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On the spectra of fermionic second quantization operators

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Abstract

We derive several formulae for the spectra of the second quantization operators in abstract fermionic Fock spaces.

1 Introduction

Abstract theory of Fock spaces [1, 2, 3, 4] provides powerful mathematical tools when one analyzes models of quantum field theory, the most promising physical theory which is expected to describe the fundamental interactions of elementary particles. This results from the fact that quantum field theory deals with a quantum system with infinitely many degrees of freedom, including particles which may be created or annihilated, and that Fock spaces are furnished with suitable structure to describe particle creation or annihilation. In mathematical physics, two different types of Fock spaces, bosonic (or symmetric) Fock space and fermionic (or antisymmetric) Fock spaces, are considered, reflecting the fact that there are two different sorts of elementary particles in Nature — bosons and fermions — .

In mathematical analyses of quantum theories, one of the most important problems includes to determine the spectra of various self-adjoint operators representing physical observables, especially, that of a Hamiltonian, which represents the total energy of the system under consideration. To each self-adjoint operator A acting in an underlying one particle Hilbert space \mathcal{H} , bosonic or fermionic second quantization is defined as an operator which naturally “lifts” A up to the bosonic or fermionic Fock space over \mathcal{H} , respectively. In a bosonic Fock space, the spectra of second quantization operators were well investigated and useful formulae for them have been available. However, as far as we know, the corresponding useful formulae in a fermionic Fock space are still missing. The main motivation of the present work is to derive such formulae in fermionic Fock spaces to fill the gap.

Let \mathcal{H} be an infinite dimensional separable Hilbert space over \mathbb{C} with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$ (we omit the subscript \mathcal{H} if there will be no danger of confusion). For a linear operator T on \mathcal{H} , we denote its domain by $D(T)$. For a subspace $D \subset D(T)$, the symbol $T \upharpoonright D$ denotes the restriction of T to D . We denote by \bar{T} the closure of T if T is closable. The spectrum (resp. the point spectrum) of T is denoted by $\sigma(T)$ (resp. $\sigma_p(T)$). The symbol $\otimes^n \mathcal{H}$ (resp. $\wedge^n \mathcal{H}$) denotes the n -fold tensor product of \mathcal{H} (resp. the n -fold antisymmetric tensor product). Let \mathfrak{S}_n be the symmetric group of order n . The antisymmetrization operator \mathcal{A}_n on $\otimes^n \mathcal{H}$ is defined to be

$$\mathcal{A}_n := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) U_{\sigma},$$

where U_{σ} is a unitary operator on $\otimes^n \mathcal{H}$ such that $U_{\sigma}(\psi_1 \otimes \cdots \otimes \psi_n) = \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(n)}$, $\psi_j \in \mathcal{H}$, $j = 1, \dots, n$, and $\text{sgn}(\sigma)$ is the signature of the permutation $\sigma \in \mathfrak{S}_n$. Then, \mathcal{A}_n is an orthogonal projection onto

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$\wedge^n \mathcal{H}$. The fermionic Fock space over \mathcal{H} is defined by

$$\mathcal{F}_f(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \wedge^n \mathcal{H} := \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \mid \Psi^{(n)} \in \wedge^n \mathcal{H}, \sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2 < \infty \right\}.$$

For a densely defined closable operator A on \mathcal{H} and $j = 1, \dots, n$, we define a linear operator \tilde{A}_j on $\otimes^n \mathcal{H}$ by

$$\tilde{A}_j := I \otimes \cdots \otimes I \otimes \overset{j\text{-th}}{\tilde{A}} \otimes I \otimes \cdots \otimes I,$$

where I denotes the identity. For each $n \in \{0\} \cup \mathbb{N}$, a linear operator $A^{(n)}$ on $\otimes^n \mathcal{H}$ is defined by

$$A^{(0)} := 0, \quad A^{(n)} := \overline{\sum_{j=1}^n \tilde{A}_j \upharpoonright \otimes_{\text{alg}}^n D(A)}, \quad n \geq 1,$$

where $\otimes_{\text{alg}}^n D(A)$ means the n -fold algebraic tensor product of $D(A)$. Denote the reduced part of $A^{(n)}$ (resp. $\otimes^n A$) to $\wedge^n \mathcal{H}$ by $A_f^{(n)}$ (resp. $\wedge^n A$). The infinite direct sum of these closed operators

$$d\Gamma_f(A) := \bigoplus_{n=0}^{\infty} A_f^{(n)}$$

is called the first type fermionic second quantization of A , while the direct sum

$$\Gamma_f(A) := \bigoplus_{n=0}^{\infty} \wedge^n A$$

is the second type.

