Hilbert Space Representations of Generalized Canonical Commutation Relations

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Abstract

We consider Hilbert space representations of a generalization of canonical commutation relations (CCR): $[X_j, X_k] := X_jX_k - X_kX_j = i\Theta_{jk}I$ $(j,k = 1,2,\ldots,2n)$, where X_j's are elements of an algebra with identity I, i is the imaginary unit, and Θ_{jk} is a real number with $\Theta_{jk} = -\Theta_{kj}$ $(j,k = 1,\ldots,2n)$. Some basic aspects on Hilbert space representations of the generalized CCR (GCCR) are discussed. We define a Schrödinger type representation of the GCCR by analogy with the usual Schrödinger representation of the CCR with n degrees of freedom. Also we introduce a Weyl type representation of the GCCR. The main result of the present paper is a uniqueness theorem on Weyl representations of the GCCR.

Keywords: canonical commutation relations; generalized canonical commutation relations ; quantum space; quantum phase space; non-commutative space-time

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1 Introduction

In this paper, we consider Hilbert space representations of a *generalized canonical commutation relations* (GCCR) with n degrees of freedom ($n \in \mathbb{N} := \{1,2,3,\ldots\}$) of the following type:

$$[X_j, X_k] = i\Theta_{jk}I \qquad (j,k = 1,\ldots,2n), \tag{1.1}$$

where X_j's are elements of an algebra with identity I, $[X_j, X_k] := X_jX_k - X_kX_j$, i is the imaginary unit, and $\Theta_{jk} \in \mathbb{R}$ (the set of real numbers) with anti-symmetry $\Theta_{jk} =$

$-\Theta_{kj}$ $(j,k=1,\ldots,2n)$ such that, for some pair (j,k), $\Theta_{jk}\neq 0$. For convenience, we call (1.1) the Θ-*GCCR with* n *degrees of freedom* and the $2n\times 2n$ matrix

$$\Theta := (\Theta_{jk})_{j,k=1,\ldots,2n} \tag{1.2}$$

the *non-commutative factor* for $\{X_j\}_{j=1}^{2n}$.

Note that, in the case where Θ is equal to

$$J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{1.3}$$

with I_n being the $n\times n$ unit matrix, (1.1) becomes the CCR with n degrees of freedom. Namely, if we put $Q_j := X_j,\quad P_j := X_{n+j}\quad (j=1,\ldots,n)$ in the present case, then we have

$$[Q_j,Q_k]=0,\quad [P_j,P_k]=0,\quad [Q_j,P_k]=i\delta_{jk}I\quad (j,k=1,\ldots,n),$$

where δ_{jk} is the Kronecker delta. Thus (1.1) is a natural generalization of the CCR with n degrees of freedom.

The GCCR also includes some of non-commutative space-times (e.g., [10, 11, 14, 20]), non-commutative spaces (e.g., [16, 17]) and non-commutative phase spaces (e.g., [5, 6, 15, 18, 19, 25, 32]). In fact, one of the motivations for the present work is to investigate general structures underlying those non-commutative objects. In this paper, however, we present only some fundamental aspects of Hilbert space representations of the GCCR. The main result is to establish a uniqueness theorem on Weyl type representations of the GCCR (for the definition, see Section 4).

In Section 2, we define Hilbert space representations of the GCCR and discuss some basic facts on them. It is shown that there exists a one-to-one correspondence between representations of the GCCR and the CCR with the same degrees of freedom. In Section 3, we introduce a Schrödinger type representation of the GCCR, whose representation space is $L^2(\mathbb{R}^n)$ as in the case of the Schrödinger representation of the CCR with n degrees of freedom. In Section 4, Weyl type representations of the GCCR are defined by analogy with Weyl representations of CCR. In the last section, we prove the uniqueness theorem mentioned above. In Appendix, we present some basic properties of self-adjoint operators obeying generalized Weyl relations, which are used in the text.

2 Basic Facts on Hilbert Space Representations of the Θ-GCCR

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ (anti-linear in the first variable and linear in the second one) and norm $\|\cdot\|$. For a linear operator A on $\mathcal{H}$, we denote its domain by $D(A)$. For linear operators $A_1, \ldots, A_p$ on $\mathcal{H}$,

$$D\left(\sum_{i=1}^{p} A_i\right) := \cap_{i=1}^{p} D(A_i), \quad D(A_1 A_2) := \{\Psi \in D(A_2) | A_2\Psi \in D(A_1)\},$$
$$D(A_1 \cdots A_p) := D((A_1 \cdots A_{p-1})A_p) \quad (p \geq 3).$$

Definition 2.1 Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ and $X_j, j = 1, \ldots, 2n$, be symmetric (not necessarily essentially self-adjoint) operators on $\mathcal{H}$. We set $\mathbf{X} := (X_1, \ldots, X_{2n})$. We say that the triple $(\mathcal{H}, \mathcal{D}, \mathbf{X})$ is a *symmetric representation* of the Θ-GCCR with n degrees of freedom if $\mathcal{D} \subset \cap_{j,k=1}^{2n} D(X_j X_k)$ and (1.1) holds on $\mathcal{D}$.

