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Stokes Resolvent Estimates in Spaces of Bounded Functions

Ken Abe ∗ Yoshikazu Giga † Matthias Hieber ‡

Abstract

We give a direct proof for the analyticity of the Stokes semigroup in spaces of bounded functions. This was recently proved by an indirect argument by the first and second authors for a class of domains called strictly admissible domains including bounded and exterior domains. Invoking the strictly admissibility, our approach is based on an adjustment of a standard resolvent estimate method for general elliptic operators introduced by K. Masuda (1972) and H. B. Stewart (1974). The resolvent approach in particular clarifies the sectorial region, \( \text{Re} \, z > 0 \) for \( z \in \mathbb{C} \) for which the Stokes semigroup has an analytic continuation in spaces of bounded functions.

Keywords Analytic semigroups; Bounded function spaces; Resolvent estimates; Stokes equations

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1 Introduction and main results

The investigation of the linear Stokes equations as well as properties and corresponding estimates are often basis for the analysis of the nonlinear Navier-Stokes

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equations. In particular, analyticity of the solution operator (called the Stokes semigroup) plays a fundamental role for studying the Navier-Stokes equations. It is well-known that the Stokes semigroup forms an analytic semigroup on $L^p_\sigma(\Omega)$ for $p \in (1, \infty)$, the space of $L^p$-solenoidal vector fields for various kind of domains $\Omega \subset \mathbb{R}^n, n \geq 2$ including bounded and exterior domains having smooth boundaries; see e.g. [35], [20]. By now, analyticity results are known for other type unbounded domains, see [16], [17], [3] ([5], [4] with variable viscosity coefficients) and Lipschitz domains [31]. An $L^p$-theory is developed in [12], [13], [14] for a general domain. Moreover, $L^p$-theory is investigated in [18] for unbounded domains, where the Helmholtz projection is bounded.

It is the aim of this paper to consider the case $p = \infty$. Note that the Helmholtz projection is no longer bounded in $L^\infty$ even if $\Omega = \mathbb{R}^n$. When $\Omega = \mathbb{R}^{n+}$, the analyticity of the semigroup is known in $L^\infty$-type spaces including $C_0,\sigma(\Omega)$, the $L^\infty$-closure of $C_\infty c,\sigma(\Omega)$, the space of all smooth solenoidal vector fields compactly supported in $\Omega$ [10](see also [36], [25]). Their approach is based on explicit calculations of the solution operator $R(\lambda) : f \mapsto v = v_\lambda$ of the corresponding resolvent problem of

\begin{align}
\lambda v - \Delta v + \nabla q &= f \quad \text{in } \Omega, \\
\div v &= 0 \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega.
\end{align}

As recently shown in [1], [2] by a blow-up argument, to the non-stationary Stokes equations, the Stokes semigroup is extendable to an analytic semigroup on $C_0,\sigma$ for admissible domains including bounded and exterior domains having boundaries of class $C^3$.

In this paper, we present a direct resolvent approach to the Stokes resolvent equations (1.1)-(1.3) and establish the a priori estimate of the form

$$M_p(v, q)(x, \lambda) = |\lambda||v(x)| + |\lambda|^{1/2}||\nabla v(x)|| + |\lambda|^{n/2p}||\nabla^2 v||_{L^p(\Omega_{\lambda,\rho-1/2})} + |\lambda|^{n/p}||\nabla q||_{L^p(\Omega_{\lambda,\rho-1/2})},$$

for $p > n$ and

$$\sup_{\lambda \in \Sigma_{\theta,\delta}} ||M_p(v, q)||_{L^\infty(\Omega)} \leq C ||f||_{L^\infty(\Omega)}$$

for some constant $C > 0$ independent of $f$. Here $\Omega_{x,\rho}$ denotes the intersection of $\Omega$ with an open ball $B_x(r)$ centered at $x \in \Omega$ with radius $r > 0$, i.e. $\Omega_{x,\rho} = B_x(r) \cap \Omega$ and $\Sigma_{\theta,\delta}$ denotes the sectorial region in the complex plane given by $\Sigma_{\theta,\delta} = \{ \lambda \in \mathbb{C} \setminus \{0\} | |\arg \lambda| < \theta, |\lambda| > \delta \}$ for $\theta \in (\pi/2, \pi)$ and $\delta > 0$. Our approach is inspired by
the corresponding approach for general elliptic operators. K. Masuda was the first to prove analyticity of the semigroup associated to general elliptic operators in $C_0(\mathbb{R}^n)$ including the case of higher orders [27], [28] ([29]). This result was then extended by H. B. Stewart to the case for the Dirichlet problem [37] and more general boundary condition [38]. This Masuda-Stewart method was applied to many other situations [6], [24], [22], [7]. However, its application to the resolvent Stokes equations (1.1)-(1.3) was unknown.

In the sequel, we prove the estimate (1.4) by invoking the $L^p$-estimates for the Stokes resolvent equations with inhomogeneous divergence-free condition [15], [16]. We invoke strictly admissibility of a domain introduced in [2] which implies an estimate of pressure $q$ in terms of the velocity by

$$\sup_{x \in \Omega} d_{\Omega}(x)|\nabla q(x)| \leq C_{\Omega}||W(v)||_{L^\infty(\partial\Omega)} \quad \text{with} \quad W(v) = -(\nabla v - \nabla^T v)n_{\Omega},$$

(1.5)

where $\nabla f$ denotes $(\partial f_i/\partial x_j)_{1 \leq i, j \leq n}$ and $\nabla^T f = (\nabla f)^T$ for a vector field $f = (f_i)_{1 \leq i \leq n}$. The estimate (1.5) plays a key role in transferring results from the elliptic situation to the situation of the Stokes system. Here $n_{\Omega}$ denotes the unit outward normal vector field on $\partial \Omega$ and $d_{\Omega}$ denotes the distance function from the boundary, $d_{\Omega}(x) = \inf_{y \in \partial \Omega} |x - y|$ for $x \in \Omega$. The estimate (1.5) can be viewed as a regularizing-type estimate for solutions to the Laplace equation $\Delta P = 0$ in $\Omega$ with the Neumann boundary condition $\partial P/\partial n_{\Omega} = \text{div}_{\partial \Omega} W$ on $\partial \Omega$ for a tangential vector field $W$ where $\text{div}_{\partial \Omega} = \text{tr} \ \nabla_{\partial \Omega}$ denotes the surface divergence and $\nabla_{\partial \Omega} = \nabla - n_{\Omega}(n_{\Omega} \cdot \nabla)$ is the gradient on $\partial \Omega$. It is known that $P = q$ solves this Neumann problem with $W = W(v)$ given by (1.5) [2, Lemma 3.1] and the estimate (1.5) holds for bounded domains [1] and exterior domains [2]. Note that when $n = 3$, $-W(v)$ is nothing but a tangential trace of vorticity, i.e. $-W(v) = \text{curl} \ v \times n_{\Omega}$.

We call $\Omega$ strictly admissible if there exists a constant $C = C_{\Omega}$ such that the a priori estimate

$$||\nabla P||_{L^\infty(\Omega)} \leq C||W||_{L^\infty(\partial\Omega)}$$

(1.6)

holds for all solutions $P$ of the Neumann problem with a tangential vector field $W \in L^\infty(\partial\Omega)$. Here $L^\infty(\Omega)$ denotes the space of all locally integrable functions $f$ such that $d_{\Omega} f$ is essentially bounded in $\Omega$ and equipped with the norm $||f||_{L^\infty(\Omega)} = \sup_{x \in \Omega} d_{\Omega}(x)|f(x)|$. The meaning of a solution is understood in the weak sense, i.e. we say $\nabla P \in L^\infty(\Omega)$ is a solution for the Neumann problem if $\int_{\Omega} P \Delta \varphi dx = - \int_{\partial \Omega} W \cdot \nabla \varphi dH^{n-1}(x)$ holds for all $\varphi \in C_c^2(\overline{\Omega})$ satisfying $\partial \varphi/\partial n_{\Omega} = 0$ on $\partial \Omega$, where $H^{n-1}$ denotes the $n - 1$ dimensional Hausdorff measure.

We are now in the position to formulate the main results of this paper.
Theorem 1.1. Let $\Omega$ be a strictly admissible, uniformly $C^2$-domain in $\mathbb{R}^n$ (with $n \geq 2$.) Let $p > n$. For $\theta \in (\pi/2, \pi)$ there exist constants $\delta$ and $C$ such that the a priori estimate (1.4) holds for all solutions $(v, \nabla q) \in W^{1,p}_{\text{loc}}(\bar{\Omega}) \times (L^p_{\text{loc}}(\bar{\Omega}) \cap L^\infty_d(\Omega))$ of (1.1)-(1.3) for $f \in C_{0,\sigma}(\Omega)$ with $\lambda \in \Sigma_{\theta, \delta}$.

