



Title	Stokes Resolvent Estimates in Spaces of Bounded Functions
Author(s)	Abe, Ken; Giga, Yoshikazu; Hieber, Matthias
Citation	Hokkaido University Preprint Series in Mathematics, 1022, 1-24
Issue Date	2012-11-12
DOI	10.14943/84168
Doc URL	http://hdl.handle.net/2115/69827
Type	bulletin (article)
File Information	pre1022.pdf



[Instructions for use](#)

Stokes Resolvent Estimates in Spaces of Bounded Functions

Ken Abe ^{*} Yoshikazu Giga [†] Matthias Hieber [‡]

Abstract

We give a direct proof for the analyticity of the Stokes semigroup in spaces of bounded functions. This was recently proved by an indirect argument by the first and second authors for a class of domains called strictly admissible domains including bounded and exterior domains. Invoking the strictly admissibility, our approach is based on an adjustment of a standard resolvent estimate method for general elliptic operators introduced by K. Masuda (1972) and H. B. Stewart (1974). The resolvent approach in particular clarifies the sectorial region, $\operatorname{Re} z > 0$ for $z \in \mathbf{C}$ for which the Stokes semigroup has an analytic continuation in spaces of bounded functions.

Keywords Analytic semigroups; Bounded function spaces; Resolvent estimates; Stokes equations

Mathematics Subject Classification (2010) 35Q35, 35K90

1 Introduction and main results

The investigation of the linear Stokes equations as well as properties and corresponding estimates are often basis for the analysis of the nonlinear Navier-Stokes

^{*}K. Abe: Graduate School of Mathematical Sciences, University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo, 153-8914, Japan, E-mail: kabe@ms.u-tokyo.ac.jp

[†]Y. Giga: Graduate School of Mathematical Sciences, University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo, 153-8914, Japan, E-mail: labgiga@ms.u-tokyo.ac.jp

[‡]M. Hieber: Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgartenstr. 7, D-64289 Darmstadt, Germany and Center of Smart Interfaces, Petersenstr. 32, D-64287 Darmstadt, E-mail: hieber@mathematik.tu-darmstadt.de

equations. In particular, analyticity of the solution operator (called the Stokes semigroup) plays a fundamental role for studying the Navier-Stokes equations. It is well-known that the Stokes semigroup forms an analytic semigroup on $L^p_\sigma(\Omega)$ for $p \in (1, \infty)$, the space of L^p -solenoidal vector fields for various kind of domains $\Omega \subset \mathbf{R}^n$, $n \geq 2$ including bounded and exterior domains having smooth boundaries; see e.g. [35], [20]. By now, analyticity results are known for other type unbounded domains, see [16], [17], [3] ([5], [4] with variable viscosity coefficients) and Lipschitz domains [31]. An \tilde{L}^p -theory is developed in [12], [13], [14] for a general domain. Moreover, L^p -theory is investigated in [18] for unbounded domains, where the Helmholtz projection is bounded.

It is the aim of this paper to consider the case $p = \infty$. Note that the Helmholtz projection is no longer bounded in L^∞ even if $\Omega = \mathbf{R}^n$. When $\Omega = \mathbf{R}^n_+$, the analyticity of the semigroup is known in L^∞ -type spaces including $C_{0,\sigma}(\Omega)$, the L^∞ -closure of $C_{c,\sigma}^\infty(\Omega)$, the space of all smooth solenoidal vector fields compactly supported in Ω [10](see also [36], [25].) Their approach is based on explicit calculations of the solution operator $R(\lambda) : f \mapsto v = v_\lambda$ of the corresponding resolvent problem of

$$\lambda v - \Delta v + \nabla q = f \quad \text{in } \Omega, \quad (1.1)$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega, \quad (1.2)$$

$$v = 0 \quad \text{on } \partial\Omega. \quad (1.3)$$

As recently shown in [1], [2] by a blow-up argument, to the non-stationary Stokes equations, the Stokes semigroup is extendable to an analytic semigroup on $C_{0,\sigma}$ for *admissible domains* including bounded and exterior domains having boundaries of class C^3 .

In this paper, we present a direct resolvent approach to the Stokes resolvent equations (1.1)-(1.3) and establish the a priori estimate of the form

$$M_p(v, q)(x, \lambda) = |\lambda| |v(x)| + |\lambda|^{1/2} |\nabla v(x)| + |\lambda|^{n/2p} \|\nabla^2 v\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})} + |\lambda|^{n/2p} \|\nabla q\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})},$$

for $p > n$ and

$$\sup_{\lambda \in \Sigma_{\vartheta,\delta}} \|M_p(v, q)\|_{L^\infty(\Omega)}(\lambda) \leq C \|f\|_{L^\infty(\Omega)} \quad (1.4)$$

for some constant $C > 0$ independent of f . Here $\Omega_{x,r}$ denotes the intersection of Ω with an open ball $B_x(r)$ centered at $x \in \Omega$ with radius $r > 0$, i.e. $\Omega_{x,r} = B_x(r) \cap \Omega$ and $\Sigma_{\vartheta,\delta}$ denotes the sectorial region in the complex plane given by $\Sigma_{\vartheta,\delta} = \{\lambda \in \mathbf{C} \setminus \{0\} \mid |\arg \lambda| < \vartheta, |\lambda| > \delta\}$ for $\vartheta \in (\pi/2, \pi)$ and $\delta > 0$. Our approach is inspired by

the corresponding approach for general elliptic operators. K. Masuda was the first to prove analyticity of the semigroup associated to general elliptic operators in $C_0(\mathbf{R}^n)$ including the case of higher orders [27], [28] ([29].) This result was then extended by H. B. Stewart to the case for the Dirichlet problem [37] and more general boundary condition [38]. This Masuda-Stewart method was applied to many other situations [6], [24], [22], [7]. However, its application to the resolvent Stokes equations (1.1)-(1.3) was unknown.

In the sequel, we prove the estimate (1.4) by invoking the L^p -estimates for the Stokes resolvent equations with inhomogeneous divergence-free condition [15], [16]. We invoke *strictly admissibility* of a domain introduced in [2] which implies an estimate of pressure q in terms of the velocity by

$$\sup_{x \in \Omega} d_\Omega(x) |\nabla q(x)| \leq C_\Omega \|W(v)\|_{L^\infty(\partial\Omega)} \quad \text{with } W(v) = -(\nabla v - \nabla^T v)n_\Omega, \quad (1.5)$$

where ∇f denotes $(\partial f_i / \partial x_j)_{1 \leq i, j \leq n}$ and $\nabla^T f = (\nabla f)^T$ for a vector field $f = (f_i)_{1 \leq i \leq n}$. The estimate (1.5) plays a key role in transferring results from the elliptic situation to the situation of the Stokes system. Here n_Ω denotes the unit outward normal vector field on $\partial\Omega$ and d_Ω denotes the distance function from the boundary, $d_\Omega(x) = \inf_{y \in \partial\Omega} |x - y|$ for $x \in \Omega$. The estimate (1.5) can be viewed as a regularizing-type estimate for solutions to the Laplace equation $\Delta P = 0$ in Ω with the Neumann boundary condition $\partial P / \partial n_\Omega = \text{div}_{\partial\Omega} W$ on $\partial\Omega$ for a tangential vector field W where $\text{div}_{\partial\Omega} = \text{tr } \nabla_{\partial\Omega}$ denotes the surface divergence and $\nabla_{\partial\Omega} = \nabla - n_\Omega(n_\Omega \cdot \nabla)$ is the gradient on $\partial\Omega$. It is known that $P = q$ solves this Neumann problem with $W = W(v)$ given by (1.5) [2, Lemma 3.1] and the estimate (1.5) holds for bounded domains [1] and exterior domains [2]. Note that when $n = 3$, $-W(v)$ is nothing but a tangential trace of vorticity, i.e. $-W(v) = \text{curl } v \times n_\Omega$. We call Ω *strictly admissible* if there exists a constant $C = C_\Omega$ such that the a priori estimate

$$\|\nabla P\|_{L_d^\infty(\Omega)} \leq C \|W\|_{L^\infty(\partial\Omega)} \quad (1.6)$$

holds for all solutions P of the Neumann problem with a tangential vector field $W \in L^\infty(\partial\Omega)$. Here $L_d^\infty(\Omega)$ denotes the space of all locally integrable functions f such that $d_\Omega f$ is essentially bounded in Ω and equipped with the norm $\|f\|_{L_d^\infty(\Omega)} = \sup_{x \in \Omega} d_\Omega(x) |f(x)|$. The meaning of a solution is understood in the weak sense, i.e. we say $\nabla P \in L_d^\infty(\Omega)$ is a solution for the Neumann problem if $\int_\Omega P \Delta \varphi dx = - \int_{\partial\Omega} W \cdot \nabla_{\partial\Omega} \varphi d\mathcal{H}^{n-1}(x)$ holds for all $\varphi \in C_c^2(\bar{\Omega})$ satisfying $\partial \varphi / \partial n_\Omega = 0$ on $\partial\Omega$, where \mathcal{H}^{n-1} denotes the $n - 1$ dimensional Hausdorff measure.

We are now in the position to formulate the main results of this paper.

