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LORENTZ SPACE ESTIMATES FOR VECTOR FIELDS WITH DIVERGENCE AND CURL IN HARDY SPACES

YOSHIKAZU GIGA AND XINGFEI XIANG

Abstract. In this note, we establish the estimate on the Lorentz space $L(3/2,1)$ for vector fields in bounded domains under the assumption that the normal or the tangential component of the vector fields on the boundary vanishing. We prove that the $L(3/2,1)$ norm of the vector field can be controlled by the norms of its divergence and curl in the atomic Hardy spaces and the L^1 norm of the vector field itself.

1. Introduction

In this note, we consider the estimate on the Lorentz space $L(3/2,1)$ for vector fields in a bounded domain Ω in $\mathbb{R}^3$ by assuming that the divergence and the curl in atomic Hardy spaces. This work originates from the problem raised by Bourgain and Brezis in [4, open problem 1], where assume the divergence-free and the curl in L^1. Our result in this note shows that the $L(3/2,1)$ estimate controlled by the divergence and curl in the atomic Hardy spaces holds if the normal or the tangential component of the vector fields on the boundary vanishing. While for the case where the divergence-free and the curl in L^1 we still don't know how to treat.

For the case where the vector $\mathbf{u} \in W_0^{1,p}(\Omega)$ with $1 < p < \infty$, the estimate on the Lorentz space with the divergence and the curl in the Hardy spaces is easy to obtain. Indeed, applying the representation from the fundamental theorem of vector analysis

$$\mathbf{u} = -\frac{1}{4\pi}\mathrm{grad}\int_\Omega \frac{1}{|x-y|}\,\mathrm{div}\,\mathbf{u}(y)dy + \frac{1}{4\pi}\,\mathrm{curl}\int_\Omega \frac{1}{|x-y|}\,\mathrm{curl}\,\mathbf{u}(y)dy, \tag{1.1}$$

we can obtain the estimate on $\nabla\mathbf{u}$ by the estimate on the singular integrals in Hardy spaces (see [19, Theorem 3.3])

$$\|\nabla\mathbf{u}\|_{L^1(\Omega)} \le C(\|\,\mathrm{div}\,\mathbf{u}\|_{\mathcal{H}(\Omega)} + \|\,\mathrm{curl}\,\mathbf{u}\|_{\mathcal{H}(\Omega)}),$$

where the norm $\|\cdot\|_{\mathcal{H}(\Omega)}$ denotes

$$\|f\|_{\mathcal{H}(\Omega)} = \|\tilde{f}\|_{\mathcal{H}(\mathbb{R}^3)},$$

$\tilde{f}$ is the zero extension of the function f outside of Ω and $\mathcal{H}(\mathbb{R}^3)$ is the usual Hardy space $H^p(\mathbb{R}^3)$ with $p = 1$ (see [19, Chapter III]); here and hereafter C denotes a positive constant independent of vector fields or functions and its numerical value may be different in each occasion. Then using the L^1 estimate for the Newtonian potential, and noting

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that $\mathcal{H}(\Omega)$ is continuously imbedded into the space $L^1(\Omega)$, we have

$$\begin{aligned}\|\mathbf{u}\|_{L^1(\Omega)} &\le C(\|\operatorname{div}\mathbf{u}\|_{L^1(\Omega)} + \|\operatorname{curl}\mathbf{u}\|_{L^1(\Omega)})\\ &\le C(\|\operatorname{div}\mathbf{u}\|_{\mathcal{H}(\Omega)} + \|\operatorname{curl}\mathbf{u}\|_{\mathcal{H}(\Omega)}).\end{aligned}$$

Noting the fact that $W^{1,1}(\Omega)$ is continuously imbedded into the Lorentz space $L(3/2,1)$, we can thus obtain that

$$\|\mathbf{u}\|_{L(3/2,1)(\Omega)} \le C(\|\operatorname{div}\mathbf{u}\|_{\mathcal{H}(\Omega)} + \|\operatorname{curl}\mathbf{u}\|_{\mathcal{H}(\Omega)}).$$

But for the vector $\mathbf{u}$ not vanishing on the boundary, two terms involving boundary integrals will be added to the representation (1.1) (see [24]). Both of the terms are not easy to deal with on the Lorentz space.

This note studies the vector fields with the normal or the tangential components on the boundary vanishing but not the zero boundary condition, in contrast to the representation (1.1), the Helmholtz-Weyl decomposition on the Lorentz spaces in our proof will be employed.

