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HARMONIC FUNCTIONS ON THE BRANCHING GRAPH ASSOCIATED WITH THE INFINITE WREATH PRODUCT OF A COMPACT GROUP

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ABSTRACT. Detailed study of the characters of $\mathfrak{S}_\infty(T)$, the wreath product of compact group T with the infinite symmetric group \mathfrak{S}_∞ , is indispensable for harmonic analysis on this big group. In preceding works, we investigated limiting behavior of characters of finite wreath product $\mathfrak{S}_n(T)$ as $n \rightarrow \infty$ and its connection with characters of $\mathfrak{S}_\infty(T)$. This paper takes a dual approach to these problems. We study harmonic functions on $\mathbb{Y}(\widehat{T})$, the branching graph of the inductive system of $\mathfrak{S}_n(T)$'s, and give a classification of the minimal nonnegative harmonic functions on it. This immediately implies a classification of the characters of $\mathfrak{S}_\infty(T)$, which is a logically independent proof of the one obtained in earlier works. We obtain explicit formulas for minimal nonnegative harmonic functions on $\mathbb{Y}(\widehat{T})$ and Martin integral expressions for harmonic functions.

INTRODUCTION

The present paper constitutes a part of our project on harmonic analysis of $\mathfrak{S}_\infty(T)$, the wreath product of compact group T with the infinite symmetric group \mathfrak{S}_∞ . Among the publications concerning this subject, we mention [7] and [9] (done jointly with E. Hirai), in which asymptotic behavior of characters of $\mathfrak{S}_n(T)$, the wreath product of T with \mathfrak{S}_n , and its connection with characters of $\mathfrak{S}_\infty(T)$ are analyzed in details from different points of view. Put together with [7] and [9], the present paper cuts a clearer figure as their companion. However, we keep its self-contained characterization having independent results and approach. Let us explain the problems we treat as well as how they are related to [7] and [9]. Several notions freely used in Introduction are given the definitions and explanations in later sections.

$\mathfrak{S}_\infty(T)$ is the semi-direct product of $D_\infty(T)$, the restricted direct product of T , and \mathfrak{S}_∞ under the canonical action of the latter on the former (see (1.13), (1.14)), equipped with the inductive limit topology. The following problems are fundamental in harmonic analysis of $\mathfrak{S}_\infty(T)$:

- (a) classification of the finite factorial (or primary) unitary representations
- (b) canonical direct integral decomposition of a finite unitary representation into factorial ones.

One knows that (b) is solved in an abstract sense as the central decomposition by virtue of von Neumann's reduction theory, though concrete computation of a measure for the superposition remains as an independent problem. Concerning (a),

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we recall that the following four objects are equivalent, in other words, there is a bijective correspondence between one another:

(a1) the quasi-equivalence classes of finite factorial unitary representations of $\mathfrak{S}_\infty(T)$

(a2) the extremal points of

$$(0.1) \quad \mathcal{K}(\mathfrak{S}_\infty(T)) = \{f : \mathfrak{S}_\infty(T) \longrightarrow \mathbb{C} \mid \\ f \text{ is continuous, positive-definite, central and normalized}\}$$

(a3) the extremal points of

$$(0.2) \quad \mathcal{H}(\mathbb{Y}(\widehat{T})) = \{\varphi : \mathbb{Y}(\widehat{T}) \longrightarrow \mathbb{C} \mid \varphi \text{ is harmonic, non-negative and normalized}\}$$

(a4) the extremal points of

$$(0.3) \quad \mathcal{M}(\mathfrak{T}(\widehat{T})) = \{\text{central probabilities on } \mathfrak{T}(\widehat{T})\}.$$

$\mathbb{Y}(\widehat{T})$ denotes the vertices of the branching graph for $\mathfrak{S}_n(T)$'s (see (1.16)). Harmonicity of φ is defined in (1.18). The term 'normalized' means $f(e) = 1$ in (0.1) and $\varphi(\emptyset) = 1$ in (0.2). $\mathfrak{T}(\widehat{T})$ denotes the set of infinite paths from \emptyset on the branching graph. Centrality in (0.3) is defined in (3.1). Correspondence between (a1) and (a2) is described in [3]. Explicit realizations of finite factorial unitary representations of $\mathfrak{S}_\infty(T)$ in terms of the classifying parameters are given in [6]. We refer to [9] for bijections between (0.1), (0.2) and (0.3). In Section 4, we review these bijections in a slightly wider context, namely in the case of a general inductive system of compact groups. Furthermore, relations between their topologies are also discussed. An extremal point of (0.1) is called a character of $\mathfrak{S}_\infty(T)$. Determination of the set of all characters of $\mathfrak{S}_\infty(T)$ had been performed by the second author and E. Hiai, which was completed in [4] and [5] in a final form. A main purpose of a series of works of [7], [9] and the present paper is to understand the above four objects (a1) – (a4) for $\mathfrak{S}_\infty(T)$ through limiting procedures from the inductive system of $\mathfrak{S}_n(T)$'s and thereby to give a sufficiently concrete answer to the fundamental problem of (a). As contrasted with one another, the present paper takes an approach from the viewpoint of (a3), while [7] and [9] did from (a2) and (a4) respectively. In the case where T is a finite group, such a character theory was developed in [2].

We give a characterization of (a3) by computing the Martin boundary $\partial\mathbb{Y}(\widehat{T})$ of the branching graph $\mathbb{Y}(\widehat{T})$. This approach is strongly motivated by those works treating the Young graph and its variations as [17], [12], [14], [1], [13]. It turns out that the minimal Martin boundary $\partial_m\mathbb{Y}(\widehat{T})$ is a proper subset of $\partial\mathbb{Y}(\widehat{T})$ if and only if compact group T is continuous. As an alternative answer to problem (b), we show the Martin integral representation for any element φ in (0.2) as

$$(0.4) \quad \varphi(\Lambda) = \int_{\Delta} \varphi_\omega(\Lambda) Q(d\omega),$$

in which Δ realizing $\partial_m\mathbb{Y}(\widehat{T})$ and kernel function $\varphi_\omega(\Lambda)$ are explicitly given together with the manner to obtain probability Q from φ as analogue of the 'radial limit' for a harmonic function on the unit disk of \mathbb{C} . Our approach is constructive without relying on Choquet's theorem.

1. PRELIMINARIES ABOUT WREATH PRODUCT GROUPS AND BRANCHING GRAPHS

Throughout the present paper, let T be a compact group with identity element e_T . $[T]$ and \widehat{T} denote the set of conjugacy classes of T and the set of equivalence classes of continuous irreducible unitary representations of T respectively. T may be non-commutative and continuous. For technical simplicity, however, we assume that \widehat{T} is at most countable. In this section, we recall necessary notions and some known results on wreath products of a compact group and associated branching graphs. In what follows, ‘continuous unitary representation’ is referred to as UR for short. Similarly, ‘irreducible UR’ is abbreviated to IUR. For $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, \mathfrak{S}_n denotes the symmetric group of degree n . \mathbb{Y}_n denotes the set of Young diagrams of size n . Set $\mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n$, the totality of all Young diagrams, where $\mathbb{Y}_0 = \{\emptyset\}$. We use the following (conventional) notations and terminologies for Young diagrams: for $\lambda \in \mathbb{Y}_n$,

- λ_j is the length of the j th row where $\lambda_1 \geq \lambda_2 \geq \dots$
- $m_i(\lambda)$ is the number of rows of length i contained in λ
- $l(\lambda) = \sum_{i=1}^{\infty} m_i(\lambda)$ is the number of rows of λ
- $|\lambda| = \sum_{j=1}^{l(\lambda)} \lambda_j = \sum_{i=1}^{\infty} im_i(\lambda) = n$ is the size of λ
- π^λ is an IUR of \mathfrak{S}_n labeled by λ .
- $\chi^\lambda = \text{tr} \pi^\lambda$ is the associated irreducible character of \mathfrak{S}_n .

(A) *Wreath product of T .* For $n \in \mathbb{N}$, \mathfrak{S}_n canonically acts on T^n , the n -fold direct product of T : $\sigma \in \mathfrak{S}_n$, $\mathbf{t} = (t_1, \dots, t_n) \in T^n$,

$$\sigma(\mathbf{t}) = (t_{\sigma^{-1}(1)}, \dots, t_{\sigma^{-1}(n)}).$$

This action causes semi-direct product $T^n \rtimes \mathfrak{S}_n$, which is denoted by $\mathfrak{S}_n(T)$ and called the wreath product of T with \mathfrak{S}_n . We have $\mathfrak{S}_1(T) = T$ from the definition, and set $\mathfrak{S}_0(T) = \{e\}$ for a notational convenience.

(B) *Standard decomposition into basic elements.* Any non-trivial element $g = (\mathbf{t}, \sigma)$ in $\mathfrak{S}_n(T)$ where $\mathbf{t} \in T^n$ and $\sigma \in \mathfrak{S}_n$ has the standard decomposition:

$$(1.1) \quad g = \xi_{q_1} \cdots \xi_{q_r} g_1 \cdots g_m$$

into two kinds of basic elements ξ_{q_i} and g_j as follows. For each $q \in \{1, 2, \dots, n\}$, ξ_q in (1.1) denotes an element in T^n with non-trivial $t_q \neq e_T$ only at the q th entry:

$$\xi_q = (e_T, \dots, e_T, t_q, e_T, \dots, e_T), \quad t_q \in T,$$

where $\{q\}$ is referred to as $\text{supp} \xi_q$ (= support of ξ_q). We use the notation as $\xi_q = (t_q, (q))$ for ξ_q above. Each g_j in (1.1) has the form (\mathbf{t}_j, σ_j) in which σ_j is a cycle permutation in \mathfrak{S}_n and \mathbf{t}_j holds possibly non-trivial element in T only at the positions of $\text{supp} \sigma_j$. Here $\text{supp} \sigma_j$ denotes the set of letters in $\{1, 2, \dots, n\}$ of which the cycle σ_j consists. Set $\text{supp} g_j = \text{supp} \sigma_j$. Moreover, all the supports

$$\text{supp} \xi_{q_1}, \dots, \text{supp} \xi_{q_r}, \text{supp} g_1, \dots, \text{supp} g_m$$

are taken to be disjoint in (1.1). Since $g = (\mathbf{t}, \sigma)$ is non-trivial in $\mathfrak{S}_n(T)$, the union of these supports is non-empty. It follows from (1.1) that $\sigma = \sigma_1 \cdots \sigma_m$ is a cycle decomposition in \mathfrak{S}_n . The standard decomposition of g in (1.1) is uniquely determined up to the orders of ξ_{q_i} 's and g_j 's.

(C) *The conjugacy classes of $\mathfrak{S}_n(T)$.* It is immediate that

$$(1.2) \quad \sigma(\mathbf{t}, (q))\sigma^{-1} = (\mathbf{t}, \sigma(q))$$

for $\sigma \in \mathfrak{S}_n$, $q \in \{1, 2, \dots, n\}$, $t \in T$. For basic element (t, σ) , $\sigma = (i_1 i_2 \cdots i_l)$, let t hold $t_{i_j} \in T$ at the position i_j , i.e.

$$t = \prod_{j=1}^l (t_{i_j}, (i_j)).$$

Using (1.2) also, we have

$$\begin{aligned} (t_{i_1}, (i_1))^{-1} t \sigma (t_{i_1}, (i_1)) &= (t_{i_1}^{-1}, (i_1)) t \sigma (t_{i_1}, (i_1)) \sigma^{-1} \sigma \\ &= \left(\prod_{j=2}^l (t_{i_j}, (i_j)) \right) (t_{i_1}, (i_2)) \sigma = (t_{i_2} t_{i_1}, (i_2)) \left(\prod_{j=3}^l (t_{i_j}, (i_j)) \right) \sigma. \end{aligned}$$

Repeating these conjugations, we see (t, σ) is conjugate to

$$(1.3) \quad ((t_{i_l} t_{i_{l-1}} \cdots t_{i_2} t_{i_1}, (i_l)), (i_1 i_2 \cdots i_{l-1} i_l)).$$

Moreover, in (1.3), conjugacy class $[t_{i_l} \cdots t_{i_2} t_{i_1}]$ of T is well-defined since it is independent of the cyclic order of t_{i_1}, \dots, t_{i_l} . A conjugacy class of T and a 1-row Young diagram (corresponding to the conjugacy class of l -cycles in \mathfrak{S}_n) are thus assigned to basic element (t, σ) . In the expression of (1.1), we can add product of

$$(e_T, (q)), \quad q \notin \bigsqcup_{i=1}^r \text{supp} \xi_{q_i} \sqcup \bigsqcup_{j=1}^m \text{supp} g_j$$

without affecting anything. Furthermore, this yields the expression (1.1) even for the identity element e in $\mathfrak{S}_n(T)$. We hence see that the conjugacy classes of $\mathfrak{S}_n(T)$ are parametrized by

$$(1.4) \quad \mathbb{Y}_n([T]) = \left\{ P = (\rho_\theta)_{\theta \in [T]} \mid \rho_\theta \in \mathbb{Y}, \sum_{\theta \in [T]} |\rho_\theta| = n \right\}.$$

Let C_P denote the conjugacy class of $\mathfrak{S}_n(T)$ labeled by $P \in \mathbb{Y}_n([T])$.

(D) *The equivalence classes of IURs of $\mathfrak{S}_n(T)$.* By virtue of a standard inducing-up method, the equivalence classes of IURs of $\mathfrak{S}_n(T)$ are parametrized by

$$(1.5) \quad \mathbb{Y}_n(\widehat{T}) = \left\{ \Lambda = (\lambda^\zeta)_{\zeta \in \widehat{T}} \mid \lambda^\zeta \in \mathbb{Y}, \sum_{\zeta \in \widehat{T}} |\lambda^\zeta| = n \right\}.$$

In fact, IUR π^Λ of $\mathfrak{S}_n(T)$ labeled by $\Lambda = (\lambda^\zeta)_{\zeta \in \widehat{T}} \in \mathbb{Y}_n(\widehat{T})$ is constructed as shown in [7, §3] (see also [9, §§1.1]). For the case where T is finite, we refer also to [11, Chapter 4]. To recall the construction, picking up a partition of $\{1, 2, \dots, n\}$ into $|\lambda^\zeta|$ -blocks:

$$\{1, 2, \dots, n\} = \bigsqcup_{\zeta \in \widehat{T}} I_{n, \zeta}, \quad |I_{n, \zeta}| = |\lambda^\zeta|,$$

we define IUR η of T^n by

$$\eta = \boxtimes_{\zeta \in \widehat{T}} (\boxtimes_{i \in I_{n, \zeta}} \pi^{\zeta_i}), \quad \pi^{\zeta_i} \in \zeta_i \in \widehat{T}, \quad \zeta_i \equiv \zeta \quad (i \in I_{n, \zeta}).$$

Under the action of $\sigma \in \mathfrak{S}_n$ on \widehat{T}^n by

$$\sigma \eta(t) = \eta(\sigma^{-1}(t)), \quad \sigma^{-1}(t) = (t_{\sigma(i)}) \quad (t = (t_i) \in T^n),$$

the stationary subgroup $S^\eta = \{\sigma \in \mathfrak{S}_n \mid \sigma\eta \cong \eta\}$ is isomorphic to $\prod_{\zeta \in \widehat{T}} \mathfrak{S}_{I_{n,\zeta}} \cong \prod_{\zeta \in \widehat{T}} \mathfrak{S}_{|\lambda^\zeta|}$. Set

$$H_n = T^n \rtimes S^\eta \cong \prod_{\zeta \in \widehat{T}} \mathfrak{S}_{|\lambda^\zeta|}(T).$$

Define IUR ρ^ζ of $\mathfrak{S}_{|\lambda^\zeta|}(T)$ by

$$\rho^\zeta(\mathbf{t}, \sigma) = (\pi^\zeta)^{\boxtimes |\lambda^\zeta|}(\mathbf{t})I(\sigma) \quad (\mathbf{t} \in T^{|\lambda^\zeta|}, \sigma \in \mathfrak{S}_{|\lambda^\zeta|})$$

where $I(\sigma)(\otimes_i v_i) = \otimes_i v_{\sigma^{-1}(i)}$ on $(V^\zeta)^{\otimes |\lambda^\zeta|}$, the representation space of $(\pi^\zeta)^{\boxtimes |\lambda^\zeta|}$. ρ^ζ 's give an IUR of H_n on the representation space of η , $V^\eta \cong \bigotimes_{\zeta \in \widehat{T}} (V^\zeta)^{\otimes |\lambda^\zeta|}$, as $\rho^\eta = \boxtimes_{\zeta \in \widehat{T}} \rho^\zeta$. On the other hand, we have $\xi^\eta = \boxtimes_{\zeta \in \widehat{T}} \pi^{\lambda^\zeta}$ as an IUR of S^η . ξ^η is regarded as an IUR of H_n by considering trivial actions of T^n . We see $\rho^\eta \otimes \xi^\eta$ is an IUR of H_n . Then, π^Λ is given by the induced representation

$$\pi^\Lambda = \text{Ind}_{H_n}^{\mathfrak{S}_n(T)} \rho^\eta \otimes \xi^\eta.$$

(E) *Branching rule for $\mathfrak{S}_n(T)$'s.* If $k < n$, canonical inclusion $\iota_{n,k} : \mathfrak{S}_k(T) \rightarrow \mathfrak{S}_n(T)$ is defined as $\iota_{n,k}(\mathbf{t}, \sigma) = (\tilde{\mathbf{t}}, \tilde{\sigma})$ where

$$\tilde{\mathbf{t}} = (\mathbf{t}, e_T, \dots, e_T) \in T^k \times T^{n-k} = T^n, \quad \tilde{\sigma} = \sigma(k+1)(k+2)\cdots(n) \in \mathfrak{S}_n.$$

Under the inclusion $\iota_{n,k}$, $\mathfrak{S}_k(T)$ is regarded as a subgroup of $\mathfrak{S}_n(T)$. If $\Lambda \in \mathbb{Y}_n(\widehat{T})$ is obtained by adding a box at an entry of $M \in \mathbb{Y}_{n-1}(\widehat{T})$, we say Λ is adjacent to M and write as $M \nearrow \Lambda$. In this situation, since the entry of M at which a box is put is uniquely determined, we denote it by $\zeta_{M,\Lambda} \in \widehat{T}$. For $\Lambda \in \mathbb{Y}_n(\widehat{T})$ and corresponding IUR π^Λ of $\mathfrak{S}_n(T)$, the restriction of π^Λ to subgroup $\mathfrak{S}_{n-1}(T)$, denoted by $\text{Res}_{\mathfrak{S}_{n-1}(T)}^{\mathfrak{S}_n(T)} \pi^\Lambda$, has the following irreducible decomposition:

$$(1.6) \quad \text{Res}_{\mathfrak{S}_{n-1}(T)}^{\mathfrak{S}_n(T)} \pi^\Lambda \cong \bigoplus_{M \in \mathbb{Y}_{n-1}(\widehat{T}) : M \nearrow \Lambda} [\dim \zeta_{M,\Lambda}] \pi^M.$$