Theorem 1.1. *Let Ω be a strictly admissible, uniformly C^2 -domain in \mathbf{R}^n (with $n \geq 2$.) Let $p > n$. For $\vartheta \in (\pi/2, \pi)$ there exist constants δ and C such that the a priori estimate (1.4) holds for all solutions $(v, \nabla q) \in W_{loc}^{2,p}(\bar{\Omega}) \times (L_{loc}^p(\bar{\Omega}) \cap L_d^\infty(\Omega))$ of (1.1)-(1.3) for $f \in C_{0,\sigma}(\Omega)$ with $\lambda \in \Sigma_{\vartheta,\delta}$.*

The a priori estimate (1.4) implies the analyticity of the Stokes semigroup in L^∞ -type spaces. Let us observe the generation of an analytic semigroup in $C_{0,\sigma}(\Omega)$. By invoking the \tilde{L}^p -theory [12], [13], [14] we verify the existence of a solution to (1.1)-(1.3), $(v, \nabla q) \in W_{loc}^{2,p}(\bar{\Omega}) \times (L_{loc}^p(\bar{\Omega}) \cap L_d^\infty(\Omega))$ for $f \in C_{c,\sigma}^\infty(\Omega)$ in a uniformly C^2 -domain Ω . The solution operator $R(\lambda)$ is then uniquely extendable to $C_{0,\sigma}(\Omega)$ by the uniform approximation together with the estimates (1.4). Here the solution operator to the pressure gradient $f \mapsto \nabla q_\lambda$ is also uniquely extended for $f \in C_{0,\sigma}$. We observe that $R(\lambda)$ is injective on $C_{0,\sigma}$ since the estimate (1.5) immediately implies that $f = 0$ for $f \in C_{0,\sigma}$ such that $v_\lambda = R(\lambda)f = 0$. The operator $R(\lambda)$ may be regarded as a surjective operator from $C_{0,\sigma}$ to the range of $R(\lambda)$. The open mapping theorem then implies the existence of a closed operator A such that $R(\lambda) = (\lambda - A)^{-1}$; see [8, Proposition B.6]. We call A the Stokes operator in $C_{0,\sigma}(\Omega)$. From Theorem 1.1 we obtain

Theorem 1.2. *Let Ω be a strictly admissible, uniformly C^2 -domain in \mathbf{R}^n . Then the Stokes operator A generates a C_0 -analytic semigroup on $C_{0,\sigma}(\Omega)$ of angle $\pi/2$.*

We next consider the space $L_\sigma^\infty(\Omega)$ defined by

$$L_\sigma^\infty(\Omega) = \left\{ f \in L^\infty(\Omega) \mid \int_\Omega f \cdot \nabla \varphi dx = 0 \text{ for all } \varphi \in \hat{W}^{1,1}(\Omega) \right\},$$

where $\hat{W}^{1,1}(\Omega)$ denotes the homogeneous Sobolev space of the form $\hat{W}^{1,1}(\Omega) = \{\varphi \in L_{loc}^1(\Omega) \mid \nabla \varphi \in L^1(\Omega)\}$. Note that $C_{0,\sigma}(\Omega) \subset L_\sigma^\infty(\Omega)$. When a domain Ω is unbounded, the space $L_\sigma^\infty(\Omega)$ includes non-decaying solenoidal vector fields at the space infinity. Actually, the a priori estimates (1.4) is also valid for $f \in L_\sigma^\infty$. In particular, (1.4) implies the uniqueness of a solution for $f \in L_\sigma^\infty$. We verify the existence of a solution by approximating $f \in L_\sigma^\infty$ with compactly supported solenoidal vector fields $\{f_m\}_{m=1}^\infty \subset C_{c,\sigma}^\infty$. Note that $f \in L_\sigma^\infty$ is no longer approximated in the uniform topology in general so we weaken the convergence topology, for example, to the pointwise convergence, i.e. $f_m \rightarrow f$ a.e. in Ω keeping $\|f_m\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}$ with some constant $C = C_\Omega$. When a domain Ω is bounded, it is known that this approximation is valid [1, Lemma 6.3]. Moreover, we are able to take $C_\Omega = 1$ without assuming any regularity of $\partial\Omega$ since $L_\sigma^\infty \subset L_\sigma^r$. Although it

is unknown whether this approximation is valid for general domains, we are able to characterize the space L_σ^∞ by this approximation at least for exterior domains [2, Lemma 5.1]. In the following, we restrict our results to bounded and exterior domains. Through the approximation argument we verify the existence of a solution to (1.1)-(1.3) for general $f \in L_\sigma^\infty$. We then define the Stokes operator on L_σ^∞ by the same way as for $C_{0,\sigma}$. Since bounded and exterior domains are strictly admissible [1, Theorem 2.5], [2, Theorem 2.9] provided that the boundary is C^3 , we have

Theorem 1.3. *Assume that Ω is a bounded or an exterior domain with C^3 -boundary. Then the Stokes operator A generates a (non C_0 -)analytic semigroup on $L_\sigma^\infty(\Omega)$ of angle $\pi/2$.*

Remarks 1.4. (i) The direct resolvent approach clarifies the angle of the analyticity of the Stokes semigroup e^{tA} on $C_{0,\sigma}$. Theorem 1.2 (and also Theorem 1.3) asserts that e^{tA} is angle $\pi/2$ on $C_{0,\sigma}$ which does not follow from a priori L^∞ -estimates for solutions to the non-stationary Stokes equations proved by blow-up arguments [1, Theorem 1.2], [2, Theorem 1.5]; see also [19] for blow-up arguments.

(ii) We observe that our argument applies to other boundary conditions, for example, to the Robin boundary condition, i.e. $B(v) = 0$ and $v \cdot n_\Omega = 0$ on $\partial\Omega$ where

$$B(v) = \alpha v_{\tan} + (D(v)n_\Omega)_{\tan} \quad \text{with } \alpha \geq 0.$$

Here $D(v) = (\nabla v + \nabla^T v)/2$ denotes the deformation tensor and f_{\tan} the tangential component of a vector field f on $\partial\Omega$. Note that the case $\alpha = \infty$ corresponds to the Dirichlet boundary condition (1.3); see [30] for generation results subject to the Robin boundary conditions on L^∞ for \mathbf{R}_+^n . The L^p -resolvent estimates for the Robin boundary condition was established in [21] for concerning analyticity and was later strengthened in [32] to non-divergence free vector fields. We shall use the generalized resolvent estimate in [32] to extend our result in spaces of bounded functions to the Robin boundary condition (Theorem 3.6). For a more detailed discussion, see Remark 3.5.

(iii) We observe that the domain of the Stokes operator $D(A)$ is dense in $C_{0,\sigma}$. In fact, by invoking the \tilde{L}^p -theory and using (1.4) we have

$$\|\lambda v - f\|_{L^\infty(\Omega)} = \|\tilde{A}_p v\|_{L^\infty(\Omega)} \leq \frac{C}{|\lambda|} \|\tilde{A}_p f\|_{L^\infty(\Omega)} \rightarrow 0, \quad |\lambda| \rightarrow \infty$$

for $f \in C_{c,\sigma}^\infty \subset D(\tilde{A}_p)$, where \tilde{A}_p is the Stokes operator in \tilde{L}^p . Thus we conclude that $D(A)$ is dense in $C_{0,\sigma}$. On the other hand, smooth functions are not dense in

L^∞ and $e^{tA}f$ is smooth for $t > 0$, $e^{tA}f \rightarrow f$ as $t \downarrow 0$ in L^∞_σ does not hold for some $f \in L^\infty_\sigma$. This means e^{tA} is a non- C_0 analytic semigroup. We refer to [34, 1.1.2] for properties of the analytic semigroup generated by non-densely defined sectorial operators; see also [8, Definition 3.2.5].

(iv) For a bounded domain Ω , $v(\cdot, t) = e^{tA}v_0$ and $\nabla q = (1 - \mathbf{P})[\Delta v]$ give a solution to the non-stationary Stokes equations, $v_t - \Delta v + \nabla q = 0$, $\operatorname{div} v = 0$ in $\Omega \times (0, \infty)$ with $v = 0$ on $\partial\Omega$ for initial data $v_0 \in L^\infty_\sigma(\Omega)$. Although for general domains the Helmholtz projection operator $\mathbf{P} : L^p(\Omega) \rightarrow L^p_\sigma(\Omega)$ is no longer bounded on L^∞ even if $\Omega = \mathbf{R}^n$, we are able to define the pressure $\nabla q = \mathbf{K}[W(v)]$ at least for exterior domains Ω by the solution operator to the Neumann problem (*harmonic-pressure operator*) $\mathbf{K} : L^\infty_{\tan}(\partial\Omega) \ni W \mapsto \nabla P \in L^\infty_d(\Omega)$ [2, Remark 2.10]. Here $L^\infty_{\tan}(\partial\Omega)$ denotes the closed subspace of all tangential vector fields in $L^\infty(\partial\Omega)$.

(v) We observe that the Masuda-Stewart method does not imply the large time behavior for e^{tA} . For a bounded domain the energy inequality implies that maximum of $v(\cdot, t) = e^{tA}v_0$ (and also v_t) decay exponentially as $t \rightarrow \infty$ [1, Remark 5.4 (i)]. In particular, e^{tA} is a bounded analytic semigroup on L^∞_σ . Recently, based on the L^∞ -estimates [1, Theorem 1.2] it was shown in [26] that e^{tA} is bounded semigroup on L^∞_σ for exterior domains by appealing to the maximum modulus theorem for the boundary-value problem of the stationary Stokes equations. Note that it is unknown whether e^{tA} is a bounded analytic semigroup on L^∞_σ .