Let $\nu(x)$ be the unit outer normal vector at $x \in \partial\Omega$. Our main result now reads:

Theorem 1.1. *Assume Ω is a bounded domain in $\mathbb{R}^3$ with C^2 boundary. Let $\mathbf{u} \in C^{1,\alpha}(\bar{\Omega})$ with $\operatorname{div}\mathbf{u} \in \mathcal{H}_{\mu_0}(\Omega)$ and $\operatorname{curl}\mathbf{u} \in \mathcal{H}_{\mu_0}(\Omega)$, where the atomic Hardy space $\mathcal{H}_{\mu_0}(\Omega)$ is defined in Definition 2.4. Then if either $\nu \cdot \mathbf{u} = 0$ or $\nu \times \mathbf{u} = 0$ on $\partial\Omega$, we have*

$$\|\mathbf{u}\|_{L(3/2,1)(\Omega)} \le C\left(\|\operatorname{div}\mathbf{u}\|_{\mathcal{H}_{\mu_0}(\Omega)} + \|\operatorname{curl}\mathbf{u}\|_{\mathcal{H}_{\mu_0}(\Omega)} + \|\mathbf{u}\|_{L^1(\Omega)}\right), \tag{1.2}$$

where the constant C depending only on μ_0 and the domain Ω, but not on the vector $\mathbf{u}$.

Our proof for proving Theorem 1.1 is based on the Helmholtz-Weyl decomposition on the Lorentz spaces (For the decomposition on L^p spaces we refer to [12, Theorem 2.1]):

$$\mathbf{u} = \nabla p_{\mathbf{u}} + \operatorname{curl}\mathbf{w}_{\mathbf{u}} + \mathscr{H}_{\mathbf{u}}, \tag{1.3}$$

where $\mathscr{H}_{\mathbf{u}}$ is the harmonic part, the function $p_{\mathbf{u}}$ satisfies the Laplace equation and the vector $\mathbf{w}_{\mathbf{u}}$ satisfies the elliptic system involving $\operatorname{curl}\mathbf{u}$. The advantage of using (1.3) is that we need not handle the terms involving boundary integral. Then our strategy is to establish the estimate on the Lorentz norm of $\nabla p_{\mathbf{u}}$ by the norm of $\operatorname{div}\mathbf{u}$ in the Hardy space, and of $\operatorname{curl}\mathbf{w}_{\mathbf{u}}$ by the norm of $\operatorname{curl}\mathbf{u}$. To obtain these estimates, the duality between several spaces will be introduced. The harmonic part $\mathscr{H}_{\mathbf{u}}$ because of its regularity can be controlled by the L^1 norm of the vector $\mathbf{u}$ itself.

We would like to mention that starting with the pioneering work in [2] by Bourgain and Brezis, related interesting L^1 estimate for vector fields have been well studied by several authors, see [2-5, 14-16, 20-23, 25] and the references therein. In particular, Bourgain and Brezis in [3, 4] showed the $L^{3/2}(\mathbb{T}^3)$ norm of the divergence-free vector $\mathbf{u}$ can be controlled by the L^1 norm of $\operatorname{curl}\mathbf{u}$; Lanzani and Stein in [14] obtained the estimate of the smooth q-forms on the $L^{3/2}$ spaces by the L^1 norms of their exterior derivative and co-exterior derivative; I. Mitrea and M. Mitrea in [16] considered these estimates in homogeneous Besov spaces; Van Schaftingen in [23] established the estimates in Besov, Triebel-Lizorkin and Lorentz spaces of differential forms on $\mathbb{R}^n$ in terms of their L^1 norm.

The organization of this paper is as follows. In Section 2, some well known spaces are introduced. The proof of Theorem 1.1 will be given in section 3. Throughout the paper, the bold typeface is used to indicate vector quantities; normal typeface will be used for vector components and for scalars.

2. Lorentz spaces, Hardy spaces and BMO spaces

In this section we will introduce several well-known spaces and show some properties of these spaces, which will be used in the proof of our theorem. These spaces can be found in many literatures and papers.

2.1. **Lorentz spaces.** Let (X, S, μ) be a $\sigma-$finite measure space and $f : X \to \mathbb{R}$ be a measurable function. We define the distribution function of f as

$$f_*(s) = \mu(\{|f| > s\}), \quad s > 0,$$

and the nonincreasing rearrangement of f as

$$f^*(t) = \inf\{s > 0, f_*(s) \leq t\}, \quad t > 0.$$

The Lorentz space is defined by

$$L(m, q) = \{f : X \to \mathbb{R} \text{ measurable}, \|f\|_{L^{m,q}} < \infty\} \quad \text{with } 1 \leq m < \infty$$

equipped with the quasi-norm

$$\|f\|_{L(m,q)} = \Big(\int_0^\infty \big(t^{1/m} f^*(t)\big)^q \frac{dt}{t}\Big)^{1/q}, \quad 1 \leq q < \infty$$

and

$$\|f\|_{L(m,\infty)} = \sup_{t>0} t^{1/m} f^*(t), \quad q = \infty.$$

From the definition of the Lorentz space, we can obtain the following properties.