The a priori estimate (1.4) implies the analyticity of the Stokes semigroup in $L^\infty$-type spaces. Let us observe the generation of an analytic semigroup in $C_{0,\sigma}(\Omega)$. By invoking the $\tilde{L}^p$-theory [12], [13], [14] we verify the existence of a solution to (1.1)-(1.3), $(v, \nabla q) \in W^{1,p}_{\text{loc}}(\bar{\Omega}) \times (L^p_{\text{loc}}(\bar{\Omega}) \cap L^\infty_d(\Omega))$ for $f \in C_{C_\sigma}(\Omega)$ in a uniformly $C^2$-domain $\Omega$. The solution operator $(\lambda)$ is then uniquely extendable to $C_{0,\sigma}(\Omega)$ by the uniform approximation together with the estimates (1.4). Here the solution operator to the pressure gradient $f \mapsto \nabla q_\lambda$ is also uniquely extended for $f \in C_{0,\sigma}$. We observe that $R(\lambda)$ is injective on $C_{0,\sigma}$ since the estimate (1.5) immediately implies that $f = 0$ for $f \in C_{0,\sigma}$ such that $v_\lambda = R(\lambda)f = 0$. The operator $R(\lambda)$ may be regarded as a surjective operator from $C_{0,\sigma}$ to the range of $R(\lambda)$. The open mapping theorem then implies the existence of a closed operator $A$ such that $R(\lambda) = (\lambda - A)^{-1}$; see [8, Proposition B.6]. We call $A$ the Stokes operator in $C_{0,\sigma}(\Omega)$. From Theorem 1.1 we obtain

Theorem 1.2. Let $\Omega$ be a strictly admissible, uniformly $C^2$-domain in $\mathbb{R}^n$. Then the Stokes operator $A$ generates a $C_0$-analytic semigroup on $C_{0,\sigma}(\Omega)$ of angle $\pi/2$.

We next consider the space $L^\infty_{\sigma}(\Omega)$ defined by

$$L^\infty_{\sigma}(\Omega) = \left\{ f \in L^\infty(\Omega) \mid \int_\Omega f \cdot \nabla \varphi dx = 0 \text{ for all } \varphi \in \hat{W}^{1,1}(\Omega) \right\},$$

where $\hat{W}^{1,1}(\Omega)$ denotes the homogeneous Sobolev space of the form $\hat{W}^{1,1}(\Omega) = \{ \varphi \in L^1_{\text{loc}}(\Omega) \mid \nabla \varphi \in L^1(\Omega) \}$. Note that $C_{0,\sigma}(\Omega) \subset L^\infty_{\sigma}(\Omega)$. When a domain $\Omega$ is unbounded, the space $L^\infty_{\sigma}(\Omega)$ includes non-decaying solenoidal vector fields at the space infinity. Actually, the a priori estimates (1.4) is also valid for $f \in L^\infty_{\sigma}$. In particular, (1.4) implies the uniqueness of a solution for $f \in L^\infty_{\sigma}$. We verify the existence of a solution by approximating $f \in L^\infty_{\sigma}$ with compactly supported solenoidal vector fields $\{f_m\}_{m=1}^{\infty} \subset C_{c,\sigma}$. Note that $f \in L^\infty_{\sigma}$ is no longer approximated in the uniform topology in general so we weaken the convergence topology, for example, to the pointwise convergence, i.e. $f_m \to f$ a.e. in $\Omega$ keeping $\|f_m\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}$ with some constant $C = C_\Omega$. When a domain $\Omega$ is bounded, it is known that this approximation is valid [1, Lemma 6.3]. Moreover, we are able to take $C_\Omega = 1$ without assuming any regularity of $\partial \Omega$ since $L^\infty_{\sigma} \subset L^r_{\sigma}$. Although it
is unknown whether this approximation is valid for general domains, we are able to characterize the space $L^\infty_{\sigma}$ by this approximation at least for exterior domains [2, Lemma 5.1]. In the following, we restrict our results to bounded and exterior domains. Through the approximation argument we verify the existence of a solution to (1.1)-(1.3) for general $f \in L^\infty_{\sigma}$. We then define the Stokes operator on $L^\infty_{\sigma}$ by the same way as for $C_0,\sigma$. Since bounded and exterior domains are strictly admissible [1, Theorem 2.5], [2, Theorem 2.9] provided that the boundary is $C^3$, we have

**Theorem 1.3.** Assume that $\Omega$ is a bounded or an exterior domain with $C^3$-boundary. Then the Stokes operator $A$ generates a (non $C_0$-)analytic semigroup on $L^\infty_{\sigma}(\Omega)$ of angle $\pi/2$.

**Remarks 1.4.** (i) The direct resolvent approach clarifies the angle of the analyticity of the Stokes semigroup $e^{tA}$ on $C_0,\sigma$. Theorem 1.2 (and also Theorem 1.3) asserts that $e^{tA}$ is angle $\pi/2$ on $C_0,\sigma$ which does not follow from a priori $L^\infty$-estimates for solutions to the non-stationary Stokes equations proved by blow-up arguments [1, Theorem 1.2], [2, Theorem 1.5]; see also [19] for blow-up arguments.

(ii) We observe that our argument applies to other boundary conditions, for example, to the Robin boundary condition, i.e. $B(v) = 0$ and $v \cdot n_\Omega = 0$ on $\partial \Omega$ where

$$B(v) = \alpha v_{\text{tan}} + (D(v)n_{\Omega})_{\text{tan}}$$

with $\alpha \geq 0$.

Here $D(v) = (\nabla v + \nabla^T v)/2$ denotes the deformation tensor and $f_{\text{tan}}$ the tangential component of a vector field $f$ on $\partial \Omega$. Note that the case $\alpha = \infty$ corresponds to the Dirichlet boundary condition (1.3); see [30] for generation results subject to the Robin boundary conditions on $L^\infty$ for $\mathbb{R}^n$. The $L^p$-resolvent estimates for the Robin boundary condition was established in [21] for concerning analyticity and was later strengthened in [32] to non-divergence free vector fields. We shall use the generalized resolvent estimate in [32] to extend our result in spaces of bounded functions to the Robin boundary condition (Theorem 3.6). For a more detailed discussion, see Remark 3.5.

(iii) We observe that the domain of the Stokes operator $D(A)$ is dense in $C_0,\sigma$. In fact, by invoking the $L^p$-theory and using (1.4) we have

$$\|\lambda v - f\|_{L^\infty(\Omega)} = \|\tilde{A} p v\|_{L^\infty(\Omega)} \leq \frac{C}{|\lambda|} \|\tilde{A} p f\|_{L^\infty(\Omega)} \to 0, \quad |\lambda| \to \infty$$

for $f \in C^\infty_{0,\sigma} \subset D(\tilde{A})$, where $\tilde{A}$ is the Stokes operator in $L^p$. Thus we conclude that $D(A)$ is dense in $C_0,\sigma$. On the other hand, smooth functions are not dense in
$L^\infty$ and $e^{tA} f$ is smooth for $t > 0$, $e^{tA} f \to f$ as $t \downarrow 0$ in $L^\infty_v$ does not hold for some $f \in L^\infty_v$. This means $e^{tA}$ is a non-$C_0$ analytic semigroup. We refer to [34, 1.1.2] for properties of the analytic semigroup generated by non-densely defined sectorial operators; see also [8, Definition 3.2.5].

(iv) For a bounded domain $\Omega$, $v(\cdot, t) = e^{tA} v_0$ and $\nabla q = (1 - P) [\Delta v]$ give a solution to the non-stationary Stokes equations, $v_t - \Delta v + \nabla q = 0$, $\text{div } v = 0$ in $\Omega \times (0, \infty)$ with $v = 0$ on $\partial \Omega$ for initial data $v_0 \in L^\infty_v(\Omega)$. Although for general domains the Helmholtz projection operator $P : L^p(\Omega) \to L^p_v(\Omega)$ is no longer bounded on $L^\infty$ even if $\Omega = \mathbb{R}^n$, we are able to define the pressure $\nabla q = K[W(v)]$ at least for exterior domains $\Omega$ by the solution operator to the Neumann problem (harmonic-pressure operator) $K : L^\infty_v(\partial \Omega) \ni W \mapsto \nabla P \in L^\infty_v(\Omega)$ [2, Remark 2.10]. Here $L^\infty_v(\partial \Omega)$ denotes the closed subspace of all tangential vector fields in $L^\infty(\partial \Omega)$.

(v) We observe that the Masuda-Stewart method does not imply the large time behavior for $e^{tA}$. For a bounded domain the energy inequality implies that maximum of $v(\cdot, t) = e^{tA} v_0$ (and also $v_t$) decay exponentially as $t \to \infty$ [1, Remark 5.4 (i)]. In particular, $e^{tA}$ is a bounded analytic semigroup on $L^\infty_v$. Recently, based on the $L^\infty$-estimates [1, Theorem 1.2] it was shown in [26] that $e^{tA}$ is bounded semigroup on $L^\infty_v$ for exterior domains by appealing to the maximum modulus theorem for the boundary-value problem of the stationary Stokes equations. Note that it is unknown whether $e^{tA}$ is a bounded analytic semigroup on $L^\infty_v$.