(F) *Irreducible character formula for $\mathfrak{S}_n(T)$.* For $\Lambda \in \mathbb{Y}_n(\widehat{T})$, let

$$(1.7) \quad \chi^\Lambda(g) = \text{tr} \pi^\Lambda(g), \quad g \in \mathfrak{S}_n(T),$$

be the character value of IUR π^Λ at g . A formula for computing the value of (1.7) was given in [7, Theorem 4.5] (see also [9, §§1.2]). For the purpose of this paper, we need to know asymptotic behavior of irreducible characters of $\mathfrak{S}_n(T)$ along with a *fixed conjugacy class type and growing IURs*. Taking this situation into account, we recall the formula as follows. Let $g \in \mathfrak{S}_k(T)$ have the standard decomposition like (1.1) where ξ_q may be $(e_T, (q))$. If $n \geq k$, g considered as an element of $\mathfrak{S}_n(T)$ under the inclusion $\iota_{n,k} : \mathfrak{S}_k(T) \rightarrow \mathfrak{S}_n(T)$ has the same standard decomposition as (1.1). As shown in [7, Theorem 4.5], we have, for $\Lambda \in \mathbb{Y}_n(\widehat{T})$,

$$(1.8) \quad \chi^\Lambda(g) = \sum_{\mathcal{Q}, \mathcal{J}} \frac{(n - |\mathcal{Q}| - \sum_{j \in \mathcal{J}} |\sigma_j|)!}{\prod_{\zeta \in \widehat{T}} (|\lambda^\zeta| - |\mathcal{Q}_\zeta| - \sum_{j \in \mathcal{J}_\zeta} |\sigma_j|)!} \prod_{\zeta \in \widehat{T}} \left\{ (\dim \zeta)^{|\lambda^\zeta| - |\mathcal{Q}_\zeta| - \sum_{j \in \mathcal{J}_\zeta} |\sigma_j|} \left(\prod_{q \in \mathcal{Q}_\zeta} \chi^\zeta(t_q) \right) \left(\prod_{j \in \mathcal{J}_\zeta} \chi^\zeta(P_{\sigma_j}(t_j)) \right) \chi_{(|\sigma_j|)_{j \in \mathcal{J}_\zeta} \cup (1^{|\lambda^\zeta| - \sum_{j \in \mathcal{J}_\zeta} |\sigma_j|})}^{\lambda^\zeta} \right\},$$

where $g = (t, \sigma)$, $\sigma = \sigma_1 \cdots \sigma_m$, $g_j = (t_j, \sigma_j)$ ($j \in \{1, 2, \dots, m\}$), $|\sigma_j| = |\text{supp}\sigma_j|$,

$$P_{\sigma_j}(t_j) = t_{i_1} t_{i_2} \cdots t_{i_l}, \quad \sigma_j = (i_1 i_2 \cdots i_l), \quad t_j = (t_i)_{i \in \text{supp}\sigma_j},$$

$\chi^\zeta = \text{tr}\pi^\zeta$ (irreducible character) for $\zeta \in \widehat{T}$, $(|\sigma_j|)_{j \in J_\zeta}$ is a Young diagram of size $\sum_{j \in J_\zeta} |\sigma_j|$, and $\mathcal{Q} = (Q_\zeta)_{\zeta \in \widehat{T}}$ and $\mathcal{J} = (J_\zeta)_{\zeta \in \widehat{T}}$ are partitions of $Q = \{q_1, \dots, q_r\}$ and $J = \{1, 2, \dots, m\}$ respectively. We note that (1.8) is valid either if ξ_q might be $(e_T, (q))$ in the standard decomposition of $g \in \mathfrak{S}_k(T)$ or if not. In other words, we can take either of the following (i) or (ii) in the expression of the right hand side of (1.8):

(i) $Q = \{q_1, q_2, \dots, q_r\}$ where $t_{q_i} \neq e_T$ ($i = 1, 2, \dots, r$)

(ii) $Q = \{1, 2, \dots, k\} \setminus \bigsqcup_{j=1}^m \text{supp}\sigma_j$.

The verification reduces to the obvious identity: for $a, b \in \mathbb{N}$ and $a_i \in \mathbb{N} \cup \{0\}$ ($i = 1, \dots, p$) such that $b \leq a$ and $a_1 + \dots + a_p = a$,

$$\frac{a!}{a_1! \cdots a_p!} = \sum_{\Delta = (\Delta_i)_{i=1}^p: \text{partition of } \{1, \dots, b\}} \frac{(a-b)!}{(a_1 - |\Delta_1|)! \cdots (a_p - |\Delta_p|)!}$$

with the convention $1/(-n)! = 0$ if $a_i < |\Delta_i|$. (Imagine dividing an a -set consisting of b whites and $a-b$ blacks into a_1 -set, \dots , a_p -set.) The formula (1.8) can be expressed also by using the notation for a conjugacy class of $\mathfrak{S}_k(T)$ as (1.4). Let $g \in C_P \subset \mathfrak{S}_k(T)$ where $P = (\rho_\theta)_{\theta \in [T]}$. Picking up nontrivial ρ_θ 's (i.e. $\neq \emptyset$), we write as $P = (\rho_{\theta_i})_{i=1, \dots, l}$. Then the standard decomposition (1.1) of g has the form of

$$(1.9) \quad g = \xi_{q_1^{(\theta_1)}} \cdots \xi_{q_{r(\theta_1)}^{(\theta_1)}} g_1^{(\theta_1)} \cdots g_{m(\theta_1)}^{(\theta_1)} \cdots \cdots \xi_{q_1^{(\theta_l)}} \cdots \xi_{q_{r(\theta_l)}^{(\theta_l)}} g_1^{(\theta_l)} \cdots g_{m(\theta_l)}^{(\theta_l)},$$

where $r(\theta_i) = m_1(\rho_{\theta_i})$, $m(\theta_i) = m_2(\rho_{\theta_i}) + m_3(\rho_{\theta_i}) + \dots$, and

$$\xi_{q_h^{(\theta_i)}} = (t_{q_h^{(\theta_i)}}, (q_h^{(\theta_i)})), \quad g_j^{(\theta_i)} = (t_j^{(\theta_i)}, \sigma_j^{(\theta_i)}), \quad \theta_i = [t_{q_h^{(\theta_i)}}] = [P_{\sigma_j^{(\theta_i)}}(t_j^{(\theta_i)})]$$

for $h \in \{1, 2, \dots, r(\theta_i)\}$, $j \in \{1, 2, \dots, m(\theta_i)\}$. Note that

$$k = \sum_{\theta \in [T]} |\rho_\theta| = \sum_{i=1}^l \left(r(\theta_i) + \sum_{j=1}^{m(\theta_i)} |\sigma_j^{(\theta_i)}| \right).$$

Let $(\rho_{\theta_i})_j^*$ denote the j th row of ρ_{θ_i} . The length of $(\rho_{\theta_i})_j^*$ is $(\rho_{\theta_i})_j$. We distinguish $(\rho_{\theta_i})_j^*$ from $(\rho_{\theta_{i'}})_{j'}^*$ even if they have the same length (i.e. if $(\rho_{\theta_i})_j = (\rho_{\theta_{i'}})_{j'}$). Decomposing each ρ_{θ_i} into rows, consider the set of the rows of P:

$$\text{rows}(P) = \{(\rho_{\theta_1})_1^*, \dots, (\rho_{\theta_1})_{l(\rho_{\theta_1})}^*, \dots, (\rho_{\theta_l})_1^*, \dots, (\rho_{\theta_l})_{l(\rho_{\theta_l})}^*\}.$$

When (1.8) is applied to $g \in \mathfrak{S}_k(T)$ decomposed as (1.9), $\mathcal{Q} = (Q_\zeta)_{\zeta \in \widehat{T}}$ and $\mathcal{J} = (J_\zeta)_{\zeta \in \widehat{T}}$ are partitions of

$$\{q_1^{(\theta_1)}, \dots, q_{r(\theta_1)}^{(\theta_1)}, \dots, q_1^{(\theta_l)}, \dots, q_{r(\theta_l)}^{(\theta_l)}\} \quad \text{and} \\ \{g_1^{(\theta_1)}, \dots, g_{m(\theta_1)}^{(\theta_1)}, \dots, g_1^{(\theta_l)}, \dots, g_{m(\theta_l)}^{(\theta_l)}\}$$

respectively. It is immediate that there exists a bijective correspondence between

$$(1.10) \quad \text{partition } (\mathcal{Q}, \mathcal{J}) \quad \longleftrightarrow \quad \text{map } : \text{rows}(P) \longrightarrow \widehat{T}.$$

In other words, partition $(\mathcal{Q}, \mathcal{J})$ is equivalent to giving the rows of P a \widehat{T} -labeling. Since π^Λ (where $\Lambda = (\lambda^\zeta)_{\zeta \in \widehat{T}} \in \mathbb{Y}_n(\widehat{T})$) is given as an induced representation from

subgroup $H_n = T^n \rtimes \prod_{\zeta \in \widehat{T}} \mathfrak{S}_{|\lambda^\zeta|}$ of $\mathfrak{S}_n(T)$, $\chi^\Lambda(g)$ is non-zero only if g is conjugate to an element of H_n . This condition for g is rephrased in terms of conjugacy class C_P containing g as: there exists a map $r : \text{rows}(P) \rightarrow \widehat{T}$ such that

$$(1.11) \quad \sum_{(\rho_{\theta_i})_j^* : r((\rho_{\theta_i})_j^*) = \zeta} (\rho_{\theta_i})_j \leq |\lambda^\zeta|, \quad \zeta \in \widehat{T}$$

(namely, sum of the length of the rows labeled by ζ does not exceed $|\lambda^\zeta|$ for any ζ). In the case of $n = k$, (1.11) is equivalent to the condition with $=$ replacing \leq . The Young diagram consisting of the rows in $r^{-1}(\zeta)$ is denoted by the same notation $r^{-1}(\zeta)$. Similarly, the Young diagram consisting of the rows both in $r^{-1}(\zeta)$ and ρ_θ is denoted by $r^{-1}(\zeta) \cap \rho_\theta$. Under the correspondence of (1.10), we have

$$(|\sigma_j|)_{j \in J_\zeta} \sqcup (1^{|\mathcal{Q}_\zeta|}) = r^{-1}(\zeta), \quad \zeta \in \widehat{T}.$$

Furthermore, we have, for $\zeta \in \widehat{T}$,

$$\left(\prod_{q \in \mathcal{Q}_\zeta} \chi^\zeta(t_q) \right) \left(\prod_{j \in J_\zeta} \chi^\zeta(P_{\sigma_j}(t_j)) \right) = \prod_{\theta \in [T]} (\chi_\theta^\zeta)^{l(r^{-1}(\zeta) \cap \rho_\theta)},$$

$$\chi_{(|\sigma_j|)_{j \in J_\zeta} \sqcup (1^{|\mathcal{Q}_\zeta| - \sum_{j \in J_\zeta} |\sigma_j|})}^{\lambda^\zeta} = \chi_{r^{-1}(\zeta) \sqcup (1^{|\lambda^\zeta| - |r^{-1}(\zeta)|})}^{\lambda^\zeta}.$$

Here a character is denoted by the notation χ with the upper and lower indexes expressing the labels of a UR and a conjugacy class respectively. Namely, χ_θ^ζ is the value at θ of the irreducible character $\chi^\zeta = \text{tr} \pi^\zeta$ of T . In (1.8), we can assume $|Q| + \sum_{j \in J} |\sigma_j| = k$ by putting elements of the form $\xi_q = (e_T, (q))$ in the standard decomposition of $g \in \mathfrak{S}_k(T)$ if necessary (see the notice following (1.8)). For $P = (\rho_\theta)_{\theta \in [T]} \in \mathbb{Y}_k([T])$, let us glue $n - k$ 1-box rows to $\rho_{\{e_T\}}$, the diagram at $\{e_T\}$ -entry of P , and let $\iota_{n,k}(P)$ denote the resulting element of $\mathbb{Y}_n([T])$, i.e. $\iota_{n,k}(P) = (\tilde{\rho}_\theta)_{\theta \in [T]}$ where

$$\tilde{\rho}_\theta = \begin{cases} \rho_\theta, & \theta \neq \{e_T\}, \\ \rho_{\{e_T\}} \sqcup (1^{n-k}), & \theta = \{e_T\}. \end{cases}$$

Then, we have $C_{\iota_{n,k}(P)} \cap \mathfrak{S}_k(T) = C_P$ under the inclusion $\mathfrak{S}_k(T) \subset \mathfrak{S}_n(T)$ by $\iota_{n,k}$. After these preparations, we get an alternative expression of (1.8) as

$$(1.12) \quad \chi_{\iota_{n,k}(P)}^\Lambda = \sum_{r \text{ satisfying (1.11)}} \frac{(n-k)!}{\prod_{\zeta \in \widehat{T}} (|\lambda^\zeta| - |r^{-1}(\zeta)|)!}$$

$$\times \prod_{\zeta \in \widehat{T}} \left\{ (\dim \zeta)^{|\lambda^\zeta| - |r^{-1}(\zeta)|} \left(\prod_{\theta \in [T]} (\chi_\theta^\zeta)^{l(r^{-1}(\zeta) \cap \rho_\theta)} \right) \chi_{r^{-1}(\zeta) \sqcup (1^{|\lambda^\zeta| - |r^{-1}(\zeta)|})}^{\lambda^\zeta} \right\}$$

for $P \in \mathbb{Y}_k([T])$ and $\Lambda \in \mathbb{Y}_n(\widehat{T})$ with $k \leq n$. If $k = n$, (1.12) has a simpler expression since r satisfying (1.11) (necessarily $=$ instead of \leq) yields $|\lambda^\zeta| = |r^{-1}(\zeta)|$ for any $\zeta \in \widehat{T}$.

(G) *Infinite wreath product.* \mathfrak{S}_∞ denotes the infinite symmetric group, the set of all finite permutations of \mathbb{N} . \mathfrak{S}_∞ acts on the restricted direct product of T :

$$(1.13) \quad D_\infty(T) = \{ \mathbf{t} = (t_1, t_2, \dots) \in T^\infty \mid t_j = e_T \text{ except finite } j\text{'s} \}$$

canonically by

$$(1.14) \quad \sigma(\mathbf{t}) = (t_{\sigma^{-1}(1)}, t_{\sigma^{-1}(2)}, \dots), \quad \sigma \in \mathfrak{S}_\infty, \mathbf{t} \in D_\infty(T).$$

The semi-direct product $D_\infty(T) \rtimes \mathfrak{S}_\infty$ caused by this action is called the wreath product of T with \mathfrak{S}_∞ and denoted by $\mathfrak{S}_\infty(T)$. Canonical inclusion $\iota_n : \mathfrak{S}_n(T) \rightarrow \mathfrak{S}_\infty(T)$ satisfies $\iota_n \circ \iota_{n,k} = \iota_k$ for $k < n$. Under this inclusion map, we regard $\mathfrak{S}_\infty(T)$ as $\bigcup_{n=0}^\infty \mathfrak{S}_n(T)$. Since the standard decomposition (1.1) is stable for n 's large enough, we get the standard decomposition into basic elements for each element in $\mathfrak{S}_\infty(T)$. The conjugacy classes of $\mathfrak{S}_\infty(T)$ are then parametrized by

$$(1.15) \quad \mathbb{Y}([T]) = \{P = (\rho_\theta)_{\theta \in [T]} \mid \rho_\theta \in \mathbb{Y}, m_1(\rho_{\{e_T\}}) = 0\}.$$

In (1.15), the trivial cycles are omitted; e.g. the trivial conjugacy class $\{e\}$ of $\mathfrak{S}_\infty(T)$ corresponds to $P = (\rho_\theta)$ with $\rho_\theta = \emptyset$ for any $\theta \in [T]$ (instead of $\rho_{\{e_T\}} = (1^\infty)$). Note that \emptyset satisfies $m_1(\emptyset) = 0$.

(H) *Branching graph.* Considering $\mathbb{Y}_n(\widehat{T})$ in (1.5), set

$$(1.16) \quad \mathbb{Y}(\widehat{T}) = \bigsqcup_{n=0}^\infty \mathbb{Y}_n(\widehat{T}).$$

$\mathbb{Y}_0(\widehat{T})$ is by definition singleton set $\{\emptyset\}$. Note that $\mathbb{Y}_1(\widehat{T}) = \widehat{T}$. Among the vertices $\mathbb{Y}(\widehat{T})$, let us define edge structure by taking the branching rule (1.6) into account. Namely, $\Lambda \in \mathbb{Y}_n(\widehat{T})$ and $M \in \mathbb{Y}_{n+1}(\widehat{T})$ are joined by an edge if and only if $\Lambda \nearrow M$. Moreover, we put multiplicity

$$(1.17) \quad \kappa(\Lambda, M) = \dim \zeta_{\Lambda, M}$$

on the edge $\Lambda \nearrow M$.

Example 1.1. Let T be \mathfrak{S}_3 , the symmetric group of degree 3. $\widehat{\mathfrak{S}}_3$ consists of $\zeta_1 (= 1)$, $\zeta_2 (= \text{sgn})$ and ζ_3 where $\dim \zeta_3 = 2$. According to this order, an element of $\mathbb{Y}(\widehat{\mathfrak{S}}_3)$ is expressed as $(\lambda^1, \lambda^2, \lambda^3)$ with $\lambda^j \in \mathbb{Y}$. The very beginning of the branching graph $\mathbb{Y}(\widehat{\mathfrak{S}}_3)$ is as drawn in Fig. 1.

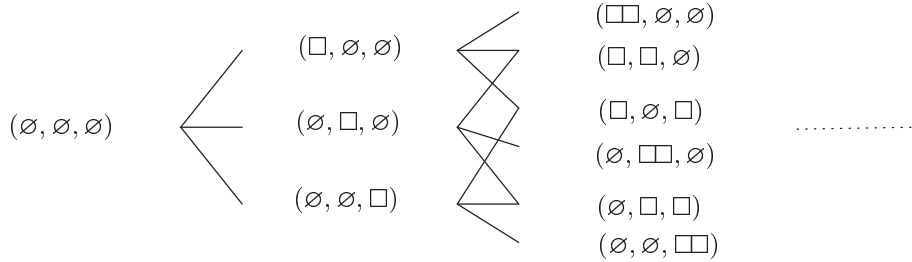


FIGURE 1. Branching graph $\mathbb{Y}(\widehat{\mathfrak{S}}_3)$

(I) *Harmonic function.* Let us mention the definition of (0.2). A \mathbb{C} -valued function φ on $\mathbb{Y}(\widehat{T})$ is said to be harmonic if

$$(1.18) \quad \varphi(\Lambda) = \sum_{M: \Lambda \nearrow M} \kappa(\Lambda, M) \varphi(M), \quad \Lambda \in \mathbb{Y}(\widehat{T})$$

holds. φ is said to be normalized if $\varphi(\emptyset) = 1$ holds. Since (1.18) reflects directly the branching rule (1.6) or equivalently

$$(1.19) \quad \text{Ind}_{\mathfrak{S}_h(T)}^{\mathfrak{S}_{h+1}(T)} \pi^\Lambda \cong \bigoplus_{M: \Lambda \nearrow M} [\dim \zeta_{\Lambda, M}] \pi^M,$$

defining harmonicity through (1.18) is canonical from the viewpoint of representation theory. On the other hand, (1.18) seems to be non-canonical from the viewpoint of probability theory or (discrete) potential theory since harmonicity is usually defined by using a transition probability as

$$(1.20) \quad \varphi(\Lambda) = \sum_M p(\Lambda, M) \varphi(M)$$

where $p(\Lambda, M) \geq 0$ and $\sum_M p(\Lambda, M) = 1$. For example, a constant function is not harmonic according to (1.18). We review basic definitions and necessary properties of Martin boundary in Appendix A. As noted there, Martin boundary theories are equivalent, based on either (1.18) or (1.20) by virtue of what is called ‘ h -transform’ method. Let us proceed with (1.18).