Lemma 2.1. [1, Embedding theorem] *Let Ω be a bounded domain in $\mathbb{R}^3$. We have the following conclusions.*

(i) *If $p > q$ and $1 < m < \infty$, then $\|f\|_{L(m,p)(\Omega)} \leq C\|f\|_{L(m,q)(\Omega)}$.*
(ii) *If $m < n$ and $1 \leq p, q \leq \infty$, then $\|f\|_{L(n,p)(\Omega)} \leq C(\Omega)\|f\|_{L(m,q)(\Omega)}$.*

Lemma 2.2. [8, 11, Duality of Lorentz spaces] *Assume that Ω is a bounded domain in $\mathbb{R}^3$, and let f be a bounded function belonging to $L(m, p)$ with $1 < m < \infty$ and $1 \leq p \leq \infty$. Then there exists a constant C independent of f satisfying*

$$\|f\|_{L(m,p)} \leq C \sup_{g \in L^{m',p'}} \frac{\int_\Omega f g dx}{\|g\|_{L(m',p')}}$$

with $m' = m/(m-1)$, $p' = p/(p-1)$.

Proof. For the case $1 \leq p < \infty$, the dual space of $L(m, p)$ is $L(m', p')$ (see [11]), the conclusion follows immediately. The dual space of $L(m, \infty)$ is $L(m', 1) \oplus S_0 \oplus S_\infty$, where the spaces S_0 and S_∞ defined in [8] annihilate all functions which are bounded and supported on any set of finite measure (see [8]). Thus, the desired estimate follows from the assumptions when $p = \infty$. □

2.2. **Hardy spaces.** There are several equivalent definitions for Hardy spaces in $\mathbb{R}^n$ and in bounded domains. In this paper we define the Hardy space in the bounded domains by the atomic decomposition.

Definition 2.3. *An $\mathcal{H}_{\mu_0}(\mu_0 > 0)$ atom with respect to the cube Q is a function $a(x)$ satisfying the following three conditions:*

(i) *the function $a(x)$ is supported in a cube Q;*
(ii) *the inequality $|a| \leq |Q|^{-1}$ holds almost everywhere;*
(iii) *there exists a constant μ with $\mu \geq \mu_0 > 0$ such that for $|Q| < 1$ we have*

$$\left| \int_Q a(x) dx \right| \leq |Q|^{\mu}.$$

We now define the Hardy spaces $\mathcal{H}_{\mu_0}(\Omega)$ appeared in Theorem 1.1.

Definition 2.4. *[Atomic Hardy spaces] A function f defined on Ω belongs to $\mathcal{H}_{\mu_0}(\Omega)$ if the function f can be expressed as*

$$f = \sum_k \lambda_k a_k, \tag{2.1}$$

where a_k is a collection of $\mathcal{H}_{\mu_0}$ atoms with respect to the cube Q_k with $Q_k \subset \Omega$ and λ_k is a sequence of complex numbers with $\sum |\lambda_k| < \infty$. Furthermore, the norm of $\|f\|_{\mathcal{H}_{\mu_0}}$ is defined by

$$\|f\|_{\mathcal{H}_{\mu_0}} = \inf \sum |\lambda_k|,$$

where the infimum is taken over all the decompositions (2.1)*.*

We need to mention that when $\mu_0 = 1/3$, Chang et al give an equivalent definition of $\mathcal{H}_{\mu_0}$ by means of a grand maximal function, see Definition 1.2 and Theorem 2.5 in [6].

2.3. **BMO spaces.** A local integrable function f will be said to belong to BMO if the inequality

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq A \tag{2.2}$$

holds for all cubes Q, here $f_Q = |Q|^{-1} \int_Q f dx$ denotes the mean value of f over the cube Q. The smallest bound A for which (2.2) is satisfied is the taken to be the semi-norm of f in this space, and is denoted by $\|f\|_{BMO}$.

Lemma 2.5. [7, Sobolev embedding into BMO] *Let $g \in BMO(\Omega)$ with $\nabla g \in L(3, \infty)(\Omega)$. Then we have*

$$\|g\|_{BMO(\Omega)} \leq \|\nabla g\|_{L(3,\infty)(\Omega)}. \tag{2.3}$$

Lemma 2.6 (Duality of BMO)**.** *Let $f \in \mathcal{H}_{\mu_0}(\Omega)$ and $g \in BMO(\Omega)$. Then for any $1/\mu_0 < p < \infty$ we have*

$$\int_\Omega f \cdot g dx \leq C(\|g\|_{BMO(\Omega)} + \|g\|_{L^p(\Omega)}) \|f\|_{\mathcal{H}_{\mu_0}(\Omega)}, \tag{2.4}$$

with a constant C depending only on μ_0, p and the domain Ω.