In the sequel, we sketch a proof for the a priori estimate (1.4). Our argument can be divided into the following three steps:

(i) (Localization) We first localize a solution (v, q) of the Stokes equations (1.1)-(1.3) in a domain $\Omega' = B_{x_0}((\eta + 1)r) \cap \Omega$ for $x_0 \in \Omega$, $r > 0$ and parameters $\eta \geq 1$ by setting $u = v\theta_0$ and $p = (q - q_c)\theta_0$ with a constant q_c and the smooth cut-off function θ_0 around $\Omega_{x_0, r}$ satisfying $\theta_0 \equiv 1$ in $B_{x_0}(r)$ and $\theta_0 \equiv 0$ in $B_{x_0}((\eta + 1)r)^c$. We then observe that (u, p) solves the Stokes resolvent equations with inhomogeneous divergence-free condition in the localized domain Ω' . Applying the L^p -estimates for the localized Stokes equations we have

$$\begin{aligned} & |\lambda| \|u\|_{L^p(\Omega')} + |\lambda|^{1/2} \|\nabla u\|_{L^p(\Omega')} + \|\nabla^2 u\|_{L^p(\Omega')} + \|\nabla p\|_{L^p(\Omega')} \\ & \leq C_p \left(\|h\|_{L^p(\Omega')} + \|\nabla g\|_{L^p(\Omega')} + |\lambda| \|g\|_{W_0^{-1,p}(\Omega')} \right), \end{aligned} \quad (1.7)$$

where $W_0^{-1,p}(\Omega')$ denotes the dual space of the Sobolev space $W^{1,p'}(\Omega')$ with $1/p + 1/p' = 1$. The external forces h and g contain error terms appearing in the cut-off

procedure and are explicitly given by

$$h = f\theta_0 - 2\nabla v \nabla \theta_0 - v \Delta \theta_0 + (q - q_c) \nabla \theta_0, \quad g = v \cdot \nabla \theta_0. \quad (1.8)$$

(ii) (Error estimates) A key step is to estimate the error terms of the pressure such as $(q - q_c) \nabla \theta_0$. We here simplify the description by disregarding the terms related to g in order to describe the essence of the proof. We will give precise estimates for the terms related to g in Section 3. Now, the error terms related to h are estimated in the form

$$\|h\|_{L^p(\Omega')} \leq Cr^{n/p} \left((\eta + 1)^{n/p} \|f\|_{L^\infty(\Omega)} + (\eta + 1)^{-(1-n/p)} \left(r^{-2} \|v\|_{L^\infty(\Omega)} + r^{-1} \|\nabla v\|_{L^\infty(\Omega)} \right) \right). \quad (1.9)$$

If we disregard the term $(q - q_c) \nabla \theta_0$ in h , the estimates (1.8) easily follows by using the estimates of the cut-off function θ_0 , i.e. $\|\theta_0\|_\infty + (\eta + 1)r \|\nabla \theta_0\|_\infty + (\eta + 1)^2 r^2 \|\nabla^2 \theta_0\|_\infty \leq K$ with some constant K . We invoke the estimate (1.5) in order to handle the pressure term by velocity through the *Poincaré-Sobolev-type inequality*:

$$\|\varphi - (\varphi)\|_{L^p(\Omega_{x_0,s})} \leq Cs^{n/p} \|\nabla \varphi\|_{L_d^\infty(\Omega)} \quad \text{for all } \varphi \in \hat{W}_d^{1,\infty}(\Omega), \quad (1.10)$$

with some constant C independent of $s > 0$, where (φ) denotes the mean value of φ in $\Omega_{x_0,s}$ and $\hat{W}_d^{1,\infty}(\Omega) = \{\varphi \in L_{\text{loc}}^1(\bar{\Omega}) \mid \nabla \varphi \in L_d^\infty(\Omega)\}$. We prove the inequality (1.10) in Section 2. By taking $q_c = (q)$ and applying (1.10) for $\varphi = q$ and $s = (\eta + 1)r$ we obtain the estimate (1.9) via (1.5).

(iii) (Interpolation) Once we establish the error estimates for h and g , it is easy to obtain the estimate (1.4) by applying the interpolation inequality,

$$\|\varphi\|_{L^\infty(\Omega_{x_0,r})} \leq Cr^{-n/p} \left(\|\varphi\|_{L^p(\Omega_{x_0,r})} + r \|\nabla \varphi\|_{L^p(\Omega_{x_0,r})} \right) \quad \text{for } \varphi \in W_{\text{loc}}^{1,p}(\bar{\Omega}), \quad (1.11)$$

for $\varphi = u$ and ∇u . Now taking $r = |\lambda|^{-1/2}$ we obtain the estimate for $M_p(v, q)(x_0, \lambda)$ with the parameters η of the form,

$$M_p(v, q)(x_0, \lambda) \leq C \left((\eta + 1)^{n/p} \|f\|_{L^\infty(\Omega)} + (\eta + 1)^{-(1-n/p)} \|M_p(v, q)\|_{L^\infty(\Omega)}(\lambda) \right) \quad (1.12)$$

for some constant C independent of η . The second term in the right-hand-side is absorbed into the left-hand-side by letting η sufficiently large provided $p > n$.

Actually, in the procedure (ii) we take q_c by the mean value of q in $\Omega_{x_0,(\eta+2)r}$ since we estimate $|\lambda| \|g\|_{W_0^{-1,p}}$. By using the equation (1.1) we reduce the estimate of $|\lambda| \|g\|_{W_0^{-1,p}}$ to the L^∞ -estimate for the boundary value of $q - q_c$ on $\partial\Omega'$. In order to

estimate $\|q - q_c\|_{L^\infty(\Omega')}$ we use a uniformly local L^p -norm bound for ∇q besides the sup-bound for ∇v . This is the reason why we need the norm $\|M_p(v, q)\|_{L^\infty(\Omega)}(\lambda)$ in the right-hand-side of (1.12). For general elliptic operators, the estimate (1.12) is valid without invoking the uniformly local L^p -norm bound for second derivatives of a solution.

This paper is organized as follows. In Section 2 we prove the inequality (1.10) for uniformly C^2 -domains. More precisely, we prove stronger estimates than (1.10) both interior and up to boundary $\Omega_{x_0, s}$ of Ω . In Section 3 we first prepare the estimates for h and g and then prove the a priori estimate (1.4) (Theorem 1.1.) After proving Theorem 1.1, we also note the estimates (1.4) subject to the Robin boundary condition.

2 Poincaré-Sobolev-type inequality

In this section we prove the inequality (1.10) in a uniformly C^2 -domain. We start with the Poincaré-Sobolev-type inequality in a bounded domain D and observe the compactness of the embedding from $\hat{W}_d^{1, \infty}(D)$ to $L^p(D)$ which is the key in proving the inequality (1.10) by *reductio ad absurdum*.

2.1 Curvilinear coordinates

Let D be a bounded domain in \mathbf{R}^n , $n \geq 2$ and $p \in [1, \infty)$. We prove an inequality of the form,

$$\|\varphi - (\varphi)\|_{L^p(D)} \leq C \|\nabla \varphi\|_{L_d^\infty(D)} \quad \text{for } \varphi \in \hat{W}_d^{1, \infty}(D) \quad (2.1)$$

where (φ) denotes the mean value of φ in D , i.e. $(\varphi) = \int_D \varphi dx$. If we replace the norm $\|\nabla \varphi\|_{L_d^\infty(D)}$ by the L^p -norm $\|\nabla \varphi\|_{L^p(D)}$, the estimate (2.1) is nothing but the Poincaré inequality [11, 5.8.1]. We observe that the boundedness of $\|\nabla \varphi\|_{L_d^\infty(\Omega)}$ implies L^p -integrability of φ in D even if $\nabla \varphi$ is not in $L^p(D)$. For example, when $D = B_0(1)$, $\varphi(x) = \log(1 - |x|)$ is in L^p although $|\nabla \varphi(x)| = d_D(x)^{-1}$ is not for any $p \in [1, \infty)$. Since the space $\hat{W}_d^{1, \infty}$ is compactly embedded to the space $C(\bar{D}')$ for each subdomain D' of D with $\bar{D}' \subset D$, we shall show a pointwise upper bound for φ near $\partial D'$ by an L^p -integrable function to conclude that the space $\hat{W}_d^{1, \infty}(D)$ is compactly embedded to $L^p(D)$ by the dominated convergence theorem. We estimate $\varphi \in \hat{W}_d^{1, \infty}(D)$ near ∂D directly by using the curvilinear coordinates. Here for a domain Ω , $\partial\Omega \neq \emptyset$, we say that $\partial\Omega$ is C^k if for each $x_0 \in \partial\Omega$, there exist

constants α, β and C^k -function h of $n - 1$ variables y' such that (up to rotation and translation if necessary) we have

$$\begin{aligned} U(x_0) \cap \Omega &= \{(y', y_n) \mid h(y') < y_n < h(y') + \beta, |y'| < \alpha\}, \\ U(x_0) \cap \partial\Omega &= \{(y', y_n) \mid y_n = h(y'), |y'| < \alpha\}, \\ \sup_{|l| \leq k, |y'| < \alpha} |\partial_{y'}^l h(y')| &\leq K, \quad \nabla' h(0) = 0, \quad h(0) = 0, \end{aligned}$$

with the constant K and the neighborhood of x_0 , $U(x_0) = U_{\alpha, \beta, h}(x_0)$, i.e.

$$U_{\alpha, \beta, h}(x_0) = \{(y', y_n) \in \mathbf{R}^n \mid h(y') - \beta < y_n < h(y') + \beta, |y'| < \alpha\}.$$

Here $\partial_x^l = \partial_{x_1}^{l_1} \cdots \partial_{x_n}^{l_n}$ for a multi-index $l = (l_1, \dots, l_n)$ and $\partial_{x_j} = \partial/\partial x_j$ as usual and ∇' denotes the gradient in \mathbf{R}^{n-1} . Moreover, if we are able to take uniform constants α, β, K independent of each $x_0 \in \partial\Omega$, we call Ω uniformly C^k -domain of type (α, β, K) as defined in [33, I.3.2].

We estimate $\varphi \in \hat{W}_d^{1,1}(\Omega)$ along the boundary using the curvilinear coordinates.

Proposition 2.1. *Let D be a bounded domain with C^k -boundary ($k \geq 2$). Let $\Gamma = \{x \in \partial D \mid x = (x', h(x')), |x'| < \alpha'\}$ be a neighborhood of $x_0 \in \partial D$. (i) There exist positive constants μ and α' such that $(\gamma, d) \mapsto X(\gamma, d) = \gamma + dn_D(\gamma)$ is a C^{k-1} diffeomorphism from $\Gamma \times (0, \mu)$ onto*

$$\mathcal{N}^\mu(\Gamma) = \{X(\gamma, d) \in U(x_0) \mid (\gamma, d) \in \Gamma \times (0, \mu)\},$$

i.e. $x \in \mathcal{N}^\mu(\Gamma)$ has a unique projection to ∂D denoted by $\gamma(x) \in \partial D$ such that

$$(\gamma(x), d_D(x)) = X^{-1}(x) \quad \text{for } x \in \mathcal{N}^\mu(\Gamma).$$

(ii) There exists a constant C_1 such that for any $x_1 \in \overline{\mathcal{N}^\mu(\Gamma)}$ and $r_1 > 0$ satisfying $D_{x_1, r_1} = B_{x_1}(r_1) \cap D \subset \mathcal{N}^\mu(\Gamma)$,

$$|\varphi(x) - \varphi(y)| \leq C_1 \left(\left| \log \frac{d_D(x)}{d_D(y)} \right| + \frac{|\gamma(x) - \gamma(y)|}{\max\{d_D(x), d_D(y)\}} \right) \sup_{z \in D_{x_1, r_1}} d_D(z) |\nabla \varphi(z)| \quad \text{for } x, y \in D_{x_1, r_1}$$

and $\varphi \in \hat{W}_d^{1,\infty}(D)$.