2 Main Results

For a linear operator T on \mathcal{H} , $\sigma_d(T)$ denotes the discrete spectrum of T . We introduce the notation

$$t(\lambda; \{\lambda_1, \dots, \lambda_n\}) := \#\{j \mid \lambda = \lambda_j\}, \quad (2.1)$$

or, in words, $t(\lambda; \{\lambda_1, \dots, \lambda_n\})$ represents how many λ 's appear in the set $\{\lambda_1, \dots, \lambda_n\}$. The main results of the present paper are summarized in the following theorems:

Theorem 2.1. *Let T be a self-adjoint operator on \mathcal{H} . Then, the following (i), (ii) hold.*

(i). *The point spectrum of $T_f^{(n)}$ is given by*

$$\sigma_p(T_f^{(n)}) = \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_p(T) (j = 1, 2, \dots, n), t(\lambda_j; \{\lambda_1, \dots, \lambda_n\}) \leq \dim \ker(T - \lambda_j) \right\}. \quad (2.2)$$

(ii). *If $0 \notin \sigma_p(T)$, then the point spectrum of $\wedge^n T$ is given by*

$$\sigma_p(\wedge^n T) = \left\{ \prod_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_p(T) (j = 1, 2, \dots, n), t(\lambda_j; \{\lambda_1, \dots, \lambda_n\}) \leq \dim \ker(T - \lambda_j) \right\}. \quad (2.3)$$

If $0 \in \sigma_p(T)$, then it is given by

$$\sigma_p(\wedge^n T) = \{0\} \cup \left\{ \prod_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_p(T) (j = 1, 2, \dots, n), t(\lambda_j; \{\lambda_1, \dots, \lambda_n\}) \leq \dim \ker(T - \lambda_j) \right\}. \quad (2.4)$$

Theorem 2.2. *Let T be a self-adjoint operator on \mathcal{H} . Then, the following (i), (ii) hold.*

(i). The spectrum of $T_{\mathbf{f}}^{(n)}$ is given by

$$\sigma(T_{\mathbf{f}}^{(n)}) = \overline{\left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma(T) (j = 1, 2, \dots, n), \text{ if } \lambda_j \in \sigma_d(T), t(\lambda_j; \{\lambda_1, \dots, \lambda_n\}) \leq \dim \ker(T - \lambda_j) \right\}}. \quad (2.5)$$

(ii). The spectrum of $\wedge^n T$ is given by

$$\sigma(\wedge^n T) = \overline{\left\{ \prod_{j=1}^n \lambda_j \mid \lambda_j \in \sigma(T) (j = 1, 2, \dots, n), \text{ if } \lambda_j \in \sigma_d(T), t(\lambda_j; \{\lambda_1, \dots, \lambda_n\}) \leq \dim \ker(T - \lambda_j) \right\}}. \quad (2.6)$$

3 Proof of the Theorems

$\mathcal{H}_p(T)$ denotes the closed linear subspace spanned by the eigenvectors of a linear operator T on \mathcal{H} .

Lemma 3.1. *If T is a self-adjoint operator, then*

(i).

$$\mathcal{H}_p(T^{(n)}) = \otimes^n \mathcal{H}_p(T), \quad (3.1)$$

(ii).

$$\mathcal{H}_p(\otimes^n T) = \left[\otimes^n (\mathcal{H}_p(T) \cap (\ker T)^\perp) \right] \oplus \ker(\otimes^n T). \quad (3.2)$$

Proof. (i) Let $\sigma_c(T) := \sigma(T) \setminus \sigma_p(T)$ and let $\mathcal{H}_c(T) := \text{Ran } E_T(\sigma_c(T))$, where $E_T(\cdot)$ is the one dimensional spectral measure of T . Then, we obtain the direct sum decomposition $T = T_p \oplus T_c$ corresponding to the decomposition $\mathcal{H} = \mathcal{H}_p(T) \oplus \mathcal{H}_c(T)$. Then, one can show that

$$\begin{aligned} T^{(n)} &= \overline{\bigoplus_{\#_j=p \text{ or } c} \sum_{j=1}^n I_{\#_1} \otimes \cdots \otimes I_{\#_{j-1}} \otimes T_{\#_j} \otimes I_{\#_{j+1}} \otimes \cdots \otimes I_{\#_n}} \\ &= \bigoplus_{\#_j=p \text{ or } c} S_{\#_1, \dots, \#_n}, \end{aligned} \quad (3.3)$$

where $I_{\#_j}$ is the identity operator in $\mathcal{H}_{\#_j}(T)$, and

$$S_{\#_1, \dots, \#_n} := \sum_{j=1}^n I_{\#_1} \otimes \cdots \otimes I_{\#_{j-1}} \otimes T_{\#_j} \otimes I_{\#_{j+1}} \otimes \cdots \otimes I_{\#_n}.$$