Proof. From the definition of the space $\mathcal{H}_{\mu_0}(\Omega)$, the integral in the left side of (2.4) can be written as

$$\int_\Omega f \cdot g dx = \sum_k \lambda_k \int_{Q_k} a_k \cdot g dx, \tag{2.5}$$

where a_k is a collection of $\mathcal{H}_{\mu_0}$ atoms and λ_k is a sequence of complex numbers. Note that

$$\int_{Q_k} a_k \cdot g dx = \int_{Q_k} a_k \cdot (g - g_{Q_k}) dx + g_{Q_k} \int_{Q_k} a_k dx, \tag{2.6}$$

where g_{Q_k} denotes the mean value of g over the cubic Q_k. From the definition of the BMO space and the condition (ii) in Definition 2.3, it follows that

$$\left| \int_{Q_k} a_k \cdot (g - g_{Q_k}) dx \right| \le \|g\|_{BMO(\Omega)}.$$

From the condition (iii) in Definition 2.3 and by Hölder's inequality, we have

$$\left| g_{Q_k} \int_{Q_k} a_k dx \right| \le \left| |Q_k|^{\mu-1} \int_{Q_k} g dx \right| \le C|Q_k|^{\mu_0 - 1 + 1/q} \|g\|_{L^p(\Omega)} \le C\|g\|_{L^p(\Omega)}$$

where p and q are conjugate exponents satisfying $1/p + 1/q = 1$, $p > 1/\mu_0$ and the constant C depends on p, μ_0 and the domain. Plugging the above two inequalities to (2.6), and then by (2.5) we have

$$\left| \int_\Omega f \cdot g dx \right| \le C(\|g\|_{BMO(\Omega)} + \|g\|_{L^p(\Omega)}) \sum_k |\lambda_k|. \tag{2.7}$$

Taking the infimum on both sides in (2.7), we obtain this lemma. □

3. Proof of the main Theorem

Before proving our main theorem, we first introduce the Dirichlet fields $\mathbb{H}_2(\Omega)$:

$$\mathbb{H}_2(\Omega) = \{\mathbf{u} \in C^2(\Omega) : \operatorname{div} \mathbf{u} = 0, \operatorname{curl} \mathbf{u} = 0 \text{ in } \Omega, \nu \times \mathbf{u} = 0 \text{ on } \partial\Omega\},$$

and the Neumann fields $\mathbb{H}_1(\Omega)$:

$$\mathbb{H}_1(\Omega) = \{\mathbf{u} \in C^2(\Omega) : \operatorname{div} \mathbf{u} = 0, \operatorname{curl} \mathbf{u} = 0 \text{ in } \Omega, \nu \cdot \mathbf{u} = 0 \text{ on } \partial\Omega\}.$$

Both of the spaces depend only on the topological structure of Ω, and

$$\dim(\mathbb{H}_2(\Omega)) = m, \quad \dim(\mathbb{H}_1(\Omega)) = N,$$

where N and m are respectively the first and the second Betti number of the domain Ω, in this note we assume both of them are finite, we refer to [9, Chapter 9] for details.

Let $1 < p < \infty$ and $1 \le q \le \infty$. Then we define

$$V(p,q) \equiv \{\mathbf{w} \in L^p(\Omega), \nabla \mathbf{w} \in L(p,q)(\Omega) \ : \ \text{div } \mathbf{w} = 0, \ \nu \times \mathbf{w} = 0 \text{ on } \partial\Omega\},$$

$$X(p,q) \equiv \{\mathbf{w} \in L^p(\Omega), \nabla \mathbf{w} \in L(p,q)(\Omega) \ : \ \text{div } \mathbf{w} = 0, \ \nu \cdot \mathbf{w} = 0 \text{ on } \partial\Omega\}.$$

We now establish the decomposition for the vector fields on the Lorentz spaces.

Lemma 3.1 (Decomposition of the Lorentz spaces). *Suppose that Ω is a bounded domain in $\mathbb{R}^3$ with C^2 boundary, and let $1 < p < \infty$ and $1 \le q \le \infty$. Then*

Case 1. Each element $\mathbf{u} \in L(p,q)(\Omega)$ has the unique decomposition:

$$\mathbf{u} = \nabla v + \operatorname{curl} \mathbf{w} + \mathbf{h}, \tag{3.1}$$

where $\nabla v \in L(p,q)(\Omega)$, $\mathbf{w} \in V(p,q)(\Omega)$ and $\mathbf{h} \in \mathbb{H}_1(\Omega)$. Also, we have the estimate:

$$\|\nabla v\|_{L(p,q)(\Omega)} + \|\nabla \mathbf{w}\|_{L(p,q)(\Omega)} + \|\mathbf{h}\|_{L(p,q)(\Omega)} \le C(p,q,\Omega)\|\mathbf{u}\|_{L(p,q)(\Omega)}. \tag{3.2}$$

Case 2. Each element $\mathbf{u} \in L(p,q)(\Omega)$ has the unique decomposition:

$$\mathbf{u} = \nabla \hat{v} + \operatorname{curl} \hat{\mathbf{w}} + \hat{\mathbf{h}}. \tag{3.3}$$

where $\nabla \hat{v} \in L(p,q)(\Omega)$, $\hat{v} \in W_0^{1,r}(\Omega)$ with $r < p$ and $\nabla \hat{\mathbf{w}} \in X(p,q)(\Omega)$ $\hat{\mathbf{h}} \in \mathbb{H}_2(\Omega)$. Also, we have the estimate:

$$\|\nabla \hat{v}\|_{L(p,q)(\Omega)} + \|\nabla \hat{\mathbf{w}}\|_{L(p,q)(\Omega)} + \|\hat{\mathbf{h}}\|_{L(p,q)(\Omega)} \le C(p,q,\Omega)\|\mathbf{u}\|_{L(p,q)(\Omega)}. \tag{3.4}$$