If all the X_j's ($j = 1, \ldots, 2n$) are self-adjoint, we say that $(\mathcal{H}, \mathcal{D}, \mathbf{X})$ is a *self-adjoint representation* of the GCCR.

Remark 2.2 In each symmetric representation $(\mathcal{H}, \mathcal{D}, \mathbf{X})$ of the Θ-GCCR, $\mathcal{H}$ is *infinite dimensional* (if $\mathcal{H}$ were finite dimensional, then, for (j, k) such that $\Theta_{jk} \neq 0$, $0 = \text{trace of } [X_j, X_k] = i\Theta_{jk} \dim \mathcal{H} \neq 0$ and hence one is led to a contradiction).

Remark 2.3 It follows from a well known fact on commutation properties of linear operators (e.g., [22, Theorem 1.2.3]) that, for (j, k) with $\Theta_{jk} \neq 0$, at least one of X_j and X_k is *unbounded*. Hence one has to be careful about domains of X_j's.

Remark 2.4 In the case of Hilbert space representations of CCR, symmetric representations, but non-self-adjoint ones, also play important roles. For example, such representations appear in mathematical theories of time operators [2, 3, 4, 8, 9, 21] (see also [29, 30] for investigations from purely operator-theoretic points of view). Thus it is expected that, in addition to self-adjoint representations of the Θ-GCCR, non-self-adjoint symmetric representations of it may have any importance in applications to quantum physics.

Remark 2.5 In the context of quantum mechanics, for a symmetric operator A and a unit vector $\psi \in D(A)$, $(\Delta A)_\psi := \|(A - \langle \psi, A\psi \rangle)\psi\|$ is called the uncertainty of A in the vector state ψ. Let $(\mathcal{H}, \mathcal{D}, \mathbf{X})$ be a symmetric representation of the Θ-GCCR with n degrees of freedom. Then one has uncertainty relations of Robertson type [26]: for all unit vectors $\psi \in \mathcal{D}$ and $j, k = 1, \ldots, 2n$,

$$(\Delta X_j)_\psi (\Delta X_k)_\psi \geq \frac{1}{2} | \langle \psi, \Theta_{jk} \psi \rangle |.$$

Let $(\mathcal{H}, \mathcal{D}, \mathbf{X})$ be a symmetric representation of the Θ-GCCR as in Definition 2.1. We assume for simplicity the following:

Assumption: The non-commutative factor Θ is regular (invertible).

Under this assumption, Θ is a regular anti-symmetric real matrix. Hence, by a well known fact in the theory of linear algebra (e.g., [27, p.173, Problem 9]), the following fact holds:

Lemma 2.6 *There exists a regular $2n \times 2n$ real matrix T_0 such that ${}^{\mathrm{t}}T_0 \Theta T_0 = J$, where ${}^{\mathrm{t}}T_0$ is the transposed matrix of T_0 and J is defined by (1.3).*

The matrix T_0 in Lemma 2.6 belongs to the set

$$M_\Theta := \{T | T \text{ is a } 2n \times 2n \text{ real matrix such that } {}^{\mathrm{t}}T\Theta T = J\}. \tag{2.1}$$

It is easy to see that, for each $T \in M_\Theta$, there exists a unique $2n \times 2n$ symplectic matrix W (i.e., ${}^{\mathrm{t}}WJW = J$) such that $T = T_0 W$. Hence

$$M_\Theta = \{T_0 W | {}^{\mathrm{t}}WJW = J\}. \tag{2.2}$$

For a $2n \times 2n$ real matrix $L = (L_{jk})_{j,k=1,\ldots,2n}$, we define

$$X_j^L := \sum_{k=1}^{2n} L_{kj} X_k, \quad j = 1, \ldots, 2n. \tag{2.3}$$

We call the correspondence $\mathbf{X} \mapsto \mathbf{X}^L := (X_1^L, \ldots, X_{2n}^L)$ the *L-transform of* $\mathbf{X}$.

Let

$$\Theta_L := {}^{\mathrm{t}}L\Theta L. \tag{2.4}$$

Proposition 2.7

(i) *For all $j = 1, \ldots, 2n$, X_j^L is a symmetric operator on $\mathcal{H}$.*

(ii) *For all $j, k = 1, \ldots, 2n$,*

$$[X_j^L, X_k^L] = i(\Theta_L)_{jk} \tag{2.5}$$

on $\mathcal{D}$.

(iii) *For each $T \in M_\Theta$ and $j, k = 1, \ldots, 2n$,*

$$[X_j^T, X_k^T] = iJ_{jk} \tag{2.6}$$

on $\mathcal{D}$.

Proof. An easy exercise. ∎

Proposition 2.7-(i) and (ii) show that $(\mathcal{H}, \mathcal{D}, \mathbf{X}^L)$ is a symmetric representation of the Θ_L-GCCR with n degrees of freedom.