In the sequel, we sketch a proof for the a priori estimate (1.4). Our argument can be divided into the following three steps:

(i) (Localization) We first localize a solution $(v, q)$ of the Stokes equations (1.1)-(1.3) in a domain $\Omega' = B_{r_0}((\eta + 1)r) \cap \Omega$ for $x_0 \in \Omega$, $r > 0$ and parameters $\eta \geq 1$ by setting $u = v\theta_0$ and $p = (q - q_0)\theta_0$ with a constant $q_0$ and the smooth cut-off function $\theta_0$ around $\Omega_{r_0}$ satisfying $\theta_0 \equiv 1$ in $B_{r_0}(r)$ and $\theta_0 \equiv 0$ in $B_{r_0}((\eta + 1)r)^c$. We then observe that $(u, p)$ solves the Stokes resolvent equations with inhomogeneous divergence-free condition in the localized domain $\Omega'$. Applying the $L^p$-estimates for the localized Stokes equations we have

$$
|\lambda||u|_{L^p(\Omega')} + |\lambda|^{1/2}||\nabla u||_{L^p(\Omega')} + ||\nabla^2 u||_{L^p(\Omega')} + ||\nabla p||_{L^p(\Omega')}
\leq C_p \left( ||h||_{L^p(\Omega')} + ||\nabla g||_{L^p(\Omega')} + ||\lambda||g||_{W^{1,p'}_0(\Omega')} \right),
$$

(1.7)

where $W^{-1,p'}(\Omega')$ denotes the dual space of the Sobolev space $W^{1,p'}(\Omega')$ with $1/p + 1/p' = 1$. The external forces $h$ and $g$ contain error terms appearing in the cut-off
procedure and are explicitly given by

\[ h = f_0 - 2\nabla v \nabla \theta_0 - v \Delta \theta_0 + (q - q_c) \nabla \theta_0, \quad g = v \cdot \nabla \theta_0. \]  

(1.8)

(ii) (Error estimates) A key step is to estimate the error terms of the pressure such as \( (q - q_c) \nabla \theta_0 \). We here simplify the description by disregarding the terms related to \( g \) in order to describe the essence of the proof. We will give precise estimates for the terms related to \( g \) in Section 3. Now, the error terms related to \( h \) are estimated in the form

\[
\| h \|_{L^p(\Omega)} \leq C r^{\eta/p} \left( (\eta + 1)^{\eta/p} \| f \|_{L^\infty(\Omega)} + (\eta + 1)^{-(1-n/p)} \left( r^{-2} \| v \|_{L^\infty(\Omega)} + r^{-1} \| \nabla v \|_{L^\infty(\Omega)} \right) \right).
\]

(1.9)

If we disregard the term \( (q - q_c) \nabla \theta_0 \) in \( h \), the estimates (1.8) easily follows by using the estimates of the cut-off function \( \theta_0 \), i.e. \( \| \theta_0 \|_{\infty} + (\eta + 1) r \| \nabla \theta_0 \|_{\infty} + (\eta + 1)^2 r^2 \| \nabla^2 \theta_0 \|_{\infty} \leq K \) with some constant \( K \). We invoke the estimate (1.5) in order to handle the pressure term by velocity through the Poincaré-Sobolev-type inequality:

\[
\| \varphi - (\varphi) \|_{L^p(\Omega_{\eta, r})} \leq C s^{\eta/p} \| \nabla \varphi \|_{L^p_{\eta, r}(\Omega)} \quad \text{for all } \varphi \in \hat{W}^{1,\infty}_d(\Omega),
\]

(1.10)

with some constant \( C \) independent of \( s > 0 \), where \( (\varphi) \) denotes the mean value of \( \varphi \) in \( \Omega_{\eta, r} \), and \( \hat{W}^{1,\infty}_d(\Omega) = \{ \varphi \in L^1_{\operatorname{loc}}(\tilde{\Omega}) \mid \nabla \varphi \in L^\infty_d(\Omega) \} \). We prove the inequality (1.10) in Section 2. By taking \( q_c = (q) \) and applying (1.10) for \( \varphi = q \) and \( s = (\eta + 1) r \) we obtain the estimate (1.9) via (1.5).

(iii) (Interpolation) Once we establish the error estimates for \( h \) and \( g \), it is easy to obtain the estimate (1.4) by applying the interpolation inequality.

\[
\| \varphi \|_{L^p(\Omega_{\eta, r})} \leq C r^{-\eta/p} \left( \| \varphi \|_{L^p(\Omega_{\eta, r})} + r \| \nabla \varphi \|_{L^p(\Omega_{\eta, r})} \right) \quad \text{for } \varphi \in W^{1,p}_{\eta, r}(\tilde{\Omega}),
\]

(1.11)

for \( \varphi = u \) and \( \nabla u \). Now taking \( r = |\lambda|^{-1/2} \) we obtain the estimate for \( M_p(v, q)(x_0, \lambda) \) with the parameters \( \eta \) of the form,

\[
M_p(v, q)(x_0, \lambda) \leq C \left( (\eta + 1)^{\eta/p} \| f \|_{L^\infty(\Omega)} + (\eta + 1)^{-(1-n/p)} \| M_p(v, q) \|_{L^\infty(\Omega)}(\lambda) \right)
\]

(1.12)

for some constant \( C \) independent of \( \eta \). The second term in the right-hand-side is absorbed into the left-hand-side by letting \( \eta \) sufficiently large provided \( p > n \).

Actually, in the procedure (ii) we take \( q_c \) by the mean value of \( q \) in \( \Omega_{|\lambda|, (\eta + 2) r} \), since we estimate \( |\lambda| \| g \|_{W^{n,1}_0} \). By using the equation (1.1) we reduce the estimate of \( |\lambda| \| g \|_{W^{n,1}_0} \) to the \( L^\infty \)-estimate for the boundary value of \( q - q_c \) on \( \partial \Omega' \). In order to
estimate \( \| q - q_c \|_{L^p(\Omega)} \) we use a uniformly local \( L^p \)-norm bound for \( \nabla q \) besides the sup-bound for \( \nabla v \). This is the reason why we need the norm \( \| M_p(v, q) \|_{L^p(\Omega)} \) in the right-hand-side of (1.12). For general elliptic operators, the estimate (1.12) is valid without invoking the uniformly local \( L^p \)-norm bound for second derivatives of a solution.

This paper is organized as follows. In Section 2 we prove the inequality (1.10) for uniformly \( C^2 \)-domains. More precisely, we prove stronger estimates than (1.10) both interior and up to boundary \( \Omega_{x_0, d} \) of \( \Omega \). In Section 3 we first prepare the estimates for \( h \) and \( g \) and then prove the a priori estimate (1.4) (Theorem 1.1.) After proving Theorem 1.1, we also note the estimates (1.4) subject to the Robin boundary condition.

2 Poincaré-Sobolev-type inequality

In this section we prove the inequality (1.10) in a uniformly \( C^2 \)-domain. We start with the Poincaré-Sobolev-type inequality in a bounded domain \( D \) and observe the compactness of the embedding from \( \hat{W}^{1,\infty}_d(D) \) to \( L^p(D) \) which is the key in proving the inequality (1.10) by reductio ad absurdum.

2.1 Curvilinear coordinates

Let \( D \) be a bounded domain in \( \mathbb{R}^n, n \geq 2 \) and \( p \in [1, \infty) \). We prove an inequality of the form,

\[
\| \varphi - (\varphi) \|_{L^p(D)} \leq C\| \nabla \varphi \|_{L^\infty(D)} \quad \text{for } \varphi \in \hat{W}^{1,\infty}_d(D) \tag{2.1}
\]

where \((\varphi)\) denotes the mean value of \( \varphi \) in \( D \), i.e. \( (\varphi) = \frac{1}{|D|} \int_D \varphi dx \). If we replace the norm \( \| \nabla \varphi \|_{L^\infty(D)} \) by the \( L^p \)-norm \( \| \nabla \varphi \|_{L^p(D)} \), the estimate (2.1) is nothing but the Poincaré inequality [11, 5.8.1]. We observe that the boundedness of \( \| \nabla \varphi \|_{L^\infty(\Omega)} \) implies \( L^p \)-integrability of \( \varphi \) in \( D \) even if \( \nabla \varphi \) is not in \( L^p(D) \). For example, when \( D = B_0(1) \), \( \varphi(x) = \log (1 - |x|) \) is in \( L^p \) although \( |\nabla \varphi(x)| = d_D(x)^{-1} \) is not for any \( p \in [1, \infty) \). Since the space \( \hat{W}^{1,\infty}_d \) is compactly embedded to the space \( C(\bar{D'}) \) for each subdomain \( D' \) of \( D \) with \( \bar{D'} \subset D \), we shall show a pointwise upper bound for \( \varphi \) near \( \partial D' \) by an \( L^p \)-integrable function to conclude that the space \( \hat{W}^{1,\infty}_d(D) \) is compactly embedded to \( L^p(D) \) by the dominated convergence theorem. We estimate \( \varphi \in \hat{W}^{1,\infty}_d(D) \) near \( \partial D \) directly by using the curvilinear coordinates. Here for a domain \( \Omega \), \( \partial \Omega \neq \emptyset \), we say that \( \partial \Omega \) is \( C^k \) if for each \( x_0 \in \partial \Omega \), there exist
constants \( \alpha, \beta \) and \( C^k \)-function \( h \) of \( n - 1 \) variables \( y' \) such that (up to rotation and translation if necessary) we have