Note that T_c have no eigenvalues. Since $\sigma_p(T)$ and $\sigma_c(T)$ are disjoint, one finds that, if $\#_j = c$ for some j , then $\mathcal{H}_p(S_{\#_1, \dots, \#_n}) = \{0\}$ by using Lemma A.1. Hence, in the direct sum decomposition in equation (3.3), only the term with $\#_j = p$, for all j , is nontrivial, which implies

$$\begin{aligned} \mathcal{H}_p(T^{(n)}) &= \bigoplus_{\#_j=p \text{ or } c} \mathcal{H}_p(S_{\#_1, \dots, \#_n}) \\ &= \mathcal{H}_p(S_{p, \dots, p}). \end{aligned}$$

Since $S_{p, \dots, p}$ is an operator in $\otimes^n \mathcal{H}_p(T)$, it immediately follows that $\mathcal{H}_p(S_{p, \dots, p}) \subset \otimes^n \mathcal{H}_p(T)$.

Conversely, let $\psi = \psi_1 \otimes \cdots \otimes \psi_n \in \otimes^n \mathcal{H}_p(T)$ with $\psi_j \in \ker(T - \lambda_j)$. Then, direct computation shows

$$\begin{aligned} T^{(n)}\psi &= \sum_{j=1}^n \psi_1 \otimes \cdots \otimes T\psi_j \otimes \cdots \otimes \psi_n \\ &= \sum_{j=1}^n \psi_1 \otimes \cdots \otimes \lambda_j \psi_j \otimes \cdots \otimes \psi_n \\ &= \sum_{j=1}^n \lambda_j \psi. \end{aligned} \quad (3.4)$$

Thus, we see $\psi \in \ker(T^{(n)} - \lambda)$ with $\lambda = \sum_j \lambda_j$, and especially $\psi \in \mathcal{H}_p(T^{(n)})$. Since the closed linear subspace spanned by such ψ 's is $\otimes^n \mathcal{H}_p(T)$ and $\mathcal{H}_p(T^{(n)})$ is closed, the converse inclusion follows. This proves (3.1).

(ii) Let T_0 and T_1 be the reduced parts of T by $\ker T$ and $(\ker T)^\perp$ respectively. Then, we have a direct sum decomposition of $\mathcal{H}_p(\otimes^n T)$:

$$\mathcal{H}_p(\otimes^n T) = \bigoplus_{b_j=0 \text{ or } 1} \mathcal{H}_p(T_{b_1} \otimes \cdots \otimes T_{b_n}).$$

From this, we learn

$$\mathcal{H}_p(\otimes^n T) = \mathcal{H}_p(\otimes^n T_1) \oplus \ker(\otimes^n T), \quad (3.5)$$

because, if $b_j = 0$ for some j , then $T_{b_1} \otimes \cdots \otimes T_{b_n}$ is a null operator.

Now, we will show, in general, that for self-adjoint operators A, B , whose point spectra do *not* contain zero,

$$\mathcal{H}_p(A \otimes B) = \mathcal{H}_p(A) \otimes \mathcal{H}_p(B). \quad (3.6)$$

In the same manner as in the proof of (i), we use the direct sum decompositions $A = A_p \oplus A_c$ and $B = B_p \oplus B_c$ to obtain a decomposition of $\mathcal{H}_p(A \otimes B)$:

$$\mathcal{H}_p(A \otimes B) = \mathcal{H}_p(A_p \otimes B_p) \oplus \mathcal{H}_p(A_p \otimes B_c) \oplus \mathcal{H}_p(A_c \otimes B_p) \oplus \mathcal{H}_p(A_c \otimes B_c). \quad (3.7)$$

