = \beta(w_o)$ and $\hat{w}_o = \hat{\eta}(\hat{v}_o)$, $\hat{y}_o = \hat{\beta}(\hat{w}_o)$. Then a local isomorphism $\psi : (R; D^1, D^2) \to (\hat{R}; \hat{D}^1, \hat{D}^2)$ such that $\psi(v_o) = \hat{v}_o$ induces a local isomorphism $\varphi : (W; C^*, N) \to (\hat{W}, \hat{C}^*, \hat{N})$ such that $\varphi(w_o) = \hat{w}_o$ and $\varphi_*(\kappa_1(v_o)) = \hat{\kappa}_1(\hat{v}_o)$, and vice versa. Furthermore a local isomorphism $\varphi : (W; C^*, N) \to (\hat{W}, \hat{C}^*, \hat{N})$ such that $\varphi(w_o) = \hat{w}_o$ induces a local isomorphism $\phi : (Y; D^*_N, D_N) \to (\hat{Y}; D^*_{\hat{N}}, D_{\hat{N}})$ such that $\phi(y_o) = \hat{y}_o$ and $\phi_*(\kappa_2(w_o)) = \hat{\kappa}_2(\hat{w}_o)$, and vice versa.

Here we remark that, when $\hat{E}$ coincides with E, i.e., when $s = r$, we have $N^* = N$ and $D^*_N = D_N$. Hence, in this case, the condition (ii) is automatically satisfied under the condition (i) and the equivalence of $(R; D^1, D^2)$ is reducible to that of (Y, D_N).

We will discuss conditions for the symbol algebra $\mathfrak{s}$, where the condition (i) or (ii) in the above Theorem is automatically satisfied, in the next section.

6. Typical Classes and their Generalizations

6.1. **Minimum subspace F satisfying $\mathfrak{f}^\perp \subset S^2(F)$.** Let $(R; D^1, D^2)$ be a PD manifold of second order, which is regular of type $\mathfrak{s}$. For the symbol algebra $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$, where

$$\mathfrak{s}_{-3} = \mathbb{R}, \quad \mathfrak{s}_{-2} = V^*, \quad \mathfrak{s}_{-1} = V \oplus \mathfrak{f}, \qquad \mathfrak{f} \subset S^2(V^*),$$

there exists a unique minimum subspace F of V such that $\mathfrak{f}^\perp \subset S^2(F) \subset S^2(V)$, under the identification: $S^2(V) \cong (S^2(V^*))^*$. This follows from

$$S^2(F_1) \cap S^2(F_2) = S^2(F_1 \cap F_2),$$

for F_1 and F_2 such that $\mathfrak{f}^\perp \subset S^2(F_i)$ $(i = 1, 2)$. F may coincide with V. Moreover, it follows from $S^2(a(F)) = a(S^2(F)) \supset a(\mathfrak{f}^\perp) = \mathfrak{f}^\perp$ for $a \in G_0(\mathfrak{s})$ that F is $G_0(\mathfrak{s})$-invariant. We put

$$\hat{F} = \{v \in F \mid v \circledcirc F \subset \mathfrak{f}^\perp\}.$$

Let r and s be the dimensions of F and $\hat{F}$ respectively. In the rest of this section, we assume that $\{0\} \subsetneqq F \subsetneqq V$.

Now we will discuss the Second Reduction Procedure for $(R; D^1, D^2)$, utilizing the minimum subspace F satisfying $\mathfrak{f}^\perp \subset S^2(F)$ $(\subset F \otimes_S V)$. Let us fix a point $v \in R$. Then, as in §5.2, there exists a coframe $\{\varpi, \varpi_1, \ldots, \varpi_n, \omega^1, \ldots, \omega^n, \varpi_{k\beta}\ (1 \leqq k \leqq r, r+1 \leqq \beta \leqq n), \varpi_{\alpha\beta}\ (r+1 \leqq$

$\alpha \leqq \beta \leqq n), \pi_\lambda\ (1 \leqq \lambda \leqq t_1)\}$ ($t_1 = \dim \mathfrak{f} - \dim F^\perp \otimes_S V^*$) and 1-forms $\{\pi_{ij}\ (s+1 \leqq i, j \leqq r)\}$ defined around $v \in R$ such that

$$D^1 = \{\varpi = 0\}, \quad D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\}, \quad \tilde{N} = \tilde{N}(F) = \{\varpi = \varpi_1 = \cdots = \varpi_r = 0\},$$

$$\tilde{N}^* = \tilde{N}(\hat{F}) = \{\varpi = \varpi_1 = \cdots = \varpi_s = 0\}, \quad \tilde{N}^\perp = \{\varpi = \varpi_1 = \cdots = \varpi_n = \omega^{r+1} = \cdots = \omega^n = 0\},$$

and the following structure equations hold;

$$(A^*) \qquad d\varpi \equiv \omega^1 \wedge \varpi_1 + \cdots + \omega^n \wedge \varpi_n \pmod{\varpi},$$

$$(B^*.1) \qquad \begin{cases} d\varpi_a \equiv \displaystyle\sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \varpi_{a\alpha}\ (1 \leqq a \leqq s), \\ d\varpi_i \equiv \displaystyle\sum_{j=s+1}^{r} \omega^j \wedge \pi_{ij} + \sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \varpi_{i\alpha}\ (s+1 \leqq i \leqq r), \\ d\varpi_\alpha \equiv \displaystyle\sum_{a=1}^{s} \omega^a \wedge \varpi_{a\alpha} + \sum_{i=s+1}^{r} \omega^i \wedge \varpi_{i\alpha} + \sum_{\beta=r+1}^{n} \omega^\beta \wedge \varpi_{\alpha\beta}\ (r+1 \leqq \alpha \leqq n), \end{cases}$$

(mod $\varpi, \varpi_1, \ldots, \varpi_n$), where $\varpi_{\alpha\beta} = \varpi_{\beta\alpha}$, $\pi_{ij} = \pi_{ji}$, and $\{\pi_\lambda (1 \leqq \lambda \leqq t_1)\}$ is a basis of $\langle\{\pi_{ij}\ (s+1 \leqq i, j \leqq r)\}\rangle$ (mod $\varpi, \varpi_1, \ldots, \varpi_n$). Moreover $\{\Omega_{s+1}, \ldots, \Omega_r\}$ are linearly independent (mod $\varpi, \varpi_1 \ldots, \varpi_n$), where $\Omega_i = \sum_{j=s+1}^{r} \omega^j \wedge \pi_{ij}$. Furthermore, we have

$$C(F) = \{\varpi = \varpi_1 = \cdots = \varpi_n = \omega^{s+1} = \cdots = \omega^n = \pi_\lambda = \varpi_{k\alpha} = 0 \\ (1 \leqq \lambda \leqq t_1, 1 \leqq k \leqq r, r+1 \leqq \alpha \leqq n)\},$$

and

$$\mathfrak{f}(F) = \{\varpi = \varpi_1 = \cdots = \varpi_n = \omega^1 = \cdots = \omega^n = \pi_\lambda = \varpi_{k\alpha} = 0 \\ (1 \leqq \lambda \leqq t_1, 1 \leqq k \leqq r, r+1 \leqq \alpha \leqq n)\}.$$

In the rest of this section, we will adopt the Einstein's convention for indices. The index ranges are as follows; $1 \leqq a, b \leqq s$, $s+1 \leqq i, j, k \leqq r$, $r+1 \leqq \alpha, \beta \leqq n$ and $1 \leqq \lambda \leqq t_1$. From $(B^*.1)$, we have

$$(B^*.2) \quad \begin{cases} d\varpi_a \equiv \omega^\alpha \wedge \varpi_{a\alpha} + A_a^{\alpha\beta}\varpi_\alpha \wedge \varpi_\beta + B_a^{\alpha\lambda}\varpi_\alpha \wedge \pi_\lambda + B_a^{\alpha b\beta}\varpi_\alpha \wedge \varpi_{b\beta} + \\ \quad B_a^{\alpha i\beta}\varpi_\alpha \wedge \varpi_{i\beta} + C_a^{\alpha\beta\gamma}\varpi_\alpha \wedge \varpi_{\beta\gamma} + D_{ab}^{\alpha}\varpi_\alpha \wedge \omega^b + E_{ai}^{\alpha}\varpi_\alpha \wedge \omega^i + F_{a\beta}^{\alpha}\varpi_\alpha \wedge \omega^\beta, \\ d\varpi_i \equiv \omega^j \wedge \pi_{ij} + \omega^\alpha \wedge \varpi_{i\alpha} + A_i^{\alpha\beta}\varpi_\alpha \wedge \varpi_\beta + B_i^{\alpha\lambda}\varpi_\alpha \wedge \pi_\lambda + B_i^{\alpha b\beta}\varpi_\alpha \wedge \varpi_{b\beta} + \\ \quad B_i^{\alpha j\beta}\varpi_\alpha \wedge \varpi_{j\beta} + C_i^{\alpha\beta\gamma}\varpi_\alpha \wedge \varpi_{\beta\gamma} + D_{ib}^{\alpha}\varpi_\alpha \wedge \omega^b + E_{ij}^{\alpha}\varpi_\alpha \wedge \omega^j + F_{i\beta}^{\alpha}\varpi_\alpha \wedge \omega^\beta, \end{cases}$$

(mod $\varpi, \varpi_1, \ldots, \varpi_r$). First, replacing $\varpi_{a\alpha}$ and $\varpi_{i\alpha}$ by $\varpi_{a\alpha} - F_{a\alpha}^{\beta}\varpi_\beta$ and $\varpi_{i\alpha} - F_{i\alpha}^{\beta}\varpi_\beta$ respectively, we may assume that $F_{a\alpha}^{\beta} = F_{i\alpha}^{\beta} = 0$. From (A^*), we have $d\varpi \equiv 0$ (mod $\mathfrak{M}_1^*$), where $\mathfrak{M}_1^* = \langle\{\varpi, \varpi_1, \ldots, \varpi_r, \omega^{r+1}, \ldots, \omega^n\}\rangle$. Hence, from $(B^*.2)$, we have

$$\begin{aligned} d\omega^\alpha \wedge \varpi_\alpha &\equiv \omega^b \wedge d\varpi_b + \omega^i \wedge d\varpi_i \\ &\equiv \omega^b \wedge (A_b^{\alpha\beta}\varpi_\alpha \wedge \varpi_\beta + B_b^{\alpha\lambda}\varpi_\alpha \wedge \pi_\lambda + B_b^{\alpha c\beta}\varpi_\alpha \wedge \varpi_{c\beta} + B_b^{\alpha i\beta}\varpi_\alpha \wedge \varpi_{i\beta} \\ &\qquad + C_b^{\alpha\beta\gamma}\varpi_\alpha \wedge \varpi_{\beta\gamma} + D_{bc}^{\alpha}\varpi_\alpha \wedge \omega^c + E_{bi}^{\alpha}\varpi_\alpha \wedge \omega^i) \\ &+ \omega^i \wedge (\omega^j \wedge \pi_{ij} + A_i^{\alpha\beta}\varpi_\alpha \wedge \varpi_\beta + B_i^{\alpha\lambda}\varpi_\alpha \wedge \pi_\lambda + B_i^{\alpha b\beta}\varpi_\alpha \wedge \varpi_{b\beta} + B_i^{\alpha j\beta}\varpi_\alpha \wedge \varpi_{j\beta} \\ &\qquad + C_i^{\alpha\beta\gamma}\varpi_\alpha \wedge \varpi_{\beta\gamma} + D_{ib}^{\alpha}\varpi_\alpha \wedge \omega^b + E_{ij}^{\alpha}\varpi_\alpha \wedge \omega^j) \end{aligned}$$

(mod $\mathfrak{M}_1^*$). Thus we get

$$
(6.1)\qquad \begin{aligned} d\omega^\alpha \equiv &- \omega^b \wedge (B_b^{\alpha\lambda}\pi_\lambda + B_b^{\alpha c\beta}\varpi_{c\beta} + B_b^{\alpha i\beta}\varpi_{i\beta} + C_b^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{bc}^\alpha\omega^c + E_{bi}^\alpha\omega^i) \\ &- \omega^i \wedge (B_i^{\alpha\lambda}\pi_\lambda + B_i^{\alpha b\beta}\varpi_{b\beta} + B_i^{\alpha j\beta}\varpi_{j\beta} + C_i^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ib}^\alpha\omega^b + E_{ij}^\alpha\omega^j) \end{aligned}
$$

(mod $\varpi, \varpi_1, \ldots, \varpi_n, \omega^{r+1}, \ldots, \omega^n$). Putting $\mathfrak{M}_2^* = \langle\{\varpi, \varpi_1, \ldots, \varpi_n, \omega^\alpha, \pi_\lambda (r+1 \leqq \alpha \leqq n, 1 \leqq \lambda \leqq t_1)\}\rangle$, we have

$$
d\varpi \equiv d\varpi_a \equiv d\varpi_i \equiv 0, \quad d\varpi_\alpha \equiv \omega^b \wedge \varpi_{b\alpha} + \omega^j \wedge \varpi_{j\alpha} \pmod{\mathfrak{M}_2^*}.
$$

Then, taking exterior derivatives of both sides of the first equation of $(B^*.2)$ and calculating mod $\mathfrak{M}_2^*$, we get

$$
d\omega^\alpha \wedge \varpi_{a\alpha} + d\varpi_\alpha \wedge (B_a^{\alpha c\beta}\varpi_{c\beta} + B_a^{\alpha i\beta}\varpi_{i\beta} + C_a^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ac}^\alpha\omega^c + E_{ai}^\alpha\omega^i) \equiv 0 \pmod{\mathfrak{M}_2^*}.
$$

Substituting (6.1) into the above equation, we get

$$
\begin{aligned}
&\omega^b \wedge \varpi_{a\alpha} \wedge (B_b^{\alpha c\beta}\varpi_{c\beta} + B_b^{\alpha i\beta}\varpi_{i\beta} + C_b^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{bc}^\alpha\omega^c + E_{bi}^\alpha\omega^i) \\
+&\omega^i \wedge \varpi_{a\alpha} \wedge (B_i^{\alpha c\beta}\varpi_{c\beta} + B_i^{\alpha j\beta}\varpi_{j\beta} + C_i^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ib}^\alpha\omega^b + E_{ij}^\alpha\omega^j) \\
+&\omega^b \wedge \varpi_{b\alpha} \wedge (B_a^{\alpha c\beta}\varpi_{c\beta} + B_a^{\alpha i\beta}\varpi_{i\beta} + C_a^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ac}^\alpha\omega^c + E_{ai}^\alpha\omega^i) \\
+&\omega^j \wedge \varpi_{j\alpha} \wedge (B_a^{\alpha c\beta}\varpi_{c\beta} + B_a^{\alpha i\beta}\varpi_{i\beta} + C_a^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ac}^\alpha\omega^c + E_{ai}^\alpha\omega^i) \equiv 0 \pmod{\mathfrak{M}_2^*}.
\end{aligned}
$$

Hence we obtain

$$
(6.a.1)\qquad \omega^i \wedge \varpi_{a\alpha} \wedge (B_i^{\alpha c\beta}\varpi_{c\beta} + C_i^{\alpha\beta\gamma}\varpi_{\beta\gamma} + E_{ij}^\alpha\omega^j) \equiv 0 \pmod{\mathfrak{M}_2^*}.
$$

$$
(6.a.2)\qquad \begin{aligned} &\omega^b \wedge \varpi_{a\alpha} \wedge (B_b^{\alpha c\beta}\varpi_{c\beta} + B_b^{\alpha i\beta}\varpi_{i\beta} + C_b^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{bc}^\alpha\omega^c + (E_{bi}^\alpha - D_{ib}^\alpha)\omega^i) \\ &+ \omega^b \wedge \varpi_{b\alpha} \wedge (B_a^{\alpha c\beta}\varpi_{c\beta} + B_a^{\alpha i\beta}\varpi_{i\beta} + C_a^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ac}^\alpha\omega^c + E_{ai}^\alpha\omega^i) \equiv 0 \\ &\hspace{20em}\pmod{\mathfrak{M}_2^*}. \end{aligned}
$$

$$
(6.a.3)\qquad B_i^{\alpha j\beta}\omega^i \wedge \varpi_{a\alpha} \wedge \varpi_{j\beta} + B_a^{\alpha c\beta}\omega^j \wedge \varpi_{j\alpha} \wedge \varpi_{c\beta} \equiv 0 \pmod{\mathfrak{M}_2^*}.
$$

$$
(6.a.4)\qquad \omega^j \wedge \varpi_{j\alpha} \wedge (B_a^{\alpha i\beta}\varpi_{i\beta} + C_a^{\alpha\beta\gamma}\varpi_{\beta\gamma} + D_{ac}^\alpha\omega^c + E_{ai}^\alpha\omega^i) \equiv 0 \pmod{\mathfrak{M}_2^*}.
$$

From, $(6.a.1)$, if $r - s > 0$, we get $B_i^{\alpha c\beta} = 0$ for $c\beta \neq a\alpha$, $C_i^{\alpha\beta\gamma} = 0$ and $E_{ij}^\alpha = E_{ji}^\alpha$. In particular, when $r - s > 0$, $B_i^{\alpha a\beta} = 0$ if $s \geqq 2$ and $B_i^{\alpha 1\beta} = B_i^{\beta 1\alpha}$ if $s = 1$. Moreover, by replacing π_{ij} by $\pi_{ij} + E_{ij}^\alpha\varpi_\alpha$, we may assume $E_{ij}^\alpha = 0$.

In case $s \geqq 2$, first let us choose any a $(1 \leqq a \leqq r)$. Since $s \geqq 2$, we can find b such that $a \neq b$. Then, from $(6.a.2)$, we see that the coefficients of $\omega^b \wedge \varpi_{b\alpha} \wedge \varpi_{i\beta}$, $\omega^b \wedge \varpi_{b\alpha} \wedge \varpi_{\beta\gamma}$, $\omega^b \wedge \varpi_{b\alpha} \wedge \omega^i$ and $\omega^b \wedge \varpi_{a\alpha} \wedge \omega^i$ are $B_a^{\alpha i\beta}$, $C_a^{\alpha\beta\gamma}$, E_{ai}^α and $(E_{bi}^\alpha - D_{ib}^\alpha)$, and those of $\omega^b \wedge \varpi_{b\alpha} \wedge \omega^c$ and $\omega^a \wedge \varpi_{a\alpha} \wedge \omega^b$ are D_{ac}^α and $2D_{ab}^\alpha - D_{ba}^\alpha$ respectively. Hence we get $B_a^{\alpha i\beta} = 0$, $C_a^{\alpha\beta\gamma} = 0$, $E_{ai}^\alpha = 0$, $D_{ia}^\alpha = 0$, $D_{ac}^\alpha = 0$ for $c \neq b$ and $2D_{ab}^\alpha = D_{ba}^\alpha$. Similarly, interchanging the role of a and b, from $(6.b.2)$, we get $D_{bc}^\alpha = 0$ for $c \neq a$ and $2D_{ba}^\alpha = D_{ab}^\alpha$. Thus we obtain $D_{ac}^\alpha = 0$ for any c. Moreover, from $(6.a.2)$, we see that the coefficients of $\omega^b \wedge \varpi_{b\alpha} \wedge \varpi_{c\beta}$ ($c\beta \neq a\alpha$ nor $b\alpha$), $\omega^b \wedge \varpi_{b\alpha} \wedge \varpi_{a\alpha}$ and $\omega^a \wedge \varpi_{a\alpha} \wedge \varpi_{b\alpha}$ are $B_a^{\alpha c\beta}$, $B_a^{\alpha a\alpha} - B_b^{\alpha b\alpha}$ and $2B_a^{\alpha b\alpha}$ respectively. Hence we get $B_a^{\alpha c\beta} = 0$ for $c\beta \neq a\alpha$ nor $b\alpha$, $B_a^{\alpha a\alpha} = B_b^{\alpha b\alpha}$ and $B_a^{\alpha b\alpha} = 0$. Moreover, from $(6.a.3)$, we get $B_i^{\alpha j\beta} = 0$ for $i \neq j$, $B_i^{\alpha i\beta} = 0$ for $\alpha \neq \beta$ and $B_i^{\alpha i\alpha} = B_a^{\alpha a\alpha} (= B^\alpha)$.

In case $s = 1$, $(6.a.2)$ reduces to

$$
\omega^1 \wedge \varpi_{1\alpha} \wedge (B_1^{\alpha 1\beta}\varpi_{1\beta} + B_1^{\alpha i\beta}\varpi_{i\beta} + C_1^{\alpha\beta\gamma}\varpi_{\beta\gamma} + (E_{1i}^\alpha - \frac{1}{2}D_{i1}^\alpha)\omega^i) \equiv 0 \pmod{\mathfrak{M}_2^*}.
$$

Hence we get $B_1^{\alpha 1\beta} = B_1^{\beta 1\alpha}$, $B_1^{\alpha i\beta} = 0$, $C_1^{\alpha\beta\gamma} = 0$ and $E_{1i}^{\alpha} = \frac{1}{2}D_{i1}^{\alpha}$. In case $r = 1$, by replacing ω^α by $\omega^\alpha + B_1^{\alpha 1\beta}\varpi_\beta$, we may assume $B_1^{\alpha 1\beta} = 0$. If $r \geqq 2$, from (6.1.3), we get $B_1^{\alpha 1\beta} = 0$ for $\alpha \neq \beta$, $B_i^{\alpha j\beta} = 0$ for $i \neq j$, $B_i^{\alpha i\beta} = 0$ for $\alpha \neq \beta$ and $B_i^{\alpha i\alpha} = B_1^{\alpha 1\alpha} (= B^\alpha)$. If we further assume $r \geqq 3$, we have $r - s \geqq 2$. Hence, from (6.1.4), we get $D_{11}^{\alpha} = 0$ and $E_{1i}^{\alpha} = 0$, which also implies $D_{i1}^{\alpha} = 0$.

Thus, in case $s \geqq 2$ or $s = 1$ and $r \geqq 3$, we see that, by replacing ω^α by $\omega^\alpha + B^\alpha\varpi_\alpha$, $(B^*.2)$ reduces to

$$(\hat{B}^*.2) \qquad \begin{cases} d\varpi_a \equiv & \omega^\alpha \wedge \varpi_{a\alpha} + A_a^{\alpha\beta}\varpi_\alpha \wedge \varpi_\beta + B_a^{\alpha\lambda}\varpi_\alpha \wedge \pi_\lambda \\ d\varpi_i \equiv \omega^j \wedge \pi_{ij} + & \omega^\alpha \wedge \varpi_{i\alpha} + A_i^{\alpha\beta}\varpi_\alpha \wedge \varpi_\beta + B_i^{\alpha\lambda}\varpi_\alpha \wedge \pi_\lambda + B_i^{\alpha 1\beta}\varpi_\alpha \wedge \varpi_{1\beta} \end{cases}$$

(mod $\varpi, \varpi_1, \ldots, \varpi_r$). Here we may assume $A_a^{\alpha\beta} = -A_a^{\beta\alpha}$ and $B_i^{\alpha 1\beta} = 0$ when $s \geqq 2$. We note that, in case $s = r = 1$, we have

$$d\varpi_1 \equiv \omega^\alpha \wedge \varpi_{1\alpha} + A_1^{\alpha\beta}\varpi_\alpha \wedge \varpi_\beta + D_{11}^{\alpha}\varpi_\alpha \wedge \omega^1 \pmod{\varpi, \varpi_1}.$$

Moreover, putting $\mathfrak{M}_3^* = \langle\{\varpi, \varpi_1, \ldots, \varpi_n, \omega^i, \omega^\alpha (s+1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\}\rangle$, we have

$$d\varpi \equiv d\varpi_a \equiv d\varpi_i \equiv 0, \quad d\varpi_\alpha \equiv \omega^b \wedge \varpi_{b\alpha} \pmod{\mathfrak{M}_3^*}.$$

Then, taking exterior derivatives of both sides of the first equation of $(\hat{B}^*.2)$ and calculating mod $\mathfrak{M}_3^*$, we get

$$d\omega^\alpha \wedge \varpi_{a\alpha} + \omega^b \wedge \varpi_{b\alpha} \wedge (B_a^{\alpha\lambda}\pi_\lambda) \equiv 0 \pmod{\mathfrak{M}_3^*}.$$

Hence, in case $s \geqq 2$, looking at the coefficient of $\omega^b \wedge \varpi_{b\alpha} \wedge \pi_\lambda$ for $b \neq a$, we get $B_a^{\alpha\lambda} = 0$. In case $s = 1$, the above equation reduces to

$$(d\omega^\alpha - B_1^{\alpha\lambda}\omega^1 \wedge \pi_\lambda) \wedge \varpi_{1\alpha} \equiv 0 \pmod{\mathfrak{M}_3^*}.$$

Then, substituting (6.1) into the above equation, we obtain $B_1^{\alpha\lambda} = 0$. Thus, if $s \geqq 2$, $s = 1$ and $r \geqq 3$ or $s = r = 1$ and $C(F)$ is completely integrable, the first equation of $(\hat{B}^*.2)$ reduces to

$$(6.2) \qquad d\varpi_a \equiv \omega^\alpha \wedge \varpi_{a\alpha} + A_a^{\alpha\beta}\varpi_\alpha \wedge \varpi_\beta \pmod{\varpi, \varpi_1, \ldots, \varpi_r}.$$

Now we will show that (6.2) further reduces to

$$d\varpi_a \equiv \omega^\alpha \wedge \varpi_{a\alpha} \pmod{\varpi, \varpi_1, \ldots, \varpi_r},$$

dividing the proof in the following three cases: (1) $s \geqq 2$, (2) $s = r = 1$ and $C(F)$ is completely integrable, (3) $s = 1$ and $r \geqq 3$.