\[
U(x_0) \cap \Omega = \{(y', y_n) \mid h(y') < y_n < h(y') + \beta, \ |y'| < \alpha\},
\]

\[
U(x_0) \cap \partial \Omega = \{(y', y_n) \mid y_n = h(y'), |y'| < \alpha\},
\]

\[
\sup_{|y| \leq k, |y'| < \alpha} |\partial_y h(y')| \leq K, \ \nabla' h(0) = 0, \ h(0) = 0,
\]

with the constant \( K \) and the neighborhood of \( x_0 \), \( U(x_0) = U_{\alpha, \beta, h}(x_0) \), i.e.

\[
U_{\alpha, \beta, h}(x_0) = \{(y', y_n) \in \mathbb{R}^n \mid h(y') - \beta < y_n < h(y') + \beta, |y'| < \alpha\}.
\]

Here \( \partial_i = \partial_{x_{i_1}} \cdots \partial_{x_{i_k}} \) for a multi-index \( l = (l_1, \ldots, l_n) \) and \( \partial_{x_i} = \partial / \partial x_j \) as usual and \( \nabla' \) denotes the gradient in \( \mathbb{R}^{n-1} \). Moreover, if we are able to take uniform constants \( \alpha, \beta, K \) independent of each \( x_0 \in \partial \Omega \), we call \( \Omega \) uniformly \( C^k \)-domain of type \( (\alpha, \beta, K) \) as defined in [33, I.3.2].

We estimate \( \varphi \in \dot{W}^{1,1}_d(\Omega) \) along the boundary using the curvilinear coordinates.

**Proposition 2.1.** Let \( D \) be a bounded domain with \( C^k \)-boundary \( (k \geq 2) \). Let \( \Gamma = \{x \in \partial D \mid x = (x', h(x'))), |x'| < \alpha' \} \) be a neighborhood of \( x_0 \in \partial D \).

(i) There exist positive constants \( \mu \) and \( \alpha' \) such that \((\gamma, d) \mapsto X(\gamma, d) = \gamma + d \mu (\gamma)\) is a \( C^{k-1} \) diffeomorphism from \( \Gamma \times (0, \mu) \) onto

\[
N^\mu(\Gamma) = \{X(\gamma, d) \in U(x_0) \mid (\gamma, d) \in \Gamma \times (0, \mu)\},
\]

i.e. \( x \in N^\mu(\Gamma) \) has a unique projection to \( \partial D \) denoted by \( \gamma(x) \in \partial D \) such that

\[
(\gamma(x), d_D(x)) = X^{-1}(x) \quad \text{for } x \in N^\mu(\Gamma).
\]

(ii) There exists a constant \( C_1 \) such that for any \( x_1 \in \overline{N^\mu(\Gamma)} \) and \( r_1 > 0 \) satisfying \( D_{x_1, r_1} = B_{x_1}(r_1) \cap D \subset \overline{N^\mu(\Gamma)} \),

\[
|\varphi(x) - \varphi(y)| \leq C_1 \left( \log \frac{d_D(x)}{d_D(y)} + \frac{|\gamma(x) - \gamma(y)|}{\max(d_D(x), d_D(y))} \right) \sup_{z \in D_{x_1, r_1}} d_D(z) |\nabla \varphi(z)| \quad \text{for } x, y \in D_{x_1, r_1}
\]

and \( \varphi \in \dot{W}^{1,\infty}_d(D) \).

**Proof.** The assertion (i) is based on the inverse function theorem [23, Lemma 4.4.7]. We shall prove the second assertion (ii). We take points \( x, y \in D_{x_1, r_1} \) for
for each fixed $y$ bounded from above by an $(i)$ We observe from the second assertion that Remarks 2.2.

We connect $x$ and $z$ by the straight line to estimate

$$|\varphi(x) - \varphi(z)| \leq |\varphi(x) - \varphi(y)| + |\varphi(y) - \varphi(z)|.$$

It remains to estimate $|\varphi(z) - \varphi(y)|$. We connect $z$ and $y$ by the curve

$$C_{z,y} = \{X(\gamma(t), d(y)) : \gamma(t) = (\gamma'(t), h(\gamma'(t))), \gamma'(t) = t\gamma'(x) + (1-t)\gamma'(y), 0 \leq t \leq 1\},$$

where $\gamma'$ denotes the $n-1$ variables of $\gamma$. We then estimate

$$|\varphi(z) - \varphi(y)| = \left| \int_0^1 \frac{d}{dt} \varphi(X(\gamma(t), d(y))) dt \right|$$

$$\leq C(1 + \mu K) \frac{|\gamma(x) - \gamma(y)|}{d(y)} \sup_{z \in D_{x,1}} d(z)|\nabla \varphi(z)|,$$

since $|d\gamma(t)/dt| \leq C|\gamma(x) - \gamma(y)|$ and $|\nabla \partial D| \leq K$ with a constant $C$ depending on $K$. The assertion (ii) thus follows. \hfill \Box

**Remarks 2.2.** (i) We observe from the second assertion that $\varphi \in \dot{W}^{1,\infty}_p(D)$ is bounded from above by an $L^p$-integrable function for all $p \in [1, \infty)$ near $\partial D$, i.e. for each fixed $y \in D_{x_1, r_1}$ such that $d_D(y) \geq \delta$ we have

$$|\varphi(x)| \leq C_2(|\log d_D(x)| + 1) \left( \sup_{z \in D_{x_1, r_1}} d_D(z)|\nabla \varphi(z)| \right) + |\varphi(y)| \quad \text{for } x \in D_{x_1, r_1},$$

(2.2)
with a constant $C_2$ depending on $\mu, \delta$.

(ii) Note that Proposition 2.1 is also valid for a uniformly $C^k$-domain $\Omega$ of type $(\alpha, \beta, K)$, i.e. there exist constants $\mu, \alpha'$, depending only on $\alpha, \beta, K$, such that for each $x_0 \in \partial \Omega$ the assertions (i) and (ii) hold. The above constants $C_1$ and $C_2$ are depending only on $\alpha, \beta, K$ and $\delta$. In the sequel, we will apply Proposition 2.1 to a uniformly $C^2$-domain to prove the inequality (1.10).

The estimate (2.2) implies the compactness from $\hat{W}^{1,\infty}_d(D)$ to $L^p(D)$.

Lemma 2.3. Let $D$ be a bounded domain in $\mathbb{R}^n, n \geq 2$ with $C^2$-boundary. Then there exists a constant $C_D$ such that the estimate (2.1) holds for all $\varphi \in \hat{W}^{1,\infty}_d(D).$ Moreover, the space $\hat{W}^{1,\infty}_d(D)$ is compactly embedded into $L^p(D)$.

Proof. We argue by contradiction. Suppose that the estimate (2.1) were false for any choice of the constant $C$. Then there would exist a sequence of functions $\{\varphi_m\}_{m=1}^{\infty} \subset \hat{W}^{1,\infty}_d(D)$ such that

$$||\varphi - (\varphi_m)||_{L^p(D)} > m||\nabla \varphi_m||_{L^\infty_d(D)}, \quad m \in \mathbb{N}. $$

We may assume $(\varphi_m) = 0$ by replacing $\varphi_m$ to $\varphi_m - (\varphi_m)$. We divide $\varphi_m$ by $M_m = ||\varphi_m||_{L^p(D)}$ to get a sequence of functions $\{\phi_m\}_{m=1}^{\infty}, \phi_m = \varphi_m/M_m$ such that

$$||\nabla \phi_m||_{L^\infty_d(D)} < 1/m,

||\phi_m||_{L^p(D)} = 1 \quad \text{with} \quad (\phi_m) = 0. $$

We now prove the compactness of $\{\phi_m\}_{m=1}^{\infty}$ in $L^p(D)$. Since $||\nabla \phi_m||_{L^\infty_d(D)}$ is bounded, $\{\phi_m\}_{m=1}^{\infty}$ subsequently converges to a limit $\tilde{\phi}$ locally uniformly in $D$. By Proposition 2.1, in particular, the estimate (2.2) implies that $\phi_m$ is uniformly bounded from above by an $L^p$-integrable function near $\partial D$. The dominated convergence theorem implies that

$$\phi_m \to \tilde{\phi} \quad \text{in} \quad L^p(D) \quad \text{as} \quad m \to \infty. $$

Since $\nabla \phi_m(x) \to 0$ as $m \to \infty$ for each $x \in D$ and $||\tilde{\phi}||_{L^p(D)} = 1$, $\tilde{\phi}$ is a non-zero constant which contradicts the fact that $(\tilde{\phi}) = 0$. We reached a contradiction.