Proposition 2.7-(iii) implies the following:

Corollary 2.8 *Let $T \in M_\Theta$ and*

$$Q_j := X_j^T, \quad P_j := X_{n+j}^T \quad (j = 1, \ldots, n).$$

Then $(\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^n)$ is a symmetric representation of the CCR with n degrees of freedom.

Corollary 2.8 means that, for each $T \in M_\Theta$, the T-transform of $\mathbf{X}$ gives a correspondence from a symmetric representation of the Θ-GCCR with n degrees of freedom to a symmetric representation of the CCR with the same degrees of freedom.

One can easily see that (2.3) with $L = T \in M_\Theta$ implies that

$$X_j = \sum_{k=1}^{2n} (T^{-1})_{kj} X_k^T \tag{2.7}$$

on $\cap_{j=1}^{2n} D(X_j)$. Thus every symmetric representation of the Θ-GCCR with n degrees of freedom is constructed from a symmetric representation of the CCR with the same degrees of freedom via (2.7).

Conversely, if a symmetric representation $(\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^n)$ of the CCR with n degrees of freedom is given and let

$$X_j(\mathbf{Q}, \mathbf{P}; T) := \sum_{k=1}^n (T^{-1})_{kj} Q_k + \sum_{k=1}^n (T^{-1})_{(n+k)j} P_k \tag{2.8}$$

with $\mathbf{Q} := (Q_1, \ldots, Q_n)$ and $\mathbf{P} := (P_1, \ldots, P_n)$. Then $(\mathcal{H}, \mathcal{D}, \mathbf{X}(\mathbf{Q}, \mathbf{P}; T))$ is a symmetric representation of the Θ-GCCR and (2.3) holds with $L = T$, $X_j^T = Q_j, X_{n+j}^T = P_j$ $(j = 1, \ldots, n)$ and $X_j = X_j(\mathbf{Q}, \mathbf{P}; T)$ $(j = 1, \ldots, 2n)$. Hence every symmetric representation of the CCR with n degrees of freedom is constructed from a symmetric representation of the Θ-GCCR with the same degrees of freedom. Thus, *for each $T \in M_\Theta$, there exists a one-to-one correspondence between a symmetric representation of the Θ-GCCR and a symmetric representation of the CCR with n degrees of freedom.*

3 Representations of Schrödinger Type

Let $T \in M_\Theta$. By the fact on $\mathbf{X}(\mathbf{Q}, \mathbf{P}; T)$ stated in the preceding section, we can define a class of representations of the Θ-GCCR. Let $(L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \{q_j, p_j\}_{j=1}^n)$ be the Schrödinger representation of the CCR with n degrees of freedom, i.e., q_j is the multiplication operator by the jth component x_j of $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $p_j := -iD_j$ with D_j being the generalized partial differential operator in x_j, acting in $L^2(\mathbb{R}^n)$. Let

$$X_j(\mathbf{q}, \mathbf{p}; T) = \sum_{k=1}^n (T^{-1})_{kj} q_k + \sum_{k=1}^n (T^{-1})_{(n+k)j} p_k,$$

which is (2.8) with $\mathbf{Q} = \mathbf{q}$ and $\mathbf{P} = \mathbf{p}$. We denote the closure of $X_j(\mathbf{q}, \mathbf{p}; T)$ by $\overline{X}_j(\mathbf{q}, \mathbf{p}; T)$ and set

$$\overline{\mathbf{X}}(\mathbf{q}, \mathbf{p}; T) := (\overline{X}_1(\mathbf{q}, \mathbf{p}; T), \ldots, \overline{X}_{2n}(\mathbf{q}, \mathbf{p}; T)).$$

We call the triple $\pi_{\mathrm{S}}^T := (L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \overline{\mathbf{X}}(\mathbf{q}, \mathbf{p}; T))$ the *T-Schrödinger representation of the Θ-GCCR*.

It is easy to see that, for all $j = 1, \ldots, 2n$, $X_j(\mathbf{q}, \mathbf{p}; T)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ (apply, e.g., the Nelson commutator theorem [24, Theorem X.37] with dominating operator $N = \sum_{j=1}^n (q_j^2 + p_j^2) + I)$[1]. Hence $\overline{X}_j(\mathbf{q}, \mathbf{p}; T)$ is self-adjoint. Thus we obtain the following:

Proposition 3.1 *For each $T \in M_\Theta$, the T-Schrödinger representation π_{S}^T is a self-adjoint representation of the Θ-GCCR.*

4 Representations of Weyl Type

Based on an analogy with Weyl representations of CCR, we introduce a concept of Weyl representation for Θ-GCCR.

Definition 4.1 Let $\{X_j\}_{j=1}^{2n}$ be a set of self-adjoint operators on a Hilbert space $\mathcal{H}$. We say that $\{X_j\}_{j=1}^{2n}$ is a *Weyl representation of the Θ-GCCR* with n degrees of freedom if, for all $s, t \in \mathbb{R}$ and $j, k = 1, \ldots, 2n$,

$$e^{itX_j} e^{isX_k} = e^{-ist\Theta_{jk}} e^{isX_k} e^{itX_j}. \tag{4.1}$$

We call these relations the Θ-*Weyl relations*.

