# ON FINITE TIME STOPPING PHENOMENA FOR ONE-HARMONIC MAP FLOW

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Abstract. For a very strong diffusion equation like total variation flow it is often observed that the solution stops at a steady state in a finite time. This phenomenon is called a finite time stopping or a finite time extinction if the steady state is zero. Such a phenomenon is also observed in one-harmonic map flow from an interval to a unit circle when initial data is piecewise constant. However, if the target manifold is a unit two-dimensional sphere, the finite time stopping may not occur. An explicit example is given in this paper.

## 1. Introduction

We are interested in phenomena of finite time extinction or stopping of solutions to a singular diffusion equation. For a diffusion equation like the heat equation, the solution tends to a steady state as time tends to infinity. However, it never reaches to a steady state in a finite time. If the diffusion is very strong like a total variation flow

$$u_t - \operatorname{div}(\nabla u/|\nabla u|) = 0 \quad \text{in} \quad \mathbf{R}^n \times (0,\infty) \tag{1}$$

or $p$-Laplace fast diffusion flow

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\, \nabla u\right) = 0 \quad \text{in} \quad \mathbf{R}^n \times (0,\infty)$$

with $1 < p < 2n/(n+1)$, it is well-known that the solution tends to zero (which is a steady solution) in finite time for a smooth initial data under some boundary conditions like periodic or Dirichlet conditions [12], [5, Chapter VII]. This phenomenon is called a finite time extinction or if one emphasizes that the solution does not move after the extinction time, one calls it a finite time stopping of the solution. This phenomenon is quite common for a singular diffusion equation where the diffusion effect is very strong for a particular slope at least. In fact, a finite time stopping is also proved for fourth order total variation flow [12].

In this note we consider the one-harmonic map flow with values in a unit circle $S^1$ or a unit sphere $S^2$. It is formally a gradient flow of a total variation energy with value constraints in $S^{N-1} \subset \mathbf{R}^N$. Its explicit form

$$\mathbf{u}_t - \operatorname{div}(\nabla \mathbf{u}/|\nabla \mathbf{u}|) - \mathbf{u}|\nabla \mathbf{u}| = 0, \tag{2}$$

where $\mathbf{u}$ is a mapping from a domain $\Omega$ in $\mathbf{R}^m$ to $S^{N-1}$. As we discuss later, the notion of a solution itself is nontrivial because of singularity at $\nabla \mathbf{u} = 0$. In [11] we proposed a notion of solution when $\Omega$ is a bounded open interval $I$ and initial data is a piecewise

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constant. In fact, it was proved in [11] that there is a unique global in time solution under the Dirichlet condition. When the target space is $S^1$, the solution stops in finite time [11] so finite time stopping phenomenon is observed for one-harmonic map flow.

The goal of this note is to give an explicit counterexample for a finite stopping phenomenon when the target manifold is

$$S^2 = \left\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

In other words, there is a direction where the diffusion is not very strong. We consider (2) in a bounded open interval $I = (0, L)$ and consider a piecewise constant initial datum

$$\mathbf{u}_0(x) = \mathbf{a}1_{(0,\ell_1)} + \mathbf{h}_0 1_{(\ell_1,\ell_2)} + \mathbf{b}1_{(\ell_2,L)} \tag{3}$$

with a given division $0 < \ell_1 < \ell_2 < L$ of $I$, when $1_J$ is the characteristic function of an internal $J$, i.e. $1_J(x) = 1$ if $x \in J$ and otherwise $1_J(x) = 0$. We impose the Dirichlet boundary condition

$$\mathbf{u}(0, t) = \mathbf{a}, \quad \mathbf{u}(L, t) = \mathbf{b}, \quad t > 0 \tag{4}$$

**Theorem 1.1.** *Assume that* $\mathbf{a}$ *and* $\mathbf{b}$ *are points on the equator of* $S^2$*, i.e.* $\mathbf{a}, \mathbf{b} \in S^2 \cap \{x_3 = 0\}$ *and that* $\mathbf{a}$ *and* $\mathbf{b}$ *are symmetric with respect to* $x_1$*-axis and stay in the region* $\{x_1 > 0\}$*. Assume that* $\mathbf{h}_0 \in S^2$ *is a point on* $x_1x_3$*-plane but not on the equator. Then the solution of (2) – (4) does not stop in a finite time. More precisely, the solution* $\mathbf{u}$ *is of the form*

$$\mathbf{u}(x, t) = \mathbf{a}1_{(0,\ell_1)} + \mathbf{h}(t)1_{(\ell_1,\ell_2)} + \mathbf{b}1_{(\ell_2,L)} \tag{5}$$

*and satisfies the property that* $\mathbf{h}(t)$ *converges to* $(1, 0, 0)$ *as* $t$ *tends to infinity but* $\mathbf{h}(t)$ *never reaches to* $(1, 0, 0)$ *in finite time.*

The proof is easy because the system is reduced to a system of an ordinary differential equation. We shall give an explicit proof in the next section.