In case (1), putting $\mathfrak{M}_4^* = \langle\{\varpi, \varpi_1, \ldots, \varpi_r, \omega^\alpha, \pi_\lambda, \varpi_\alpha \wedge \varpi_\beta\ (1 \leqq \lambda \leqq t_1, r+1 \leqq \alpha \leqq \beta \leqq n)\}\rangle$, we have

$$d\varpi \equiv d\varpi_a \equiv d\varpi_i \equiv 0, \quad d\varpi_\alpha \equiv \omega^b \wedge \varpi_{b\alpha} + \varpi_\beta \wedge \eta_{\alpha\beta} \pmod{\mathfrak{M}_4^*},$$

for some 1-forms $\eta_{\alpha\beta}$. Then, taking exterior derivatives of both sides of (6.2) and calculating mod $\mathfrak{M}_4^*$, we get

$$d\omega^\alpha \wedge \varpi_{a\alpha} + A_a^{\alpha\beta}(d\varpi_\alpha \wedge \varpi_\beta - \varpi_\alpha \wedge d\varpi_\beta) \equiv 0 \pmod{\mathfrak{M}_4^*}.$$

Hence we obtain

$$d\omega^\alpha \wedge \varpi_{a\alpha} + 2A_a^{\alpha\beta}\omega^b \wedge \varpi_{b\alpha} \wedge \varpi_\beta \equiv 0 \pmod{\mathfrak{M}_4^*}.$$

Thus, since $s \geqq 2$, first choose any a $(1 \leqq a \leqq r)$, and find b such that $a \neq b$. Then, looking at the coefficient of $\omega^b \wedge \varpi_{b\alpha} \wedge \varpi_\beta$, we obtain $A_a^{\alpha\beta} = 0$.

In case (2), putting $\mathfrak{M}_5^* = \langle\{\varpi, \varpi_1, \omega^\alpha, \varpi_\alpha \wedge \varpi_\beta\ (2 \leqq \alpha \leqq \beta \leqq n)\}\rangle$, we have

$$d\varpi \equiv d\varpi_1 \equiv 0, \quad d\varpi_\alpha \equiv \omega^1 \wedge \varpi_{1\alpha} + \varpi_\beta \wedge \eta_{\alpha\beta} \pmod{\mathfrak{M}_5^*},$$

for some 1-forms $\eta_{\alpha\beta}$. Then, taking exterior derivatives of both sides of (6.2) and calculating mod $\mathfrak{M}_5^*$, we get

$$(d\omega^\alpha - 2A_1^{\alpha\beta}\omega^1 \wedge \varpi_\beta) \wedge \varpi_{1\alpha} \equiv 0 \pmod{\mathfrak{M}_5^*}.$$

Thus we obtain

$$d\omega^\alpha \equiv 2A_1^{\alpha\beta}\omega^1 \wedge \varpi_\beta \pmod{\varpi, \varpi_1, \omega^\alpha, \varpi_{1\alpha}, \varpi_\alpha \wedge \varpi_\beta \ (2 \leqq \alpha \leqq \beta \leqq n)}. \tag{6.3}$$

From (A^*), we have $d\varpi \equiv 0 \pmod{\varpi, \varpi_1, \omega^2, \ldots, \omega^n}$. Hence we have

$$d\omega^\alpha \wedge \varpi_\alpha \equiv \omega^1 \wedge d\varpi_1 \equiv \omega^1 \wedge (A_1^{\alpha\beta}\varpi_\alpha \wedge \varpi_\beta) \pmod{\varpi, \varpi_1, \omega^2, \ldots, \omega^n}.$$

Then, substituting (6.3) into the above equation, we obtain $A_1^{\alpha\beta}\omega^1 \wedge \varpi_\alpha \wedge \varpi_\beta \equiv 0 \pmod{\varpi, \varpi_1, \omega^\alpha, \varpi_{1\alpha}(2 \leqq \alpha \leqq n)}$, which implies $A_1^{\alpha\beta} = 0$.

In case (3), putting $\mathfrak{M}_6^* = \langle\{\varpi, \varpi_1, \ldots, \varpi_r, \omega^\alpha, \varpi_{1\alpha}, \pi_\lambda, \varpi_\alpha \wedge \varpi_\beta \ (r+1 \leqq \alpha \leqq \beta \leqq n, 1 \leqq \lambda \leqq t_1)\}\rangle$, we have

$$d\varpi \equiv d\varpi_1 \equiv d\varpi_i \equiv 0, \quad d\varpi_\alpha \equiv \omega^i \wedge \varpi_{\alpha i} + \varpi_\beta \wedge \eta_{\alpha\beta},$$

for some 1-forms $\eta_{\alpha\beta}$. Then, taking exterior derivatives of both sides of (6.2) and calculating mod $\mathfrak{M}_6^*$, we get

$$A_1^{\alpha\beta}(d\varpi_\alpha \wedge \varpi_\beta - \varpi_\alpha \wedge d\varpi_\beta) \equiv 2A_1^{\alpha\beta}\omega^i \wedge \varpi_{\alpha i} \wedge \varpi_\beta \equiv 0 \pmod{\mathfrak{M}_6^*}.$$

This implies $A_1^{\alpha\beta} = 0$.

Summarizing the discussion above, we obtain

Proposition 6.1. *Let $(R; D^1, D^2)$ be a PD manifold of second order, which is regular of type $\mathfrak{s}$. Let F be the minimum subspace of V satisfying $\mathfrak{f}^\perp \subset S^2(F) \subset F \otimes_S V$.*

Assume that $\{0\} \subsetneqq \hat{F} \subset F \subsetneqq V$. Let s and r be the dimensions of $\hat{F}$ and F respectively. Then, if $s \geqq 2$, $s = 1$ and $r \geqq 3$ or $s = r = 1$ and $C(F)$ is completely integrable, for the covariant system $\tilde{N}^* = \{\varpi = \varpi_1 = \cdots = \varpi_s = 0\}$, the following structure equation holds

$$\begin{cases} d\varpi \equiv \omega^1 \wedge \varpi_1 + \cdots + \omega^n \wedge \varpi_n \pmod{\varpi} \\ d\varpi_1 \equiv \omega^{r+1} \wedge \varpi_{1r+1} + \cdots + \omega^n \wedge \varpi_{1n} \\ \qquad \ldots \qquad\qquad\qquad\qquad \pmod{\varpi, \varpi_1, \ldots, \varpi_r} \\ d\varpi_s \equiv \omega^{r+1} \wedge \varpi_{sr+1} + \cdots + \omega^n \wedge \varpi_{sn} \end{cases}$$

where $\tilde{N} = \{\varpi = \varpi_1 = \cdots = \varpi_r = 0\}$

We note here that, if $s = 1$ and $r \leqq 2$, we have $\mathfrak{f}^\perp = \langle\{e_1 \circledcirc e_1\}\rangle$ or $\langle\{e_1 \circledcirc e_1, e_1 \circledcirc e_2\}\rangle$. Furthermore we note that in case $s < r$, by Lemma 5.6, if $C(F) = \mathrm{Ch}\,(\tilde{N})$, then $C(F) \subset \mathrm{Ch}\,(\tilde{N}^*)$. Hence, by Proposition 6.1 and Lemma 5.3, we obtain

Proposition 6.2. *Let $(R; D^1, D^2)$ be a PD manifold of second order, which is regular of type $\mathfrak{s}$. Let F be the minimum subspace of V satisfying $\mathfrak{f}^\perp \subset S^2(F) \subset F \otimes_S V$.*

Assume that $\{0\} \subsetneqq \hat{F} \subset F \subsetneqq V$. Then, except for the cases $\mathfrak{f}^\perp = \langle\{e_1 \circledcirc e_1\}\rangle$ or $\langle\{e_1 \circledcirc e_1, e_1 \circledcirc e_2\}\rangle$, $C(F)$ is completely integrable and $C(F) = Ch\,(\tilde{N})$. Moreover $C(F) \subset Ch\,(\tilde{N}^*)$ when $s < r$.

Remark 6.1. By Proposition 6.1, for the minimum subspace F of V such that $\hat{F} \neq \{0\}$, except for the cases when $\mathfrak{f}^\perp = \langle\{e_1 \circledcirc e_1\}\rangle$ or $\langle\{e_1 \circledcirc e_1, e_1 \circledcirc e_2\}\rangle$, the assumptions (i) and (ii) for the Two Step Reduction Theorem (Theorem 5.2) are automatically satisfied. For the case when $\mathfrak{f}^\perp = \langle\{e_1 \circledcirc e_1\}\rangle$, see Theorem 6.1 in the next subsection.

6.2. **Typical Class of Type $\mathfrak{f}^2(r)$ and its Generalization.** Let $(R; D^1, D^2)$ be a PD manifold of second order satisfying the condition (C), which is regular of type $\mathfrak{f}^2(r)$. Namely $(R; D^1, D^2)$ is a PD manifold of second order such that symbol algebra $\mathfrak{s}(v)$ at each point $v \in R$ is isomorphic to $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$, where

$$\mathfrak{s}_{-3} = \mathbb{R}, \quad \mathfrak{s}_{-2} = V^* \quad \text{and} \quad \mathfrak{s}_{-1} = V \oplus \mathfrak{f}^2(r).$$

Here $\mathfrak{f}^2(r)$ is given by $(\mathfrak{f}^2(r))^\perp = S^2(F) \subset S^2(V)$, for a subspace F of V of dimension r.

Then, by Proposition 6.1, as the case $s = r \geqq 2$ and the case $s = r = 1$ and $C(F)$ is completely integrable, we obtain the structure equations for $\tilde{N} = \{\varpi = \varpi_1 = \cdots = \varpi_r = 0\}$ as follows ;

$$\begin{cases} d\varpi \equiv \ \omega^\alpha \wedge \varpi_\alpha \\ d\varpi_a \equiv \ \omega^\alpha \wedge \varpi_{a\alpha} \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_r). \end{cases}$$

Hence, by Theorem 5.2, the equivalence of $(R; D^1, D^2)$, which is regular of type $\mathfrak{f}^2(r)$, is reducible to that of (Y, D_N) such that (Y, D_N) is a regular differential system of type $\mathfrak{c}^1(n-r, r+1)$, where

$$\mathfrak{c}^1(n-r, r+1) = \mathfrak{c}_{-2} \oplus \mathfrak{c}_{-1} \quad \mathfrak{c}_{-2} = W, \quad \mathfrak{c}_{-1} = \hat{V} \oplus W \otimes \hat{V}^*$$

is the symbol algebra of the canonical system on the first order jet space of $n-r$ independent and $r+1$ dependent variables (see §2.5 [27]). Here W and $\hat{V}$ are vector spaces of dimension $r+1$ and $n-r$ respectively.

Summarizing the discussion above, by the Second Reduction Theorem (Theorem 5.2), we obtain (Proposition 5.1 and Theorem 5.3 in [25] and §3 in [23])

Theorem 6.1. *Let $(R; D^1, D^2)$ be a PD manifold of second order, which is regular of type $\mathfrak{f}^2(r)$.*

(1) *If $r = 1$, assume that $C(F)$ is completely integrable, then the equivalence of $(R; D^1, D^2)$ is reducible to the equivalence of a regular differential system (Y, D_N) of type $\mathfrak{c}^1(n-1, 2)$.*

(2) *If $r \geqq 2$, the equivalence of $(R; D^1, D^2)$ is reducible to the equivalence of a regular differential system (Y, D_N) of type $\mathfrak{c}^1(n-r, r+1)$, which is locally a space of 1-jets for $n-r$ independent and $r+1$ dependent variables.*

Remark 6.2. (1) In case $r = 1$, $(R; D^1, D^2)$, which is of type $\mathfrak{f}^2(1)$, is called of (weak) parabolic type and $C(F)$ coincides with the Monge characteristic system. Hence under the assumption that $C(F)$ is completely integrable, $(R; D^1, D^2)$ is called a equation of Goursat type in [25]. Utilizing the above reduction theorem, we discussed the contact equivalences of classes of Goursat type equations (G_2-geometry of second order), which are related to Parabolic Geometries (geometry of (Y, D_N)) of each exceptional simple Lie Groups (see §6 in [25]).

(2) In case $r \geqq 2$, since a regular differential system (Y, D_N) of type $\mathfrak{c}^1(n-r, r+1)$ $(r+1 \geqq 3)$ is isomorphic to $(J(M, n-r), C)$, where $\dim M = n+1$ (cf. Theorem 1.6 [22]), $(R; D^1, D^2)$ can be transformed, by a contact transformation, to the linear (model) system $R = \{p_{ab} = 0\ (1 \leqq a, b \leqq r)\}$ (see §3 in [23] for the detail).

Now, as the generalization of the Typical Class of Type $\mathfrak{f}^2(r)$, we will consider a PD manifold $(R; D^1, D^2)$ of second order, which is regular of type $\mathfrak{f}^2(r, s)$. Here $\mathfrak{f}^2(r, s)$ is given by $(\mathfrak{f}^2(r, s))^\perp = \hat{F} \otimes_S F \subset S^2(F)$, where $\hat{F} \subset F$ are subspaces of V of dimension s and r respectively. Namely let us fix a point $v \in R$. Then, as in §6.1, there exists a coframe $\{\varpi, \varpi_1, \ldots, \varpi_n, \omega^1, \ldots, \omega^n, \varpi_{a\beta}\ (1 \leqq a \leqq s, r+1 \leqq \beta \leqq n), \varpi_{kl}\ (s+1 \leqq k \leqq l \leqq n)\}$ defined around $v \in R$ such that

$$D^1 = \{\varpi = 0\}, \quad D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\}, \quad \tilde{N} = \tilde{N}(F) = \{\varpi = \varpi_1 = \cdots = \varpi_r = 0\},$$

$$\tilde{N}^* = \tilde{N}(\hat{F}) = \{\varpi = \varpi_1 = \cdots = \varpi_s = 0\}, \ \tilde{N}^\perp = \{\varpi = \varpi_1 = \cdots = \varpi_n = \omega^{r+1} = \cdots = \omega^n = 0\},$$

and the following structure equations hold;

$$(\tilde{A}) \qquad d\varpi \equiv \omega^1 \wedge \varpi_1 + \cdots + \omega^n \wedge \varpi_n \pmod{\varpi},$$

$$(\tilde{B}.1) \qquad \begin{cases} d\varpi_a \equiv \displaystyle\sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \varpi_{a\alpha} \ (1 \leqq a \leqq s), \\ d\varpi_i \equiv \displaystyle\sum_{j=s+1}^{r} \omega^j \wedge \varpi_{ij} + \sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \varpi_{i\alpha} \ (s+1 \leqq i \leqq r), \\ d\varpi_\alpha \equiv \displaystyle\sum_{a=1}^{s} \omega^a \wedge \varpi_{a\alpha} + \sum_{i=s+1}^{r} \omega^i \wedge \varpi_{i\alpha} + \sum_{\beta=r+1}^{n} \omega^\beta \wedge \varpi_{\alpha\beta} \ (r+1 \leqq \alpha \leqq n), \end{cases}$$

(mod $\varpi, \varpi_1, \ldots, \varpi_n$), where $\varpi_{kl} = \varpi_{lk}$. Furthermore, we have

$$C(F) = \{\varpi = \varpi_1 = \cdots = \varpi_n = \omega^{s+1} = \cdots = \omega^n = \varpi_{ij} = \varpi_{k\alpha} = 0$$
$$(s+1 \leqq i \leqq j \leqq r, 1 \leqq k \leqq r, r+1 \leqq \alpha \leqq n)\},$$

By the calculation in §6.1, in case $s \geqq 2$, we have

$$(\tilde{B}.2) \qquad \begin{cases} d\varpi_a \equiv \omega^\alpha \wedge \varpi_{a\alpha} \\ d\varpi_i \equiv \omega^j \wedge \varpi_{ij} + \omega^\alpha \wedge \varpi_{i\alpha} + A_i^{\alpha\beta} \varpi_\alpha \wedge \varpi_\beta + B_i^{\alpha jk} \varpi_\alpha \wedge \varpi_{jk} \end{cases}$$

(mod $\varpi, \varpi_1, \ldots, \varpi_r$), where we may assume $A_i^{\alpha\beta} = -A_i^{\beta\alpha}$. Putting $\tilde{\mathfrak{M}}_1 = \langle\{\varpi, \varpi_1, \ldots, \varpi_n, \omega^i, \omega^\alpha \ (s+1 \leqq i \leqq r, r+1 \leqq \alpha \leqq \beta \leqq n)\}\rangle$ and by (6.1), we have

$$d\varpi \equiv d\varpi_a \equiv d\varpi_i \equiv d\omega^\alpha \equiv 0, \quad d\varpi_\alpha \equiv \omega^b \wedge \varpi_{b\alpha} \pmod{\tilde{\mathfrak{M}}_1}.$$

Then, taking exterior derivatives of both sides of the second equation of ($\tilde{B}$.2) and calculating mod $\tilde{\mathfrak{M}}_1$, we get

$$d\omega^j \wedge \varpi_{ij} + \omega^b \wedge \varpi_{b\alpha} \wedge (B_i^{\alpha jk} \varpi_{jk}) \equiv 0 \pmod{\tilde{\mathfrak{M}}_1}.$$

Hence we have $B_i^{\alpha jk} = 0$ if $i \notin \{j, k\}$ and $(d\omega^j + B_i^{\alpha ij} \omega^b \wedge \varpi_{b\alpha}) \wedge \varpi_{ij} \equiv 0 \pmod{\tilde{\mathfrak{M}}_1}$. Thus we obtain

$$d\omega^j \equiv -B_i^{\alpha ij} \omega^b \wedge \varpi_{b\alpha} \pmod{\tilde{\mathfrak{M}}_1} \quad \text{for} \quad i = s+1, \ldots, r.$$

Then, putting $B^{\alpha j} = B_i^{\alpha ij}$ $(i = s+1, \ldots, r)$, and replacing ω^j and ω^α by $\omega^j + B^{\alpha j} \varpi_\alpha$ and $\omega^\alpha + B^{\alpha j} \varpi_j$ respectively, we may assume $B_i^{\alpha jk} = 0$.

Moreover, putting $\tilde{\mathfrak{M}}_2 = \langle\{\varpi, \varpi_1, \ldots, \varpi_r, \omega^i, \omega^\alpha, \varpi_{ij}, \varpi_{i\alpha}, \varpi_\alpha \wedge \varpi_\beta \ (s+1 \leqq i \leqq j \leqq r, r+1 \leqq \alpha \leqq \beta \leqq n)\}\rangle$, we have

$$d\varpi \equiv d\varpi_a \equiv d\varpi_i \equiv 0, \quad d\varpi_\alpha \equiv \omega^b \wedge \varpi_{b\alpha} + \varpi_\beta \wedge \eta_{\alpha\beta} \pmod{\tilde{\mathfrak{M}}_2},$$

for some 1-forms $\eta_{\alpha\beta}$. Then, taking exterior derivatives of both sides of the second equation of ($\tilde{B}$.2) and calculating mod $\tilde{\mathfrak{M}}_2$, we get

$$A_i^{\alpha\beta}(d\varpi_\alpha \wedge \varpi_\beta - \varpi_\alpha \wedge d\varpi_\beta) \equiv 2A_i^{\alpha\beta} \omega^b \wedge \varpi_{b\alpha} \wedge \varpi_\beta \equiv 0 \pmod{\tilde{\mathfrak{M}}_2},$$

which implies $A_i^{\alpha\beta} = 0$.

Thus, by the Second Reduction Theorem (Theorem 5.2), we obtain

Theorem 6.2. *Let $(R; D^1, D^2)$ be a PD manifold of second order, which is regular of type $\mathfrak{f}^2(r,s)$.*

Then, if $2 \leqq s < r$, the equivalence of $(R; D^1, D^2)$ is reducible, by Theorem 5.2, to the equivalence of a regular differential system (Y, D_N) , where $D_N = \{\varpi = \varpi_1 = \cdots = \varpi_r = 0\}$ such that the following structure equation holds;

$$\begin{cases} d\varpi \equiv & \displaystyle\sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \varpi_\alpha, \\ d\varpi_a \equiv & \displaystyle\sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \varpi_{a\alpha} \ (1 \leqq a \leqq s), \qquad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_r) \\ d\varpi_i \equiv \displaystyle\sum_{j=s+1}^{r} \omega^j \wedge \varpi_{ij} + & \displaystyle\sum_{\alpha=r+1}^{n} \omega^\alpha \wedge \varpi_{i\alpha} \ (s+1 \leqq i \leqq r), \end{cases}$$

6.3. **Typical Class of Type $\mathfrak{f}^1(r)$ and its Generalization.** In this subsection, as the generalization of the Typical Class of Type $\mathfrak{f}^1(r)$, we will consider a PD manifold $(R; D^1, D^2)$ of second order, which is regular of type $\mathfrak{s}$ such that $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$ satisfies the following condition: For $\mathfrak{s}_{-3} = \mathbb{R}$, $\mathfrak{s}_{-2} = V^*$ and $\mathfrak{s}_{-1} = V \oplus \mathfrak{f}$, $\mathfrak{f} \subset S^2(V^*)$,

$(F.1)$ There exist subspaces E and H of V of dimension r and $n-r$ respectively such that

$$V = E \oplus H, \quad E \otimes H \subset \mathfrak{f}^\perp \subset E \otimes_S V.$$

Here $\mathfrak{f}^1(r)$ is given by $(\mathfrak{f}^1(r))^\perp = E \otimes H$. Namely let us fix a point $v \in R$. Then, as in §6.1, there exists a coframe $\{\varpi, \varpi_1, \ldots, \varpi_n, \omega^1, \ldots, \omega^n, \pi_\lambda, \varpi_{\alpha\beta}\ (1 \leqq \lambda \leqq t, r+1 \leqq \alpha \leqq \beta \leqq n)\}$ $(t = \dim \mathfrak{f} - \dim S^2(E^\perp))$ and 1-forms $\{\pi_{ij}\ (1 \leqq i,j \leqq r)\}$ defined around $v \in R$ such that

$$D^1 = \{\varpi = 0\}, \quad D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\},$$

$\tilde{N} = \tilde{N}(E) = \{\varpi = \varpi_1 = \cdots = \varpi_r = 0\}$, $\tilde{N}^\perp = \{\varpi = \varpi_1 = \cdots = \varpi_n = \omega^{r+1} = \cdots = \omega^n = 0\}$, and the following structure equations hold;

$$(\hat{A}) \qquad d\varpi \equiv \omega^1 \wedge \varpi_1 + \cdots + \omega^n \wedge \varpi_n \quad (\mathrm{mod}\ \varpi),$$

$$(\hat{B}.1) \qquad \begin{cases} d\varpi_i \equiv \displaystyle\sum_{j=1}^{r} \omega^j \wedge \pi_{ij} & (1 \leqq i \leqq r), \\ d\varpi_\alpha \equiv \displaystyle\sum_{\beta=r+1}^{n} \omega^\beta \wedge \varpi_{\alpha\beta} & (r+1 \leqq \alpha \leqq n), \end{cases}$$

$(\ \mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_n)$, where $\varpi_{\alpha\beta} = \varpi_{\beta\alpha}$, $\pi_{ij} = \pi_{ji}$, and $\{\pi_\lambda (1 \leqq \lambda \leqq t)\}$ is a basis of $\langle\{\pi_{ij}\ (1 \leqq i,j \leqq r)\}\rangle$ $(\mathrm{mod}\ \varpi, \varpi_1, \ldots, \varpi_n)$. Furthermore, we have

$$\mathfrak{f}(E) = \{\varpi = \varpi_1 = \cdots = \varpi_n = \omega^1 = \cdots = \omega^n = \pi_{ij} = 0\ (1 \leqq i \leqq j \leqq r)\}.$$

From $(\hat{B}.1)$, we have

$$(\hat{B}.2) \qquad \begin{aligned} d\varpi_i \equiv \omega^j \wedge \pi_{ij} &+ A_i^{\alpha\beta} \varpi_\alpha \wedge \varpi_\beta + B_i^{\alpha\lambda} \varpi_\alpha \wedge \pi_\lambda + C_i^{\alpha\beta\gamma} \varpi_\alpha \wedge \varpi_{\beta\gamma} \\ &+ D_{ij}^{\alpha} \varpi_\alpha \wedge \omega^j + E_{i\beta}^{\alpha} \varpi_\alpha \wedge \omega^\beta \quad (\mathrm{mod}\ \varpi, \varpi_1, \ldots, \omega_r). \end{aligned}$$

First, replacing ϖ_i by $\varpi - E_{ir+1}^{r+1} \varpi$, we may assume $E_{ir+1}^{r+1} = 0$. Putting $\hat{\mathfrak{M}}_1 = \langle\{\varpi, \varpi_1, \ldots, \varpi_n, \omega^i, \pi_\lambda\ (1 \leqq i \leqq r, 1 \leqq \lambda \leqq t)\}\rangle$, we have

$$d\varpi \equiv d\varpi_i \equiv 0, \quad d\varpi_\alpha \equiv \omega^\beta \wedge \varpi_{\alpha\beta} \quad (\mathrm{mod} \quad \hat{\mathfrak{M}}_1).$$

Then, taking exterior derivatives of both sides of $(\hat{B}.2)$ and calculating mod $\hat{\mathfrak{M}}_1$, we get

$$\omega^\beta \wedge \varpi_{\alpha\beta} \wedge (C_i^{\alpha\beta\gamma}\varpi_{\beta\gamma} + E^\alpha_{i\beta}\omega^\beta) \equiv 0 \pmod{\hat{\mathfrak{M}}_1}.$$

Under the assumption $r \leqq n-2$, this implies $C_i^{\alpha\beta\gamma} = E^\alpha_{i\beta} = 0$ (see the proof of Lemma 2.1 in [23] for the detail). Hence, by Lemma 5.3. (1), we obtain $\mathfrak{f}(E) = \mathrm{Ch}\,(D^1) \cap \mathrm{Ch}\,(\tilde{N})$. Let $(W; C^*, N)$ be the IG manifold of corank r associated with $(R; D^1, D^2, \tilde{N}(E))$ and $(R_W; D^1_W, D^2_W)$ be the Lagrange Grassmann bundle over $(W; C^*, N)$. Then, by Theorem 5.1. (1), $\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ is a local isomorphism. Thus the local equivalence of $(R; D^1, D^2)$ is reducible to the local equivalence of $(W; C^*, N)$. By the condition $(F.1)$ of the symbol algebra, we have

$$S^2(E^\perp) \subset \mathfrak{f} \subset S^2(H^\perp) \oplus S^2(E^\perp).$$

Hence we get $\mathfrak{f} \cap (E^\perp \otimes_S V^*) = S^2(E^\perp) = \mathfrak{f}(E) = \mathrm{Ker}\,\eta_*$, which implies, by Lemma 3.1 (4), $S(N) = \{0\}$ on W.