But, by Lemma A.1, we have $\mathcal{H}_p(A_c \otimes B_c) = \{0\}$. Moreover, by the same Lemma, we also have $\mathcal{H}_p(A_p \otimes B_c) = \{0\}$ and $\mathcal{H}_p(A_c \otimes B_p) = \{0\}$, because we have assumed that $0 \notin \sigma_p(A)$ and $0 \notin \sigma_p(B)$. Therefore, one finds

$$\mathcal{H}_p(A \otimes B) = \mathcal{H}_p(A_p \otimes B_p). \quad (3.8)$$

Since the operator $A_p \otimes B_p$ acts in $\mathcal{H}_p(A) \otimes \mathcal{H}_p(B)$, it is clear that $\mathcal{H}_p(A_p \otimes B_p) \subset \mathcal{H}_p(A) \otimes \mathcal{H}_p(B)$, which means

$$\mathcal{H}_p(A \otimes B) \subset \mathcal{H}_p(A) \otimes \mathcal{H}_p(B).$$

The converse inclusion follows from the similar discussion given in (i). Thus, we prove (3.6).

The above general discussion shows that $\mathcal{H}_p(\otimes^n T_1) = \otimes^n \mathcal{H}_p(T_1)$ since T_1 does not have zero eigenvalue. Substituting this equation in (3.5), we obtain (3.2). ▀

Proof of Theorem 2.1. (i) For vectors $\zeta_1, \dots, \zeta_n \in \mathcal{H}$, we define the wedge product of these vectors by

$$\zeta_1 \wedge \cdots \wedge \zeta_n := \sqrt{n!} \mathcal{A}_n(\zeta_1 \otimes \cdots \otimes \zeta_n).$$

Let $\{\xi_k\}_k$ be a complete orthonormal system (CONS) of $\mathcal{H}_p(T)$ consisting of eigenvectors of T . Then, as is well known, the family

$$\Lambda := \{\xi_{k_1} \wedge \cdots \wedge \xi_{k_n} \mid k_1 < \cdots < k_n\} \quad (3.9)$$

forms a CONS of $\wedge^n \mathcal{H}_p(T)$, and each element is obviously an eigenvector of $T^{(n)}$. By the reducibility and Lemma 3.1 (i), we have

$$\mathcal{H}_p(T_f^{(n)}) = \mathcal{A}_n \mathcal{H}_p(T^{(n)}) = \mathcal{A}_n \otimes^n \mathcal{H}_p(T) = \wedge^n \mathcal{H}_p(T). \quad (3.10)$$

We claim

$$\sigma_p(T_f^{(n)}) = \left\{ \lambda \in \mathbb{R} \mid \text{There exists } \eta \in \Lambda \text{ such that } T_f^{(n)} \eta = \lambda \eta \right\}. \quad (3.11)$$

The right hand side is clearly included by the left hand side. To prove the converse, let $\lambda \in \sigma_p(T_f^{(n)})$ with an eigenvector ψ :

$$T_f^{(n)} \psi = \lambda \psi.$$

Take the inner product with $\xi_{k_1} \wedge \cdots \wedge \xi_{k_n}$ to obtain

$$\left(\sum_{j=1}^n \lambda_{k_j} - \lambda \right) \langle \xi_{k_1} \wedge \cdots \wedge \xi_{k_n}, \psi \rangle = 0, \quad (3.12)$$

for all $k_1 < \cdots < k_n$, where λ_{k_j} is an eigenvalue of T to which ξ_{k_j} belongs. By equation (3.10), Λ is a CONS of $\mathcal{H}_p(T_f^{(n)})$, and thus, for at least one choice of (k_1, \dots, k_n) , $\langle \xi_{k_1} \wedge \cdots \wedge \xi_{k_n}, \psi \rangle \neq 0$. This and equation (3.12) imply

$$\lambda = \sum_{j=1}^n \lambda_{k_j},$$

for such (k_1, \dots, k_n) . Since $\sum_{j=1}^n \lambda_{k_j}$ is an eigenvalue of $T_f^{(n)}$ to which the eigenvector $\xi_{k_1} \wedge \cdots \wedge \xi_{k_n}$ belongs, λ is an element of the right hand side of (3.11). This proves (3.11), but the right hand side of (3.11) is exactly the same set that appears in the right hand side of (2.2). Then, the proof is completed.

(ii) In the case where $0 \notin \sigma_p(T)$, we can prove (2.3) by using Lemma 3.1 (ii) in the same way as in the proof of (2.2).