(*J*) *Dimension function.* On the branching graph $\mathbb{Y}(\widehat{T})$, a path u joining $\Lambda \in \mathbb{Y}_l(\widehat{T})$ to $M \in \mathbb{Y}_m(\widehat{T})$, where $l < m$, is expressed as

$$u = (u(l) \nearrow u(l+1) \nearrow \cdots \nearrow u(m-1) \nearrow u(m)), \quad u(l) = \Lambda, \quad u(m) = M.$$

We set

$$(1.21) \quad w_u = \prod_{i=l}^{m-1} \kappa(u(i), u(i+1))$$

and call w_u the weight of path u . Set also

$$(1.22) \quad d(\Lambda, M) = \sum_{\text{path } u: \Lambda \nearrow \cdots \nearrow M} w_u,$$

which means the number of weighted paths from Λ to M . In particular, set

$$(1.23) \quad d(\Lambda) = d(\emptyset, \Lambda) = \dim \pi^\Lambda, \quad \Lambda \in \mathbb{Y}(\widehat{T}).$$

The second equality of (1.23) follows from iterating (1.6) and then using (1.22). The functions d in (1.22) and (1.23) are called dimension functions on $\mathbb{Y}(\widehat{T})$. Although dimension functions are defined on a general branching through weights of paths as (1.22), we see that, in the case of branching graph $\mathbb{Y}(\widehat{T})$ for wreath product groups, weight w_u in (1.21) depends only on initial $\Lambda = (\lambda^\zeta)$ and terminal $M = (\mu^\zeta)$, and is expressed as

$$(1.24) \quad w_u = \prod_{\zeta \in \widehat{T}} (\dim \zeta)^{|\mu^\zeta| - |\lambda^\zeta|},$$

which is directly seen from the branching rule (1.6). Hence (1.22) is reduced to the case of a simple (i.e. multiplicity free) branching graph. In [2], Boyer pointed out this phenomenon and applied it to develop character theory of the infinite wreath product of a finite group. According to the cardinality $|T|$ of compact group T , set

$$(1.25) \quad T^\natural = \begin{cases} \mathbb{T} \text{ (torus } \cong S^1), & |T| = \infty, \\ \mathbb{Z}/p\mathbb{Z}, & |T| = p < \infty. \end{cases}$$

d^{\natural} denotes the dimension function on the branching graph for $\mathfrak{S}_n(T^{\natural})$'s. Putting (1.24) into (1.22), we have

$$(1.26) \quad d(\Lambda, M) = \prod_{\zeta \in \widehat{T}} (\dim \zeta)^{|\mu^{\zeta}| - |\lambda^{\zeta}|} \sum_{\text{path } u: \Lambda \nearrow \dots \nearrow M} 1 = \prod_{\zeta \in \widehat{T}} (\dim \zeta)^{|\mu^{\zeta}| - |\lambda^{\zeta}|} d^{\natural}(\Lambda, M).$$

(K) *Martin kernel.* We set

$$(1.27) \quad K(\Lambda, M) = \frac{d(\Lambda, M)}{d(M)}, \quad \Lambda, M \in \mathbb{Y}(\widehat{T})$$

under the convention that $d(\Lambda, M) = 0$ if there are no paths from Λ to M on $\mathbb{Y}(\widehat{T})$. (1.27) agrees with a common terminology in Markov chain theory in which a Martin kernel is defined as a ratio of potential kernels (see Appendix A). K^{\natural} denoting the Martin kernel in accordance with (1.25), (1.26) yields

$$(1.28) \quad K(\Lambda, M) = \prod_{\zeta \in \widehat{T}} \frac{1}{(\dim \zeta)^{|\lambda^{\zeta}|}} K^{\natural}(\Lambda, M).$$

We introduce a distance on $\mathbb{Y}(\widehat{T})$ by

$$(1.29) \quad D(\Lambda, M) = \sum_{N \in \mathbb{Y}(\widehat{T})} C_N (|K(N, \Lambda) - K(N, M)| + |\delta_{N, \Lambda} - \delta_{N, M}|), \quad \Lambda, M \in \mathbb{Y}(\widehat{T}),$$

where C_N is an appropriately chosen positive coefficient. Then, as noted in Appendix A, the following (i) and (ii) are equivalent for sequence

$$\{M^{(n)} = (\mu^{(n)\zeta})_{\zeta \in \widehat{T}}\}_{n \in \mathbb{N}}$$

in $\mathbb{Y}(\widehat{T})$:

- (i) $\{M^{(n)}\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to D .
- (ii) Either there exists $M \in \mathbb{Y}(\widehat{T})$ such that

$$(1.30) \quad M^{(n)} = M \quad \text{for sufficiently large } n,$$

or else

$$(1.31) \quad |M^{(n)}| = \sum_{\zeta \in \widehat{T}} |\mu^{(n)\zeta}| \xrightarrow{n \rightarrow \infty} \infty \quad \text{and}$$

$$(1.32) \quad \{K(\Lambda, M^{(n)})\}_{n \in \mathbb{N}} \text{ is a Cauchy sequence in } \mathbb{R} \text{ for any } \Lambda \in \mathbb{Y}(\widehat{T}).$$

We obtain Martin compactification $\overline{\mathbb{Y}(\widehat{T})}$ by taking the completion of $\mathbb{Y}(\widehat{T})$ with respect to distance D . As seen from (1.30) and (1.31), $\mathbb{Y}(\widehat{T})$ is an open subset of $\overline{\mathbb{Y}(\widehat{T})}$, on which the relative topology is discrete.

(L) *Martin boundary of $\mathbb{Y}(\widehat{T})$.* The compact set $\overline{\mathbb{Y}(\widehat{T})} \setminus \mathbb{Y}(\widehat{T})$ is called the Martin boundary of $\mathbb{Y}(\widehat{T})$ and denoted by $\partial \mathbb{Y}(\widehat{T})$ in this paper. Martin kernel K in (1.27) is extended as a continuous function on $\mathbb{Y}(\widehat{T}) \times \overline{\mathbb{Y}(\widehat{T})}$:

$$(1.33) \quad K(\Lambda, \omega) = \lim_{n \rightarrow \infty} K(\Lambda, M^{(n)}), \quad (\Lambda, \omega) \in \mathbb{Y}(\widehat{T}) \times \overline{\mathbb{Y}(\widehat{T})},$$

along with $M^{(n)} \in \mathbb{Y}(\widehat{T}) \xrightarrow[n \rightarrow \infty]{} \omega \in \overline{\mathbb{Y}(\widehat{T})}$, which is again called a Martin kernel.

We have

$$(1.34) \quad K(\emptyset, \omega) = 1, \quad K(\Lambda, \omega) \geq 0.$$

Consider the minimal Martin boundary of $\mathbb{Y}(\widehat{T})$

(1.35)

$$\partial_m \mathbb{Y}(\widehat{T}) = \{\omega \in \partial \mathbb{Y}(\widehat{T}) \mid K(\cdot, \omega) \text{ is a minimal harmonic function on } \mathbb{Y}(\widehat{T})\}.$$

Note that Martin kernel $K(\cdot, \omega)$ is not necessarily harmonic on $\mathbb{Y}(\widehat{T})$ for $\omega \in \partial \mathbb{Y}(\widehat{T})$ as we actually see later in the case of continuous T . Here nonnegative harmonic function φ is said to be minimal if harmonic ψ such that $0 \leq \psi \leq \varphi$ is necessarily a constant multiple of φ . Under the definition of (0.2), $\varphi \in \mathcal{H}(\mathbb{Y}(\widehat{T}))$ is extremal if and only if it is minimal. In fact, extremality of φ implies

$$\begin{aligned} 0 \leq \psi \leq \varphi &\implies \varphi = \psi(\emptyset) \cdot \frac{\psi}{\psi(\emptyset)} + (1 - \psi(\emptyset)) \frac{\varphi - \psi}{1 - \psi(\emptyset)} \\ &\implies \varphi = \frac{\psi}{\psi(\emptyset)} \iff \psi = \psi(\emptyset)\varphi. \end{aligned}$$

Conversely, minimality of φ implies

$$\begin{aligned} \varphi &= \alpha\psi_1 + (1 - \alpha)\psi_2, \quad 0 \leq \alpha \leq 1, \quad \psi_1, \psi_2 \in \mathcal{H}(\mathbb{Y}(\widehat{T})) \\ &\implies \alpha\psi_1 = \text{constant} \cdot \varphi, \quad (1 - \alpha)\psi_2 = \text{constant} \cdot \varphi \implies \psi_1 = \psi_2 = \varphi. \end{aligned}$$

Comparing (0.2) and (1.35), we hence have

$$\{K(\cdot, \omega) \mid \omega \in \partial_m \mathbb{Y}(\widehat{T})\} \subset \text{the extremal points of } \mathcal{H}(\mathbb{Y}(\widehat{T})).$$

Actually, we shall see that the equality holds (Theorem 3.3).

2. COMPUTATION OF THE MARTIN BOUNDARY OF $\mathbb{Y}(\widehat{T})$

In order to compute the Martin boundary of $\mathbb{Y}(\widehat{T})$, we need a formula for the Martin kernel $K(\Lambda, M)$ from which its asymptotic behavior can be read out efficiently. Let us set some further notations on characters of \mathfrak{S}_n . The normalized irreducible character of \mathfrak{S}_n corresponding to $\lambda \in \mathbb{Y}_n$ is denoted by $\tilde{\chi}^\lambda = \chi^\lambda / \dim \lambda$. For $\rho, \lambda \in \mathbb{Y}$, set

$$(2.1) \quad \Sigma_\rho(\lambda) = \begin{cases} |\lambda|^{\downarrow|\rho|} \tilde{\chi}_{(\rho, 1^{|\lambda|-|\rho|})}^\lambda, & |\rho| \leq |\lambda|, \\ 0 & |\rho| > |\lambda| \end{cases}$$

where $n^{\downarrow k} = n(n-1)\cdots(n-k+1)$ is a descending power for $n, k \in \mathbb{N}$, $k \leq n$. We use a conventional notation

$$(2.2) \quad z_\rho = \prod_{i=1}^{\infty} i^{m_i(\rho)} m_i(\rho)!, \quad \rho = (1^{m_1(\rho)} 2^{m_2(\rho)} \dots) \in \mathbb{Y}.$$

Theorem 2.1. *Let $k, n \in \mathbb{N}$ and $k \leq n$. For $\Lambda = (\lambda^\zeta)_{\zeta \in \widehat{T}} \in \mathbb{Y}_k(\widehat{T})$ and $M = (\mu^\zeta)_{\zeta \in \widehat{T}} \in \mathbb{Y}_n(\widehat{T})$, the Martin kernel is expressed as*

$$(2.3) \quad K(\Lambda, M) = \frac{n^k}{n^{\downarrow k}} \prod_{\zeta \in \widehat{T}} \left\{ \frac{1}{(\dim \zeta)^{|\lambda^\zeta|}} \sum_{\rho \in \mathbb{Y}_{|\lambda^\zeta|}} \frac{1}{z_\rho} \chi_\rho^{\lambda^\zeta} \frac{1}{n^{|\lambda^\zeta|}} \Sigma_\rho(\mu^\zeta) \right\}.$$

Proof. Note that the right hand side of (2.3) is actually a finite product since $\lambda^\zeta = \emptyset$ holds except finite number of ζ 's. If no paths connect Λ to M , there exists $\zeta \in \widehat{T}$ such that $|\lambda^\zeta| > |\mu^\zeta|$. Then, $\Sigma_\rho(\mu^\zeta) = 0$ holds for any $\rho \in \mathbb{Y}_{|\lambda^\zeta|}$ by (2.1), and hence both sides of (2.3) are 0. Assuming that $|\lambda^\zeta| \leq |\mu^\zeta|$ holds for any $\zeta \in \widehat{T}$, we deduce (2.3) by combining the character formula for $\mathfrak{S}_k(T)$ and the Fourier inversion on $\mathfrak{S}_k(T)$. Recall (1.28) with (1.25). Then, it suffices to show (2.3) for the case of T^{\natural} :

$$K^{\natural}(\Lambda, M) = \frac{n^k}{n^{\vee k}} \prod_{\zeta \in \widehat{T}^{\natural}} \left(\sum_{\rho \in \mathbb{Y}_{|\lambda^\zeta|}} \frac{1}{z_\rho} \chi_\rho^{\lambda^\zeta} \frac{1}{n^{|\lambda^\zeta|}} \Sigma_\rho(\mu^\zeta) \right).$$

In this proof below, we write T, d, k instead of $T^{\natural}, d^{\natural}, K^{\natural}$ for simplicity. For basic element $(\mathbf{t}, \sigma) = \mathbf{t}\sigma$, where $\sigma = (i_1 i_2 \cdots i_l)$, and $\zeta \in \widehat{T} (= \widehat{T}^{\natural})$, we simply write

$$\zeta(\mathbf{t}) = \zeta(t_{i_1} \cdots t_{i_l}).$$

Since $T (= T^{\natural})$ is commutative, the order of product of t_{i_j} 's are not involved.

[Step 1] Using

$$\frac{1}{d(M)} \chi^M |_{\mathfrak{S}_k(T)} = \sum_{N \in \mathbb{Y}_k(\widehat{T})} \frac{d(N, M)}{d(M)} \chi^N$$

and complete orthonormality of the irreducible characters with respect to the normalized Haar measure dg on $\mathfrak{S}_k(T)$, we have

$$(2.4) \quad \begin{aligned} \frac{d(\Lambda, M)}{d(M)} &= \int_{\mathfrak{S}_k(T)} \frac{1}{d(M)} \chi^M(g) \overline{\chi^\Lambda(g)} dg \\ &= \sum_{\sigma \in \mathfrak{S}_k} \frac{1}{k!} \int_{T^k} \frac{1}{d(M)} \chi^M(\mathbf{t}\sigma) \overline{\chi^\Lambda(\mathbf{t}\sigma)} d\mathbf{t} \end{aligned}$$

where $\mathbf{t} = (t_1, \dots, t_k)$, $d\mathbf{t} = dt_1 \cdots dt_k$, and $d\mathbf{t}$ is the normalized Haar measure on T . Let $\sigma \in \mathfrak{S}_k$ be fixed (until the end of Step 3) to have cycle decomposition $\sigma = \sigma_1 \cdots \sigma_l$. Decompose $\mathbf{t} \in T^k$ as $\mathbf{t} = \mathbf{t}_0 \mathbf{t}_1 \cdots \mathbf{t}_l$ so that $\text{supp } t_j = \text{supp } \sigma_j$ ($j = 1, \dots, l$) and $\text{supp } \mathbf{t}_0 = \{1, \dots, k\} \setminus \bigsqcup_{j=1}^l \text{supp } t_j$. If $\sigma = e_{\mathfrak{S}_k}$, we simply set $\mathbf{t}_0 = \mathbf{t}$. Letting $\mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_l$ vary independently, we compute the integral over T^k for each fixed $\sigma \in \mathfrak{S}_k$ in (2.4).

[Step 2] Applying (1.8) to $\Lambda = (\lambda^\zeta)_{\zeta \in \widehat{T}} \in \mathbb{Y}_k(\widehat{T})$ and $g = \mathbf{t}\sigma$, we have

$$(2.5) \quad \chi^\Lambda(\mathbf{t}\sigma) = \sum_{\mathcal{Q}, \mathcal{J}} \prod_{\zeta \in \widehat{T}} \left\{ \left(\prod_{q \in Q_\zeta} \zeta(t_q) \right) \left(\prod_{j \in J_\zeta} \zeta(t_j) \right) \chi_{(|\sigma_j|)_{j \in J_\zeta} \sqcup (1^{|\mathcal{Q}_\zeta|})}^{\lambda^\zeta} \right\},$$

where $(\mathcal{Q}, \mathcal{J})$ satisfies

$$(2.6) \quad |\lambda^\zeta| = |\mathcal{Q}_\zeta| + \sum_{j \in J_\zeta} |\sigma_j|, \quad \zeta \in \widehat{T}.$$

Note that partitions \mathcal{Q} of Q and \mathcal{J} of J vary independently of \mathbf{t} in (2.5). This assures that integration in \mathbf{t} over T^k and summation in \mathcal{Q}, \mathcal{J} commute. Applying

(1.8) (with the notice following there) again to $M = (\mu^\zeta)_{\zeta \in \widehat{T}} \in \mathbb{Y}_n(\widehat{T})$, we have

$$(2.7) \quad \chi^M|_{\mathfrak{S}_k(T)}(t\sigma) = \sum_{\mathcal{Q}, \mathcal{J}} \frac{(n-k)!}{\prod_{\zeta \in \widehat{T}} (|\mu^\zeta| - |Q_\zeta| - \sum_{j \in J_\zeta} |\sigma_j|)!} \prod_{\zeta \in \widehat{T}} \left\{ \left(\prod_{q \in Q_\zeta} \zeta(t_q) \right) \left(\prod_{j \in J_\zeta} \zeta(t_j) \right) \chi_{(\sigma_j)_{j \in J_\zeta} \sqcup (1^{|\mu^\zeta| - \sum_{j \in J_\zeta} |\sigma_j|})}^{\mu^\zeta} \right\},$$

where $(\mathcal{Q}, \mathcal{J})$ satisfies

$$(2.8) \quad |\mu^\zeta| \geq |Q_\zeta| + \sum_{j \in J_\zeta} |\sigma_j|, \quad \zeta \in \widehat{T}.$$

$(\mathcal{Q}, \mathcal{J})$'s in (2.5) and (2.7) are partitions of the same object determined by σ with different constraints of (2.6) and (2.8) respectively. (2.6) is stronger since $|\lambda^\zeta| \leq |\mu^\zeta|$ for any $\zeta \in \widehat{T}$. Combining (2.5) with (2.7) and taking integration over T^k , we write

$$(2.9) \quad \int_{T^k} \chi^M(t\sigma) \overline{\chi^\Lambda(t\sigma)} dt \\ = \sum_{\mathcal{Q}, \mathcal{J}} \sum_{\mathcal{Q}', \mathcal{J}'} \frac{(n-k)!}{\prod_{\zeta \in \widehat{T}} (|\mu^\zeta| - |Q_\zeta| - \sum_{j \in J_\zeta} |\sigma_j|)!} \prod_{\zeta \in \widehat{T}} \chi_{(\sigma_j)_{j \in J_\zeta} \sqcup (1^{|\mu^\zeta| - \sum_{j \in J_\zeta} |\sigma_j|})}^{\mu^\zeta} \chi_{(\sigma'_j)_{j \in J'_\zeta} \sqcup (1^{|\mu^\zeta| - \sum_{j \in J'_\zeta} |\sigma'_j|})}^{\lambda^\zeta} \\ \int_{T^k} \prod_{\zeta \in \widehat{T}} \left\{ \left(\prod_{q \in Q_\zeta} \zeta(t_q) \right) \left(\prod_{q \in Q'_\zeta} \overline{\zeta(t_q)} \right) \left(\prod_{j \in J_\zeta} \zeta(t_j) \right) \left(\prod_{j \in J'_\zeta} \overline{\zeta(t_j)} \right) \right\} dt.$$

[Step 3] We verify that the integral in the right hand side of (2.9), temporally labeled by (*) here, is equal to $\delta_{(\mathcal{Q}, \mathcal{J}), (\mathcal{Q}', \mathcal{J}')}$. Recall that we are considering a partition induced by σ . \mathcal{Q} and \mathcal{Q}' give \widehat{T} -labeling to the singleton blocks while \mathcal{J} and \mathcal{J}' give \widehat{T} -labeling to the other blocks. If $\mathcal{Q} \neq \mathcal{Q}'$, there exists a block $\{q\}$ given distinct labels, say ζ_1 and ζ_2 , by \mathcal{Q} and \mathcal{Q}' respectively. When we compute (*) by using Fubini's theorem, we find

$$\int_T \zeta_1(t_q) \overline{\zeta_2(t_q)} dt_q, \quad \zeta_1 \neq \zeta_2$$

inside the integral, which vanishes by orthogonality of (irreducible) characters of T . If $\mathcal{J} \neq \mathcal{J}'$, there exists a block $\{i_1, i_2, \dots, i_p\}$ with distinct \widehat{T} -labeling by \mathcal{J} and \mathcal{J}' . Letting $\sigma_j = (i_1 i_2 \dots i_p)$ and $dt_j = dt_{i_1} \dots dt_{i_p}$, we have

$$\int_{T^p} \zeta_1(t_j) \overline{\zeta_2(t_j)} dt_j, \quad \zeta_1 \neq \zeta_2$$

inside the integral, which vanishes also. If $(\mathcal{Q}, \mathcal{J}) = (\mathcal{Q}', \mathcal{J}')$, we get

$$(*) = \prod_{\zeta \in \widehat{T}} \left\{ \prod_{q \in Q_\zeta} \left(\int_T \zeta(t_q) \overline{\zeta(t_q)} dt_q \right) \prod_{j \in J_\zeta} \left(\int_{T^{|\sigma_j|}} \zeta(t_j) \overline{\zeta(t_j)} dt_j \right) \right\} = 1.$$