Summarizing the discussion above, we obtain

Proposition 6.3. *Let $(R; D^1, D^2)$ be a PD manifold of second order, which is regular of type $\mathfrak{s}$ such that the symbol subspace $\mathfrak{f} \subset S^2(V^*)$ satisfies the condition $(F.1)$. Let $(W; C^*, N)$ be the IG manifold of corank r associated with $(R; D^1, D^2, \tilde{N}(E))$ and $(R_W; D^1_W, D^2_W)$ be the Lagrange Grassmann bundle over $(W; C^*, N)$. Then $\kappa_1 : (R; D^1, D^2) \to (R(W); D^1_W, D^2_W)$ is a local isomorphism and $S(N) = \{0\}$ on W.*

Conversely we will now consider an IG manifold $(W; C^*, N)$ of corank r satisfying $S(N) = \{0\}$. We assume that W is regular with respect to $\mathrm{Ch}\,(C^*)$ as in §4.1. Let $(R(W); D^1_W, D^2_W)$ be the Lagrange Grassmann bundle over $(W; C^*, N)$. Then, by Lemma 4.1 and 4.2, $(R(W); D^1_W, D^2_W)$ is, globally, a PD manifold of second order and $\zeta : R(W) \to L(J)$ is an immersion.

Moreover, by Lemma 3.1 (4) (see also the last paragraph of §4.2), the condition $S(N) = \{0\}$ implies that

$$S(N_W)(\hat{v}) = \mathfrak{f}(\hat{v}) \cap (E(\hat{v})^\perp \otimes_S V(\hat{v})^*) = \mathrm{Ker}\,\tau_* = S^2(E(\hat{v})^\perp),$$

at each $\hat{v} \in R(W)$, where $E(\hat{v}) = V(\hat{v}) \cap N^\perp_W(\hat{v})$. Equivalently we have

$$\mathfrak{f}(\hat{v})^\perp + S^2(E(\hat{v})) = E(\hat{v}) \otimes_S V(\hat{v}) \qquad \text{at each} \quad \hat{v} \in R(W).$$

For a complimentary subspace $H(\hat{v})$, $V(\hat{v}) = E(\hat{v}) \oplus H(\hat{v})$, we have $E(\hat{v}) \otimes_S V(\hat{v}) = S^2(E(\hat{v}) \oplus E(\hat{v}) \otimes H(\hat{v})$. Thus the condition $(F.1)$ in this case is the existence of a complimentary subspace $H(\hat{v})$ such that $\mathfrak{f}(\hat{v})^\perp \supset E(\hat{v}) \otimes H(\hat{v})$.

7. Construction of $(W(Y); C^*_Y, N_Y)$ and $(R(Y); D^1_Y, D^2_Y)$

7.1. **Case $N^* = N$.** Starting from a regular differential system (Y, D_N), we will construct an IG manifold $(W(Y); C^*_Y, N_Y)$ and the Lagrange Grassmann bundle $R(Y) = R(W(Y))$ over $(W(Y); C^*_Y, N_Y)$ and will examine the condition when $(R(Y); D^1_Y, D^2_Y)$ becomes a PD manifold of second order, where D^1_Y and D^2_Y are canonical systems on $R(Y)$.

Let (Y, D_N) be a differential system satisfying the following condition;

$(Y.1)$ D_N is a differential system of codimension $r+1$ such that $\mathrm{Ch}\,(D_N)$ is trivial.

We assume that D_N is of constant Engel half-rank (see [4] II §4) and let t be the Engel half-rank of D_N. Let $\{\varpi_0, \ldots, \varpi_r\}$ be a local defining 1-forms of (Y, D_N) on a neighborhood U of $y_o \in Y$. Then, for a section $\varpi \in \Gamma(D^\perp_N)$, we have

$$(d\varpi)^{t+1} \equiv 0 \qquad \pmod{\varpi_0, \ldots, \varpi_r},$$

on U and $(d\hat{\varpi})^t \not\equiv 0$ for some $\hat{\varpi} \in \Gamma(D^\perp_N)$. Here we may assume $(d\varpi_0)^t \not\equiv 0$ around $y_o \in Y$.

Now let us consider the collection $\hat{W}(Y)$ of hyperplanes w in each tangent space $T_y(Y)$ at $y \in Y$ which contains the fibre $D_N(y)$ of D_N;

$$\hat{W}(Y) = \bigcup_{y \in Y} \hat{W}_y \subset J(Y, m-1),$$

$$\hat{W}_y = \{w \in \mathrm{Gr}(T_y(Y), m-1) \mid w \supset D_N(y)\} \cong \mathbb{P}(T_y(Y)/D_N(y)) = \mathbb{P}^r,$$

where $m = \dim Y$. Moreover C^*_Y is the canonical system obtained by the Grassmannian construction and N_Y is the lift of D_N. Precisely, C^*_Y and N_Y are given by

$$C^*_Y(w) = \nu_*^{-1}(w) \supset N_Y(w) = \nu_*^{-1}(D_N(y)),$$

for each $w \in \hat{W}(Y)$ and $y = \nu(w)$, where $\nu : \hat{W}(Y) \to Y$ is the projection.

We will now examine the condition for $(\hat{W}(Y); C^*_Y, N_Y)$ to be an IG manifold of corank r. Let us consider

$$\varpi = \varpi_0 + \lambda_1 \varpi_1 + \cdots + \lambda_r \varpi_r$$

on U. Namely we consider a point $w \in \hat{W}(Y)$ such that $w = \{\varpi = 0\} \subset T_y(Y)$, where $y = \nu(w) \in U$. Here $(\lambda_1, \ldots, \lambda_r)$ constitutes an inhomogeneous coordinate of the fibres of $\nu : \hat{W}(Y) \to Y$. Denoting the pullback on $\hat{W}(Y)$ of 1-forms on Y by the same symbol, we have

$$C^*_Y = \{\varpi = 0\},$$

and

$$d\varpi = d\varpi_0 + \sum_{i=1}^{r} \lambda_i d\varpi_i + \sum_{i=1}^{r} d\lambda_i \wedge \varpi_i.$$

on $\nu^{-1}(U)$. By the Engel half-rank condition, we have, around y_0,

$$\begin{aligned} d\varpi_0 + \sum_{i=1}^{r} \lambda_i d\varpi_i &\equiv \sum_{\alpha=1}^{t} \tilde{\omega}^\alpha \wedge \tilde{\varpi}^\alpha \pmod{\varpi_0, \ldots, \varpi_r}, \\ &\equiv \sum_{\alpha=1}^{t} \tilde{\omega}^\alpha \wedge \tilde{\varpi}^\alpha + \sum_{i=1}^{r} \gamma_i \wedge \varpi_i \pmod{\varpi}, \end{aligned}$$

where $\tilde{\omega}^\alpha, \tilde{\varpi}^\alpha$ $(1 \leqq \alpha \leqq t)$, γ_i $(1 \leqq i \leqq r)$ are 1-forms on U defined around y_0 such that $\{\varpi_0, \ldots, \varpi_r, \tilde{\omega}^\alpha, \tilde{\varpi}^\alpha \ (1 \leqq \alpha \leqq t)\}$ are linearly independent at each point. Then we have

$$d\varpi \equiv \sum_{\alpha=1}^{t} \tilde{\omega}^\alpha \wedge \tilde{\varpi}^\alpha + \sum_{i=1}^{r} (d\lambda_i + \gamma_i) \wedge \varpi_i \pmod{\varpi}.$$

around $w_0 = \{\varpi_0 = 0\} \in \nu^{-1}(U)$. Hence we have

$$\mathrm{Ch}\,(C^*_Y)(w) = \{\varpi = \varpi_i = d\lambda_i + \gamma_i = \tilde{\omega}^\alpha = \tilde{\varpi}^\alpha = 0 \quad (1 \leqq i \leqq r, 1 \leqq \alpha \leqq t)\}. \tag{7.1}$$

around $w_0 \in \nu^{-1}(U)$. Thus the following subset $W(Y)$ of $\hat{W}(Y)$ is an open (dense) subset of $\hat{W}(Y)$;

$$W(Y) = \{w \in \hat{W}(Y) \mid \text{corank } \mathrm{Ch}\,(C^*_Y)(w) = 2n+1\}$$

where $n = r + t$.

Now we claim

Proposition 7.1. *Let (Y, D_N) be a differential system satisfying $(Y.1)$ and let t be the Engel half-rank of D_N. Then $(W(Y); C^*_Y, N_Y)$ is an IG manifold of corank r, where $n = r + t$. Moreover*

(1) $N_Y^\perp = Ch(N_Y) \oplus Ch(C^*_Y)$ *and* $S(N_Y) = Ch(C^*_Y)$.

(2) $\partial N_Y^\perp \subset N_Y$, *hence* $N^*_Y = N_Y$, *where* $N^*_Y = \partial N_Y^\perp + N_Y$.

Proof. Notations being as above, we have on a neighborhood of $w_0 \in \nu^{-1}(U)$

$$C_Y^* = \{\varpi = 0\}, \qquad N_Y = \{\varpi = \varpi_1 = \cdots = \varpi_r = 0\},$$

and

$$d\varpi \equiv \sum_{\alpha=1}^{t} \tilde{\omega}^\alpha \wedge \tilde{\varpi}^\alpha + \sum_{i=1}^{r} (d\lambda_i + \gamma_i) \wedge \varpi_i \qquad (\mathrm{mod} \quad \varpi).$$

Hence we get

$$\mathrm{Ch}\,(C_Y^*) = \{\varpi = \varpi_i = \tilde{\varpi}^\alpha = \tilde{\omega}^\alpha = d\lambda_i + \gamma_i = 0 \qquad (1 \leqq i \leqq r, 1 \leqq \alpha \leqq t)\},$$

and

$$N_Y^\perp = \{\varpi = \varpi_i = \tilde{\varpi}^\alpha = \tilde{\omega}^\alpha = 0 \qquad (1 \leqq i \leqq r, 1 \leqq \alpha \leqq t)\}.$$

Moreover, from $\mathrm{Ch}\,(D_N) = \{0\}$, we have $\mathrm{Ch}\,(N_Y) = \mathrm{Ker}\,\nu_*$, which implies $\mathrm{Ch}\,(C_Y^*) \cap \mathrm{Ch}\,(N_Y) = \{0\}$. Thus $(W(Y); C_Y^*, N_Y)$ is an IG manifold of corank r.

(1) From $\mathrm{Ch}\,(N_Y) \subset N_Y^\perp$ and the rank count, we have

$$N_Y^\perp = \mathrm{Ch}\,(N_Y) \oplus \mathrm{Ch}\,(C_Y^*).$$

Then it follows that

$$S(N_Y)(w) = \{X \in \mathrm{Ch}\,(C_Y^*)(w) \mid [X, \Gamma(N_Y^\perp)] \subset \Gamma(N_Y)\} \supset \mathrm{Ch}\,(C_Y^*)(w).$$

(2) $\partial N_Y^\perp \subset N_Y$ follows immediately from $N_Y^\perp = \mathrm{Ch}\,(N_Y) \oplus \mathrm{Ch}\,(C_Y^*)$. This completes the proof of Proposition. □

Here we observe that, when (Y, D_N) is obtained from a PD manifold of second order such that $E = \hat{E}$ as in Theorem 5.2, the Engel-half rank of (Y, D_N) equals to $t = n - r$. This can be checked as follows; Let $(W; C^*, N)$ be the associated IG manifold of corank r such that $Y = W/\mathrm{Ch}\,(N)$ is the leaf space. Then we have a map $\kappa_2 : W \to \hat{W}(Y)$ as in §5.3, which is an immersion. Hence we see $\kappa_2(W) \subset W(Y)$ and $\kappa_2 : (W; C^*, N) \to (W(Y); C_Y^*, N_Y)$ is a local isomorphism of IG manifolds.

Now we consider the Lagrange Grassmann bundle $R(Y) = R(W(Y))$ over $(W(Y); C_Y^*, N_Y)$:

$$R(Y) = \bigcup_{w \in W(Y)} R_w, \qquad R_w = \{\hat{v} \subset N_Y(w) \mid \gamma_w \mid_{\hat{v}} = 0, \quad \hat{v} \text{ is maximal}\}.$$

Let D_Y^2 be the canonical system on $R(Y)$ and let D_Y^1 and $\hat{N}_Y$ be the lifts of C_Y^* and N_Y respectively, i.e.,

$$D_Y^1(\hat{v}) = \tau_*^{-1}(C_Y^*(w)), \qquad D_Y^2(\hat{v}) = \tau_*^{-1}(\hat{v}), \qquad \hat{N}_Y(\hat{v}) = \tau_*^{-1}(N_Y(w)),$$

where $\tau : R(Y) \to W(Y)$ is the projection and $w = \tau(\hat{v})$.

In general, in order to see whether $(R(Y); D_Y^1, D_Y^2)$ is a PD manifold of second order, we must check the condition $A(\hat{v}) = \{0\}$ for each $\hat{v} \in R(Y)$, utilizing, e.g., Lemma 4.2 together with the structure equation of D_N. Here we will discuss two extreme cases in the following.

First we impose the following condition for (Y, D_N);

$(Y.2)$ rank $D_N = 2t$, where t is the Engel half-rank of D_N.

This condition is equivalent to the condition that $\{\varpi_0, \ldots, \varpi_r, \tilde{\varpi}^1, \ldots, \tilde{\varpi}^t, \tilde{\omega}^1, \ldots, \tilde{\omega}^t\}$ forms a coframe on a neighborhood around $y_o \in Y$ in the above notation. Then we have

Proposition 7.2. *Let (Y, D_N) be a differential system satisfying $(Y.1)$, and let t be the Engel half-rank of D_N. Then*

(1) *$(W(Y), C^*_Y)$ is a contact manifold of dimension $2n+1$ if and only if $(Y.2)$ holds.*

(2) *If $(Y.2)$ holds, $(W(Y); C^*_Y, N_Y)$ is an IG manifold of corank r, where $n = r+t$, such that $N_Y^\perp = Ch\,(N_Y)$ and that $(R(Y); D^1_Y, D^2_Y)$ is, globally, a PD manifold of second order satisfying $\mathfrak{f}(\hat v) \cong S^2(E^\perp)$ for some r-dimensional subspace E of V at each $\hat v \in R(Y)$.*

Proof. (1) $(W(Y), C^*_Y)$ is a contact manifold of dimension $2n+1$ if and only if $\mathrm{Ch}\,(C^*_Y) = \{0\}$. By (7.1), this is equivalent to the condition that $\{\varpi, \varpi_i, d\lambda_i + \gamma_i, \tilde\omega^\alpha, \tilde\varpi^\alpha (1 \leqq i \leqq r, 1 \leqq \alpha \leqq t)\}$ is a coframe around $w_o \in \nu^{-1}(U)$, which is also equivalent to the condition that $\{\varpi_0, \ldots, \varpi_r, \tilde\varpi^1, \ldots, \tilde\varpi^t, \tilde\omega^1, \ldots, \tilde\omega^t\}$ is a coframe on a neighborhood of $y_o \in Y$.

(2) If $(Y.2)$ holds, $N_Y^\perp = \mathrm{Ch}\,(N_Y)$ follows from Proposition 7.1 and the last assertion follows from $S(N_Y) = \mathrm{Ch}\,(C^*_Y) = \{0\}$ and Lemma 4.2. $\square$

Unfortunately, in view of Case (3) of Theorem in [23] (see also §6.1 [27]), these $R(Y)$ are inevitably incompatible systems, i.e., $R(Y)$ does not satisfy the compatibility condition (C) in §3.1 (or §4.2). We will give some examples of this case in §8.3.

Secondly we assume the following condition for (Y, D_N) (see [4] II §4);

$(Y.3)$ The Cartan rank of D_N coincides with the Engel half-rank of D_N.

Under this condition, we have a local defining 1-forms $\{\varpi_0, \ldots, \varpi_r\}$ of (Y, D_N) on a neighborhood U of $y_0 \in Y$ such that the structure equation of the following form holds:

$$d\varpi_i \equiv \omega^1 \wedge \pi_{i1} + \cdots + \omega^t \wedge \pi_{it} \qquad (\mathrm{mod} \quad \varpi_0, \ldots, \varpi_r) \qquad \text{for} \quad i = 0, \ldots, r,$$

where $\omega^\alpha, \pi_{i\alpha}$ $(1 \leqq i \leqq r, 1 \leqq \alpha \leqq t)$ are 1-forms defined around $y_0 \in Y$ such that $\{\varpi_0, \ldots, \varpi_r, \omega^1, \ldots, \omega^t\}$ are linearly independent on U.

Moreover, from $d\varpi_i \equiv \sum_{\alpha=1}^t \omega^\alpha \wedge \pi_{i\alpha} + \sum_{j=1}^r \gamma_i^j \wedge \varpi_j \quad (\mathrm{mod} \quad \varpi)$ for $i = 0, \ldots, r$, we calculate

$$d\varpi \equiv \sum_{\alpha=1}^t \omega^\alpha \wedge (\pi_{0\alpha} + \sum_{i=1}^r \lambda_i \pi_{i\alpha}) + \sum_{i=1}^r (d\lambda_i + \gamma_0^i + \sum_{j=1}^r \lambda_j \gamma_j^i) \wedge \varpi_i \quad (\mathrm{mod} \quad \varpi).$$

By the Engel half-rank condition, we may assume $(d\varpi_0)^t \not\equiv 0 \quad (\mathrm{mod} \quad \varpi_0, \ldots, \varpi_r)$ as before, i.e., $\{\varpi_0, \ldots, \varpi_r, \omega^1, \ldots, \omega^t, \pi_{01}, \ldots, \pi_{0t}\}$ are linearly independent around $y_o \in Y$. Then, putting $n = r+t$, $\varpi^\alpha = \pi_{0\alpha} + + \sum_{i=1}^r \lambda_i \pi_{i\alpha}$ and $\Omega_i = \gamma_0^i + \sum_{j=1}^r \lambda_j \gamma_j^i$, we see that corank of $\mathrm{Ch}\,(C^*_Y)(w)$ equals $2n+1$ if and only if $\{\varpi^1, \ldots, \varpi^t\}$ are linearly independent (mod $\varpi, \varpi_1, \ldots, \varpi_r, \omega^1, \ldots, \omega^t, d\lambda_1 + \Omega_1, \ldots, d\lambda_r + \Omega_r$) at each point w. Hence, on a neighborhood of w_0 in $W(Y)$, where $\{\varpi, \varpi_i, d\lambda_i, \varpi^\alpha, \omega^\alpha (1 \leqq i \leqq r, 1 \leqq \alpha \leqq t)\}$ are linearly independent at each point, we have

$$C^*_Y = \{\varpi = 0\}, \qquad N_Y = \{\varpi = \varpi_1 = \cdots = \varpi_r = 0\},$$

and

$$d\varpi \equiv \sum_{\alpha=1}^t \omega^\alpha \wedge \varpi^\alpha + \sum_{i=1}^r (d\lambda_i + \Omega_i) \wedge \varpi_i \qquad (\mathrm{mod} \quad \varpi).$$

Thus

$$\mathrm{Ch}\,(C^*_Y) = \{\varpi = \varpi_i = \varpi^\alpha = \omega^\alpha = d\lambda_i + \Omega_i = 0 \qquad (1 \leqq i \leqq r, 1 \leqq \alpha \leqq t)\},$$

and

$$N_Y^\perp = \{\varpi = \varpi_i = \varpi^\alpha = \omega^\alpha = 0 \qquad (1 \leqq i \leqq r, 1 \leqq \alpha \leqq t)\}.$$

Then, from $N_Y^{\perp}(w) \subset \hat{v} \subset N_Y(w)$ and $d\varpi\mid_{\hat{v}}=0$, if $\omega^1 \wedge \cdots \wedge \omega^t \mid_{\hat{v}} \neq 0$, it follows that

$$\hat{v} = \{X \in N_Y(w) \mid (\varpi^\alpha - \sum_{\beta=1}^{t} p^*_{\alpha\beta}\omega^\beta)(X) = 0 \quad (1 \leqq \alpha \leqq t)\},$$

where $p^*_{\alpha\beta} = p^*_{\beta\alpha}$ $(1 \leqq \alpha, \beta \leqq t)$. For these $\hat{v} \in R_w$, we have

Lemma 7.1. $A(\hat{v}) = \{0\}$ *for* $\hat{v} = \{\varpi_0 = \cdots = \varpi_r = \varpi^\alpha - \sum_{\beta=1}^{t} p^*_{\alpha\beta}\omega^\beta = 0 \quad (1 \leqq \alpha \leqq t)\}$.

Proof. First we have

$$d\varpi_i \equiv \omega^1 \wedge \pi_{i1} + \cdots + \omega^t \wedge \pi_{it} \qquad (\mathrm{mod} \quad \varpi_0, \ldots, \varpi_r) \qquad \text{for} \quad i = 0, \ldots, r,$$

Hence, for $X \in S(N_Y)(w) = \mathrm{Ch}\,(C^*_Y)(w)$, we have

$$X \rfloor d\varpi_i \equiv -\sum_{\alpha=1}^{t} \pi_{i\alpha}(X)\omega^\alpha \qquad (\mathrm{mod} \quad (\hat{v})^{\perp}),$$

for $i = 0, \ldots, r$. Then, from Lemma 4.2 (2), we get

$$\begin{aligned} A(\hat{v}) &= \{X \in S(N_Y)(w) \mid \pi_{i\alpha}(X) = 0 \quad (0 \leqq i \leqq r, 1 \leqq \alpha \leqq t)\} \\ &= \mathrm{Ch}\,(C^*_Y)(w) \cap \mathrm{Ch}\,(N_Y)(w) = \{0\}. \end{aligned}$$

□

Thus we obtain

Proposition 7.3. *Let* (Y, D_N) *be a differential system satisfying* $(Y.1)$ *and* $(Y.3)$.
Then $(R(Y); D^1_Y, D^2_Y)$ *is a PD manifold of second order on an open subset of* $R(Y)$.

Here we note that , for $\hat{v}_1 = \{\varpi_0 = \cdots = \varpi_r = \omega^\alpha = 0 \quad (1 \leqq \alpha \leqq t)\} \in R(Y)$, we have $A(\hat{v}_1) = \mathrm{Ch}\,(C^*_Y)$, which follows from $d\varpi_i \equiv 0 \quad (\mathrm{mod} \quad \varpi_0, \ldots, \varpi_r, \omega^\alpha (1 \leqq \alpha \leqq t))$ for $i = 0, \ldots, r$. Hence $(R(Y); D^1_Y, D^2_Y)$ is not a PD manifold of second order globally in this case.

When $(R(Y); D^1_Y, D^2_Y)$ becomes a PD manifold of second order satisfying the compatibility condition (C), $\hat{N}_Y$ defines a subspace $\pi_{-2}(\hat{N}_Y(\hat{v}))$ of $\mathfrak{s}_{-2}(\hat{v}) = D^1_Y(\hat{v})/D^2_Y(\hat{v})$ at each point $\hat{v} \in R(Y)$, where $\pi_{-2} : D^1_Y(\hat{v}) \to \mathfrak{s}_{-2}(\hat{v})$ is the projection. This subspace defines the subspace $E(\hat{v})^{\perp}$ of $V(\hat{v})^*$, through the symbol algebra identifications of $(R(Y); D^1_Y, D^2_Y)$ at $\hat{v} \in R(Y)$: $\mathfrak{s}_{-2}(\hat{v}) \cong V(\hat{v})^*$, $\mathfrak{s}_{-1} = V(\hat{v}) \oplus \mathfrak{f}(\hat{v})$ and $\mathfrak{f}(\hat{v}) \subset S^2(V(\hat{v})^*)$, where $V(\hat{v})$ is an integral element of $(R(Y), D^2_Y)$ at $\hat{v}$. Thus we obtain the subspace $E(\hat{v}) \subset V(\hat{v})$. Then, by the construction of $R(Y)$ in §6.1, we have $\mathfrak{f}(\hat{v})^{\perp} \subset E(\hat{v}) \otimes_S V(\hat{v})$. Moreover we have $\hat{E}(\hat{v}) = E(\hat{v})$, which follows from Proposition 7.1 (2). Thus $E(\hat{v})$ satisfies $S^2(E(\hat{v})) \subset \mathfrak{f}(\hat{v})^{\perp} \subset E(\hat{v}) \otimes_S V(\hat{v})$. Hence $E(\hat{v})$ is $G_0(\mathfrak{s}(\hat{v}))$-invariant. Thus $\hat{N}_Y$ is a (first order) covariant system of $(R(Y); D^1_Y, D^2_Y)$.

7.2. **General Case.** Now, starting from a pair of differential systems $(Y; D^*_N, D_N)$, we will construct an IG manifold $(W(Y); C^*_Y, N_Y)$ over $(Y; D^*_N, D_N)$ such that $N_Y = \mu^{-1}(D_N)$ and $\bar{N}_Y = \mu^{-1}(D^*_N)$ where $\mu : W(Y) \to Y$ is the projection, and the Lagrange Grassmann bundle $R(Y) = R(W(Y))$ over $(W(Y); C^*_Y, N_Y)$ and will examine the conditions for $\bar{N}_Y = N^*_Y$ and when $(R(Y); D^1_Y, D^2_Y)$ becomes a PD manifold of second order.

Let $(Y; D^*_N, D_N)$ be a pair of differential systems on Y satisfying the following condition;

$(\hat{Y}.1)$ D^*_N and D_N are differential systems of codimension $s + 1$ and $r + 1$ on Y such that

$$D^*_N \supset D_N \supset \mathrm{Ch}\,(D^*_N) \supset \mathrm{Ch}\,(D_N) = \{0\}.$$

We assume that D^*_N is of constant Engel half-rank and let t_1 be the Engel half-rank of D^*_N. Let

$\{\varpi_0, \ldots, \varpi_s\}$ be a local defining 1-forms of (Y, D_N^*) on a neighborhood U of $y_o \in Y$. Then, for a section $\varpi \in \Gamma((D_N^*)^\perp)$, we have

$$(d\varpi)^{t_1+1} \equiv 0 \qquad (\text{mod} \quad \varpi_0, \ldots, \varpi_s),$$

on U and $(d\hat{\varpi})^{t_1} \not\equiv 0$ for some $\hat{\varpi} \in \Gamma((D_N^*)^\perp)$. Here we may assume $(d\varpi_0)^{t_1} \not\equiv 0$ around $y_o \in Y$.