For a linear operator A on a Hilbert space, we denote its spectrum by $\sigma(A)$.

Proposition 4.2 *Let $\{X_j\}_{j=1}^{2n}$ be a Weyl representation of the Θ-GCCR on $\mathcal{H}$. Then there is a dense subspace $\mathcal{D}_0 \subset \mathcal{H}$ left invariant by each X_j $(j = 1, \ldots, 2n)$ such that $(\mathcal{H}, \mathcal{D}_0, \mathbf{X})$ is a self-adjoint representation of the Θ-GCCR. Moreover, for every pair (X_j, X_k) such that $\Theta_{jk} \neq 0$, X_j and X_k are purely absolutely continuous with*

$$\sigma(X_j) = \sigma(X_k) = \mathbb{R}, \quad j = 1, \ldots, 2n. \tag{4.2}$$

Proof. By (4.1), we can apply the results described in Appendix A of the present paper. In the present context, we need only to take, in the notation in Appendix A, $N = 2n$, $a_{jk} = \Theta_{jk}$ and $A_j = X_j$. By Proposition A.4-(iii) and Corollary A.5, there exists a dense subspace $\mathcal{D}_0$ left invariant by X_j $(j = 1, \ldots, 2n)$ and $[X_j, X_k] = i\Theta_{jk}$ on $\mathcal{D}_0$. Thus the first half of the proposition is derived. The second half follows from Proposition A.1. ∎

[1]This can be proved also by applying Proposition 4.6 below.

Remark 4.3 As in the case of self-adjoint representations of CCR (e.g., [1, 13, 30]), the converse of Proposition 4.2 does not hold (i.e., a self-adjoint representation of the Θ-GCCR is not necessarily a Weyl one).

We recall that a set $\{Q_j, P_j\}_{j=1}^n$ of self-adjoint operators on $\mathcal{H}$ is a Weyl representation of the CCR with n degrees of freedom if, for all $s, t \in \mathbb{R}$ and $j, k = 1, \ldots, n$, the following Wey relations hold:

$$e^{itQ_j} e^{isP_k} = e^{-ist\delta_{jk}} e^{isP_k} e^{itQ_j}, \quad e^{itQ_j} e^{isQ_k} = e^{isQ_k} e^{itQ_j}, \qquad e^{itP_j} e^{isP_k} = e^{isP_k} e^{itP_j}.$$

Remark 4.4 A set $\{Q_j, P_j\}_{j=1}^n$ of self-adjoint operators on $\mathcal{H}$ is a Weyl representation of the CCR with n degrees of freedom if and only if $\{X_j\}_{j=1}^{2n}$ with $X_j := Q_j, X_{n+j} = P_j$ ($j = 1, \ldots, n$) is a Weyl representation of the J-GCCR, where J is given by (1.3).

Let $T \in M_\Theta$ be arbitrarily fixed. The next proposition shows that the T-transform of each Weyl representation of the Θ-GCCR is a Weyl representation of the CCR with n degrees of freedom:

Proposition 4.5 *Let $\{X_j\}_{j=1}^{2n}$ be a Weyl representation of the Θ-GCCR on $\mathcal{H}$ and let $\mathbf{X}^T$ be the T-transform of $\mathbf{X}$. Then each X_j^T is essentially self-adjoint and $\{\overline{X_j^T}\}_{j=1}^{2n}$ is a Weyl representation of the J-GCCR.*

Proof. The essential self-adjointness of X_j^T follows from a simple application of Theorem A.6 in Appendix. Corollary A.7 in Appendix and the relation ${}^{\mathrm{t}}T\Theta T = J$ imply that $\{\overline{X_j^T}\}_{j=1}^{2n}$ satisfies the J-Weyl relations. ∎

In the same way as in the proof of Proposition 4.5, we can prove the following proposition:

Proposition 4.6 *Let $\{Q_j, P_j\}_{j=1}^n$ be a Weyl representation of the CCR with n degrees of freedom on a Hilbert space $\mathcal{H}$. Let $X_j(\mathbf{Q}, \mathbf{P}; T)$ ($j = 1, \ldots, 2n$) be defined by (2.8). Then each $X_j(\mathbf{Q}, \mathbf{P}; T)$ is essentially self-adjoint and $\{\overline{X}_j(\mathbf{Q}, \mathbf{P}; T)\}_{j=1}^{2n}$ is a Weyl representation of Θ-GCCR with n degrees of freedom.*

This proposition shows that the converse of Proposition 4.5 holds too. Thus, *for each $T \in M_\Theta$, there exists a one-to-one correspondence between a Weyl representation of the CCR with n degrees of freedom and that of the Θ-GCCR with the same degrees of freedom.*