Moreover, (2.6) with $(\mathcal{Q}', \mathcal{J}')$ yields

$$|\lambda^\zeta| = |Q'_\zeta| + \sum_{j \in J'_\zeta} |\sigma_j| = |Q_\zeta| + \sum_{j \in J_\zeta} |\sigma_j|.$$

Hence (2.9) now implies

$$(2.10) \quad \int_{T^k} \chi^M(t\sigma) \overline{\chi^\Lambda(t\sigma)} dt = \sum_{\mathcal{Q}, \mathcal{J}} \frac{(n-k)!}{\prod_{\zeta \in \widehat{T}} (|\mu^\zeta| - |\lambda^\zeta|)!} \prod_{\zeta \in \widehat{T}} \chi_{(|\sigma_j|)_{j \in \mathcal{J}_\zeta} \sqcup (1^{|\mathcal{Q}_\zeta|}) \sqcup (1^{|\mu^\zeta| - |\lambda^\zeta|})}^{\mu^\zeta} \chi_{(|\sigma_j|)_{j \in \mathcal{J}_\zeta} \sqcup (1^{|\mathcal{Q}_\zeta|})}^{\lambda^\zeta}.$$

[Step 4] Combine (2.4) and (2.10) with

$$d(M) = \dim \pi^M = \frac{n!}{\prod_{\zeta \in \widehat{T}} |\mu^\zeta|!} \prod_{\zeta \in \widehat{T}} \dim \mu^\zeta, \quad M \in \mathbb{Y}_n(\widehat{T}).$$

We write $\mathcal{Q}^\sigma, \mathcal{J}^\sigma$ for partitions in (2.10) since they depend on σ . Then, we see

$$(2.11) \quad K(\Lambda, M) = \frac{1}{k! n^{\downarrow k}} \sum_{\sigma \in \mathfrak{S}_k} \sum_{\mathcal{Q}^\sigma, \mathcal{J}^\sigma} \prod_{\zeta \in \widehat{T}} \left\{ \frac{|\mu^\zeta|!}{(|\mu^\zeta| - |\lambda^\zeta|)!} \frac{1}{\dim \mu^\zeta} \chi_{(|\sigma_j|)_{j \in \mathcal{J}^\sigma} \sqcup (1^{|\mathcal{Q}_\zeta^\sigma|}) \sqcup (1^{|\mu^\zeta| - |\lambda^\zeta|})}^{\mu^\zeta} \chi_{(|\sigma_j|)_{j \in \mathcal{J}^\sigma} \sqcup (1^{|\mathcal{Q}_\zeta^\sigma|})}^{\lambda^\zeta} \right\}.$$

Note that $\sum_{\mathcal{Q}^\sigma, \mathcal{J}^\sigma} \cdots$ in (2.11) actually depends only on the conjugacy class of σ . We hence continue (2.11) as

$$(2.12) \quad K(\Lambda, M) = \frac{1}{k! n^{\downarrow k}} \sum_{\rho \in \mathbb{Y}_k} \frac{k!}{z_\rho} \sum_{\mathcal{Q}^\sigma, \mathcal{J}^\sigma} \prod_{\zeta \in \widehat{T}} \left\{ \frac{|\mu^\zeta|!}{(|\mu^\zeta| - |\lambda^\zeta|)!} \frac{1}{\dim \mu^\zeta} \chi_{(|\sigma_j|)_{j \in \mathcal{J}^\sigma} \sqcup (1^{|\mathcal{Q}_\zeta^\sigma|}) \sqcup (1^{|\mu^\zeta| - |\lambda^\zeta|})}^{\mu^\zeta} \chi_{(|\sigma_j|)_{j \in \mathcal{J}^\sigma} \sqcup (1^{|\mathcal{Q}_\zeta^\sigma|})}^{\lambda^\zeta} \right\}$$

where, in the inner sum for each $\rho \in \mathbb{Y}_k$, σ is a fixed representative in the conjugacy class $C_\rho \subset \mathfrak{S}_k$. Recall (2.2) for z_ρ .

[Step 5] Let $\{\lambda^{\zeta_1}, \dots, \lambda^{\zeta_p}\}$ be the set of nontrivial (i.e. $\neq \emptyset$) entries of Λ . We have $\sum_{i=1}^p |\lambda^{\zeta_i}| = k$. Considering the cycles (including trivial ones) of representative σ in conjugacy class C_ρ labeled by $\zeta_1, \dots, \zeta_p \in \widehat{T}$ (where the cycles are all distinguished), we associate to each ζ_i Young diagram

$$(2.13) \quad \rho^{(i)} \in \mathbb{Y}_{|\lambda^{\zeta_i}|} \quad \text{such that} \quad \rho^{(1)} \sqcup \cdots \sqcup \rho^{(p)} = \rho.$$

Namely, labeling which destroys (2.13) is not allowed because of (2.6). Set $\phi : \mathbb{Y}_{|\lambda^{\zeta_1}|} \times \cdots \times \mathbb{Y}_{|\lambda^{\zeta_p}|} \rightarrow \mathbb{Y}_k$ by

$$\phi(\rho^{(1)}, \dots, \rho^{(p)}) = \rho^{(1)} \sqcup \cdots \sqcup \rho^{(p)}.$$

It suffices to collect ρ 's belonging to the range of ϕ in the sum $\sum_{\rho \in \mathbb{Y}_k}$ in (2.12). The inner sum $\sum_{\mathcal{Q}^\sigma, \mathcal{J}^\sigma}$ in (2.12) is divided into partial sums according to the points of fiber $\phi^{-1}(\rho)$, where it is obvious that different points in fiber $\phi^{-1}(\rho)$ never admit a common partition $(\mathcal{Q}^\sigma, \mathcal{J}^\sigma)$. Rearranging the right hand side of (2.12), we have

$$(2.14) \quad \sum_{\rho \in \mathbb{Y}_k} \frac{1}{z_\rho} \sum_{\mathcal{Q}^\sigma, \mathcal{J}^\sigma} \cdots = \sum_{(\rho^{(1)}, \dots, \rho^{(p)}) \in \mathbb{Y}_{|\lambda^{\zeta_1}|} \times \cdots \times \mathbb{Y}_{|\lambda^{\zeta_p}|}} \sum_{(\mathcal{Q}, \mathcal{J}) : (**)} (**)$$

where the range (**) over which partition $(\mathcal{Q}, \mathcal{J})$ runs depends on $(\rho^{(1)}, \dots, \rho^{(p)})$. On the other hand, (***) does not depend on $(\mathcal{Q}, \mathcal{J})$ in (**) once $(\rho^{(1)}, \dots, \rho^{(p)})$ is chosen, which is equal to

$$(2.15) \quad \frac{1}{\prod_{j=1}^{\infty} j^{m_j(\rho^{(1)}) + \dots + m_j(\rho^{(p)})} (m_j(\rho^{(1)}) + \dots + m_j(\rho^{(p)}))!} \cdot \prod_{i=1}^p \left\{ \frac{|\mu^{\zeta_i}|!}{(|\mu^{\zeta_i}| - |\lambda^{\zeta_i}|)!} \frac{1}{\dim \mu^{\zeta_i} \chi_{\rho^{(i)} \sqcup (1|\mu^{\zeta_i} - |\lambda^{\zeta_i}|)} \chi_{\rho^{(i)}}^{\lambda^{\zeta_i}}} \right\}.$$

The cardinality of the range (**) for a given $(\rho^{(1)}, \dots, \rho^{(p)})$ is

$$(2.16) \quad \prod_{j=1}^{\infty} \frac{(m_j(\rho^{(1)}) + \dots + m_j(\rho^{(p)}))!}{m_j(\rho^{(1)})! \dots m_j(\rho^{(p)})!}$$

since the cycles of length j admit labeling by ζ_1, \dots, ζ_p in the j th factor ways in (2.16).

[Step 6] Combining (2.12) with (2.14) – (2.16), we have

$$(2.17) \quad K(\Lambda, M) = \frac{1}{n^{\downarrow k}} \sum_{(\rho^{(1)}, \dots, \rho^{(p)}) \in \mathbb{Y}_{|\lambda^{\zeta_1}|} \times \dots \times \mathbb{Y}_{|\lambda^{\zeta_p}|}} \frac{1}{z_{\rho^{(1)}} \dots z_{\rho^{(p)}}} \prod_{i=1}^p \left\{ \frac{|\mu^{\zeta_i}|!}{(|\mu^{\zeta_i}| - |\lambda^{\zeta_i}|)!} \frac{1}{\dim \mu^{\zeta_i} \chi_{\rho^{(i)} \sqcup (1|\mu^{\zeta_i} - |\lambda^{\zeta_i}|)} \chi_{\rho^{(i)}}^{\lambda^{\zeta_i}}} \right\},$$

then, by using the notation of (2.1) and noting $k = \sum_{i=1}^p |\lambda^{\zeta_i}|$, we continue (2.17) as

$$\begin{aligned} &= \frac{n^k}{n^{\downarrow k}} \sum_{(\rho^{(1)}, \dots, \rho^{(p)}) \in \mathbb{Y}_{|\lambda^{\zeta_1}|} \times \dots \times \mathbb{Y}_{|\lambda^{\zeta_p}|}} \frac{1}{z_{\rho^{(1)}} \dots z_{\rho^{(p)}}} \prod_{i=1}^p \left\{ \frac{\Sigma_{\rho^{(i)}}(\mu^{\zeta_i})}{n^{|\lambda^{\zeta_i}|}} \chi_{\rho^{(i)}}^{\lambda^{\zeta_i}} \right\} \\ &= \frac{n^k}{n^{\downarrow k}} \prod_{i=1}^p \left(\sum_{\rho \in \mathbb{Y}_{|\lambda^{\zeta_i}|}} \frac{1}{z_{\rho}} \frac{\Sigma_{\rho}(\mu^{\zeta_i})}{n^{|\lambda^{\zeta_i}|}} \chi_{\rho}^{\lambda^{\zeta_i}} \right) = \frac{n^k}{n^{\downarrow k}} \prod_{\zeta \in \widehat{\mathcal{T}}} \left(\sum_{\rho \in \mathbb{Y}_{|\lambda^{\zeta}|}} \frac{1}{z_{\rho}} \chi_{\rho}^{\lambda^{\zeta}} \frac{\Sigma_{\rho}(\mu^{\zeta})}{n^{|\lambda^{\zeta}|}} \right). \end{aligned}$$

This completes the proof of (2.3). \square

We investigate asymptotic behavior of (2.3) with $k = |\Lambda| = \sum_{\zeta \in \widehat{\mathcal{T}}} |\lambda^{\zeta}|$ fixed as $n = |M| = \sum_{\zeta \in \widehat{\mathcal{T}}} |\mu^{\zeta}|$ tends to ∞ . A suitable framework for analysis of the function $n^{-|\lambda^{\zeta}|} \Sigma_{\rho}(\mu^{\zeta})$ is given by the Kerov–Olshanski algebra \mathbb{A} consisting of polynomial functions of Young diagrams, which we briefly review now.

We set the Frobenius coordinates of $\mu = (\mu_1 \geq \mu_2 \geq \dots) \in \mathbb{Y}$ by

$$(2.18) \quad a_i(\mu) = \mu_i - i + \frac{1}{2}, \quad b_i(\mu) = \mu'_i - i + \frac{1}{2}, \quad i = 1, \dots, d(\mu)$$

where $d = d(\mu)$ is the diagonal length (number of boxes) of μ , and write as $\mu = (a_i(\mu) | b_i(\mu))_{i=1, \dots, d}$. For $k \in \mathbb{N}$, (super-symmetric) power sum in μ is defined by

$$(2.19) \quad p_k = p_k(\mu) = \sum_{i=1}^d a_i(\mu)^k + (-1)^{k-1} b_i(\mu)^k, \quad \mu \in \mathbb{Y},$$

and by

$$(2.20) \quad p_{\rho} = p_{\rho}(\mu) = p_{\rho_1}(\mu) p_{\rho_2}(\mu) \dots p_{\rho_l}(\mu), \quad \mu \in \mathbb{Y}$$

for arbitrary $\rho \in \mathbb{Y}$ with $l = l(\rho)$. Power sums $\{p_k\}_{k \in \mathbb{N}}$, regarded as functions in μ , are algebraically independent over \mathbb{R} . The algebra generated by $\{p_k\}_{k \in \mathbb{N}}$ is called the Kerov–Olshanski algebra and denoted by \mathbb{A} . Declaring p_k to be homogeneous of degree k , written by $\deg p_k = k$, we equip \mathbb{A} with the canonical filtration. The canonical degree of any element $f \in \mathbb{A}$ is also denoted by $\deg f$. For example, $\deg p_\rho = \rho_1 + \cdots + \rho_l = |\rho|$ for $\rho \in \mathbb{Y}$. Σ_ρ in (2.1) also belongs to \mathbb{A} . It is related to a power sum as follows.

Proposition 2.2. *For any $\rho \in \mathbb{Y}$, there exists $R_\rho \in \mathbb{A}$ such that*

$$(2.21) \quad \Sigma_\rho = p_\rho + R_\rho, \quad \deg R_\rho < |\rho|$$

hold in \mathbb{A} .

We refer to [10, §3–§4] for the proof of Proposition 2.2.

(2.21) combined with (2.19) and (2.20) suggests that convergence of $K(\Lambda, M)$ in (2.3) as $n = |M| \rightarrow \infty$ is characterized by asymptotics of rescaled Frobenius coordinates

$$\frac{a_i(\mu^\zeta)}{n}, \quad \frac{b_i(\mu^\zeta)}{n}, \quad \zeta \in \widehat{T}, \quad i \in \mathbb{N}.$$

We are thus in a position to get concrete expression of the Martin boundary and the Martin kernels (1.33) of the branching graph $\mathbb{Y}(\widehat{T})$. Set

$$(2.22) \quad \widetilde{\Delta} = \left\{ (\alpha, \beta, c) \mid \begin{aligned} &\alpha = (\alpha_{\zeta, i})_{\zeta \in \widehat{T}, i \in \mathbb{N}}, \quad \beta = (\beta_{\zeta, i})_{\zeta \in \widehat{T}, i \in \mathbb{N}}, \quad c = (c_\zeta)_{\zeta \in \widehat{T}}; \\ &\alpha_{\zeta, 1} \geq \alpha_{\zeta, 2} \geq \cdots \geq 0, \quad \beta_{\zeta, 1} \geq \beta_{\zeta, 2} \geq \cdots \geq 0, \quad c_\zeta \geq 0, \\ &\sum_{i=1}^{\infty} (\alpha_{\zeta, i} + \beta_{\zeta, i}) \leq c_\zeta \text{ for } \forall \zeta \in \widehat{T}; \quad \sum_{\zeta \in \widehat{T}} c_\zeta \leq 1 \end{aligned} \right\}$$

and

$$(2.23) \quad \Delta = \left\{ (\alpha, \beta, c) \in \widetilde{\Delta} \mid \sum_{\zeta \in \widehat{T}} c_\zeta = 1 \right\}.$$

We have

$$(2.24) \quad \Delta \not\subseteq \widetilde{\Delta} \subset B(\ell^1(\widehat{T} \times \mathbb{N}))^2 \times B(\ell^1(\widehat{T}))$$

where $B(\cdot)$ denotes the closed unit ball with center 0 in an ℓ^1 -space. The closed unit balls in the rightmost set of (2.24) are compact with respect to the weak* topologies $\sigma(\ell^1, c_0)$, where c_0 consists of the sequences converging to 0 at infinity. In this case, the product of the weak* topologies is equivalent to the topology of pointwise converging on $(\widehat{T} \times \mathbb{N}) \sqcup (\widehat{T} \times \mathbb{N}) \sqcup \widehat{T}$. Since $\widetilde{\Delta}$ is closed with respect to this pointwise converging topology as directly seen from the definition (2.22), $\widetilde{\Delta}$ is a compact set. On the other hand, we should note that Δ of (2.23) is not closed in $\widetilde{\Delta}$ with respect to the given topology if \widehat{T} is an infinite set. As (super-symmetric) power sums and Schur functions on $\widetilde{\Delta}$, let us set

$$(2.25) \quad p_k^\zeta(\alpha, \beta, c) = \begin{cases} \sum_{i=1}^{\infty} (\alpha_{\zeta, i}^k + (-1)^{k-1} \beta_{\zeta, i}^k), & k \geq 2, \quad k \in \mathbb{N}, \\ c_\zeta, & k = 1, \end{cases}$$

$$(2.26) \quad p_\rho^\zeta(\alpha, \beta, c) = p_{\rho_1}^\zeta(\alpha, \beta, c) \cdots p_{\rho_l}^\zeta(\alpha, \beta, c), \quad \rho = (\rho_1 \geq \cdots \geq \rho_l) \in \mathbb{Y}$$

with $p_{\emptyset}^{\zeta}(\alpha, \beta, c) = 1$, and

$$(2.27) \quad s_{\lambda}^{\zeta}(\alpha, \beta, c) = \sum_{\rho \in \mathbb{Y}_{|\lambda|}} \frac{1}{z_{\rho}} \chi_{\rho}^{\lambda} p_{\rho}^{\zeta}(\alpha, \beta, c), \quad \lambda \in \mathbb{Y}$$

for $\zeta \in \widehat{T}$ and $(\alpha, \beta, c) \in \widetilde{\Delta}$.

The definition of (2.27) obviously comes from the well-known relation (or Frobenius character formula) between Schur functions and power sums. Note that

$$p_1^{\zeta}(\alpha, \beta, c) \neq \sum_{i=1}^{\infty} (\alpha_{\zeta, i} + \beta_{\zeta, i})$$

in general.

Lemma 2.3. (I) *Let T be finite.*

(1) $p_k^{\zeta} \in C(\Delta; \mathbb{R}) (= \{\mathbb{R}\text{-valued continuous functions on } \Delta\})$. Hence $p_{\rho}^{\zeta}, s_{\lambda}^{\zeta} \in C(\Delta; \mathbb{R})$ also.

(2) $\{p_k^{\zeta} \mid \zeta \in \widehat{T}, k \in \mathbb{N}\}$ separates arbitrary two points in Δ .

(II) *Let T be infinite. (1) and (2) above hold with $\widehat{\Delta}$ replacing Δ .*

Proof. (I) (1) Continuity of p_k^{ζ} with respect to the pointwise converging topology on Δ is easy to see, which is similar to the well-known argument on the Thoma simplex (i.e. Δ for $T = \{e\}$). See also Lemma 4.6 in [9].