For the compactness of $\{\phi_m\}_{m=1}^{\infty}$ in $L^p(D)$ we here only invoke the boundedness of $||\nabla \phi_m||_{L^\infty_d(D)}$. This means that the embedding from $\hat{W}^{1,\infty}_d(D)$ into $L^p(D)$ is compact.

The proof is now complete. \(\square\)
2.2 Estimates near the boundary

We now prove the inequality (1.10) for uniformly $C^2$-domains $\Omega$. When the ball $B_{x_0}(r)$ locates interior of $\Omega$, i.e. $\Omega_{x_0,r} = B_{x_0}(r)$, applying (2.1) to $\varphi_r(x) = \varphi(x_0 + rx)$ in $D = \text{int} B_0(1)$ implies the estimate

$$
\|\varphi - (\varphi)\|_{L^p(\Omega_{x_0,r})} \leq C r^{n/p} \sup_{z \in \Omega_{x_0,r}} d\Omega_{x_0,r}(z)|\nabla \varphi(z)|, \quad r > 0. \tag{2.3}
$$

Since $d\Omega_{x_0,r}(x) \leq d\Omega(x)$ for $x \in \Omega_{x_0,r}$, the assertion (1.10) follows. However, if $B_{x_0}(r)$ involves $\partial \Omega$, the boundary of $\Omega_{x_0,r}$ may not have $C^1$-regularity. We thus prove

$$
\|\varphi - (\varphi)\|_{L^p(\Omega_{x_0,r})} \leq C r^{n/p} \sup_{z \in \Omega_{x_0,r}} d\Omega(z)|\nabla \varphi(z)| \quad \text{for } \varphi \in \hat{W}^{1,\infty}_d(\Omega) \tag{2.4}
$$

for $x_0 \in \Omega$ and $r > 0$ satisfying $d\Omega(x_0) < r$, which is weaker than (2.3).

**Proposition 2.4.** Let $\Omega$ be a uniformly $C^2$-domain. There exist constants $r_0$ and $C$ such that for $x_0 \in \Omega$ and $r < r_0$ satisfying $d\Omega(x_0) < r$, the estimate (2.4) holds for all $\varphi \in \hat{W}^{1,\infty}_d(\Omega)$ with a constant $C$ independent of $x_0$ and $r$.

The inequality (1.10) easily follows from Proposition 2.4.

**Lemma 2.5.** The inequality (1.10) holds for $\varphi \in \hat{W}^{1,\infty}_d(\Omega)$ for all $x_0 \in \Omega$ and $r < r_0$ with a constant $C$ independent of $x_0$ and $r$.

**Proof.** For $r < r_0$, combining (2.3) for $d\Omega(x_0) > r$ with (2.4) for $d\Omega(x_0) < r$, the assertion (1.10) follows. \hfill \Box

**Proof of Proposition 2.4.** We argue by contradiction. Suppose that the estimate (2.4) were false for any choice of constants $r_0$ and $C$. Then, there would exist a sequence of functions $\{\varphi_m\}_{m=1}^{\infty} \subset \hat{W}^{1,\infty}_d(\Omega)$ and a sequence of points $\{x_m\}_{m=1}^{\infty} \subset \Omega$ satisfying $d\Omega(x_m) < r_m \downarrow 0$ such that

$$
\|\varphi_m - (\varphi_m)\|_{L^p(\Omega_{x_m,r_m})} > m r_m^{n/p} \sup_{z \in \Omega_{x_m,r_m}} d\Omega(z)|\nabla \varphi_m(z)|, \quad m \in \mathbb{N}.
$$

Replacing $\varphi_m$ by $\varphi_m - (\varphi_m)$ and dividing $\varphi_m$ by $r_m^{-n/p}\|\varphi_m\|_{L^p(\Omega_{x_m,r_m})}$ (still denoted by $\varphi_m$) we observe that $\varphi_m$ satisfies $r_m^{-n/p}\|\varphi_m\|_{L^p(\Omega_{x_m,r_m})} = 1$ with $(\varphi_m) = 0$ and $\sup_{z \in \Omega_{x_m,r_m}} d\Omega(z)|\nabla \varphi_m(z)| < 1/m$. Since the points $\{x_m\}_{m=1}^{\infty}$ accumulates at the boundary $\partial \Omega$, we may assume by rotation and translation of $\Omega$ that $x_m = (0, d_m)$ with
d_m = d_\Omega(x_m) which subsequently converges to the origin located on the boundary \( \partial \Omega \). Here the neighborhood of the origin is denoted by \( \Omega_{\text{loc}} = U(0) \cap \Omega \) with constants \( \alpha, \beta \) and \( C^2 \)-function \( h \), i.e.

\[
\Omega_{\text{loc}} = \{(x', x_n) \in \mathbb{R}^n_+ | h(x') < x_n < h(x') + \beta, \ |x'| < \alpha \}.
\]

We rescale \( \varphi_m \) around the point \( x_m \) by setting

\[
\phi_m(x) = \varphi_m(x_m + r_m x) \quad \text{for} \quad x \in \Omega^m,
\]

where \( \Omega^m = \{x \in \Omega | x = (y - x_m)/r_m, y \in \Omega \} \) is the rescaled domain. Since \( c_m = d_m/r_m < 1 \), by taking a subsequence we may assume \( \lim_{m \to \infty} c_m = c_0 \leq 1 \).

We then observe that the rescaled domain \( \Omega^m \) expands to a half space \( \mathbb{R}^r_{+ - c_0} = \{(x', x_n) \in \mathbb{R}^n | x_n > -c_0 \} \). In fact, the neighborhood \( \Omega_{\text{loc}} \subset \Omega \) is rescaled to the domain,

\[
\Omega_{\text{loc}}^m = \left\{(x', x_n) \in \mathbb{R}^n \mid \frac{1}{r_m} h(r_m x') - c_m < x_n < \frac{1}{r_m} h(r_m x') + \frac{\beta}{r_m}, \ |x'| < \alpha \right\}
\]

which converges to \( \mathbb{R}^r_{+ - c_0} \) by letting \( m \to \infty \). Note that constants of uniformly regularity of \( \partial \Omega_m \) are uniformly bounded under this rescaling procedure. Moreover, for any constants \( \mu \) and \( \alpha' \), the curvilinear neighborhood of the origin \( \mathcal{N}^{\mu}(\Gamma) \) is in \( \Omega_{\text{loc}}^m \) for sufficiently large \( m \geq 1 \), where \( \Gamma = \Gamma_{\alpha'}(0) \) is the neighborhood of the origin on \( \partial \Omega^m \). Then the estimates for \( \varphi_m \) are inherited to the estimates for \( \phi_m \), i.e.

\[
\sup_{z \in \Omega_{\alpha,1}^m} d_{\Omega^m}(z)|\nabla \phi_m(z)| < 1/m, \quad m \in \mathbb{N},
\]

\[
||\phi_m||_{L^p(\Omega_{\alpha,1}^m)} = 1 \quad \text{with} \quad (\phi_m) = \int_{\Omega_{\alpha,1}^m} \phi_m = 0,
\]

where \( \Omega_{\alpha,1}^m = B_0(1) \cap \Omega^m \). From above bound for \( \phi_m \) the sequence \( \{\phi_m\}_{m=1}^{\infty} \) subsequently converges to a limit \( \bar{\phi} \) locally uniformly in \( (\mathbb{R}^n_{+ - c_0})_{0,1} = \mathbb{R}^n_{+ - c_0} \cap B_0(1) \).

We now observe the compactness of the sequence \( \{\phi_m\}_{m=1}^{\infty} \) in \( L^p((\mathbb{R}^n_{+ - c_0})_{0,1}) \). By Remark 2.2 (ii) applying Proposition 2.1 to \( \Omega^m \), the estimate (2.2) with \( x_1 = 0, r = 1 \) and a fixed \( y \in \Omega_{\alpha,1}^m \) satisfying \( d_{\Omega^m}(y) \geq \delta \) yield

\[
|\varphi_m(x)| \leq C(|\log d_{\Omega^m}(x)| + 1) \left( \sup_{z \in \Omega_{\alpha,1}^m} d_{\Omega^m}(z)|\nabla \phi_m(z)| \right) + |\phi_m(y)| \quad \text{for} \quad x \in \Omega_{\alpha,1}^m,
\]

for sufficiently large \( m \geq 1 \). Here the constant \( C \) is independent of \( m \geq 1 \). Since \( \phi_m \) is uniformly bounded from above by an \( L^p \)-integrable function in \( \Omega_{\alpha,1}^m \), the dominated convergence theorem implies that \( \phi_m \) converges to a limit \( \bar{\phi} \) in \( L^p((\mathbb{R}^n_{+ - c_0})_{0,1}) \).
Since $\nabla \phi_m(x) \to 0$ as $m \to \infty$ for each $x \in (\mathbb{R}^n_{+,-c_0})_{0,1}$ and $||\tilde{\phi}||_{L^p(\mathbb{R}^n_{+,-c_0})_{0,1}} = 1$, $\tilde{\phi}$ is a non-zero constant which contradicts the fact that $(\tilde{\phi}) = 0$. We reached a contradiction and the proof is now complete.