It is well known [31] that the Schrödinger representation $\{q_j, p_j\}_{j=1}^n$ is a Weyl representation o the CCR with n degrees of freedom. Hence we obtain the following result:

Corollary 4.7 *For each $T \in M_\Theta$, the T-Schrödinger representation $\{\overline{X}_j(\mathbf{q}, \mathbf{p}; T)\}_{j=1}^{2n}$ is a Weyl representation of the Θ-GCCR.*

We say that a Weyl representation $\{X_j\}_{j=1}^{2n}$ of the Θ-GCCR on $\mathcal{H}$ is *irreducible* if every closed subspace $\mathcal{M}$ of $\mathcal{H}$ which is invariant under the action of e^{itX_j} $(t \in \mathbb{R}, j = 1, \dots, 2n)$ is $\{0\}$ or $\mathcal{H}$.

Proposition 4.8 *Let $T \in M_\Theta$. Then the T-Schrödinger representation $\{\overline{X}_j(\mathbf{q}, \mathbf{p}; T)\}_{j=1}^{2n}$ as a Weyl representation of the Θ-GCCR is irreducible.*

Proof. Let $\mathcal{M}$ be an invariant closed subspace of $e^{it\overline{X}_j(\mathbf{q},\mathbf{p};T)}$ $(t \in \mathbb{R}, j = 1, \dots, 2n)$. We have

$$q_j = \sum_{k=1}^{2n} T_{kj} X_k(\mathbf{q}, \mathbf{p}; T), \quad p_j = \sum_{k=1}^{2n} T_{k(n+j)} X_k(\mathbf{q}, \mathbf{p}; T).$$

on $\cap_{j=1}^{2n} D(X_j(\mathbf{q}, \mathbf{p}; T)) = \cap_{j=1}^{n} D(q_j) \cap D(p_j)$. Hence, by an application of Theorem A.6 in Appendix, e^{itq_j} and e^{itp_j} $(t \in \mathbb{R})$ can be written respectively as a scalar multiple of $e^{it\overline{X}_1(\mathbf{q},\mathbf{p};T)} \dots e^{it\overline{X}_{2n}(\mathbf{q},\mathbf{p};T)}$. Hence $\mathcal{M}$ is invariant under the action of e^{itq_j} and e^{itp_j} $(t \in \mathbb{R}, j = 1, \dots, n)$. It is well known that $\{e^{itq_j}, e^{itp_j} | t \in \mathbb{R}, j = 1, \dots, n\}$ is irreducible. Thus $\mathcal{M} = \{0\}$ or $\mathcal{H}$. ∎

5 Uniqueness Theorem on Weyl Representations of the Θ-GCCR

In this section, we prove the main result of the present paper, i.e., a uniqueness theorem on Weyl representations of the Θ-GCCR, which may be regarded as a GCCR version of the celebrated von Neumann uniqueness theorem of Weyl representations of CCR ([22, Theorem 4.11.1], [23, Theorem VIII.14], [31]).

Theorem 5.1 *Let $\{X_j\}_{j=1}^{2n}$ be a Weyl representation of the Θ-GCCR on a separable Hilbert space $\mathcal{H}$. Then, for each $T \in M_\Theta$, there exist mutually orthogonal closed subspaces $\mathcal{H}_\ell$ ($\ell = 1, \dots, N$; $N \in \mathbb{N}$ or ∞) such that the following* (i)–(iii) *hold:*

(i) $\mathcal{H} = \oplus_{\ell=1}^{N} \mathcal{H}_\ell$.

(ii) *For each $j = 1, \dots, 2n$, X_j is reduced by each $\mathcal{H}_\ell, \ell = 1, \dots, N$. We denote by $X_j^{(\ell)}$ the reduced part of X_j to $\mathcal{H}_\ell$.*

(iii) *For each ℓ, there exists a unitary operator $U_\ell : \mathcal{H}_\ell \to L^2(\mathbb{R}^n)$ such that*

$$U_\ell X_j^{(\ell)} U_\ell^{-1} = \overline{X}_j(\mathbf{q}, \mathbf{p}; T), \quad j = 1, \dots, 2n, \tag{5.1}$$

where $\{\overline{X}_j(\mathbf{q}, \mathbf{p}; T)\}_{j=1}^{2n}$ is the T-Schrödinger representation of the Θ-GCCR.

Proof. Let $T \in M_\Theta$, $\mathbf{X}^T$ be the T-transform of $\mathbf{X}$ and $Q_j := \overline{X_j^T}$, $P_j := \overline{X_{n+j}^T}$ ($j = 1, \ldots, n$). Then, by Proposition 4.5 and Remark 4.4, $\{Q_j, P_j\}_{j=1}^n$ is a Weyl representation of the CCR with n degrees of freedom. Hence, by the von Neumann uniqueness theorem mentioned above, there exist mutually orthogonal closed subspaces $\mathcal{H}_\ell$ such that (i) given above and the following (a) and (b) hold:

(a) For each $j = 1, \ldots, n$ and all $t \in \mathbb{R}$, e^{itQ_j} and e^{itP_j} leave each $\mathcal{H}_\ell$ invariant ($\ell = 1, \ldots, N$).