(2) This separation property is also similar to the case of Thoma simplex. Let $(\alpha, \beta, c), (\alpha', \beta', c') \in \Delta$ satisfy

$$p_k^{\zeta}(\alpha, \beta, c) = p_k^{\zeta}(\alpha', \beta', c'), \quad \zeta \in \widehat{T}, k \in \mathbb{N}.$$

Setting $k = 1$, we have $(c_{\zeta})_{\zeta \in \widehat{T}} = (c'_{\zeta})_{\zeta \in \widehat{T}}$. For each $\zeta \in \widehat{T}$, $\{p_k^{\zeta}(\alpha, \beta, c) \mid k \in \mathbb{N}\}$ completely determines $(\alpha_{\zeta, i})_{i \in \mathbb{N}}$ and $(\beta_{\zeta, i})_{i \in \mathbb{N}}$ through the equality

$$\begin{aligned} \exp\left\{\sum_{k=2}^{\infty} \frac{p_k^{\zeta}(\alpha, \beta, c)}{k} z^k\right\} &= \exp\left\{\sum_{k=2}^{\infty} \sum_{i=1}^{\infty} (\alpha_{\zeta, i}^k + (-1)^{k-1} \beta_{\zeta, i}^k) \frac{z^k}{k}\right\} \\ &= \exp\left\{-\sum_{i=1}^{\infty} (\alpha_{\zeta, i} + \beta_{\zeta, i}) z\right\} \prod_{i=1}^{\infty} \frac{1 + \beta_{\zeta, i} z}{1 - \alpha_{\zeta, i} z}. \end{aligned}$$

Proof in the case of (II) is quite similar. \square

Lemma 2.4. *Let X be a compact set and $\{h_{\alpha}\}_{\alpha \in A} \subset C(X; \mathbb{R})$ separate arbitrary two points in X . Then, the topology on X determined by pointwise converging on A through $x(\alpha) = h_{\alpha}(x)$ coincides with the original one on X .*

Proof. The assertion is immediate from compactness of X , continuity of h_{α} and the fact that the pointwise converging topology is Hausdorff. \square

Theorem 2.5. *A homeomorphic characterization of the Martin boundary $\partial\mathbb{Y}(\widehat{T})$ of the branching graph $\mathbb{Y}(\widehat{T})$ is given by the following:*

- (i) if T is finite, $\partial\mathbb{Y}(\widehat{T}) \cong \Delta$,
- (ii) if T is infinite, $\partial\mathbb{Y}(\widehat{T}) \cong \widetilde{\Delta}$.

The Martin kernel is expressed by using Schur functions (2.27) as

$$(2.28) \quad K(\Lambda, \omega) = \prod_{\zeta \in \widehat{T}} \frac{1}{(\dim \zeta)^{|\lambda^{\zeta}|}} s_{\lambda^{\zeta}}^{\zeta}(\omega), \quad \Lambda = (\lambda^{\zeta})_{\zeta \in \widehat{T}} \in \mathbb{Y}(\widehat{T}), \quad \omega = (\alpha, \beta, c)$$

where $\omega \in \Delta$ for finite T and $\omega \in \tilde{\Delta}$ for infinite T .

Proof. The right hand side of (2.28) is actually a finite product by the definitions of (2.25) – (2.27).

[Step 1] A point in the Martin boundary $\partial\mathbb{Y}(\widehat{T})$ is given by an equivalence class of Cauchy sequences $\{\mathbb{M}^{(n)}\}_{n \in \mathbb{N}}$ satisfying (1.31) and (1.32). $k \in \mathbb{N}$ and $\Lambda \in \mathbb{Y}_k(\widehat{T})$ being fixed, let $K(\Lambda, \mathbb{M}^{(n)})$ converge with $m = |\mathbb{M}^{(n)}| \rightarrow \infty$ as $n \rightarrow \infty$ where $\mathbb{M}^{(n)} = (\mu^{(n)\zeta})_{\zeta \in \widehat{T}} \in \mathbb{Y}_m(\widehat{T})$. For simplicity of notations, however, we omit superscript $^{(n)}$ of $\mathbb{M}^{(n)}$, $\mu^{(n)\zeta}$ and consider the limit with $m = |\mathbb{M}| = \sum_{\zeta \in \widehat{T}} |\mu^\zeta| \rightarrow \infty$. Theorem 2.1 yields that

$$(2.29) \quad \prod_{\zeta \in \widehat{T}} \left\{ \sum_{\rho \in \mathbb{Y}_{|\lambda^\zeta|}} \frac{1}{z_\rho} \chi_\rho^{\lambda^\zeta} \frac{1}{m^{|\lambda^\zeta|}} \Sigma_\rho(\mu^\zeta) \right\}$$

converges with $m \rightarrow \infty$. We verify that the expression obtained by replacing Σ_ρ by p_ρ in (2.29) also converges as $m \rightarrow \infty$. Take $R_\rho \in \mathbb{A}$ satisfying (2.21) for each $\rho \in \mathbb{Y}$ and consider

$$(2.30) \quad \left| \prod_{\zeta \in \widehat{T}} \left\{ \sum_{\rho \in \mathbb{Y}_{|\lambda^\zeta|}} \frac{1}{z_\rho} \chi_\rho^{\lambda^\zeta} \frac{1}{m^{|\lambda^\zeta|}} (p_\rho(\mu^\zeta) + R_\rho(\mu^\zeta)) \right\} - \prod_{\zeta \in \widehat{T}} \left\{ \sum_{\rho \in \mathbb{Y}_{|\lambda^\zeta|}} \frac{1}{z_\rho} \chi_\rho^{\lambda^\zeta} \frac{1}{m^{|\lambda^\zeta|}} p_\rho(\mu^\zeta) \right\} \right|.$$

We have $n^{-|\lambda^\zeta|} |p_\rho(\mu^\zeta)| \leq 1$ for $\rho \in \mathbb{Y}_{|\lambda^\zeta|}$. For given $k \in \mathbb{N}$, the range of $|\lambda^\zeta|$ and $\rho \in \mathbb{Y}_{|\lambda^\zeta|}$ is a finite set. R_ρ is a linear combination of p_σ 's with $|\sigma| < |\rho| = |\lambda^\zeta|$. Hence there exists $c_k > 0$, independent of $\zeta \in \widehat{T}$, such that

$$\begin{aligned} \left| \sum_{\rho \in \mathbb{Y}_{|\lambda^\zeta|}} \frac{1}{z_\rho} \chi_\rho^{\lambda^\zeta} \frac{1}{m^{|\lambda^\zeta|}} p_\rho(\mu^\zeta) \right| &\leq \dim \lambda^\zeta \sum_{\rho \in \mathbb{Y}_{|\lambda^\zeta|}} \frac{1}{z_\rho} \leq c_k, \\ \left| \sum_{\rho \in \mathbb{Y}_{|\lambda^\zeta|}} \frac{1}{z_\rho} \chi_\rho^{\lambda^\zeta} \frac{1}{m^{|\lambda^\zeta|}} R_\rho(\mu^\zeta) \right| &\leq \frac{\dim \lambda^\zeta}{m} \sum_{\rho \in \mathbb{Y}_{|\lambda^\zeta|}} \frac{1}{z_\rho m^{|\lambda^\zeta|-1}} |R_\rho(\mu^\zeta)| \leq \frac{c_k}{m}. \end{aligned}$$

Taking constant $C_k > 0$ depending only on k , we have

$$(2.31) \quad (2.30) \leq \frac{1}{m} C_k.$$

This together with (2.29) implies

$$(2.32) \quad \prod_{\zeta \in \widehat{T}} \left\{ \sum_{\rho \in \mathbb{Y}_{|\lambda^\zeta|}} \frac{1}{z_\rho} \chi_\rho^{\lambda^\zeta} \frac{1}{m^{|\lambda^\zeta|}} p_\rho(\mu^\zeta) \right\}$$

converges with $m = |\mathbb{M}| \rightarrow \infty$.

[Step 2] Since (2.32) converges with $m \rightarrow \infty$ for any $k \in \mathbb{N}$ and any $\Lambda = (\lambda^\zeta)_{\zeta \in \widehat{T}} \in \mathbb{Y}_k(\widehat{T})$, we have, equivalently, the convergence of

$$\begin{aligned} \frac{1}{m} p_1(\mu^\zeta) &= \frac{|\mu^\zeta|}{m}, \\ \frac{1}{m^k} p_k(\mu^\zeta) &= \sum_{i=1}^{\infty} \left\{ \left(\frac{a_i(\mu^\zeta)}{m} \right)^k + (-1)^{k-1} \left(\frac{b_i(\mu^\zeta)}{m} \right)^k \right\}, \quad k \geq 2 \end{aligned}$$

with $m = |M| \rightarrow \infty$. Or, in terms of p_k^ζ in (2.25),

$$p_k^\zeta \left(\left(\frac{a_i(\mu^\eta)}{m} \right)_{\eta,i}, \left(\frac{b_i(\mu^\eta)}{m} \right)_{\eta,i}, \left(\frac{|\mu^\eta|}{m} \right)_\eta \right)$$

converges for any $\zeta \in \widehat{T}$ and any $k \in \mathbb{N}$. Lemma 2.3 and Lemma 2.4 then ensure that

$$(2.33) \quad \left(\left(\frac{a_i(\mu^\zeta)}{m} \right)_{\zeta,i}, \left(\frac{b_i(\mu^\zeta)}{m} \right)_{\zeta,i}, \left(\frac{|\mu^\zeta|}{m} \right)_\zeta \right)$$

converges in $\widetilde{\Delta}$ [resp. Δ] for infinite [resp. finite] T with $m = |M| \rightarrow \infty$. Set its limit as $((\alpha_{\zeta,i}), (\beta_{\zeta,i}), (c_\zeta)) \in \widetilde{\Delta}$ [resp. Δ]. We thus obtain a map from $\partial\mathbb{Y}(\widehat{T})$ into $\widetilde{\Delta}$ [resp. Δ] which is homeomorphic again by Lemma 2.3 and Lemma 2.4.

[Step 3] We show that the map $\partial\mathbb{Y}(\widehat{T}) \rightarrow \widetilde{\Delta}$ [resp. Δ] defined in Step 2 is surjective. By virtue of Lemma 2.3 and Lemma 2.4, it suffices to verify that, given $(\alpha, \beta, c) \in \widetilde{\Delta}$ [resp. Δ] where $\alpha = (\alpha_{\zeta,i}), \beta = (\beta_{\zeta,i}), c = (c_\zeta)$, there exists a sequence $(M = (\mu^\zeta)_{\zeta \in \widehat{T}}), M \in \mathbb{Y}_n(\widehat{T})$, satisfying

$$(2.34) \quad \lim_{n \rightarrow \infty} \left(\left(\frac{a_i(\mu^\zeta)}{n} \right)_{\zeta,i}, \left(\frac{b_i(\mu^\zeta)}{n} \right)_{\zeta,i}, \left(\frac{|\mu^\zeta|}{n} \right)_\zeta \right) = (\alpha, \beta, c).$$

For simplicity of expressions, we omit to describe precise estimates of the effects caused by taking integer parts. First, according to c_ζ , assign nc_ζ ($\lfloor nc_\zeta \rfloor$, precisely denoted) boxes to each $\zeta \in \widehat{T}$ to construct Young diagram μ^ζ . If T is finite, it is easy to handle the non-integer part $nc_\zeta - \lfloor nc_\zeta \rfloor$. If T is infinite, there may remain boxes of order n because of $\sum_{\zeta \in \widehat{T}} c_\zeta \leq 1$. To deal with the remaining boxes, assign \sqrt{n} order number of boxes to \sqrt{n} order number of ζ 's. Then, we have $|\mu^\zeta|/n \rightarrow c_\zeta$ as $n \rightarrow \infty$. For determining $a_i(\mu^\zeta)$ and $b_i(\mu^\zeta)$ to construct μ^ζ , we assign $n\alpha_{\zeta,i}$ and $n\beta_{\zeta,i}$ respectively to $(\zeta, i) \in \widehat{T} \times \mathbb{N}$ by using $\lfloor nc_\zeta \rfloor$ boxes assigned to $\zeta \in \widehat{T}$ above. This can be done since $\sum_{i=1}^\infty (\alpha_{\zeta,i} + \beta_{\zeta,i}) \leq c_\zeta$ holds. Possible remaining boxes at $\zeta \in \widehat{T}$ is at most of order n . Again, we consume the remainders by assigning \sqrt{n} order boxes to \sqrt{n} order i 's. The construction immediately yields the convergence of (2.34) in $\widetilde{\Delta}$ [resp. Δ]. This completes the proof of $\partial\mathbb{Y}(\widehat{T}) \cong \Delta$ for finite T and $\partial\mathbb{Y}(\widehat{T}) \cong \widetilde{\Delta}$ for infinite T .

[Step 4] We deduce (2.28) from (2.3). Since we established a homeomorphic characterization of $\partial\mathbb{Y}(\widehat{T})$, it suffices to show that, for $\Lambda = (\lambda^\zeta)_{\zeta \in \widehat{T}} \in \mathbb{Y}_k(\widehat{T})$ and $\omega = (\alpha, \beta, c) \in \Delta$ for finite T and $\in \widetilde{\Delta}$ for infinite T , (2.32) converges to $\prod_{\zeta \in \widehat{T}} s_{\lambda^\zeta}^\zeta(\alpha, \beta, c)$ under the convergence of (2.34) as $m = |M| = \sum_{\zeta \in \widehat{T}} |\mu^\zeta| \rightarrow \infty$. Recall

$$\begin{aligned} \frac{1}{m} p_1(\mu^\zeta) &= \frac{|\mu^\zeta|}{m}, \\ \frac{1}{m^k} p_k(\mu^\zeta) &= \sum_{i=1}^\infty \left\{ \left(\frac{a_i(\mu^\zeta)}{m} \right)^k + (-1)^{k-1} \left(\frac{b_i(\mu^\zeta)}{m} \right)^k \right\}, \quad k \geq 2. \end{aligned}$$

As the argument in Lemma 2.3 (1), (2.34) yields

$$\frac{1}{m^k} p_k(\mu^\zeta) \rightarrow p_k^\zeta(\alpha, \beta, c), \quad k \in \mathbb{N}, \quad \zeta \in \widehat{T}.$$

Since the ζ 's with $|\lambda^\zeta| > 0$ are finite, the desired convergence then follows from the definition of $s_{\lambda^\zeta}^\zeta$ in (2.27). \square

Remark 2.6. If T is a trivial group $\{e_T\}$, Δ of (2.23) reduces to the Thoma simplex, which is homeomorphic to the Martin boundary of the usual Young graph. Theorem 2.5 under restriction of $T = \{e\}$ is of course consistent to the well-known result for the Young graph.

We note the following superharmonicity of $K(\cdot, \omega)$ and characterization of its harmonicity for later use (in the proof of Theorem 3.1).

Lemma 2.7. *The Martin kernel $K(\Lambda, \omega)$ is superharmonic in Λ :*

$$(2.35) \quad K(\Lambda, \omega) \geq \sum_{N: \Lambda \nearrow N} \kappa(\Lambda, N) K(N, \omega), \quad \Lambda \in \mathbb{Y}(\widehat{T}), \quad \omega \in \partial \mathbb{Y}(\widehat{T}).$$

Proof. We have

$$K(\Lambda, \omega) = \lim_{m \rightarrow \infty} K(\Lambda, M), \quad \Lambda \in \mathbb{Y}(\widehat{T})$$

for an appropriate sequence such that $M \rightarrow \omega$ with $m = |M| \rightarrow \infty$. Then,

$$\begin{aligned} \sum_{N: \Lambda \nearrow N} \kappa(\Lambda, N) K(N, \omega) &= \sum_{N: \Lambda \nearrow N} \kappa(\Lambda, N) \lim_{m \rightarrow \infty} K(N, M) \\ &\leq \liminf_{m \rightarrow \infty} \sum_{N: \Lambda \nearrow N} \kappa(\Lambda, N) K(N, M) = \liminf_{m \rightarrow \infty} K(\Lambda, M) = K(\Lambda, \omega). \end{aligned}$$

□

Lemma 2.8. *The Martin kernel $K(\cdot, \omega)$ is harmonic if and only if $\omega \in \Delta$.*

Proof. Definitions (2.25) – (2.27) imply

$$s_{\emptyset}^{\zeta}(\omega) = p_{\emptyset}^{\zeta}(\omega) = 1, \quad s_{(1)}^{\zeta}(\omega) = p_1^{\zeta}(\omega) = c_{\zeta}, \quad \omega = (\alpha, \beta, c), \quad c = (c_{\zeta})_{\zeta \in \widehat{T}}$$

for $\zeta \in \widehat{T}$. Moreover, in a similar way to the Pieri's formula for Schur functions, we have

$$(2.36) \quad s_{(1)}^{\zeta}(\omega) s_{\lambda}^{\zeta}(\omega) = \sum_{\mu \in \mathbb{Y}: \lambda \nearrow \mu} s_{\mu}^{\zeta}(\omega), \quad \lambda \in \mathbb{Y}$$

for $\zeta \in \widehat{T}$. First, setting $\Lambda = \emptyset$ in (2.28), we have $K(\emptyset, \omega) = 1$ and

$$\sum_{N \in \mathbb{Y}(\widehat{T}): \emptyset \nearrow N} \kappa(\emptyset, N) K(N, \omega) = \sum_{\zeta \in \widehat{T}} s_{(1)}^{\zeta}(\omega) = \sum_{\zeta \in \widehat{T}} c_{\zeta}.$$

Hence harmonicity of $K(\cdot, \omega)$ yields $\omega \in \Delta$ in particular. More generally, setting $\text{supp}\Lambda = \{\zeta \in \widehat{T} \mid \lambda^\zeta \neq \emptyset\}$ for $\Lambda = (\lambda^\zeta)_{\zeta \in \widehat{T}} \in \mathbb{Y}(\widehat{T})$, we see from (2.28) and (2.36)

$$\begin{aligned}
& \sum_{N: \Lambda \nearrow N} \kappa(\Lambda, N) K(N, \omega) \\
&= \sum_{\zeta \notin \text{supp}\Lambda} \dim \zeta \frac{s_{(1)}^\zeta(\omega)}{\dim \zeta} K(\Lambda, \omega) \\
&\quad + \sum_{\zeta \in \text{supp}\Lambda} \dim \zeta \left(\prod_{\eta \in \text{supp}\Lambda \setminus \{\zeta\}} \frac{s_{\lambda^\eta}^\eta(\omega)}{(\dim \eta)^{|\lambda^\eta|}} \right) \left(\sum_{\mu: \lambda^\zeta \nearrow \mu} \frac{s_\mu^\zeta(\omega)}{(\dim \zeta)^{|\mu|}} \right) \\
&= \sum_{\zeta \notin \text{supp}\Lambda} c_\zeta K(\Lambda, \omega) + \sum_{\zeta \in \text{supp}\Lambda} \frac{1}{(\dim \zeta)^{|\lambda^\zeta|}} \left(\prod_{\eta \in \text{supp}\Lambda \setminus \{\zeta\}} \frac{s_{\lambda^\eta}^\eta(\omega)}{(\dim \eta)^{|\lambda^\eta|}} \right) \sum_{\mu: \lambda^\zeta \nearrow \mu} s_\mu^\zeta(\omega) \\
&= \sum_{\zeta \notin \text{supp}\Lambda} c_\zeta K(\Lambda, \omega) + \sum_{\zeta \in \text{supp}\Lambda} c_\zeta K(\Lambda, \omega) = K(\Lambda, \omega) \left(\sum_{\zeta \in \widehat{T}} c_\zeta \right).
\end{aligned}$$

This yields the assertion. \square

3. MARTIN INTEGRAL ON Δ

In this section, we formulate and prove the Martin integral representation (0.4), and then discuss some resulting facts. Let us begin with recalling the bijective correspondence between (0.2) and (0.3). A cylindrical subset of $\mathfrak{T}(\widehat{T})$ (= the set of infinite paths from \emptyset on the branching graph $\mathbb{Y}(\widehat{T})$) is associated with finite path $u = (u(0) \nearrow \cdots \nearrow u(n))$ as

$$C_u = \{t \in \mathfrak{T}(\widehat{T}) \mid t(k) = u(k), k = 0, 1, \dots, n\}.$$

Consider the Borel σ -field \mathfrak{B} generated by the cylindrical subsets of $\mathfrak{T}(\widehat{T})$. $\mathcal{P}(\mathfrak{T}(\widehat{T}))$ denotes the set of probabilities on measurable space $(\mathfrak{T}(\widehat{T}), \mathfrak{B})$. $M \in \mathcal{P}(\mathfrak{T}(\widehat{T}))$ is said to be central (as already anticipated in (0.3)) if it satisfies

$$(3.1) \quad M(C_u)/w_u = M(C_v)/w_v$$

for arbitrary finite paths u and v from \emptyset whenever they share a common terminating vertex. Recall (1.21) for the definition of weight w_u . In the case of a branching graph for wreath product groups, since (1.24) yields $w_u = w_v$ for these u and v , (3.1) simply reduces to

$$M(C_u) = M(C_v).$$

$\mathcal{M}(\mathfrak{T}(\widehat{T}))$ denotes the set of central probabilities on $(\mathfrak{T}(\widehat{T}), \mathfrak{B})$ (see (0.3)). Then, the bijective correspondence between $\varphi \in \mathcal{H}(\mathbb{Y}(\widehat{T}))$ and $M \in \mathcal{M}(\mathfrak{T}(\widehat{T}))$ is given by the equation

$$(3.2) \quad \varphi(\Lambda) = \frac{M(C_u)}{w_u}, \quad u = (\emptyset \nearrow \cdots \nearrow \Lambda).$$

We refer to [9, §2.2] for central probabilities and their correspondence to harmonic functions on a general branching graph. As noted there, centrality of $M \in \mathcal{P}(\mathfrak{T}(\widehat{T}))$ is characterized by a certain invariance of M with respect to a transformation group

on $\mathfrak{T}(\widehat{T})$. Then, ergodicity of M is naturally considered with respect to the transformation group¹. Ergodicity characterizes extremality for a central probability.