Proof. The assertion (i) is based on the inverse function theorem [23, Lemma 4.4.7]. We shall prove the second assertion (ii). We take points $x, y \in D_{x_1, r_1}$ for

$x_1 \in \overline{\mathcal{N}^\mu(\Gamma)}$ and $r_1 > 0$ satisfying $D_{x_1, r_1} \subset \mathcal{N}^\mu(\Gamma)$. We may assume $d_D(y) = d(y) > d(x)$. By setting $z = (\gamma(x), d(y))$ we estimate

$$|\varphi(x) - \varphi(y)| \leq |\varphi(x) - \varphi(z)| + |\varphi(z) - \varphi(y)|.$$

We connect x and z by the straight line to estimate

$$\begin{aligned} |\varphi(x) - \varphi(z)| &= \left| \int_0^1 \frac{d}{dt} \varphi(X(\gamma(x), td(x) + (1-t)d(y))) dt \right| \\ &= \left| \int_0^1 (d(y) - d(x)) (\nabla \varphi)(X(\gamma(x), td(x) + (1-t)d(y))) \cdot n_D(\gamma(x)) dt \right| \\ &\leq (d(y) - d(x)) \int_0^1 \frac{dt}{t(d(x) - d(y)) + d(y)} \sup_{z \in D_{x_1, r_1}} d(z) |\nabla \varphi(z)| \\ &= \left| \log \frac{d(y)}{d(x)} \right| \sup_{z \in D_{x_1, r_1}} d(z) |\nabla \varphi(z)|. \end{aligned}$$

It remains to estimate $|\varphi(z) - \varphi(y)|$. We connect z and y by the curve

$$C_{z,y} = \{X(\gamma(t), d(y)) \mid \gamma(t) = (\gamma'(t), h(\gamma'(t))), \gamma'(t) = t\gamma'(x) + (1-t)\gamma'(y), 0 \leq t \leq 1\},$$

where γ' denotes the $n-1$ variables of γ . We then estimate

$$\begin{aligned} |\varphi(z) - \varphi(y)| &= \left| \int_0^1 \frac{d}{dt} \varphi(X(\gamma(t), d(y))) dt \right| \\ &= \left| \int_0^1 \frac{d\gamma}{dt}(t) (1 + d(y) \nabla_{\partial D} n_D(\gamma(t))) \nabla \varphi(X(\gamma(t), d(y))) dt \right| \\ &\leq C(1 + \mu K) \frac{|\gamma(x) - \gamma(y)|}{d(y)} \sup_{z \in D_{x_1, r_1}} d(z) |\nabla \varphi(z)|, \end{aligned}$$

since $|d\gamma(t)/dt| \leq C|\gamma(x) - \gamma(y)|$ and $|\nabla_{\partial D} n_D| \leq K$ with a constant C depending on K . The assertion (ii) thus follows. \square

Remarks 2.2. (i) We observe from the second assertion that $\varphi \in \hat{W}_d^{1, \infty}(D)$ is bounded from above by an L^p -integrable function for all $p \in [1, \infty)$ near ∂D , i.e. for each fixed $y \in D_{x_1, r_1}$ such that $d_D(y) \geq \delta$ we have

$$|\varphi(x)| \leq C_2(|\log d_D(x)| + 1) \left(\sup_{z \in D_{x_1, r_1}} d_D(z) |\nabla \varphi(z)| \right) + |\varphi(y)| \quad \text{for } x \in D_{x_1, r_1} \quad (2.2)$$

with a constant C_2 depending on μ, δ .

(ii) Note that Proposition 2.1 is also valid for a uniformly C^k -domain Ω of type (α, β, K) , i.e. there exist constants μ, α' , depending only on α, β, K , such that for each $x_0 \in \partial\Omega$ the assertions (i) and (ii) hold. The above constants C_1 and C_2 are depending only on α, β, K and δ . In the sequel, we will apply Proposition 2.1 to a uniformly C^2 -domain to prove the inequality (1.10).

The estimate (2.2) implies the compactness from $\hat{W}_d^{1,\infty}(D)$ to $L^p(D)$.

Lemma 2.3. *Let D be a bounded domain in $\mathbf{R}^n, n \geq 2$ with C^2 -boundary. Then there exists a constant C_D such that the estimate (2.1) holds for all $\varphi \in \hat{W}_d^{1,\infty}(D)$. Moreover, the space $\hat{W}_d^{1,\infty}(D)$ is compactly embedded into $L^p(D)$.*

Proof. We argue by contradiction. Suppose that the estimate (2.1) were false for any choice of the constant C . Then there would exist a sequence of functions $\{\varphi_m\}_{m=1}^\infty \subset \hat{W}_d^{1,\infty}(D)$ such that

$$\|\varphi_m - (\varphi_m)\|_{L^p(D)} > m \|\nabla \varphi_m\|_{L_d^\infty(D)}, \quad m \in \mathbf{N}.$$

We may assume $(\varphi_m) = 0$ by replacing φ_m to $\varphi_m - (\varphi_m)$. We divide φ_m by $M_m = \|\varphi_m\|_{L^p(D)}$ to get a sequence of functions $\{\phi_m\}_{m=1}^\infty, \phi_m = \varphi_m/M_m$ such that

$$\begin{aligned} \|\nabla \phi_m\|_{L_d^\infty(D)} &< 1/m, \\ \|\phi_m\|_{L^p(D)} &= 1 \quad \text{with } (\phi_m) = 0. \end{aligned}$$

We now prove the compactness of $\{\phi_m\}_{m=1}^\infty$ in $L^p(D)$. Since $\|\nabla \phi_m\|_{L_d^\infty(D)}$ is bounded, $\{\phi_m\}_{m=1}^\infty$ subsequently converges to a limit $\bar{\phi}$ locally uniformly in D . By Proposition 2.1, in particular, the estimate (2.2) implies that ϕ_m is uniformly bounded from above by an L^p -integrable function near ∂D . The dominated convergence theorem implies that

$$\phi_m \rightarrow \bar{\phi} \quad \text{in } L^p(D) \quad \text{as } m \rightarrow \infty.$$

Since $\nabla \phi_m(x) \rightarrow 0$ as $m \rightarrow \infty$ for each $x \in D$ and $\|\bar{\phi}\|_{L^p(D)} = 1, \bar{\phi}$ is a non-zero constant which contradicts the fact that $(\bar{\phi}) = 0$. We reached a contradiction.

For the compactness of $\{\phi_m\}_{m=1}^\infty$ in $L^p(D)$ we here only invoke the boundedness of $\|\nabla \phi_m\|_{L_d^\infty(D)}$. This means that the embedding from $\hat{W}_d^{1,\infty}(D)$ into $L^p(D)$ is compact. The proof is now complete. \square

2.2 Estimates near the boundary

We now prove the inequality (1.10) for uniformly C^2 -domains Ω . When the ball $B_{x_0}(r)$ locates interior of Ω , i.e. $\Omega_{x_0,r} = B_{x_0}(r)$, applying (2.1) to $\varphi_r(x) = \varphi(x_0 + rx)$ in $D = \text{int } B_0(1)$ implies the estimate

$$\|\varphi - (\varphi)\|_{L^p(\Omega_{x_0,r})} \leq Cr^{n/p} \sup_{z \in \Omega_{x_0,r}} d_{\Omega_{x_0,r}}(z) |\nabla \varphi(z)|, \quad r > 0. \quad (2.3)$$

Since $d_{\Omega_{x_0,r}}(x) \leq d_{\Omega}(x)$ for $x \in \Omega_{x_0,r}$, the assertion (1.10) follows. However, if $B_{x_0}(r)$ involves $\partial\Omega$, the boundary of $\Omega_{x_0,r}$ may not have C^1 -regularity. We thus prove

$$\|\varphi - (\varphi)\|_{L^p(\Omega_{x_0,r})} \leq Cr^{n/p} \sup_{z \in \Omega_{x_0,r}} d_{\Omega}(z) |\nabla \varphi(z)| \quad \text{for } \varphi \in \hat{W}_d^{1,\infty}(\Omega) \quad (2.4)$$

for $x_0 \in \Omega$ and $r > 0$ satisfying $d_{\Omega}(x_0) < r$, which is weaker than (2.3).

Proposition 2.4. *Let Ω be a uniformly C^2 -domain. There exist constants r_0 and C such that for $x_0 \in \Omega$ and $r < r_0$ satisfying $d_{\Omega}(x_0) < r$, the estimate (2.4) holds for all $\varphi \in \hat{W}_d^{1,\infty}(\Omega)$ with a constant C independent of x_0 and r .*

The inequality (1.10) easily follows from Proposition 2.4.

