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TOPOLOGICAL INSTABILITY OF LAMINAR FLOWS FOR THE TWO-DIMENSIONAL NAVIER-STOKES EQUATION WITH CIRCULAR ARC NO-SLIP BOUNDARY CONDITIONS

TSUYOSHI YONEDA

ABSTRACT. In general, before separating from a boundary, the flow moves toward reverse direction near the boundary against the laminar flow direction. Here in this paper, creation of the reverse flow phenomena (in a mathematical sense) is observed. More precisely, in the non-stationary two-dimensional Navier-Stokes equation with circular arc no-slip boundary conditions and diffusing laminar initial data (we define them rigorously in the paper), topologically changing flow, namely, instability is observed.

Key words: Navier-Stokes equation, laminar flow, no-slip boundary condition

AMS Subject Classification (2010): 35Q30, 76D05, 76D10, 53A04

1. INTRODUCTION

Ohya and Karasudani [14] developed a new wind turbine system that consists of a diffuser shroud with a broad-ring at the exit periphery and a wind turbine inside it. Their experiments show that a diffuser-shaped (not nozzle-shaped) structure can accelerate the wind at the entrance of the body (we say “wind-lends phenomena”). A strong vortex formation with a low-pressure region is created behind the broad brim. Accordingly, the wind flows into a low-pressure region, the wind velocity is accelerated further near the entrance of the diffuser. In general, creation of a vortex needs separation phenomena near a boundary (namely, topologically changing phenomena), and before separating from the boundary, the flow moves toward reverse direction near the boundary against the laminar flow direction.

In “boundary layer theory” (BLT) point of view, such phenomena itself is well studied. Our main purpose is just propose “local pressure analysis method” through (well-known) separation phenomena. In the beginning of 20th century, Prandtl proposed BLT, and it has been developing extensively (see Rosenhead [15] and Bakker [1] for example). Basically, BLT equations can be deduced from the Navier-Stokes equations. Van Dommelen and Shen [3] made a key observation of shock singularities, which helps us to analyze separation phenomena deeply. Ma and Wang [10] provided a characterization of the boundary layer separation of 2-D incompressible viscous fluids. They considered a separation equation linking a separation location and a time with the Reynolds number, the external forcing and the initial velocity field. Due to the limitation of space and the vast literature in the BLT, we do not try to do a complete survey here.

However, we need to mention the results related to the BLT (in other words, wake region) in pure mathematics. Using the Oseen system is one of the mathematical approach to analyze the wake region. For the detailed discussion of the Oseen system, we refer the

reader to [7]. In a convex obstacle case, the character of the system is elliptic in front of the obstacle. To the contrary, its character changes into parabolic type (wake region) behind the obstacle (see [9] for example). Maekawa [11] considered the two-dimensional Navier-Stokes equations in a half plane under the no-slip boundary condition. He established a solution formula for the vorticity equations and got a sufficient condition on the initial data for the vorticity to blow up to the inviscid limit.

In this paper we show that a diffuser-shaped boundary induces the reverse flow even near the entrance of the diffuser (by using “local pressure analysis method”). Let us be more precise. We consider the two-dimensional Navier-Stokes equation in $\Omega \subset \mathbb{R}^2$ (define Ω later) with no-slip and inflow-outflow conditions on $\partial\Omega$. We need to handle a shape of the boundary $\partial\Omega$ precisely, thus we set a parametrized smooth boundary $\varphi : [0, S] \rightarrow \mathbb{R}^2$ as $|\partial_s \varphi(s)| = 1$, $|\partial_s^2 \varphi(s)| = \kappa$ (curvature), $\varphi(0) = (0, 0)$, $\partial_s \varphi(0) = (1, 0)$, $\partial_s^2 \varphi(0) = (0, -\kappa)$. We choose S later (should be sufficiently small). We define $n = n(s) := (\partial_s \varphi(s))^\perp$ as a unit normal vector and $\tau = \tau(s) = \partial_s \varphi(s)$ as a unit tangent vector, where \perp represents upward direction. In order to define the domain Ω , we need the following coordinate.

Definition 1.1. (Normal coordinate.) For $s \in [0, S]$ and $r \in [0, R]$, let

$$\Phi(s, r) = \Phi_\varphi(s, r) := n(s)r + \varphi(s).$$

Remark 1.2. Since $\partial_s n(s) = \kappa \tau(s)$ (Frenet-Serret formulas), we see that

$$(\partial_r \Phi)(s, r) = n(s) \quad \text{and} \quad (\partial_s \Phi)(s, r) = (r\kappa + 1)\tau(s).$$

Now we define the domain Ω as follows:

$$\Omega = \Omega_{S,R} := \{\Phi(s, r) \in \mathbb{R}^2 : s \in (0, S), r \in (0, R)\}.$$

Note that we will take S and R to be sufficiently small depending on the initial data and the inflow condition (see Remark 1.6). The non-stationary two-dimensional Navier-Stokes equation is expressed as

$$(1.1) \quad \begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla)u = -\nabla p, & \text{div } u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u|_{\cup_{s=0}^S \varphi(s)} = 0, \end{cases}$$

where $u = u(x) = u(x, t) = (u^1(x_1, x_2, t), u^2(x_1, x_2, t))$. In this paper we sometimes abbreviate the time t not x .

Definition 1.3. (Inflow condition) Let $u_{in}(r) := (u \cdot \tau)(\Phi(0, r))$ be a (smooth) inflow with rightward direction, namely, $\partial_r u_{in}(r) > 0$. Also assume $(u \cdot n)(\Phi(0, r)) = 0$ (this condition is just for a technical reason, expressing parallel profile to the boundary). Let $\alpha_1, \alpha_2, \alpha_3 \in C^\infty([0, \infty))$ be coefficients of the time dependent inflow condition, namely,

$$u_{in}(r) = (u \cdot \tau)(\Phi(0, r)) = \alpha_1(t)r - \frac{\alpha_2(t)}{2!}r^2 + \frac{\alpha_3(t)}{3!}r^3 + O(r^4).$$

Since $\partial_r u_{in}(r) > 0$, we see that $\alpha_1(t) > 0$. Assume also the inflow does not grow polynomially for r direction (this is due to the observation of “boundary layer”, since the inflow profile should be a uniform one away from the boundary), thus, it is reasonable to focus on the following two cases:

- (Poiseuille type profile) $\alpha_1(t) > 0$, $\alpha_2(t) > 0$ and $\alpha_3(t)$ is small compare with $\alpha_1(t)$ and $\alpha_2(t)$,

- (Before separation profile) $\alpha_2(t) < 0$, $\alpha_3(t) < 0$ and $\alpha_1(t)$ is small compare with $\alpha_2(t)$ and $\alpha_3(t)$.

In this point of view, the following assumption is acceptable:

$$\kappa^2\alpha_1(t) + 2\kappa\alpha_2(t) - \alpha_3(t) > C > 0 \quad (\text{see Theorem 1.10}).$$

Remark 1.4. The case $\alpha_1(t) \searrow 0$ ($t \rightarrow t_0$) expresses nothing more than ∂ -singular (see [10]) which represents separation at the origin and a time t_0 . Namely, if separation occurs, $\alpha_1(t_0)$ must be zero.

We assume that there exists a smooth solution except for the origin, namely, assume that there exists a pair of solution (u, p) to (1.1) in

$$u, p \in C^\infty([0, T] \times D) \cap C^\infty((0, T] \times (\bar{\Omega} \setminus B_\epsilon)) \quad \text{for any } D \Subset \Omega \quad \text{and } \epsilon > 0,$$

where $B_\epsilon = \{x \in \mathbb{R}^2 : |x| < \epsilon\}$. In the physical point of view, finite energy should be required. More precisely,

$$u \in L^\infty(0, T; L^2(\Omega)) \quad \text{and} \quad \nabla u \in L^2(0, T; L^2(\Omega))$$

should be required. Since we only consider the flow near the origin, “local finite energy near the origin” must be the condition we need to check. In this case, we need to classify “non-smoothness at the origin” (in Definition 1.8) more and more. If L (in Definition 2.2) is not bijective near the boundary, then the vector field (for fixed t) exceeds L^2 -singularity (see Remark 2.3) at the origin. Thus L must be bijective. By Remark 2.7, the flow u is bounded (for fixed t) near the origin if the flow u is “diffusing laminar flow” (in Definition 1.8). Thus our conjecture should be the following:

Conjecture. If the flow is “laminar flow” (in Definition 1.7), and L (in Definition 2.2) is bijective, then the vector field u (for fixed t) exceeds L^2 -singularity at the origin if u is highly oscillating near the origin, namely, highly mixing “diffusing laminar flow” and “concentrating laminar flow” near the origin. Otherwise, the vector field u (for fixed t) never exceed L^2 -singularity. The similar thing also holds in the “topologically changing flow” case (like Moffatt vortices [13] situation is one of the possibility).

The above conjecture is out of the main topic in this paper. Thus we do not mention more about it.

Remark 1.5. Combining a result of Navier-Stokes initial value problem in Lipschitz domain [12], a boundary regularity result [8] (We believe we can generalize their result to various smooth domains) and an inhomogeneous boundary result [5] (see also [4]), the above existence and smoothness assumptions should become true. However, regularity at the origin should be more delicate. If the origin is smooth, the following ODE (which comes from the inflow condition) must have a solution (see Remark 1.11 also):

$$\partial_t \alpha_1(t) = -\nu(4\alpha_1(t)\beta(t) + \kappa^2\alpha_1(t) + 2\kappa\alpha_2(t) - \alpha_3(t)),$$

where $\beta(t)$ is a quantified geometrical behavior of the laminar flow (see Definition 1.8). The point is that the coefficient $\alpha_1(t)$ is determined by $\alpha_2(t)$, $\alpha_3(t)$, $\beta(t)$ and κ . This means that we cannot set arbitrary smooth inflow in order to have the smoothness at the origin.

Remark 1.6. We can avoid interior blow-up by taking sufficiently small R . Thus we only need to care boundary regularity not interior regularity. Moreover we can also avoid boundary blow-up except for the origin by taking sufficiently small S . Thus it is reasonable to assume T to be sufficiently large (for sufficiently small S and R).

Definition 1.7. (Laminar flow.) u is “laminar flow” (near the origin) iff u is smooth (including the origin) in $\bar{\Omega}$, $|u(x)| \neq 0$ for $x \in \Omega$ and the flow u is to the rightward direction (laminar flow direction), namely,

$$(u \cdot \tau)(x) > 0$$

for $x \in \Omega$.

We mainly consider a geometrical shape of the laminar flow near the origin. In this case, one of the five situations only occur (for fixed time t): (geometrically) diffusing, almost parallel, concentrating laminar flows, topologically changing flow (inducing the reverse flow) or non-smoothness (singularity) at the origin. Sometimes we write $u \cdot \tau = (u \cdot \tau)(s, r) = (u \cdot \tau)(s, r, t) = (u \cdot \tau)(x, t)$ with $x = \Phi(s, r)$ unless confusion occurs.