$M \in \mathcal{M}(\mathfrak{T}(\widehat{T}))$ yields n th marginal distribution $M^{(n)}$ on $\mathbb{Y}_n(\widehat{T})$: for $\Lambda \in \mathbb{Y}_n(\widehat{T})$,

$$(3.3) \quad \begin{aligned} M^{(n)}(\Lambda) &= M(\{t \in \mathfrak{T}(\widehat{T}) \mid t(n) = \Lambda\}) \\ &= \sum_{\text{finite path } u: u(n)=\Lambda} M(C_u) = \sum_u w_u \varphi(\Lambda) = d(\Lambda) \varphi(\Lambda) \end{aligned}$$

by recalling (3.2) and (1.22). Consider injective map $\iota^{(n)} : \mathbb{Y}_n(\widehat{T}) \rightarrow \Delta$ defined through the rescaled Frobenius coordinates by

$$(3.4) \quad \Lambda = (\lambda^\zeta)_{\zeta \in \widehat{T}} \in \mathbb{Y}_n(\widehat{T}) \mapsto \left(\left(\frac{a_i(\lambda^\zeta)}{n} \right)_{\zeta, i}, \left(\frac{b_i(\lambda^\zeta)}{n} \right)_{\zeta, i}, \left(\frac{|\lambda^\zeta|}{n} \right)_\zeta \right) \in \Delta.$$

Recall that $+1/2$ is added in (2.18), which ensures that the image of $\iota^{(n)}$ is included in Δ .

Theorem 3.1. *Any $\varphi \in \mathcal{H}(\mathbb{Y}(\widehat{T}))$ yields an integral representation*

$$(3.5) \quad \varphi(\Lambda) = \int_{\Delta} K(\Lambda, \omega) Q(d\omega), \quad \Lambda \in \mathbb{Y}(\widehat{T}).$$

Here kernel $K(\Lambda, \omega)$ is given by (2.28). $Q \in \mathcal{P}(\Delta)$ is uniquely determined by φ . Taking $M \in \mathcal{M}(\mathfrak{T}(\widehat{T}))$ corresponding to φ through (3.2), we have

$$(3.6) \quad Q = \lim_{n \rightarrow \infty} \iota_*^{(n)} M^{(n)}$$

as a weakly converging limit in $\mathcal{P}(\Delta)$.

Remark 3.2. $\widetilde{\Delta}$ is equipped with the metrizable topology of the product of pointwise converging one mentioned after (2.24), which is separable, and also with the associated Borel structure. Δ is a Borel subset of $\widetilde{\Delta}$. Recall that, if S is a separable metric space, $\mathcal{P}(S)$ is equipped with a separable metrizable topology characterized by weak convergence of a sequence of probabilities:

$$\lim_{n \rightarrow \infty} \int_S f(s) \mu_n(ds) = \int_S f(s) \mu(ds)$$

for any bounded continuous function f on S . The procedure of (3.6) taking a weak limit after pushing forward the marginal distribution to the boundary seems to be analogous to the classical radial limit for a harmonic function on the unit disk.

Proof. [Step 1] First we show uniqueness of $Q \in \mathcal{P}(\Delta)$ in (3.5). Let \mathcal{S} be the set of \mathbb{R} -linear combinations of $\{K(\Lambda, \cdot) \mid \Lambda \in \mathbb{Y}(\widehat{T})\}$ in (2.28). We verify that \mathcal{S} is dense in $C(\Delta; \mathbb{R})$ for finite T and in $C(\widetilde{\Delta}; \mathbb{R})$ for infinite T by using the Stone–Weierstrass theorem. \mathcal{S} contains constant 1, which corresponds to $\Lambda = \emptyset$. Lemma 2.3 assures that p_k^ζ 's and hence s_λ^ζ 's also separate arbitrary two points in Δ or $\widetilde{\Delta}$. By (2.27), the \mathbb{R} -linear combinations of $\{s_\lambda^\zeta\}_{\lambda \in \mathbb{Y}}$ coincide with those of $\{p_\rho^\zeta\}_{\rho \in \mathbb{Y}}$, the latter clearly forming an algebra. Hence $K(\Lambda, \cdot)K(M, \cdot)$ for $\Lambda, M \in \mathbb{Y}(\widehat{T})$ is linearized and belongs to \mathcal{S} . This completes the proof of density of \mathcal{S} . Then, integration on \mathcal{S} determines that on the whole $C(\Delta; \mathbb{R})$ or $C(\widetilde{\Delta}; \mathbb{R})$. $Q \in \mathcal{P}(\Delta)$ or $\mathcal{P}(\widetilde{\Delta})$ is thus

¹Allow us to make a correction to §2.2 of [9] here. In pp.1200–1201, the transformation group on $\mathfrak{T}(\widehat{T})$ mentioned above should be written as $\langle \mathfrak{S}_{\mathfrak{T}(\alpha)} \mid \alpha \in \mathbb{G} \rangle$, the group generated by $\mathfrak{S}_{\mathfrak{T}(\alpha)}$'s, instead of $\bigcup_{\alpha \in \mathbb{G}} \mathfrak{S}_{\mathfrak{T}(\alpha)}$ since the inclusion $\mathfrak{S}_{\mathfrak{T}(\alpha)} \subset \mathfrak{S}_{\mathfrak{T}(\beta)}$ does not hold even if $\alpha \nearrow \dots \nearrow \beta$.

uniquely determined. Note that, if T is infinite, an element of $\mathcal{P}(\tilde{\Delta})$ supported by Δ is identified with that of $\mathcal{P}(\Delta)$.

[Step 2] We show that, if there exists a limit

$$Q_0 = \lim_{n \rightarrow \infty} \iota_*^{(n)} M^{(n)}$$

in $\mathcal{P}(\Delta)$ for finite T or in $\mathcal{P}(\tilde{\Delta})$ for infinite T , φ is expressed as

$$(3.7) \quad \varphi(\Lambda) = \int_{\Delta} K(\Lambda, \omega) Q_0(d\omega) \quad \text{or} \quad \varphi(\Lambda) = \int_{\tilde{\Delta}} K(\Lambda, \omega) Q_0(d\omega)$$

for $\Lambda \in \mathbb{Y}(\hat{T})$. Let T be infinite, since the case of finite T is treated in the same way. Combining

$$\varphi(\Lambda) = \sum_{N \in \mathbb{Y}_n(\hat{T})} d(\Lambda, N) \varphi(N) = \sum_{N \in \mathbb{Y}_n(\hat{T})} K(\Lambda, N) M^{(n)}(N)$$

which is seen from (3.3), and

$$\int_{\tilde{\Delta}} K(\Lambda, \omega) \iota_*^{(n)} M^{(n)}(d\omega) = \sum_{N \in \mathbb{Y}_n(\hat{T})} K(\Lambda, \iota^{(n)} N) M^{(n)}(N),$$

we write as

$$(3.8) \quad \begin{aligned} & \left| \varphi(\Lambda) - \int_{\tilde{\Delta}} K(\Lambda, \omega) Q_0(d\omega) \right| \\ & \leq \left| \varphi(\Lambda) - \int_{\tilde{\Delta}} K(\Lambda, \omega) \iota_*^{(n)} M^{(n)}(d\omega) \right| \\ & \quad + \left| \int_{\tilde{\Delta}} K(\Lambda, \omega) \iota_*^{(n)} M^{(n)}(d\omega) - \int_{\tilde{\Delta}} K(\Lambda, \omega) Q_0(d\omega) \right| \\ & \leq \sum_{N \in \mathbb{Y}_n(\hat{T})} |K(\Lambda, \omega) - K(\Lambda, \iota^{(n)} N)| M^{(n)}(N) \\ & \quad + \left| \int_{\tilde{\Delta}} K(\Lambda, \omega) \iota_*^{(n)} M^{(n)}(d\omega) - \int_{\tilde{\Delta}} K(\Lambda, \omega) Q_0(d\omega) \right|. \end{aligned}$$

Comparing (2.3) with

$$\begin{aligned} K(\Lambda, \iota^{(n)} N) &= \prod_{\zeta \in \hat{T}} \frac{1}{(\dim \zeta)^{|\lambda^\zeta|}} s_{\lambda^\zeta}^\zeta(\iota^{(n)} N) \\ &= \prod_{\zeta \in \hat{T}} \frac{1}{(\dim \zeta)^{|\lambda^\zeta|}} \sum_{\rho \in \mathbb{Y}_{|\lambda^\zeta|}} \frac{1}{z_\rho} \chi_\rho^\lambda p_\rho^\zeta \left(\left(\frac{a_i(\nu^\zeta)}{n} \right)_{\zeta, i}, \left(\frac{b_i(\nu^\zeta)}{n} \right)_{\zeta, i}, \left(\frac{|\nu^\zeta|}{n} \right)_\zeta \right) \\ &= \prod_{\zeta \in \hat{T}} \frac{1}{(\dim \zeta)^{|\lambda^\zeta|}} \sum_{\rho \in \mathbb{Y}_{|\lambda^\zeta|}} \frac{1}{z_\rho} \chi_\rho^\lambda \frac{1}{n^{|\rho|}} p_\rho(\nu^\zeta) \end{aligned}$$

and using the estimate of (2.31), we have

$$|K(\Lambda, N) - K(\Lambda, \iota^{(n)} N)| \leq \frac{1}{n} C_k, \quad \Lambda \in \mathbb{Y}_k(\hat{T}), \quad N \in \mathbb{Y}_n(\hat{T}).$$

Hence the first term in the rightmost side of (3.8) is bounded by C_k/n . The second term converges to 0 as $n \rightarrow \infty$ since $K(\Lambda, \cdot) \in C(\tilde{\Delta}; \mathbb{R})$. We thus obtained (3.7).

[Step 3] Since $\mathcal{P}(\Delta)$ for finite T and $\mathcal{P}(\tilde{\Delta})$ for infinite T are compact (and metrizable) with respect to the topology of weak convergence, $\{\iota_*^{(n)} M^{(n)}\}_{n \in \mathbb{N}}$ contains a

converging subsequence. The argument in [Step 2] implies that its limit Q_0 satisfies (3.7) and hence is unique. Hence the whole sequence $\{\iota_*^{(n)}M^{(n)}\}_{n \in \mathbb{N}}$ converges as $n \rightarrow \infty$. This completes the proof in the case of finite T . If T is infinite, we now have

$$Q = \lim_{n \rightarrow \infty} \iota_*^{(n)}M^{(n)} \text{ in } \mathcal{P}(\tilde{\Delta}) \quad \text{and} \quad \varphi(\Lambda) = \int_{\tilde{\Delta}} K(\Lambda, \omega)Q(d\omega).$$

Using harmonicity of φ , we have

$$\begin{aligned} 0 &= \varphi(\Lambda) - \sum_{N: \Lambda \nearrow N} \kappa(\Lambda, N)\varphi(N) \\ &= \int_{\tilde{\Delta}} K(\Lambda, \omega)Q(d\omega) - \sum_{N: \Lambda \nearrow N} \kappa(\Lambda, N) \int_{\tilde{\Delta}} K(N, \omega)Q(d\omega) \\ &= \int_{\tilde{\Delta}} \left(K(\Lambda, \omega) - \sum_{N: \Lambda \nearrow N} \kappa(\Lambda, N)K(N, \omega) \right) Q(d\omega). \end{aligned}$$

Superharmonicity of $K(\cdot, \omega)$ (Lemma 2.7) then yields

$$K(\Lambda, \omega) = \sum_{N: \Lambda \nearrow N} \kappa(\Lambda, N)K(N, \omega), \quad Q\text{-a.e. } \omega.$$

Taking a common exceptional set for $\Lambda \in \mathbb{Y}(\hat{T})$, we verify that $K(\cdot, \omega)$ is harmonic on $\mathbb{Y}(\hat{T})$ for a.e. ω . In other words, Q is supported by Δ and regarded as an element of $\mathcal{P}(\Delta)$. We note that

$$Q = \lim_{n \rightarrow \infty} \iota_*^{(n)}M^{(n)} \text{ in } \mathcal{P}(\tilde{\Delta}) \quad \text{and} \quad \iota_*^{(n)}M^{(n)}, Q \in \mathcal{P}(\Delta)$$

imply the convergence in $\mathcal{P}(\Delta)$, which is seen e.g. from the characterization of weak convergence of probabilities in terms of convergence on any Borel set with boundary of probability 0. \square

Recall that the minimal Martin boundary is defined in (1.35) as

$$\partial_m \mathbb{Y}(\hat{T}) = \{\omega \in \partial \mathbb{Y}(\hat{T}) \mid K(\cdot, \omega) \text{ is minimal harmonic}\}.$$

For $\omega \in \partial \mathbb{Y}(\hat{T})$, $K(\emptyset, \omega) = 1$ and $K(\Lambda, \omega) \geq 0$ hold. Letting $\text{ex}C$ denote the set of extremal points of convex set C , we always have

$$(3.9) \quad \{K(\cdot, \omega) \mid \omega \in \partial_m \mathbb{Y}(\hat{T})\} \subset \text{ex}\mathcal{H}(\mathbb{Y}(\hat{T})).$$

On the other hand, the Martin integral representation (Theorem 3.1) ensures

$$(3.10) \quad \text{ex}\mathcal{H}(\mathbb{Y}(\hat{T})) \subset \{K(\cdot, \omega) \mid \omega \in \Delta\}.$$

Actually, we have equalities in (3.9) and (3.10) as follows.

Theorem 3.3. *An extremal point of $\mathcal{H}(\mathbb{Y}(\hat{T}))$ is exactly a Martin kernel $K(\cdot, \omega)$ for $\omega \in \Delta$. The minimal Martin boundary $\partial_m \mathbb{Y}(\hat{T})$ of the branching graph $\mathbb{Y}(\hat{T})$ is homeomorphically isomorphic to Δ for any T .*

Proof. Let $\omega \in \Delta$. $K(\cdot, \omega)$ is harmonic by Lemma 2.8. If it is expressed as a convex combination in $\mathcal{H}(\mathbb{Y}(\hat{T}))$:

$$K(\cdot, \omega) = c\varphi_1 + (1-c)\varphi_2, \quad \varphi_1, \varphi_2 \in \mathcal{H}(\mathbb{Y}(\hat{T})), \quad 0 \leq c \leq 1,$$

Theorem 3.1 applied to φ_1 and φ_2 yields

$$K(\Lambda, \omega) = c \int_{\Delta} K(\Lambda, \omega') Q_1(d\omega') + (1-c) \int_{\Delta} K(\Lambda, \omega') Q_2(d\omega'), \quad \Lambda \in \mathbb{Y}(\widehat{T})$$

for $Q_1, Q_2 \in \mathcal{P}(\Delta)$. The uniqueness part of Theorem 3.1 implies

$$\delta_{\omega} = cQ_1 + (1-c)Q_2$$

holds in $\mathcal{P}(\Delta)$, which is possible only when $c = 0$ or $c = 1$ or $Q_1 = Q_2$. This shows extremality (= minimality) of $K(\cdot, \omega)$ in $\mathcal{H}(\mathbb{Y}(\widehat{T}))$, in other words,

$$(3.11) \quad \{K(\cdot, \omega) \mid \omega \in \Delta\} \subset \{K(\cdot, \omega) \mid \omega \in \partial_m \mathbb{Y}(\widehat{T})\}.$$

Combining (3.11) with (3.9) and (3.10), we have equality between them. Homeomorphic property is already shown in Theorem 2.5. \square

Combining Theorem 2.5 and Theorem 3.3, we can translate the obtained results into those for the other objects in (a2) – (a4) described in Introduction. Recall the bijective correspondence

$$f \in \mathcal{K}(\mathfrak{S}_{\infty}(T)) \longleftrightarrow \varphi \in \mathcal{H}(\mathbb{Y}(\widehat{T}))$$

given by

$$(3.12) \quad f|_{\mathfrak{S}_k(T)} = \sum_{\Lambda \in \mathbb{Y}_k(\widehat{T})} \varphi(\Lambda) \chi^{\Lambda}, \quad k \in \mathbb{N}.$$

An extremal point of $\mathcal{K}(\mathfrak{S}_{\infty}(T))$ is simply called a character of $\mathfrak{S}_{\infty}(T)$. It is known that a character of $\mathfrak{S}_{\infty}(T)$ is factorizable (or multiplicative) with respect to the standard decomposition (1.1) into basic elements. We can now compute a character of $\mathfrak{S}_{\infty}(T)$ directly by using (2.28) under the correspondence of Theorem 3.3 without knowing a priori the factorizability. Recall (1.15) for the structure of the conjugacy classes of $\mathfrak{S}_{\infty}(T)$.

Theorem 3.4. *Let f be a character of $\mathfrak{S}_{\infty}(T)$ and $\omega \in \Delta$ the corresponding parameter in Theorem 3.3. The value of $f = f_{\omega}$ at an element in the conjugacy class of $\mathfrak{S}_{\infty}(T)$ corresponding to $P = (\rho_{\theta})_{\theta \in [T]} \in \mathbb{Y}([T])$, denoted by $f_{\omega}(P)$, is given by*

$$(3.13) \quad f_{\omega}(P) = \prod_{j=1}^{\infty} \prod_{\theta \in [T]} \left(\sum_{\zeta \in \widehat{T}} p_j^{\zeta}(\omega) \frac{\chi_{\theta}^{\zeta}}{(\dim \zeta)^j} \right)^{m_j(\rho_{\theta})}.$$

In particular, f is factorizable.