Next, suppose $0 \in \sigma_p(T)$. By the reducibility and Lemma 3.1 (ii), we have

$$\mathcal{H}_p(\wedge^n T) = (\wedge^n \mathcal{H}_p(T_1)) \oplus \mathcal{A}_n \ker(\otimes^n T). \quad (3.13)$$

Let $\{\xi_k\}_k$ be a CONS of $\mathcal{H}_p(T_1)$ and

$$\Lambda_1 := \{\xi_{k_1} \wedge \cdots \wedge \xi_{k_n} \mid k_1 < \cdots < k_n\},$$

then Λ_1 forms a CONS of $\wedge^n \mathcal{H}_p(T_1)$. We claim

$$\sigma_p(\wedge^n T) \setminus \{0\} = \left\{ \lambda \in \mathbb{R} \setminus \{0\} \mid \text{There exists } \eta \in \Lambda_1 \text{ such that } \wedge^n T \eta = \lambda \eta \right\}. \quad (3.14)$$

Since the right hand side is clearly included by the left, it suffices to prove that, for each $\lambda \in \sigma_p(\wedge^n T) \setminus \{0\}$, there is an $\eta \in \Lambda_1$ such that $\wedge^n T \eta = \lambda \eta$. Let $\lambda \in \sigma_p(\wedge^n T) \setminus \{0\}$. Then, there exists a $\psi \in \mathcal{H}_p(\wedge^n T)$ satisfying

$$\wedge^n T \psi = \lambda \psi.$$

But since $\lambda \neq 0$, we may assume $\psi \in \wedge^n \mathcal{H}_p(T_1)$ by (3.13). Taking the inner product with $\xi_{k_1} \wedge \cdots \wedge \xi_{k_n}$ on both sides, one obtains

$$\left(\prod_{j=1}^n \lambda_{k_j} - \lambda \right) \langle \xi_{k_1} \wedge \cdots \wedge \xi_{k_n}, \psi \rangle = 0, \quad (3.15)$$

for all $k_1 < \cdots < k_n$, where λ_{k_j} is a non-zero eigenvalue of T to which ξ_{k_j} belongs. By noting that $\psi \in \wedge^n \mathcal{H}_p(T_1)$, and Λ_1 is a CONS of $\wedge^n \mathcal{H}_p(T_1)$, we learn that for at least one of (k_1, \dots, k_n) 's $\langle \xi_{k_1} \wedge \cdots \wedge \xi_{k_n}, \psi \rangle \neq 0$. Thus, we have

$$\lambda = \prod_{j=1}^n \lambda_{k_j}$$

for such (k_1, \dots, k_n) 's. But since $\prod_{j=1}^n \lambda_{k_j}$ is an eigenvalue to which the eigenvector $\xi_{k_1} \wedge \cdots \wedge \xi_{k_n} \in \Lambda_1$ belongs, we proved the claim (3.14).

The right hand side of (3.14) is rewritten as

$$\left\{ \prod_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_p(T) \setminus \{0\}, t(\lambda_j; \{\lambda_1, \dots, \lambda_n\}) \leq \dim \ker(T - \lambda_j) \right\}, \quad (3.16)$$

and therefore, we have proved

$$\sigma_p(\wedge^n T) \setminus \{0\} = \left\{ \prod_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_p(T) \setminus \{0\}, t(\lambda_j; \{\lambda_1, \dots, \lambda_n\}) \leq \dim \ker(T - \lambda_j) \right\}, \quad (3.17)$$

which implies (2.4). ▮

Proof of Theorem 2.2. (i) Let

$$\Sigma_n := \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma(T), \text{ if } \lambda_j \in \sigma_d(T), t(\lambda_j; \{\lambda_1, \dots, \lambda_n\}) \leq \dim \ker(T - \lambda_j) \right\}$$

be the right hand side of (2.5) without closure.

First, we prove the left hand side includes the right. Let $\lambda = \sum_{j=1}^n \lambda_j$, $\lambda_j \in \sigma(T)$, and if $\lambda_j \in \sigma_d(T)$, $t(\lambda_j; \{\lambda_1, \dots, \lambda_n\}) \leq \dim \ker(T - \lambda_j)$. Choose $\varepsilon_0 > 0$ in such a way that $\lambda_j \neq \lambda_k$ implies $U_{\varepsilon_0}(\lambda_j) \cap U_{\varepsilon_0}(\lambda_k) = \emptyset$, where $U_\varepsilon(\lambda)$ is the ε -neighborhood of λ . Then, for all ε with $0 < \varepsilon \leq \varepsilon_0$, there exists an orthonormal set $\{\psi_j\}_{j=1}^n \subset D(T)$ such that $\psi_j \in \text{Ran } E_T(U_\varepsilon(\lambda_j))$, $\|(T - \lambda_j)\psi\| < \varepsilon$. Then, we obtain

$$\begin{aligned} & \| (T_f^{(n)} - \lambda)\psi_1 \wedge \dots \wedge \psi_n \| \\ & \leq \left\| \sum_{j=1}^n \psi_1 \wedge \dots \wedge (T - \lambda_j)\psi_j \wedge \dots \wedge \psi_n \right\| \\ & \leq \sqrt{n!} \varepsilon. \end{aligned}$$

Hence, $\lambda \in \sigma(T_f^{(n)})$.