Definition 1.8. (Classification of Navier-Stokes flow for fixed time.) Let

$$\mathcal{L}_t(s, r) = \mathcal{L}(s, r) := (r\kappa + 1) \frac{u \cdot n}{u \cdot \tau} \quad (\text{slope of the velocity with Riemannian metric})$$

and let β be a quantified geometrical behavior of the laminar flow (near the origin):

$$\beta = \beta(t) := \lim_{s, r \rightarrow 0} \partial_s \partial_r \mathcal{L}(s, r).$$

- Diffusing laminar flow: We call (geometrically) diffusing laminar flow iff $u, p \in C^\infty(\Omega)$ and

$$\beta(t) > 0.$$

- Almost parallel laminar flow: We call (geometrically) almost parallel laminar flow iff $u, p \in C^\infty(\Omega)$ and

$$\beta(t) = 0.$$

- Concentrating laminar flow: We call (geometrically) concentrating laminar flow iff $u, p \in C^\infty(\Omega)$ and

$$\beta(t) < 0.$$

- Topologically changing flow (not laminar flow case): We say topologically changing flow iff $u, p \in C^\infty(\Omega)$ and there is $x \in \Omega$ such that $|u(x)| = 0$ or $(u \cdot \tau)(x) \leq 0$.
- Non-smoothness at the origin: We say non-smoothness at the origin (for fixed t) iff

$$u(\cdot, t) \notin C^\infty(\Omega \cap B_\epsilon) \quad \text{or} \quad p(\cdot, t) \notin C^\infty(\Omega \cap B_\epsilon) \quad \text{for} \quad \epsilon > 0.$$

In order to give the main theorem, we need to define “trajectory”.

Definition 1.9. (Trajectory.) Let $\tilde{\gamma}_X : [0, T] \rightarrow \Omega$ be such that

$$\partial_t \tilde{\gamma}_X(t) = u(\tilde{\gamma}_X(t), t), \quad \tilde{\gamma}_X(0) = X \in \Omega.$$

Note that the equation (1.1) can be rewritten to $\partial_t(u(\tilde{\gamma}(t), t)) = (\Delta u - \nabla p)(\tilde{\gamma}(t), t)$.

The following is the main theorem.

Theorem 1.10. (*Horizontally stopping particles phenomena.*) Let the initial datum u_0 satisfies the diffusing laminar flow condition, namely, $\beta(0) > 0$. For any given smooth inflow $u_{in}(r)$ with $\kappa^2\alpha_1 + 2\kappa\alpha_2 - \alpha_3 > C > 0$ (see Definition 1.3 and (3.5)) then the topologically changing flow (or non-smoothness at the origin) must occur in finite time. In other words, particles near the boundary slow down and finally stop horizontally in finite time. More precisely, there is $\bar{R} < R$ such that if $\bar{r} < \bar{R}$, then

$$\lim_{t \rightarrow \tilde{T}} (u \cdot \tau)(\tilde{\gamma}_{\Phi(0, \bar{r})}(t), t) = 0,$$

where $\tilde{T} (< T)$ is depending on \bar{r} , ν , κ , α_1 , α_2 and α_3 .

Remark 1.11. In order to keep the smoothness at the origin, $\alpha_1(t)$ must satisfy

$$\partial_t \alpha_1(t) = -\nu(4\alpha_1(t)\beta(t) + \kappa^2\alpha_1(t) + 2\kappa\alpha_2(t) - \alpha_3(t)).$$

Otherwise, non-smoothness immediately occurs. This is due to the “breaking effect” (Remark 3.2). In this case, we have $\alpha_1(t) \searrow 0$ as $t \rightarrow \tilde{T}$ (this expresses ∂ -singular, see [10]).

Remark 1.12. There are direct and indirect evidences for the validity of the “Kutta condition” in restricted regions (see [2]). The method used in the above theorem may give another support for the validity of the Kutta condition in pure mathematical sense. Moreover, we may be able to apply the method to “Taylor vortices” (see Chapter II, Section 4 in [16]) which is closely related to the bifurcation theory.

Now we give outline of the proof briefly. Basically, we need to estimate trajectory of a particle near the boundary. In order to do so, we need to estimate each Δu and ∇p near the boundary. First we construct “streamline coordinate” and then we can estimate Δu directly. Next we construct “pressure coordinate” based on level set of the pressure and no-slip boundary condition. In this case, $\Delta u = \nabla p$ on the boundary is the crucial point. Third we calculate some kind of Riemannian metric of the “pressure coordinate” at the origin (the pressure is nonlocal operator, nevertheless we can estimate it by using orders of approximation).

2. STREAMLINE COORDINATE

In this section we define a streamline coordinate in the laminar flow case, and give its properties. Throughout this section, we assume smoothness at the origin.

Definition 2.1. (Streamline.) For fixed t , let $\gamma_X : [0, \bar{S}) \rightarrow \Omega$ be such that (\bar{S} is depending on each X , but not so important value, since we only consider near the origin)

$$\partial_{s'} \gamma_X(s') = u(\gamma_X(s'), t), \quad \gamma_X(0) = X \in \Omega.$$

Definition 2.2. (Streamline coordinate L .) For fixed s , s_0 and r_0 , let $s' = s'(s, s_0, r_0)$ and $r = r(s') = r(s, s_0, r_0)$ be such that $\Phi(s, r(s')) = \gamma_{\Phi(s_0, r_0)}(s')$. Note that s' and r are uniquely determined in the laminar flow case. Let $L_t = L = L(s) = L(s; s_0, r_0) = r(s')$. In particular, let $L(s, r_0) := L(s; 0, r_0)$.

Please keep in mind that s' is a parameter for the streamline, s is for the normal coordinate. s is depending on s' and vice versa.

Remark 2.3. If $L(s, \cdot)$ is not bijective (near the boundary) for some $s > 0$, then the vector field u is not in $L^2(\Omega)$. It is in contradiction to the smoothness. Thus $L(s, \cdot)$ is bijective (near the boundary) for any $s > 0$. In fact, if $L(s, \cdot)$ is not bijective for some $s > 0$, then L should be strictly positive, namely,

$$\tilde{R}_s := \inf_{0 < r_0 < R} L(s, r_0) > 0.$$

In this case, there is $\tilde{S}_r > 0$ (depending on each r) and $s_b \geq 0$ such that

$$\gamma_{\Phi(s,r)}(s') \rightarrow (s_b, 0) \quad (s' \rightarrow -\tilde{S}_r)$$

for $r \in (0, \tilde{R}_s)$. This means that a streamlines-bundle is concentrating to a point $(s_b, 0)$ in the leftward direction. Note that we need to take a negative value $-\tilde{S}_r$, since u is to the rightward direction. By Corollary 2.6 (to the 2D case) in [6], we have $u \notin L^2(\Omega)$.

Remark 2.4. Due to Taylor expansion, we see

$$L_t(s, r_0) = L(s, r_0) = c_1(s)r_0 + \frac{c_2(s)}{2!}r_0^2 + \frac{c_3(s)}{3!}r_0^3 + O(r_0^4),$$

where $c_j(s) = c_j(s, t)$, $c_j(s) \in C^\infty([0, S])$, $c_1(0) = 1$, $c_j(0) = 0$ for $j \geq 2$. By Remark 2.3, there is a L^{-1} (for fixed s) such that $r_0 = L^{-1}(r) = L^{-1}(s, r)$ and

$$(2.2) \quad L^{-1}(r) = L^{-1}(s, r) = \frac{r}{c_1(s)} - \frac{c_2(s)}{2c_1^3(s)}r^2 + O(r^3).$$

Sometimes we denote $L^{-1} = L^{-1}(x) = L^{-1}(s, r)$ with $x = \Phi(s, r)$ unless confusion occurs.

Remark 2.5. We see the relation between streamline coordinate L and slope of the velocity with Riemannian metric \mathcal{L} :

$$(2.3) \quad (\partial_s L)(s; s_0, r_0)|_{(s_0, r_0)=(s, r)} = (\partial_s L)(s, L^{-1}(s, r)) = \frac{u \cdot n}{u \cdot \tau}(r\kappa + 1) = \mathcal{L}(s, r).$$

A direct calculation shows that $\beta(t) = \lim_{s, r \rightarrow 0} \partial_s \partial_r \mathcal{L}(s, r) = \partial_s^2 c_1(0)$. Moreover, since $\lim_{s \rightarrow 0} \mathcal{L}(s, r) = 0$ for $r > 0$, $\partial_s c_j(0) = 0$ for $j \geq 1$. In order to obtain (2.3), we need a re-parametrized streamline based on a parameter s . Recall $L(s) = r(s')$. Since

$$(2.4) \quad \begin{aligned} \partial_s(\Phi(s, L(s))) &= (\partial_s \Phi)(s, L(s)) + (\partial_r \Phi)(s, L(s))\partial_s L(s), \\ (\partial_s \Phi)(s, L(s)) \cdot n(s) &= 0, \quad (\partial_r \Phi)(s, L(s)) \cdot \tau(s) = 0, \\ (\partial_s \Phi)(s, L(s)) \cdot \tau(s) &= L(s)\kappa + 1, \quad (\partial_r \Phi)(s, L(s)) \cdot n(s) = 1 \end{aligned}$$

(the above second to fifth equalities are corresponding to Riemannian metric of the normal coordinate),

$$\Phi(s, L(s)) = \gamma(s'(s))$$

and

$$\begin{aligned} \partial_s(\Phi(s, L(s))) &= (\partial_s \Phi)(s, L(s)) + (\partial_r \Phi)(s, L(s))\partial_s L(s) \\ &= \partial_s \gamma(s'(s)) = \partial_{s'} \gamma(s') \partial_s s'(s) = u(\Phi(s, L(s))) \partial_s s'(s), \end{aligned}$$

we see that

$$(u \cdot \tau) \partial_s s'(s) = L(s)\kappa + 1 \quad \text{and then} \quad \partial_s s'(s) = \frac{L(s)\kappa + 1}{u \cdot \tau}.$$

Thus we have a re-parametrized streamline based on the parameter s :

$$\partial_s(\Phi(s, L(s))) = (L(s)\kappa + 1) \left(\frac{u}{u \cdot \tau} \right) (x)|_{x=\Phi(s, L(s))}.$$

From the above expression and Remark 1.2, we have

$$\partial_s L(s) = (r\kappa + 1) \frac{u \cdot n}{u \cdot \tau}.$$

Thus

$$(\partial_s L)(s, L^{-1}(s, r)) = (r\kappa + 1) \frac{u \cdot n}{u \cdot \tau} = \mathcal{L}(s, r).$$

Remark 2.6. By divergence free, we see that (construction of u by using L , L^{-1} , \mathcal{L} and the inflow condition u_{in})

$$(u \cdot \tau)(s, r) = \frac{u_{in}(0, L^{-1}(s, r))}{(\partial_{r_0} L)(s, L^{-1}(s, r))}, \quad (u \cdot n)(s, r) = \frac{(u \cdot \tau)\mathcal{L}}{r\kappa + 1}.$$

We give a calculation to obtain the left equality. In the calculation, the main idea is just using divergence free and Gauss's divergence theorem. Let

$$D_\epsilon := \cup_{r_0 < \bar{r}_0 \leq r_0 + \epsilon} \cup_{0 < \bar{s} \leq s} \Phi(\bar{s}, L(\bar{s}, \bar{r}_0)).$$

By Gauss's divergence theorem,

$$0 = \int_{D_\epsilon} \operatorname{div} u = - \int_{r_0}^{r_0 + \epsilon} (u \cdot \tau)(0, \bar{r}_0) d\bar{r}_0 + \int_{L(s, r_0)}^{L(s, r_0 + \epsilon)} (u \cdot \tau)(s, \bar{r}) d\bar{r}.$$