Lemma 2.5. *The inequality (1.10) holds for $\varphi \in \hat{W}_d^{1,\infty}(\Omega)$ for all $x_0 \in \Omega$ and $r < r_0$ with a constant C independent of x_0 and r .*

Proof. For $r < r_0$, combining (2.3) for $d_{\Omega}(x_0) > r$ with (2.4) for $d_{\Omega}(x_0) < r$, the assertion (1.10) follows. \square

Proof of Proposition 2.4. We argue by contradiction. Suppose that the estimate (2.4) were false for any choice of constants r_0 and C . Then, there would exist a sequence of functions $\{\varphi_m\}_{m=1}^{\infty} \subset \hat{W}_d^{1,\infty}(\Omega)$ and a sequence of points $\{x_m\}_{m=1}^{\infty} \subset \Omega$ satisfying $d_{\Omega}(x_m) < r_m \downarrow 0$ such that

$$\|\varphi_m - (\varphi_m)\|_{L^p(\Omega_{x_m,r_m})} > mr_m^{n/p} \sup_{z \in \Omega_{x_m,r_m}} d_{\Omega}(z) |\nabla \varphi_m(z)|, \quad m \in \mathbf{N}.$$

Replacing φ_m by $\varphi_m - (\varphi_m)$ and dividing φ_m by $r_m^{-n/p} \|\varphi_m\|_{L^p(\Omega_{x_m,r_m})}$ (still denoted by φ_m) we observe that φ_m satisfies $r_m^{-n/p} \|\varphi_m\|_{L^p(\Omega_{x_m,r_m})} = 1$ with $(\varphi_m) = 0$ and $\sup_{z \in \Omega_{x_m,r_m}} d_{\Omega}(z) |\nabla \varphi_m(z)| < 1/m$. Since the points $\{x_m\}_{m=1}^{\infty}$ accumulates at the boundary $\partial\Omega$, we may assume by rotation and translation of Ω that $x_m = (0, d_m)$ with

$d_m = d_\Omega(x_m)$ which subsequently converges to the origin located on the boundary $\partial\Omega$. Here the neighborhood of the origin is denoted by $\Omega_{\text{loc}} = U(0) \cap \Omega$ with constants α, β and C^2 -function h , i.e.

$$\Omega_{\text{loc}} = \{(x', x_n) \in \mathbf{R}_+^n \mid h(x') < x_n < h(x') + \beta, |x'| < \alpha\}.$$

We rescale φ_m around the point x_m by setting

$$\phi_m(x) = \varphi_m(x_m + r_m x) \quad \text{for } x \in \Omega^m,$$

where $\Omega^m = \{x \in \Omega \mid x = (y - x_m)/r_m, y \in \Omega\}$ is the rescaled domain. Since $c_m = d_m/r_m < 1$, by taking a subsequence we may assume $\lim_{m \rightarrow \infty} c_m = c_0 \leq 1$. We then observe that the rescaled domain Ω^m expands to a half space $\mathbf{R}_{+,-c_0}^n = \{(x', x_n) \in \mathbf{R}^n \mid x_n > -c_0\}$. In fact, the neighborhood $\Omega_{\text{loc}} \subset \Omega$ is rescaled to the domain,

$$\Omega_{\text{loc}}^m = \left\{ (x', x_n) \in \mathbf{R}^n \mid \frac{1}{r_m} h(r_m x') - c_m < x_n < \frac{1}{r_m} h(r_m x') + \frac{\beta}{r_m}, |x'| < \frac{\alpha}{r_m} \right\}$$

which converges to $\mathbf{R}_{+,-c_0}^n$ by letting $m \rightarrow \infty$. Note that constants of uniform regularity of $\partial\Omega_m$ are uniformly bounded under this rescaling procedure. Moreover, for any constants μ and α' , the curvilinear neighborhood of the origin $\mathcal{N}^\mu(\Gamma)$ is in Ω_{loc}^m for sufficiently large $m \geq 1$, where $\Gamma = \Gamma_{\alpha'}(0)$ is the neighborhood of the origin on $\partial\Omega^m$. Then the estimates for φ_m are inherited to the estimates for ϕ_m , i.e.

$$\sup_{z \in \Omega_{0,1}^m} d_{\Omega^m}(z) |\nabla \phi_m(z)| < 1/m, \quad m \in \mathbf{N},$$

$$\|\phi_m\|_{L^p(\Omega_{0,1}^m)} = 1 \quad \text{with } (\phi_m) = \int_{\Omega_{0,1}^m} \phi_m = 0,$$

where $\Omega_{0,1}^m = B_0(1) \cap \Omega^m$. From above bound for ϕ_m the sequence $\{\phi_m\}_{m=1}^\infty$ subsequently converges to a limit $\bar{\phi}$ locally uniformly in $(\mathbf{R}_{+,-c_0}^n)_{0,1} = \mathbf{R}_{+,-c_0}^n \cap B_0(1)$. We now observe the compactness of the sequence $\{\phi_m\}_{m=1}^\infty$ in $L^p((\mathbf{R}_{+,-c_0}^n)_{0,1})$. By Remark 2.2 (ii) applying Proposition 2.1 to Ω^m , the estimate (2.2) with $x_1 = 0, r = 1$ and a fixed $y \in \Omega_{0,1}^m$ satisfying $d_{\Omega^m}(y) \geq \delta$ yield

$$|\varphi_m(x)| \leq C(|\log d_{\Omega^m}(x)| + 1) \left(\sup_{z \in \Omega_{0,1}^m} d_{\Omega^m}(z) |\nabla \phi_m(z)| \right) + |\phi_m(y)| \quad \text{for } x \in \Omega_{0,1}^m,$$

for sufficiently large $m \geq 1$. Here the constant C is independent of $m \geq 1$. Since ϕ_m is uniformly bounded from above by an L^p -integrable function in $\Omega_{0,1}^m$, the dominated convergence theorem implies that ϕ_m converges to a limit $\bar{\phi}$ in $L^p((\mathbf{R}_{+,-c_0}^n)_{0,1})$.

Since $\nabla\phi_m(x) \rightarrow 0$ as $m \rightarrow \infty$ for each $x \in (\mathbf{R}_{+,-c_0}^n)_{0,1}$ and $\|\bar{\phi}\|_{L^p((\mathbf{R}_{+,-c_0}^n)_{0,1})} = 1$, $\bar{\phi}$ is a non-zero constant which contradicts the fact that $(\bar{\phi}) = 0$. We reached a contradiction and the proof is now complete.

3 A priori estimates for the Stokes equations

The goal of this section is to prove the a priori estimate (1.4) by using the inequality (1.10). A key step is to establish the estimates for h and g in the procedure (ii) as explained in the introduction. We first recall the L^p -estimates to the Stokes equations (1.7) and the interpolation inequality (1.11). Note that the constants C_p and C_I in (1.7) and (1.11) respectively are independent of the volume of domains $\Omega', \Omega_{x_0,r}$.

3.1 L^p -estimates for localized equations

Let Ω' be a bounded domain with C^2 -boundary. For the a priori estimate (1.4) we invoke the L^p -estimates (1.7) to the Stokes resolvent equations with inhomogeneous divergence-free condition,

$$\lambda u - \Delta u + \nabla p = h \quad \text{in } \Omega', \quad (3.1)$$

$$\operatorname{div} u = g \quad \text{in } \Omega', \quad (3.2)$$

$$u = 0 \quad \text{on } \partial\Omega', \quad (3.3)$$

for $h \in L^p(\Omega')$, $g \in W^{1,p}(\Omega') \cap L_{\text{av}}^p(\Omega')$ and $\lambda \in \Sigma_{\vartheta,0}$ with $\vartheta \in (\pi/2, \pi)$. Here $L_{\text{av}}^p(\Omega')$ denotes the space of all functions g in $L^p(\Omega')$ satisfying average zero, i.e. $\int_{\Omega'} g dx = 0$. The estimate (1.7) is proved by a perturbation argument [15], [16] with the constant C_p independent of the volume of Ω' .

Proposition 3.1. ([15], [16]) *Let $\vartheta \in (\pi/2, \pi)$ and $\lambda \in \Sigma_{\vartheta,0}$. For $h \in L^p(\Omega')$ and $g \in W^{1,p}(\Omega') \cap L_{\text{av}}^p(\Omega')$ there exists a unique solution of (3.1)-(3.3) satisfying the estimates (1.7) with the constant C_p independent of the volume of Ω' and depending on ϑ, p, n and the C^2 -regularity of $\partial\Omega'$.*

We estimate the L^∞ -norms of a solution up to first derivatives via the Sobolev embeddings together with the L^p -estimates (1.7) for $p > n$. In order to estimate the L^∞ -norms of a solution we apply the interpolation inequality (1.11) [24, Chapter 3. Lemma 3.1.4] in $\Omega_{x_0,r} = B_{x_0}(r) \cap \Omega$ for $x_0 \in \bar{\Omega}$ and $r < r_0$ with a constant r_0 . In what follows, we fix the constant r_0 by taking the same constant r_0 given by Lemma 2.5. The constant C_I is also independent of the radius r .

3.2 Estimates in the localization procedure

We prepare the estimates for h and g in the procedure (ii). The estimate for $|\lambda|||g||_{W_0^{-1,p}}$ is different from that of $||h||_{L^p}$. In order to estimate $|\lambda|||g||_{W_0^{-1,p}}$ we use the uniformly local L^p -norm bound for ∇q besides the sup-bound of ∇v as in (3.8). After establishing these estimates, we will put the procedures (i)-(iii) together in the next subsection.

Let Ω be a uniformly C^2 -domain. Let θ be a smooth cut-off function satisfying $\theta \equiv 1$ in $[0, 1/2]$ and $\theta \equiv 0$ in $[1, \infty)$. For $x_0 \in \Omega$ and $r > 0$ we set $\theta_0(x) = \theta(|x - x_0|/(\eta + 1)r)$ with parameters $\eta \geq 1$ and observe that $\theta_0 \equiv 1$ in $B_{x_0}(r)$ and $\theta_0 \equiv 0$ in $B_{x_0}((\eta + 1)r)^c$. The cut-off function θ_0 is uniformly bounded by a constant K , i.e.

$$\|\theta_0\|_\infty + (\eta + 1)r\|\nabla\theta_0\|_\infty + (\eta + 1)^2r^2\|\nabla^2\theta_0\|_\infty \leq K, \quad \eta \geq 1 \quad (3.4)$$