Proof. Take $x \in C_P \subset \mathfrak{S}_{\infty}(T)$ and then $k \in \mathbb{N}$ such that $x \in \mathfrak{S}_k(T)$. Let $\text{supp} P$, i.e. the θ 's with nontrivial ρ_{θ} 's, be $\{\theta_1, \theta_2, \dots, \theta_l\}$. For computing $f_{\omega}(P)$ through (3.12), we have formulas for $\varphi(\Lambda) = K(\Lambda, \omega)$ in (2.28) and for χ^{Λ} in (1.12) (in the case of $k = n$). Noting that the condition (1.11) is reduced to

$$(3.14) \quad \sum_{(\rho_{\theta_i})_j^* : r((\rho_{\theta_i})_j^*) = \zeta} (\rho_{\theta_i})_j = |\lambda^{\zeta}|, \quad \zeta \in \widehat{T},$$

we have

$$(3.15) \quad f_\omega(\mathbf{P}) = \sum_{\Lambda \in \mathbb{Y}_k(\widehat{T})} \sum_{r \text{ satisfying (3.14)}} \prod_{\zeta \in \widehat{T}} \left\{ \frac{1}{(\dim \zeta)^{|\lambda^\zeta|}} s_{\lambda^\zeta}^\zeta(\omega) \left(\prod_{i=1}^l (\chi_{\theta_i}^\zeta)^{l(r^{-1}(\zeta) \cap \rho_{\theta_i})} \right) \chi_{r^{-1}(\zeta)}^{\lambda^\zeta} \right\}.$$

In (3.15), let $\Lambda = (\lambda^\zeta)_{\zeta \in \widehat{T}}$ range over $\mathbb{Y}_k(\widehat{T})$ in two stages. First take finite subset F of \widehat{T} as $\text{supp} \Lambda$, i.e. the ζ 's with nontrivial λ^ζ 's, and then take $n^\zeta \in \mathbb{N}$ for $\zeta \in F$ as $n^\zeta = |\lambda^\zeta|$. We necessarily have $\sum_{\zeta \in F} n^\zeta = k$. Next, for each $(F, (n^\zeta)_{\zeta \in F})$, let $\Lambda \in \mathbb{Y}_k(\widehat{T})$ range under the constraint of

$$(3.16) \quad \text{supp} \Lambda = F \quad \text{and} \quad |\lambda^\zeta| = n^\zeta.$$

For fixed $(F, (n^\zeta)_{\zeta \in F})$, note that the sum in Λ in the above second stage and the sum \sum_r in (3.15) commute. Setting the condition

$$(3.17) \quad \sum_{(\rho_{\theta_i})_j^* : r((\rho_{\theta_i})_j^*) = \zeta} (\rho_{\theta_i})_j = n^\zeta, \quad \zeta \in \widehat{T}$$

instead of (3.14), we hence continue (3.15) as

$$(3.15) = \sum_{(F, (n^\zeta)_{\zeta \in F})} \sum_{r : (3.17)} \sum_{\Lambda : (3.16)} \prod_{\zeta \in F} \left\{ \frac{1}{(\dim \zeta)^{n^\zeta}} \left(\prod_{i=1}^l (\chi_{\theta_i}^\zeta)^{l(r^{-1}(\zeta) \cap \rho_{\theta_i})} \right) \chi_{r^{-1}(\zeta)}^{\lambda^\zeta} s_{\lambda^\zeta}^\zeta(\omega) \right\} \\ = \sum_{(F, (n^\zeta)_{\zeta \in F})} \sum_{r : (3.17)} \prod_{\zeta \in F} \left\{ p_{r^{-1}(\zeta)}^\zeta \frac{1}{(\dim \zeta)^{n^\zeta}} \left(\prod_{i=1}^l (\chi_{\theta_i}^\zeta)^{l(r^{-1}(\zeta) \cap \rho_{\theta_i})} \right) \right\}$$

by using inversion of (2.27). Let us verify that this expression is equal to

$$(3.18) \quad \prod_{i=1}^l \prod_{j=1}^{\infty} \left(\sum_{\zeta \in \widehat{T}} p_j^\zeta(\omega) \frac{1}{(\dim \zeta)^j} \chi_{\theta_i}^\zeta \right)^{m_j(\rho_{\theta_i})}.$$

In (3.18), since two kinds of products are finite ones and the infinite sum in ζ converges absolutely, no problems occur in changing the terms. Let n^ζ be the degree of the factors labeled by ζ in each term of the development of (3.18), where $p_j^\zeta(\omega)$ is regarded to have degree j . We have

$$\sum_{\zeta \in \widehat{T}} n^\zeta = \sum_{i=1}^l \sum_{j=1}^{\infty} j m_j(\rho_{\theta_i}) = \sum_{i=1}^l |\rho_{\theta_i}| = k.$$

For a term in the development of (3.18), let F be the ζ 's which actually appear and $(n^\zeta)_{\zeta \in F}$ the set of degrees labeled by ζ . Then, the terms having given $(F, (n^\zeta)_{\zeta \in F})$ are exactly counted by the maps r satisfying (3.17). We observe that the term corresponding to $(F, (n^\zeta)_{\zeta \in F}, r)$ is expressed as a product of $p_j^\zeta(\omega)$, $(\dim \zeta)^{-j}$, $\chi_{\theta_i}^\zeta$ in the desired form seen in the rightmost side of the above continuation of (3.15). This completes the proof of (3.13). \square

Theorem 3.5. *Any $f \in \mathcal{K}(\mathfrak{S}_\infty(T))$ yields an integral representation*

$$(3.19) \quad f(\mathbf{P}) = \int_{\Delta} f_\omega(\mathbf{P})Q(d\omega), \quad \mathbf{P} \in \mathbb{Y}([T]).$$

Here character f_ω is given by (3.13). $Q \in \mathcal{P}(\Delta)$ is the same with that of Theorem 3.1 under the correspondence $f \in \mathcal{K}(\mathfrak{S}_\infty(T)) \leftrightarrow \varphi \in \mathcal{H}(\mathbb{Y}(\widehat{T}))$ through (3.12).

Proof. For given $\mathbf{P} \in \mathbb{Y}([T])$, take $x \in C_{\mathbf{P}} \subset \mathfrak{S}_\infty(T)$ and then $k \in \mathbb{N}$ such that $x \in \mathfrak{S}_k(T)$. The integral representation for an element of $\mathcal{H}(\mathbb{Y}(\widehat{T}))$ (Theorem 3.1) is easily transformed to the one for the corresponding element of $\mathcal{K}(\mathfrak{S}_\infty(T))$. More precisely, we have

$$\begin{aligned} f(x) &= \sum_{\Lambda \in \mathbb{Y}_k(\widehat{T})} \varphi(\Lambda) \chi^\Lambda(x) = \sum_{\Lambda \in \mathbb{Y}_k(\widehat{T})} \left(\int_{\Delta} K(\Lambda, \omega) Q(d\omega) \right) \chi^\Lambda(x) \\ &= \int_{\Delta} \left(\sum_{\Lambda \in \mathbb{Y}_k(\widehat{T})} K(\Lambda, \omega) \chi^\Lambda(x) \right) Q(d\omega) \\ &= \int_{\Delta} f_\omega(\mathbf{P}) Q(d\omega), \end{aligned}$$

where changing sum and integral is justified by

$$\sum_{\Lambda \in \mathbb{Y}_k(\widehat{T})} \int_{\Delta} |K(\Lambda, \omega) \chi^\Lambda(x)| Q(d\omega) \leq \sum_{\Lambda \in \mathbb{Y}_k(\widehat{T})} d(\Lambda) \varphi(\Lambda) = 1.$$

□

4. CHARACTERS, HARMONIC FUNCTIONS AND CENTRAL PROBABILITIES ASSOCIATED WITH AN INDUCTIVE SYSTEM OF COMPACT GROUPS

We note bijective correspondences between the spaces of (0.1) – (0.3) for an inductive system of compact groups instead of restricting to wreath products $\mathfrak{S}_n(T)$. Let us consider a sequence of compact groups

$$\{e\} = G_0 \subset G_1 \subset \cdots \subset G_{n-1} \subset G_n \subset \cdots$$

in which the embedding $G_{n-1} \subset G_n$ is a continuous homomorphism for any $n \in \mathbb{N}$. Set $G_\infty = \lim_{n \rightarrow \infty} G_n = \bigcup_{n=0}^{\infty} G_n$ (inductive limit group). \widehat{G}_n denoting the set of equivalence classes of IURs of G_n , we have a branching graph with $\mathbb{G} = \bigsqcup_{n=0}^{\infty} \widehat{G}_n$ as the vertex set by introducing edge $\alpha \nearrow \beta$ and its multiplicity $\kappa(\alpha, \beta)$ for $\alpha \in \widehat{G}_{n-1}$ and $\beta \in \widehat{G}_n$ through the irreducible decomposition

$$\text{Res}_{G_{n-1}}^{G_n} \pi^\beta \cong \bigoplus_{\alpha \in \widehat{G}_{n-1}: \alpha \nearrow \beta} \kappa(\alpha, \beta) \pi^\alpha, \quad \pi^\beta \in \beta.$$

(Recall subsection (H) in Section 1 for $G_n = \mathfrak{S}_n(T)$.)

For a while (until the end of the proof of Proposition 4.2), we forget the dual countability assumption, in other words, do not assume that \widehat{G}_n is countable.

Equipping G_∞ with the inductive limit topology, set

$$(4.1) \quad \mathcal{K}(G_\infty) = \{f : G_\infty \rightarrow \mathbb{C} \mid f \text{ is continuous, positive-definite, central, } f(e) = 1\}.$$

We modify (0.2) by adding to be countably supported and set

$$(4.2) \quad \mathcal{H}(\mathbb{G}) = \left\{ \varphi : \mathbb{G} \longrightarrow \mathbb{C} \mid \begin{array}{l} \varphi \text{ is harmonic, non-negative, } \varphi(\emptyset) = 1, \\ \text{supp}\varphi \text{ is an at most countable set} \end{array} \right\}.$$

Here harmonicity of φ is defined by

$$\varphi(\alpha) = \sum_{\beta: \alpha \nearrow \beta} \kappa(\alpha, \beta) \varphi(\beta), \quad \alpha \in \mathbb{G}$$

similarly to (1.18). If a graph consisting of subset $\mathbb{G}^0 \subset \mathbb{G}$ and the edges inherited from \mathbb{G} satisfies that, for any $\beta \in \mathbb{G}^0$, all vertices lying on the finite paths from \emptyset to β belong to \mathbb{G}^0 , then we refer to \mathbb{G}^0 simply as a subgraph of \mathbb{G} . It is immediate that $\text{supp}\varphi$ is a subgraph of \mathbb{G} for harmonic and non-negative φ . $\mathfrak{T}(\mathbb{G})$ denotes the set of all infinite paths $t = (t(0) \nearrow t(1) \nearrow t(2) \nearrow \cdots), t(n) \in \widehat{G}_n$, on branching graph \mathbb{G} . $\mathfrak{T}(\mathbb{G})$ is equipped with the σ -field generated by its cylindrical subsets. We consider $\mathfrak{T}(\mathbb{G}^0)$ also for subgraph $\mathbb{G}^0 \subset \mathbb{G}$, where $t(n) \in \mathbb{G}_n^0 = \mathbb{G}^0 \cap \widehat{G}_n$. We modify (0.3) to set

$$(4.3) \quad \mathcal{M}(\mathfrak{T}(\mathbb{G})) = \left\{ M \in \mathcal{P}(\mathfrak{T}(\mathbb{G})) \mid \begin{array}{l} M \text{ is central and supported by } \mathfrak{T}(\mathbb{G}^0) \\ \text{for some countable subgraph } \mathbb{G}^0 \subset \mathbb{G} \end{array} \right\}.$$

Here centrality of M is defined similarly to (3.1) by

$$M(C_u)/w_u = M(C_v)/w_v$$

for finite paths u and v from \emptyset terminating at a common vertex, where w_u is the weight of path u defined by (1.21) and C_u is the cylindrical set associated with u . Note that $\mathcal{K}(G_\infty)$, $\mathcal{H}(\mathbb{G})$ and $\mathcal{M}(\mathfrak{T}(\mathbb{G}))$ are all convex sets.

Proposition 4.1. *There exist bijective affine maps between $\mathcal{K}(G_\infty)$, $\mathcal{H}(\mathbb{G})$ and $\mathcal{M}(\mathfrak{T}(\mathbb{G}))$. In particular, their extremal points have bijective correspondences.*

Proof. Recall that function $a : C_1 \rightarrow C_2$, where C_1 and C_2 are convex sets, is said to be affine if it satisfies $a(sx + (1-s)y) = sa(x) + (1-s)a(y)$ for any $x, y \in C_1$ and $0 \leq s \leq 1$. A bijection between $\mathcal{H}(\mathbb{G})$ and $\mathcal{M}(\mathfrak{T}(\mathbb{G}))$ is given in the same way with (3.2): for $\varphi \in \mathcal{H}(\mathbb{G})$ and $M \in \mathcal{M}(\mathfrak{T}(\mathbb{G}))$,

$$(4.4) \quad \varphi(\alpha) = M(C_u)/w_u, \quad \alpha \in \mathbb{G}$$

where α is the terminal vertex of finite path u . This fact is fully shown in [9, Lemma 2.9] for a general branching graph.

As noted in (3.12), a bijection between $\mathcal{K}(G_\infty)$ and $\mathcal{H}(\mathbb{G})$ is given by using Fourier expansion on each compact group G_k : for $f \in \mathcal{K}(G_\infty)$ and $\varphi \in \mathcal{H}(\mathbb{G})$,

$$(4.5) \quad f|_{G_k} = \sum_{\alpha \in \widehat{G}_k} \varphi(\alpha) \chi^\alpha, \quad k \in \mathbb{N}$$

where χ^α denotes the irreducible character for α ; $\chi^\alpha(x) = \text{tr}\pi^\alpha(x)$. This fact is shown in [9, Theorem 4.2] for $G_n = \mathfrak{S}_n(T)$ with an arbitrary compact group T , and there needs no modification in its proof to apply general G_n . Either bijection given by (4.4) or (4.5) is clearly affine. \square

$\text{ex}\mathcal{K}(G_\infty)$, the set of extremal elements in $\mathcal{K}(G_\infty)$, is denoted by $\mathcal{E}(G_\infty)$, whose element is called a character of G_∞ . Given an extremal element in $\mathcal{K}(G_\infty)$, $\mathcal{H}(\mathbb{G})$ or $\mathcal{M}(\mathfrak{T}(\mathbb{G}))$, it is approximated by objects at finite levels in the following manners.

Proposition 4.2. *Let f be a character of G_∞ and M the corresponding probability in $\text{ex}\mathcal{M}(\mathfrak{T}(\mathbb{G}))$ determined in Proposition 4.1. For M -a.s. path $t \in \mathfrak{T}(\mathbb{G})$, convergence of the normalized irreducible characters*

$$\lim_{n \rightarrow \infty} \tilde{\chi}^{(t(n))} = f$$

holds uniformly on each G_k , $k \in \mathbb{N}$.

Proof. This fact is shown in [9, Theorem 4.3] for $G_k = \mathfrak{S}_k(T)$ with an arbitrary compact group T . No modification is needed in its proof to apply general G_k . $\varphi \in \text{ex}\mathcal{H}(\mathbb{G})$ being taken correspondingly, the proof is based on the M -a.s. convergence

$$\varphi(\alpha) = \lim_{n \rightarrow \infty} K(\alpha, t(n)), \quad \alpha \in \text{supp}\varphi \text{ (subgraph of } \mathbb{G}\text{)},$$

which is shown via martingale convergence theorem in [9]. \square

Let us look at the topologies given the convex sets (4.1), (4.2) and (4.3). Hereafter again, we assume that each \hat{G}_n is at most countable for technical conveniences. Generally speaking, we consider compact-open topology for a space of functions and weak convergence topology for a space of probabilities. To deal with $\mathcal{K}(G_\infty)$, recall that any compact set K of G_∞ , which is given the inductive limit topology, is included in G_n for sufficiently large n . See [8] for this fact (§6.3) and other properties of an inductive limit topology. Hence the compact-open topology, or topology of uniform convergence on every compact set, on $\mathcal{K}(G_\infty)$ admits

$$\left\{ \{g \in \mathcal{K}(G_\infty) \mid \max_{x \in G_n} |f(x) - g(x)| < \epsilon_k\} \mid n, k \in \mathbb{N} \right\}$$

as a fundamental system of neighborhoods of f , where $\epsilon_k > 0$ and $\epsilon_k \searrow 0$ as $k \rightarrow \infty$. Since $\mathbb{G} = \bigsqcup_{n=0}^{\infty} \hat{G}_n$ is discrete, the compact-open topology on $\mathcal{H}(\mathbb{G})$ is just the one of pointwise converging on \mathbb{G} . The path space $\mathfrak{T}(\mathbb{G})$ is equipped with the relative topology of $\prod_{n=0}^{\infty} \hat{G}_n$ having the product (or weak) topology of discrete ones on \hat{G}_n . Since $\mathfrak{T}(\mathbb{G})$ is a separable metric space (by virtue of countability assumption for \hat{G}_n 's, so is $\mathcal{P}(\mathfrak{T}(\mathbb{G}))$ with respect to the topology of weak convergence of a sequence of probabilities (recall Remark 3.2). The topology on $\mathfrak{T}(\mathbb{G})$ is also generated by cylindrical sets $\{C_u \mid u : \text{finite path on } \mathbb{G}\}$.

Lemma 4.3. *Assume \hat{G}_n be at most countable for any $n \in \mathbb{N}$. The bijective map*

$$(4.6) \quad \mathcal{M}(\mathfrak{T}(\mathbb{G})) \longrightarrow \mathcal{H}(\mathbb{G})$$

induced by the correspondence through (4.4) between $M \in \mathcal{M}(\mathfrak{T}(\mathbb{G}))$ and $\varphi \in \mathcal{H}(\mathbb{G})$ is a homeomorphism.

Proof. Note that cylindrical set C_u is open and closed. Then, since 1_{C_u} is a bounded continuous function on $\mathfrak{T}(\mathbb{G})$, weak convergence of $\{M_n\}_{n \in \mathbb{N}}$ to M in $\mathcal{P}(\mathfrak{T}(\mathbb{G}))$ implies convergence on every cylindrical set.

To show continuity of the inverse map $\mathcal{H}(\mathbb{G}) \longrightarrow \mathcal{M}(\mathfrak{T}(\mathbb{G}))$, let a sequence $\{\varphi_n\}$ converge to φ in $\mathcal{H}(\mathbb{G})$. Take $M_n \in \mathcal{M}(\mathfrak{T}(\mathbb{G}))$ and $M \in \mathcal{M}(\mathfrak{T}(\mathbb{G}))$ corresponding to φ_n and φ respectively. Recall a well-known fact that $\{M_n\}$ converges weakly to M in $\mathcal{M}(\mathfrak{T}(\mathbb{G}))$ if and only if

$$(4.7) \quad M(O) \leq \liminf_{n \rightarrow \infty} M_n(O)$$

holds for any open subset O of $\mathfrak{T}(\mathbb{G})$. The topology on $\mathfrak{T}(\mathbb{G})$ yields that, for any open $O \subset \mathfrak{T}(\mathbb{G})$, there exists a disjoint family of cylindrical sets $\{O_j\}$ such that

$O = \bigsqcup_j O_j$. Note that there are only countable number of cylindrical sets. We have $M(O_j) = \lim_{n \rightarrow \infty} M_n(O_j)$ for j by the assumption of convergence of φ_n to φ . Then, (4.7) follows from

$$M(O) = \sum_j M(O_j) = \sum_j \lim_{n \rightarrow \infty} M_n(O_j) \leq \liminf_{n \rightarrow \infty} \sum_j M_n(O_j) = \liminf_{n \rightarrow \infty} M_n(O).$$

This completes the proof of the desired continuity. \square

Remark 4.4. The argument in the first paragraph of the above proof yields also that $\mathcal{M}(\mathfrak{T}(\mathbb{G}))$ is a closed subset of $\mathcal{P}(\mathfrak{T}(\mathbb{G}))$.