Next, in order to prove the converse inclusion, we will show that there exists a CONS $\{\Psi_k\}_{k=1}^\infty \subset \bigwedge^n \mathcal{H}$ satisfying the following condition:

$$\text{supp } \mu_{\Psi_k} \subset \overline{\Sigma_n}, \quad k = 1, 2, \dots, \quad (3.18)$$

where $\mu_{\Psi_k}(B) := \|E_{T_f^{(n)}}(B)\Psi_k\|^2$ for a one dimensional borel set $B \in \mathcal{B}^1$. Then, it immediately follows that $\bigwedge^n \mathcal{H} = \overline{\mathcal{L}(\{\Psi_k\}_k)} \subset \text{Ran } E_{T_f^{(n)}}(\overline{\Sigma_n})$. Here, $\overline{\mathcal{L}(\{\Psi_k\}_k)}$ denotes the closed linear subspace spanned by $\{\Psi_k\}_k$. Hence, we obtain $E_{T_f^{(n)}}(\overline{\Sigma_n}) = I$, and this implies $\sigma(T_f^{(n)}) = \text{supp } E_{T_f^{(n)}} \subset \overline{\Sigma_n}$ by definition of support of a spectral measure.

Now, we shall show (3.18). Denote the essential spectrum of T by $\sigma_{\text{ess}}(T)$ and introduce the notations $\mathcal{H}_d(T) := \text{Ran } E_f^{(n)}(\sigma_d(E_f^{(n)}))$ and $\mathcal{H}_{\text{ess}}(T) := \text{Ran } E_f^{(n)}(\sigma_{\text{ess}}(E_f^{(n)}))$. Let $\{\zeta_k\}_k$ be a CONS of $\mathcal{H}_d(T)$ consisting of eigenvectors of T , and let $\{\eta_l\}_j$ be a CONS of $\mathcal{H}_{\text{ess}}(T)$. Then, the set

$$\Lambda := \left\{ \zeta_{k_1} \wedge \dots \wedge \zeta_{k_N} \wedge \eta_{l_1} \wedge \dots \wedge \eta_{l_{N'}} \mid k_1 < \dots < k_N, l_1 < \dots < l_{N'}, N + N' = n \right\}$$

forms a CONS of $\bigwedge^n \mathcal{H}$. Fix some $\Psi = \zeta_{k_1} \wedge \dots \wedge \zeta_{k_N} \wedge \eta_{l_1} \wedge \dots \wedge \eta_{l_{N'}} \in \Lambda$. In addition, let $\nu_{k_i}(B) := \|E_T(B)\zeta_{k_i}\|^2$ ($i = 1, \dots, N$), $\nu_{l_j}(B) := \|E_T(B)\eta_{l_j}\|^2$ ($j = 1, \dots, N'$) for $B \in \mathcal{B}^1$, and $B_i := \text{supp } \nu_{k_i}$, $B_{N+j} := \text{supp } \nu_{l_j}$. Let

$$\begin{aligned} J & := \bigcup_{\sigma \in \mathfrak{S}_n} B_{\sigma(1)} \times \dots \times B_{\sigma(n)}, \\ J_{\Sigma_n} & := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1 + \dots + \lambda_n \in \Sigma_n\}, \end{aligned}$$

and $E_T^n(\cdot)$ be an n -dimensional spectral measure acting in $\otimes^n \mathcal{H}$ defined by the relation

$$E_T^n(B_1 \times \dots \times B_n) = E_T(B_1) \otimes \dots \otimes E_T(B_n), \quad B_1, \dots, B_n \in \mathcal{B}^1.$$

Then, we see that $J \subset J_{\Sigma_n}$ by the construction of Λ . By direct computation, we have $E_T^n(J)\Psi = \Psi$. Therefore, it follows that $E_T^n(J_{\Sigma_n})\Psi = \Psi$ and one finds

$$\begin{aligned} E_{T_f^{(n)}}(\Sigma_n)\Psi & = E_{T^{(n)}}(\Sigma_n)\mathcal{A}_n\Psi = E_T^n(J_{\Sigma_n})\Psi \\ & = \Psi, \end{aligned}$$

where Lemma A.2 is used to obtain the second equality. This shows that $\text{supp } \mu_\Psi \subset \overline{\Sigma_n}$, and therefore, Λ is a desired CONS satisfying (3.18), completing the proof of (2.5).