Now let us consider the collection $\tilde{W}(Y)$ of hyperplanes w in each tangent space $T_y(Y)$ at $y \in Y$ which contains the fibre $D_N^*(y)$ of D_N^*;

$$\tilde{W}(Y) = \bigcup_{y \in Y} \tilde{W}_y \subset J(Y, m-1),$$

$$\tilde{W}_y = \{w \in \mathrm{Gr}(T_y(Y), m-1) \mid w \supset D_N^*(y)\} \cong \mathbb{P}(T_y(Y)/D_N^*(y)) = \mathbb{P}^s,$$

where $m = \dim Y$. Moreover C_Y^* is the canonical system obtained by the Grassmannian construction, $\bar{N}_Y$ and N_Y are the lifts of D_N^* and D_N respectively. Precisely, C_Y^*, $\bar{N}_Y$ and N_Y are given by

$$C_Y^*(w) = \mu_*^{-1}(w) \supset \bar{N}_Y(w) = \mu_*^{-1}(D_N^*(y)) \supset N_Y(w) = \mu_*^{-1}(D_N(y)),$$

for each $w \in \tilde{W}(Y)$ and $y = \mu(w)$, where $\mu : \tilde{W}(Y) \to Y$ is the projection.

We will now examine the condition for $(\tilde{W}(Y); C_Y^*, N_Y)$ to be an IG manifold of corank r. Let us consider

$$\varpi = \varpi_0 + \lambda_1 \varpi_1 + \cdots + \lambda_s \varpi_s$$

on U. Namely we consider a point $w \in \tilde{W}(Y)$ such that $w = \{\varpi = 0\} \subset T_y(Y)$, where $y = \mu(w) \in U$. Here $(\lambda_1, \ldots, \lambda_s)$ constitutes an inhomogeneous coordinate of the fibres of $\mu : \hat{W}(Y) \to Y$. Denoting the pullback on $\tilde{W}(Y)$ of 1-forms on Y by the same symbol, we have

$$C_Y^* = \{\varpi = 0\},$$

and

$$d\varpi = d\varpi_0 + \sum_{i=1}^{s} \lambda_i d\varpi_i + \sum_{i=1}^{s} d\lambda_i \wedge \varpi_i.$$

on $\mu^{-1}(U)$. Then, as in §7.1, by the Engel half-rank condition, we see that the following subset $W(Y)$ of $\tilde{W}(Y)$ is an open (dense) subset of $\tilde{W}(Y)$;

$$W(Y) = \{w \in \tilde{W}(Y) \mid \text{corank Ch}\,(C_Y^*)(w) = 2n+1\}$$

where $n = s + t_1$. Assume that $(W(Y); C_Y^*, N_Y)$ is an IG manifold of corank r. Then, by the condition $(W.3)$ in §2.2, we have $\ \text{rank}\, d\varpi\mid_{N_Y(w)} = 2(n-r)$ at each $w \in W(Y)$. Namely we have, at each point $w = \{\varpi = \varpi_0 + \sum_{i=1}^{s} \lambda_i \varpi_i = 0\} \in W(Y)$,

$$d\varpi \equiv d\varpi_0 + \sum_{i=1}^{s} \lambda_i d\varpi_i \equiv \sum_{\alpha=1}^{n-r} \tilde{\omega}^\alpha \wedge \tilde{\varpi}^\alpha \quad (\text{mod} \quad \varpi_0, \ldots, \varpi_r),$$

where $D_N^* = \{\varpi_0 = \cdots = \varpi_s = 0\}$ and $D_N = \{\varpi_0 = \cdots = \varpi_r = 0\}$. Thus $(Y; D_N^*, D_N)$ satisfies the following condition;

$(\hat{Y}.2)$ The Engel half-rank of D_N^* (mod $\ D_N^\perp$) equals to $t_2 = t_1 - (r - s) = n - r$.

Conversely, under this condition, there exists a local defining 1-forms $\{\varpi_0, \ldots, \varpi_r\}$ of (Y, D_N) on a neighborhood U of $y_o \in Y$ such that

$$D_N^* = \{\varpi_0 = \cdots = \varpi_s = 0\}, \qquad D_N = \{\varpi_0 = \cdots = \varpi_r = 0\},$$

and

$$
(7.2)\qquad \begin{aligned} d\varpi_0 + \sum_{i=1}^{s} \lambda_i d\varpi_i &\equiv \sum_{\alpha=1}^{t_2} \tilde{\omega}^\alpha \wedge \tilde{\varpi}^\alpha \pmod{\varpi_0, \ldots, \varpi_r}, \\ &\equiv \sum_{j=s+1}^{r} \omega_j \wedge \varpi_j + \sum_{\alpha=1}^{t_2} \tilde{\omega}^\alpha \wedge \tilde{\varpi}^\alpha \pmod{\varpi_0, \ldots, \varpi_s}, \end{aligned}
$$

where $\tilde{\omega}^\alpha, \tilde{\varpi}^\alpha (1 \leqq \alpha \leqq t_2)$ and $\omega_j (s+1 \leqq j \leqq r)$ are 1-forms on U defined around y_0 such that $\{\varpi_0, \ldots, \varpi_r, \tilde{\omega}^\alpha, \tilde{\varpi}^\alpha \ (1 \leqq \alpha \leqq t_2)\}$ are linearly independent at each point. Moreover, by the Engel half-rank condition for D_N^*, we see that $\{\varpi_0, \ldots, \varpi_r, \omega_j (s+1 \leqq j \leqq r), \tilde{\omega}^\alpha, \tilde{\varpi}^\alpha \ (1 \leqq \alpha \leqq t_2)\}$ are linearly independent around $y_0 \in Y$.

Now we claim

Proposition 7.4. *Let $(Y; D_N^*, D_N)$ be a pair of differential systems satisfying $(\hat{Y}.1)$ and $(\hat{Y}.2)$ and let t_1 be the Engel half-rank of D_N^*. Then $(W(Y); C_Y^*, N_Y)$ is an IG manifold of corank r , where $n = r + t_2 = s + t_1$. Moreover*

$$
\bar{N}_Y \supset N_Y^* = \partial N_Y^\perp + N_Y \qquad \text{and} \qquad N_Y \supset \partial \bar{N}_Y^\perp.
$$

Proof. Notations being as above, putting

$$
d\varpi_0 + \sum_{i=1}^{s} \lambda_i d\varpi_i \equiv \sum_{a=1}^{s} \gamma_a \wedge \varpi_a + \sum_{j=s+1}^{r} \omega_j \wedge \varpi_j + \sum_{\alpha=1}^{t_2} \tilde{\omega}^\alpha \wedge \tilde{\varpi}^\alpha \pmod{\varpi},
$$

we get

$$
d\varpi \equiv \sum_{a=1}^{s} (d\lambda_a + \gamma_a) \wedge \varpi_a + \sum_{j=s+1}^{r} \omega_j \wedge \varpi_j + \sum_{\alpha=1}^{t_2} \tilde{\omega}^\alpha \wedge \tilde{\varpi}^\alpha \pmod{\varpi},
$$

on a neiborghhood of $w_0 = \{\varpi_0 = 0\} \in W(Y)$. Thus we have

$$
C_Y^* = \{\varpi = 0\}, \quad \bar{N}_Y = \{\varpi = \varpi_1 = \cdots = \varpi_s = 0\}, \quad N_Y = \{\varpi = \varpi_1 = \cdots = \varpi_r = 0\},
$$

$$
\mathrm{Ch}\,(C_Y^*) = \{\varpi = \varpi_a = \varpi_j = \tilde{\varpi}^\alpha = \tilde{\omega}^\alpha = \omega_j = d\lambda_a + \gamma_a = 0 \quad (1 \leqq a \leqq s, s+1 \leqq j \leqq r, 1 \leqq \alpha \leqq t_2)\}.
$$

and

$$
N_Y^\perp = \{\varpi = \varpi_1 = \cdots = \varpi_r = \tilde{\varpi}^\alpha = \tilde{\omega}^\alpha = 0 \quad (1 \leqq \alpha \leqq t_2)\},
$$

$$
\bar{N}_Y^\perp = \{\varpi = \varpi_1 = \cdots = \varpi_r = \tilde{\varpi}^\alpha = \tilde{\omega}^\alpha = \omega_j = 0 \quad (1 \leqq \alpha \leqq t_2, s+1 \leqq j \leqq r)\}.
$$

Moreover, from $\mathrm{Ch}\,(D_N) = \{0\}$, we have $\mathrm{Ch}\,(N_Y) = \mathrm{Ker}\,\mu_*$, which implies $\mathrm{Ch}\,(C_Y^*) \cap \mathrm{Ch}\,(N_Y) = \{0\}$. Thus $(W(Y); C_Y^*, N_Y)$ is an IG manifold of corank r.

$\bar{N}_Y \supset \partial N_Y^\perp$ follows from

$$
d\varpi_a \equiv 0 \qquad (\mathrm{mod} \quad \varpi_0, \ldots, \varpi_r, \tilde{\varpi}^\alpha, \tilde{\omega}^\alpha \quad (1 \leqq \alpha \leqq t_2)),
$$

for $a = 0, \ldots, s$, which follows from (7.2) and [4] II Proposition 4.1.

Moreover, from $d\varpi \equiv \sum_{j=s+1}^{r} \omega_j \wedge \varpi_j + \sum_{\alpha=1}^{t_2} \tilde{\omega}^\alpha \wedge \tilde{\varpi}^\alpha \pmod{\varpi_0, \ldots, \varpi_s}$, we have $\bar{N}_Y^\perp \supset \mathrm{Ch}\,(\bar{N}_Y) \supset \mathrm{Ch}\,(N_Y)$. Hence, by rank comparison, we get $\bar{N}_Y^\perp = \mathrm{Ch}\,(N_Y) \oplus \mathrm{Ch}\,(C_Y^*)$, which implies that $N_Y \supset \partial \bar{N}_Y^\perp$. □

As for the criteria to the condition $\bar{N}_Y = N_Y^*$, utilizing Proposition 2.1 in §2.3 (3), we add the following ;

Proposition 7.5. *Let $(Y; D_N^*, D_N)$ be a pair of differential systems satisfying $(\hat{Y}.1)$ and $(\hat{Y}.2)$. Assume that $(W(Y); C_Y^*, N_Y)$ satisfies the compatibility condition (C^*). Then $\bar{N}_Y = N_Y^*$ holds if and only if $H(N_Y) = Ch\,(N_Y) \oplus S(N_Y)$.*

Proof. By Proposition 7.4, we have $\bar{N}_Y \supset N_Y^*$. Hence $\bar{N}_Y = N_Y^*$ if and only if $\dim C_Y^*(w)/N_Y^*(w) = s$ at each $w \in W(Y)$. By Proposition 2.1, we have $\dim C_Y^*(w)/N_Y^*(w) = \dim H(N_Y)(w)/S(N_Y)(w)$. On the other hand, $H(N_Y) \supset \mathrm{Ch}\,(N_Y) \oplus S(N_Y)$ and $\mathrm{rank}\,\mathrm{Ch}\,(N_Y) = s$. Hence $\bar{N}_Y = N_Y^*$ holds if and only if $H(N_Y) = \mathrm{Ch}\,(N_Y) \oplus S(N_Y)$. □

Now we consider the Lagrange Grassmann bundle $R(Y) = R(W(Y))$ over $(W(Y); C_Y^*, N_Y)$:

$$R(Y) = \bigcup_{w \in W(Y)} R_w, \qquad R_w = \{\hat{v} \subset N_Y(w) \mid \gamma_w\,|_{\hat{v}} = 0, \qquad \hat{v} \text{ is maximal}\}.$$

Let D_Y^2 be the canonical system on $R(Y)$ and let D_Y^1 and $\hat{N}_Y$ be the lifts of C_Y^* and N_Y respectively, i.e.,

$$D_Y^1(\hat{v}) = \tau_*^{-1}(C_Y^*(w)), \qquad D_Y^2(\hat{v}) = \tau_*^{-1}(\hat{v}), \qquad \hat{N}_Y(\hat{v}) = \tau_*^{-1}(N_Y(w)),$$

where $\tau : R(Y) \to W(Y)$ is the projection and $w = \tau(\hat{v})$.

In order to see whether $(R(Y); D_Y^1, D_Y^2)$ is a PD manifold of second order, we must check the condition $A(\hat{v}) = \{0\}$ for each $\hat{v} \in R(Y)$, utilizing, e.g., Lemma 4.2 together with the structure equation of D_N. Assuming the conditions $(\hat{Y}.1)$ and $(\hat{Y}.2)$, by Proposition 7.4, we see that the structure quation of N_Y takes the following form;

$$\begin{cases} d\varpi \equiv & \sum_{\alpha=1}^{t_2} \tilde{\omega}^\alpha \wedge \tilde{\varpi}_\alpha, \\ d\varpi_a \equiv & \sum_{\alpha=1}^{t_2} \tilde{\omega}^\alpha \wedge \pi_{a\alpha} + \sum_{\alpha=1}^{t_2} \tilde{\varpi}^\alpha \wedge \hat{\pi}_{a\alpha} \quad (1 \leqq a \leqq s), \quad (\mathrm{mod}\ \varpi, \varpi_1, \dots, \varpi_r) \\ d\varpi_i \equiv \sum_{j=s+1}^{r} \omega^j \wedge \pi_{ij} + & \sum_{\alpha=1}^{t_2} \tilde{\omega}^\alpha \wedge \pi_{i\alpha} + \sum_{\alpha=1}^{t_2} \tilde{\varpi}^\alpha \wedge \hat{\pi}_{i\alpha} \quad (s+1 \leqq i \leqq r), \end{cases}$$

Here, by Lemma 4.2 (2), we note

$$S(N_Y)(w) = \{X \in \mathrm{Ch}\,(C_Y^*)(w) \mid \pi_{ij}(X) = 0\ (s+1 \leqq i, j \leqq r)\},$$

$$\begin{aligned} A(\hat{v}_1) = \{X \in S(N_Y)(w)\ |(\pi_{a\alpha} + \sum_{\beta=1}^{t_2} p_{\alpha\beta}\hat{\pi}_{a\beta})(X) = \\ (\pi_{i\alpha} + \sum_{\beta=1}^{t_2} p_{\alpha\beta}\hat{\pi}_{i\beta})(X) = 0\ (1 \leqq a \leqq s, s+1 \leqq i \leqq r, 1 \leqq \alpha \leqq t_2)\} \end{aligned}$$

for $\hat{v}_1 = \{\varpi = \varpi_1 = \cdots = \varpi_r = \tilde{\varpi}^\alpha - \sum_{\beta=1}^{t_2} p_{\alpha\beta}\tilde{\omega}^\beta = 0\ (1 \leqq \alpha \leqq t_2)\} \in R(Y)$ and

$$\begin{aligned} A(\hat{v}_2) = \{X \in S(N_Y)(w)\ |(\hat{\pi}_{a\alpha} + \sum_{\beta=1}^{t_2} p_{\alpha\beta}\pi_{a\beta})(X) = \\ (\hat{\pi}_{i\alpha} + \sum_{\beta=1}^{t_2} p_{\alpha\beta}\pi_{i\beta})(X) = 0\ (1 \leqq a \leqq s, s+1 \leqq i \leqq r, 1 \leqq \alpha \leqq t_2)\} \end{aligned}$$

for $\hat{v}_2 = \{\varpi = \varpi_1 = \cdots = \varpi_r = \tilde{\omega}^\alpha - \sum_{\beta=1}^{t_2} p_{\alpha\beta}\tilde{\varpi}^\beta = 0\ (1 \leqq \alpha \leqq t_2)\} \in R(Y)$. Thus we need some more information on the structure of D_N to conclude $A(\hat{v}_1) = \{0\}$ or $A(\hat{v}_2) = \{0\}$. We will examine several examples of this case in §8.2.

8. Examples of Second Reduction Theorem

8.1. **Case $n = 3$.** In [6], for involutive systems of second order partial differential equations for a scalar function with 3 independent variables, E.Cartan first classified involutive subspaces $\mathfrak{f}$ of $S^2(V^*)$, over the complex number field $\mathbb{C}$, when $\dim V = 3$. In this subsection, following his classification, we will here indicate $G_0(\mathfrak{s})$-invariant subspaces E and $\hat{E}$, satisfying $\mathfrak{f}^\perp \subset E \otimes_S V$, for each involutive subspace $\mathfrak{f} \subset S^2(V^*)$, when $\dim V = 3$, which shows the applicability of Theorem 5.1 or 5.2 in each case.

(1) codim $\mathfrak{f} = 1$.

In this case $\dim \mathfrak{f}^\perp = 1$. Hence we can classify a generator f of $\mathfrak{f}^\perp$ as a quadratic form and obtain the following classification by the rank of f into three cases over $\mathbb{C}$, i.e., there exista a basis $\{e_1, e_2, e_3\}$ of V such that

$$\mathfrak{f}^\perp = \langle\{e_1 \circledcirc e_1 + e_2 \circledcirc e_2 + e_3 \circledcirc e_3\}\rangle, \quad \langle\{e_1 \circledcirc e_2\}\rangle, \quad \text{or} \quad \langle\{e_1 \circledcirc e_1\}\rangle.$$

In the first case, we have no $G_0(\mathfrak{s})$-invariant subspace. In the second case, $E = \langle\{e_1, e_2\}\rangle$, $\langle\{e_1\}\rangle$ or $\langle\{e_2\}\rangle$ and $\hat{E} = \{0\}$ in either case. In the third case, $E = \hat{E} = \langle\{e_1\}\rangle$. The third case corrresponds to the Goursat type equation (see Theorem 6.1 (1)).

(2) codim $\mathfrak{f} = 2$.

In this case, there exists a basis $\{e_1, e_2, e_3\}$ of V such that (see Proposition 3.1 [27])

$$\mathfrak{f}^\perp = \langle\{e_1 \circledcirc e_2, e_1 \circledcirc e_3\}\rangle, \quad \text{or} \quad \langle\{e_1 \circledcirc e_1, e_1 \circledcirc e_2\}\rangle.$$

In the first case, $E = \langle\{e_2, e_3\}\rangle$ or $\langle\{e_1\}\rangle$ and $\hat{E} = \{0\}$ in either case. In the second case, $E_1 = \hat{E}_1 = \langle\{e_1\}\rangle$ or $E_2 = \langle\{e_1, e_2\}\rangle$ and $\hat{E}_2 = \langle\{e_1\}\rangle$ (see §8.3 for E_2-case).

In fact, E.Cartan showed $C(E)$ is completely integrable for $E = \langle\{e_1\}\rangle$ in the first case so that Theorem 5.1 is applicable. In the second case, he showed $C(E_2)$ is completely integrable in case of (b_1) and $C(E_1)$ is completely integrable in case of (b_2) and (b_3) so that Theorem 5.2 is applicable (cf. [21]).

(3) codim $\mathfrak{f} = 3$.

In this case, there exists a basis $\{e_1, e_2, e_3\}$ of V such that (see IV [6])

$$\mathfrak{f}^\perp = \langle\{e_1 \circledcirc e_2, e_1 \circledcirc e_3, e_2 \circledcirc e_3\}\rangle, \quad \langle\{e_1 \circledcirc e_1, e_1 \circledcirc e_3, e_2 \circledcirc e_3\}\rangle, \quad \langle\{e_1 \circledcirc e_1, e_1 \circledcirc e_2, e_2 \circledcirc e_2 - e_1 \circledcirc e_3\}\rangle,$$

$$\langle\{e_1 \circledcirc e_1, e_1 \circledcirc e_2, e_2 \circledcirc e_2\}\rangle, \quad \text{or} \quad \langle\{e_1 \circledcirc e_1, e_1 \circledcirc e_2, e_1 \circledcirc e_3\}\rangle.$$

In the first case, $E = \langle\{e_1, e_2\}\rangle$, $\langle\{e_1, e_3\}\rangle$ or $\langle\{e_2, e_3\}\rangle$ and $\hat{E} = \{0\}$ in either case. In the second case, $E = \langle\{e_1, e_2\}\rangle$ or $\langle\{e_1, e_3\}\rangle$ and $\hat{E} = \langle\{e_1\}\rangle$ in either case. In the third case, $E = \langle\{e_1, e_2\}\rangle$ and $\hat{E} = \langle\{e_1\}\rangle$. In [6], the case when $C(E)$ is completely integrable, has been discussed in detail. In the fourth case, $E = \hat{E} = \langle\{e_1, e_2\}\rangle$ and $\mathfrak{f} \cong \mathfrak{f}^2(2)$ (see Theorem 6.2 (2)). In the fifth case, $E = \hat{E} = \langle\{e_1\}\rangle$ corresponds to the Cauchy characteristics and $\mathfrak{f} \cong \mathfrak{f}^3(1)$ (see §6.1 [27]).

(4) codim $\mathfrak{f} = 4$.

In this case, there exists a basis $\{e_1, e_2, e_3\}$ of V such that (see V [6])

$$\mathfrak{f}^\perp = \langle\{e_1 \circledcirc e_1, e_1 \circledcirc e_2, e_1 \circledcirc e_3, e_2 \circledcirc e_3\}\rangle, \quad \text{or} \quad \langle\{e_1 \circledcirc e_1, e_1 \circledcirc e_2, e_2 \circledcirc e_2, e_1 \circledcirc e_3\}\rangle.$$

In the first case, $E = \langle\{e_1, e_2\}\rangle$ or $\langle\{e_1, e_3\}\rangle$ and $\hat{E} = \langle\{e_1\}\rangle$ in either case. $\hat{E}$ corresponds to the Cauchy characteristics. Hence Theorem 5.1 or 5.2 is not applicable to this case. In the second case, $E = \hat{E} = \langle\{e_1, e_2\}\rangle$ and $\langle\{e_1\}\rangle$ corresponds to the Cauchy characteristics (see §8.2).

(5) codim $\mathfrak{f} = 5$.

In this case, there exists a basis $\{e_1, e_2, e_3\}$ of V such that (see VI [6])

$$\mathfrak{f}^{\perp} = \langle \{e_1 \circledcirc e_1, e_1 \circledcirc e_2, e_1 \circledcirc e_3, e_2 \circledcirc e_2, e_2 \circledcirc e_3\} \rangle.$$

In this case, $E = \hat{E} = \langle \{e_1, e_2\} \rangle$, $\mathfrak{f} \cong \mathfrak{f}^3(2)$ and $\operatorname{rank} D^2 = 4$. E corresponds to the Cauchy characteristics. Hence, by the First Reduction Theorem, this case is reduced to the geometry of (X, D), where $\operatorname{rank} D = 2$ and $\dim X = 6$.

8.2. **Case** $N^* = N$**.** We exhibit here several examples of simple graded Lie algebras $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of depth 2, such that we can construct a PD manifold $R(Y)$ of second order from a regular differential system (Y, D_N) of type $\mathfrak{m}$, through the Second Reduction Theorem, where $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the negative part of $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$.

The first example is of type $(B_3, \{\alpha_3\})$. The standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$ in this case is given as follows; $M(\mathfrak{m}) = \mathbb{R}^6$ is endowed with a coordinate $(x_1, x_2, x_3, y_1, y_2, y_3)$ such that $D_{\mathfrak{m}}$ is given by

$$D_{\mathfrak{m}} = \{\bar{\theta}_1 = \bar{\theta}_2 = \bar{\theta}_3 = 0 \},$$

where

$$\bar{\theta}_1 = dy_1 - x_2 dx_3, \bar{\theta}_2 = dy_2 - x_3 dx_1 \quad \text{and} \quad \bar{\theta}_3 = dy_3 - x_1 dx_2.$$

Thus the symbol algebra $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of type $(B_3, \{\alpha_3\})$ is described by

$$\begin{cases} d\theta_1 \equiv \omega_3 \wedge \omega_2 \\ d\theta_2 \equiv \omega_1 \wedge \omega_3 \qquad (\mathrm{mod} \quad \theta_1, \theta_2, \theta_3) \\ d\theta_3 \equiv \omega_2 \wedge \omega_1 \end{cases} \tag{8.1}$$

i.e., $\mathfrak{g}_{-2} = \wedge^2 V$ for $\mathfrak{g}_{-1} = V$, where $\dim V = 3$. Namely $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the free Lie algebra of the second kind (cf. §5.3 [15], p.245 [24]). Let (Y, D_N) be a regular differential system of type $\mathfrak{m}$ such that D_N is locally defined by

$$D_N = \{\theta_1 = \theta_2 = \theta_3 = 0\}.$$

Here $\{\theta_1, \theta_2, \theta_3, \omega_1, \omega_2, \omega_3\}$ forms a coframe on Y satisfying (8.1). Then, putting $\varpi = \theta_1 + \lambda_1 \theta_2 + \lambda_2 \theta_3$, we calculate

$$\begin{aligned} d\varpi &\equiv \omega_3 \wedge \omega_2 + \lambda_1 \omega_1 \wedge \omega_3 + \lambda_2 \omega_2 \wedge \omega_1 + (d\lambda_1 + \gamma_1) \wedge \theta_2 + (d\lambda_2 + \gamma_2) \wedge \theta_3 \\ &\equiv (\omega_3 - \lambda_2 \omega_1) \wedge (\omega_2 - \lambda_1 \omega_1) + (d\lambda_1 + \gamma_1) \wedge \theta_2 + (d\lambda_2 + \gamma_2) \wedge \theta_3 \\ &\equiv \tilde{\omega}_3 \wedge \tilde{\omega}_2 + (d\lambda_1 + \gamma_1) \wedge \theta_2 + (d\lambda_2 + \gamma_2) \wedge \theta_3 \quad (\mathrm{mod} \quad \varpi) \end{aligned}$$

for some 1-forms γ_1, γ_2 on Y, where we put $\tilde{\omega}_2 = \omega_2 - \lambda_1 \omega_1$ and $\tilde{\omega}_3 = \omega_3 - \lambda_2 \omega_1$. Thus, by symmetry in the indices $1, 2, 3$ in (8.1), we see

$$W(Y) = \hat{W}(Y) = \bigcup_{y \in Y} \tilde{W}_y, \quad \tilde{W}_y = \{w \in \mathrm{Gr}(T_y(Y), 5) \mid w \supset D_N(y)\} \cong \mathbb{P}(T_y(Y)/D_N(y)) = \mathbb{P}^2$$

and we have on $W(Y)$,

$$C_Y^* = \{\varpi = o\}, \quad N_Y = \{\varpi = \theta_2 = \theta_3 = o\}, \quad N_Y^{\perp} = \{\varpi = \theta_2 = \theta_3 = \tilde{\omega}_2 = \tilde{\omega}_3 = o\}.$$

Here $r = 2$, $t = 1$, $n = r + t = 3$ and $\dim W(Y) = 8$. Moreover, for the Lagrange Grassmann bundle;

$$R(Y) = \bigcup_{w \in W(Y)} R_w, \quad R_w = \{\hat{v} \subset N_Y(w) \mid d\varpi \mid_{\hat{v}} = 0, \quad \hat{v} \text{ is maximal}\}.$$

we have

$$\hat{v} = \{\varpi = \theta_2 = \theta_3 = \tilde{\omega}_3 - p\tilde{\omega}_2 = 0\} \text{ or } \{\varpi = \theta_2 = \theta_3 = \tilde{\omega}_2 - q\tilde{\omega}_3 = 0\}.$$

Then, by Lemma 4.2 (2) and (8.1), we get $A(\hat{v}) = \{0\}$ in any case. Hence $(R(Y); D^1_Y, D^2_Y)$ is globally a PD manifold of second order, when (Y, D_N) is a regular differential system of type $\mathfrak{m}$, where $\mathfrak{m}$ is the negative part of the simple graded Lie algebra of type $(B_3, \{\alpha_3\})$.