Proof. The decompositions for vector fields and the estimate in the Sobolev L^p spaces have been obtained earlier by Kozono and Yanagisawa in [12]. Hence, it suffices to show the estimate (3.2) and (3.4). We shall use the fact that the Lorentz space $L(p,q)$ is the real interpolation space between Lebesgue spaces L^{p_1} and L^{p_2} with $p_1 < p < p_2$ to obtain the estimate. We only prove the inequality (3.2), since (3.4) can be treated in a similar way.

From Simader and Sohr in [17] we see that, for any $\mathbf{u} \in L^p(\Omega)$ with $1 < p < \infty$ there exists $v_{\mathbf{u}} \in W^{1,p}$ being the weak solution of the Neumann problem and satisfying $\int v_{\mathbf{u}} dx = 0$ such that

$$\Delta v_{\mathbf{u}} = \operatorname{div} \mathbf{u} \quad \text{in } \Omega, \qquad \frac{\partial v_{\mathbf{u}}}{\partial \nu} = \nu \cdot \mathbf{u} \quad \text{on } \partial\Omega. \tag{3.5}$$

That is $v_{\mathbf{u}}$ satisfying the following weak form

$$\int_\Omega \nabla v_{\mathbf{u}} \cdot \nabla \phi dx = \int_\Omega \mathbf{u} \cdot \nabla \phi dx \quad \text{for any } \phi \in W^{1,p}(\Omega).$$

Define the linear operator T_1 from $L^p(\Omega)$ to $L^p(\Omega)$ by

$$T_1 : \mathbf{u} \to T_1 \mathbf{u} = \nabla v_{\mathbf{u}}.$$

Then we can get the estimate

$$\|T_1 \mathbf{u}\|_{L^p(\Omega)} \le C\|\mathbf{u}\|_{L^p(\Omega)}.$$

Noting that the Lorentz space $L(p,q)(\Omega)$ can be expressed by the real interpolation between L^{p_1} and L^{p_2} with $p_1 < p < p_2$ (see [1, Corollary 7.27]), and then applying the interpolation theorem (see [1, section 7.23]) for any $\mathbf{u} \in L(p,q)(\Omega)$ we have

$$\|T_1 \mathbf{u}\|_{L(p,q)(\Omega)} \le C\|\mathbf{u}\|_{L(p,q)(\Omega)}.$$

Similarly, there exists $\mathbf{w}_{\mathbf{u}}$ satisfying the weak form of the system

$$\begin{cases} \operatorname{curl}\operatorname{curl} \mathbf{w}_{\mathbf{u}} = \operatorname{curl} \mathbf{u} & \text{in } \Omega, \\ \operatorname{div} \mathbf{w}_{\mathbf{u}} = 0 & \text{in } \Omega, \\ \nu \times \mathbf{w}_{\mathbf{u}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

That is

$$\int_\Omega \operatorname{curl}\mathbf{w_u}\cdot\operatorname{curl}\mathbf{\Psi}dx=\int_\Omega \mathbf{u}\cdot\operatorname{curl}\mathbf{\Psi}dx \quad \text{for any } \mathbf{\Psi}\in V(p,p)(\Omega).$$

Define the linear operator T_2 from $L^p(\Omega)$ to $L^p(\Omega)$ by

$$T_2:\mathbf{u}\to T_2\mathbf{u}=\nabla\mathbf{w_u}.$$

Then we can get the estimate

$$\|T_2\mathbf{u}\|_{L^p(\Omega)}\le C\|\mathbf{u}\|_{L^p(\Omega)}.$$

Applying the interpolation theorem, for any $\mathbf{u}\in L(p,q)(\Omega)$ we have

$$\|T_2\mathbf{u}\|_{L(p,q)(\Omega)}\le C\|\mathbf{u}\|_{L(p,q)(\Omega)}.$$

The estimate

$$\|\mathbf{h_u}\|_{L(p,q)(\Omega)}\le C\|\mathbf{u}\|_{L(p,q)(\Omega)}.$$

is directly from the fact that $\mathbf{h_u}$ can be expressed by

$$\mathbf{h_u}=\sum_{i=1}^N(\mathbf{u},\mathbf{h}_i)\mathbf{h}_i,$$

where $\mathbf{h}_i\in\mathbb{H}_1(\Omega)$. Thus we get the estimate (3.2) and the proof is now complete. □

Lemma 3.2. *Under the assumption in Lemma 3.1, for any vectors $\mathbf{u}\in L(3,\infty)(\Omega)$,*