Lemma 4.5. *Assume \widehat{G}_n be at most countable for any $n \in \mathbb{N}$. The bijective map*

$$(4.8) \quad \mathcal{K}(G_\infty) \longrightarrow \mathcal{H}(\mathbb{G})$$

given by the correspondence through (4.5) between $f \in \mathcal{K}(G_\infty)$ and $\varphi \in \mathcal{H}(\mathbb{G})$ is a homeomorphism.

Proof. Let a sequence $\{f_n\}$ converge to f in $\mathcal{K}(G_\infty)$. (4.5) implies

$$(4.9) \quad \varphi(\alpha) = \int_{G_k} f(x) \overline{\chi^\alpha(x)} dx, \quad \alpha \in \widehat{G}_k,$$

and also a similar equality for f_n and corresponding φ_n . Then we have for any $k \in \mathbb{N}$ and any $\alpha \in \widehat{G}_k$

$$(4.10) \quad \begin{aligned} |\varphi_n(\alpha) - \varphi(\alpha)| &\leq \int_{G_k} |f_n(x) - f(x)| |\chi^\alpha(x)| dx \\ &\leq \dim \alpha \cdot \max_{x \in G_k} |f_n(x) - f(x)| \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

This means convergence of $\{\varphi_n\}$ to φ in $\mathcal{H}(\mathbb{G})$.

To show continuity of the inverse map, let a sequence $\{\varphi_n\}$ converge to φ in $\mathcal{H}(\mathbb{G})$, noting that $\mathcal{H}(\mathbb{G})$ obviously enjoys the first countability since \mathbb{G} has only countable number of finite subsets. Let $k \in \mathbb{N}$ be fixed, and take arbitrary $\epsilon > 0$. Since

$$\sum_{\alpha \in \widehat{G}_k} d(\alpha) \varphi(\alpha) = \varphi(\emptyset) = f(e) = 1$$

holds, there exists a finite subset $K_1 \subset \widehat{G}_k$ such that

$$0 \leq \sum_{\alpha \in \widehat{G}_k \setminus K_1} d(\alpha) \varphi(\alpha) < \epsilon.$$

Moreover, since we have

$$\begin{aligned} \sum_{\alpha \in \widehat{G}_k \setminus K_1} d(\alpha) \varphi_n(\alpha) &= 1 - \sum_{\alpha \in K_1} d(\alpha) \varphi_n(\alpha) \\ &\xrightarrow{n \rightarrow \infty} 1 - \sum_{\alpha \in K_1} d(\alpha) \varphi(\alpha) = \sum_{\alpha \in \widehat{G}_k \setminus K_1} d(\alpha) \varphi(\alpha), \end{aligned}$$

there exists $n_1 \in \mathbb{N}$ such that

$$n > n_1 \implies \sum_{\alpha \in \widehat{G}_k \setminus K_1} d(\alpha) \varphi_n(\alpha) < \epsilon.$$

Then, letting $f_n, f \in \mathcal{K}(G_\infty)$ correspond to φ_n, φ respectively, we have for $x \in G_k$ and $n > n_1$

(4.11)

$$\begin{aligned} & |f_n(x) - f(x)| \\ & \leq \left| \sum_{\alpha \in \widehat{G}_k \setminus K_1} \varphi_n(\alpha) \chi^\alpha(x) \right| + \left| \sum_{\alpha \in \widehat{G}_k \setminus K_1} \varphi(\alpha) \chi^\alpha(x) \right| + \left| \sum_{\alpha \in K_1} (\varphi_n(\alpha) - \varphi(\alpha)) \chi^\alpha(x) \right| \\ & \leq \sum_{\alpha \in \widehat{G}_k \setminus K_1} d(\alpha) \varphi_n(\alpha) + \sum_{\alpha \in \widehat{G}_k \setminus K_1} d(\alpha) \varphi(\alpha) + \sum_{\alpha \in K_1} d(\alpha) |\varphi_n(\alpha) - \varphi(\alpha)| \\ & < 2\epsilon + \sum_{\alpha \in K_1} d(\alpha) |\varphi_n(\alpha) - \varphi(\alpha)|, \end{aligned}$$

which yields

$$\limsup_{n \rightarrow \infty} \max_{x \in G_k} |f_n(x) - f(x)| \leq 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have uniform convergence of f_n to f on G_k . Since $k \in \mathbb{N}$ is arbitrary, we have $\lim_{n \rightarrow \infty} f_n = f$ in $\mathcal{K}(G_\infty)$. \square

Let us now return to the case of $G_n = \mathfrak{S}_n(T)$, wreath product of compact group T with at most countable \widehat{T} .

Theorem 4.6. *$\text{ex}\mathcal{H}(\mathbb{Y}(\widehat{T}))$, $\text{ex}\mathcal{M}(\mathfrak{T}(\widehat{T}))$ and $\text{ex}\mathcal{K}(\mathfrak{S}_\infty(T))$ are all homeomorphically isomorphic to Δ of (2.23). In particular, they are not compact if T is infinite.*

Proof. We showed in Lemma 4.3 and Lemma 4.5 that the bijective maps between $\mathcal{H}(\mathbb{Y}(\widehat{T}))$, $\mathcal{M}(\mathfrak{T}(\widehat{T}))$ and $\mathcal{K}(\mathfrak{S}_\infty(T))$ are all homeomorphisms. Since these maps are affine, they give homeomorphic isomorphisms between the sets of extremal points. By Theorem 3.3 we already know a bijective correspondence between $\text{ex}\mathcal{H}(\mathbb{Y}(\widehat{T}))$ and Δ . Its homeomorphic property is read out through the explicit formula (2.28) for $K(\Lambda, \omega)$ in a similar argument to Step 2 of the proof of Theorem 2.5. \square

In contrast with Proposition 4.2, we consider the following question.

Question 4.7. Let the convergence of the normalized irreducible characters

$$(4.12) \quad \lim_{n \rightarrow \infty} \tilde{\chi}^{\alpha^{(n)}} = f$$

hold compact-uniformly on G_∞ along a sequence $(\alpha^{(n)})_{n=0,1,2,\dots}$, $\alpha^{(n)} \in \widehat{G}_n$. Then, is the limit function f a character of G_∞ ?

Note that taking limit on the left hand side of (4.12) is well-defined since every compact subset of G_∞ is included in all sufficiently large G_n 's. In the following statement, the Martin distance on \mathbb{G} is given in a similar way to (1.29). See also (A.16) in Appendix.

Lemma 4.8. *Assume \widehat{G}_n be at most countable for any $n \in \mathbb{N}$. For a sequence $(\alpha^{(n)})_{n=0,1,2,\dots}$, $\alpha^{(n)} \in \widehat{G}_n$, the following (i) and (ii) are equivalent.*

(i) *There exists $\omega \in \partial\mathbb{G}$ such that*

- $\alpha^{(n)}$ converges to ω as $n \rightarrow \infty$,
- $K(\cdot, \omega)$ is harmonic on \mathbb{G} .

(ii) *Normalized irreducible character $\tilde{\chi}^{\alpha^{(n)}}$ converges compact-uniformly on G_∞ as $n \rightarrow \infty$.*

Proof. To deduce (ii) from (i), we begin with

$$(4.13) \quad \tilde{\chi}^{\alpha^{(n)}}|_{G_k} = \sum_{\alpha \in \hat{G}_k} K(\alpha, \alpha^{(n)})\chi^\alpha.$$

Harmonicity of $K(\cdot, \omega)$ ensures that $f \in \mathcal{K}(G_\infty)$ is well-defined by

$$(4.14) \quad f|_{G_k} = \sum_{\alpha \in \hat{G}_k} K(\alpha, \omega)\chi^\alpha.$$

In (4.13) and (4.14), (i) implies $\lim_{n \rightarrow \infty} K(\alpha, \alpha^{(n)}) = K(\alpha, \omega)$ exists for any $\alpha \in \mathbb{G}$. Then, we estimate the supremum norm $\|f - \tilde{\chi}^{\alpha^{(n)}}\|$ in the same way as (4.11). This yields (ii).

Assuming (ii), we have

$$\lim_{n \rightarrow \infty} \tilde{\chi}^{\alpha^{(n)}} = f \in \mathcal{K}(G_\infty).$$

Take $\varphi \in \mathcal{H}(\mathbb{G})$ corresponding to f via Proposition 4.1. Taking Fourier coefficient (4.9) and following (4.10), we see

$$(4.15) \quad K(\alpha, \alpha^{(n)}) \xrightarrow{n \rightarrow \infty} \varphi(\alpha), \quad \alpha \in \mathbb{G}.$$

Hence $(\alpha^{(n)})$ converges to a point in the Martin boundary of \mathbb{G} , denoted by $\omega \in \partial\mathbb{G}$. Furthermore, $K(\cdot, \omega) = \varphi$ is harmonic. \square

Remark 4.9. In Lemma 4.8, if the normalized irreducible characters $\tilde{\chi}^{\alpha^{(n)}}$ are assumed to converge pointwise on G_∞ instead of compact-uniform one, the limit function f is still positive-definite, central and normalized. If we assume further that f is continuous, the convergence actually proves to be compact-uniform. Indeed, taking $\varphi \in \mathcal{H}(\mathbb{G})$ corresponding to $f \in \mathcal{K}(G_\infty)$, considering Fourier coefficient as in (4.9), and then applying Lebesgue's convergence theorem, we have (4.15). It remains to repeat the estimate of (4.11) as in the first half of the proof of Lemma 4.8.

The answer to Question 4.7 is affirmative for our wreath product case.

Proposition 4.10. *In the case of $G_n = \mathfrak{S}_n(T)$, condition (i) in Lemma 4.8 is equivalent to*

(i)' *There exists $\omega \in \partial_m \mathbb{Y}(\hat{T})$ such that $\alpha^{(n)}$ converges to ω as $n \rightarrow \infty$ (with respect to the Martin distance (1.29) on $\mathbb{Y}(\hat{T})$).*

Under (i), (i)', and/or (ii), limit function

$$f = \lim_{n \rightarrow \infty} \tilde{\chi}^{\alpha^{(n)}}$$

is a character of $\mathfrak{S}_\infty(T)$. The correspondence $f \longleftrightarrow \omega$ is the one given in Theorem 3.3 and Theorem 4.6.

Proof. We check (i)' follows from apparently weak (i). Under (i), Lemma 2.8 ensures $\omega \in \Delta$. Then, $\omega \in \partial_m \mathbb{Y}(\hat{T})$ follows by Theorem 3.3. \square

Remark 4.11. Question 4.7 is affirmed in the case of infinite-dimensional unitary group $U(\infty) = \lim_{n \rightarrow \infty} U(n)$. See [15, Proposition 10.9] and the references therein. On the other hand, let each G_n be a finite group in $G_\infty = \lim_{n \rightarrow \infty} G_n$. In Condition (i) of Lemma 4.8, harmonicity of Martin kernel $K(\cdot, \omega)$ then follows automatically from the other convergence condition. In [13, §§1.5, Chapter 0], it is suggested

that $K(\cdot, \omega)$ obtained here is not necessarily extremal. This implies that the corresponding element in $\mathcal{K}(G_\infty)$, which is the limit function of $\tilde{\chi}^{\alpha^{(n)}}$ in Condition (ii), need not be a character of G_∞ .

APPENDIX A. MARKOV CHAIN AND MARTIN BOUNDARY

This appendix is devoted to a brief review on the Martin boundary associated with a Markov chain. Such a review is supplemented because the definition (1.18) of harmonicity might seem to be strange from probabilistic viewpoints. We take [16] as a main reference, which is well written and most suitable for our purpose. Let S be a countable set. A transition probability on S is by definition a function $p(x, y)$ on $S \times S$ satisfying $p(x, y) \geq 0$ ($x, y \in S$) and $\sum_{y \in S} p(x, y) = 1$ ($x \in S$). Any transition probability induces a random motion on S , called a (temporally homogeneous) Markov chain on state space S , in which $p(x, y)$ is interpreted as the probability that the chain moves from x to y by one unit time. We have transition matrix $P = [p(x, y)]_{x, y \in S}$ by giving S a total order. For $n \in \mathbb{N} \cup \{0\}$ and $x, y \in S$, set

$$(A.1) \quad p_n(x, y) = (P^n)_{x, y} = \sum_{z_1, \dots, z_{n-1} \in S} p(x, z_1) p(z_1, z_2) \cdots p(z_{n-1}, y),$$

$$(A.2) \quad G_p(x, y) = \sum_{n=0}^{\infty} p_n(x, y).$$

Subscript p indicates dependence on the transition probability $p(x, y)$. Probability \mathbb{P} on S^∞ governing the Markov chain $(X_n)_{n=0,1,2,\dots}$ is constructed via well-known extension theorem so that

$$p_n(x, y) = \mathbb{P}(X_n = y \mid X_0 = x)$$

holds. \mathbb{E} denoting the expectation with respect to \mathbb{P} ,

$$G_p(x, y) = \mathbb{E}\left[\sum_{n=0}^{\infty} 1_{\{X_n=y\}} \mid X_0 = x\right]$$

is the expected number for the chain to visit y starting from x . Let us assume that a reference vertex o is fixed and that

$$(A.3) \quad G_p(o, y) > 0, \quad y \in S.$$

This is the case if the chain starting from o can visit any vertex with positive probability. Set

$$(A.4) \quad K_p(x, y) = \frac{G_p(x, y)}{G_p(o, y)}, \quad x, y \in S$$

and call it a Martin kernel. We define a distance on S by

$$(A.5) \quad D_p(x, y) = \sum_{z \in S} C(z) (|K_p(z, x) - K_p(z, y)| + |\delta_{z,x} - \delta_{z,y}|), \quad x, y \in S$$

where $C(z)$ is a positive coefficient for the sake of uniform convergence of the series in x and y . By virtue of the uniform convergence of the series in (A.5), we see that, for any $\epsilon > 0$, there exists a finite subset S_ϵ of S such that

$$(A.6) \quad D_p(x, y) \leq \epsilon \quad \text{whenever} \quad x \notin S_\epsilon \text{ and } y \notin S_\epsilon.$$

We can then conclude that a sequence $\{x_n\}_{n \in \mathbb{N}}$ in S is of Cauchy if and only if either of the following holds:

$$(A.7) \quad x_n \equiv x \in S \quad \text{for sufficiently large } n,$$

$$(A.8) \quad x_n \rightarrow \infty \text{ and, for any } z \in S, \{K_p(z, x_n)\}_{n \in \mathbb{N}} \text{ is a Cauchy sequence in } \mathbb{R}.$$

Moreover, any sequence in S contains a Cauchy subsequence with respect to D_p , as seen from property (A.6) and the so-called diagonal argument. This implies that S is totally bounded. The completion of S with respect to D_p is hence compact, which is called the Martin compactification of S and denoted by \bar{S} . The construction based on (A.5) assures that S is an open subset of \bar{S} and that the restriction of the topology on \bar{S} to S is discrete. The compact set $\bar{S} \setminus S = \partial S$ is the Martin boundary of S . Martin kernel (A.4) is extended as a continuous function $K_p(x, \omega)$ on $S \times \bar{S}$.

Let us now assume that non-negative function $q(x, y)$ on $S \times S$ is given instead of a transition probability. Provided that there exists a positive q -harmonic function h on S i.e.

$$(A.9) \quad h(x) > 0, \quad h(x) = \sum_{y \in S} q(x, y)h(y), \quad x \in S,$$

we set a transition probability on S by

$$(A.10) \quad p(x, y) = \frac{1}{h(x)} q(x, y)h(y), \quad x, y \in S.$$

Consider analogue of (A.1) and (A.2) for $q(x, y)$:

$$(A.11) \quad \begin{aligned} q_n(x, y) &= \sum_{z_1, \dots, z_{n-1} \in S} q(x, z_1)q(z_1, z_2) \cdots q(z_{n-1}, y), \\ G_q(x, y) &= \sum_{n=0}^{\infty} q_n(x, y). \end{aligned}$$

Comparing (A.11) with (A.2), we have

$$(A.12) \quad G_p(x, y) = \frac{1}{h(x)} G_q(x, y)h(y), \quad x, y \in S.$$

In particular, (A.3) is equivalent to the analogous condition

$$(A.13) \quad G_q(o, y) > 0, \quad y \in S.$$

Furthermore, defining a Martin kernel by

$$(A.14) \quad K_q(x, y) = \frac{G_q(x, y)}{G_q(o, y)}, \quad x, y \in S,$$

we have

$$(A.15) \quad K_p(x, y) = \frac{h(o)}{h(x)} K_q(x, y), \quad x, y \in S.$$

Set now as (A.5)

$$(A.16) \quad D_q(x, y) = \sum_{z \in S} C'(z) (|K_q(z, x) - K_q(z, y)| + |\delta_{z,x} - \delta_{z,y}|), \quad x, y \in S$$

with positive factor $C'(z)$ so as for the series to converge uniformly in x, y . (A.15) obviously yields that the Cauchy property of a sequence in S is equivalent with respect to either D_p or D_q . We thus arrive at the same Martin boundary theory starting from transition probability $p(x, y)$ or (non-transition probability) $q(x, y)$

related as (A.10). Note that function φ on S is p -harmonic if and only if $h\varphi$ is q -harmonic.

Let us turn to the branching graph $\mathbb{Y}(\widehat{T})$. Based on (non-transition probability)

$$(A.17) \quad q(\Lambda, M) = \begin{cases} \kappa(\Lambda, M), & \text{if } \Lambda \nearrow M, \\ 0, & \text{otherwise,} \end{cases} \quad \Lambda, M \in \mathbb{Y}(\widehat{T}),$$

the Martin boundary of $\mathbb{Y}(\widehat{T})$ is constructed as above. In particular, taking \emptyset as the reference vertex o , we have $G_q(\emptyset, M) = d(M) > 0$ as (A.13). An example of a q -harmonic function in (A.9) is given by the special case of $\omega = (0, 0, c) \in \Delta$ in (2.28), namely

$$(A.18) \quad h(\Lambda) = \prod_{\zeta \in \widehat{T}} \frac{c_\zeta^{|\lambda^\zeta|} \dim \lambda^\zeta}{(\dim \zeta)^{|\lambda^\zeta|} |\lambda^\zeta|!}, \quad \Lambda = (\lambda^\zeta)_{\zeta \in \widehat{T}} \in \mathbb{Y}(\widehat{T}).$$

It is straightforward to verify q -harmonicity of (A.18) with respect to q defined by (A.17), with $c_\zeta > 0$ satisfying $\sum_{\zeta \in \widehat{T}} c_\zeta = 1$. In the special case where T is trivial, (A.18) reduces to

$$(A.19) \quad h(\lambda) = \frac{\dim \lambda}{|\lambda|!}, \quad \lambda \in \mathbb{Y}$$

and hence agree with the harmonic function on \mathbb{Y} corresponding to the Plancherel measure as well as the regular character of \mathfrak{S}_∞ . Applying (A.10) and (A.17) to (A.19), we have

$$p(\lambda, \mu) = \frac{1}{n+1} \frac{\dim \mu}{\dim \lambda}, \quad \lambda \in \mathbb{Y}_n \nearrow \mu \in \mathbb{Y}_{n+1}.$$

This induces a chain called the Plancherel growth process. If we take h in (A.9) to be minimal q -harmonic (as is the case of (A.18) under (A.17)), then a bounded non-negative p -harmonic function with respect to p in (A.10) is necessarily constant, which follows from minimality of h as a q -harmonic function.

Martin kernel (A.14) is denoted by K in (1.27). A straightforward translation of the discussion above in this appendix leads to (1.29) – (1.34).

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