3 A priori estimates for the Stokes equations

The goal of this section is to prove the a priori estimate (1.4) by using the inequality (1.10). A key step is to establish the estimates for $h$ and $g$ in the procedure (ii) as explained in the introduction. We first recall the $L^p$-estimates to the Stokes equations (1.7) and the interpolation inequality (1.11). Note that the constants $C_p$ and $C_I$ in (1.7) and (1.11) respectively are independent of the volume of domains $\Omega'$, $\Omega_{x_0,r}$.

3.1 $L^p$-estimates for localized equations

Let $\Omega'$ be a bounded domain with $C^2$-boundary. For the a priori estimate (1.4) we invoke the $L^p$-estimates (1.7) to the Stokes resolvent equations with inhomogeneous divergence-free condition,

$$\lambda u - \Delta u + \nabla p = h \quad \text{in } \Omega', \tag{3.1}$$
$$\text{div } u = g \quad \text{in } \Omega', \tag{3.2}$$
$$u = 0 \quad \text{on } \partial \Omega', \tag{3.3}$$

for $h \in L^p(\Omega')$, $g \in W^{1,p}(\Omega') \cap L^p_{av}(\Omega')$ and $\lambda \in \sum_{\theta,0}$ with $\theta \in (\pi/2, \pi)$. Here $L^p_{av}(\Omega')$ denotes the space of all functions $g$ in $L^p(\Omega')$ satisfying average zero, i.e. $\int_{\Omega'} g dx = 0$. The estimate (1.7) is proved by a perturbation argument [15], [16] with the constant $C_p$ independent of the volume of $\Omega'$.

**Proposition 3.1.** ([15], [16]) Let $\theta \in (\pi/2, \pi)$ and $\lambda \in \sum_{\theta,0}$. For $h \in L^p(\Omega')$ and $g \in W^{1,p}(\Omega') \cap L^p_{av}(\Omega')$ there exists a unique solution of (3.1)-(3.3) satisfying the estimates (1.7) with the constant $C_p$ independent of the volume of $\Omega'$ and depending on $\theta$, $p$, $n$ and the $C^2$-regularity of $\partial \Omega'$.

We estimate the $L^\infty$-norms of a solution up to first derivatives via the Sobolev embeddings together with the $L^p$-estimates (1.7) for $p > n$. In order to estimate the $L^\infty$-norms of a solution we apply the interpolation inequality (1.11) [24, Chapter 3, Lemma 3.1.4] in $\Omega_{x_0,r} = B_{x_0}(r) \cap \Omega$ for $x_0 \in \bar{\Omega}$ and $r < r_0$ with a constant $r_0$. In what follows, we fix the constant $r_0$ by taking the same constant $r_0$ given by Lemma 2.5. The constant $C_I$ is also independent of the radius $r$. 

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3.2 Estimates in the localization procedure

We prepare the estimates for \( h \) and \( g \) in the procedure (ii). The estimate for \( |\lambda||g|_{W_0^{1,p}} \) is different from that of \( ||h||_{L^p} \). In order to estimate \( |\lambda||g|_{W_0^{1,p}} \), we use the uniformly local \( L^p \)-norm bound for \( \nabla q \) besides the sup-bound of \( \nabla v \) as in (3.8). After establishing these estimates, we will put the procedures (i)-(iii) together in the next subsection.

Let \( \Omega \) be a uniformly \( C^2 \)-domain. Let \( \theta \) be a smooth cut-off function satisfying \( \theta \equiv 1 \) in \([0,1/2] \) and \( \theta \equiv 0 \) in \([1,\infty) \). For \( x_0 \in \Omega \) and \( r > 0 \) we set \( \theta_0(x) = \theta(|x - x_0|/(\eta + 1)r) \) with parameters \( \eta \geq 1 \) and observe that \( \theta_0 \equiv 1 \) in \( B_{x_0}(r) \) and \( \theta_0 \equiv 0 \) in \( B_{x_0}((\eta + 1)r) \). The cut-off function \( \theta_0 \) is uniformly bounded by a constant \( K \), i.e.

\[
\|\theta_0\|_{\infty} + (\eta + 1)r\|\nabla \theta_0\|_{\infty} + (\eta + 1)^2r^2\|\nabla^2 \theta_0\|_{\infty} \leq K, \quad \eta \geq 1 \tag{3.4}
\]

Let \((v,\nabla q) \in W^{2,p}_{\text{loc}}(\bar{\Omega}) \times L^p_{\text{loc}}(\bar{\Omega})\) be a solution of (1.1)-(1.3) for \( f \in L^p_\omega(\Omega) \) and \( \lambda \in \Sigma_{\theta,0} \). We localize a solution \((v,\nabla q)\) in a domain \( \Omega' = \Omega_{x_0,(\eta + 1)r} \) by setting \( u = v\theta_0 \) and \( p = \hat{q}\theta_0 \) with \( \hat{q} = q - q_c \) and a constant \( q_c \). We then observe that \((u,\nabla p)\) solves the localized equation (3.1)-(3.3) in the domain \( \Omega' \) with \( h \) and \( g \) given by (1.8). We shall show the following estimates for \( h \) and \( g \),

\[
\|\nabla g\|_{L^p(\Omega')} \leq C_1 r^{n/p}(\eta + 1)^{1 - n/p}(r^{-1}\|\nabla v\|_{L^\infty(\Omega)} + r^{-2}\|v\|_{L^\infty(\Omega)}), \tag{3.5}
\]

\[
|\lambda||g|_{W_0^{1,p}(\Omega')} \leq C_2 r^{n/p}(\eta + 1)^{n/p}\|f\|_{L^\infty(\Omega)} + (\eta + 1)^{-1 - n/p}(r^{-1}\|\nabla v\|_{L^\infty(\Omega)} + r^{-2}\|v\|_{L^\infty(\Omega)}), \tag{3.6}
\]

\[
|\lambda||g|_{W_0^{1,p}(\Omega')} \leq C_3 r^{n/p}(\eta + 1)^{n/p}\|f\|_{L^\infty(\Omega)} + (\eta + 1)^{1 - n/p}(r^{-1}\|\nabla v\|_{L^\infty(\Omega)} + r^{-n/p}\sup_{z \in \Omega}\|\nabla q\|_{L^\infty(\Omega')}). \tag{3.7}
\]

with constants \( C_1, C_2 \) and \( C_3 \) independent of \( r \) and \( \eta \geq 1 \). For the estimates of the terms of \( f, v \) and \( \nabla v \) we use the estimates

\[
\|f\theta_0\|_{L^p(\Omega')} \leq KC_4^1 r^{n/p}(\eta + 1)^{n/p}\|f\|_{L^\infty(\Omega)}, \tag{3.8}
\]

\[
\|\nabla v\nabla \theta_0\|_{L^p(\Omega')} \leq KC_4^1 r^{n/p}(\eta + 1)^{-1 - n/p}r^{-1}\|\nabla v\|_{L^\infty(\Omega)}, \tag{3.9}
\]

\[
\|v\nabla^2 \theta_0\|_{L^p(\Omega')} \leq KC_4^1 r^{n/p}(\eta + 1)^{-(1 - n/p)r^{-2}\|v\|_{L^\infty(\Omega)}}, \tag{3.10}
\]

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for all $r > 0$ and $\eta \geq 1$, where the constant $C_n$ denotes the volume of the $n$-dimensional unit ball. Since $\nabla g = \nabla v \nabla \theta_0 + v \nabla^2 \theta_0$ does not contain the pressure, the estimate (3.5) easily follows from the estimates (3.9) and (3.10).

For the estimates (3.6) and (3.7) we apply the inequality (1.10). We choose a constant $q_c$ by a mean value of $q$ in $\Omega_{(\eta+2)r}$, i.e.

$$q_c = \frac{1}{\Omega_{(\eta+2)r}} \int_{\Omega_{(\eta+2)r}} q(x) dx.$$  \hspace{1cm} (3.11)

We then observe that the inequality (1.10) implies the estimate

$$\|q\|_{L^p(\Omega_{\eta+2r})} \leq C r^{n/p}(\eta + 2)^{n/p}\|\nabla q\|_{L_\infty(\Omega)}$$ \hspace{1cm} (3.12)

for $r > 0$ and $\eta \geq 1$ satisfying $(\eta + 2)r \leq r_0$, where $\hat{q} = q - q_c$.