Thus

$$\frac{\epsilon}{L(s, r_0 + \epsilon) - L(s, r_0)} \frac{1}{\epsilon} \int_{r_0}^{r_0 + \epsilon} (u \cdot \tau)(0, \bar{r}_0) d\bar{r}_0 = \frac{1}{L(s, r_0 + \epsilon) - L(s, r_0)} \int_{L(s, r_0)}^{L(s, r_0 + \epsilon)} (u \cdot \tau)(s, \bar{r}) d\bar{r}.$$

Then we have

$$\frac{(u \cdot \tau)(0, r_0)}{(\partial_{r_0} L)(s, r_0)} = (u \cdot \tau)(s, L(s, r_0)) \quad \text{as } \epsilon \rightarrow 0.$$

Remark 2.7. Diffusing laminar flow induces decreasing velocity along streamlines. To the contrary, concentrating laminar flow induces increasing one. Since $(\partial_r \mathcal{L})(s, r) = (\partial_r \partial_s L)(s, L^{-1}(s, r))(\partial_r L^{-1})(s, r)$, $(\partial_{r_0} L)(0, r_0) = 1$ and $r = L(s, r_0)$, we see that

$$(\partial_{r_0} L)(s, r_0) = \int_0^{\bar{s}} \partial_{r_0} L(\bar{s}, r_0) d\bar{s} + \partial_{r_0} L(0, r_0) = 1 + \int_0^s \frac{(\partial_r \mathcal{L})(\bar{s}, L(\bar{s}, r_0))}{(\partial_r L^{-1})(\bar{s}, L(\bar{s}, r_0))} d\bar{s}.$$

The above left hand side term essentially expresses $u \cdot \tau$ (see Remark 2.6), $\partial_r \mathcal{L}$ in right hand side is controlled by $\beta = \lim \partial_s \partial_r \mathcal{L}$. Since $\lim_{s \rightarrow 0} \partial_r \mathcal{L} = 0$ for $r \geq 0$ (just apply Taylor expansion of L in Remark 2.4), $\partial_r \mathcal{L} \geq 0$ for sufficiently small s and r if $\lim_{s, r \rightarrow 0} \partial_s \partial_r \mathcal{L} > 0$. By (2.2), $(\partial_r L^{-1})(s, r)$ is strictly positive for sufficiently small s and r . Thus, if u has diffusing-almost parallel laminar flow, namely, $\beta \geq 0$, then $(u \cdot \tau)(0, L^{-1}(s, r)) \geq (u \cdot \tau)(s, r)$ for sufficiently small s and r . (To the contrary, if u has concentrating laminar flow, namely, $\beta < 0$, then $(u \cdot \tau)(0, L^{-1}(s, r)) < (u \cdot \tau)(s, r)$ for sufficiently small s and r .) Therefore diffusing-almost parallel laminar flow cannot create any boundary blow-up.

Remark 2.8. We now give direct differentiations of $L(s, r_0)$, $L^{-1}(s, r)$ and $\mathcal{L}(s, r)$ which will be needed to prove the main theorem. For the sake of generality, we assume $\partial_s c_1(0)$, $\partial_s c_2(0)$ and $\partial_s c_3(0)$ are not always zero (in this paper, $\partial_s c_1(0)$, $\partial_s c_2(0)$ and $\partial_s c_3(0)$ are zero due to the inflow condition). Recall that $c_1(0) = 1$ and $c_j(0) = 0$ ($j \geq 2$). We see that

$$\begin{aligned}
\partial_s L^{-1}|_{s=0} &= -\partial_s c_1(0)r + O(r^2), \\
\partial_s^2 L^{-1}|_{s=0} &= (2(\partial_s c_1(0))^2 - \partial_s^2 c_1(0))r + O(r^2), \\
\partial_r L^{-1}|_{s=0} &= 1, \\
\partial_r^2 L^{-1}|_{s=0} &= 0 + O(r), \\
\partial_s L|_{s=0} &= \partial_s c_1(0)r_0 + O(r_0^2), \\
\partial_s^2 L|_{s=0} &= \partial_s^2 c_1(0)r_0 + O(r_0^2), \\
\partial_s^3 L|_{s=0} &= \partial_s^3 c_1(0)r_0 + O(r_0^2), \\
\partial_{r_0} L|_{s=0} &= 1, \\
\partial_{r_0}^2 L|_{s=0} &= 0, \\
\partial_{r_0}^3 L|_{s=0} &= 0, \\
\partial_s \partial_{r_0} L|_{s=0} &= \partial_s c_1(0) + \partial_s c_2(0)r_0 + O(r_0^2), \\
\partial_s^2 \partial_{r_0} L|_{s=0} &= \partial_s^2 c_1(0) + \partial_s^2 c_2(0)r_0 + O(r_0^2), \\
\partial_s \partial_{r_0}^2 L|_{s=0} &= \partial_s c_2(0) + \partial_s c_3(0)r_0 + O(r_0^2),
\end{aligned}$$

$$\begin{aligned}
\partial_s L^{-1}|_{r=0} &= 0, \\
\partial_s^2 L^{-1}|_{r=0} &= 0, \\
\partial_r L^{-1}|_{r=0} &= \frac{1}{c_1(s)}, \\
\partial_r^2 L^{-1}|_{r=0} &= -\frac{c_2(s)}{c_1(s)^2}, \\
\partial_s L|_{r_0=0} &= 0, \\
\partial_s^2 L|_{r_0=0} &= 0, \\
\partial_s^3 L|_{r_0=0} &= 0, \\
\partial_{r_0} L|_{r_0=0} &= c_1(s), \\
\partial_{r_0}^2 L|_{r_0=0} &= c_2(s), \\
\partial_{r_0}^3 L|_{r_0=0} &= c_3(s), \\
\partial_s \partial_{r_0} L|_{r_0=0} &= \partial_s c_1(s), \\
\partial_s^2 \partial_{r_0} L|_{r_0=0} &= \partial_s^2 c_1(s), \\
\partial_s \partial_{r_0}^2 L|_{r_0=0} &= \partial_s c_2(s),
\end{aligned}$$

and also

$$\begin{aligned}
\mathcal{L} &= \partial_s L, \\
\partial_s \mathcal{L} &= \partial_s^2 L + \partial_{r_0} \partial_s L \partial_s L^{-1}, \\
\partial_s^2 \mathcal{L} &= \partial_s^3 L + 2\partial_{r_0} \partial_s^2 L \partial_s L^{-1} + \partial_{r_0}^2 \partial_s L (\partial_s L^{-1})^2 + \partial_{r_0} \partial_s L \partial_s^2 L^{-1}, \\
\partial_r \mathcal{L} &= \partial_{r_0} \partial_s L \partial_r L^{-1}, \\
\partial_r^2 \mathcal{L} &= \partial_{r_0}^2 \partial_s L (\partial_r L^{-1})^2 + \partial_{r_0} \partial_s L \partial_r^2 L^{-1},
\end{aligned}$$

which gives us that

$$\begin{aligned}
\mathcal{L}|_{s=0} &= \partial_s c_1(0)r + O(r^2), \\
\partial_s \mathcal{L}|_{s=0} &= (\partial_s^2 c_1(0) - (\partial_s c_1(0))^2) r + O(r^2), \\
\partial_s^2 \mathcal{L}|_{s=0} &= \partial_s^3 c_1(0)r - 2\partial_s^2 c_1(0)\partial_s c_1(0)r + \partial_s c_1(0) (2(\partial_s c_1(0))^2 - \partial_s^2 c_1(0)) r + O(r^2), \\
\partial_r \mathcal{L}|_{s=0} &= \partial_s c_1(0) + \partial_s c_2(0)r + O(r^2), \\
\partial_r^2 \mathcal{L}|_{s=0} &= \partial_s c_2(0) + \partial_s c_3(0)r + O(r^2),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}|_{r=0} &= 0, \\
\partial_s \mathcal{L}|_{r=0} &= 0, \\
\partial_s^2 \mathcal{L}|_{r=0} &= 0, \\
\partial_r \mathcal{L}|_{r=0} &= \frac{\partial_s c_1(s)}{c_1(s)}, \\
\partial_r^2 \mathcal{L}|_{r=0} &= \frac{\partial_s c_2(s)}{c_1^2(s)} - \frac{\partial_s c_1(s)c_2(s)}{c_1(s)}.
\end{aligned}$$

3. PROOF OF THE MAIN THEOREM

First we give a time line observation.

Definition 3.1. We define ‘‘Persisting laminar flow time’’ T_l as

$$T_l := \inf\{t > 0 : u(\cdot, t) \text{ is topologically changing flow or non-smoothness at the origin or blowup}\}.$$

We also define ‘‘Persisting diffusing-almost parallel laminar flow time’’ T_d as

$$T_d := \inf\{t > 0 : u(\cdot, t) \text{ is concentrating laminar flow or topologically changing flow or non-smoothness at the origin or blowup}\}.$$

Note that $0 \leq T_d \leq T_l \leq T \leq \infty$ (T is a possible blowup time). It suffices to show $T_l < \infty$ in order to prove the main theorem. In order to do so, we need to show the following two assertions:

- u cannot keep diffusing-almost parallel laminar flow for infinite time, namely $T_d \neq \infty$.
- u cannot create concentrating laminar flow from almost parallel laminar flow, namely, $T_d = T_l$.

To prove the above two assertions, the following is the key estimate (we say ‘‘breaking effect’’):

$$\begin{aligned}
(3.5) \quad \lim_{r, s \rightarrow 0} \partial_r [(\Delta u - \nabla p) \cdot \tau](\Phi(s, r)) &= -\nu (4\alpha_1 \partial_s^2 c_1(0) + \kappa^2 \alpha_1 + 2\kappa \alpha_2 - \alpha_3) \\
&= -\nu (4\alpha_1 \beta(t) + \kappa^2 \alpha_1 + 2\kappa \alpha_2 - \alpha_3) \\
&=: \rho(\nu, \kappa, \beta, \alpha_1, \alpha_2, \alpha_3) < 0 \quad \text{for } t \in (0, T_l).
\end{aligned}$$

Note that $\kappa^2\alpha_1 + 2\kappa\alpha_2 - \alpha_3 > C > 0$ by the assumption in the main theorem. Intuitively, the above estimate says that the acceleration of particles near the boundary is leftward direction (breaking effect).