Here we note : In case $Y = G/P_3$; R-space of type $(B_3, \{\alpha_3\})$, $W(Y)$ is identified with G/P_1 of type $(B_3, \{\alpha_2, \alpha_3\})$, where C^*_Y and N_Y correspond to $\mathfrak{g}_{-3}\oplus\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$ respectively for $\mathfrak{m}_1 = \mathfrak{g}_{-4}\oplus\mathfrak{g}_{-3}\oplus\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$. Here $J = G/P_2$ of type $(B_3, \{\alpha_2\})$ is the standard contact manifold of B_3 type. Moreover $R(Y)$ is identified with G/B of type $(B_3, \{\alpha_1.\alpha_2, \alpha_3\})$, where D^1_Y and D^2_Y correspond to $\mathfrak{g}_{-4}\oplus\mathfrak{g}_{-3}\oplus\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$ respectively for $\mathfrak{m}_0 = \mathfrak{g}_{-5}\oplus\mathfrak{g}_{-4}\oplus\mathfrak{g}_{-3}\oplus\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$.

To obtain an explicit description of the model equation in this case, we calculate

$$\begin{aligned}
\varpi &= dy_1 - x_2dx_3 + \lambda_1(dy_2 - x_3dx_1) + \lambda_2(dy_3 - x_1dx_2) \\
&= d(y_1 + \lambda_1y_2 + \lambda_2y_3) - y_2d\lambda_1 - y_3d\lambda_2 - x_2dx_3 - \lambda_1x_3dx_1 - \lambda_2x_1dx_2 \\
&= d(y_1 + \lambda_1y_2 + \lambda_2y_3 - \lambda_1x_1x_3) - (y_2 - x_1x_3)d\lambda_1 - y_3d\lambda_2 - \lambda_2x_1dx_2 - (x_2 - \lambda_1x_1)dx_3 \\
&= d(y_1 + \lambda_1y_2 + \lambda_2y_3 - \lambda_1x_1x_3 - \lambda_2x_1x_2) - (y_2 - x_1x_3)d\lambda_1 - (y_3 - x_1x_2)d\lambda_2 \\
&\qquad - (x_2 - \lambda_1x_1)(dx_3 - \lambda_2dx_1) + \lambda_1\lambda_2x_1dx_1 \\
&= d(y_1 + \lambda_1y_2 + \lambda_2y_3 - \lambda_1x_1x_3 - \lambda_2x_1x_2) - (y_2 - x_1x_3)d\lambda_1 - (y_3 - \lambda_1x_1^2)d\lambda_2 \\
&\qquad - (x_2 - \lambda_1x_1)d(x_3 - \lambda_2x_1) + \frac{1}{2}\lambda_1\lambda_2dx_1^2 \\
&= dZ - P_1dX_1 - P_2dX_2 - P_3dX_3
\end{aligned}$$

Thus, putting

$$\begin{cases} Z = y_1 + \lambda_1y_2 + \lambda_2y_3 - \lambda_1x_1x_3 - \lambda_2x_1x_2 + \dfrac{1}{2}\lambda_1\lambda_2x_1^2, X_1 = \lambda_1,\ X_2 = \lambda_2,\ X_3 = x_3 - \lambda_2x_1, \\ P_1 = y_2 - x_1x_3 + \dfrac{1}{2}\lambda_2x_1^2,\ P_2 = y_3 - \dfrac{1}{2}\lambda_1x_1^2,\ P_3 = x_2 - \lambda_1x_1, \end{cases}$$

we obtain a canonical coordinate $(X_1, X_2, X_3, Z, P_1, P_2, P_3)$ of $J = W(Y)/\mathrm{Ch}\,(C^*)$.

Conversely we calculate

$$\begin{cases} x_1 = a, \lambda_1 = X_1, \lambda_2 = X_2, x_3 = X_3 + aX_2, x_2 = P_3 + aX_1, \\ y_3 = P_2 + \dfrac{1}{2}a^2X_1, y_2 = P_1 + aX_3 + \dfrac{1}{2}a^2X_2 \end{cases}$$

Hence we have

$$\begin{cases} \bar{\theta}_2 = dy_2 - x_3dx_1 = dP_1 + adX_3 + \dfrac{1}{2}a^2dX_2 \\ \bar{\theta}_3 = dy_3 - x_1dx_2 = dP_2 - \dfrac{1}{2}a^2dX_1 - adP_3 \end{cases} \tag{8.2}$$

Substituting $dP_i = P_{i1}dP_1 + P_{i2}dP_2 + P_{i3}dP_3$ for $i = 1, 2, 3$ into (8.2), we obtain the following description of the model equation of type $(B_3, \{\alpha_3\})$;

$$P_{11} = 0,\ P_{12} = -\frac{1}{2}P_{13}^2,\ P_{22} = P_{13}^2 \cdot P_{33},\ P_{23} = -P_{13} \cdot P_{33}.$$

More generally, our second example is of type $(BD_\ell, \{\alpha_3\})$ for $\ell \geqq 4$. Explicitly, put

$$S = \begin{pmatrix} 0 & 0 & K \\ 0 & E_p & 0 \\ K & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where $K = K_3$ is the anti-diagonal unit 3×3 matrix and E_p is the unit $p \times p$ matrix. On $\hat{U} = \mathbb{R}^{p+6}$, we give an inner product $(,)$ by $(\boldsymbol{x}, \boldsymbol{y}) = {}^t\boldsymbol{x}\, S\boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{p+6}$. Then the signature of $(\hat{U}, (\,,\,))$ is $(p+3, 3)$. Moreover, on $U = \mathbb{R}^p$, we have an inner product $(\,,\,)$ by $(\mathbf{a}, \mathbf{b}) = {}^t\mathbf{a}\mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p$.

We put

$$\mathfrak{g} = \mathfrak{o}(\hat{U}) = \mathfrak{o}(p+3, 3) = \{\ X \in \mathfrak{gl}(p+6, \mathbb{R}) \mid {}^tXS + SX = 0\}.$$

We will introduce the gradation of $\mathfrak{g} = \mathfrak{o}(\hat{U})$ by subdividing $X \in \mathfrak{g}$ as follows:

$$\begin{array}{c} \\ 3 \\ p \\ 3 \end{array} \!\! \begin{array}{c} \begin{array}{ccc} 3 & p & 3 \end{array} \\ \begin{pmatrix} A & -\hat{F} & D \\ B & G & F \\ C & -\hat{B} & -A' \end{pmatrix} \end{array},$$

where $C = -C'$, $D = -D'$, $G \in \mathfrak{o}(p)$ $\hat{B} = K^tB$ and $\hat{F} = K^tF$. Here we write $Y' = K\,{}^tY\,K \in M(3,3)$ for $Y \in M(3,3)$. Y' is the "transposed" matrix of Y with respect to the anti-diagonal line. Then the Lie algebra $\mathfrak{g}$ has the gradation

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$$

where

$$\mathfrak{g}_{-2} = \langle C \rangle,\ \mathfrak{g}_{-1} = \langle B \rangle,\ \mathfrak{g}_0 = \langle A \rangle \oplus \langle G \rangle,\ \mathfrak{g}_1 = \langle F \rangle,\ \mathfrak{g}_2 = \langle D \rangle.$$

and $\dim \mathfrak{g}_{-2} = \dim \mathfrak{g}_2 = 3$, $\dim \mathfrak{g}_{-1} = \dim \mathfrak{g}_1 = 3p$. Precisely, this gradation is of type $(D_4, \{\alpha_3, \alpha_4\})$ when $p = 2$, of type $(B_\ell, \{\alpha_3\})$ when $p = 2\ell - 5 \geqq 1$ and of type $(D_\ell, \{\alpha_3\})$ when $p = 2\ell - 6 \geqq 4$. The structure of $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ can be described as follows (see §4 [12]); Let V be a vector space of dimension 3 and U be a vector space with the inner product $(\,,\,)$ of dimension p. Then $\mathfrak{m}$ is isomorphic to $\mathfrak{m}^1(U, V)$, where

$$\mathfrak{m}^1(U, V) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, \quad \mathfrak{g}_{-2} = \wedge^2 V, \mathfrak{g}_{-1} = U \otimes V.$$

The bracket product is defined by

$$[u_1 \otimes v_1, u_2 \otimes v_2] = (u_1, u_2) v_1 \wedge v_2 \qquad \text{for} \quad u_1, u_2 \in U, v_1, v_2 \in V.$$

Moreover the standard differential system $(M(\mathfrak{m}), D_\mathfrak{m})$ of type $\mathfrak{m}$ in this case is given as follows; $M(\mathfrak{m}) = \mathbb{R}^{3p+3}$ is endowed with a coordinate $(x_1^\alpha, x_2^\alpha, x_3^\alpha, y_1, y_2, y_3)$ $(1 \leqq \alpha \leqq p)$ such that $D_\mathfrak{m}$ is given by

$$D_\mathfrak{m} = \{\bar{\theta}_1 = \bar{\theta}_2 = \bar{\theta}_3 = 0\ \},$$

where

$$\bar{\theta}_1 = dy_1 - \sum_{\alpha=1}^p x_2^\alpha dx_3^\alpha, \quad \bar{\theta}_2 = dy_2 - \sum_{\alpha=1}^p x_3^\alpha dx_1^\alpha \quad \text{and} \quad \bar{\theta}_3 = dy_3 - \sum_{\alpha=1}^p x_1^\alpha dx_2^\alpha.$$

Thus the symbol algebra $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathfrak{m}^1(U, V)$ is described by

$$(8.3) \qquad \begin{cases} d\theta_1 \equiv \displaystyle\sum_{\alpha=1}^p \omega_3^\alpha \wedge \omega_2^\alpha \\ d\theta_2 \equiv \displaystyle\sum_{\alpha=1}^p \omega_1^\alpha \wedge \omega_3^\alpha \qquad (\text{mod} \quad \theta_1, \theta_2, \theta_3) \\ d\theta_3 \equiv \displaystyle\sum_{\alpha=1}^p \omega_2^\alpha \wedge \omega_1^\alpha \end{cases}$$

In fact, taking the dual basis $\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, X_1^\alpha, X_2^\alpha, X_3^\alpha\ (1 \leqq \alpha \leqq p)\}$ of the coframe $\{\bar\theta_1, \bar\theta_2, \bar\theta_3, dx_1^\alpha, dx_2^\alpha, dx_3^\alpha\ (1 \leqq \alpha \leqq p)\}$ on $M(\mathfrak{m})$, we have

$$X_1^\alpha = \frac{\partial}{\partial x_1^\alpha} + x_3^\alpha \frac{\partial}{\partial y_2}, \quad X_2^\alpha = \frac{\partial}{\partial x_2^\alpha} + x_1^\alpha \frac{\partial}{\partial y_3}, \quad \text{and} \quad X_3^\alpha = \frac{\partial}{\partial x_3^\alpha} + x_2^\alpha \frac{\partial}{\partial y_1}.$$

Thus $\{X_1^\alpha, X_2^\alpha, X_3^\alpha\ (1 \leqq \alpha \leqq p)\}$ constitutes a free basis of the sections $\Gamma(D_\mathfrak{m})$ of $D_\mathfrak{m}$, and we obtain

$$[X_1^\alpha, X_2^\beta] = \delta_\beta^\alpha \frac{\partial}{\partial y_3},\ [X_2^\alpha, X_3^\beta] = \delta_\beta^\alpha \frac{\partial}{\partial y_1},\ [X_3^\alpha, X_1^\beta] = \delta_\beta^\alpha \frac{\partial}{\partial y_2},\ [X_i^\alpha, X_i^\beta] = 0 \quad (i = 1, 2, 3),$$

for $1 \leqq \alpha, \beta \leqq p$. Hence $\mathfrak{m}$ is isomorphic to $\mathfrak{m}^1(U, V)$.

Let (Y, D_N) be a regular differential system of type $\mathfrak{m}$ such that D_N is locally defined by

$$D_N = \{\theta_1 = \theta_2 = \theta_3 = 0\}.$$

Here $\{\theta_1, \theta_2, \theta_3, \omega_1^\alpha, \omega_2^\alpha, \omega_3^\alpha (1 \leqq \alpha \leqq p)\}$ forms a coframe on Y satisfying (8.3). Then, putting $\varpi = \theta_1 + \lambda_1\theta_2 + \lambda_2\theta_3$, we calculate

$$\begin{aligned} d\varpi &\equiv \sum_{\alpha=1}^p \omega_3^\alpha \wedge \omega_2^\alpha + \lambda_1 \sum_{\alpha=1}^p \omega_1^\alpha \wedge \omega_3^\alpha + \lambda_2 \sum_{\alpha=1}^p \omega_2^\alpha \wedge \omega_1^\alpha + (d\lambda_1 + \gamma_1) \wedge \theta_2 + (d\lambda_2 + \gamma_2) \wedge \theta_3 \\ &\equiv \sum_{\alpha=1}^p (\omega_3^\alpha - \lambda_2\omega_1^\alpha) \wedge (\omega_2^\alpha - \lambda_1\omega_1^\alpha) + (d\lambda_1 + \gamma_1) \wedge \theta_2 + (d\lambda_2 + \gamma_2) \wedge \theta_3 \\ &\equiv \sum_{\alpha=1}^p \tilde\omega_3^\alpha \wedge \tilde\omega_2^\alpha + (d\lambda_1 + \gamma_1) \wedge \theta_2 + (d\lambda_2 + \gamma_2) \wedge \theta_3 \pmod{\varpi} \end{aligned}$$

for some 1-forms γ_1, γ_2 on Y, where we put $\tilde\omega_2^\alpha = \omega_2^\alpha - \lambda_1\omega_1^\alpha$ and $\tilde\omega_3^\alpha = \omega_3^\alpha - \lambda_2\omega_1^\alpha$ for $\alpha = 1, \ldots, p$. Thus, by symmetry in the indices $1, 2, 3$ in (8.3), we see

$$W(Y) = \hat W(Y) = \bigcup_{y \in Y} \tilde W_y, \quad \tilde W_y = \{w \in \mathrm{Gr}(T_y(Y), 3p{+}2) \mid w \supset D_N(y)\} \cong \mathbb{P}(T_y(Y)/D_N(y)) = \mathbb{P}^2$$

and we have on $W(Y)$,

$$C_Y^* = \{\varpi = o\},\ N_Y = \{\varpi = \theta_2 = \theta_3 = o\},\ N_Y^\perp = \{\varpi = \theta_2 = \theta_3 = \tilde\omega_2^\alpha = \tilde\omega_3^\alpha = o\ (1 \leqq \alpha \leqq p)\}.$$

Here $r = 2$, $t = p$, $n = r + t = p + 2$ and $\dim W(Y) = 3p + 5$. Moreover we have

$$\mathrm{Ch}\,(C_Y^*) = \{\varpi = \theta_2 = \theta_3 = d\lambda_1 + \gamma_1 = d\lambda_2 + \gamma_2 = \tilde\omega_2^\alpha = \tilde\omega_3^\alpha = o\ (1 \leqq \alpha \leqq p)\}.$$

Now we consider the Lagrange Grassmann bundle $R(Y) = R(W(Y))$ over $(W(Y); C_Y^*, N_Y)$:

$$R(Y) = \bigcup_{w \in W(Y)} R_w, \quad R_w = \{\hat v \subset N_Y(w) \mid d\varpi\ |_{\hat v} = 0, \quad \hat v \text{ is maximal}\}.$$

From $N_Y^\perp(w) \subset \hat v \subset N_Y(w)$ and $d\varpi\ |_{\hat v} = 0$, if $\tilde\omega_2^1 \wedge \cdots \wedge \tilde\omega_2^p\ |_{\hat v} \neq 0$, it follows that

$$\hat v = \{X \in N_Y(w) \mid \varpi_\alpha(X) = 0 \quad (1 \leqq \alpha \leqq p)\},$$

where $\varpi_\alpha = \tilde\omega_3^\alpha - \sum_{\beta=1}^p p_{\alpha\beta}^*\tilde\omega_2^\beta$ for $1 \leqq \alpha \leqq p$ and $p_{\alpha\beta}^* = p_{\beta\alpha}^*$ $(1 \leqq \alpha, \beta \leqq p)$. For these $\hat v \in R_w$, we claim

$$A(\hat v) = \{0\} \quad \textit{for} \quad \hat v = \{\varpi = \theta_2 = \theta_3 = \varpi_\alpha = 0\ (1 \leqq \alpha \leqq p)\}.$$

In fact, first we have

$$d\theta_2 \equiv \sum_{\alpha=1}^p \omega_1^\alpha \wedge \tilde\omega_3^\alpha, \quad d\theta_3 \equiv \sum_{\alpha=1}^p \tilde\omega_2^\alpha \wedge \omega_1^\alpha \pmod{\varpi, \theta_2, \theta_3}.$$

Thus we get

$$d\varpi \equiv 0, \quad d\theta_2 \equiv \sum_{\alpha,\beta=1}^{p} p^*_{\alpha\beta}\omega_1^\alpha \wedge \tilde{\omega}_2^\beta, \quad d\theta_3 \equiv \sum_{\alpha=1}^{p} \tilde{\omega}_2^\alpha \wedge \omega_1^\alpha \pmod{(\hat{v})^\perp},$$

Hence, for $X \in S(N_Y)(w) = \mathrm{Ch}\,(C_Y^*)(w)$, we have

$$X \rfloor d\theta_3 \equiv -\sum_{\alpha=1}^{p} \omega_1^\alpha(X)\tilde{\omega}_2^\alpha \qquad \pmod{(\hat{v})^\perp},$$

Then, from Lemma 4.2 (2), we obtain

$$\begin{aligned} A(\hat{v}) &= \{X \in \mathrm{Ch}\,(C_Y^*)(w) \mid \omega_1^\alpha(X) = 0 \quad (1 \leqq \alpha \leqq p)\} \\ &= \mathrm{Ch}\,(C_Y^*)(w) \cap \mathrm{Ch}\,(N_Y)(w) = \{0\}. \end{aligned}$$

Hence $(R(Y); D_Y^1, D_Y^2)$ is a PD manifold of second order on an open subset of $R(Y)$, when (Y, D_N) is a regular differential system of type $\mathfrak{m}$, where $\mathfrak{m}$ is the negative part of the simple graded Lie algebra of type $(BD_\ell, \{\alpha_3\})$.

To obtain an explicit description of the model equation in this case, we calculate

$$\begin{aligned} \varpi &= dy_1 - \sum_{\alpha=1}^{p} x_2^\alpha dx_3^\alpha + \lambda_1(dy_2 - \sum_{\alpha=1}^{p} x_3^\alpha dx_1^\alpha) + \lambda_2(dy_3 - \sum_{\alpha=1}^{p} x_1^\alpha dx_2^\alpha) \\ &= d(y_1 + \lambda_1 y_2 + \lambda_2 y_3) - y_2 d\lambda_1 - y_3 d\lambda_2 - \sum_{\alpha=1}^{p} x_2^\alpha dx_3^\alpha - \lambda_1 \sum_{\alpha=1}^{p} x_3^\alpha dx_1^\alpha - \lambda_2 \sum_{\alpha=1}^{p} x_1^\alpha dx_2^\alpha \\ &= d(y_1 + \lambda_1 y_2 + \lambda_2 y_3 - \lambda_1 \sum_{\alpha=1}^{p} x_1^\alpha x_3^\alpha) - (y_2 - \sum_{\alpha=1}^{p} x_1^\alpha x_3^\alpha) d\lambda_1 - y_3 d\lambda_2 - \lambda_2 \sum_{\alpha=1}^{p} x_1^\alpha dx_2^\alpha \\ &\qquad\qquad - \sum_{\alpha=1}^{p} (x_2^\alpha - \lambda_1 x_1^\alpha) dx_3^\alpha \\ &= d(y_1 + \lambda_1 y_2 + \lambda_2 y_3 - \lambda_1 \sum_{\alpha=1}^{p} x_1^\alpha x_3^\alpha - \lambda_2 \sum_{\alpha=1}^{p} x_1^\alpha x_2^\alpha) - (y_2 - \sum_{\alpha=1}^{p} x_1^\alpha x_3^\alpha) d\lambda_1 \\ &\qquad - (y_3 - \sum_{\alpha=1}^{p} x_1^\alpha x_2^\alpha) d\lambda_2 - \sum_{\alpha=1}^{p} (x_2^\alpha - \lambda_1 x_1^\alpha)(dx_3^\alpha - \lambda_2 dx_1^\alpha) + \lambda_1\lambda_2 \sum_{\alpha=1}^{p} x_1^\alpha dx_1^\alpha \\ &= d(y_1 + \lambda_1 y_2 + \lambda_2 y_3 - \lambda_1 \sum_{\alpha=1}^{p} x_1^\alpha x_3^\alpha - \lambda_2 \sum_{\alpha=1}^{p} x_1^\alpha x_2^\alpha) - (y_2 - \sum_{\alpha=1}^{p} x_1^\alpha x_3^\alpha) d\lambda_1 \\ &\qquad - (y_3 - \lambda_1 \sum_{\alpha=1}^{p} (x_1^\alpha)^2) d\lambda_2 - \sum_{\alpha=1}^{p} (x_2^\alpha - \lambda_1 x_1^\alpha) d(x_3^\alpha - \lambda_2 x_1^\alpha) + \frac{1}{2}\lambda_1\lambda_2 \sum_{\alpha=1}^{p} d(x_1^\alpha)^2 \\ &= dZ - P_1 dX_1 - P_2 dX_2 - \sum_{\alpha=1}^{p} P_{\alpha+2} dX_{\alpha+2} \end{aligned}$$

Thus, putting

$$\begin{cases} Z = y_1 + \lambda_1 y_2 + \lambda_2 y_3 - \lambda_1 \sum_{\alpha=1}^{p} x_1^\alpha x_3^\alpha - \lambda_2 \sum_{\alpha=1}^{p} x_1^\alpha x_2^\alpha + \frac{1}{2}\lambda_1\lambda_2 \sum_{\alpha=1}^{p} (x_1^\alpha)^2, \\ X_1 = \lambda_1,\ X_2 = \lambda_2,\ X_{\alpha+2} = x_3^\alpha - \lambda_2 x_1^\alpha, \\ P_1 = y_2 - \sum_{\alpha=1}^{p} x_1^\alpha x_3^\alpha + \frac{1}{2}\lambda_2 \sum_{\alpha=1}^{p} (x_1^\alpha)^2,\ P_2 = y_3 - \frac{1}{2}\lambda_1 \sum_{\alpha=1}^{p} (x_1^\alpha)^2,\ P_{\alpha+2} = x_2^\alpha - \lambda_1 x_1^\alpha, \end{cases}$$

we obtain a canonical coordinate $(X_1, \ldots, X_{p+2}, Z, P_1, \ldots, P_{p+2})$ of $J = W(Y)/\mathrm{Ch}\,(C_Y^*)$.