(b) For each ℓ, there exists a unitary operator $U_\ell : \mathcal{H}_\ell \to L^2(\mathbb{R}^n)$ such that

$$U_\ell e^{itQ_j} U_\ell^{-1} = e^{itq_j}, \quad U_\ell e^{itP_j} U_\ell^{-1} = e^{itp_j}, \quad t \in \mathbb{R}, j = 1, \ldots, n. \tag{5.2}$$

By (2.7), we have $X_j = X_j(\mathbf{Q}, \mathbf{P}; T)$ on $\cap_{j=1}^{2n} D(X_j)$. Hence $X_j \subset \overline{X}_j(\mathbf{Q}, \mathbf{P}; T)$. By Proposition 4.6, $\overline{X}_j(\mathbf{Q}, \mathbf{P}; T)$ is self-adjoint. Hence $X_j = \overline{X}_j(\mathbf{Q}, \mathbf{P}; T)$. Therefore, by Theorem A.6 in Appendix, we obtain

$$e^{itX_j} = e^{it^2 \sum_{k<m}^{2n} J_{km}(T^{-1})_{kj}(T^{-1})_{mj}/2} e^{itQ_1} \cdots e^{itQ_n} e^{itP_1} \cdots e^{itP_n}, \quad j = 1, \ldots, 2n.$$

Hence each e^{itX_j} leaves $\mathcal{H}_\ell$ invariant ($\ell = 1, \ldots, N$). Therefore X_j is reduced by each $\mathcal{H}_\ell$. We denote the reduced part of X_j to $\mathcal{H}_\ell$ by $X_j^{(\ell)}$. Then, we have by (5.2)

$$U_\ell e^{itX_j^{(\ell)}} U_\ell^{-1} = e^{it^2 \sum_{k<m}^{2n} J_{km}(T^{-1})_{kj}(T^{-1})_{mj}/2} e^{itq_1} \cdots e^{itq_n} e^{itp_1} \cdots e^{itp_n} = e^{it\overline{X}_j(\mathbf{q}, \mathbf{p}; T)},$$

Thus (5.1) follows. ∎

Theorem 5.1 tells us that every Weyl representation of the Θ-GCCR on a *separable* Hilbert space is unitarily equivalent to a direct sum of the T-Schrödinger representation of the Θ-GCCR, where $T \in M_\Theta$ is arbitrary.

The next corollary immediately follows from Theorem 5.1:

Corollary 5.2 *Let $\{X_j\}_{j=1}^{2n}$ be an irreducible Weyl representation of the Θ-GCCR on a separable Hilbert space $\mathcal{H}$. Then, for each $T \in M_\Theta$, there exists a unitary operator $U : \mathcal{H} \to L^2(\mathbb{R}^n)$ such that*

$$U X_j U^{-1} = \overline{X}_j(\mathbf{q}, \mathbf{p}; T), \quad j = 1, \ldots, 2n. \tag{5.3}$$

The following result shows that the arbitrariness of the choice of T in the T-Schrödinger representation of the Θ-GCCR is implemented by unitary operators.

Corollary 5.3 *Let $S, T \in M_\Theta$. Then there exists a unitary operator V on $L^2(\mathbb{R}^n)$ such that*

$$V \overline{X}_j(\mathbf{q}, \mathbf{p}; S) V^{-1} = \overline{X}_j(\mathbf{q}, \mathbf{p}; T), \quad j = 1, \ldots, 2n. \tag{5.4}$$

Proof. We need only to apply Corollary 5.2 to the case where $X_j = \overline{X}_j(\mathbf{q}, \mathbf{p}; S)$. ∎

Remark 5.4 As in the case of non-Weyl representations of CCR, for non-Weyl representations of the Θ-GCCR, the conclusion of Theorem 5.1 does not hold in general. Examples of such representations of the Θ-GCCR can be constructed from non-Weyl representations of CCR (e.g., [1, 13, 29, 30]). A detailed description of some examples is given in [5].

A Some Properties of Self-Adjoint Operators Satisfying Relations of Weyl Type

Let $N \geq 2$ be an integer and A_j $(j = 1, \ldots, N)$ be self-adjoint operators on a Hilbert space $\mathcal{H}$ satisfying relations of Weyl type:

$$e^{itA_j} e^{isA_k} = e^{-itsa_{jk}} e^{isA_k} e^{itA_j}, \quad t, s \in \mathbb{R},\ j, k = 1, \ldots, N, \tag{A.1}$$

where a_{jk}'s are real constants. It follows that a_{jk} is anti-symmetric in (j, k):

$$a_{jk} = -a_{kj}, \quad j, k = 1, \ldots, N. \tag{A.2}$$

The unitarity of e^{itA_j} and functional calculus imply that

$$\exp(ise^{itA_j} A_k e^{-itA_j}) = \exp(is(A_k - ta_{jk})), \quad s, t \in \mathbb{R}.$$

Hence we have the operator equality

$$e^{itA_j} A_k e^{-itA_j} = A_k - ta_{jk}, \quad t \in \mathbb{R}, j, k = 1, \ldots, N. \tag{A.3}$$

For a linear operator A on a Hilbert space, we denote the spectrum of A by $\sigma(A)$.