Remark 3.2. We see that

$$\partial_t\alpha_1(t) = \lim_{r,s \rightarrow 0} \partial_r [(\Delta u - \nabla p) \cdot \tau] (\Phi(s, r)).$$

We will show the key estimate (3.5) later. We now prove the main theorem by using (3.5). By the smoothness and $(\Delta u - \nabla p)(\Phi(s, r)) \cdot \tau(s)|_{r=0} = 0$ for $s > 0$ (due to the no-slip boundary condition), there are \tilde{S} and \tilde{R} (sufficiently small, independent of t) s.t.

$$(3.6) \quad [(\Delta u - \nabla p) \cdot \tau] (\Phi(s, r)) \leq \frac{\rho r}{2} < 0$$

for $s \in [0, \tilde{S}]$ and $r \in [0, \tilde{R}]$. Recall $\tilde{\gamma}$ is a trajectory. Let $s(t)$ and $r(t)$ be such that

$$\tilde{\gamma}(t) = \Phi(s(t), r(t)).$$

By the fundamental theorem of calculations ($0 < t < T_l$),

$$(u \cdot \tau)(\tilde{\gamma}(t), t) = \int_0^t \partial_{\bar{t}} \left(u(\tilde{\gamma}(\bar{t}), \bar{t}) \right) \cdot \tau(s(\bar{t})) d\bar{t} + \int_0^t u(\tilde{\gamma}(\bar{t}), \bar{t}) \cdot \partial_t \tau(s(\bar{t})) d\bar{t} + (u \cdot \tau)(\tilde{\gamma}(0), 0).$$

$\partial_t(\tau(s(\bar{t})))$ is always downward direction due to the curvature, since

$$\partial_{\bar{t}} \tau(s(\bar{t})) = \partial_s \tau(s) \partial_{\bar{t}} s = -\kappa n(s) \partial_{\bar{t}} s(\bar{t}) \quad (\partial_{\bar{t}} s(\bar{t}) \geq 0) \quad (\text{Frenet-Serret formulas}).$$

Since u is diffusing- almost parallel laminar flow, u is expressed as $u = (u \cdot \tau)\tau + (u \cdot n)n$ with $(u \cdot \tau) > 0$, $(u \cdot n) \geq 0$ (just use $L = r_0(1 + (\beta/2)s^2 + O(s^3)) + O(r_0^2)$ and the formula (2.6), then we immediately obtain the sign of $(u \cdot n)$). By the right above two observations, we see

$$u(\tilde{\gamma}(t'), t') \cdot \partial_t \tau(s(t')) \leq 0.$$

We estimate the right hand side of the first term $\int_0^t (\partial_t u \cdot \tau)$ by using (3.6). First prove the first assertion. We assume $T_d = \infty$. By taking sufficiently small $r(0)$, we see

$$(u \cdot \tau)(\tilde{\gamma}(t), t) \leq \frac{t\rho(\inf_{0 \leq t' \leq t} r(t'))}{2} + (u \cdot \tau)(\tilde{\gamma}(0), 0) \leq \frac{t\rho r(0)}{2} + \alpha_1 r(0).$$

Note that $r(t)$ is increasing in time t . By taking an appropriate t , then

$$(u \cdot \tau)(\tilde{\gamma}(t), t) < 0.$$

By the above upper bound, there is a \tilde{T} such that

$$\lim_{t \rightarrow \tilde{T}} (u \cdot \tau)(\tilde{\gamma}(t), t) = 0.$$

This is a contradiction, since the flow is topologically changing at a time \tilde{T} . Thus $T_d < \infty$. We now prove the second assertion $T_d = T_l$.

Remark 3.3. Since $\Phi(s(t), r(t)) = \tilde{\gamma}(t)$, (2.4) and

$$u(\tilde{\gamma}(t), t) = \partial_t \tilde{\gamma}(t) = \partial_t \Phi(s(t), r(t)) = (\partial_s \Phi)(s, r) \partial_t s(t) + (\partial_r \Phi)(s, r) \partial_t r(t),$$

we see $u \cdot n = \partial_t r(t)$ and $u \cdot \tau = (r(t)\kappa + 1) \partial_t s(t)$. Thus

$$\mathcal{L}_t(s(t), r(t)) = \frac{\partial_t r(t)}{\partial_t s(t)}.$$

Assume $T_d < T_l$. Roughly saying, in the streamline (for each time) consideration, concentrating streamline structure gives accelerating particles (see Remark 2.7). The breaking effect (3.5) is nothing to do with this observation. However, by the breaking effect (3.5), the particle must be breaking (even if $\beta(t)$ is strictly negative, namely, concentrating streamlines, ρ can be negative due to the assumption in the main theorem). This observation brings a contradiction, namely, concentrating streamline never occur. This means that T_d is equal to T_l . We now give the rigorous proof. We see $\partial_t \partial_s \mathcal{L}|_{s,r=0} = \partial_t \partial_r \mathcal{L}|_{s,r=0} = \partial_t \partial_s^2 \mathcal{L}|_{s,r=0} = \partial_t \partial_r^2 \mathcal{L}|_{s,r=0} = \partial_t \mathcal{L}|_{s,r=0} = 0$ and $\partial_t \beta(t) = \partial_t \partial_s \partial_r \mathcal{L}_t|_{s,r=0} = \partial_t \partial_s^2 c_1(0, t) < 0$ ($\partial_s^2 c_1$ is changing its sign at T_d) for some small time interval in $(T_d, T_d + \epsilon]$ with sufficiently small $\epsilon > 0$. Assume t_1, t_2 and t ($t_1 < t_2 < t$) are in the small time interval, then we see that (monotone decreasing and strictly negative):

$$(3.7) \quad 0 > \mathcal{L}_{t_1}(s, r) > \mathcal{L}_{t_2}(s, r) > \mathcal{L}_t(s, r) \quad \text{for sufficiently small } s \text{ and } r$$

due to the Taylor expansion of $\partial_t \mathcal{L}$ in s and r . Note that ρ is still strictly negative if t_1, t_2 and t are sufficiently close to T_d (due to the assumption $\kappa^2 \alpha_1 + 2\alpha_2 \kappa - \alpha_3 > 0$). Let $s(t_1) = 0$, then there is $r_0 > 0$ (to be sufficiently small if $r(t_1)$ is sufficiently small) such that

$$\begin{aligned} \alpha_1 r(t_1) &\approx \alpha_1 r(t_1) - \frac{\alpha_2}{2} r(t_1)^2 + \frac{\alpha_3}{3!} r(t_1)^3 + O(r^4) \\ &= (u \cdot \tau)(0, r(t_1), t_1) \\ &\geq (u \cdot \tau)(s(t_2), r(t_2), t_2) \\ &= (u \cdot \tau)(s(t_2), L_{t_2}(s(t_2), r_0), t_2) \\ &\geq (u \cdot \tau)(0, r_0, t_2) \\ &= \alpha_1 r_0 - \frac{\alpha_2}{2} r_0^2 + \frac{\alpha_3}{3!} r_0^3 + O(r_0^4) \\ &\approx \alpha_1 r_0. \end{aligned}$$

The third inequality is due to (3.5), the fifth inequality is due to Remark 2.7. Thus $r(t_1) \geq r_0$. However it is in contradiction. The reason is in what follows: Assume

$$r(t_1) - L_t(s(t_1); s(t_2), r(t_2)) \geq 0.$$

Then the simple calculation yields

$$\frac{r(t_2) - r(t_1)}{s(t_2) - s(t_1)} - \frac{r(t_2) - L_t(s(t_1); s(t_2), r(t_2))}{s(t_2) - s(t_1)} \leq 0.$$

This means that

$$\frac{\partial_t r(t_2)}{\partial_t s(t_2)} - \lim_{t_1 \rightarrow t_2} \frac{L_t(s(t_2); s(t_2), r(t_2)) - L_t(s(t_1); s(t_2), r(t_2))}{s(t_2) - s(t_1)} \leq 0.$$

By Remark 3.3 and (2.3), then

$$\mathcal{L}_{t_2}(s, r) - \mathcal{L}_t(s, r) \leq 0.$$

It is in contradiction to (3.7). Therefore

$$L_t(s(t_1); s(t_2), r(t_2)) > r(t_1).$$

By the property of concentrating laminar flow (more precisely, the concentrating laminar flow can be expressed as $u = (u \cdot \tau)\tau + (u \cdot n)n$ with $(u \cdot \tau) > 0$ and $(u \cdot n) < 0$),

$$L_t(s(t_1); s(t_2), r(t_2)) \leq L_t(0; s(t_2), r(t_2)) \rightarrow r_0 \quad (t \rightarrow t_2),$$

we have $r(t_1) < r_0$.

Remark 3.4. By $\limsup_{t \rightarrow \tilde{T}} \partial_t(u(\tilde{\gamma}(t), t)) \cdot \tau < 0$ and the smoothness, the particle at $\tilde{\gamma}(\tilde{T})$ will move to the leftward direction (reverse flow direction).

4. THE KEY ESTIMATE: BREAKING EFFECT

In this section we show the following key estimate:

$$\lim_{r, s \rightarrow 0} \partial_r [(\Delta u - \nabla p) \cdot \tau](s, r) = -\nu(4\alpha_1\beta + \kappa^2\alpha_1 + 2\kappa\alpha_2 - \alpha_3).$$

Recall that $\beta = \partial_s^2 c_1(0)$. Our first aim is to show

$$(4.8) \quad \begin{aligned} \Delta u \cdot \tau|_{s=0} &= \kappa\alpha_1 - \alpha_2 - (2\alpha_1\partial_s^2 c_1(0) + 2\kappa^2\alpha_1 + \kappa\alpha_2 - \alpha_3)r + O(r^2), \\ \Delta u \cdot \tau|_{r=0} &= \kappa\alpha_1 - \alpha_2 + O(s^2), \\ \Delta u \cdot n|_{s=0} &= 0 + O(r^2), \\ \Delta u \cdot n|_{r=0} &= 2\alpha_1\partial_s^2 c_1(0)s + O(s^2). \end{aligned}$$

For the sake of generality, we take $\bar{s}, \bar{r} > 0$ and calculate $(\Delta u)(\Phi(\bar{s}, \bar{r}))$ (finally, we take $\bar{s} \rightarrow 0$ or $\bar{r} \rightarrow 0$). We call the point $\Phi(\bar{s}, \bar{r})$ to be Q . That is $Q = \Phi(\bar{s}, \bar{r})$. Then, we use the orthonormal frame $e_1 = \tau(\bar{s})$, $e_2 = n(\bar{s})$ at the point Q to construct a cartesian coordinate system with the new y_1 -axis to be the straight line which passes through Q and parallel to the vector e_1 , and the new y_2 -axis to be the straight line which passes through Q and is parallel to the vector e_2 . Then, for the given vector field u in an open neighborhood near $\partial\Omega$, we define for each $x = \Phi(s, r) \in \Omega$ near the boundary, the two components of $u(x)$ with respect to the e_1 direction and the e_2 direction. It is expressed as

$$u(x) = v^1(x)e_1 + v^2(x)e_2,$$

where $v^1(x) := u(x) \cdot e_1$ and $v^2(x) := u(x) \cdot e_2$. Since $\varphi([0, S])$ is known to be a circular arc with radius κ^{-1} , let C to be the point at which the center of the circular arc $\varphi([0, S])$ is located. Let $y = (y_1, y_2)$ to be the coordinate representation of the point x in the coordinate system based at Q which is specified by the orthonormal frame $\{e_1, e_2\}$. That is, the point x can be realized as $x = Q + y_1e_1 + y_2e_2$ and we also regard $v^j(y)$ as $v^j(Q + y_1e_1 + y_2e_2)$ ($j = 1, 2$). Then we see that

$$(\Delta u)(\Phi(\bar{s}, \bar{r})) = (\partial_{y_1}^2 v^1(0) + \partial_{y_2}^2 v^1(0)) e_1 + (\partial_{y_1}^2 v^2(0) + \partial_{y_2}^2 v^2(0)) e_2.$$

In order to compute $\partial_{y_1}^2 v^1$, $\partial_{y_2}^2 v^1$, $\partial_{y_1}^2 v^2$ and $\partial_{y_2}^2 v^2$, we need an coordinate transformation. We now consider the case when $y_2 > 0$ (the case $y_2 < 0$ is similar so we omit it). A curve $\cup_{s' > 0} \Phi(s' + s, r)$ which passes through x should be a circular arc with radius $|\overline{Cx}|$ and centered at C also. Here, $|\overline{Cx}|$ is length of the line segment \overline{Cx} . That is, the distance between C and x . Let L_{y_1} to be the straight line which passes through Q and is parallel to the e_1 direction. That is L_{y_1} is the new y_1 -axis of the new coordinate system. First, let A to be the unique point on L_{y_1} such that the line segment \overline{QA} is perpendicular to \overline{Ax} . Let D to be the point of intersection between the line segment \overline{Cx} and the line L_{y_1} . Let $x = \Phi(s, r)$. Observe that $|\overline{Cx}| = \kappa^{-1} + r$, $\tan(\kappa(s - \bar{s})) = \frac{|\overline{QD}|}{|\overline{CQ}|}$, $|\overline{Ax}| = y_2$, $|\overline{QA}| = y_1$, and $|\overline{CQ}| = \kappa^{-1} + \bar{r}$. We first compute $|\overline{QD}|$ and $|\overline{DA}|$ through the following observation.