Let $(v, \nabla q) \in W_{\text{loc}}^{2,p}(\bar{\Omega}) \times L_{\text{loc}}^p(\bar{\Omega})$ be a solution of (1.1)-(1.3) for $f \in L_c^\infty(\Omega)$ and $\lambda \in \Sigma_{\vartheta,0}$. We localize a solution $(v, \nabla q)$ in a domain $\Omega' = \Omega_{x_0,(\eta+1)r}$ by setting $u = v\theta_0$ and $p = \hat{q}\theta_0$ with $\hat{q} = q - q_c$ and a constant q_c . We then observe that $(u, \nabla p)$ solves the localized equation (3.1)-(3.3) in the domain Ω' with h and g given by (1.8). We shall show the following estimates for h and g ,

$$\|\nabla g\|_{L^p(\Omega')} \leq C_1 r^{n/p} (\eta + 1)^{-(1-n/p)} \left(r^{-1} \|\nabla v\|_{L^\infty(\Omega)} + r^{-2} \|v\|_{L^\infty(\Omega)} \right), \quad (3.5)$$

$$\begin{aligned} \|h\|_{L^p(\Omega')} &\leq C_2 r^{n/p} \left((\eta + 1)^{n/p} \|f\|_{L^\infty(\Omega)} \right. \\ &\quad \left. + (\eta + 1)^{-(1-n/p)} \left(r^{-1} \|\nabla v\|_{L^\infty(\Omega)} + r^{-2} \|v\|_{L^\infty(\Omega)} \right) \right), \end{aligned} \quad (3.6)$$

$$\begin{aligned} |\lambda|||g||_{W_0^{-1,p}(\Omega')} &\leq C_3 r^{n/p} \left((\eta + 1)^{n/p} \|f\|_{L^\infty(\Omega)} \right. \\ &\quad \left. + (\eta + 1)^{-(1-2n/p)} \left(r^{-1} \|\nabla v\|_{L^\infty(\Omega)} + r^{-n/p} \sup_{z \in \Omega} \|\nabla q\|_{L^p(\Omega_{z,r})} \right) \right), \end{aligned} \quad (3.7)$$

with constants C_1, C_2 and C_3 independent of r and $\eta \geq 1$. For the estimates of the terms of f, v and ∇v we use the estimates

$$\|f\theta_0\|_{L^p(\Omega')} \leq KC_n^{1/p} r^{n/p} (\eta + 1)^{n/p} \|f\|_{L^\infty(\Omega)}, \quad (3.8)$$

$$\|\nabla v \nabla \theta_0\|_{L^p(\Omega')} \leq KC_n^{1/p} r^{n/p} (\eta + 1)^{-(1-n/p)} r^{-1} \|\nabla v\|_{L^\infty(\Omega)}, \quad (3.9)$$

$$\|v \nabla^2 \theta_0\|_{L^p(\Omega')} \leq KC_n^{1/p} r^{n/p} (\eta + 1)^{-(1-n/p)} r^{-2} \|v\|_{L^\infty(\Omega)}, \quad (3.10)$$

for all $r > 0$ and $\eta \geq 1$, where the constant C_n denotes the volume of the n -dimensional unit ball. Since $\nabla g = \nabla v \nabla \theta_0 + v \nabla^2 \theta_0$ does not contain the pressure, the estimate (3.5) easily follows from the estimates (3.9) and (3.10).

For the estimates (3.6) and (3.7) we apply the inequality (1.10). We choose a constant q_c by a mean value of q in $\Omega_{x_0, (\eta+2)r}$, i.e.

$$q_c = \int_{\Omega_{x_0, (\eta+2)r}} q(x) dx. \quad (3.11)$$

We then observe that the inequality (1.10) implies the estimate

$$\|\hat{q}\|_{L^p(\Omega_{x_0, (\eta+2)r})} \leq C r^{n/p} (\eta + 2)^{n/p} \|\nabla q\|_{L_d^\infty(\Omega)} \quad (3.12)$$

for $r > 0$ and $\eta \geq 1$ satisfying $(\eta + 2)r \leq r_0$, where $\hat{q} = q - q_c$.

In order to estimate (3.7) we estimate the L^∞ -norm of \hat{q} on Ω' since by using the equation $\lambda v = f + \Delta v - \nabla q$ we reduce (3.7) to the estimate of the boundary value of \hat{q} on $\partial\Omega'$. This is the reason why we take q_c by (3.11). We apply the inequality (1.11) in $\Omega_{x_1, r} \subset \Omega_{x_0, (\eta+2)r}$ for $x_1 \in \Omega'$ and $r > 0$ with $p > n$ to estimate

$$\begin{aligned} \|\hat{q}\|_{L^\infty(\Omega_{x_1, r})} &\leq C_I r^{-n/p} \left(\|\hat{q}\|_{L^p(\Omega_{x_1, r})} + r \|\nabla q\|_{L^p(\Omega_{x_1, r})} \right) \\ &\leq C_I r^{-n/p} \left(\|\hat{q}\|_{L^p(\Omega_{x_0, (\eta+2)r})} + r \sup_{z \in \Omega} \|\nabla q\|_{L^p(\Omega_{z, r})} \right). \end{aligned} \quad (3.13)$$

Combining the estimate (3.13) with (3.12) and taking a supremum for $x_1 \in \Omega'$, we have

$$\|\hat{q}\|_{L^\infty(\Omega')} \leq C \left((\eta + 2)^{n/p} \|\nabla q\|_{L_d^\infty(\Omega)} + r^{1-n/p} \sup_{z \in \Omega} \|\nabla q\|_{L^p(\Omega_{z, r})} \right). \quad (3.14)$$

We now invoke the strictly admissibility of a domain Ω to estimate the norm $\|\nabla q\|_{L_d^\infty(\Omega)}$ by the sup-norm of ∇v in Ω via (1.5).

Proposition 3.2. *Let Ω be a uniformly C^2 -domain. Assume that Ω is strictly admissible, then the estimate*

$$\|\hat{q}\|_{L^p(\Omega')} \leq C_4 r^{n/p} (\eta + 2)^{n/p} \|\nabla v\|_{L^\infty(\Omega)} \quad (3.15)$$

holds for all $r > 0$ and $\eta \geq 1$ satisfying $(\eta + 2)r \leq r_0$ and $p \in [1, \infty)$. If in addition $p > n$, then the estimate

$$\|\hat{q}\|_{L^\infty(\Omega')} \leq C_5 \left((\eta + 2)^{n/p} \|\nabla v\|_{L^\infty(\Omega)} + r^{1-n/p} \sup_{z \in \Omega} \|\nabla q\|_{L^p(\Omega_{z, r})} \right) \quad (3.16)$$

holds. The constants C_4 and C_5 are independent of r and η .

Proof. By (1.5), (3.12) and (3.14), the assertion follows. \square

By using the estimates (3.15) and (3.16) we obtain the estimates (3.6) and (3.7).

Lemma 3.3. *Let Ω be a strictly admissible, uniformly C^2 -domain. Let $(v, \nabla q) \in W_{loc}^{2,p}(\bar{\Omega}) \times (L_{loc}^p(\bar{\Omega}) \cap L_d^\infty(\Omega))$ be a solution of (1.1)-(1.3) for $f \in L_\sigma^\infty(\Omega)$ and $\lambda \in \Sigma_{\theta,0}$ with $p > n$. Then the estimates (3.5)-(3.7) hold for $\Omega' = \text{int } B_{x_0}((\eta + 1)r) \cap \Omega$ with $x_0 \in \Omega$, $r > 0$ and $\eta \geq 1$ satisfying $(\eta + 2)r \leq r_0$ with the constants C_1 , C_2 and C_3 independent of x_0 , r and η .*

Proof. As mentioned before, (3.5) follows from (3.9) and (3.10). The estimate (3.6) follows from the estimates (3.8)-(3.10) and (3.15). We shall show the estimate (3.7). By using the equation $\lambda g = \lambda v \cdot \nabla \theta_0 = (f + \Delta v - \nabla q) \cdot \nabla \theta_0$, we estimate

$$|\lambda| \|g\|_{W_0^{-1,p}(\Omega')} \leq \|f \cdot \nabla \theta_0\|_{W_0^{-1,p}(\Omega')} + \|\Delta v \cdot \nabla \theta_0\|_{W_0^{-1,p}(\Omega')} + \|\nabla q \cdot \nabla \theta_0\|_{W_0^{-1,p}(\Omega')}.$$

Since $\|f \cdot \nabla \theta_0\|_{W_0^{-1,p}(\Omega')} = \|f \theta_0\|_{L^p(\Omega')}$ for $f \in L_\sigma^\infty(\Omega)$, it suffices to show the estimates

$$\|\Delta v \cdot \nabla \theta_0\|_{W_0^{-1,p}(\Omega')} \leq C_6 r^{n/p} (\eta + 1)^{-(1-n/p)} r^{-1} \|\nabla v\|_{L^\infty(\Omega)}, \quad (3.17)$$

$$\|\nabla q \cdot \nabla \theta_0\|_{W_0^{-1,p}(\Omega')} \leq C_7 r^{n/p} (\eta + 1)^{-(1-2n/p)} \left(r^{-1} \|\nabla v\|_{L^\infty(\Omega)} + r^{-n/p} \sup_{z \in \Omega} \|\nabla q\|_{L^p(\Omega_{z,r})} \right). \quad (3.18)$$

We first show (3.17). Take $\varphi \in W^{1,p'}(\Omega')$ satisfying $\|\varphi\|_{W^{1,p'}(\Omega')} \leq 1$. By using $\text{div } v = 0$ integration by parts yields that

$$\sum_{i,j=1}^n \int_{\Omega'} \partial_j^2 v^i \partial_i \theta_0 \varphi dx = \sum_{i,j=1}^n \int_{\Omega'} (\partial_j v^i - \partial_i v^j) \partial_i \theta_0 \partial_j \varphi dx - \int_{\partial \Omega'} (\partial_j v^i - \partial_i v^j) \partial_i \theta_0 \varphi n_\Omega^i d\mathcal{H}^{n-1}(x).$$

We estimate the second term in the right-hand side by the $W^{1,1}$ -norm of φ in Ω' [11, 5.5 Theorem 1.1] to estimate

$$\|\varphi\|_{L^1(\partial \Omega)} \leq C_T \|\varphi\|_{W^{1,1}(\Omega')} \leq 2C_T |\Omega'|^{1/p} \quad (3.19)$$

with the constant C_T depending on the C^1 -regularity of the boundary $\partial \Omega$ but independent of $|\Omega'|$, the volume of Ω' . We thus obtain

$$\begin{aligned} \left| \int_{\Omega'} \partial_j^2 v^i \partial_i \theta_0 \varphi dx \right| &\leq (1 + 2C_T) \|(\partial_j v^i - \partial_i v^j) \partial_j \theta_0\|_{L^\infty(\Omega')} |\Omega'|^{1/p} \\ &\leq 2(1 + 2C_T) K C_n^{1/p} r^{n/p} (\eta + 1)^{-(1-n/p)} r^{-1} \|\nabla v\|_{L^\infty(\Omega)}. \end{aligned}$$