In order to estimate (3.7) we estimate the $L^\infty$-norm of $\hat{q}$ on $\Omega'$ since by using the equation $\lambda v = f + \Delta v - \nabla q$ we reduce (3.7) to the estimate of the boundary value of $\hat{q}$ on $\partial \Omega'$. This is the reason why we take $q_c$ by (3.11). We apply the inequality (1.11) in $\Omega_{x_1, r} \subset \Omega_{(\eta+2)r}$ for $x_1 \in \Omega'$ and $r > 0$ with $p > n$ to estimate

$$\|\hat{q}\|_{L^\infty(\Omega_{x_1, r})} \leq C r^{-n/p}(\|\hat{q}\|_{L^p(\Omega_{x_1, r})} + r\|\nabla q\|_{L^p(\Omega_{x_1, r})})$$

$$\leq C r^{-n/p}(\|\hat{q}\|_{L^p(\Omega_{(\eta+2)r})} + r \sup_{z \in \Omega} \|\nabla q\|_{L^p(\Omega_{z, r})}).$$ \hspace{1cm} (3.13)

Combining the estimate (3.13) with (3.12) and taking a supremum for $x_1 \in \Omega'$, we have

$$\|\hat{q}\|_{L^\infty(\Omega')} \leq C (\eta + 2)^{n/p}\|\nabla q\|_{L_\infty(\Omega)} + r^{1-n/p} \sup_{z \in \Omega} \|\nabla q\|_{L^p(\Omega_{z, r})}.$$ \hspace{1cm} (3.14)

We now invoke the strictly admissibility of a domain $\Omega$ to estimate the norm $\|\nabla q\|_{L_\infty(\Omega)}$ by the sup-norm of $\nabla v$ in $\Omega$ via (1.5).

**Proposition 3.2.** Let $\Omega$ be a uniformly $C^2$-domain. Assume that $\Omega$ is strictly admissible, then the estimate

$$\|\hat{q}\|_{L^p(\Omega')} \leq C_4 r^{n/p}(\eta + 2)^{n/p}\|\nabla v\|_{L^\infty(\Omega)}$$ \hspace{1cm} (3.15)

holds for all $r > 0$ and $\eta \geq 1$ satisfying $(\eta + 2)r \leq r_0$ and $p \in [1, \infty)$. If in addition $p > n$, then the estimate

$$\|\hat{q}\|_{L^\infty(\Omega')} \leq C_5 (\eta + 2)^{n/p}\|\nabla v\|_{L^\infty(\Omega)} + r^{1-n/p} \sup_{z \in \Omega} \|\nabla q\|_{L^p(\Omega_{z, r})}.$$ \hspace{1cm} (3.16)

holds. The constants $C_4$ and $C_5$ are independent of $r$ and $\eta$. 

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Lemma 3.3. Let \( \Omega \) be a strictly admissible, uniformly \( C^2 \)-domain. Let \( (v, \nabla q) \in W^{2,p}_0(\Omega) \times L^p_0(\Omega) \) be a solution of (1.1)-(1.3) for \( f \in L^p_0(\Omega) \) and \( \lambda \in \mathbb{R} \). By (1.5), (3.12) and (3.14), the assertion follows.

Proof. By using the estimates (3.15) and (3.16) we obtain the estimates (3.6) and (3.7).

We first show (3.17). Take \( \varphi \in W^{1,\nu}(\Omega') \) satisfying \( \|\varphi\|_{W^{1,\nu}(\Omega')} \leq 1 \). By using \( \text{div } v = 0 \) integration by parts yields that

\[
\sum_{i,j=1}^n \int_{\Omega'} \partial_i v_j \partial_i \varphi dx = \sum_{i,j=1}^n \int_{\Omega'} (\partial_i v_j - \partial_j v_i) \partial_i \vartheta_0 \varphi dx - \int_{\partial\Omega'} (\partial_i v_j - \partial_j v_i) \partial_i \vartheta_0 \varphi n_i d\mathcal{H}^{n-1}(x).
\]

We estimate the second term in the right-hand side by the \( W^{1,1} \)-norm of \( \varphi \) in \( \Omega' \) [11, 5.5 Theorem 1.1] to estimate

\[
\|\varphi\|_{L^1(\partial\Omega)} \leq C_T \|\varphi\|_{W^{1,1}(\Omega')} \leq 2 C_T |\Omega'|^{1/p}
\]

with the constant \( C_T \) depending on the \( C^1 \)-regularity of the boundary \( \partial\Omega \) but independent of \( |\Omega'| \), the volume of \( \Omega' \). We thus obtain

\[
\left| \int_{\Omega'} \partial_i^2 v_j \partial_i \vartheta_0 \varphi dx \right| \leq (1 + 2 C_T) ||(\partial_i v_j - \partial_j v_i) \partial_i \vartheta_0||_{L^p(\Omega')} |\Omega'|^{1/p} \\
\leq 2(1 + 2 C_T) K C_n^{1/p} |\Omega'|^{1/p} ||\nabla v||_{L^p(\Omega)}.
\]
Thus the estimate (3.17) holds with the constant 

\[ C \]

remaining to show the estimate (3.18). Since \( \nabla q = \nabla \hat{q} \), integration by parts yields that

\[
\int_{\Omega} \nabla q \cdot \nabla \theta \varphi \, dx = - \int_{\Omega} \hat{q}(\Delta \theta \varphi + \nabla \theta \cdot \nabla \varphi) \, dx + \int_{\partial \Omega} \hat{q} \nabla \theta \cdot n \, dH^{n-1}(x)
\]

\[
= I + II + III.
\]

Combining (3.4), (3.19) with (3.16), we obtain

\[
II + III \leq (1 + 2C_T)\|\nabla \theta \|_{L^\infty(\Omega')}|\Omega'|^{1/p}
\]

\[
\leq (1 + 2C_T)KC^{-1/p}r^{1/p}(\eta + 1)^{-(1-n/p)f^{1-1}|\hat{q}|_{L^\infty(\Omega')}
\]

\[
\leq Cr^{n/p}(\eta + 1)^{-(1-2n/p)}(r^{1-n/p}\|\nabla \varphi\|_{L^\infty(\Omega)} + r^{n/p} \sup_{z \in \Omega} \|\nabla q\|_{L^p(\Omega,z)}),
\]

with the constant \( C \) depending on \( C_T, K, C, p, C_4 \) and \( C_5 \) but independent of \( r \) and \( \eta \). We complete the proof by showing the estimate for \( I \). Applying the Hölder inequality, for \( s, s' \in (1, \infty) \) with \( 1/s + 1/s' = 1 \) we have

\[
I \leq K(\eta + 1)^2r^{-2}\|\varphi\|_{L^s(\Omega')}\|\hat{q}\|_{L^{s'}(\Omega')}.
\]

Since \( p > n \), the conjugate exponent \( p' \) is strictly smaller than \( n/(n - 1) \) for \( n \geq 2 \). By setting \( 1/s = 1/p' + 1/n \) we apply the Sobolev inequality [11, 5.6 Theorem 2] to estimate \( \|\varphi\|_{L^s(\Omega')} \leq C_s\|\varphi\|_{W^{1,p'}(\Omega')} \leq C_s \) with the constant \( C_s \) independent of \( |\Omega'| \). Applying the estimate (3.15) to \( \hat{q} \) yields

\[
I \leq Cr^{n/p}(\eta + 2)^{n/p}\|\nabla \varphi\|_{L^\infty(\Omega)}
\]

\[
\leq Cr^{n/p}(\eta + 2)^{-(1-n/p)}r^{-1}\|\nabla \varphi\|_{L^\infty(\Omega)},
\]

since \( 1/s' = 1 - 1/s = 1/p + 1/n \). The constant \( C \) is independent of \( r \) and \( \eta \). The proof is now complete. \( \square \)

**Remark 3.4.** From the estimate (3.7) we observe that the exponent \(-(1 - 2n/p)\) of \( (\eta + 1) \) in front of the term \((r^{1-n/p}\|\nabla \varphi\|_{L^\infty(\Omega)} + r^{n/p} \sup_{z \in \Omega} \|\nabla q\|_{L^p(\Omega,z)})\) is negative provided that \( p > 2n \). We thus first prove the a priori estimate (1.4) for \( p > 2n \). Once we obtain the estimate \( \|\lambda\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)} \) it is easy to replace the estimate (3.7) to

\[
\|\lambda\|_{W^{1,p}(\Omega)} \leq CKC_n^{\frac{1/n}{p}r^{(1-n/p)}(\eta + 1)^{n/p}}\|f\|_{L^\infty(\Omega)}
\]

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for $p > n$ since

$$|\lambda||v \cdot \nabla \theta_0||_{W_0^{-1, p}(\Omega')} = |\lambda||v \theta_0||_{L^p(\Omega)}$$

$$\leq C||\theta_0||_{L^p(\Omega')}||f||_{L^\infty(\Omega)}$$

$$\leq CKC_\mu^{1/p}p^{\eta/p}(\eta + 1)^{n/p}||f||_{L^\infty(\Omega)}.$$

### 3.3 Interpolation

We now prove the a priori estimate (1.4) for $p > n$. The size of the parameter $\eta$ and the constant $\delta$ are determined only through the constants $C_p, C_I$ and $C_1-C_3$. Although we eventually obtain the estimate (1.12) for all $p > n$, firstly we prove the case $p > 2n$ as observed by Remark 3.4. The case $p > 2n$ is enough for analyticity but for the completeness we prove the estimate (1.4) for all $p > n$.