Proposition A.1 *Suppose that there exists a pair (j, k) such that $a_{jk} \neq 0$ (hence $j \neq k$). Then*

$$\sigma(A_j) = \mathbb{R}, \quad \sigma(A_k) = \mathbb{R}. \tag{A.4}$$

Moreover, A_j and A_k are purely absolutely continuous.

Proof. By (A.3) and the unitary invariance of spectrum, we have $\sigma(A_k) = \sigma(A_k - ta_{jk})$ for all $t \in \mathbb{R}$. Since $a_{jk} \neq 0$, this implies the second equation of (A.4). By (A.2), we have $a_{kj} \neq 0$. Hence, by considering the case of (j, k) replaced by (k, j), we obtain the first equation of (A.4).

Relation (A.3) means that (A_k, A_j) is a weak Weyl representation of the CCR with one degree of freedom ([2]–[4], [29]). Hence A_j is purely absolutely continuous [2, 21, 29]. Similarly we can show that A_k is purely absolutely continuous. ∎

Proposition A.2 *Let j and k be fixed. Then, for all $\psi \in D(A_j) \cap D(A_j A_k)$, ψ is in $D(A_k A_j)$ and*

$$[A_j, A_k]\psi = ia_{jk}\psi. \tag{A.5}$$

Proof. An easy exercise (use (A.3)). ∎

For each function $f \in C_0^\infty(\mathbb{R}^N)$ and each vector $\psi \in \mathcal{H}$, we define a vector ψ_f by

$$\psi_f := \int_{\mathbb{R}^N} f(\mathbf{t}) e^{it_1 A_1} \cdots e^{it_N A_N} \psi d\mathbf{t}, \tag{A.6}$$

where $\mathbf{t} = (t_1, \ldots, t_N) \in \mathbb{R}^N$ and the integral on the right hand side is taken in the strong sense. We introduce

$$\mathcal{D}_0 := \mathrm{Span}\{\psi_f | \psi \in \mathcal{H}, f \in C_0^\infty(\mathbb{R}^N)\}, \tag{A.7}$$

where $\mathrm{Span}\{\cdots\}$ denotes the subspace algebraically spanned by the vectors in the set $\{\cdots\}$. It is easy to see that $\mathcal{D}_0$ is dense in $\mathcal{H}$.

For $f : \mathbb{R}^N \to \mathbb{C}$ (the set of complex numbers), we set $\|f\|_1 := \int_{\mathbb{R}^N} |f(\mathbf{t})| d\mathbf{t}$.

Lemma A.3 *Let $f_n, f \in C_0^\infty(\mathbb{R}^N)$ such that $\|f_n - f\|_1 \to 0\,(n \to \infty)$. Then $\|\psi_{f_n} - \psi_f\| \to 0\,(n \to \infty)$.*

Proof. Since $e^{it_j A_j}$ is unitary, we have $\|\psi_{f_n} - \psi_f\| \leq \|f_n - f\|_1 \|\psi\|$. Thus the desired result follows. ∎

For each $j = 1, \ldots, N$, we define a function g_j on $\mathbb{R}^N$ by

$$g_j(\mathbf{t}) := \begin{cases} 0 & \text{for } j = 1 \\ \sum_{k=1}^{j-1} a_{jk} t_k & \text{for } 2 \leq j \leq N \end{cases}, \quad \mathbf{t} \in \mathbb{R}^N. \tag{A.8}$$

Proposition A.4

(i) *For all $t \in \mathbb{R}$ and $j = 1, \ldots, N$, e^{itA_j} leaves $\mathcal{D}_0$ invariant.*

(ii) *For each $j = 1, \ldots, N$, A_j leaves $\mathcal{D}_0$ invariant (i.e., $A_j \mathcal{D}_0 \subset \mathcal{D}_0$) and, for all $\ell \in \mathbb{N}$,*

$$A_j^\ell \psi_f = (-i)^\ell \psi_{F_j^\ell(f)}, \quad f \in C_0^\infty(\mathbb{R}^N), \tag{A.9}$$

where $F_j : C_0^\infty(\mathbb{R}^N) \to C_0^\infty(\mathbb{R}^N)$ is defined by

$$F_j(f) := -\partial_j f - ig_j f, \quad f \in C_0^\infty(\mathbb{R}^N) \tag{A.10}$$

and F_j^ℓ is the ℓ times composition of F_j with $F_j^0 := I$ (identity).