(ii) The proof is very similar to that of (i), and we omit it. ▮

As a corollary of these theorems, we can derive the formulae for spectra of the fermionic second quantization operators. In what follows, we will use the simpler notation $t(\lambda_j)$ in place of $t(\lambda_j; \{\lambda_1, \dots, \lambda_n\})$ for notational simplicity.

Corollary 3.2. *Let T be a self-adjoint operator. Then, the following (i), (ii) hold.*

(i). *The spectrum and the point spectrum of the first type fermionic second quantization operator of T are given by*

$$\begin{aligned}\sigma(\mathrm{d}\Gamma_f(T)) &= \{0\} \cup \overline{\left(\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma(T), j = 1, \dots, n, \text{ if } \lambda_j \in \sigma_d(T), t(\lambda_j) \leq \dim \ker(T - \lambda_j) \right\} \right)}, \\ \sigma_p(\mathrm{d}\Gamma_f(T)) &= \{0\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_p(T), j = 1, \dots, n, t(\lambda_j) \leq \dim \ker(T - \lambda_j) \right\} \right).\end{aligned}$$

(ii). *As to the second type fermionic second quantization operator of T , one has*

$$\sigma(\Gamma_f(T)) = \{1\} \cup \overline{\left(\bigcup_{n=1}^{\infty} \left\{ \prod_{j=1}^n \lambda_j \mid \lambda_j \in \sigma(T), j = 1, \dots, n, \text{ if } \lambda_j \in \sigma_d(T), t(\lambda_j) \leq \dim \ker(T - \lambda_j) \right\} \right)}.$$

If $0 \notin \sigma_p(T)$, then

$$\sigma_p(\Gamma_f(T)) = \{1\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_p(T), j = 1, \dots, n, t(\lambda_j) \leq \dim \ker(T - \lambda_j) \right\} \right).$$

If $0 \in \sigma_p(T)$, then

$$\sigma_p(\Gamma_f(T)) = \{0\} \cup \{1\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_p(T), j = 1, \dots, n, t(\lambda_j) \leq \dim \ker(T - \lambda_j) \right\} \right).$$

4 Example — Kinetic energy of free fermions in a finite box —

Let \mathcal{F} be the fermionic Fock space over $\mathcal{H} = l^2(\Gamma_L; \mathbb{C}^4)$, where

$$\Gamma_L := \frac{2\pi}{L} \mathbb{Z}^3, \quad L > 0.$$

This Hilbert space \mathcal{H} consists of quantum mechanical state vectors of one Dirac fermion in momentum representation living in a finite volume box $[-L/2, L/2]^3 \subset \mathbb{R}^3$. As a one particle Hamiltonian, we adopt a multiplication operator by a function E_M :

$$E_M(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M^2}, \quad \mathbf{p} \in \Gamma_L,$$

where $\mathbf{p} \in \mathbb{R}^3$ is a spacial momentum of a Dirac particle and $M \geq 0$ is a constant representing a bare mass of a Dirac particle. On a spinor space \mathbb{C}^4 , E_M acts as a diagonal matrix.

The spectrum of E_M is given as follows:

Lemma 4.1. (i). *The spectrum of E_M is given by*

$$\sigma(E_M) = \sigma_d(E_M) = \left\{ \sqrt{\mathbf{p}^2 + M^2} \mid \mathbf{p} \in \Gamma_L \right\}. \quad (4.1)$$

(ii). *The multiplicity of eigenvalue λ is given by*

$$\dim \ker(E_M - \lambda) = 4r \left(\frac{L^2}{4\pi^2} (\lambda^2 - M^2) \right), \quad (4.2)$$

where $r(N)$ denotes

$$r(N) := \#\{\mathbf{n} \in \mathbb{Z}^3 \mid N = \mathbf{n}^2\}.$$

Proof. Since (i) is well known, we will prove only (ii).