**Proof of Theorem 1.1.** We set $\delta = \delta_\eta = (\eta + 2)/r_0^2$ and now take $r = 1/|\lambda|^{1/2}$ for $\lambda \in \Sigma_{\theta, \delta}$. We then observe that $r = 1/|\lambda|^{1/2}$ and $\eta \geq 1$ automatically satisfy $r(\eta + 2) \leq r_0$ for $\lambda \in \Sigma_{\theta, \delta}$. We may assume that the boundary of $\Omega' = B_{r_0}((\eta + 1)r) \cap \Omega$ is $C^2$ because the localized equations (3.1)-(3.3) can be regarded as the equation in a subdomain $\Omega''$ of $\Omega$ by taking $\Omega''$ with $C^2$-boundary so that $\Omega' \subset \Omega''$ and $\Omega''$ preserves an order of the volume of $\Omega'$, i.e. $|\Omega''|$ is bounded from above by $C(\eta + 1)^{p/r^2}$ with a constant $C$ independent of $r > 0$ and $\eta \geq 1$. We first prove

**Case (I) $p > 2n$.** By applying the $L^p$-estimates (1.7) to $u = v \theta_0$ and $p = \hat{q} \theta_0$ in $\Omega'$ and combining the estimates (3.5)-(3.7) with (1.7), we obtain

$$|\lambda||u||_{L^p(\Omega')} + |\lambda|^{1/2}||\nabla u||_{L^p(\Omega')} + ||\nabla^2 u||_{L^p(\Omega')} + ||\nabla p||_{L^p(\Omega')}$$

$$\leq C_8|\lambda|^{n/2}p((\eta + 1)^{n/p}||f||_{L^\infty(\Omega)} + (\eta + 1)^{-1/2}||u||_{L^\infty(\Omega)}), \quad (3.20)$$

with the constants $C_8$ independent of $r = 1/|\lambda|^{1/2}$ and $\eta \geq 1$. We next estimate the $L^\infty$-norms of $u$ and $\nabla u$ in $\Omega$ by interpolation. Applying the interpolation inequality (1.11) for $\varphi = u$ and $\nabla u$ implies the estimates

$$||u||_{L^\infty(\Omega_{\varphi})} \leq C_{I}r^{-n/p}\left(||u||_{L^p(\Omega_{\varphi})} + r||\nabla u||_{L^p(\Omega_{\varphi})}\right),$$

$$||\nabla u||_{L^\infty(\Omega_{\varphi})} \leq C_{I}r^{-n/p}\left(||\nabla u||_{L^p(\Omega_{\varphi})} + r||\nabla^2 u||_{L^p(\Omega_{\varphi})}\right).$$

Summing up these norms together with $|\lambda|^{n/2}||\nabla^2 u||_{L^p(\Omega_{\varphi})}$ and $|\lambda|^{n/2}||\nabla p||_{L^p(\Omega_{\varphi})}$,
we have

\[ M_p(u, p)(x_0, \lambda) \leq C\eta^{-n/p} \left( |\lambda||u||_{L^p(\Omega_{x_0,r})} + |\lambda|^{1/2}||\nabla u||_{L^p(\Omega_{x_0,r})} + ||\nabla^2 u||_{L^p(\Omega_{x_0,r})} + ||\nabla p||_{L^p(\Omega_{x_0,r})} \right) \]  

(3.21)

with the constant \( C_0 \) independent of \( r \) and \( \eta \geq 1 \). Since \((u, \nabla p)\) agrees with \((v, \nabla q)\) in \( \Omega_{x_0,r} \), combining (3.20) with (3.21) yields

\[ M_p(v, q)(x_0, \lambda) \leq C_{10} \left( (\eta + 1)^{n/p}||f||_{L^\infty(\Omega)} + (\eta + 1)^{-(1-n/p)}||M_p(v, q)||_{L^\infty(\Omega)}(\lambda) \right) \]

(3.22)

with \( C_{10} = C_8C_9 \). By taking a supremum for \( x_0 \in \Omega \) and letting \( \eta \geq 1 \) large so that \( C_{10}(\eta + 1)^{-(1-n/p)} < 1/2 \) we obtain (1.4) with \( p > 2n \).

We shall complete the proof by showing the uniformly local \( L^p \)-bound for second derivatives of \((v, q)\) for all \( p > n \).

**Case (II) \( p > n \).** Since \( |\lambda||g||_{W^{-1,p}_0} \) is bounded for \( \rho > 2n \), we may assume \((v, \nabla q)\) \( \in W^{2,\rho}_0(\tilde{\Omega}) \times L^{\rho}_0(\tilde{\Omega}) \) with \( \rho > 2n \). By using \(|\lambda||v||_{L^\infty(\Omega)} \leq C||f||_{L^\infty(\Omega)} \) for \( \lambda \in \Sigma_{\theta, \delta} \) with \( \delta = \delta_p \) we replace the estimate (3.7) to

\[ |\lambda||g||_{W^{-1,p}_0(\tilde{\Omega})} \leq CKC_n^{1/p}p^{1/p}(\eta + 1)^{n/p}||f||_{L^\infty(\Omega)} \]

by Remark 3.4. Then we are able to replace the estimate (3.22) to

\[ ||M_p(v, q)||_{L^\infty(\Omega)}(\lambda) \leq C_{11} \left( (\eta + 1)^{p/n}||f||_{L^\infty(\Omega)} + (\eta + 1)^{-(1-n/p)}||M_p(v, q)||_{L^\infty(\Omega)}(\lambda) \right) . \]

Letting \( \eta \geq 1 \) large so that \( C_{11}(\eta + 1)^{-(1-n/p)} < 1/2 \), we obtain (1.4) for all \( p > n \). The proof is now complete.

**Remark 3.5.** (Robin boundary condition) Concerning the Robin boundary condition we replace the Dirichlet boundary condition for the localized equations (3.3) to the inhomogeneous boundary condition with a tangential vector field \( k \),

\[ B(u) = k, \quad u \cdot n_{\partial \Omega} = 0 \quad \text{on} \ \partial \Omega'. \]

Instead of the estimate (1.7) we apply the \( L^p \)-estimate of the form,

\[ |||u|||_{L^p(\Omega')} + |||\lambda|||_{L^p(\Omega')} + |||\nabla u|||_{L^p(\Omega')} + |||\nabla^2 u|||_{L^p(\Omega')} + |||\nabla p|||_{L^p(\Omega')} \]

\[ \leq C(|||u|||_{L^p(\Omega')} + |||\nabla g|||_{L^p(\Omega')} + |||\lambda|||_{W^{-1,p}_0(\Omega')} + |||\lambda|^{1/2}||k||L^p(\Omega') + |||\nabla k|||_{L^p(\Omega')}), \]

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where $k$ is identified with its arbitrary extension to $\Omega'$. Since $k = v_{\tan}\partial\theta_0/\partial n_{\Omega'}$ for $u = v\theta_0$ and $p = \hat{q}\theta_0$, we observe that the norms of $k$ in the right-hand-side are estimated by the same way with $\|\nabla g\|_{L^p}$ where $g = v\cdot\nabla\theta_0$. The $L^p$-estimates for the Robin boundary condition is proved by [32] for bounded and exterior domains by generalizing the perturbation argument to the Dirichlet boundary condition [16].

We thus observe that the constant $C$ is also independent of the volume $\Omega'$. After proving the a priori estimate (1.4) for $f \in L^\infty_\sigma$ subject to the Robin boundary condition, we verify the existence of a solution of (1.1) and (1.2). In particular, $v \in L^\infty_\sigma$ (not in $C_{0,\sigma}$). Then we are able to define the Stokes operator $A = A_R$ in $L^\infty_\sigma$ in the same way as we did for the Dirichlet boundary condition. Our observations may be summarized as following

**Theorem 3.6.** Assume that $\Omega$ is a bounded or an exterior domain with $C^3$!-boundary in $\mathbb{R}^n$. Then the Stokes operator $A = A_R$ subject to the Robin boundary condition generates an analytic semigroup on $L^\infty_\sigma(\Omega)$ of angle $\pi/2$.

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**References**