Since $\triangle DAx$ and $\triangle DQC$ are similar triangles, it follows that

$$\frac{|\overline{DA}|}{y_2} = \frac{|\overline{QD}|}{\kappa^{-1} + \bar{r}}.$$

Hence, by substituting $|\overline{DA}| = (y_2/(\kappa^{-1} + \bar{r}))|\overline{QD}|$ into the equation $y_1 = |\overline{QD}| + |\overline{DA}|$, it follows that

$$|\overline{DA}| = \frac{y_1 y_2}{\kappa^{-1} + \bar{r} + y_2} \quad \text{and} \quad |\overline{QD}| = \frac{(\kappa^{-1} + \bar{r})y_1}{\kappa^{-1} + \bar{r} + y_2}.$$

By Pythagorean theorem, we see that

$$|\overline{Dx}| = y_2 \left\{ 1 + \left(\frac{y_1}{\kappa^{-1} + \bar{r} + y_2} \right)^2 \right\}^{\frac{1}{2}} \quad \text{and} \quad |\overline{DC}| = (\kappa^{-1} + \bar{r}) \left\{ 1 + \left(\frac{y_1}{\kappa^{-1} + \bar{r} + y_2} \right)^2 \right\}^{\frac{1}{2}}.$$

Hence, we have

$$|\overline{Cx}| = |\overline{Dx}| + |\overline{DC}| = \{(\kappa^{-1} + \bar{r} + y_2)^2 + y_1^2\}^{\frac{1}{2}}.$$

Let $u^s(s, r) := (u \cdot \tau)(s, r)$ and $u^r(s, r) := (u \cdot n)(s, r)$. As a result, we have (coordinate transformation between the normal coordinate and the cartesian one)

$$(4.9) \quad \begin{cases} r = r(y_1, y_2) = \sqrt{y_1^2 + (\kappa^{-1} + \bar{r} + y_2)^2} - \kappa^{-1}, \\ s = s(y_1, y_2) = \kappa^{-1} \arctan \left(\frac{y_1}{\kappa^{-1} + \bar{r} + y_2} \right) + \bar{s}, \\ v^1(y) = u^s(s, r) \cos \theta + u^r(s, r) \sin \theta, \\ v^2(y) = -u^s(s, r) \sin \theta + u^r(s, r) \cos \theta, \end{cases}$$

where, $\theta = \theta(y)$ is defined by the angle between \overline{Dx} and \overline{Ax} , namely,

$$(4.10) \quad \theta(y) = \kappa(s - \bar{s}) = \arctan \left(\frac{y_1}{\kappa^{-1} + \bar{r} + y_2} \right).$$

Since $\triangle DAx$ is a right angled triangle, it follows that

$$(4.11) \quad \begin{aligned} \sin(\theta(y)) &= \frac{|\overline{DA}|}{|\overline{Dx}|} = \frac{y_1}{\{y_1^2 + (\kappa^{-1} + \bar{r} + y_2)^2\}^{\frac{1}{2}}}, \\ \cos(\theta(y)) &= \frac{\kappa^{-1} + \bar{r} + y_2}{\{y_1^2 + (\kappa^{-1} + \bar{r} + y_2)^2\}^{\frac{1}{2}}}. \end{aligned}$$

By coordinate transformation (4.9), construction of u by using L , L^{-1} and \mathcal{L} (Remark 2.6), and direct differentiations of L , L^{-1} and \mathcal{L} (Remark 2.8), we can calculate Δu near the origin directly, and then we have (4.8). For the detailed calculation, see Appendix. By the no-slip boundary condition,

$$(4.12) \quad \begin{aligned} (\nabla p)(\varphi(s)) &= (\Delta u)(\varphi(s)) \\ &= \left[\kappa \alpha_1 - \alpha_2 + O(s^2) \right] \tau(s) + \left[2\alpha_1 \partial_s^2 c_1(0) s + O(s^2) \right] n(s). \end{aligned}$$

Note that we cannot figure out entire ∇p in Ω , since p (which is depending on u) is a nonlocal operator. However, to prove the main theorem, we do not need entire ∇p in Ω ,

only need a first order part of ∇p . Let us explain more precisely. Since p is smooth, we can use Taylor expansion in r (for fixed s):

$$\begin{aligned} (\nabla p \cdot \tau)(\Phi(s, r)) &= (\nabla p \cdot \tau)(\varphi(s)) + a_1(s)r + \frac{a_2(s)}{2!}r^2 + \frac{a_3(s)}{3!}r^3 + \dots \\ &= (\Delta u \cdot \tau)(\varphi(s)) + a_1(s)r + O(s^3). \end{aligned}$$

In this case, we cannot figure out the higher order coefficients $a_2(s)$, $a_3(s)$, \dots , since p is a nonlocal operator. Nevertheless, we can figure out the exact $a_1(s)$ by using the idea of “pressure coordinate” and (4.12). In order to construct the pressure coordinate only using smoothness and (4.12), we use “gradient of pressure-curve” and “level set of the pressure”. In this setting, the key is to estimate $\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{|Z(\epsilon, s, r)|}$ (a sort of Riemannian metric, we explain it later). Let $q_X : [0, \bar{S}) \rightarrow \Omega$ and $q_X^\perp : [0, \bar{R}) \rightarrow \Omega$ be such that (\bar{S} and \bar{R} are depending on each X , but not so important value, since we only consider near the origin)

$$\partial_{s'} q_X(s') = \nabla p(q_X(s')), \quad \partial_{r'} q_X^\perp(r') = (\nabla p)^\perp(q_X^\perp(r'))$$

with $q_X(0) = X$ and $q_X^\perp(0) = X$. Note that

$$(4.13) \quad \partial_{r'} q_{\varphi(s)}^\perp(r')|_{r'=0} = (\Delta u)^\perp(\varphi(s)) = -(\Delta u \cdot n)\tau + (\Delta u \cdot \tau)n.$$

Definition 4.1. (Pressure coordinate.) We construct a sufficiently small distorted rectangular (near the origin) with four curves composed by “gradient of the pressure-curve q ” and “level set of the pressure q^\perp ”.

- (Upward-direction-left-side curve.) Let $\hat{s} = \hat{s}(s, r)$ and $r' = r'(s, r)$ (with $r'(s, 0) = 0$) be such that (note that the meaning of r and r' are different)

$$q_{\varphi(s)}^\perp(r') = \Phi(\hat{s}, r).$$

Note that $\hat{s}(s, 0) = s$.

- (Rightward-direction-upper-side curve.) For sufficiently small $\epsilon > 0$, let s' and $\ell = \ell(s')$ (s' is depending on s , r and ϵ) be such that

$$q_F(s') = q_{\varphi(s+\epsilon)}^\perp(\ell(s')),$$

where $F = q_{\varphi(s)}^\perp(r')$. Note that s' and ℓ are uniquely determined near the boundary (since $|\nabla p| \neq 0$). We denote the key curve $\cup_{\bar{\epsilon}=0}^\epsilon q_{\varphi(s+\bar{\epsilon})}^\perp(\ell(s'))$ as $Z(\epsilon, s, r)$.

Our aim is to estimate $\epsilon/|Z(\epsilon, s, r)|$ (upper-side curve vs lower-side curve) which is related to a sort of Riemannian metric. The four points $\varphi(s)$, $\varphi(s+\epsilon)$, $q_{\varphi(s)}^\perp(r')$ and $q_{\varphi(s+\epsilon)}^\perp(\ell(s'))$ with four curves create a rectangular by taking $s, r, \epsilon \rightarrow 0$ which will be explained later. We rephrase the upward-direction-right-side curve $\cup_{r''>0} q_{\varphi(s+\epsilon)}^\perp(r'')$ in the normal coordinate as follows:

- Let $\hat{s} = \hat{s}(\epsilon, s, r)$ and $\hat{r} = \hat{r}(\epsilon, s, r)$ be such that $q_{\varphi(s+\epsilon)}^\perp(\ell(s')) = \Phi(\hat{s}, \hat{r})$.

Summarize the indexes. s' and r' are for parametrized curves. s , r and ϵ are for the normal coordinate. \hat{s} is determined by the intersection of the parametrized level set pressure and a curve (circular arc portion) parallel to the boundary. Up to here, we can express three points and two curves explicitly. In order to construct a distorted small rectangular (later we denote the distorted rectangular by $\hat{D}_{\epsilon, r}(s)$), we need to express the last point and curves explicitly by \hat{s} and \hat{r} .

Remark 4.2. We estimate \hat{s} and $\hat{\hat{s}}$. First we show

$$(4.14) \quad (\partial_r \hat{s})(s, 0) = \tilde{\mathcal{L}}(s), \quad \text{where} \quad \tilde{\mathcal{L}}(s) := -\frac{(\Delta u \cdot n)(\varphi(s))}{(\Delta u \cdot \tau)(\varphi(s))}.$$

By upward-direction-left-side curve in Definition 4.1, we see that

$$\begin{aligned} \partial_r q_{\varphi(s)}^\perp(r') &= \partial_r (\Phi(\hat{s}, r)) \\ &= \partial_s \Phi(\hat{s}, r) \partial_r \hat{s}(s, r) + (\partial_r \Phi)(\hat{s}, r) \\ &= (1 + \kappa r) \tau(\hat{s})(\partial_s \hat{s})(s, r) + n(\hat{s}). \end{aligned}$$

On the other hand,

$$\partial_r q_{\varphi(s)}^\perp(r') = \partial_{r'} q_{\varphi(s)}^\perp(r') (\partial_r r')(s, r).$$

Thus we have for $r = 0$ that (in this case, $\hat{s} = s$ and $r' = 0$)

$$\begin{aligned} \left(-(\Delta u \cdot n(s)) \tau(s) + (\Delta u \cdot \tau(s)) n(s) \right) (\partial_r r')(s, 0) \\ = \partial_{r'} q_{\varphi(s)}^\perp(0) (\partial_r r')(s, 0) = \tau(s) (\partial_s \hat{s})(s, 0) + n(s). \end{aligned}$$

We multiply $n(s)$ and $\tau(s)$ on both sides. Then we have

$$-\Delta u \cdot n(s) \partial_r r' = (\partial_r \hat{s})(s, 0) \quad \text{and} \quad \Delta u \cdot \tau(s) \partial_r r' = 1.$$

Thus we have (4.14). By Taylor expansion, we see that

$$\hat{s}(s, r) = s + \tilde{\mathcal{L}}(s)r + \frac{(\partial_r^2 \hat{s})(s, 0)}{2!} r^2 + O(r^3),$$

more precisely,

$$(4.15) \quad s + \tilde{\mathcal{L}}(s)r - Cr^2 \leq \hat{s} \leq s + \tilde{\mathcal{L}}(s)r + Cr^2,$$

where C is a positive constant independent of ϵ , s and r . By the similar calculation to $\hat{\hat{s}}$, we also have

$$s + \epsilon + \tilde{\mathcal{L}}(s + \epsilon)\hat{r} - C\hat{r}^2 \leq \hat{\hat{s}} \leq s + \epsilon + \tilde{\mathcal{L}}(s + \epsilon)\hat{r} + C\hat{r}^2.$$

In what follows, we mainly handle the first order coefficient $\tilde{\mathcal{L}}(s)$.