We now discuss several well-posedness results for (2). Even for the total variation flow (1) one cannot interpret a solution in a classical sense because the equation has a nonlocal character at the place where $\nabla u = 0$. Fortunately, this problem can be rigorously formulated as a gradient flow of a convex energy, which is in this case a total variation energy; see [16] for an intuitive explanation and [1] for a thorough study of this problem. The unique existence is guaranteed by the theory of maximal monotone operators [17], [3]. However, if one imposes the constraint $|\mathbf{u}| = 1$, the problem is not viewed as a gradient flow of a convex energy. So the theory of maximal monotone operators does not apply to the equation (2).

In [11] we proposed a notion of solution for a piecewise constant initial data in an interval to a sphere and established a unique global-in-time solution in a class of piecewise constant functions. Here the jump is measured in a metric of the ambient space where the sphere is embedded not in a geodesic distance of the sphere. See also [14], [15] for further development.

In [10] Kashima, Yamazaki and the first author constructed a local-in-time solution when initial energy is small and initial data is small but it is not clear whether their solution is unique. As shown in [13] a classical solution may breakdown in a finite time for a map from a disk to $S^2$. Actually, bubbling phenomena occurs in rotational symmetry [4], [9].

More recently, a notion of $BV$-solution was introduced in [2] and [6] for the Neumann problem for (2). However, as pointed out by [7], [8], their argument seems to have a flaw at least where the solution has a real jump. The difficult issue is how to interpret the term $\mathbf{u}|\nabla \mathbf{u}|$ where $\mathbf{u}$ is in $BV$. A new notion of a solution is introduced in [7], [8]. When the target sphere is one-dimensional, it is shown in [7] that a global-in-time solution exists uniquely if the target space is a semicircle and that it exists if the target space is a circle and the initial datum has no angular jumps larger than $\pi$. Their source space is a general multi-dimensional bounded domain and the Neumann condition is imposed. They extended this result when the target sphere has higher dimensions.

Note that they measured the jumps in a geodesic distance of the sphere not in a metric of the ambient space where the sphere is embedded. So, our solution in Theorem 1.1 is not their solution. However, since both metrics agree in infinitesimal sense, in other words, the metric derivative is the same, the same initial data gives a counter example of finite time stopping for a solution in the sense of [7], [8].

The problem (2) with the homogeneous Neumann boundary condition was proposed as a tool to de-noise two-dimensional image or optical flow where the target sphere is one dimensional [19]. It is also proposed to de-noise color images by smoothing the chromaticity [18]. In their case the dimension of the target sphere is two and also the image is restricted in an octant of the sphere.

## 2. Non-finite time stopping phenomenon

We shall give a proof of our main theorem. Since the solution $\mathbf{u}$ is of the form (5), the total variation $\varphi$ of $\mathbf{u}$ is of the form

$$\varphi[\mathbf{u}] = \int_0^L |\nabla \mathbf{u}| = |\mathbf{h}(t) - \mathbf{a}| + |\mathbf{h}(t) - \mathbf{b}|$$

Thus its subdifferential ($L^2$ sense) is of the form

$$\partial\varphi\left(\mathbf{u}(t)\right) = \frac{1}{c}\left(\frac{\mathbf{h}(t)-\mathbf{a}}{|\mathbf{h}(t)-\mathbf{a}|} + \frac{\mathbf{h}(t)-\mathbf{b}}{|\mathbf{h}(t)-\mathbf{b}|}\right)$$

with $c = x_2 - x_1$ so the equation (2) is reduced to

$$\frac{d\mathbf{h}}{dt}(t) = -P_{\mathbf{u}(t)}\left(\partial\varphi\left(\mathbf{u}(t)\right)\right) = -\frac{1}{c}P_{\mathbf{u}(t)}\left(\frac{\mathbf{h}(t)-\mathbf{a}}{|\mathbf{h}(t)-\mathbf{a}|} + \frac{\mathbf{h}(t)-\mathbf{b}}{|\mathbf{h}(t)-\mathbf{b}|}\right) \tag{6}$$

where $P_v$ is the orthogonal projection of $\mathbf{R}^3$ to the tangent space of $S^2$ at $v \in S^2$, i.e. $P_v(w) = w - (v \cdot w)v$.