For each $(\mathbf{p}, l) \in \Gamma_L \times \{1, 2, 3, 4\}$, let

$$\delta_{\mathbf{p}}^l(\mathbf{q}, m) = \delta_{\mathbf{p}\mathbf{q}}\delta_{lm}, \quad \mathbf{q} \in \Gamma_L, m = 1, 2, 3, 4.$$

Then, $\{\delta_{\mathbf{p}}^l\}_{\mathbf{p}, l}$ forms a CONS of \mathcal{H} under natural identification $\mathcal{H} = l^2(\Gamma_L \times \{1, 2, 3, 4\})$. From a general theory of multiplication operators, we have

$$\ker(E_M - \lambda) = \{\psi \in D(E_M) \mid \psi(\mathbf{p}) \neq 0 \text{ implies } \sqrt{\mathbf{p}^2 + M^2} = \lambda\}. \quad (4.3)$$

This means that ψ is an eigenvector of E_M if and only if it belongs to the linear subspace spanned by

$$\{\delta_{\mathbf{p}}^l \in \mathcal{H} \mid \sqrt{\mathbf{p}^2 + M^2} = \lambda, l = 1, 2, 3, 4\}.$$

Since the above vectors are linearly independent, we find

$$\begin{aligned} \dim \ker(E_M - \lambda) &= 4 \cdot \#\{\mathbf{p} \in \Gamma_L \mid \sqrt{\mathbf{p}^2 + M^2} = \lambda\} \\ &= 4 \cdot r \left(\frac{L^2}{4\pi^2} (\lambda^2 - M^2) \right). \end{aligned} \quad (4.4)$$

■

From Lemma 4.1 and Corollary 3.2, we finally arrive at the formula for the second quantization operator $d\Gamma_{\mathbf{f}}(E_M)$ acting in \mathcal{F} :

Theorem 4.2.

$$\sigma(d\Gamma_{\mathbf{f}}(E_M)) = \{0\} \cup \left(\overline{\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^n \sqrt{\frac{4\pi^2}{L^2} N_j + M^2} \mid 1 \leq r(N_j), t(N_j; \{N_1, \dots, N_n\}) \leq 4r(N_j), j = 1, \dots, n \right\}} \right). \quad (4.5)$$

The crucial difference from the bosonic case is the existence of the restriction $t(N_j; \{N_1, \dots, N_n\}) \leq 4r(N_j)$ reflecting Pauli's exclusion principle. It would be interesting to note that, if Dirac fermions are not contained in a finite box but live in \mathbb{R}^3 , then there is no restriction because the spectrum of E_M in this case consists only of essential spectra.

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A Appendix

We will collect well known facts in abstract Fock spaces used in this paper. Detailed proofs will be found in [1].

Lemma A.1. *Let A_j ($j = 1, \dots, n$) be self-adjoint operators on a separable Hilbert space \mathcal{H}_i . Then,*

(i)

$$\sigma_{\mathbf{p}} \left(\overline{\sum_{j=1}^n \tilde{A}_j} \right) = \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_{\mathbf{p}}(A_j), j = 1, \dots, n \right\}.$$

(ii) *If $0 \notin \sigma_{\mathbf{p}}(A_j)$ for all A_j , then*

$$\sigma_{\mathbf{p}} \left(\bigotimes_{j=1}^n A_j \right) = \left\{ \prod_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_{\mathbf{p}}(A_j), j = 1, \dots, n \right\}.$$

If $0 \in \sigma_{\mathbb{P}}(A_j)$ for a A_j , then

$$\sigma_{\mathbb{P}}\left(\bigotimes_{j=1}^n A_j\right) = \{0\} \cup \left\{ \prod_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_{\mathbb{P}}(A_j), j = 1, \dots, n \right\}.$$

Lemma A.2. Let A_j ($j = 1, \dots, n$) be self-adjoint operators on a Hilbert space \mathcal{H}_i . Let $E := E_{\tilde{A}_1} \times \dots \times E_{\tilde{A}_n}$ be the product measure and \mathcal{B}_1 be the Borel field of \mathbb{R} . Then,

(i)

$$E_{\Sigma}(J) := E\left(\left\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \sum_{j=1}^n \lambda_j \in J\right\}\right), \quad J \in \mathcal{B}^1$$

is the spectral measure of $\overline{\sum_{j=1}^n \tilde{A}_j}$.

(ii)

$$E_{\otimes}(J) := E\left(\left\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \prod_{j=1}^n \lambda_j \in J\right\}\right), \quad J \in \mathcal{B}^1$$

is the spectral measure of $\otimes_{j=1}^n A_j$.

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