Remark 4.3. We see that

$$\begin{aligned} \frac{\tilde{\mathcal{L}}(\epsilon) - \tilde{\mathcal{L}}(0)}{\epsilon} &= (\partial_s \tilde{\mathcal{L}})(0) + O(\epsilon) \\ &= \frac{(\Delta u \cdot n)(\varphi(0)) \partial_s (\Delta u \cdot \tau)(\varphi(s))|_{s=0}}{(\Delta u \cdot \tau)(\varphi(0))^2} - \frac{\partial_s (\Delta u \cdot n)(\varphi(s))|_{s=0}}{(\Delta u \cdot \tau)(\varphi(0))} + O(\epsilon). \end{aligned}$$

Thus

$$(\partial_s \tilde{\mathcal{L}})(0) + O(\epsilon) = \frac{2\alpha_1 \partial_s^2 c_1(0)}{\alpha_2 - \kappa \alpha_1} + O(\epsilon).$$

Let us set a distorted rectangular $\hat{D}_{\epsilon, r}(s)$ as follows:

$$\hat{D}_{\epsilon, r} := \cup_{\bar{\epsilon}=0}^\epsilon \cup_{\bar{r}=0}^r q_{\varphi(s+\bar{\epsilon})}^\perp(\ell(s')).$$

Note that s' is implicitly depending only on s , \bar{r} and $\bar{\epsilon}$. Since all of the four angles tend to $\pi/2$ when $\epsilon, s, r \rightarrow 0$,

$$\lim_{s \rightarrow 0} \lim_{\substack{\epsilon, r \rightarrow 0 \\ r/\epsilon = \eta}} \frac{\hat{D}_{\epsilon, r}}{|\hat{D}_{\epsilon, r}|}$$

is a rectangular (note that if $\partial_s c_1(0)$ is not zero, we cannot obtain such rectangular). Thus $r/\hat{r} \rightarrow 1$ as $s, \epsilon, r \rightarrow 0$. This means that for sufficiently small $\hat{\epsilon} > 0$, there is $\hat{R} = \hat{R}(\hat{\epsilon}, \eta)$ such that

$$(4.16) \quad (1 - \hat{\epsilon})r \leq \hat{r} \leq (1 + \hat{\epsilon})r$$

(expressing almost the same length on the right-left side curves) for $r < \hat{R}$ with $r/\epsilon = \eta$. From (4.15), we have

$$\begin{aligned} & s + \epsilon + (1 - \hat{\epsilon})\tilde{\mathcal{L}}(s + \epsilon)r - C(1 - \hat{\epsilon})^2 r^2 \\ & \leq \hat{\hat{s}} \\ & \leq s + \epsilon + \tilde{\mathcal{L}}(s + \epsilon)(1 + \hat{\epsilon})r + C(1 + \hat{\epsilon})^2 r^2 \end{aligned}$$

and then

$$\begin{aligned} & \epsilon + \left(\tilde{\mathcal{L}}(s + \epsilon) - \tilde{\mathcal{L}}(s) \right) r - \hat{\epsilon} \tilde{\mathcal{L}}(s + \epsilon)r + C(1 - (1 - \hat{\epsilon})^2)r^2 \\ & \leq \hat{\hat{s}} - \hat{s} \\ & \leq \epsilon + \left(\tilde{\mathcal{L}}(s + \epsilon) - \tilde{\mathcal{L}}(s) \right) r + \hat{\epsilon} \tilde{\mathcal{L}}(s + \epsilon)r + C((1 + \hat{\epsilon})^2 - 1)r^2, \end{aligned}$$

which we can rewrite to the following (unless confusion occurs):

$$\hat{\hat{s}} - \hat{s} = \epsilon + (\tilde{\mathcal{L}}(s + \epsilon) - \tilde{\mathcal{L}}(s))r + \text{error term}.$$

The right above error term is composed by $O(\hat{\epsilon}sr)$ and $O(r^2)$ (note that $\tilde{\mathcal{L}}(s) = 0 + O(s)$). We now estimate $|Z(\epsilon, s, r)|$. We re-define $\hat{r} = \hat{r}(\tilde{s})$ and $s' = s'(\tilde{s})$ as follows:

$$q_F(s'(\tilde{s})) = \Phi(\tilde{s}, \hat{r}(\tilde{s})).$$

Note that the meaning of s and \tilde{s} are different (opposite setting to Definition 4.1. In Definition 4.1, first, take upward-leftside curve, second, take rightward-upperside curve. In this definition, first, take rightward-lowerside (boundary) curve, second, take upward-rightside curve). By the well-known formula in geometry, we see

$$|Z(\epsilon, s, r)| = \int_{\hat{s}}^{\hat{\hat{s}}} \left| \partial_{\tilde{s}} \left(\Phi(\tilde{s}, \hat{r}(\tilde{s})) \right) \right| d\tilde{s}$$

Implicitly, \hat{s} and $\hat{\hat{s}}$ are depending on ϵ, s and r . We see that

$$\partial_{\tilde{s}} \left(\Phi(\tilde{s}, \hat{r}(\tilde{s})) \right) = \kappa \tau(\tilde{s}) \hat{r}(\tilde{s}) + n(\tilde{s}) (\partial_{\tilde{s}} \hat{r})(\tilde{s}) + \tau(\tilde{s}).$$

On the other hand,

$$\partial_{\tilde{s}} \left(\Phi(\tilde{s}, \hat{r}(\tilde{s})) \right) = \partial_{\tilde{s}} (q_F(s'(\tilde{s}))) = (\nabla p)(q_F(s'(\tilde{s}))) \partial_{\tilde{s}} s' = (\nabla p)(\Phi(\tilde{s}, \hat{r}(\tilde{s}))) \partial_{\tilde{s}} s'.$$

Multiply $n(\tilde{s})$ and $\tau(\tilde{s})$ on both sides, then we have

$$\partial_{\tilde{s}} \hat{r} = (\nabla p \cdot n)(\Phi(\tilde{s}, \hat{r}(\tilde{s}))) =: \nabla p \cdot n \quad \text{and} \quad \partial_{\tilde{s}} s' = \frac{\kappa \hat{r} + 1}{\nabla p \cdot \tau}.$$

This gives us that

$$\partial_{\tilde{s}} \hat{r} = (\kappa \hat{r} + 1) \frac{\nabla p \cdot n}{\nabla p \cdot \tau}.$$

Thus we have

$$\begin{aligned} |Z(\epsilon, s, r)| &= \int_{\hat{s}}^{\hat{\hat{s}}} \sqrt{(\kappa \hat{r} + 1)^2 + (\partial_{\tilde{s}} \hat{r}(\tilde{s}))^2} d\tilde{s} = \int_{\hat{s}}^{\hat{\hat{s}}} \frac{|\nabla p \cdot \tau|}{\sqrt{(\nabla p \cdot n)^2 + (\nabla p \cdot \tau)^2}} (\kappa \hat{r} + 1) d\tilde{s} \\ &= \int_{\hat{s}}^{\hat{\hat{s}}} \left(\cos \theta(\tilde{s}, \hat{r}) \right)^{-1} (\kappa \hat{r} + 1) d\tilde{s} \quad (\text{note that } \hat{r} \text{ is depending on } \tilde{s}), \end{aligned}$$

where $\theta(\Phi(\tilde{s}, \hat{r}))$ is defined by the angle between $(\nabla p / |\nabla p|)(\Phi(\tilde{s}, \hat{r}))$ and $\tau(\tilde{s})$. Due to the smoothness of p , we see that (just using Taylor expansion of the cosine function)

$$(4.17) \quad 1 - C\hat{r}^2 \leq \cos \theta(\Phi(\tilde{s}, \hat{r})) = \left(\frac{\nabla p}{|\nabla p|} \cdot \tau \right) (\Phi(\tilde{s}, \hat{r})) \leq 1.$$

Therefore we can estimate $|Z|$ as follows:

$$(\kappa(1 - \hat{\epsilon})r + 1) (\hat{\hat{s}} - \hat{s}) \leq |Z(\epsilon, s, r)| \leq \frac{\kappa(1 + \hat{\epsilon})r + 1}{1 - C(1 + \hat{\epsilon})^2 r^2} (\hat{\hat{s}} - \hat{s}).$$

By the above estimate, we have the following lower and upper bound of $|Z(\epsilon, s, r)|$:

$$(4.18) \quad \begin{aligned} &((1 - \hat{\epsilon})r\kappa + 1) \left[\epsilon + \left(\tilde{\mathcal{L}}(s + \epsilon) - \tilde{\mathcal{L}}(s) \right) r - \hat{\epsilon} \tilde{\mathcal{L}}(s + \epsilon) r + C(1 - (1 - \hat{\epsilon})^2) r^2 \right] \\ &\leq |Z(\epsilon, s, r)| \\ &\leq \frac{(1 + \hat{\epsilon})r\kappa + 1}{1 - C(1 + \hat{\epsilon})^2 r^2} \left[\epsilon + \left(\tilde{\mathcal{L}}(s + \epsilon) - \tilde{\mathcal{L}}(s) \right) r - \hat{\epsilon} \tilde{\mathcal{L}}(s + \epsilon) r + C((1 + \hat{\epsilon})^2 - 1) r^2 \right]. \end{aligned}$$

The above estimate tells us that

$$|Z(\epsilon, s, r)| = (r\kappa + 1) \left(\epsilon + (\tilde{\mathcal{L}}(s + \epsilon) - \tilde{\mathcal{L}}(s)) r \right) + \text{error term}.$$

The right above error term is composed by $O(\hat{\epsilon}sr)$ and $O(r^2)$. Thus we have

$$\begin{aligned} \frac{|Z(\epsilon, s, r)|}{\epsilon} &= (r\kappa + 1)(1 + \partial_s \tilde{\mathcal{L}}(s)r + O(\epsilon)r) + \text{error term} \\ &= 1 + \kappa r + \partial_s \tilde{\mathcal{L}}(s)r + \text{error term} \end{aligned}$$

The first and second errors are different. The second error is composed by $O(\epsilon r)$, $O(\hat{\epsilon}sr)$ and $O(r\eta)$. By the following fundamental formula,

$$\frac{1}{f(r)} = \frac{1}{f(0)} - \frac{f'(0)}{(f(0))^2} r + O(r^2),$$

we have

$$\begin{aligned} \frac{\epsilon}{|Z(\epsilon, s, r)|} &= \frac{1}{1 + O(\hat{\epsilon}sr)} - (\kappa + \partial_s \tilde{\mathcal{L}}(s))r + \text{error term} \\ &= 1 - (\kappa + \partial_s \tilde{\mathcal{L}}(s))r + \text{error term}. \end{aligned}$$