We shall write it by coordinate. We write $\mathbf{a} = (a_1, a_2, 0),\ \mathbf{b} = (a_1, -a_2, 0)$ with $a_1^2 + a_2^2 = 1,\ a_1 > 0$. If we write $\mathbf{h}(t) = (h_1(t), h_2(t), h_3(t))$, it is clear that $h_2(t) = 0$ because of the symmetry. Thus

$$\begin{aligned}\mathbf{h}(t) - \mathbf{a} &= \left(h_1(t) - a_1, -a_2, h_3(t)\right),\\ \mathbf{h}(t) - \mathbf{b} &= \left(h_1(t) - a_1, a_2, h_3(t)\right).\end{aligned}$$

Since

$$|\mathbf{h}(t) - \mathbf{a}| = |\mathbf{h}(t) - \mathbf{b}| = \sqrt{2 - 2a_1 h_1(t)},$$

we have

$$\frac{\mathbf{h}(t)-\mathbf{a}}{|\mathbf{h}(t)-\mathbf{a}|}+\frac{\mathbf{h}(t)-\mathbf{b}}{|\mathbf{h}(t)-\mathbf{b}|}=\frac{\sqrt{2}}{\sqrt{1-a_1h_1(t)}}\left(h_1(t)-a_1,\ 0,\ h_3(t)\right).$$

Since $P_v(w)=w-(v\cdot w)v$, one observes that

$$\begin{aligned}
cP_{\mathbf{u}(t)}\left(\partial\varphi\left(\mathbf{u}(t)\right)\right) &= \sqrt{2\left(1-a_1h_1(t)\right)^{-1}}\big\{\left(h_1(t)-a_1,0,h_3(t)\right)\\
&\quad -\big(\left(h_1(t),0,h_3(t)\right)\cdot\left(h_1(t)-a_1,0,h_3(t)\right)\big)\left(h_1(t),0,h_3(t)\right)\big\}\\
&= \sqrt{2\left(1-a_1h_1(t)\right)^{-1}}\big(\left(h_1(t)-a_1\right)-h_1(t)\left(1-a_1h_1(t)\right),0,h_3(t)-h_3(t)\left(1-a,h_1(t)\right)\big)\\
&= \sqrt{2\left(1-a_1h_1(t)\right)^{-1}}\left(a_1\left(h_1(t)^2-1\right),0,a_1h_1(t)h_3(t)\right)
\end{aligned}$$

where we invoke $(h_1(t),0,h_3(t))\cdot(h_1(t)-a_1,0,h_3(t))=1-a_1h_1$ by $h_1^2+h_3^2=1$. Thus the equation (6) is reduced to

$$(\dot h_1,\dot h_3)=-\frac{\sqrt{2}\,a_1}{c\sqrt{1-a_1h_1(t)}}\left(h_1(t)^2-1,h_1(t)h_3(t)\right) \tag{7}$$

where $\dot h_i=dh_i/dt$. We shall analyze the system (7). Since $h_1\leq 1$, we observe from (7) that $\dot h_1\geq 0$. If one sets $y=h_1(t)$, the equation (7) together with the monotonicity of $h_1$ yields

$$\frac{1}{2}c_1(1-y^2)\leq\dot y\leq\frac{1}{2}c_2(1-y^2) \tag{8}$$

with

$$c_1=\frac{2\sqrt{2}\,a_1}{c\sqrt{1-a_1h_{01}}},\quad c_2=\frac{2\sqrt{2}\,a_1}{c\sqrt{1-a_1}}$$

since $-1<h_1(0)=h_{01}<1$. Divide (8) by $1-y^2$ and integrate over $(0,t)$ to get

$$c_1t\leq\log\frac{1+y}{1-y}-\log\frac{1+h_{01}}{1-h_{01}}\leq c_2t$$

or

$$ce^{c_1t}\leq\frac{1+y}{1-y}\leq ce^{c_2t}\quad\text{with}\quad c=\frac{1+h_{01}}{1-h_{01}}>0.$$

In other words,

$$\frac{ce^{c_1t}-1}{ce^{c_1t}+1}\leq y=h_1(t)\leq\frac{ce^{c_2t}-1}{ce^{c_2t}+1}.$$

Since $c_1,c_2>0$ this inequality implies $h_1(t)<1$ for all $t>0$ and $h_1(t)\to 1$ as $t\to\infty$. Thus, $\dot h_1$ does not equal zero in finite time by (7) so $\mathbf{u}$ does not stop in finite time. (Since $h_3^2=1-h_1^2$, $\mathbf{h}$ converges to $(1,0,0)$ as $t\to\infty$ exponentially fast but does not stop in finite.)

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