(iii) *For all $\ell_1, \ldots, \ell_N \in \mathbb{N} \cup \{0\}$,*

$$A_1^{\ell_1} A_2^{\ell_2} \cdots A_N^{\ell_N} \psi_f = \psi_{F_1^{\ell_1} \cdots F_N^{\ell_N}(f)}, \quad f \in C_0^\infty(\mathbb{R}^N). \tag{A.11}$$

Proof. (i) Let ψ_f be as above. Then we have $e^{itA_j}\psi_f = \int_{\mathbb{R}^N} f(\mathbf{t})e^{itA_j}e^{it_1A_1}\cdots e^{it_NA_N}\psi d\mathbf{t}$. By (A.1), we have

$$e^{itA_j}e^{it_1A_1}\cdots e^{it_NA_N} = e^{-itg_j(\mathbf{t})}e^{it_1A_1}\cdots e^{it_{j-1}A_{j-1}}e^{i(t_j+t)A_j}e^{it_{j+1}A_{j+1}}\cdots e^{it_NA_N}.$$

Hence

$$e^{itA_j}\psi_f = \psi_{f_j^{(t)}}. \tag{A.12}$$

with

$$f_j^{(t)}(\mathbf{t}) := f(t_1,\ldots,t_{j-1},t_j-t,t_{j+1},\ldots,t_N)e^{-itg_j(\mathbf{t})}.$$

It is easy to see that $f_j^{(t)}$ is in $C_0^\infty(\mathbb{R}^N)$. Hence $\psi_{f_j^{(t)}} \in \mathcal{D}_0$. Thus e^{itA_j} leaves $\mathcal{D}_0$ invariant.

(ii) By (A.12), we have for all $t \in \mathbb{R}\setminus\{0\}$ $(e^{itA_j}-1)\psi_f/t = \psi_{(f_j^{(t)}-f)/t}$. It is easy to see that $\|(f_j^{(t)}-f)/t - F_j(f)\|_1 \to 0 (t\to 0)$. Hence, by Lemma A.3,

$$\lim_{t\to 0}\frac{(e^{itA_j}-1)\psi_f}{t} = \psi_{F_j(f)}.$$

Therefore ψ_f is in $D(A_j)$ and $iA_j\psi_f = \psi_{F_j(f)}$. Hence (A.9) with $\ell = 1$ holds. Then one can prove (A.9) by induction.

(iii) This easily follows from (ii). ∎

Propositions A.2 and A.4 immediately yield the following result:

Corollary A.5 *For all $j,k = 1,\ldots,N$, $[A_j,A_k] = ia_{jk}$ on $\mathcal{D}_0$.*

Theorem A.6 *For all $c_j \in \mathbb{R}, j = 1,\ldots,N$, $\sum_{j=1}^N c_jA_j$ is essentially self-adjoint on $\mathcal{D}_0$ and*

$$e^{it\overline{\sum_{j=1}^N c_jA_j}} = e^{it^2\sum_{j<k}^N a_{jk}c_jc_k/2}e^{itc_1A_1}e^{itc_2A_2}\cdots e^{itc_NA_N}, \tag{A.13}$$

where, for a closable operator C, $\overline{C}$ denotes the closure of C.

Proof. For each $t\in\mathbb{R}$, we define an operator $U(t)$ by

$$U(t) := e^{it^2\sum_{j<k}^N a_{jk}c_jc_k/2}e^{itc_1A_1}e^{itc_2A_2}\cdots e^{itc_NA_N}.$$

By using (A.1), one can show that $\{U(t)\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group. Hence, by the Stone theorem, there exists a unique self-adjoint operator A on $\mathcal{H}$ such that $U(t) = e^{itA}$, $\quad t\in\mathbb{R}$. By Proposition A.4, $U(t)$ leaves $\mathcal{D}_0$ invariant and strongly differentiable on $\mathcal{D}_0$ with

$$\left.\frac{dU(t)\psi}{dt}\right|_{t=0} = i\sum_{j=1}^N c_jA_j\psi, \quad \psi\in\mathcal{D}_0.$$

Hence $\mathcal{D}_0$ is a core of A (e.g., [23, Theorem VIII.10]). Hence $A\psi = \sum_{j=1}^N c_j A_j \psi,\ \psi \in \mathcal{D}_0$. Thus the desired result follows. ∎

For all $c_j \in \mathbb{R}, j = 1, \ldots, N$, we set

$$A(\mathbf{c}) := \overline{\sum_{j=1}^N c_j A_j}, \quad \mathbf{c} = (c_1, \ldots, c_N) \in \mathbb{R}^N. \tag{A.14}$$

Corollary A.7 *For all $\mathbf{c}, \mathbf{d} \in \mathbb{R}^N$ and $t, s \in \mathbb{R}$,*

$$e^{itA(\mathbf{c})} e^{isA(\mathbf{d})} = e^{-its \sum_{j,k=1}^N a_{jk} c_j d_k} e^{isA(\mathbf{d})} e^{itA(\mathbf{c})}. \tag{A.15}$$

Proof. By direct computations using (A.13) and (A.1). ∎

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