The right above error term is composed by $O(\epsilon r)$, $O(\eta r)$ and $O(\hat{\epsilon} s \eta)$. Now we estimate the gradient of the pressure at the point $\Phi(\hat{s}, r)$,

$$\begin{aligned}
(\nabla p \cdot \tau)(\Phi(\hat{s}, r)) &= |\nabla p(\Phi(\hat{s}, r))| \left(\frac{\nabla p}{|\nabla p|} \cdot \tau \right) (\Phi(\hat{s}, r)) \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{|Z(\epsilon, s, r)|} \left| \int_{Z(\epsilon, s, r)} \nabla p(x) \cdot \vec{t} \, dx \right| \left(\frac{\nabla p}{|\nabla p|} \cdot \tau \right) (\Phi(\hat{s}, r)) \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{|Z(\epsilon, s, r)|} |p(\Phi(\hat{s}, \hat{r})) - p(\Phi(\hat{s}, r))| \left(\frac{\nabla p}{|\nabla p|} \cdot \tau \right) (\Phi(\hat{s}, r)) \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{|Z(\epsilon, s, r)|} |p(\varphi(s + \epsilon)) - p(\varphi(s))| \left(\frac{\nabla p}{|\nabla p|} \cdot \tau \right) (\Phi(\hat{s}, r)),
\end{aligned}$$

where \vec{t} is a unit tangent vector along $Z(\epsilon, s, r)$. Note that

$$(4.19) \quad \left(\frac{\nabla p}{|\nabla p|} \cdot \tau \right) (\Phi(\hat{s}, r)) = \cos \theta(\Phi(\hat{s}, r))$$

can be estimated as (4.17), and

$$\begin{aligned}
\frac{|p(\varphi(s + \epsilon)) - p(\varphi(s))|}{\epsilon} &= |\partial_s(p(\varphi(s))) + O(\epsilon)| = |(\nabla p \cdot \tau)(\varphi(s)) + O(\epsilon)| \\
&= |(\Delta u \cdot \tau)(\varphi(s)) + O(\epsilon)| = |\kappa \alpha_1 - \alpha_2 + O(s^2) + O(\epsilon)|.
\end{aligned}$$

Let us consider the case $\kappa \alpha_1 - \alpha_2 \geq 0$ (we can obtain the same conclusion even in the case $\kappa \alpha_1 - \alpha_2 < 0$, since (4.19) becomes negative). We have

$$(4.20) \quad \frac{|p(\varphi(s + \epsilon)) - p(\varphi(s))|}{\epsilon} = \kappa \alpha_1 - \alpha_2 + O(s^2) + O(\epsilon) = (\Delta u \cdot \tau)(\varphi(0)) + O(s^2) + O(\epsilon).$$

Therefore we have

$$(\nabla p \cdot \tau)(\Phi(\hat{s}, r)) = (\Delta u \cdot \tau)(\varphi(0)) - (\Delta u \cdot \tau)(\varphi(0))(\kappa + \partial_s \tilde{\mathcal{L}}(0))r + \text{error term}$$

for $r < \hat{R}$. The right above error term is composed by $O(\epsilon r)$, $O(\eta r)$ and $O(\hat{\epsilon} s \eta)$. Therefore we have the following estimate (more precisely, in the limit procedure, we take $s \rightarrow 0$, $r \rightarrow 0$ and $\epsilon \rightarrow 0$ with $r/\epsilon = \eta$ and $s/r \rightarrow 0$):

$$\begin{aligned}
&\lim_{s \rightarrow 0} \partial_r [(\Delta u - \nabla p) \cdot \tau](\Phi(\hat{s}, 0)) \\
&= \lim_{s, r \rightarrow 0} \frac{[(\Delta u - \nabla p) \cdot \tau](\Phi(\hat{s}, r)) - [(\Delta u - \nabla p) \cdot \tau](\Phi(\hat{s}, 0))}{r} \\
&= \lim_{s, r \rightarrow 0} \frac{[(\Delta u - \nabla p) \cdot \tau](\Phi(\hat{s}, r))}{r} \\
&= \lim_{s, r \rightarrow 0} \frac{1}{r} \left((\Delta u \cdot \tau)(\varphi(\hat{s})) + [\partial_s(\Delta u \cdot \tau)](\varphi(\hat{s}))r + O(r^2) \right. \\
&\quad \left. - (\Delta u \cdot \tau)(\varphi(0)) - (\Delta u \cdot \tau)(\varphi(0))(\kappa + \partial_s \tilde{\mathcal{L}}(0))r + \text{error term} \right).
\end{aligned}$$

The right above error term is composed by $O(\epsilon r)$, $O(\eta r)$ and $O(\hat{\epsilon} s \eta)$. Therefore we have

$$\begin{aligned}
& \lim_{s \rightarrow 0} \partial_r [(\Delta u - \nabla p) \cdot \tau](\Phi(\hat{s}, 0)) \\
&= -2\alpha_1 \partial_s^2 c_1(0) - 2\kappa^2 \alpha_1 - \kappa \alpha_2 + \alpha_3 + (\Delta u \cdot \tau)(\varphi(0))(\kappa + \partial_s \tilde{\mathcal{L}}(0)) + \text{error term} \\
&= -2\alpha_1 \partial_s^2 c_1(0) - 2\kappa^2 \alpha_1 - \kappa \alpha_2 + \alpha_3 + (\kappa \alpha_1 - \alpha_2)(\kappa - \partial_s(\Delta u \cdot n)(\varphi(0))) + \text{error term} \\
&= -(4\alpha_1 \partial_s^2 c_1(0) + \kappa^2 \alpha_1 + 2\alpha_2 \kappa - \alpha_3) + \text{error term}.
\end{aligned}$$

The right above error term is composed by $O(\eta)$. Since η can be taken arbitrary small numbers, we have the desired estimate.

5. APPENDIX (LAPLACIAN FORMULA)

Now we give a detailed calculation of the Laplacian formula. It is fundamental but important (only using Leibnitz rule and chain rule of differentiation). Let θ be as (4.10). First we calculate $\partial_{y_1} \cos(\theta(y))$, $\partial_{y_1}^2 \cos(\theta(y))$, $\partial_{y_2} \cos(\theta(y))$ and $\partial_{y_2}^2 \cos(\theta(y))$. We see that

$$\begin{aligned}
\partial_{y_1} \cos(\theta(y)) &= -\sin \theta(y) \partial_{y_1} \theta(y), \\
\partial_{y_1}^2 \cos(\theta(y)) &= -\cos \theta(y) (\partial_{y_1} \theta(y))^2 - \sin \theta(y) \partial_{y_1}^2 \theta(y), \\
\partial_{y_2} \cos(\theta(y)) &= -\sin \theta(y) \partial_{y_2} \theta(y), \\
\partial_{y_2}^2 \cos(\theta(y)) &= -\cos \theta(y) (\partial_{y_2} \theta(y))^2 - \sin \theta(y) \partial_{y_2}^2 \theta(y).
\end{aligned}$$

Thus

$$\begin{aligned}
\partial_{y_1} \cos(\theta(y))|_{y=0} &= 0, \\
\partial_{y_1}^2 \cos(\theta(y))|_{y=0} &= -\frac{1}{(\kappa^{-1} + \bar{r})^2}, \\
\partial_{y_2} \cos(\theta(y))|_{y=0} &= 0, \\
\partial_{y_2}^2 \cos(\theta(y))|_{y=0} &= 0.
\end{aligned}$$

Second we calculate $\partial_{y_1} \sin(\theta(y))$, $\partial_{y_1}^2 \sin(\theta(y))$, $\partial_{y_2} \sin(\theta(y))$ and $\partial_{y_2}^2 \sin(\theta(y))$. We see that

$$\begin{aligned}
\partial_{y_1} \sin(\theta(y)) &= \cos \theta(y) \partial_{y_1} \theta(y), \\
\partial_{y_1}^2 \sin(\theta(y)) &= -\sin \theta(y) (\partial_{y_1} \theta(y))^2 + \cos \theta(y) \partial_{y_1}^2 \theta(y), \\
\partial_{y_2} \sin(\theta(y)) &= \cos \theta(y) \partial_{y_2} \theta(y), \\
\partial_{y_2}^2 \sin(\theta(y)) &= -\sin \theta(y) (\partial_{y_2} \theta(y))^2 + \cos \theta(y) \partial_{y_2}^2 \theta(y).
\end{aligned}$$

Thus

$$\begin{aligned}
\partial_{y_1} \sin(\theta(y))|_{y=0} &= \frac{1}{\kappa^{-1} + \bar{r}}, \\
\partial_{y_1}^2 \sin(\theta(y))|_{y=0} &= 0, \\
\partial_{y_2} \sin(\theta(y))|_{y=0} &= 0, \\
\partial_{y_2}^2 \sin(\theta(y))|_{y=0} &= 0.
\end{aligned}$$

Recall that $\partial_{y_1}^2 v^1(0) + \partial_{y_2}^2 v^1(0) = (\Delta u \cdot \tau)(\bar{s}, \bar{r})$ and $\partial_{y_1}^2 v^2(0) + \partial_{y_2}^2 v^2(0) = (\Delta u \cdot n)(\bar{s}, \bar{r})$. By the coordinate transformation (4.9), we have that $(u^s := (u \cdot \tau)(s, r), u^r := (u \cdot n)(s, r))$

$$(5.21) \quad \begin{cases} \partial_{y_1}^2 v^1(y)|_{y=0} = \left(\partial_{y_1}^2 u^s - \frac{u^s}{(\kappa^{-1} + \bar{r})^2} + \frac{2\partial_{y_1} u^r}{\kappa^{-1} + \bar{r}} \right) \Big|_{y=0}, \\ \partial_{y_2}^2 v^1(y)|_{y=0} = \partial_{y_2}^2 u^s|_{y=0}, \\ \partial_{y_1}^2 v^2(y)|_{y=0} = \left(\partial_{y_1}^2 u^r - \frac{u^r}{(\kappa^{-1} + \bar{r})^2} - \frac{2\partial_{y_1} u^s}{\kappa^{-1} + \bar{r}} \right) \Big|_{y=0}, \\ \partial_{y_2}^2 v^2(y)|_{y=0} = \partial_{y_2}^2 u^r|_{y=0}. \end{cases}$$

In order to calculate (5.21), we use the following estimates:

$$\begin{aligned} \partial_{y_1} r &= \frac{y_1}{\{y_1^2 + (\kappa^{-1} + \bar{r} + y_2)^2\}^{\frac{1}{2}}} (= \sin(\theta(y))), \\ \partial_{y_1}^2 r &= \frac{1}{\{y_1^2 + (\kappa^{-1} + \bar{r} + y_2)^2\}^{\frac{1}{2}}} - \frac{y_1^2}{\{y_1^2 + (\kappa^{-1} + \bar{r} + y_2)^2\}^{\frac{3}{2}}}, \\ \partial_{y_2} r &= \frac{\kappa^{-1} + \bar{r} + y_2}{\{y_1^2 + (\kappa^{-1} + \bar{r} + y_2)^2\}^{\frac{1}{2}}} (= \cos(\theta(y))), \\ \partial_{y_2}^2 r &= \frac{1}{\{y_1^2 + (\kappa^{-1} + \bar{r} + y_2)^2\}^{\frac{1}{2}}} - \frac{(\kappa^{-1} + \bar{r} + y_2)^2}{\{y_1^2 + (\kappa^{-1} + \bar{r} + y_2)^2\}^{\frac{3}{2}}}, \\ \partial_{y_1} s &= \frac{\kappa^{-1}(\kappa^{-1} + \bar{r} + y_2)}{y_1^2 + (\kappa^{-1} + \bar{r} + y_2)^2}, \\ \partial_{y_1}^2 s &= -\frac{2\kappa^{-1}y_1(\kappa^{-1} + \bar{r} + y_2)}{\{y_1^2 + (\kappa^{-1} + \bar{r} + y_2)^2\}^2}, \\ \partial_{y_2} s &= -\frac{\kappa^{-1}y_1}{y_1^2 + (\kappa^{-1} + \bar{r} + y_2)^2}, \\ \partial_{y_2}^2 s &= \frac{2\kappa^{-1}y_1(\kappa^{-1} + \bar{r} + y_2)}{\{y_1^2 + (\kappa^{-1} + \bar{r} + y_2)^2\}^2}. \end{aligned}$$