Thus the estimate (3.17) holds with the constant C_6 independent of r and η . It remains to show the estimate (3.18). Since $\nabla q = \nabla \hat{q}$, integration by parts yields that

$$\begin{aligned} \int_{\Omega'} \nabla q \cdot \nabla \theta_0 \varphi dx &= - \int_{\Omega'} \hat{q} (\Delta \theta_0 \varphi + \nabla \theta_0 \cdot \nabla \varphi) dx + \int_{\partial \Omega'} \hat{q} \varphi \nabla \theta_0 \cdot n_{\Omega'} d\mathcal{H}^{n-1}(x) \\ &= I + II + III. \end{aligned}$$

Combining (3.4), (3.19) with (3.16), we obtain

$$\begin{aligned} II + III &\leq (1 + 2C_T) \|\hat{q} \nabla \theta_0\|_{L^\infty(\Omega')} |\Omega'|^{1/p} \\ &\leq (1 + 2C_T) K C_n^{1/p} r^{n/p} (\eta + 1)^{-(1-n/p)} r^{-1} \|\hat{q}\|_{L^\infty(\Omega')} \\ &\leq C r^{n/p} (\eta + 1)^{-(1-2n/p)} \left(r^{-1} \|\nabla v\|_{L^\infty(\Omega)} + r^{-n/p} \sup_{z \in \Omega} \|\nabla q\|_{L^p(\Omega_{z,r})} \right), \end{aligned}$$

with the constant C depending on C_T, K, C_n, p, C_4 and C_5 but independent of r and η . We complete the proof by showing the estimate for I . Applying the Hölder inequality, for $s, s' \in (1, \infty)$ with $1/s + 1/s' = 1$ we have

$$I \leq K (\eta + 1)^{-2} r^{-2} \|\varphi\|_{L^s(\Omega')} \|\hat{q}\|_{L^{s'}(\Omega')}.$$

Since $p > n$, the conjugate exponent p' is strictly smaller than $n/(n-1)$ for $n \geq 2$. By setting $1/s = 1/p' + 1/n$ we apply the Sobolev inequality [11, 5.6 Theorem 2] to estimate $\|\varphi\|_{L^s(\Omega')} \leq C_S \|\varphi\|_{W^{1,p'}(\Omega')} \leq C_S$ with the constant C_S independent of $|\Omega'|$. Applying the estimate (3.15) to \hat{q} yields

$$\begin{aligned} I &\leq C r^{n/s'-2} (\eta + 2)^{n/s'-2} \|\nabla v\|_{L^\infty(\Omega)} \\ &\leq C r^{n/p} (\eta + 2)^{-(1-n/p)} r^{-1} \|\nabla v\|_{L^\infty(\Omega)}, \end{aligned}$$

since $1/s' = 1 - 1/s = 1/p + 1/n$. The constant C is independent of r and η . The proof is now complete. \square

Remark 3.4. From the estimate (3.7) we observe that the exponent $-(1 - 2n/p)$ of $(\eta + 1)$ in front of the term $(r^{-1} \|\nabla v\|_{L^\infty(\Omega)} + r^{-n/p} \sup_{z \in \Omega} \|\nabla q\|_{L^p(\Omega_{z,r})})$ is negative provided that $p > 2n$. We thus first prove the a priori estimate (1.4) for $p > 2n$. Once we obtain the estimate $|\lambda| \|v\|_{L^\infty(\Omega)} \leq C \|f\|_{L^\infty(\Omega)}$ it is easy to replace the estimate (3.7) to

$$|\lambda| \|g\|_{W_0^{-1,p}(\Omega')} \leq C K C_n^{1/n} r^{n/p} (\eta + 1)^{n/p} \|f\|_{L^\infty(\Omega)}$$

for $p > n$ since

$$\begin{aligned} |\lambda| \|v \cdot \nabla \theta_0\|_{W_0^{-1,p}(\Omega')} &= |\lambda| \|v \theta_0\|_{L^p(\Omega)} \\ &\leq C \|\theta_0\|_{L^p(\Omega')} \|f\|_{L^\infty(\Omega)} \\ &\leq CK C_n^{1/p} r^{n/p} (\eta + 1)^{n/p} \|f\|_{L^\infty(\Omega)}. \end{aligned}$$

3.3 Interpolation

We now prove the a priori estimate (1.4) for $p > n$. The size of the parameter η and the constant δ are determined only through the constants C_p, C_I and C_1-C_3 . Although we eventually obtain the estimate (1.12) for all $p > n$, firstly we prove the case $p > 2n$ as observed by Remark 3.4. The case $p > 2n$ is enough for analyticity but for the completeness we prove the estimate (1.4) for all $p > n$.

Proof of Theorem 1.1. We set $\delta = \delta_\eta = (\eta + 2)^2 / r_0^2$ and now take $r = 1/|\lambda|^{1/2}$ for $\lambda \in \Sigma_{\vartheta, \delta}$. We then observe that $r = 1/|\lambda|^{1/2}$ and $\eta \geq 1$ automatically satisfy $r(\eta + 2) \leq r_0$ for $\lambda \in \Sigma_{\vartheta, \delta}$. We may assume that the boundary of $\Omega' = B_{x_0}((\eta + 1)r) \cap \Omega$ is C^2 because the localized equations (3.1)-(3.3) can be regarded as the equation in a subdomain Ω'' of Ω by taking Ω'' with C^2 -boundary so that $\Omega' \subset \Omega''$ and Ω'' preserves an order of the volume of Ω' , i.e. $|\Omega''|$ is bounded from above by $C(\eta + 1)^n r^n$ with a constant C independent of $r > 0$ and $\eta \geq 1$. We first prove

Case (I) $p > 2n$. By applying the L^p -estimates (1.7) to $u = v\theta_0$ and $p = \hat{q}\theta_0$ in Ω' and combining the estimates (3.5)-(3.7) with (1.7), we obtain

$$\begin{aligned} &|\lambda| \|u\|_{L^p(\Omega')} + |\lambda|^{1/2} \|\nabla u\|_{L^p(\Omega')} + \|\nabla^2 u\|_{L^p(\Omega')} + \|\nabla p\|_{L^p(\Omega')} \\ &\leq C_8 |\lambda|^{-n/2p} \left((\eta + 1)^{n/p} \|f\|_{L^\infty(\Omega)} + (\eta + 1)^{-(1-2n/p)} \|M_p(v, q)\|_{L^\infty(\Omega)}(\lambda) \right), \end{aligned} \quad (3.20)$$

with the constant C_8 independent of $r = 1/|\lambda|^{1/2}$ and $\eta \geq 1$. We next estimate the L^∞ -norms of u and ∇u in Ω by interpolation. Applying the interpolation inequality (1.11) for $\varphi = u$ and ∇u implies the estimates

$$\begin{aligned} \|u\|_{L^\infty(\Omega_{x_0, r})} &\leq C_I r^{-n/p} \left(\|u\|_{L^p(\Omega_{x_0, r})} + r \|\nabla u\|_{L^p(\Omega_{x_0, r})} \right), \\ \|\nabla u\|_{L^\infty(\Omega_{x_0, r})} &\leq C_I r^{-n/p} \left(\|\nabla u\|_{L^p(\Omega_{x_0, r})} + r \|\nabla^2 u\|_{L^p(\Omega_{x_0, r})} \right). \end{aligned}$$

Summing up these norms together with $|\lambda|^{n/2p} \|\nabla^2 u\|_{L^p(\Omega_{x_0, r})}$ and $|\lambda|^{n/2p} \|\nabla p\|_{L^p(\Omega_{x_0, r})}$,

we have

$$\begin{aligned} & M_p(u, p)(x_0, \lambda) \\ & \leq C_9 r^{-n/p} \left(|\lambda| \|u\|_{L^p(\Omega_{x_0, r})} + |\lambda|^{1/2} \|\nabla u\|_{L^p(\Omega_{x_0, r})} + \|\nabla^2 u\|_{L^p(\Omega_{x_0, r})} + \|\nabla p\|_{L^p(\Omega_{x_0, r})} \right) \end{aligned} \quad (3.21)$$

with the constant C_9 independent of r and $\eta \geq 1$. Since $(u, \nabla p)$ agrees with $(v, \nabla q)$ in $\Omega_{x_0, r}$, combining (3.20) with (3.21) yields

$$M_p(v, q)(x_0, \lambda) \leq C_{10} \left((\eta + 1)^{n/p} \|f\|_{L^\infty(\Omega)} + (\eta + 1)^{-(1-2n/p)} \|M_p(v, q)\|_{L^\infty(\Omega)}(\lambda) \right) \quad (3.22)$$

with $C_{10} = C_8 C_9$. By taking a supremum for $x_0 \in \Omega$ and letting $\eta \geq 1$ large so that $C_{10}(\eta + 1)^{-(1-2n/p)} < 1/2$ we obtain (1.4) with $p > 2n$.

We shall complete the proof by showing the uniformly local L^p -bound for second derivatives of (v, q) for all $p > n$.

Case (II) $p > n$. Since $|\lambda| \|g\|_{W_0^{-1, \tilde{p}}}$ is bounded for $\tilde{p} > 2n$, we may assume $(v, \nabla q) \in W_{\text{loc}}^{2, \tilde{p}}(\bar{\Omega}) \times L_{\text{loc}}^{\tilde{p}}(\bar{\Omega})$ with $\tilde{p} > 2n$. By using $|\lambda| \|v\|_{L^\infty(\Omega)} \leq C \|f\|_{L^\infty(\Omega)}$ for $\lambda \in \Sigma_{\theta, \delta}$ with $\delta = \delta_{\tilde{p}}$ we replace the estimate (3.7) to

$$|\lambda| \|g\|_{W_0^{-1, p}(\Omega')} \leq CK C_n^{1/p} r^{n/p} (\eta + 1)^{n/p} \|f\|_{L^\infty(\Omega)}$$

by Remark 3.4. Then we are able to replace the estimate (3.22) to

$$\|M_p(v, q)\|_{L^\infty(\Omega)}(\lambda) \leq C_{11} \left((\eta + 1)^{n/p} \|f\|_{L^\infty(\Omega)} + (\eta + 1)^{-(1-n/p)} \|M_p(v, q)\|_{L^\infty(\Omega)}(\lambda) \right).$$

Letting $\eta \geq 1$ large so that $C_{11}(\eta + 1)^{-(1-n/p)} < 1/2$, we obtain (1.4) for all $p > n$. The proof is now complete.