Case 1. the decomposition (3.1) *holds and for $1\le p<\infty$ we have the estimate*

$$\|v\|_{L^p(\Omega)}+\|\mathbf{w}\|_{L^p(\Omega)}\le C(p,\Omega)\|\mathbf{u}\|_{L(3,\infty)(\Omega)}. \tag{3.7}$$

Case 2. the decomposition (3.3) *holds and for $1\le p<\infty$ we have the estimate*

$$\|\hat{v}\|_{L^p(\Omega)}+\|\hat{\mathbf{w}}\|_{L^p(\Omega)}\le C(p,\Omega)\|\mathbf{u}\|_{L(3,\infty)(\Omega)}. \tag{3.8}$$

Proof. It suffices to prove the estimate (3.7) and (3.8). We first prove (3.7). Applying the estimate in [13, Proposition 2.1] and from Lemma 2.1 (ii), we see that for p_1 with $p_1<3$ we have

$$\|v\|_{L^{p_1}(\Omega)}\le C(p_1,\Omega)\|\mathbf{u}\|_{L^{p_1}(\Omega)}\le C(p_1,\Omega)\|\mathbf{u}\|_{L(3,\infty)(\Omega)}.$$

Then applying Lemma 2.1 (ii) for ∇v and by the inequality (3.2), we have

$$\|\nabla v\|_{L^{p_1}(\Omega)}\le C(p_1,\Omega)\|\nabla v\|_{L(3,\infty)(\Omega)}\le C(p_1,\Omega)\|\mathbf{u}\|_{L(3,\infty)(\Omega)}.$$

Since W^{1,p_1} is continuously embedded into the space L^p with $p<3p_1/(3-p_1)$, for any $1<p<\infty$ we can choose suitable p_1 such that the following holds

$$\|v\|_{L^p(\Omega)}\le C(p,\Omega)\|v\|_{W^{1,p_1}(\Omega)}\le C(p,\Omega)\|\mathbf{u}\|_{L(3,\infty)(\Omega)}.$$

Other inequalities can be obtained by a similar way. □

We are now in the position to prove our main theorem.

Proof of Theorem 1.1. We first prove the case where $\nu \cdot \mathbf{u} = 0$. For $\mathbf{u} \in C^{1,\alpha}(\bar{\Omega})$ we take the decomposition

$$\mathbf{u} = \nabla p_{\mathbf{u}} + \operatorname{curl} \mathbf{w}_{\mathbf{u}} + \mathbf{h}_{\mathbf{u}},$$

where $p_{\mathbf{u}} \in C^{2,\alpha}(\Omega)$ satisfying

$$\Delta p_{\mathbf{u}} = \operatorname{div} \mathbf{u} \quad \text{in } \Omega, \qquad \frac{\partial p_{\mathbf{u}}}{\partial \nu} = 0 \quad \text{on } \partial\Omega; \tag{3.9}$$

$\mathbf{w}_{\mathbf{u}} \in V^{2,\alpha}_{\tau}(\Omega)$ with

$$V^{2,\alpha}_{\tau}(\Omega) \equiv \left\{ \mathbf{w} \in C^{2,\alpha}(\Omega) \ : \ \operatorname{div} \ \mathbf{w} = 0 \text{ in } \Omega,\ \nu \times \mathbf{w} = 0 \text{ on } \partial\Omega \right\}$$

and $\mathbf{w}_{\mathbf{u}}$ satisfying

$$\begin{cases} \operatorname{curl}\operatorname{curl} \mathbf{w}_u = \operatorname{curl} \mathbf{u} & \text{in } \Omega, \\ \operatorname{div} \mathbf{w}_u = 0 & \text{in } \Omega, \\ \nu \times \mathbf{w}_{\mathbf{u}} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.10}$$

and $\mathbf{h}_{\mathbf{u}} \in \mathbb{H}_1(\Omega)$.

Let $p_{\mathbf{u}}$ be defined by (3.9). For any vector $\mathbf{F} \in L(3,\infty)(\Omega)$, we use the decomposition (3.1) in Lemma 3.1 for the vector $\mathbf{F}$, then

$$(\nabla p_{\mathbf{u}}, \mathbf{F}) = (\nabla p_{\mathbf{u}}, \nabla p_{\mathbf{F}}) = (\operatorname{div} \mathbf{u}, p_{\mathbf{F}}).$$

From Lemma 2.2 and the above equality we see that

$$\|\nabla p_{\mathbf{u}}\|_{L(3/2,1)(\Omega)} \le C \sup_{\mathbf{F} \in L(3,\infty)(\Omega)} \frac{(\nabla p_{\mathbf{u}}, \mathbf{F})}{\|\mathbf{F}\|_{L(3,\infty)(\Omega)}} \le C \sup_{\mathbf{F} \in L(3,\infty)(\Omega)} \frac{(\operatorname{div} \mathbf{u}, p_{\mathbf{F}})}{\|\mathbf{F}\|_{L(3,\infty)(\Omega)}}. \tag{3.11}$$