In particular,

$$\begin{aligned} \partial_{y_1} r|_{y=0} &= 0, \\ \partial_{y_1}^2 r|_{y=0} &= \frac{1}{\kappa^{-1} + \bar{r}}, \\ \partial_{y_2} r|_{y=0} &= 1, \\ \partial_{y_2}^2 r|_{y=0} &= 0, \\ \partial_{y_1} s|_{y=0} &= \frac{\kappa^{-1}}{\kappa^{-1} + \bar{r}}, \\ \partial_{y_1}^2 s|_{y=0} &= 0, \\ \partial_{y_2} s|_{y=0} &= 0, \\ \partial_{y_2}^2 s|_{y=0} &= 0. \end{aligned}$$

Therefore we have that

$$\begin{aligned}
\partial_{y_1} u^s|_{y=0} &= \frac{\kappa^{-1}}{\kappa^{-1} + \bar{r}} \partial_s u^s(\bar{s}, \bar{r}), \\
\partial_{y_1}^2 u^s|_{y=0} &= \left(\frac{\kappa^{-1}}{\kappa^{-1} + \bar{r}} \right)^2 \partial_s^2 u^s(\bar{s}, \bar{r}) + \frac{1}{\kappa^{-1} + \bar{r}} \partial_r u^s(\bar{s}, \bar{r}), \\
\partial_{y_2} u^s|_{y=0} &= \partial_r u^s(\bar{s}, \bar{r}), \\
\partial_{y_2}^2 u^s|_{y=0} &= \partial_r^2 u^s(\bar{s}, \bar{r}), \\
\partial_{y_1} u^r|_{y=0} &= \frac{\kappa^{-1}}{\kappa^{-1} + \bar{r}} \partial_s u^r(\bar{s}, \bar{r}), \\
\partial_{y_1}^2 u^r|_{y=0} &= \left(\frac{\kappa^{-1}}{\kappa^{-1} + \bar{r}} \right)^2 \partial_s^2 u^r(\bar{s}, \bar{r}) + \frac{1}{\kappa^{-1} + \bar{r}} \partial_r u^r(\bar{s}, \bar{r}), \\
\partial_{y_2} u^r|_{y=0} &= \partial_r u^r(\bar{s}, \bar{r}), \\
\partial_{y_2}^2 u^r|_{y=0} &= \partial_r^2 u^r(\bar{s}, \bar{r}).
\end{aligned}$$

First we calculate $\partial_s u^s|_{s=0}$ and $\partial_s^2 u^s|_{s=0}$. Here we use the following notations for the sake of simplicity: $u^s(L^{-1}) := u^s(0, L^{-1}(s, r))$ and $(\partial_{r_0} L)(L^{-1}) := (\partial_{r_0} L)(s, L^{-1}(s, r))$. Recall that

$$u^s(s, r) = \frac{u^s(0, L^{-1}(s, r))}{(\partial_{r_0} L)(s, L^{-1}(s, r))} = \frac{u^s(L^{-1})}{(\partial_{r_0} L)(L^{-1})}.$$

Since (left hand side of u^s is $u^s(s, r)$, right hand side of $u^s(L^{-1})$ is $u^s(0, L^{-1}(s, r))$)

$$\begin{aligned}
\partial_s u^s &= \frac{\partial_s (u^s(L^{-1}))}{(\partial_{r_0} L)(L^{-1})} + u^s(L^{-1}) \partial_s \left(\frac{1}{(\partial_{r_0} L)(L^{-1})} \right), \\
\partial_s^2 u^s &= \frac{\partial_s^2 (u^s(L^{-1}))}{(\partial_r L)(L^{-1})} + 2 \partial_s (u^s(L^{-1})) \partial_s \left(\frac{1}{(\partial_{r_0} L)(L^{-1})} \right) + u^s(L^{-1}) \partial_s^2 \left(\frac{1}{(\partial_{r_0} L)(L^{-1})} \right),
\end{aligned}$$

and (here we use $L := L(L^{-1})$ and $u^s := u^s(L^{-1})$)

$$\begin{aligned}
\partial_s (u^s(L^{-1})) &= (\partial_r u^s) \partial_s L^{-1}, \\
\partial_s^2 (u^s(L^{-1})) &= (\partial_r^2 u^s) (\partial_s L^{-1})^2 + (\partial_r u^s) (\partial_s^2 L^{-1}), \\
\partial_s \left(\frac{1}{(\partial_{r_0} L)(L^{-1})} \right) &= - \frac{(\partial_s \partial_{r_0} L) + (\partial_{r_0}^2 L) (\partial_s L^{-1})}{(\partial_{r_0} L)^2}, \\
\partial_s^2 \left(\frac{1}{(\partial_{r_0} L)(L^{-1})} \right) &= \frac{2 ((\partial_s \partial_{r_0} L) + (\partial_{r_0}^2 L) (\partial_s L^{-1}))^2}{(\partial_{r_0} L)^3} \\
&\quad - \frac{(\partial_s^2 \partial_{r_0} L) + 2(\partial_s \partial_{r_0}^2 L) (\partial_s L^{-1}) + (\partial_{r_0}^3 L) (\partial_s L^{-1})^2 + (\partial_{r_0}^2 L) (\partial_s^2 L^{-1})}{(\partial_{r_0} L)^2},
\end{aligned}$$

we have (by Remark 2.8, and here we take $\partial_s c_1(0) = \partial_s c_2(0) = \partial_s c_3(0) = 0$)

$$\begin{aligned}
\partial_s u^s|_{s=0} &= 0 + O(r^2), \\
\partial_s^2 u^s|_{s=0} &= -2\alpha_1 \partial_s^2 c_1(0) r + O(r^2). \\
\partial_s u^s|_{r=0} &= 0, \\
\partial_s^2 u^s|_{r=0} &= 0.
\end{aligned}$$

Second we estimate $\partial_r u^s|_{s=0}$ and $\partial_r^2 u^s|_{s=0}$. Since

$$\begin{aligned}\partial_r u^s &= \frac{\partial_r (u^s(L^{-1}))}{(\partial_{r_0} L)(L^{-1})} + u^s(L^{-1}) \partial_r \left(\frac{1}{(\partial_{r_0} L)(L^{-1})} \right), \\ \partial_r^2 u^s &= \frac{\partial_r^2 (u^s(L^{-1}))}{(\partial_r L)(L^{-1})} + 2\partial_r (u^s(L^{-1})) \partial_r \left(\frac{1}{(\partial_{r_0} L)(L^{-1})} \right) + u^s(L^{-1}) \partial_r^2 \left(\frac{1}{(\partial_{r_0} L)(L^{-1})} \right)\end{aligned}$$

and (here we also use $L := L(L^{-1})$)

$$\begin{aligned}\partial_r (u^s(L^{-1})) &= (\partial_r u^s) \partial_r L^{-1}, \\ \partial_r^2 (u^s(L^{-1})) &= (\partial_r^2 u^s) (\partial_r L^{-1})^2 + (\partial_r u^s) \partial_r^2 L^{-1}, \\ \partial_r \left(\frac{1}{(\partial_{r_0} L)(L^{-1})} \right) &= -\frac{(\partial_{r_0}^2 L) \partial_r L^{-1}}{(\partial_{r_0} L)^2}, \\ \partial_r^2 \left(\frac{1}{(\partial_{r_0} L)(L^{-1})} \right) &= \frac{2(\partial_{r_0}^2 L)^2 \partial_r L^{-1}}{(\partial_{r_0} L)^3} - \frac{(\partial_{r_0}^3 L) \partial_r L^{-1}}{(\partial_{r_0} L)^2} - \frac{(\partial_{r_0}^2 L) \partial_r^2 L^{-1}}{(\partial_{r_0} L)^2},\end{aligned}$$

we have (by Remark 2.8)

$$(5.22) \quad \begin{aligned}\partial_r u^s|_{s=0} &= \alpha_1 - \alpha_2 r + O(r^2), \\ \partial_r^2 u^s|_{s=0} &= -\alpha_2 + \alpha_3 r + O(r^2), \\ \partial_r u^s|_{r=0} &= \frac{\alpha_1}{c_1^2(s)}, \\ \partial_r^2 u^s|_{r=0} &= -\frac{\alpha_2}{c_1^3(s)} - \frac{\alpha_1 c_2(s)}{c_1^2(s)} - \frac{2\alpha_1 c_2(s)}{c_1^4(s)}.\end{aligned}$$

Note that $c_2(s) = c_2(0) + \partial_s c_2(0)s + O(s^2) = 0 + O(s^2)$ (here we take $\partial_s c_2(0) = 0$).

Third we estimate $\partial_s u^r$, $\partial_s^2 u^r$, $\partial_r u^r$ and $\partial_r^2 u^r$. Recall that $u^r(s, r) = \frac{u^s(s, r) \mathcal{L}(s, r)}{1 + r\kappa}$. Since (we use $\mathcal{L} := \mathcal{L}(s, r)$)

$$\begin{aligned}\partial_s u^r &= (\partial_s u^s \mathcal{L} + u^s \partial_s \mathcal{L}) \frac{1}{1 + r\kappa}, \\ \partial_s^2 u^r &= (\partial_s^2 u^s \mathcal{L} + 2\partial_s u^s \partial_s \mathcal{L} + u^s \partial_s^2 \mathcal{L}) \frac{1}{1 + r\kappa}, \\ \partial_r u^r &= (\partial_r u^s \mathcal{L} + u^s \partial_r \mathcal{L}) \frac{1}{1 + r\kappa} - u^s \mathcal{L} \frac{\kappa}{(1 + r\kappa)^2}, \\ \partial_r^2 u^r &= (\partial_r^2 u^s \mathcal{L} + 2\partial_r u^s \partial_r \mathcal{L} + u^s \partial_r^2 \mathcal{L}) \frac{1}{1 + r\kappa} - (u^s \partial_r \mathcal{L} + \partial_r u^s \mathcal{L}) \frac{\kappa}{(1 + r\kappa)^2} + u^s \mathcal{L} \frac{2\kappa}{(1 + r\kappa)^3},\end{aligned}$$

we have (by Remark 2.8)

$$\begin{aligned}
\partial_s u^r|_{s=0} &= 0 + O(r^2), \\
\partial_s^2 u^r|_{s=0} &= 0 + O(r^2). \\
\partial_s u^r|_{r=0} &= 0, \\
\partial_s^2 u^r|_{r=0} &= 0, \\
\partial_r u^r|_{s=0} &= 0 + O(r^2), \\
\partial_r^2 u^r|_{s=0} &= 0 + O(r^2). \\
\partial_r u^r|_{r=0} &= 0, \\
\partial_r^2 u^r|_{r=0} &= \frac{2\alpha_1 \partial_s c_1(s)}{c_1^3(s)}.
\end{aligned}$$

Note that $\partial_s c_1(s) = \partial_s^2 c_1(0)s + O(s^2)$. Combining these estimates with $\bar{s} = 0$ or $\bar{r} = 0$, we have (4.8).

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