Remark 3.5. (Robin boundary condition) Concerning the Robin boundary condition we replace the Dirichlet boundary condition for the localized equations (3.3) to the inhomogeneous boundary condition with a tangential vector field k ,

$$B(u) = k, \quad u \cdot n_{\Omega'} = 0 \quad \text{on } \partial\Omega'.$$

Instead of the estimate (1.7) we apply the L^p -estiamte of the form,

$$\begin{aligned} & |\lambda| \|u\|_{L^p(\Omega')} + |\lambda|^{1/2} \|\nabla u\|_{L^p(\Omega')} + \|\nabla^2 u\|_{L^p(\Omega')} + \|\nabla p\|_{L^p(\Omega')} \\ & \leq C (\|h\|_{L^p(\Omega')} + \|\nabla g\|_{L^p(\Omega')} + |\lambda| \|g\|_{W_0^{-1, p}(\Omega')} + |\lambda|^{1/2} \|k\|_{L^p(\Omega')} + \|\nabla k\|_{L^p(\Omega')}), \end{aligned}$$

where k is identified with its arbitrary extension to Ω' . Since $k = \nu_{\tan} \partial \theta_0 / \partial n_{\Omega'}$ for $u = \nu \theta_0$ and $p = \hat{q} \theta_0$, we observe that the norms of k in the right-hand-side are estimated by the same way with $\|\nabla g\|_{L^p}$ where $g = \nu \cdot \nabla \theta_0$. The L^p -estimates for the Robin boundary condition is proved by [32] for bounded and exterior domains by generalizing the perturbation argument to the Dirichlet boundary condition [16]. We thus observe that the constant C is also independent of the volume Ω' . After proving the a priori estimate (1.4) for $f \in L^\infty_\sigma$ subject to the Robin boundary condition, we verify the existence of a solution of (1.1) and (1.2). In particular, $\nu \in L^\infty_\sigma$ (not in $C_{0,\sigma}$.) Then we are able to define the Stokes operator $A = A_R$ in L^∞_σ in the same way as we did for the Dirichlet boundary condition. Our observations may be summarized as following

Theorem 3.6. *Assume that Ω is a bounded or an exterior domain with C^3 -boundary in \mathbf{R}^n . Then the Stokes operator $A = A_R$ subject to the Robin boundary condition generates an analytic semigroup on $L^\infty_\sigma(\Omega)$ of angle $\pi/2$.*

Acknowledgments

The work of first author is supported by Grant-in-aid for Scientific Research of JSPS Fellow No. 24-8019. The work of second author is partially supported by Grant-in-aid for Scientific Research, No. 21224001 (Kiban S), No. 23244015 (Kiban A), the Japan Society for the Promotion of Science (JSPS). This work is supported in part by the DFG-JSPS International Research Training Group 1529 on Mathematical Fluid Dynamics.

References

- [1] K. Abe, Y. Giga, Analyticity of the Stokes semigroup in spaces of bounded functions, Acta Math., to appear
- [2] K. Abe, Y. Giga, The L^∞ -Stokes semigroup in exterior domains, Hokkaido University Preprint Series in Mathematics, no. 1011 (2012): <http://eprints3.math.sci.hokudai.ac.jp/2211/1/pre1011.pdf>
- [3] T. Abe, Y. Shibata, On a resolvent estimate of the Stokes equation on an infinite layer, J. Math. Soc. Japan, **55**, 469-497 (2003)

- [4] H. Abels, Nonstationary Stokes system with variable viscosity in bounded and unbounded domains, *Discrete Contin. Dyn. Syst. Ser. S*, **3**, 141-157 (2010)
- [5] H. Abels, Y. Terasawa, On Stokes operators with variable viscosity in bounded and unbounded domains, *Math. Ann.*, **334**, 381-429 (2009)
- [6] P. Acquistapace, B. Terreni, Hölder classes with boundary conditions as interpolation spaces, *Math. Z.*, **195**, 451-471 (1987)
- [7] L. Angiuli, D. Pallara, F. Paronetto, Analytic semigroups generated in $L^1(\Omega)$ by second order elliptic operators via duality methods, *Semigroup Forum*, **80**, 255-271 (2010)
- [8] W. Arendt, Ch. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Birkhäuser, Basel 2011
- [9] M. E. Bogovskiĭ, Solution of the first boundary value problem for the equation of continuity of an incompressible medium, *Dokl. Akad. Nauk. SSSR*, **248**, 1037-1040 (1979); translation: *Soviet Math. Dokl.*, **20**, 1094-1098 (1979)
- [10] W. Desch, M. Hieber, J. Prüss, L^p -theory of the Stokes equation in a half space, *J. Evol. Equ.*, **1**, 115-142 (2001)
- [11] L. C. Evans, *Partial differential equations*, Amer. Math. Soc., Providence. R. I. 2010
- [12] R. Farwig, H. Kozono, H. Sohr, An L^q -approach to Stokes and Navier-Stokes equations in general domains, *Acta Math.*, **195**, 21-53 (2005)
- [13] R. Farwig, H. Kozono, H. Sohr, On the Helmholtz decomposition in general unbounded domains, *Arch Math.*, **88**, 239-248 (2007)
- [14] R. Farwig, H. Kozono, H. Sohr, On the Stokes operator in general unbounded domains, *Hokkaido Math. J.*, **38**, 111-136 (2009)
- [15] R. Farwig, H. Sohr, An approach to resolvent estimates for the Stokes equations in L^q -spaces, *The Navier-Stokes equations II -theory and numerical methods* (Oberwolfach, 1991), 97-110, Springer, Berlin (1992)

- [16] R. Farwig, H. Sohr, Generalized resolvent estimates for the Stokes system in bounded and unbounded domains, *J. Math. Soc. Japan*, **46**, 607-643 (1994)
- [17] R. Farwig, H. Sohr, Helmholtz decomposition and Stokes resolvent system for aperture domains in L^q spaces, *Analysis*, **16**, 1-26 (1996)
- [18] M. Geissert, H. Heck, M. Hieber, O. Sawada, Weak Neumann implies Stokes, *J. Reine. angew. Math.* **669**, 75-100 (2012)
- [19] M.-H. Giga, Y. Giga, J. Saal, Nonlinear partial differential equations: Asymptotic behavior of solutions and self-similar solutions, Birkhäuser, Boston-Basel-Berlin 2010
- [20] Y. Giga, Analyticity of the semigroup generated by the Stokes operator in L_r spaces, *Math. Z.*, **178**, 297-329 (1981)
- [21] Y. Giga, The nonstationary Navier-Stokes system with some first order boundary condition, *Proc. Japan Acad.*, **58**, 101-104 (1982)
- [22] H. Heck, M. Hieber, K. Stavrakidis, L^∞ -estimates for parabolic systems with VMO-coefficients, *Discrete Contin. Dyn. Syst. Ser. S*, **3**, 299-309 (2010)
- [23] S. G. Krantz, H. R. Parks, The implicit function theorem. History, theory, and applications, Birkhäuser, Boston-Basel-Berlin 2002
- [24] A. Lunardi, Analytic semigroup and optimal regularity in parabolic problems, Birkhäuser, Basel 1995
- [25] P. Maremonti, G. Starita, Nonstationary Stokes equations in a half-space with continuous initial data, *Zapiski Nauchnykh Seminarov POMI*, **295**, 118-167 (2003); translation: *J. Math. Sci. (N.Y.)*, **127**, 1886-1914 (2005)
- [26] P. Maremonti, On the Stokes problem in exterior domains: the maximum modulus theorem, preprint
- [27] K. Masuda, On the generation of analytic semigroups of higher-order elliptic operators in spaces of continuous functions (in Japanese), *Proc. Katata Symposium on Partial Differential Equations*, 144-149 (1972)
- [28] K. Masuda, On the generation of analytic semigroups by elliptic differential operators with unbounded coefficients, unpublished note (1972)

- [29] K. Masuda, Evolution equations (in Japanese), Kinokuniya Shoten, Tokyo, 1975
- [30] J. Saal, The Stokes operator with Robin boundary conditions in solenoidal subspaces of $L^1(\mathbb{R}_+^n)$ and $L^\infty(\mathbb{R}_+^n)$, *Comm. Partial Differential Equations*, **32**, 343-373 (2007)
- [31] Z. Shen, Resolvent estimates in L^p for the Stokes operator in Lipschitz domains, *Arch. Ration. Mech. Anal.*, **205**, 395-424 (2012)
- [32] Y. Shibata, R. Shimada, On a generalized resolvent for the Stokes system with Robin boundary condition, *J. Math. Soc. Japan*, **59**, 469-519 (2007)
- [33] H. Sohr, *The Navier-Stokes equations*, Birkhäuser, Basel 2001
- [34] E. Sinestrari, On the abstract Cauchy problem of parabolic type in spaces of continuous functions, *J. Math. Anal. Appl.*, **107**, 16-66 (1985)
- [35] V. A. Solonnikov, Estimates for solutions of nonstationary Navier-Stokes equations, *J. Soviet Math.*, **8**, 467-529 (1977)
- [36] V. A. Solonnikov, On nonstationary Stokes problem and Navier-Stokes problem in a half-space with initial data nondecreasing at infinity, *J. Math. Sci. (N. Y.)*, **114**, 1726-1740 (2003)
- [37] H. B. Stewart, Generation of analytic semigroups by strongly elliptic operators, *Trans. Amer. Math. Soc.*, **199**, 141-162 (1974)
- [38] H. B. Stewart, Generation of analytic semigroups by strongly elliptic operators under general boundary conditions, *Trans. Amer. Math. Soc.*, **259**, 299-310 (1980)