Conversely we calculate

$$\begin{cases} x_1^\alpha = a_\alpha, \lambda_1 = X_1, \lambda_2 = X_2, x_3^\alpha = X_{\alpha+2} + a_\alpha X_2, x_2^\alpha = P_{\alpha+2} + a_\alpha X_1, \\ y_3 = P_2 + \frac{1}{2}\sum_{\alpha=1}^{p} (a_\alpha)^2 X_1, y_2 = P_1 + \sum_{\alpha=1}^{p} a_\alpha X_{\alpha+2} + \frac{1}{2}\sum_{\alpha=1}^{p} (a_\alpha)^2 X_2 \end{cases}$$

Hence we have

$$\begin{cases} \bar\theta_2 = dy_2 - \sum_{\alpha=1}^{p} x_3^\alpha dx_1^\alpha = dP_1 + \sum_{\alpha=1}^{p} a_\alpha dX_{\alpha+2} + \frac{1}{2}\sum_{\alpha=1}^{p} (a_\alpha)^2 dX_2 \\ \bar\theta_3 = dy_3 - \sum_{\alpha=1}^{p} x_1^\alpha dx_2^\alpha = dP_2 - \frac{1}{2}\sum_{\alpha=1}^{p} (a_\alpha)^2 dX_1 - \sum_{\alpha=1}^{p} a_\alpha dP_{\alpha+2} \end{cases} \tag{8.4}$$

Substituting $dP_i = \sum_{j=1}^{p+2} P_{ij} dX_j$ for $i = 1, \ldots, p+2$ into (8.4), we obtain the following description of the model equation of type $(BD_\ell, \{\alpha_3\})$;

$$P_{11} = 0, \quad P_{12} = -\frac{1}{2}\sum_{\alpha=1}^{p} (P_{1\ \alpha+2})^2, \quad P_{2i} = -\sum_{\alpha=1}^{p} P_{1\ \alpha+2} \cdot P_{i\ \alpha+2} \quad (i = 2, \ldots, p+2).$$

8.3. **General Case.** In this subsection, we will first consider the third reduction step in Theorem 5.2 in general and will discuss the reconstruction procedure of this third step. As in Theorem 5.2, let $(R; D^1, D^2)$ be a PD manifold of second order, which is regular of type $\mathfrak{s}$. Assume that there exists a $G_0(\mathfrak{s})$-invariant subspace E of V of dimension r satisfying $\mathfrak{f}^\perp \subset E \otimes_S V$ and $\dim \hat{E} = s > 0$, where $\hat{E} = \{v \in E \mid v \circledcirc E \subset \mathfrak{f}^\perp\}$. Moreover assume the following two conditions:

(i) $C(E)$ is completely integrable (when $r = n-1$, assume further $\mathrm{rank}\,\mathrm{Ch}\,(D^2) < s$).

(ii) $C(E) \subset \mathrm{Ch}\,(\tilde{N}^*)$.

Let $(W; C^*, N)$ be the associated IG manifolds of corank r of $(R; D^1, D^2)$. Assume that W is regular with respect to $\mathrm{Ch}\,(N)$ and let $(Y; D_N^*, D_N)$ be the leaf space, where $\beta : W \to Y = W/\mathrm{Ch}\,(N)$ is the projection, $\beta^{-1}(D_N^*) = N^*$ and $\beta^{-1}(D_N) = N$. Then, as in §5.3, we have $\mathrm{Ch}\,(D_N^*) \subset D_N$ and $\mathrm{Ch}\,(D_N) = \{0\}$. Now we further assume that $\mathrm{Ch}\,(D_N^*)$ is a non-trivial subbundle of D_N such that Y is regular with respect to $\mathrm{Ch}\,(D_N^*)$, i.e., the space $Z = Y/\mathrm{Ch}\,(D_N^*)$ of leaves of this foliation is a manifold and that each fibre of the projection $\gamma : Y \to Z$ is connected and γ is a submersion. Then there exists a differential system F on Z of codimension $s+1$ such that $\gamma_*^{-1}(F) = D_N^*$ and $\mathrm{Ch}\,(F)$ is trivial. We consider here a Grassmann bundle $\tilde{Y}(Z)$ over Z consisting of subspaces of codimension $r-s$ in each fibre $F(z)$ of (Z, F).

$$\tilde{Y}(Z) = \bigcup_{z \in Z} \tilde{Y}_z, \qquad \tilde{Y}_z = \mathrm{Gr}\,(F(z), t_0),$$

where $t_0 + r - s = \operatorname{rank} F$. Then we have two differential systems F_N^*, F_N on $\tilde{Y}(Z)$ given by

$$F_N^*(\tilde{y}) = \xi_*^{-1}(F(z)) \supset F_N(\tilde{y}) = \xi_*^{-1}(\tilde{y}),$$

for each $\tilde{y} \in \tilde{Y}(Z)$ and $z = \xi(\tilde{y})$, where $\xi : \tilde{Y}(Z) \to Z$ is the projection. Hence F_N^* and F_N are differential systems on $\tilde{Y}(Z)$ of codimension $s+1$ and $r+1$ respectively. In this situation, we have a map κ_3 of Y into $\tilde{Y}(Z)$ given by

$$\kappa_3(y) = \gamma_*(D_N(y)) \subset F(z) = \gamma_*(D_N^*(y)),$$

for each $y \in Y$ and $z = \gamma(y)$. By the Realization Lemma for (Y, D_N, γ, Z), κ_3 is a map of constant rank such that

$$\operatorname{Ker}(\kappa_3)_* = \operatorname{Ch}(D_N) \cap \operatorname{Ker}\gamma_* = \{0\}.$$

Thus κ_3 is an immersion. Moreover we have

$$(\kappa_3)_*^{-1}(F_N^*) = D_N^* \qquad \text{and} \qquad (\kappa_3)_*^{-1}(F_N) = D_N.$$

Namely κ_3 is an immersion of $(Y; D_N^*, D_N)$ into $(\tilde{Y}(Z); F_N^*, F_N)$. Hence $(Y; D_N^*, D_N)$ can be constructed from (Z, F), at least locally, as a submanifold of $(\tilde{Y}(Z); F_N^*, F_N)$.

Furthermore, starting from $\gamma : Y \to Z$, we have the following general picture: Starting from (Z, F), by the construction in §7.1, we have an IG manifold $(W(Z); C_Z^*, N_Z)$ of corank s and the Lagrange Grassmann bundle $R(Z) = R(W(Z))$ over $(W(Z); C_Z^*, N_Z)$ as follows. Put

$$\hat{W}(Z) = \bigcup_{z\in Z} \hat{W}_z, \qquad \hat{W}_z = \{w \in \operatorname{Gr}(T_z(Z), \hat{m}-1) \mid w \supset F(z)\} \cong \mathbb{P}(T_z(Z)/F(z)) = \mathbb{P}^s,$$

where $\hat{m} = \dim Z$, C_Z^* is the canonical system obtained by the Grassmannian construction and N_Z is the lift of F. Moreover we put

$$W(Z) = \{w \in \hat{W}(Z) \mid \operatorname{corank}\operatorname{Ch}(C_Z^*)(w) = 2n+1\},$$

where $n = s + t$ and t is the Engel half-rank of F. Here we note that t is also the Engel half-rank of D_N^*. Then, by Proposition 7.1, $(W(Z); C_Z^*, N_Z)$ is an IG manifold of corank s. Let $(R(Z); D_Z^1, D_Z^2)$ be the Lagrange Grassmann bundle over $(W(Z); C_Z^*, N_Z)$, i.e.,

$$R(Z) = \bigcup_{w\in W(Z)} R_w, \qquad R_W = \{\hat{v} \subset N_Z(w) \mid \gamma_w\,|_{\hat{v}} = 0, \quad \hat{v} \text{ is maximal}\},$$

where D_Z^2 is the canonical system on $R(Z)$ and D_Z^1 and $\hat{N}_Z$ are the lifts of C_Z^* and N_Z respectively. Assume that $W(Z)$ is regular with respect to $\operatorname{Ch}(C_Z^*)$. Then we have a contact manifold (J, C) such that $J = W(Z)/\operatorname{Ch}(C_Z^*)$ and $C_Z^* = q_*^{-1}(C)$, where $q : W(Z) \to J$ is the projection. Here we have a map $\hat{\zeta} : R(Z) \to L(J)$ given by $\hat{\zeta}(\hat{v}) = q_*(\hat{v})$. $\hat{\zeta}$ is an immersion when $(R(Z); D_Z^1, D_Z^2)$ is a PD manifold of second order.

On the other hand, starting from $(Y; D_N^*, D_N)$, by the construction in §7.2, we have an IG manifold $(W(Y); C_Y^*, N_Y)$ of corank r and the Lagrange Grassmann bundle $R(Y) = R(W(Y))$ over $(W(Y); C_Y^*.N_Y)$. In case $(Y; D_N^*, D_N)$ is obtained from a PD manifold $(R; D^1, D^2)$ of second order of type $\mathfrak{s}$ as above, we have local isomorphisms $\kappa_2 : (W; C^*, N) \to (W(Y), C_Y^*, N_Y)$ and $\kappa_1 : (R; D^1, D^2) \to (R(W); D_W^1, D_W^2)$ given by $\kappa_2(w) = \beta_*(C^*(w))$ and $\kappa_1(v) = \eta_*(D^2(v))$ respectively (see §5.1 and §5.3), so that we have a local isomorphism $\hat{\kappa}_1 = \hat{\kappa}_2 \cdot \kappa_1 : (R; D^1, D^2) \to (R(Y); D_Y^1, D_Y^2)$, where the local isomorphism $\hat{\kappa}_2 : (R(W); D_W^1, D_W^2) \to (R(Y); D_Y^1, D_Y^2)$ is induced by κ_2. In this situation, $\gamma : Y \to Z$ induces the following commutative diagram:

$$\begin{array}{ccc} R(Y) & \xrightarrow{\gamma_2} & R(Z) \\ {\scriptstyle \tau_1}\downarrow & & \downarrow{\scriptstyle \tau_2} \\ W(Y) & \xrightarrow{\gamma_1} & W(Z) \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \nu} \\ Y & \xrightarrow{\gamma} & Z \end{array}$$

In fact, for $w \in W(Y)$, w is a hyperplane in $T_y(Y)$ containing $D_N^*(y)$, where $y = \mu(w)$. Then, since $D_N^*(y) \supset \mathrm{Ch}\,(D_N^*)(y) = \mathrm{Ker}\,\gamma_*(y)$ and $\gamma_*(D_N^*(y)) = F(z)$, $\gamma_*(w)$ is a hyperplane in $T_z(Z)$ containing $F(z)$, where $z = \gamma(y)$. Hence $\gamma_1 : W(Y) \to W(Z)$ is defined by $\gamma_1(w) = \gamma_*(w) \in W(Z)$ for $w \in W(Y)$. Actually $\gamma_1 : \tilde{W}(Y) \to \hat{W}(Z)$ is a $\mathbb{P}^s$-bundle homomorphism. Passing to the tangent map, since $\mu_*^{-1}(w) = C_Y^*(w)$ and $\nu_*^{-1}(\gamma_1(w)) = C_Z^*(\gamma_1(w))$, we have $(\gamma_1)_*(C_Y^*(w)) = C_Z^*(\gamma_1(w))$ and $(\gamma_1)_*(\mathrm{Ch}\,(C_Y^*)(w)) = \mathrm{Ch}\,(C_Z^*)(\gamma_1(w))$ for $w \in W(Y)$. Moreover, since $\mu_*^{-1}(D_N^*(y)) = N_Y^*(w)$ and $\nu_*^{-1}(F(\gamma(y))) = N_Z(\gamma_1(w))$, we have $(\gamma_1)_*(N_Y^*(w)) = N_Z(\gamma_1(w))$. Hence γ_1 naturally induces the map $\gamma_2 : R(Y) \to R(Z)$ by $\gamma_2(\hat{v}) = (\gamma_1)_*(\hat{v})$ for $\hat{v} \in R(Y)$. In fact, since $C_Z^* = q_*^{-1}(C)$ for the projection $q : W(Z) \to J$, we have $C_Y^* = (q \cdot \gamma_1)_*^{-1}(C)$ and $\mathrm{Ker}(q \cdot \gamma_1)_* = \mathrm{Ch}\,(C_Y^*)$, which implies that $J = W(Y)/\mathrm{Ch}\,(C_Y^*)$ at least locally. Thus we see that, for a subspace $\hat{v}$ of $T_w(W(Y))$, $\hat{v} \in R(Y)$ if and only if $\mathrm{Ch}\,(C_Y^*)(w) \subset \hat{v} \subset N_Y(w)$ and $(q \cdot \gamma_1)_*(\hat{v})$ is a legendrian subspace of (J, C). Similarly, for a subspace $\tilde{v}$ of $T_{\hat{w}}(W(Z))$, $\tilde{v} \in R(Z)$ if and only if $\mathrm{Ch}\,(C_Z^*)(\hat{w}) \subset \tilde{v} \subset N_Z(\hat{w})$ and $q_*(\tilde{v})$ is a legendrian subspace of (J, C). Hence, from $(\gamma_1)_*(N_Y(w)) \subset (\gamma_1)_*(N_Y^*(w)) = N_Z(\gamma_1(w))$, we have $\gamma_2(\hat{v}) = (\gamma_1)_*(\hat{v}) \in R(Z)$. Here we observe that both $(q \cdot \gamma_1)_*(N_Y(w))$ and $q_*(N_Z(\gamma_1(w)))$ are involutive subspaces of $C(q \cdot \gamma_1(w))$ of codimension r and s respectively such that $(q \cdot \gamma_1)_*(N_Y(w)) \subset q_*(N_Z(\gamma_1(w)))$.

Now we will give examples of constructions in §7.2. Our starting point here is a regular differential system (Z, D) of type $\mathfrak{m}_3$, where $\mathfrak{m}_3 = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the negative part of the simple graded Lie algebra of type $(BD_\ell, \{\alpha_1, \alpha_3\})$, which is discussed in §6.2 [27] related with the First Reduction Theorem. We will show that, starting from (Z, F), where $F = \partial D$ is the derived system of D, we can construct an involutive system $(R(Y); D_Y^1, D_Y^2)$ of second order of codimensin 2 in the above picture, by suitably constructing $(Y; D_N^*, D_N)$ over (Z, F).

For this purpose, let us first describe the structure of the symbol algebra $\mathfrak{m}_3$. Explicitly, as in §8.1, put

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & K & 0 \\ 0 & 0 & E_p & 0 & 0 \\ 0 & K & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $K = K_2$ is the anti-diagonal unit 2×2 matrix and E_p is the unit $p \times p$ matrix. We put

$$\mathfrak{g} = \mathfrak{o}(p+3, 3) = \{\ X \in \mathfrak{gl}(p+6, \mathbb{R}) \mid {}^tXS + SX = 0\}.$$

We will introduce the gradation of $\mathfrak{g}$ by subdividing $X \in \mathfrak{g}$ as follows:

$$\begin{array}{c} \\ 1 \\ 2 \\ p \\ 2 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & p & 2 & 1 \end{array} \\ \begin{pmatrix} a & -\eta_4' & -{}^t\xi_2 & -\eta_3' & 0 \\ \eta_1 & A & -\hat{B}_2 & C_2 & \eta_3 \\ \xi_1 & B_1 & G & B_2 & \xi_2 \\ \eta_2 & C_1 & -\hat{B}_1 & -A' & \eta_4 \\ 0 & -\eta_2' & -{}^t\xi_1 & -\eta_1' & -a \end{pmatrix} \end{array},$$

where $a \in \mathbb{R}$, $\xi_i \in \mathbb{R}^p$ $(i = 1, 2)$, $\eta_i \in \mathbb{R}^2$ $(i = 1, 2, 3, 4)$, $\eta_i' = (a_2, a_1)$ for $\eta_i = {}^t(a_1, a_2)$, $C_i = -C_i'$ $(i = 1, 2)$, $G \in \mathfrak{o}(p)$ and $\hat{B}_i = {}^t({}^t\mathfrak{b}_2, {}^t\mathfrak{b}_1)$ for $B_i = (\mathfrak{b}_1, \mathfrak{b}_2)$ $(i = 1, 2)$, where $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathbb{R}^p$. Here we write $Y' = K\,{}^tY\,K \in M(2, 2)$ for $Y \in M(2, 2)$. Y' is the "transposed" matrix of Y with respect to the anti-diagonal line. Thus $C_i = \begin{pmatrix} c_i & 0 \\ 0 & -c_i \end{pmatrix}$ for $c_i \in \mathbb{R}$. Then the Lie algebra $\mathfrak{g}$ has the gradation

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

where

$$\mathfrak{g}_{-3} = \langle \eta_2 \rangle,\ \mathfrak{g}_{-2} = \langle \xi_1 \rangle \oplus \langle C_1 \rangle,\ \mathfrak{g}_{-1} = \langle \eta_1 \rangle \oplus \langle B_1 \rangle,\ \mathfrak{g}_0 = \langle a \rangle \oplus \langle A \rangle \oplus \langle G \rangle,$$
$$\mathfrak{g}_3 = \langle \eta_3 \rangle,\ \mathfrak{g}_2 = \langle \xi_2 \rangle \oplus \langle C_2 \rangle,\ \mathfrak{g}_1 = \langle \eta_4 \rangle \oplus \langle B_2 \rangle,$$

and $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_3 = 2$, $\dim \mathfrak{g}_{-2} = \dim \mathfrak{g}_2 = p{+}1$, $\dim \mathfrak{g}_{-1} = \dim \mathfrak{g}_1 = 2(p{+}1)$. Precisely, this gradation is of type $(D_4, \{\alpha_1, \alpha_3, \alpha_4\})$ when $p = 2$, of type $(B_\ell, \{\alpha_1, \alpha_3\})$ when $p = 2\ell{-}5 \geqq 1$ and of type $(D_\ell, \{\alpha_1, \alpha_3\})$ when $p = 2\ell{-}6 \geqq 4$. Compared with the gradation of $(BD_\ell, \{\alpha_1, \alpha_2, \alpha_3\})$ or $(D_4, \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})$;

$$\mathfrak{g} = \hat{\mathfrak{g}}_{-5} \oplus \hat{\mathfrak{g}}_{-4} \oplus \hat{\mathfrak{g}}_{-3} \oplus \hat{\mathfrak{g}}_{-2} \oplus \hat{\mathfrak{g}}_{-1} \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_1 \oplus \hat{\mathfrak{g}}_2 \oplus \hat{\mathfrak{g}}_3 \oplus \hat{\mathfrak{g}}_4 \oplus \hat{\mathfrak{g}}_5,$$

which is obtained by further subdividing above matrix $X \in \mathfrak{g}$ by 2nd and $(p{+}4)$-th intermediate lines (see §6.2 [27]), we have

$$\mathfrak{g}_{-3} = \hat{\mathfrak{g}}_{-5} \oplus \hat{\mathfrak{g}}_{-4}, \quad \mathfrak{g}_{-2} = \hat{\mathfrak{g}}_{-3} \quad \text{and} \quad \mathfrak{g}_{-1} = V_2 \oplus V_1 \subset \hat{\mathfrak{g}}_{-2} \oplus \hat{\mathfrak{g}}_{-1},$$

where $V_2 = \hat{\mathfrak{g}}_{-2}$ and $V_1 = \hat{\mathfrak{g}}_{-1} \cap \mathfrak{g}_{-1}$. Explicitly we have

$$\hat{\mathfrak{g}}_{-5} = \langle {}^t(0, a_2) \rangle, \quad \hat{\mathfrak{g}}_{-4} = \langle {}^t(a_1, 0) \rangle, \quad V_2 = \langle {}^t(0, a_2) \rangle \oplus \langle (\mathfrak{b}_1, 0) \rangle, \quad V_1 = \langle {}^t(a_1, 0) \rangle \oplus \langle (0, \mathfrak{b}_2) \rangle.$$

Hence we get

$$[V_i, V_i] = 0\ (i = 1, 2), \quad [V_1, V_2] \subset \mathfrak{g}_{-2}, \quad [\mathfrak{g}_{-2}, V_2] = \hat{\mathfrak{g}}_{-5}, \quad [\mathfrak{g}_{-2}, V_1] = \hat{\mathfrak{g}}_{-4}.$$

Actually, by matrices calculation, we obtain

$$[v_2, v_1] = (a_1\mathfrak{b}_1 - a_2\mathfrak{b}_2) \oplus \begin{pmatrix} {}^t\mathfrak{b}_2\mathfrak{b}_1 & 0 \\ 0 & -{}^t\mathfrak{b}_1\mathfrak{b}_2 \end{pmatrix} \in \mathfrak{g}_{-2}, \quad [\xi \oplus C, \eta \oplus B] = \begin{pmatrix} a_1 c + {}^t\mathfrak{b}_2\xi \\ -ca_2 + {}^t\mathfrak{b}_1\xi \end{pmatrix} \in \mathfrak{g}_{-3},$$

for $v_2 = {}^t(0, a_2) \oplus (\mathfrak{b}_1, 0) \in V_2$, $v_1 = {}^t(a_1, 0) \oplus (0, \mathfrak{b}_2) \in V_1$ and for $\xi \oplus C \in \mathfrak{g}_{-2}$, $\eta \oplus B \in \mathfrak{g}_{-1}$, where $C = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$, $\eta = {}^t(a_1, a_2)$ and $B = (\mathfrak{b}_1, \mathfrak{b}_2)$. Thus, putting,

$$Z_1 = {}^t(1, 0) \in \hat{\mathfrak{g}}_{-4} \subset \mathfrak{g}_{-3}, \quad Z_2 = {}^t(0, 1) \in \hat{\mathfrak{g}}_{-5} \subset \mathfrak{g}_{-3}, \quad Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{g}_{-2},$$

$$Y_{i+1} = \boldsymbol{e}_i \in \mathfrak{g}_{-2}, \quad X_1^1 = {}^t(1, 0), \quad X_1^{i+1} = (0, \boldsymbol{e}_i) \in V_1, \quad X_2^1 = {}^t(0, -1), \quad X_2^{i+1} = (\boldsymbol{e}_i, 0) \in V_2,$$

for $i = 1, \ldots, p$, where $\boldsymbol{e}_i$ is the vector in $\mathbb{R}^p$, whose i-th component is 1 and other components are 0. We obtain a basis $\{Z_1, Z_2\}$ of $\mathfrak{g}_{-3}$, a basis $\{Y_1, \ldots, Y_{p+1}\}$ of $\mathfrak{g}_{-2}$, a basis $\{X_1^1, \ldots, X_1^{p+1}\}$ of V_1 and a basis $\{X_2^1, \ldots, X_2^{p+1}\}$ of V_2. Moreover we have the bracket relation among these vectors;

$$Z_1 = \delta_{j_1}^{j_2} \cdot [Y_{j_1}, X_1^{j_2}], \quad Z_2 = \delta_{j_1}^{j_2} \cdot [Y_{j_1}, X_2^{j_2}] \quad Y_1 = \delta_{k_1}^{k_2} \cdot [X_2^{k_1}, X_1^{k_2}],$$
$$Y_k = [X_2^1, X_1^k] = [X_2^k, X_1^1], \quad [X_1^{j_1}, X_1^{j_2}] = [X_2^{j_1}, X_2^{j_2}] = 0,$$

for $j_1, j_2 = 1, 2, \ldots, p{+}1$, $k_1, k_2, k = 2, \ldots, p{+}1$. Hence, taking the dual basis $\{\hat{\varpi}_1, \hat{\varpi}_2, \hat{\omega}_1, \ldots, \hat{\omega}_{p+1},$ $\hat{\varpi}_1^1 \ldots, \hat{\varpi}_1^{p+1}, \hat{\varpi}_2^1, \ldots, \hat{\varpi}_2^{p+1}\}$ in $\mathfrak{m}_3^*$ of the above basis $\{Z_1, Z_2, Y_1, \ldots, Y_{p+1}, X_1^1, \ldots, X_1^{p+1}, X_2^1, \ldots,$

$X_2^{p+1}\}$ in $\mathfrak{m}_3 = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, we have the following structure equation of the Lie algebra $\mathfrak{m}_3$:

$$\begin{cases} d\hat{\varpi}_1 = \hat{\varpi}_1^1 \wedge \hat{\omega}_1 + \cdots + \hat{\varpi}_1^{p+1} \wedge \hat{\omega}_{p+1}, \\ d\hat{\varpi}_2 = \hat{\varpi}_2^1 \wedge \hat{\omega}_1 + \cdots + \hat{\varpi}_2^{p+1} \wedge \hat{\omega}_{p+1}, \\ d\hat{\omega}_1 = \hat{\varpi}_1^2 \wedge \hat{\varpi}_2^2 + \cdots + \hat{\varpi}_1^{p+1} \wedge \hat{\varpi}_2^{p+1}, \\ d\hat{\omega}_k = \hat{\varpi}_1^1 \wedge \hat{\varpi}_2^k + \hat{\varpi}_1^k \wedge \hat{\varpi}_2^1, \quad (k = 2, \dots, p+1), \\ d\hat{\varpi}_1^j = d\hat{\varpi}_2^j = 0 \quad (j = 1, 2, \dots, p+1). \end{cases}$$

These being prepared, let (Z, D) be a regular differential system of type $\mathfrak{m}_3$. Let us fix a point $z \in Z$. Then there exists a coframe $\{\varpi_0, \varpi_1, \omega_1, \dots, \omega_{p+1}, \varpi_0^1, \dots, \varpi_0^{p+1}, \varpi_1^1, \dots, \varpi_1^{p+1}\}$ defined on a neighborhood U of $z \in Z$ such that

$$F = \partial D = \{\varpi_0 = \varpi_1 = 0\}, \qquad D = \{\varpi_0 = \varpi_1 = \omega_1 = \cdots = \omega_{p+1} = 0\},$$

and that the following holds:

$$\begin{cases} d\varpi_0 \equiv \varpi_0^1 \wedge \omega_1 + \cdots + \varpi_0^{p+1} \wedge \omega_{p+1}, \\ d\varpi_1 \equiv \varpi_1^1 \wedge \omega_1 + \cdots + \varpi_1^{p+1} \wedge \omega_{p+1} \ \ (\text{mod } \varpi_0, \varpi_1, \omega_{j_1} \wedge \omega_{j_2} (1 \leqq j_1 < j_2 \leqq p+1)), \end{cases} \tag{8.5}$$

$$\begin{cases} d\omega_1 \equiv \varpi_0^2 \wedge \varpi_1^2 + \cdots + \varpi_0^{p+1} \wedge \varpi_1^{p+1}, \\ d\omega_k \equiv \varpi_0^1 \wedge \varpi_1^k + \varpi_0^k \wedge \varpi_1^1, \qquad (k = 2, \dots, p+1) \qquad (\text{mod } \varpi_0, \varpi_1, \omega_1, \dots, \omega_{p+1}). \end{cases} \tag{8.6}$$

Adjusting ϖ_0^j, ϖ_1^j (mod $\omega_1, \dots, \omega_{p+1}$), we actually have

$$\begin{cases} d\varpi_0 \equiv \varpi_0^1 \wedge \omega_1 + \cdots + \varpi_0^{p+1} \wedge \omega_{p+1}, \\ d\varpi_1 \equiv \varpi_1^1 \wedge \omega_1 + \cdots + \varpi_1^{p+1} \wedge \omega_{p+1}. \qquad (\text{mod } \varpi_0, \varpi_1) \end{cases} \tag{8.7}$$

This equation describes the structure equation of (Z, F), where $F = \partial D$. Namely (Z, F) is a regular differential system of type $\mathfrak{c}^1(p+1, 2)$. Hence D is a covariant system of (Z, F) (see Theorem 1.4 [22]). Moreover, by Theorem 6.1, $(R(Z); D_Z^1, D_Z^2)$ is a PD manifold of second order and, in fact, $\hat{\zeta}(R(Z))$ is an equation of Goursat type [25].