The duality (2.6), for any $1/\mu_0 < p < \infty$, implies that

$$|(\operatorname{div} \mathbf{u}, p_{\mathbf{F}})| \le C(\mu_0, p, \Omega)(\|p_{\mathbf{F}}\|_{BMO(\Omega)} + \|p_{\mathbf{F}}\|_{L^p(\Omega)})\|\operatorname{div} \mathbf{u}\|_{\mathcal{H}_{\mu_0}(\Omega)}.$$

The inequalities (3.2) and (3.7) show that

$$\|\nabla p_{\mathbf{F}}\|_{L(3,\infty)(\Omega)} + \|p_{\mathbf{F}}\|_{L^p(\Omega)} \le C(p, \Omega)\|\mathbf{F}\|_{L(3,\infty)(\Omega)}.$$

Let p now be fixed. Then from the above two inequalities and by the inequality in Lemma 2.5, we have

$$|(\operatorname{div} \mathbf{u}, p_{\mathbf{F}})| \le C(\mu_0, \Omega)\|\mathbf{F}\|_{L(3,\infty)(\Omega)}\|\operatorname{div} \mathbf{u}\|_{\mathcal{H}_{\mu_0}(\Omega)}, \tag{3.12}$$

The inequalities (3.11) and (3.12) give

$$\|\nabla p_{\mathbf{u}}\|_{L(3/2,1)(\Omega)} \le C(\mu_0, \Omega)\|\operatorname{div} \mathbf{u}\|_{\mathcal{H}_{\mu_0}(\Omega)}. \tag{3.13}$$

Let $\mathbf{w}_{\mathbf{u}}$ be defined by (3.10). Using the decomposition (3.1) in Lemma 3.1 for the vector $\mathbf{\Phi}$, we have, by duality,

$$\|\operatorname{curl} \mathbf{w}_{\mathbf{u}}\|_{L(3/2,1)(\Omega)} \le C \sup_{\mathbf{\Phi} \in L(3,\infty)(\Omega)} \frac{(\operatorname{curl} \mathbf{w}_{\mathbf{u}}, \mathbf{\Phi})}{\|\mathbf{\Phi}\|_{L(3,\infty)(\Omega)}} \le C \sup_{\mathbf{\Phi} \in L(3,\infty)(\Omega)} \frac{(\operatorname{curl} \mathbf{u}, \mathbf{w}_{\mathbf{\Phi}})}{\|\mathbf{\Phi}\|_{L(3,\infty)(\Omega)}}.$$

Similar to the estimate for $\nabla p_{\mathbf{u}}$, we apply Lemma 2.6, Lemma 3.1 and Lemma 3.2 and get

$$\|\operatorname{curl} \mathbf{w}_{\mathbf{u}}\|_{L(3/2,1)(\Omega)} \le C(\mu_0, \Omega)\|\operatorname{curl} \mathbf{u}\|_{\mathcal{H}_{\mu_0}(\Omega)}. \tag{3.14}$$

Since $\mathbf{h_u}$ can be expressed by

$$\mathbf{h_u} = \sum_{i=1}^{N} (\mathbf{u}, \mathbf{h}_i)\mathbf{h}_i,$$

where $\mathbf{h}_i \in \mathbb{H}_1(\Omega)$, then combining the inequalities (3.13) and (3.14) we get (1.2).

We now prove the case where $\nu \times \mathbf{u} = 0$. From [12, Theorem 2.1] we see that for every $\hat{\mathbf{u}} \in C^{1,\alpha}(\Omega)$ there exists a decomposition

$$\hat{\mathbf{u}} = \nabla \hat{p}_{\mathbf{u}} + \operatorname{curl} \hat{\mathbf{w}}_{\mathbf{u}} + \hat{\mathbf{h}}_{\mathbf{u}}, \tag{3.15}$$

where $\hat{p}_{\mathbf{u}} \in C^{2,\alpha}(\Omega)$ satisfying

$$\Delta \hat{p}_{\mathbf{u}} = \operatorname{div} \hat{\mathbf{u}} \quad \text{in } \Omega, \qquad \hat{p}_{\mathbf{u}} = 0 \quad \text{on } \partial\Omega; \tag{3.16}$$

$\hat{\mathbf{w}}_{\mathbf{u}} \in X_n^{2,\alpha}(\Omega)$ with $X_n^{2,\alpha}$ defined by

$$X_n^{2,\alpha}(\Omega) \equiv \left\{ \mathbf{w} \in C^{2,\alpha}(\Omega) \ : \ \operatorname{div} \mathbf{w} = 0, \ \nu \cdot \mathbf{w} = 0 \text{ on } \partial\Omega \right\}$$

and $\hat{\mathbf{w}}_{\mathbf{u}}$ satisfying

$$\begin{cases} \operatorname{curl}\operatorname{curl} \hat{\mathbf{w}}_u = \operatorname{curl} \hat{\mathbf{u}} & \text{in } \Omega, \\ \operatorname{div} \hat{\mathbf{w}}_u = 0 & \text{in } \Omega, \\ \nu \times \operatorname{curl} \hat{\mathbf{w}}_{\mathbf{u}} = \nu \times \hat{\mathbf{u}} = 0 & \text{on } \partial\Omega, \\ \nu \cdot \hat{\mathbf{w}}_{\mathbf{u}} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.17}$$

and $\hat{\mathbf{h}}_{\mathbf{u}} \in \mathbb{H}_2(\Omega)$.