Now we will consider the following submanifold $\bar{Y} = \bar{Y}(Z)$ of $(\tilde{Y}(Z); F_N^*, F_N)$:

$$\bar{Y} = \bar{Y}(Z) = \bigcup_{z \in Z} \bar{Y}_z, \quad \bar{Y}_z = \{y \in \mathrm{Gr}\,(F(z), 3p+2) \mid y \supset D(z)\} \cong \mathbb{P}(F(z)/D(z)) \cong \mathbb{P}^p.$$

On $\bar{Y}$, we have a pair of differential systems D_N^* and D_N defined by

$$D_N^*(y) = \gamma_*^{-1}(F(z)) \supset D_N(y) = \gamma_*^{-1}(y),$$

where $\gamma : \bar{Y} \to Z$ is the projection and $z = \gamma(y)$. D_N^* is a lift of F and D_N is the canonical system by this Grassmannian construction and is a subbundle of D_N^* of codimension 1. D_N^* is a differential system of codimension 2 and its Engel-half rank is p+1. Since D is a covariant system of (Z, F), an isomorphism $\varphi : (Z, F) \to (Z, F)$ induces the isomorphism $\varphi_1 : (\bar{Y}; D_N^*, D_N) \to (\bar{Y}; D_N^*, D_N)$ by $\varphi_1(y) = \varphi_*(y) \in \mathrm{Gr}\,(F(\varphi(z)), 3p+2)$ for $z = \gamma(y)$. Conversely, since D_N is the canonical system by the Grassmannian construction and $\mathrm{Ker}\,\gamma_* = \mathrm{Ch}\,(D_N^*)$, an isomorphism $\Phi : (\bar{Y}; D_N^*, D_N) \to (\bar{Y}; D_N^*, D_N)$ induces the isomorphism $\varphi : (Z, F) \to (Z, F)$ such that $\Phi = \varphi_1$ (cf. the proof of Theorem 4.1 [27]).

We will show that $(\bar{Y}; D_N^*, D_N)$ satisfies the conditions $(\hat{Y}.1)$ and $(\hat{Y}.2)$ in §7.2 with $s = 1$, $r = 2$ and $t_1 = p+1$ on an open dense subset Y of $\bar{Y}$. For this purpose, let us introduce a fibre coordinate $(\mu_1, \dots, \mu_p)$ of $\gamma : \bar{Y} \to Z$ by putting

$$\varpi_2 = \omega_1 + \mu_1 \omega_2 + \cdots + \mu_p \omega_{p+1},$$

where

$$D_N^* = \{\varpi_0 = \varpi_1 = 0\}, \qquad \text{and} \quad D_N = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} \quad \text{on } \gamma^{-1}(U).$$

Here we denote the pullback on $\bar{Y}$ of 1-forms on Z by the same symbol. Then, by (8.6), we calculate

$$\begin{aligned} d\varpi_2 &= d\omega_1 + \sum_{\alpha=1}^{p}\{\mu_\alpha d\omega_{\alpha+1} + d\mu_\alpha \wedge \omega_{\alpha+1}\}, \\ &\equiv \sum_{\alpha=1}^{p}\{\varpi_0^{\alpha+1} \wedge \varpi_1^{\alpha+1} + \mu_\alpha(\varpi_0^1 \wedge \varpi_1^{\alpha+1} + \varpi_0^{\alpha+1} \wedge \varpi_1^1) + (d\mu_\alpha + \beta_\alpha) \wedge \omega_{\alpha+1}\}, \\ &\equiv \sum_{\alpha=1}^{p}\{(\varpi_0^{\alpha+1} + \mu_\alpha\varpi_0^1) \wedge (\varpi_1^{\alpha+1} + \mu_\alpha\varpi_1^1) + (d\mu_\alpha + \beta_\alpha) \wedge \omega_{\alpha+1}\} - (\sum_{\alpha=1}^{p} \mu_\alpha^2)\varpi_0^1 \wedge \varpi_1^1, \\ &(\bmod \ \ \varpi_0, \varpi_1, \varpi_2) \end{aligned} \tag{8.8}$$

where $\beta_1, \ldots, \beta_p$ are 1-forms on $\gamma^{-1}(U)$. Moreover, from (8.7), we have

$$\begin{cases} d\varpi_0 \equiv (\varpi_0^2 - \mu_1\varpi_0^1) \wedge \omega_2 + \cdots + (\varpi_0^{p+1} - \mu_p\varpi_0^1) \wedge \omega_{p+1}, \\ d\varpi_1 \equiv (\varpi_1^2 - \mu_1\varpi_1^1) \wedge \omega_2 + \cdots + (\varpi_1^{p+1} - \mu_p\varpi_1^1) \wedge \omega_{p+1}. \qquad (\bmod\ \varpi_0, \varpi_1, \varpi_2) \end{cases} \tag{8.9}$$

These three equations describes the structure equation of $(\bar{Y}, D_N)$. In particular, we obtain $\mathrm{Ch}\,(D_N) = \{0\}$ on an open dense subset satisfying $\sum_{\alpha=1}^{p} \mu_\alpha^2 \neq 0$. We put

$$Y = \{y \in \bar{Y} \mid \mathrm{Ch}\,(D_N)(y) = \{0\}\ \}.$$

Then, by (8.9), $(Y; D_N^*, D_N)$ satisfies the conditions $(\hat{Y}.1)$ and $(\hat{Y}.2)$ in §7.2 with $s = 1$, $r = 2$ and $t_1 = p + 1$.

We will consider now the IG manifold $(W(Y); C_Y^*, N_Y)$ of corank 2 given in Proposition 7.4. Let us consider

$$\varpi = \varpi_0 + \lambda\varpi_1$$

on $\hat{U} = \gamma^{-1}(U)$. Namely we consider a point $w \in \tilde{W}(Y)$ such that $w = \{\varpi = 0\} \subset T_y(Y)$, where $y = \mu(w) \in \hat{U}$. Here (λ) constitutes an inhomogeneous coordinate of the fibres of $\mu : \hat{W}(Y) \to Y$. By (8.7) and substituting $\omega_1 = \varpi_2 - \sum_{\alpha=1}^{p} \mu_\alpha\omega_{\alpha+1}$, we calculate

$$\begin{aligned} d\varpi &= d\varpi_0 + \lambda d\varpi_1 + d\lambda \wedge \varpi_1, \\ &\equiv \sum_{\alpha=1}^{p+1}(\varpi_0^\alpha + \lambda\varpi_1^\alpha) \wedge \omega_\alpha + (d\lambda + \beta) \wedge \varpi_1, \quad (\bmod\ \ \varpi) \\ &\equiv \omega_1^* \wedge (-\sum_{\alpha=1}^{p} \mu_\alpha\omega_{\alpha+1}) + \sum_{\alpha=2}^{p+1}(\varpi_0^\alpha + \lambda\varpi_1^\alpha) \wedge \omega_\alpha + \omega_1^* \wedge \varpi_2 + (d\lambda + \beta) \wedge \varpi_1, \\ &\equiv \sum_{\alpha=1}^{p}\{\varpi_0^{\alpha+1} + \lambda\varpi_1^{\alpha+1} - \mu_\alpha\omega_1^*\} \wedge \omega_{\alpha+1} + \omega_1^* \wedge \varpi_2 + (d\lambda + \beta) \wedge \varpi_1, \\ &\equiv \sum_{\alpha=1}^{p} \varpi_{\alpha+2}^* \wedge \omega_{\alpha+1} + \omega_1^* \wedge \varpi_2 + (d\lambda + \beta) \wedge \varpi_1, \qquad (\bmod\ \varpi) \end{aligned} \tag{8.10}$$

for some 1-form β on $\mu^{-1}(\hat{U})$, where we put $\omega_1^* = \varpi_0^1 + \lambda\varpi_1^1$ and $\varpi_{\alpha+1}^* = \varpi_0^\alpha + \lambda\varpi_1^\alpha - \mu_{\alpha-1}\omega_1^*$ for $\alpha = 2, \ldots, p+1$. Hence we see $W(Y) = \hat{W}(Y)$. Here, again, we denote the pullback on

$W(Y)$ of 1-forms on Y by the same symbol. By (8.10), we have

$C^*_Y = \{\varpi = 0\}, \quad \mathrm{Ch}\,(C^*_Y) = \{\varpi = \varpi_1 = \varpi_2 = \varpi^*_{\alpha+2} = d\lambda + \beta = \omega^*_1 = \omega_{\alpha+1} = 0 \ (1 \leqq \alpha \leqq p)\}.$

Moreover, by (8.9), we have

$$d\varpi_1 \equiv (\varpi^2_1 - \mu_1\varpi^1_1) \wedge \omega_2 + \cdots + (\varpi^{p+1}_1 - \mu_p\varpi^1_1) \wedge \omega_{p+1} \qquad (\mathrm{mod}\ \varpi, \varpi_1, \varpi_2).$$

Also, by (8.8), we have

$$\begin{aligned} d\varpi_2 \equiv & \sum_{\alpha=1}^{p}\{(\varpi^*_{\alpha+2} - \lambda\varpi^{\alpha+1}_1 + \mu_\alpha\omega^*_1 + \mu_\alpha\varpi^1_0) \wedge (\varpi^{\alpha+1}_1 + \mu_\alpha\varpi^1_1) + (d\mu_\alpha + \beta_\alpha) \wedge \omega_{\alpha+1}\} \\ & - (\sum_{\alpha=1}^{p}\mu^2_\alpha)\varpi^1_0 \wedge \varpi^1_1 \quad (\mathrm{mod} \quad \varpi, \varpi_1, \varpi_2), \\ \equiv & \sum_{\alpha=1}^{p}\{\varpi^*_{\alpha+2} \wedge (\varpi^{\alpha+1}_1 + \mu_\alpha\varpi^1_1) + (d\mu_\alpha + \beta_\alpha) \wedge \omega_{\alpha+1}\} + \omega^*_1 \wedge \{(\sum_{\alpha=1}^{p}\mu^2_\alpha)\varpi^1_1 + \sum_{\alpha=1}^{p}2\mu_\alpha\varpi^{\alpha+1}_1\} \\ & (\ \mathrm{mod} \quad \varpi, \varpi_1, \varpi_2). \end{aligned}$$

Thus we obtain the following structure equation for $(W(Y), N_Y)$:

$$N_Y = \{\varpi = \varpi_1 = \varpi_2 = 0\}, \quad N^\perp_Y = \{\varpi = \varpi_1 = \varpi_2 = \varpi^*_{\alpha+2} = \omega_{\alpha+1} = 0 \ (1 \leqq \alpha \leqq p)\},$$

$$(8.11) \qquad \begin{cases} d\varpi \equiv \displaystyle\sum_{\alpha=1}^{p}\varpi^*_{\alpha+2} \wedge \omega_{\alpha+1}, \\ d\varpi_1 \equiv \displaystyle\sum_{\alpha=1}^{p}(\varpi^{\alpha+1}_1 - \mu_\alpha\varpi^1_1) \wedge \omega_{\alpha+1}, \qquad\qquad (\mathrm{mod} \quad \varpi, \varpi_1, \varpi_2) \\ d\varpi_2 \equiv \displaystyle\sum_{\alpha=1}^{p}\{\varpi^*_{\alpha+2} \wedge (\varpi^{\alpha+1}_1 + \mu_\alpha\varpi^1_1) + (d\mu_\alpha + \beta_\alpha) \wedge \omega_{\alpha+1}\} \\ \qquad\qquad + \omega^*_1 \wedge \{(\displaystyle\sum_{\alpha=1}^{p}\mu^2_\alpha)\varpi^1_1 + \displaystyle\sum_{\alpha=1}^{p}2\mu_\alpha\varpi^{\alpha+1}_1\} \end{cases}$$

Utilizing this structure equation, we first calculate, for $X \in \mathrm{Ch}\,(C^*_Y)(w)$, $w \in W(Y)$,

$$X \rfloor d\varpi \equiv 0, \quad X \rfloor d\varpi_1 \equiv 0, \quad X \rfloor d\varpi_2 \equiv -\hat{\varpi}^1_1(X)\ \omega^*_1 \quad (\mathrm{mod} \quad N^\perp_Y),$$

where we put $\hat{\varpi}^1_1 = (\sum_{\alpha=1}^{p}\mu^2_\alpha)\varpi^1_1 + \sum_{\alpha=1}^{p}2\mu_\alpha\varpi^{\alpha+1}_1$. Hence, by (2.3), we get

$$S(N_Y)(w) = \{\ X \in \mathrm{Ch}\,(C^*_Y)(w) \mid \hat{\varpi}^1_1(X) = 0\}.$$

Now we consider the Lagrange Grassmann bundle $R(Y) = R(W(Y))$ over $(W(Y); C^*_Y, N_Y)$:

$$R(Y) = \bigcup_{w \in W(Y)} R_w, \quad R_w = \{\hat{v} \subset N_Y(w) \mid d\varpi\ |_{\hat{v}}= 0, \quad \hat{v} \text{ is maximal}\}.$$

From $N^\perp_Y(w) \subset \hat{v} \subset N_Y(w)$ and $d\varpi\ |_{\hat{v}}= 0$, if $(d\lambda + \beta) \wedge \omega^*_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{p+1}\ |_{\hat{v}} \neq 0$, it follows that

$$\hat{v} = \{X \in N_Y(w) \mid \varpi_{\alpha+2}(X) = 0 \quad (1 \leqq \alpha \leqq p)\},$$

where $\varpi_{\alpha+2} = \varpi^*_{\alpha+2} - \sum_{\beta=1}^{p} p^*_{\alpha\beta}\omega_{\beta+1}$ for $1 \leqq \alpha \leqq p$ and $p^*_{\alpha\beta} = p^*_{\beta\alpha}$ $(1 \leqq \alpha, \beta \leqq p)$. For these $\hat{v} \in R_w$, we claim

$$A(\hat{v}) = \{0\} \quad \textit{for} \quad \hat{v} = \{\varpi = \varpi_1 = \varpi_2 = \varpi_{\alpha+2} = 0 \ (1 \leqq \alpha \leqq p)\}.$$

In fact, by the structure equation (8.11), we get

$$(8.12)\quad \begin{cases} d\varpi \equiv 0, \quad d\varpi_1 \equiv \displaystyle\sum_{\alpha=1}^{p}(\varpi_1^{\alpha+1} - \mu_\alpha \varpi_1^1) \wedge \omega_{\alpha+1}, \\ d\varpi_2 \equiv \displaystyle\sum_{\alpha=1}^{p}\{d\mu_\alpha + \beta_\alpha - \sum_{\beta=1}^{p} p^*_{\alpha\beta}(\varpi_1^{\beta+1} + \mu_\beta \varpi_1^1)\} \wedge \omega_{\alpha+1} + \omega_1^* \wedge \hat{\varpi}_1^1. \quad (\mathrm{mod} \quad (\hat{v})^\perp) \end{cases}$$

Hence, for $X \in S(N_Y)(w)$, we have

$$X \rfloor d\varpi \equiv 0, \quad X \rfloor d\varpi_1 \equiv \sum_{\alpha=1}^{p}(\varpi_1^{\alpha+1} - \mu_\alpha \varpi_1^1)(X)\ \omega_{\alpha+1} \qquad (\mathrm{mod} \quad (\hat{v})^\perp),$$

and

$$X \rfloor d\varpi_2 \equiv \sum_{\alpha=1}^{p}\{d\mu_\alpha + \beta_\alpha - \sum_{\beta=1}^{p} p^*_{\alpha\beta}(\varpi_1^{\beta+1} + \mu_\beta \varpi_1^1)\}(X)\ \omega_{\alpha+1} \qquad (\mathrm{mod} \quad (\hat{v})^\perp).$$

Then, from Lemma 4.2 (2), we obtain

$$\begin{aligned} A(\hat{v}) = \{X \in S(N_Y)(w) \mid (\varpi_1^{\alpha+1} - \mu_\alpha \varpi_1^1)(X) = \\ \{d\mu_\alpha + \beta_\alpha - \sum_{\beta=1}^{p} p^*_{\alpha\beta}(\varpi_1^{\beta+1} + \mu_\beta \varpi_1^1)\}(X) = 0 \quad (1 \leqq \alpha \leqq p)\} \\ = \mathrm{Ch}\,(C_Y^*)(w) \cap \mathrm{Ch}\,(N_Y)(w) = \{0\}. \end{aligned}$$

Hence $(R(Y); D_Y^1, D_Y^2)$ is a PD manifold of second order on an open subset of $R(Y)$. Moreover, by (8.10) and (8.12), we have the structure equation for $(R(Y); D_Y^1, D_Y^2)$ as follows;

$$D_Y^1 = \{\varpi = 0\}, \qquad D_Y^2 = \{\varpi = \varpi_1 = \cdots = \varpi_{p+2} = 0\},$$

$$d\varpi \equiv \omega_0^* \wedge \varpi_1 + \omega_1^* \wedge \varpi_2 + \sum_{\alpha=1}^{p} \varpi_{\alpha+2} \wedge \omega_{\alpha+1}, \qquad (\mathrm{mod}\ \varpi)$$

$$\begin{cases} d\varpi_1 \equiv \displaystyle\sum_{\alpha=1}^{p} \pi_1^{\alpha+2} \wedge \omega_{\alpha+1}, \\ d\varpi_2 \equiv \pi_2^2 \wedge \omega_1^* + \displaystyle\sum_{\alpha=1}^{p} \pi_2^{\alpha+2} \wedge \omega_{\alpha+1}, \qquad (\mathrm{mod} \quad \varpi, \varpi_1, \ldots, \varpi_{p+2}) \end{cases}$$

where we put $\omega_0^* = d\lambda + \beta$, $\pi_1^{\alpha+2} = \varpi_1^{\alpha+1} - \mu_\alpha \varpi_1^1$, $\pi_2^2 = -\hat{\varpi}_1^1$ and $\pi_2^{\alpha+2} = d\mu_\alpha + \beta_\alpha - \sum_{\beta=1}^{p} p^*_{\alpha\beta}(\varpi_1^{\beta+1} + \mu_\beta \varpi_1^1)$ for $\alpha = 1, \ldots, p$. This shows that $(R(Y); D_Y^1, D_Y^2)$ is a PD manifold of second order, which is of type $\mathfrak{s}$, where $\mathfrak{s}$ is given by

$$\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}, \qquad \mathfrak{s}_{-3} = \mathbb{R}, \quad \mathfrak{s}_{-2} = V^*, \quad \mathfrak{s}_{-1} = V \oplus \mathfrak{f}, \quad \mathfrak{f} \subset S^2(V^*),$$

such that

$$(\mathfrak{f})^\perp = \langle \{e_1 \circledcirc e_1, e_1 \circledcirc e_2\} \rangle,$$

for a base $\{e_1, \ldots, e_{p+2}\}$ of V. Thus $\zeta(R(Y))$ is an involutive system of second order of codimension 2 (see Proposition 3.3 [27]).

To obtain an explicit description of the model equation in this case, we will first construct the standard differential system $(M(\mathfrak{m}_3), D_{\mathfrak{m}_3})$ of type $\mathfrak{m}_3$, where $\mathfrak{m}_3 = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the negative part of the simple graded Lie algebra of type $(BD_\ell, \{\alpha_1, \alpha_3\})$, by virtue of the formula given by N.Tanaka in §2.3 [16]. Let us take the basis $\{Z_1, Z_2, Y_j, X_1^j, X_2^j (1 \leqq j \leqq p+1)\}$ of

$\mathfrak{m}_3$ as above. We introduce a coordinate system $(z^1, z^2, y^1, \dots, y^{p+1}, x_1^1, \dots, x_{p+1}^1, x_1^2, \dots, x_{p+1}^2)$ of $\mathfrak{m}_3$ by putting

$$u^{-3} = z^1 Z_1 + z^2 Z_2, \qquad u^{-2} = y^1 Y_1 + \cdots + y^{p+1} Y_{p+1},$$

and

$$u^{-1} = x_1^1 X_1^1 + \cdots + x_{p+1}^1 X_1^{p+1} + x_1^2 X_2^1 + \cdots + x_{p+1}^2 X_2^{p+1},$$

where $u^p : \mathfrak{m}_3 = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \to \mathfrak{g}_p$ is the projection for $p = -1, -2, -3$. Then, by the formula in §2.3 [16], we calculate

$$\begin{aligned} du^{-2} - \frac{1}{2}[u^{-1}, du^{-1}] &= \sum_{k=1}^{p+1} dy^k Y_k - \frac{1}{2}[\ \sum_{k=1}^{p+1}(x_k^1 X_1^k + x_k^2 X_2^k), \sum_{k=1}^{p+1}(dx_k^1 X_1^k + dx_k^2 X_2^k)\ ], \\ &= \bar{\omega}_1 Y_1 + \bar{\omega}_2 Y_2 + \cdots + \bar{\omega}_{p+1} Y_{p+1}, \end{aligned}$$

where

$$\text{(8.13)} \quad \bar{\omega}_1 = dy^1 + \frac{1}{2}\sum_{k=2}^{p+1}(x_k^1 dx_k^2 - x_k^2 dx_k^1),\ \bar{\omega}_k = dy^k + \frac{1}{2}(x_1^1 dx_k^2 - x_k^2 dx_1^1) + \frac{1}{2}(x_k^1 dx_1^2 - x_1^2 dx_k^1),$$

for $k = 2, \dots, p+1$, so that

$$d\bar{\omega}_1 = dx_2^1 \wedge dx_2^2 + \cdots + dx_{p+1}^1 \wedge dx_{p+1}^2, \quad d\bar{\omega}_k = dx_1^1 \wedge dx_k^2 + dx_k^1 \wedge dx_1^2 \quad (k = 2, \dots, p+1).$$

Moreover, we calculate

$$\begin{aligned} & du^{-3} - \frac{1}{3}[u^{-2}, du^{-1}] - \frac{2}{3}[u^{-1}, du^{-2}] + \frac{1}{6}[u^{-1}, [u^{-1}, du^{-1}]] \\ &= dz^1 Z_1 + dz^2 Z_2 \\ &\quad - \frac{1}{3}[\ \sum_{k=1}^{p+1} y^k Y_k, \sum_{k=1}^{p+1}(dx_k^1 X_1^k + dx_k^2 X_2^k)\ ] - \frac{2}{3}[\ \sum_{k=1}^{p+1}(x_k^1 X_1^k + x_k^2 X_2^k, \sum_{k=1}^{p+1} dy^k Y_k\ ] \\ &\quad + \frac{1}{6}[\ \sum_{k=1}^{p+1}(x_k^1 X_1^k + x_k^2 X_2^k), [\ \sum_{k=1}^{p+1}(x_k^1 X_1^k + x_k^2 X_2^k), \sum_{k=1}^{p+1}(dx_k^1 X_1^k + dx_k^2 X_2^k)\ ]] \\ &= \bar{\varpi}_1 Z_1 + \bar{\varpi}_2 Z_2 \end{aligned}$$

where

$$\text{(8.14)} \qquad \begin{cases} \bar{\varpi}_1 = d\hat{z}^1 - (y^1 + \dfrac{1}{2}\displaystyle\sum_{k=2}^{p+1} x_k^1 x_k^2) dx_1^1 - \displaystyle\sum_{k=2}^{p+1}\{y^k + \dfrac{1}{2}(x_1^1 x_k^2 + x_k^1 x_1^2)\} dx_k^1, \\ \bar{\varpi}_2 = d\hat{z}^2 - (y^1 - \dfrac{1}{2}\displaystyle\sum_{k=2}^{p+1} x_k^1 x_k^2) dx_1^2 - \displaystyle\sum_{k=2}^{p+1}\{y^k - \dfrac{1}{2}(x_1^1 x_k^2 + x_k^1 x_1^2)\} dx_k^2, \end{cases}$$

and

$$\begin{cases} \hat{z}^1 = z^1 + \dfrac{2}{3}\displaystyle\sum_{k=1}^{p+1} x_k^1 y^k + \dfrac{1}{3}\displaystyle\sum_{k=2}^{p+1} x_1^1 x_k^1 x_k^2 + \dfrac{1}{6}\displaystyle\sum_{k=2}^{p+1}(x_k^1)^2 x_1^2, \\ \hat{z}^2 = z^2 + \dfrac{2}{3}\displaystyle\sum_{k=1}^{p+1} x_k^2 y^k - \dfrac{1}{3}\displaystyle\sum_{k=2}^{p+1} x_1^2 x_k^1 x_k^2 - \dfrac{1}{6}\displaystyle\sum_{k=2}^{p+1} x_1^1 (x_k^2)^2, \end{cases}$$

so that

$$d\bar{\varpi}_1 = dx_1^1 \wedge \hat{\omega}_1 + \cdots + dx_{p+1}^1 \wedge \hat{\omega}_{p+1}, \quad d\bar{\varpi}_2 = dx_1^2 \wedge \hat{\omega}_1 + \cdots + dx_{p+1}^2 \wedge \hat{\omega}_{p+1}.$$

Thus, $M(\mathfrak{m}_3) \cong \mathfrak{m}_3$ is endowed with a coordinate $(\hat{z}^1, \hat{z}^2, y^1, \dots, y^{p+1}, x_1^1, \dots, x_{p+1}^1, x_1^2, \dots, x_{p+1}^2)$ such that $D_{\mathfrak{m}_3}$ and $\partial D_{\mathfrak{m}_3}$ are given by

$$D_{\mathfrak{m}_3} = \{\bar{\varpi}_1 = \bar{\varpi}_2 = \bar{\omega}_1 = \cdots = \bar{\omega}_{p+1} = 0\}, \quad \text{and} \quad \partial D_{\mathfrak{m}_3} = \{\bar{\varpi}_1 = \bar{\varpi}_2 = 0\}.$$

Now put $(Z, D) = (M(\mathfrak{m}_3), D_{\mathfrak{m}_3})$ and $F = \partial D_{\mathfrak{m}_3}$.

$$F = \{\varpi_0 = \varpi_1 = 0\}, \qquad D = \{\varpi_0 = \varpi_1 = \omega_1 = \cdots = \omega_{p+1}\},$$

where $\varpi_0 = \bar{\varpi}_1$, $\varpi_1 = \bar{\varpi}_2$ and $\omega_j = \bar{\omega}_j$ $(j = 1, \dots, p+1)$. Let $\bar{Y} = \bar{Y}(Z)$ be the projective bundle over Z and $\gamma : \bar{Y} \to Z$ be the projection. We introduce a fibre coordinate $(\mu_1, \dots, \mu_p)$ of $\gamma : \bar{Y} \to Z$ by putting

$$\varpi_2 = \omega_1 + \mu_1\omega_2 + \cdots + \mu_p\omega_{p+1}. \tag{8.15}$$

Then we have

$$D_N^* = \{\varpi_0 = \varpi_1 = 0\} \qquad \text{and} \qquad D_N = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}.$$

Here we denote the pullback on $\bar{Y}$ of 1-forms on Z by the same symbol. We put

$$Y = \{y \in \bar{Y} \mid \mathrm{Ch}\,(D_N)(y) = \{0\}\ \}.$$

Starting from $(Y; D_N^*, D_N)$, we construct the IG manifold $(W(Y); C_Y^*, N_Y)$ of corank 2 and the Lagrange Grassmann bundle $(R(Y); D_Y^1, D_Y^2)$. We introduce a fibre coordinate (λ) of $\mu : W(Y) \to Y$ and calculate

$$\begin{aligned}
\varpi &= \varpi_0 + \lambda\varpi_1 \\
&= d\hat{z}^1 - (y^1 + \frac{1}{2}\sum_{k=2}^{p+1} x_k^1 x_k^2) dx_1^1 - \sum_{k=2}^{p+1}\{y^k + \frac{1}{2}(x_1^1 x_k^2 + x_k^1 x_1^2)\} dx_k^1 \\
&\qquad + \lambda[d\hat{z}^2 - (y^1 - \frac{1}{2}\sum_{k=2}^{p+1} x_k^1 x_k^2) dx_1^2 - \sum_{k=2}^{p+1}\{y^k - \frac{1}{2}(x_1^1 x_k^2 + x_k^1 x_1^2)\} dx_k^2] \\
&= d\hat{z}^1 + \lambda d\hat{z}^2 - (y^1 + \frac{1}{2}\sum_{k=2}^{p+1} x_k^1 x_k^2)(dx_1^1 + \lambda dx_1^2) - \sum_{k=2}^{p+1}\{y^k + \frac{1}{2}(x_1^1 x_k^2 + x_k^1 x_1^2)\}(dx_k^1 + \lambda dx_k^2) \\
&\qquad + \lambda\{(\sum_{k=2}^{p+1} x_k^1 x_k^2) dx_1^2 + \sum_{k=2}^{p+1}(x_1^1 x_k^2 + x_k^1 x_1^2) dx_k^2\} \\
&= d(\hat{z}^1 + \lambda\hat{z}^2) - (y^1 + \frac{1}{2}\sum_{k=2}^{p+1} x_k^1 x_k^2) d(x_1^1 + \lambda x_1^2) - \sum_{k=2}^{p+1}\{y^k + \frac{1}{2}(x_1^1 x_k^2 + x_k^1 x_1^2)\} d(x_k^1 + \lambda x_k^2) \\
&\qquad - [\hat{z}^2 - (y^1 + \frac{1}{2}\sum_{k=2}^{p+1} x_k^1 x_k^2) x_1^2 - \sum_{k=2}^{p+1}\{y^k + \frac{1}{2}(x_1^1 x_k^2 + x_k^1 x_1^2)\} x_k^2] d\lambda \\
&\qquad + \lambda\{d(\sum_{k=2}^{p+1} x_1^2 x_k^1 x_k^2) - \sum_{k=2}^{p+1} x_1^2 x_k^2 dx_k^1 + \sum_{k=2}^{p+1} x_1^1 x_k^2 dx_k^2\} \\
&= d\hat{Z} - \sum_{k=1}^{p+1} Y^k d(x_k^1 + \lambda x_k^2) - P_1 d\lambda = dZ - \sum_{k=1}^{p+2} P_k dX_k,
\end{aligned}$$

where

$$\begin{cases} \hat{Z} = \hat{z}^1 + \lambda\{\hat{z}^2 + \sum_{k=2}^{p+1} x_1^2 x_k^1 x_k^2 + \frac{1}{2}(x_1^1 + \lambda x_1^2) \sum_{k=2}^{p+1} (x_k^2)^2\}, \quad Y^1 = y^1 + \frac{1}{2} \sum_{k=2}^{p+1} (x_k^1 + \lambda x_k^2) x_k^2, \\ Y^k = y^k + \frac{1}{2}\{(x_1^1 + \lambda x_1^2) x_k^2 + (x_k^1 + \lambda x_k^2) x_1^2\}, \quad P_1 = \hat{z}^2 - \sum_{k=1}^{p+1} y^k x_k^2 - \frac{1}{2}\lambda x_1^2 \sum_{k=2}^{p+1} (x_k^2)^2, \end{cases}$$

for $k = 2, \ldots, p+1$. Thus, putting

$$\begin{cases} Z = \hat{Z} - \sum_{k=2}^{p+1} Y^k (x_k^1 + \lambda x_k^2), X_1 = \lambda, X_2 = x_1^1 + \lambda x_1^2, P_2 = Y^1, \\ X_{k+1} = Y^k, P_{k+1} = -(x_k^1 + \lambda x_k^2) \quad (k = 2, \ldots, p+1), \end{cases}$$

we obtain a canonical coordinate $(X_1, \ldots, X_{p+2}, Z, P_1, \ldots, P_{p+2})$ of $J = W(Y)/\mathrm{Ch}\,(C_Y^*) = W(Z)/\mathrm{Ch}\,(C_Z^*)$.

Conversely we calculate

$$\begin{cases} x_1^2 = c_1, x_k^2 = c_k, \lambda = X_1, x_1^1 = X_2 - c_1 X_1, x_k^1 = -(P_{k+1} + c_k X_1), \\ y^1 = P_2 + \frac{1}{2} \sum_{k=2}^{p+1} c_k P_{k+1}, y^k = X_{k+1} + \frac{1}{2}(c_1 P_{k+1} - c_k X_2), \quad (k = 2, \ldots, p+1) \\ \hat{z}^2 = P_1 + c_1 P_2 + \sum_{k=2}^{p+1} (X_{k+1} + c_1 P_{k+1}) c_k + \frac{1}{2}(c_1 X_1 - X_2)(\sum_{k=2}^{p+1} c_k^2). \end{cases}$$

Hence, by (8.13), (8.14) and (8.15), we have

$$(8.16) \quad \begin{cases} \varpi_1 = dP_1 + c_1\{dP_2 + \sum_{k=2}^{p+1} c_k dP_{k+1} + \frac{1}{2}(\sum_{k=2}^{p+1} c_k^2) dX_1\} - \frac{1}{2}(\sum_{k=2}^{p+1} c_k^2) dX_2 + \sum_{k=2}^{p+1} c_k dX_{k+1}, \\ \varpi_2 = dP_2 + \sum_{k=2}^{p+1} (c_k + c_1 \mu_{k-1}) dP_{k+1} + \frac{1}{2} \sum_{k=2}^{p+1} (c_k + 2c_1 \mu_{k-1}) c_k dX_1 \\ \qquad\qquad - (\sum_{k=2}^{p+1} c_k \mu_{k-1}) dX_2 + \sum_{k=2}^{p+1} \mu_{k-1} dX_{k+1}. \end{cases}$$

Here $(X_1, \ldots, X_{p+2}, Z, P_1, \ldots, P_{p+2}, c_1, \ldots, c_{p+1}, \mu_1, \ldots, \mu_p)$ constitute a coordinate of $W(Y)$ and $(X_1, \ldots, X_{p+2}, Z, P_1, \ldots, P_{p+2}, c_1, \ldots, c_{p+1})$ constitutes a coordinate of $W(Z)$. Now, from the canonical coordinate $(X_1, \ldots, X_{p+2}, Z, P_1, \ldots, P_{p+2})$ of J, we introduce the coordinate $(X_i, X_\alpha, Z, P_i, P_\alpha, A_i^\alpha, B_i^\alpha, S_{ij})$ $(1 \leqq i \leqq j \leqq 2, 3 \leqq \alpha \leqq p+2)$ of $I^2(J)$ as in §2.1. First we calculate

$$\begin{aligned} \varpi_1^* = \varpi_1 - c_1 \varpi_2 = & dP_1 - \sum_{k=2}^{p+1} c_1^2 \mu_{k-1} dP_{k+1} + \sum_{k=2}^{p+1} (c_k - c_1 \mu_{k-1}) dX_{k+1} \\ & - \sum_{k=2}^{p+1} c_1^2 c_k \mu_{k-1} dX_1 - \frac{1}{2} \sum_{k=2}^{p+1} (c_k - 2c_1 \mu_{k-1}) c_k dX_2. \end{aligned}$$

Then, since $N_Y = \{\varpi = \varpi_1^* = \varpi_2 = 0\}$, the canonical inclusion $\iota : W(Y) \to I^2(J)$ is given by

$$A_1^\alpha = c_1^2 \mu_{\alpha-2}, B_1^\alpha = -(c_{\alpha-1} - c_1 \mu_{\alpha-2}), A_2^\alpha = -(c_{\alpha-1} + c_1 \mu_{\alpha-2}), B_2^\alpha = -\mu_{\alpha-2},$$

$$S_{11} = c_1^3 \sum_{k=1}^{p} \mu_k^2,\ S_{12} = \frac{1}{2}\sum_{k=1}^{p+1} c_k^2 - c_1 \sum_{k=2}^{p+1} c_k\mu_{k-1} - c_1^2 \sum_{k=1}^{p} \mu_k^2,\ S_{22} = c_1 \sum_{k=1}^{p} \mu_k^2 + 2\sum_{k=2}^{p+1} c_k\mu_{k-1}.$$

Thus, by (4.3), introducing a coordinate $(X_1, \ldots, X_{p+2}, Z, P_1, \ldots, P_{p+2}, c_1, \ldots, c_{p+1}, \mu_1, \ldots, \mu_p, P^*_{\alpha,\beta})$ $(3 \leqq \alpha \leqq \beta \leqq p+2)$ of $R(Y)$ as in §4.1, the immersion $\zeta : R(Y) \to L(J)$ is expressed by

$$\begin{cases} \zeta^* P_{\alpha\beta} = P^*_{\alpha\beta}, \quad \zeta^* P_{\alpha 1} = -(c_{\alpha-1} - c_1\mu_{\alpha-2}) + c_1^2 \sum_{\beta=3}^{p+2} P^*_{\alpha\beta}\mu_{\beta-2}, \quad (3 \leqq \alpha \leqq \beta \leqq p+2) \\ \zeta^* P_{\alpha 2} = -\mu_{\alpha-2} - \sum_{\beta=3}^{p+2} P^*_{\alpha\beta}(c_{\beta-1} + c_1\mu_{\beta-2}), \quad \zeta^* P_{11} = c_1^3 \sum_{k=1}^{p} \mu_k^2 + c_1^4 \sum_{\alpha,\beta=3}^{p+2} P^*_{\alpha\beta}\mu_{\alpha-2}\mu_{\beta-2}, \\ \zeta^* P_{12} = \frac{1}{2}\sum_{k=1}^{p+1} c_k^2 - c_1 \sum_{k=2}^{p+1} c_k\mu_{k-1} - c_1^2 \sum_{k=1}^{p} \mu_k^2 - c_1^2 \sum_{\alpha,\beta=3}^{p+2} P^*_{\alpha\beta}\mu_{\alpha-2}(c_{\beta-1} + c_1\mu_{\beta-2}), \\ \zeta^* P_{22} = c_1 \sum_{k=1}^{p} \mu_k^2 + 2\sum_{k=2}^{p+1} c_k\mu_{k-1} + \sum_{\alpha,\beta=3}^{p+2} P^*_{\alpha\beta}(c_{\alpha-1} + c_1\mu_{\alpha-2})(c_{\beta-1} + c_1\mu_{\beta-2}). \end{cases}$$

This is the description of the model involutive system of codimension 2 of type $(BD_\ell, \{\alpha_1, \alpha_3\})$.

Moreover, from the canonical coordinate $(X_1, \ldots, X_{p+2}, Z, P_1, \ldots, P_{p+2})$ of J, we introduce the coordinate $(X_1, X_a, Z, P_1, P_a, \hat{A}_1^a, \hat{B}_1^a, \hat{S}_{11})$ $(2 \leqq a \leqq p+2)$ of $I^1(J)$ as in §2.1. Then, from the first equation of (8.16), since $N_Z = \{\varpi = \varpi_1 = 0\}$, the canonical inclusion $\iota : W(Z) \to I^1(J)$ is given by

$$\hat{A}_1^2 = -c_1, \quad \hat{A}_1^a = -c_1 c_{a-1}, \quad \hat{B}_1^2 = \frac{1}{2}(\sum_{k=2}^{p+1} c_k^2), \quad \hat{B}_1^a = -c_{a-1}, \quad \hat{S}_{11} = 0.$$

Thus, by (4.3), introducing a coordinate $(X_1, \ldots, X_{p+2}, Z, P_1, \ldots, P_{p+2}, c_1, \ldots, c_{p+1}, P^*_{ab})$ $(2 \leqq a \leqq b \leqq p+2)$ of $R(Z)$ as in §4.1, the immersion $\zeta : R(Z) \to L(J)$ is expressed by

$$\begin{cases} \zeta^* P_{ab} = P^*_{ab} (2 \leqq a \leqq b \leqq p+2), \quad \zeta^* P_{a1} = -c_{a-1} - c_1(P^*_{a2} + \sum_{b=3}^{p+2} P^*_{ab}c_{b-1}) \quad (3 \leqq a \leqq p+2), \\ \zeta^* P_{21} = \frac{1}{2}(\sum_{k=2}^{p+1} c_k^2) - c_1(P^*_{22} + \sum_{b=3}^{p+2} P^*_{2b}c_{b-1}), \zeta^* P_{11} = c_1^2(P^*_{22} + 2\sum_{a=3}^{p+2} P^*_{2a}c_{a-1} + \sum_{a,b=3}^{p+2} P^*_{ab}c_{a-1}c_{b-1}). \end{cases}$$

This is the description of the model Goursat type equation of type $(BD_\ell, \{\alpha_1, \alpha_3\})$ (see Remark 6.2 (1)).

8.4. **Other Examples.** In this subsection, we will treat the case of Proposition 7.2. For this purpose, we will here exhibit an example of type $(C_3, \{\alpha_2\})$.

From §5 [12], the structure of the symbol algebra $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of type $(C_3, \{\alpha_2\})$ can be described as follows; Let V be a vector space of dimension 2 and $(U, \langle, \rangle)$ be a symplectic vector space of dimension 2. Then $\mathfrak{m}$ is isomorphic to $\mathfrak{m}^2(U, V)$,

$$\mathfrak{m}^2(U, V) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, \quad \mathfrak{g}_{-2} = S^2(V), \mathfrak{g}_{-1} = U \otimes V.$$

The bracket product is defined by

$$[u_1 \otimes v_1, u_2 \otimes v_2] = \langle u_1, u_2 \rangle v_1 \circledcirc v_2 \qquad \text{for} \quad u_1, u_2 \in U, v_1, v_2 \in V.$$

Moreover the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$ in this case is given as follows; $M(\mathfrak{m}) = \mathbb{R}^7$ is endowed with a coordinate $(x_1, x_2, x_3, x_4, y_1, y_2, y_3)$ such that $D_{\mathfrak{m}}$ is given by

$$D_{\mathfrak{m}} = \{\bar{\theta}_1 = \bar{\theta}_2 = \bar{\theta}_3 = 0\ \},$$

where

$$\bar{\theta}_1 = dy_1 + x_4 dx_1 + x_3 dx_2, \quad \bar{\theta}_2 = dy_2 + x_3 dx_1 \quad \text{and} \quad \bar{\theta}_3 = dy_3 + x_4 dx_2.$$

Thus the symbol algebra $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathfrak{m}^2(U, V)$ is described by

$$\begin{cases} d\theta_1 \equiv \omega_4 \wedge \omega_1 + \omega_3 \wedge \omega_2 \\ d\theta_2 \equiv \omega_3 \wedge \omega_1 \qquad (\mathrm{mod} \quad \theta_1, \theta_2, \theta_3) \\ d\theta_3 \equiv \omega_4 \wedge \omega_2 \end{cases} \tag{8.17}$$

In fact, taking the dual basis $\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, X_1, X_2, X_3, X_4\}$ of the coframe $\{\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3, dx_1, dx_2, dx_3, dx_4\}$ on $M(\mathfrak{m})$, we have

$$X_1 = \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial y_1} - x_3 \frac{\partial}{\partial y_2}, \quad X_2 = \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial y_1} - x_4 \frac{\partial}{\partial y_3}, \quad \text{and} \quad X_3 = \frac{\partial}{\partial x_3}, X_4 = \frac{\partial}{\partial x_4}.$$

Thus $\{X_1, X_2, X_3, X_4\}$ constitutes a free basis of the sections $\Gamma(D_{\mathfrak{m}})$ of $D_{\mathfrak{m}}$, and we obtain

$$[X_2, X_3] = [X_1, X_4] = \frac{\partial}{\partial y_1}, \ [X_1, X_3] = \frac{\partial}{\partial y_2}, \ [X_2, X_4] = \frac{\partial}{\partial y_3}, \ [X_1, X_2] = [X_3, X_4] = 0.$$

Here, for a basis $\{v_1, v_2\}$ of V and a symplectic basis $\{u_1, u_2\}$ of U, X_1, X_2, X_3 and X_4 correspond to $u_1 \otimes v_1$, $u_1 \otimes v_2$, $u_2 \otimes v_1$ and $u_2 \otimes v_2$ respectively. Moreover $\frac{\partial}{\partial y_1}$, $\frac{\partial}{\partial y_2}$ and $\frac{\partial}{\partial y_3}$ correspond to $v_1 \circledcirc v_2$, $v_1 \circledcirc v_1$ and $v_2 \circledcirc v_2$ respectively. Thus $\mathfrak{m}$ is isomorphic to $\mathfrak{m}^2(U, V)$.

Let (Y, D_N) be a regular differential system of type $\mathfrak{m}$ such that D_N is locally defined by

$$D_N = \{\theta_1 = \theta_2 = \theta_3 = 0\},$$

Here $\{\theta_1, \theta_2, \theta_3, \omega_1, \omega_2, \omega_3, \omega_4\}$ forms a coframe on Y satisfying (8.17).

Then, putting $\varpi = \theta_1 + \lambda_1 \theta_2 + \lambda_2 \theta_3$, we calculate

$$\begin{aligned} d\varpi &\equiv \omega_4 \wedge \omega_1 + \omega_3 \wedge \omega_2 + \lambda_1 \omega_3 \wedge \omega_1 + \lambda_2 \omega_4 \wedge \omega_2 + (d\lambda_1 + \gamma_1) \wedge \theta_2 + (d\lambda_2 + \gamma_2) \wedge \theta_3 \\ &\equiv (\omega_4 + \lambda_1 \omega_3) \wedge \omega_1 + (\omega_3 + \lambda_2 \omega_4) \wedge \omega_2 + (d\lambda_1 + \gamma_1) \wedge \theta_2 + (d\lambda_2 + \gamma_2) \wedge \theta_3 \\ &\equiv \tilde{\omega}_4 \wedge \omega_1 + \tilde{\omega}_3 \wedge \omega_2 + (d\lambda_1 + \gamma_1) \wedge \theta_2 + (d\lambda_2 + \gamma_2) \wedge \theta_3 \quad (\mathrm{mod} \quad \varpi) \end{aligned}$$

for some 1-forms γ_1, γ_2 on Y, where we put $\tilde{\omega}_4 = \omega_4 + \lambda_1 \omega_3$ and $\tilde{\omega}_3 = \omega_3 + \lambda_2 \omega_4$. Thus we see

$$W(Y) = \{w \in \hat{W}(Y) \mid \lambda_1 \lambda_2 - 1 \neq 0\ \}$$

and we have on $W(Y)$,

$$C^*_Y = \{\varpi = o\}, \quad N_Y = \{\varpi = \theta_2 = \theta_3 = o\}, \quad N^{\perp}_Y = \{\varpi = \theta_2 = \theta_3 = \omega_1 = \omega_2 = \tilde{\omega}_3 = \tilde{\omega}_4 = o\}.$$

Here $r = 2$, $t = 2$, $n = r + t = 4$ and $\dim W(Y) = 9$. We see that (Y, D_N) satisfies the condition $(Y.2)$ in §7.1 and $(W(Y), C^*_Y)$ is a contact manifold of dimension 9. Let $(R(Y); D^1_Y, D^2_Y)$ be the Lagrange Grassmann bundle over $(W(Y); C^*_Y, N_Y)$. Then, by Proposition 7.2, $(R(Y); D^1_Y, D^2_Y)$ is a PD manifold of second order, globally over $W(Y)$. Put

$$\hat{v} = \{X \in N_Y(w) \mid \varpi_3(X) = \varpi_4(X) = 0\ \} \in R_w(Y),$$

where $\varpi_3 = \tilde{\omega}_3 - r\omega_2 - s\omega_1$ and $\varpi_4 = \tilde{\omega}_4 - s\omega_2 - t\omega_1$. Here (r, s, t) is a fibre coordinate of $R_w(Y)$. Hence, around $\hat{v} \in R(Y)$, we have

$$D^2_Y = \{\varpi = \theta_2 = \theta_3 = \varpi_3 = \varpi_4 = 0\}.$$

Then, from

$$\begin{cases} \varpi_3 - \lambda_2\varpi_4 = (1-\lambda_1\lambda_2)\omega_3 - (r\omega_2 + s\omega_1) - \lambda_2(s\omega_2 + t\omega_1), \\ \varpi_4 - \lambda_2\varpi_3 = (1-\lambda_1\lambda_2)\omega_4 - (s\omega_2 + t\omega_1) - \lambda_1(r\omega_2 + s\omega_1), \end{cases}$$

and by (8.17), we have

$$\begin{cases} d\varpi \equiv 0 \\ d\theta_2 \equiv (1-\lambda_1\lambda_2)^{-1}(r + \lambda_2 s)\ \omega_2 \wedge \omega_1 \qquad (\mathrm{mod} \quad \varpi, \theta_2, \theta_3, \varpi_3, \varpi_4) \\ d\theta_3 \equiv (1-\lambda_1\lambda_2)^{-1}(t + \lambda_1 s)\ \omega_1 \wedge \omega_2 \end{cases}$$

This shows that $(R(Y); D^1_Y, D^2_Y)$ does not satisfy the compatibility condition (C) in §3.1 on an open subset.

To obtain an explicit description of the model equation in this case, we calculate

$$\begin{aligned} \varpi &= dy_1 + x_4dx_1 + x_3dx_2 + \lambda_1(dy_2 + x_3dx_1) + \lambda_2(dy_3 + x_4dx_2) \\ &= d(y_1 + \lambda_1y_2 + \lambda_2y_3) - y_2d\lambda_1 - y_3d\lambda_2 + (x_4 + \lambda_1x_3)dx_1 + (x_3 + \lambda_2x_4)dx_2 \\ &= dZ - P_1dX_1 - P_2dX_2 - P_3dX_3 - P_4dX_4. \end{aligned}$$

Thus, putting

$$\begin{cases} Z = y_1 + \lambda_1y_2 + \lambda_2y_3, X_1 = \lambda_1,\ X_2 = \lambda_2,\ X_3 = x_1, X_4 = x_2 \\ P_1 = y_2,\ P_2 = y_3,\ P_3 = -(x_4 + \lambda_1x_3),\ P_4 = -(x_3 + \lambda_2x_4) \end{cases}$$

we obtain a canonical coordinate $(X_1, X_2, X_3, X_4, Z, P_1, P_2, P_3, P_4)$ of $(W(Y), C^*_Y)$.

Conversely we calculate

$$\begin{cases} \lambda_1 = X_1, \lambda_2 = X_2, x_1 = X_3, x_2 = X_4, x_3 = (X_1X_2 - 1)^{-1}(P_4 - X_2P_3), \\ x_4 = (X_1X_2 - 1)^{-1}(P_3 - X_1P_4), y_3 = P_2, y_2 = P_1. \end{cases}$$

Hence we have

$$\begin{cases} \bar\theta_2 = dy_2 + x_3dx_1 = dP_1 + (X_1X_2 - 1)^{-1}(P_4 - X_2P_3)dX_3 \\ \bar\theta_3 = dy_3 + x_4dx_2 = dP_2 + (X_1X_2 - 1)^{-1}(P_3 - X_1P_4)dX_4. \end{cases}$$

Thus we obtain the following description of the model equation of type $(C_3, \{\alpha_2\})$;

$$\begin{cases} P_{11} = P_{12} = P_{14} = P_{22} = P_{23} = 0, \\ P_{13} = (X_1X_2 - 1)^{-1}(X_2P_3 - P_4), P_{24} = (X_1X_2 - 1)^{-1}(X_1P_4 - P_3). \end{cases}$$

Other than $(C_3, \{\alpha_2\})$, we note here that regular differential systems of type $\mathfrak{m}$ satify the condition $(Y.2)$ in §7.1, when $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the negative part of the simple graded Lie algebra of type $(C_\ell, \{\alpha_2\})$, $(C_\ell, \{\alpha_{\ell-1}\})$ $(\ell \geqq 3)$, $(B_{2m}, \{\alpha_{2m}\})$ $(p \geqq 2)$ or $(F_4, \{\alpha_4\})$ (cf. §6 [12], §5.3 [24]).

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