We shall estimate each term in (3.15). Let $\hat{p}_{\mathbf{u}}$ be defined by (3.16). Using the decomposition (3.3) in lemma 3.1 for the vector $\mathbf{A}$, we have

$$(\nabla \hat{p}_{\mathbf{u}}, \mathbf{A}) = (\nabla \hat{p}_{\mathbf{u}}, \nabla \hat{\mathbf{v}}_{\mathbf{A}}) = (\operatorname{div} \hat{\mathbf{u}}, \hat{\mathbf{v}}_{\mathbf{A}}).$$

By the estimate (3.4), we have

$$\|\nabla \hat{p}_{\mathbf{u}}\|_{L(3/2,1)(\Omega)} \le C \sup_{\mathbf{A} \in L(3,\infty)(\Omega), \mathbf{A} \neq 0} \frac{(\nabla \hat{p}_{\mathbf{u}}, \mathbf{A})}{\|\mathbf{A}\|_{L(3,\infty)(\Omega)}} \le C \sup_{\mathbf{A} \in L(3,\infty)(\Omega), \mathbf{A} \neq 0} \frac{(\operatorname{div} \hat{\mathbf{u}}, \hat{\mathbf{v}}_{\mathbf{A}})}{\|\mathbf{A}\|_{L(3,\infty)(\Omega)}}.$$

Similar to the estimate for $\nabla p_{\mathbf{u}}$, it follows that

$$\|\nabla \hat{p}_{\mathbf{u}}\|_{L(3/2,1)(\Omega)} \le C(\mu_0, \Omega) \|\operatorname{div} \hat{\mathbf{u}}\|_{\mathcal{H}(\Omega)}.$$

For the second term in the right side of (3.15), using the decomposition (3.3) in Lemma 3.1 for the vector $\mathbf{\Phi}$, we have

$$(\operatorname{curl} \hat{\mathbf{w}}_{\mathbf{u}}, \mathbf{\Phi}) = (\operatorname{curl} \hat{\mathbf{w}}_{\mathbf{u}}, \operatorname{curl} \hat{\mathbf{w}}_{\mathbf{\Phi}}) = (\operatorname{curl} \hat{\mathbf{u}}, \hat{\mathbf{w}}_{\mathbf{\Phi}}).$$

Similar to the estimate for $\operatorname{curl} \mathbf{w}_{\mathbf{u}}$ (see (3.14)), from the above equality we get

$$\|\operatorname{curl} \hat{\mathbf{w}}_{\mathbf{u}}\|_{L(3/2,1)(\Omega)} \le C \sup_{\mathbf{\Phi} \in L(3,\infty)(\Omega), \mathbf{\Phi} \neq 0} \frac{|(\operatorname{curl} \hat{\mathbf{w}}_{\mathbf{u}}, \mathbf{\Phi})|}{\|\mathbf{\Phi}\|_{L(3,\infty)(\Omega)}} \le C(\mu_0, \Omega) \|\operatorname{curl} \hat{\mathbf{u}}\|_{\mathcal{H}_{\mu_0}(\Omega)}.$$

Since we have

$$\hat{\mathbf{h}}_{\mathbf{u}} = \sum_{i=1}^{m} (\hat{\mathbf{u}}, \hat{\mathbf{h}}_i)\hat{\mathbf{h}}_i,$$

where $\hat{\mathbf{h}}_i \in \mathbb{H}_2(\Omega)$, combing the above two inequalities we get the estimate (1.2). $\square$

In view of the proof of Theorem 1.1, we can easily see that

Corollary 3.3. *Under the assumption in Theorem 1.1, if either*

(i) $\nu \cdot \mathbf{u} = 0$ *on* $\partial\Omega$ *and the first Betti number* $N = 0$*; or*

(ii) $\nu \times \mathbf{u} = 0$ *on* $\partial\Omega$ *and the second Betti number* $m = 0$,

then it holds that

$$\|\mathbf{u}\|_{L(3/2,1)(\Omega)} \leq C(\mu_0, \Omega) \left(\| \operatorname{div} \mathbf{u}\|_{\mathcal{H}_{\mu_0}(\Omega)} + \| \operatorname{curl} \mathbf{u}\|_{\mathcal{H}_{\mu_0}(\Omega)}\right),$$

where the constant C *depends only on* μ_0 *and the domain* Ω *but not on the vector* $\mathbf{u}$.

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